Calculus III Math 153 Learning Objectives and Questions

Sandi Xhumari

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Introduction

This document is a list of TILT-ed (Transparency In Teaching and Learning) learning objectives and several corresponding questions for Math 153 Calculus III challenges, based on Hoffman OER textbook content [1]. Many questions or ideas for questions were taken from the resources available at MathQUEST [3] and the Cornell GoodQuestions project [4]. The purpose is for both instructors and students to be transparent on the exact goals of the course. CORE Learning Objectives are especially important for student success, and they will be tested on the Final Part 1.

Learning Objectives

1. **Vectors.** I understand what vectors are, their various representations in 2 or 3 dimensions, and can perform vector addition and scalar multiplication in different contexts. If two or more forces are applied to an object, it is useful to know how to add them so that we can determine the direction and amount of movement. Vectors can also be used to model playing car racing simulation games such as Racetrack, or to represent force applied on an object among many other applications.

   Criteria for Success: I can

   - represent vectors in 2 or 3 dimensions as ordered pairs/triples, as arrows, and by specifying magnitude and direction
   - give "physical" examples and illustrate the distinction between vector and scalar quantities
   - add/subtract two vectors both geometrically and symbolically
   - multiply a vector given both geometrically and symbolically by a scalar
   - use properties of vector operations to manipulate vector expressions and apply them to motion questions

   Sample questions:

   - Let $P = (6, 0, -6)$ be a point, and suppose we are given the plane $x - 2y - 2z = 5$.
     (a) Pick any point A on the plane $x - 2y - 2z = 5$, and find the component form of the vector $\overrightarrow{AP}$.
     (b) Let O be the origin. Write the vector $\overrightarrow{AP}$ in terms of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OP}$ using vector operations.

- In the racetrack game, a player moves from point $A = (1, 3)$ to the point $B = (4, -1)$.
  (a) Sketch a graph displaying the vector $\overrightarrow{AB}$ representing this move, and then find the component form of the vector $\overrightarrow{AB}$.
  (b) What are the magnitude and angle representing the vector $\overrightarrow{AB}$?
  (c) Assuming O is the origin, show geometrically and explain how the vector $2\overrightarrow{OA} - 2\overrightarrow{OP}$ is related to the vector $\overrightarrow{AB}$.
- A plane is flying with a speed of 500 mph northeast at an altitude of 10,000 miles shortly after taking off.
  (a) Sketch its velocity vector $\vec{v}$, and find its coordinate representation.
  (b) Suppose that suddenly the pilot realizes that the plane does not have enough fuel for the trip and turns around driving with a speed of 600 mph. Sketch the new velocity vector $\vec{w}$, and express it as a scalar multiple of $\vec{v}$.
- Assume you throw a ball upwards with a force F_1 of magnitude 20 lbs, while wind is blowing North-East parallel to the surface of the Earth with a force F_2 of magnitude 10 lbs. Find the resulting force acting on the ball in component form, as well as its magnitude. Assume that all vectors here have 3 dimensions.

2. **Dot and Cross Products.** I can calculate and interpret the dot and cross products in various contexts. The dot product allows us to find the projection of one vector onto another, angles between vectors, and the correlation of two variables among other things. The cross product can be used to find a vector perpendicular to two others, areas, and gives us a way to find the torque when rotating a screw, among other useful applications.

Criteria for Success: I can

- calculate the dot product of two vectors given by various representations both geometrically and symbolically
- use the dot product to solve various application questions such as finding the angle between two vectors, the projection of one onto the other, and work
- calculate the cross product of two vectors given by various representations both geometrically and symbolically
- use the cross product to solve various application questions such as finding a vector perpendicular to two given vectors, finding the area of the parallelogram determined by two vectors in 3 dimensions, and torque

Sample questions:

- Let $\vec{v} = \langle 5, 12 \rangle$, be the velocity of a train, and suppose that a gust of wind blows with velocity $\vec{u} = \langle -3, 4 \rangle$. What is the amount of speed the train gains due to the gust of wind? Note that the train is restricted to move on its tracks in the direction of $\vec{v}$.
- A force is given by the vector $F = \langle 2, 3 \rangle$ and moves an object from the point $(1, 3)$ to the point $(5, 9)$. Find the work done and sketch a picture of the movement.
- Let $P = (6, 0, -6)$ be a point, and suppose we are given the plane $x - 2y - 2z = 5$. Use the dot and/or cross product to find the distance of this point to the given plane.
- Let PQR be the triangle formed by the points P= $(3, 3, -3)$, Q= $(-1, 3, -1)$, and R= $(5, -3, 0)$. Find its area using the dot and/or cross product.
- Use the dot and/or cross product to find the equation of the plane passing through the origin, and containing the vectors $\vec{u} = \langle 5, 12, 0 \rangle$ and $\vec{v} = \langle -3, 0, 5 \rangle$.
- Consider the following two force vectors acting at the end of a 10 inch wrench. Which vector produces the larger torque?

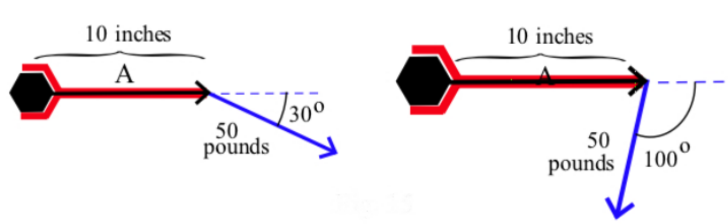


Figure 1: Two torques with specific lengths and angles. Source: [1]

3. **Sequences [CORE].** I know various ways to represent sequences and extract information from them. Excel rows/columns of numbers encoding a variety of information, average temperatures on Earth for consecutive years, and many other phenomena in life can be broken up into steps and visualized as a sequence of numbers whose behavior can shed light on those phenomena.

