Linear Algebra Math 208 Learning Objectives and Questions

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Introduction

This document is a list of TILT-ed (Transparency In Teaching and Learning) challenges (groups of learning objectives) and several corresponding questions for Math 208 Linear Algebra. The purpose is for both instructors and students to be transparent on the exact goals of the course. CORE Learning Objectives are especially important for student success, and they will be tested on the Final Part 1. Part of this content was generated with the assistance of ChatGPT, a large language model developed by OpenAI. The sample questions were inspired mainly by exercises in the OER textbooks by Hoffman [2], Austin [3], Clontz et al. [4], and the set of activities and exercises by Wawro et al. [5].

Challenge 1: Linear System Exploration

Use matrix methods to set up, solve, and analyze linear systems for applied and general situations. In the modern world of technology, we often come across questions that boil down to solving a system of equations with anywhere from one to millions of equations and as many or more variables, and linear algebra provides the main mathematical tools we use for such a task.

Learning Objectives: I can

1.1 [CORE] model real-world scenarios using various linear algebraic structures, including systems of linear equations, vector equations, matrix equations, and augmented matrices. Additionally, I can reinterpret these scenarios as intersections of lines, planes, or hyperplanes, as linear combinations of vectors, or as transformations mapping input vectors to output vectors, and formulate the appropriate questions from these viewpoints.

1.2 explain whether a matrix is in Reduced Row Echelon Form (RREF) or not, and row reduce a matrix of numbers or variables until it