Criteria for Success: I can

- use the exact formula or recurrence relation of a sequence to list its terms
- find the exact formula and/or recurrence relation for a sequence
- recognize and analyze arithmetic and geometric sequences
- determine if a sequence is monotonic or not, and bounded or unbounded

Sample questions:

- Consider the sequence given by $a_n = 5\,(-2)^n$ for $n \geq 0$.
  (a) Write down the first four terms in the sequence, and find its recurrence relation.
  (b) Select all the properties that are true for this sequence, and support your answer with an explanation: arithmetic, geometric, bounded, monotonic.
- Consider the sequence $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$, where $a_1 = \frac{2}{3}$.
  (a) Write down the next two terms in the sequence, and then find both a recurrence relation and exact formula describing it.
  (b) For each of the following properties of the sequence explain why they're true or false: arithmetic, geometric, bounded, monotonic.
- Consider the sequence given by the recurrence relation: $a_0 = 5$, and $a_n = \frac{2}{3}a_{n-1}$ for $n \geq 1$.
  (a) Write down the first four terms in the sequence, and find its exact formula.
  (b) For each of the following properties of the sequence explain why they're true or false: arithmetic, geometric, bounded, monotonic.
- Consider the sequence $-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \ldots$, where $a_1 = -\frac{2}{3}$.
  (a) Write down the next two terms in the sequence, and find a recurrence relation describing it.
  (b) For each of the following properties of the sequence explain why they're true or false: arithmetic, geometric, bounded, monotonic.
- Suppose that a_n is the amount of dollars you have in a bank account after the n-th year of opening the account. Suppose you open the account with \$100 in this account, i.e., $a_0 = 100$. Each year thereafter, you get an additional \$0.2 interest.
  (a) Write down the first four terms in the sequence a_n, and find its exact formula.
  (b) Is this bank account given by simple or compound interest? What is the yearly interest rate? Explain.
  (c) How long would you have to wait to reach \$1000?

4. **Sequences Convergence [CORE].** I can tell when a sequence converges or diverges. The ultimate behavior of a sequence modeling disease spread such as COVID-19, average yearly temperatures on Earth impacting climate change, or any other real-life phenomena is often the most important information we need to know.

   Criteria for Success: I can

   - use Calculus 1 techniques such as L'Hospital's Rule and Squeeze Theorem to find limits of sequences
   - determine convergence or divergence of a sequence numerically and graphically
   - given a bound ($\epsilon > 0$) for the difference between the terms of a convergent sequence and its limit, determine the minimal index that achieves that bound

   Sample questions:

   - Consider the sequence given by $a_n = \dfrac{n}{n+1}$ for $n \geq 0$.
     (a) Look at its graph and the numerical pattern of its terms to determine its limit as n tends to ∞. Listing terms and/or a sketch of a graph here are appropriate justifications.
     (b) Confirm your observation by using Calculus 1 limit properties and theorems. Note that heuristic arguments derived by looking at the degrees of the numerator and denominator, while very valuable for intuition, are not good enough justification here.
     (c) Let L be the limit you found above. Find the minimal index N so that for $n \geq N$, we have $|a_n - L| < 0.02$.
   - Consider the sequence given by $a_n = 5\,(-2)^n$ for $n \geq 0$. Is it convergent or divergent? Carefully explain why.
   - Consider the sequence $\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{8}{27}, \ldots$, where $a_1 = \dfrac{2}{3}$. Is it convergent or divergent? Carefully explain why.
   - When COVID-19 started being tracked at the beginning of 2019, each day the number of cases recorded outside of mainland China increased by 15-25%. Assuming the number of cases increased by exactly 20% each day starting at day 0 with 1 person infected, find an exact formula for this sequence c_n and determine how long it would take to infect one billion people (1,000,000,000)? If no precautions are taken, is it reasonable that this trend will continue in the long run, or will c_n definitely converge?

5. **Vectors and Sequences Challenge.** I can work with vectors and sequences creatively in new situations that require a deep understanding of them. Linear Algebra is a class dedicated completely to vectors, where you may also encounter abstract vector spaces. Abstract vector spaces have been further generalized to modules among other spaces for instance, and there's so much more to be discovered/invented. Sequences can sometimes follow unexpected patterns such as "A Curious Pattern Indeed", or are connected to unsolved problems such as the "Collatz Conjecture".

   Criteria for Success: I can solve conceptual questions related to vectors that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

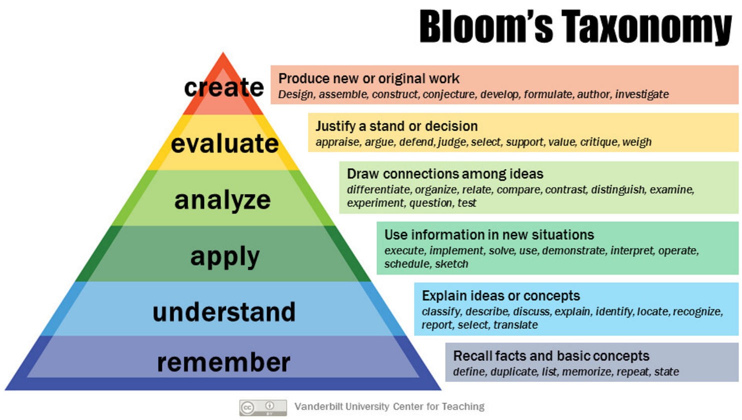


Figure 2: Bloom's Taxonomy. Source: [2]

   Sample questions:

   - Find two vectors that have a dot product of 10, and explain what the dot product means geometrically about these two vectors.
   - Find 2 nonzero vectors that when added give us $\vec{i} = \langle 1, 0\rangle$, and when subtracted give us $\vec{j} = \langle 0, 1\rangle$.
   - Find two vectors that have a cross product of $\langle 0, 3, 0\rangle$, and explain what the cross product means geometrically about these two vectors.
   - True or False. It is possible for a divergent sequence to be increasing and bounded.
   - True or False. The sequence of digits of π $(3, 1, 4, \dots)$ converges to π.

6. **Series [CORE].** I can extract information about series and their corresponding sequences of partial sums. The definition of convergence or divergence of a series depends on the convergence or divergence of the corresponding sequence of partial sums. The decimal representation of real numbers is a familiar example of a series where each digit of the number is a term of the series. Many application processes such as modeling annuities or saving plans also rely on series.

Criteria for Success: I can

- convert a series between expanded notation and sigma notation
- determine the sequence of partial sums for a given series
- find exact formulas for sequences of partial sums of arithmetic, geometric, and telescoping series
- determine the sum of a series from the definition as the limit of its partial sum
- use the formulas for both the finite and infinite sum of geometric or telescoping series

Sample questions:

- Consider the series $1 - 1 + 1 - 1 + \dots$.
  (a) Write the series using sigma notation.
  (b) Write down the first four terms in the sequence of partial sums, and then find the exact formula for the N-th partial sum.

  First Four Terms: Exact formula:

  (c) Use the exact formula to determine if the series converges or diverges directly from the limit definition, without using any tests of convergence/divergence. If it converges, find the value it converges to.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
  (a) Write the series in expanded notation.
  (b) Write down the first three terms in its sequence of partial sums, and find an exact formula for the N-th partial sum..

  First Three Terms: Exact Formula:

  (c) Use the exact formula to determine if the series converges or diverges directly from the limit definition, without using any tests of convergence/divergence. If it converges, find the value it converges to.
- Consider the series $\sum_{k=-1}^{\infty} (5k + 1)$.
  (a) Write the series in expanded notation.
  (b) Find the sum S_{100} of its first 100 terms, and then find an exact formula for the N-th partial sum.

  $S_{100} =$ Exact Formula:

(c) Use the exact formula to determine if the series converges or diverges directly from the limit definition, without using any tests of convergence/diveregence. If it converges, find the value it converges to.

- Consider the periodic number $0.\overline{123}$.

  (a) Write it as a series of fractions with powers of 10 in the denominator.

  (b) Write the series using sigma notation.

  (c) Find an exact formula for the N-th partial sum.

  (d) Use the exact formula to find the representation of $0.\overline{123}$ as a fraction.

7. **Series Tests [CORE].** I can determine convergence or divergence of a series by selecting an appropriate convergence test and applying it. Series are often used by computers to determine an answer up to an appropriate precision, so it's crucial to know if the series in question converges or not.

   Criteria for Success: I can

   - select and apply an appropriate convergence test: nth-term (divergence), geometric series, integral, comparison, limit comparison, p-series, alternating series, ratio, root, or absolute convergence
   - distinguish between an absolutely convergent, conditionally convergent, or divergent series.

   Sample questions:

   - Clearly state the name of the convergence/divergence tests you're using, and check that their conditions are satisfied. The series $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ is
     (a) absolutely convergent
     (b) conditionally convergent
     (c) divergent.
   - Clearly state the name of the convergence/divergence tests you're using, and check that their conditions are satisfied. The series $\sum_{k=1}^{\infty} \frac{-k}{k^5+2}$ is
     (a) absolutely convergent
     (b) conditionally convergent
     (c) divergent.
   - Clearly state the name of the convergence/divergence tests you're using, and check that their conditions are satisfied. The series $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ is
     (a) absolutely convergent
     (b) conditionally convergent
     (c) divergent.
   - Clearly state the name of the convergence/divergence tests you're using, and check that their conditions are satisfied. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{3k-2}}$ is
     (a) absolutely convergent
     (b) conditionally convergent
     (c) divergent.

8. **Series Approximation.** I can use partial sums to estimate the sum of a convergent series, and find error bounds where appropriate. Correctly bounding the error of a calculation can make the difference between a spaceship landing on the moon safely or falling apart in the process.

Criteria for Success: I can

- estimate the sum of a convergent series
- find error bounds using integrals
- find error bounds of alternating series
- determine how many terms must be used in order to have an approximation with an error no greater than a given value

Sample questions:

- Consider the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k}$. How many terms must be added in order to have an approximation with an error no greater than 10^{-12}? Check the conditions of any theorem/test you use.
- Use $\sum_{k=1}^{15} \frac{1}{k^2} \approx 1.58$ and the resulting error bounds to find a lower and upper bound for the value of the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Check the conditions of any theorem/test you use. You may use technology to compute a finite sum, and find the decimal expansion of the final answer. **Hint:** Conditions of any test just need to be eventually true, i.e., there is an a such that for $k >= a$ the conditions hold. It is quite interesting that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.645$.

  Solution: First, we note that the series in question is not alternating, so we have to use integrals to bound the error. We should always start by checking the conditions of the Integral Test. Let $f(x) = \frac{1}{x^2}$.

  (a) Eventually Positive. $f(x)$ is positive for all $x \geq 1$ since squaring a positive number and taking its reciprocal gives a positive number.

  (b) Eventually Continuous. $f(x)$ is a rational function, and therefore continuous on its domain, which includes $x \geq 1$.

  (c) Eventually Decreasing. $f(x)$ is decreasing since when x gets larger, the denominator gets larger, which makes the whole fraction smaller.

  (d) Eventually Convergent Integral. For any $a > 0$, we have

  $$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b x^{-2}dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_a^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{a} = \frac{1}{a},$$

  which means that the integral converges.

Therefore, by the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. Note that we could tell this immediately from the p-test as well, but we need the conditions of the integral test to be satisfied for the following argument to work.

The Remainder Estimate for the Integral Test Theorem gives us that the exact error(remainder) $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{15} \frac{1}{k^2} = \sum_{k=16}^{\infty} \frac{1}{k^2}$ is bounded by improper integrals as follows

$$\int_{16}^{\infty} \frac{1}{x^2} dx \leq \sum_{k=16}^{\infty} \frac{1}{k^2} \leq \int_{15}^{\infty} \frac{1}{x^2} dx.$$

From the work in the last condition of the integral test, we can easily evaluate the integrals above, and get

$$\frac{1}{16} \leq \sum_{k=16}^{\infty} \frac{1}{k^2} \leq \frac{1}{15}.$$

Adding $\sum_{k=1}^{15} \frac{1}{k^2} \approx 1.58$ to all sides of this inequality, we get the required lower and upper bounds:

$$1.58 + \frac{1}{16} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1.58 + \frac{1}{15}.$$

Therefore, we have

$$1.6425 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1.64\bar{6},$$

which seems accurate based on the hint.

- Find the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to within .001 of its exact value. Check the conditions of any theorem/test you use. You may use technology to compute a finite sum, and find the decimal expansion of the final answer. **Hint:** Conditions of any test just need to be eventually true, i.e., there is an a such that for $k >= a$ the conditions hold.

Solution: First, we note that the series in question is not alternating, so we have to use integrals to bound the error. We should always start by checking the conditions of the Integral Test. Let $f(x) = \frac{1}{x^2}$.

(a) Eventually Positive. $f(x)$ is positive for all $x \geq 1$ since squaring a positive number and taking its reciprocal gives a positive number.

(b) Eventually Continuous. $f(x)$ is a rational function, and therefore continuous on its domain, which includes $x \geq 1$.

(c) Eventually Decreasing. $f(x)$ is decreasing since when x gets larger, the denominator gets larger, which makes the whole fraction smaller.

(d) Eventually Convergent Integral. For any $a > 0$, we have

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b x^{-2}dx = \lim_{b\to\infty} -\frac{1}{x}\bigg|_a^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{a} = \frac{1}{a},$$

which means that the integral converges.

Therefore, by the Integral Test, $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent. Note that we could tell this immediately from the p-test as well, but we need the contidions of the intgeral test to be satisfied for the following argument to work.

Using the procedure outlined in this website, we have that

$$\int_N^{N+1}\frac{1}{x^2}dx = -\frac{1}{x}\bigg|_N^{N+1} = -\frac{1}{N+1} + \frac{1}{N} = \frac{1}{N(N+1)} \leq .001.$$

Rearranging terms, this inequality is equivalent to $N^2 + N - 1000 \geq 0$. Solving for the equality using the quadratic formula, we get two solutions $N = \frac{-1\pm\sqrt{1+4000}}{2} \approx 31.13$ and $N = \frac{-1\pm\sqrt{1+4000}}{2} \approx -32.13$. Since the parabola is concave up and we're only interested in natural numbers $N \geq 1$, we have that the inequality is true for $N \geq 32$. From the work in the last condition of the integral test, and (by desmos) $\sum_{k=1}^{32}\frac{1}{k^2} \approx 1.61416726283$ (we could actually round this to 4 decimals considering the error we're concerned is at most .001), we have

$$\begin{gathered}
\int_{33}^\infty \frac{1}{x^2}dx \leq \sum_{k=33}^{\infty}\frac{1}{k^2} \leq \int_{32}^\infty \frac{1}{x^2}dx \\
\sum_{k=1}^{32}\frac{1}{k^2} + \frac{1}{33} \leq \sum_{k=1}^{32}\frac{1}{k^2} + \sum_{k=33}^{\infty}\frac{1}{k^2} \leq \sum_{k=1}^{32}\frac{1}{k^2} + \frac{1}{32} \\
1.64447029313 \leq \sum_{k=1}^{\infty}\frac{1}{k^2} \leq 1.64541726283
\end{gathered}$$

Taking the average of the upper and lower bound we arrive at the approximation:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} \approx 1.64494377798,$$

which is guaranteed to be within 0.001 of the exact value of the series. In fact, we know that the exact value of this series is $\frac{\pi^2}{6} \approx 1.64493406685$, which we can check that it matches up to the first 4 decimals with our approximation (better than the expected 3 decimals!).

9. **Power Series [CORE].** I can manipulate and analyze power series. Power series are mainly used to represent functions, store an infinite sequence of numbers (in which case they're called generating functions), and solve recurrence relations (important in computer science).

Criteria for Success: I can

- determine the radius and interval of convergence of a power series
- add, subtract, differentiate, integrate power series

Sample questions:

- Consider the power series $f(x) = \sum_{k=0}^{\infty} \frac{2}{3^{k+1}}(x-1)^k$. Find the interval and radius of convergence of its derivative $f'(x)$.
- A power series centered at 3 of some function $f(x)$ is $\sum_{k=a}^{\infty} a_k(x-3)^k$, for some integer a, and real number constants a_k for $k \geq a$. Assuming that it converges at $x = 5$ and diverges at $x = 0$, answer the following questions. No need for explanations here, but a picture of the interval may help you out.
  (a) For what values of x it is guaranteed that the series converges?
  (b) For what values of x it is guaranteed that the diverges?
  (c) How big and small could the radius of convergence possibly be?
- Consider the power series $f(x) = \sum_{k=0}^{\infty} \frac{2}{3^{k+1}}(x-1)^k$. Find the interval and radius of convergence of $\int_1^x f(t)dt$.

10. **Power Series Representations [CORE].** I can represent functions as power series and vice versa represent power series as functions in order to solve various application questions. Many functions can be analyzed by using their power series representations. Calculators and computer software are programmed to find good approximations of values of many complicated functions using for the most part their power series representations.

Criteria for Success: I can

- use geometric series, Taylor series, binomial series, and formulas derived from them to go back and forth between a function and a power series representation
- find a series exact sum by plugging into the appropriate power series representation
- find a Taylor polynomial of a specified degree to approximate a given function
- solve limit or integral questions using power series

Sample questions:

- Consider the power series $\sum_{k=0}^{\infty} \frac{2}{3^{k+1}}(x-1)^k$. Write down the first four terms of it, and find a rational function representation of it.
- Find a representation of the function $\frac{10x}{1+5x^2}$ as a power series centered at 0, and then apply it to determine a power series for
  $\ln|1+5x^2| = \int_0^x \frac{10t}{1+5t^2}dt.$
- Find a representation of the function $f(x) = \frac{\ln(1+x)}{x}$ as a power series centered at 0, and then apply it to determine $\lim_{x\to 0} \frac{\ln(1+x)}{x}$.
- Suppose g is a function which has continuous derivatives, and that $g(0) = 8$, $g'(0) = -5$, $g''(0) = -9$ and $g'''(0) = 18$. What is the Taylor polynomial of degree 2 for g, centered at 0?

  $T_2(x) =$
- Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{16}\right)^{n-1}}{2n+1}$.
- Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^{5n+1}}{n!}$.
- Find the binomial series for $\sqrt{9-x}$, and write down the first four terms of it.
- Find the probability that a trait following the standard normal distribution is within one standard deviation of the mean by computing the following integral using the Maclaurin series of the inside function.

  $\int_{-1}^{1} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt =$

11. **Taylor Series Approximation.** I can use partial sums of Taylor series to estimate the sum of a corresponding series, and find error bounds. Bounding error is crucial to achieve a guaranteed accuracy in any application.

Criteria for Success: I can

- estimate the sum of a convergent series
- find error bounds using the Taylor Remainder Theorem
- determine the degree of the Taylor polynomial that must be used in order to have an approximation with an error no greater than a given value

Sample questions:

- Suppose g is a function which has continuous derivatives, and that $g(0) = 8$, $g'(0) = -5$, $g''(0) = -9$ and $g'''(0) = 18$. In addition, we know that its Taylor polynomial of degree 2 centered at 0 is $T_2(x) = 8 - 5x - 4.5x^2$, and all the derivatives of g are in absolute value less than 20 for any x-values. Use $T_2(x)$ to approximate $g(0.2)$, and find a bound for the error using the Taylor Remainder Theorem.

  $g(0.2) \approx$ Error $= |g(0.2) - T_2(0.2)| \leq$

- What degree Taylor polynomial of e^x centered at 0 is needed to get an error of less than 10^{-8} for computing e? You may use an electronic device to solve for the degree.
- Approximate $\sqrt{7} = \sqrt{9-2} = 3\sqrt{1-2/9}$ using the 3rd degree Taylor polynomial of $f(x) = 3\sqrt{1-x}$, and then determine the bound of the error from the Taylor Remainder Theorem assuming that we already know that $|f^{(4)}(x)| \leq 6.778$ for all $x \in [0, 2/9]$.

  Solution: By the Binomial Theorem, we get

  $$f(x) = 3(1-x)^{1/2} = 3\sum_{k=0}^{\infty}\binom{1/2}{k}(-x)^k = 3\left(1 - (1/2)x + \frac{(1/2)(-1/2)}{2}x^2 - \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots\right)$$

  Thus we have

  $$\sqrt{7} = f(2/9) \approx 3\left(1 - \frac{1}{9} - \frac{1}{2\cdot 9^2} - \frac{1}{2\cdot 9^3}\right) \approx 2.6461$$

  and since

  $$\text{Error} = |f(2/9) - T_3(2/9)| \leq \frac{6.778}{4!}|2/9|^4 \approx 0.00068871615.$$

  Note that all these calculations involve whole number exponents, addition, subtraction, multiplication, and division, which can be computed quite fast by a computer or calculator, and therefore enable square root calculations to any desired precision quite well.

12. **Series Challenge.** I can work with series and power series creatively in new situations that require a deep understanding of them. The Riemann Hypothesis is related to understanding the zeros of a function defined initially as a series, and it's perhaps the hardest and most mysterious question in all of mathematics to date. What does it feel like to invent math? Power Series on their own can be used to represent functions we already know (Taylor Series), or represent totally new functions. Using power series, your instructor, Sandi Xhumari, was able to simplify certain number theory results, as well as prove and conjecture new ones in his disertation (check out how many times the words power series shows up in the abstract).

    Criteria for Success: I can solve conceptual questions related to series and power series that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

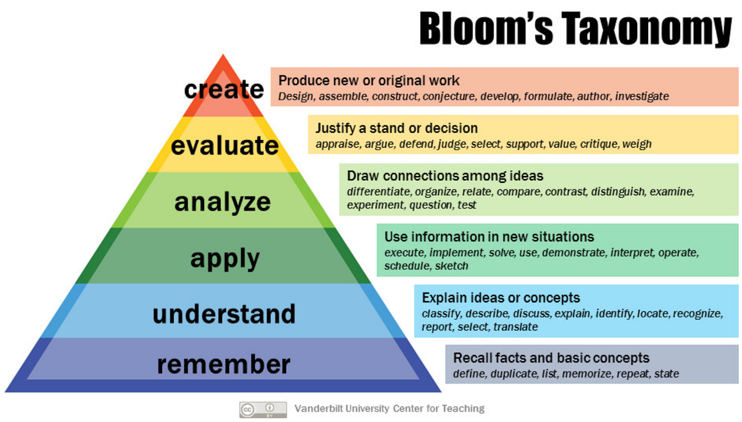


Figure 3: Bloom's Taxonomy. Source: [2]

Sample questions:

- Select all that apply. The series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2^k}+\frac{1}{k}\right)$
    (a) diverges by the divergence test
    (b) converges by the integral test
    (c) converges by the p-series test
    (d) converges by using both the p-series and and geometric series tests.
    (e) none of the above
- True or False. The p-series test can be derived using the ratio test.
- True or False. If $\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=0}^{\infty} a_n$ must converge since we keep adding terms that are getting closer and closer to 0.
- Give an example of a power series with interval of convergence $[-1, 1]$.
- True or False. A function $f(x)$ and its Taylor polynomials $T_n(x)$ are generally not equal to each other, but $\lim_{n\to\infty} T_n(x) = f(x)$ for any value of x.
- True or False. Any Taylor Series of x^3 centered at any point is equal to x^3 for all values of x.

13. **Polar Coordinates.** I am fluent with polar coordinates and can do Calculus with them. We are so used to using Cartesian coordinates (x, y) in mathematics, even though they are not always the best way to describe locations of objects or graphs of curves. Being able to switch to polar coordinates provides a better alternate point of view when flying an airplane or sailing a boat for instance. Polar equations can be useful for modeling movement of objects, and finding the lengths and areas that are more easily expressed in polar coordinates.

Criteria for Success: I can

- describe points and graphs in the plane using polar coordinates
- convert points and graphs between polar and rectangular coordinates
- use derivatives to calculate slopes and rates of change of polar equations
- find intersection points between polar equations
- find arclengths and areas using integrals in polar coordinates

Sample questions:

- SOS! You've just received a distress signal from a ship located at point A on your radar screen below.

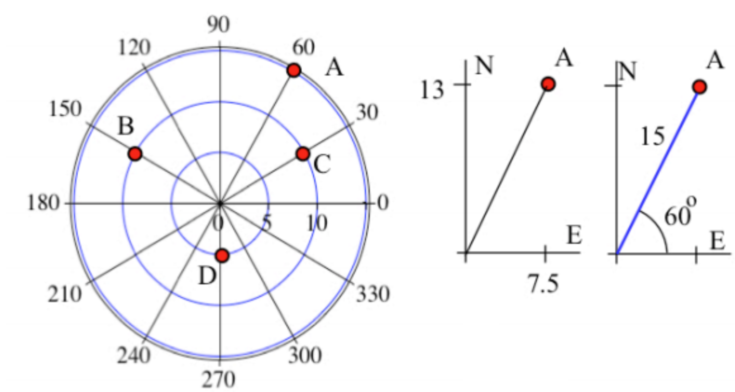


Figure 4: Four points A, B, C, and D displayed on a polar grid. Source: [1]

  (a) Describe its location to your captain. Use both Polar and Cartesian coordinates, and explain which coordinate system is more suitable for the situation.
  (b) Your captain realizes that a rescue team at point C on the radar screen is located closer to point A, so asks them to make the rescue instead. Find both the polar and rectangular equations for the straight line path that the emergency team at point C needs to use to get to point A.
  (c) Write down the appropriate integral in polar coordinates that can be used to find the arclength of the straight path that the rescue team needs to follow from point C to point A. You do not need to solve the integral.

- Consider the following table.

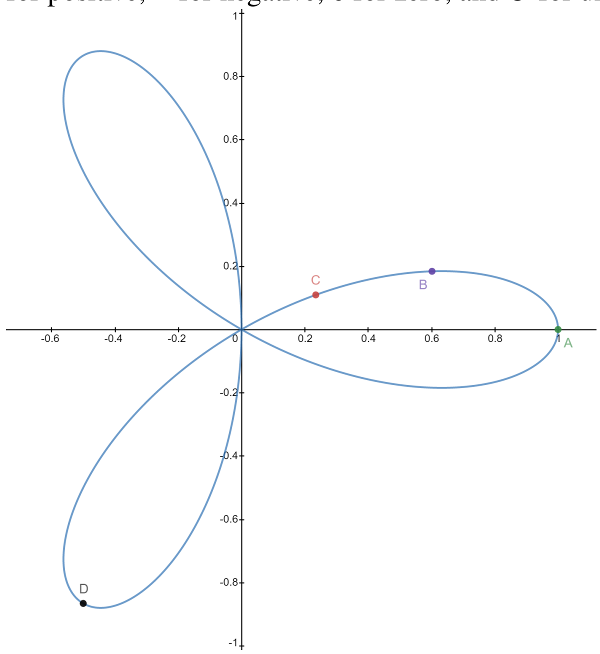
Polar	Cartesian
	$3x + 2y = 1$
	$(-3, -4)$
$\theta = \frac{\pi}{6}$	
$r = 2\sin(\theta)$	

(a) Complete the missing items in the table of examples above to convert between Polar and Cartesian points or equations. Whenever possible write the equations in the form $y = f(x)$, $x = f(y)$, $r = f(\theta)$, or $\theta = f(r)$ for some function f. Check that you got the correct answer using Desmos.

(b) Write down the appropriate integral in polar coordinates that can be used to find the area of half of the circle $r = 2\sin(\theta)$. You do not need to solve the integral.

- Consider the graph of $r = \cos(3\theta)$. Check it out on Desmos here: `https://www.desmos.com/calculator/hftgcykkyj`.

  (a) Fill out the following table of derivatives for the given points on the following curve using + for positive, − for negative, 0 for zero, and U for undefined.



Point	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				

  (b) For the point A above find two polar representations: one that uses a negative radius, and the other that uses a negative angle.

- Consider the polar equation $r = 5\cos(3\theta)$.

  (a) Find the area within this graph from $\theta = 0$ to $\theta = \pi/6$. Write down the appropriate integral without evaluating it.

  (b) How is the area within one leaf of this graph related to the area you found in part (a)?

  (c) Find the arclength of the perimeter of one leaf of this graph. Write down the appropriate integral without evaluating it.

- Consider the region inside $r = 5\cos(\theta)$, but outside $r = 3$.

  (a) Find the polar and cartesian coordinates of the two intersection points between the two curves.

  (b) Find the area of the region inside $r = 5\cos(\theta)$, but outside $r = 3$. Write down the appropriate integral without evaluating it.

14. **Parametric Equations.** I can extract information from parametric equations and their graphs, and model motion using them. Among other things, parametric equations are used to model motion of characters in an animated movie, or track location when using a GPS over time.

Criteria for Success: I can

- model linear, rational and other motion using parametric equations
- convert between parametric equations and rectangular equations
- use derivatives to calculate slopes of tangent lines of parametric equations
- find and interpret various rates of change (horizontal, vertical, speed) related to the movement of an object through a parametric equation
- find distance measured along a curve (arclength) and area enclosed by parametric equations

Sample questions:

- Consider the ellipse given by the equation $(x-1)^2 + 4y^2 = 1$.
  (a) Find a parametric equation modeling a particle moving on the top half of this ellipse so that at $t = 0$ it's located at $(2, 0)$, and at $t = 1$ it's located at $(0, 0)$.
  (b) Find the length of the trip from $t = 0$ to $t = 1$ using the above parametric equation. Setup but don't solve the definite integral.
  (c) Find the area of the top half of the ellipse using the above parametric equation. Setup but don't solve the definite integral.
- Consider the line segment connecting the points $(3, -2)$ and $(-4, 6)$.
  (a) Find a parametric equation modeling a particle moving on the line segment starting at $(3, -2)$ at $t = 0$, and ending at $(-4, 6)$ at $t = 1$. Assume time is measured in seconds, and coordinates in inches.
  (b) Convert the above parametric equation to a rectangular/Cartesian equation.
  (c) Compute the following quantities at $t = 0.5$:

  (i) $x(t) =$
  (ii) $y(t) =$
  (iii) $x'(t) =$
  (iv) $y'(t) =$
  (v) $\dfrac{dy}{dx} =$
  (vi) Speed =

  (d) Interpret the meaning of the above rates of change related to the movement of the particle at $t = 0.5$. Assume time is measured in seconds, and coordinates in inches.
- A space rocket is moving with coordinates $x(t) = -2t^3$, $y(t) = -1 + 2t$, where t is measured in hours and the coordinates in miles.
  (a) Eliminate the parameter t to find the Cartesian equation in the form $x = f(y)$
  (b) Compute the following quantities at $t = 1$:

(i) $x(t) =$

(ii) $y(t) =$

(iii) $x'(t) =$

(iv) $y'(t) =$

(v) $\dfrac{dy}{dx} =$

(vi) Speed =

(c) Interpret the meaning of the above rates of change related to the movement of the particle at $t = 1$.

(d) Suppose that at $t = 1$ hour, the space rocket runs out of fuel and no force is acting on it. Find the Cartesian equation of the path it's going to move about after $t = 1$ hour.

15. **Conic Sections.** I know the definitions of conic sections, their reflecting properties, eccentricity, and I can analyze them through both Cartesian and Polar coordinates. Telescopes, microscopes, satellite dishes, movements of cosmic objects, shadows of a ball, planar cuts of a cone, and much more are related through equations of conic sections. Being able to use one equation in polar coordinates for all conic sections reveals a nice relation between them, which can be useful in modeling the movement of bodies in space.

Criteria for Success: I can

- describe conic sections as loci of points related to foci points
- apply reflection properties of conics in physical and geometric contexts
- find the eccentricity of conic sections and understand its meaning
- analyze conic sections using polar equations

Sample questions:

- Consider a mirror in the shape of the conic section given by the equation $y = 3x^2 + 5$.
  (a) Find the x and y intercept(s), the focus(foci), directrix (if applicable), asymptotes (if applicable), and eccentricity.
  (b) Suppose you are standing at a focus of the given conic section in the dark. Towards which point of the parabola should you point the flashlight in order to illuminate the point $(1, 20)$?
- A ball rolls at a constant speed following the arrows first towards the focus at A, and then bounces off of the hyperbola.

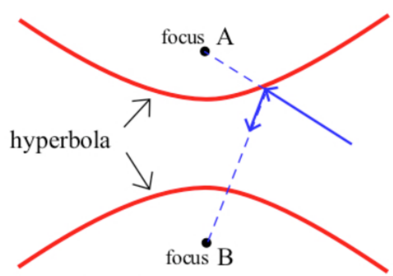


Figure 5: Four points A, B, C, and D displayed on a polar grid. Source: [1]

  (a) Sketch the path of the ball for the first 5 bounces it makes off of the hyperbola, and describe it in relation to the other focus. What does the path of the ball look like after "a long time?"
  (b) Come up with an equation of a hyperbola that approximately looks like the above picture, and then find the x and y intercept(s), the focus(foci), directrix (if applicable), asymptotes (if applicable), and eccentricity.
- Consider a mirror in the shape of the conic section given by the equation $r = \dfrac{8}{3 + 6\cos(\theta)}$.
  (a) Find the x and y intercept(s), the focus(foci), directrix (if applicable), asymptotes (if applicable), and eccentricity.
  (b) Suppose you are standing at a focus of the given conic section in the dark, and shoot a light ray vertically straight up. Draw the path of the light ray, and describe it in relation to the other focus.

16. **Polar/Parametric and Conic Sections Challenge.** I can use polar, parametric and conic sections creatively in new situations that require a deep understanding of them. Wrapping a wave function modeling sound (such as $y = \sin(x) + \cos(x)$) around a circle instead of on top of the x-axis can be done by viewing it as a polar function (such as $r = \sin(\theta) + \cos(\theta)$), and can then be leveraged to filter out noise (click "Fourier Transform" for more details). Parametric equations are crucial for modeling movement of objects such as spaceships in space with respect to time. Conic sections are curves with so many interesting properties that have been leveraged for glasses, telescopes, microscopes, and satellite dishes among many other objects.

    Criteria for Success: I can solve conceptual questions related to Polar and Parametric equations that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

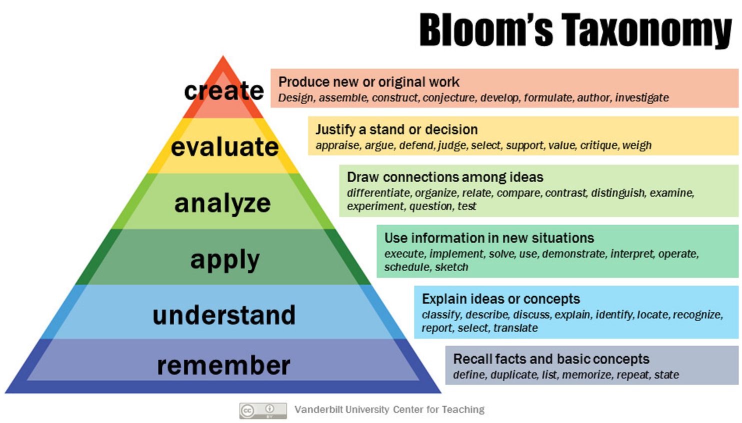


Figure 6: Bloom's Taxonomy. Source: [2]

Sample questions:

- The area of a sector of angle 30 degrees in a circle of radius r is
  (a) $\frac{\text{area of circle}}{2\pi} \cdot 30$
  (b) $\frac{1}{2}r^2(30)$
  (c) $\frac{\pi r^2}{30}$
  (d) some of the above
  (e) none of the above
- Consider the path given by the entire line $y = 2x + 1$. A parametric equation for this line is given by
  (a) $(5t, 2t + 1)$
  (b) $(\cos(t), 2\cos(t) + 1)$
  (c) $(t^4 - 1, 2t^4 - 1)$
  (d) none of the above
- Show that for any point $P = (c, d)$ on the ellipse given by $\frac{x^2}{9} + 4y^2 = 1$, the sum of the distances from this points to the two foci is 6, namely $F_1P + F_2P = 6$, where F_1 and F_2 are the two foci of this ellipse.

References

[1] Dale Hoffman. *Contemporary Calculus*. Washington State Colleges. Retrieved July 15th, 2024, from https://www.contemporarycalculus.com/dh/Calculus_all/Calculus_all.html

[2] Armstrong, P. (2010). *Bloom's Taxonomy*. Vanderbilt University Center for Teaching. Retrieved July 15th, 2024, from https://cft.vanderbilt.edu/guides-sub-pages/blooms-taxonomy/.

[3] MathQUEST: Math Questions to Engage Students (2006-2009) and MathVote: Teaching Mathematics with Classroom Voting (2010-2013). Retrieved July 15th, 2024, from http://mathquest.carroll.edu/.

[4] GoodQuestions Project, Cornell University. Retrieved July 15th, 2024, from https://pi.math.cornell.edu/~GoodQuestions/index.html.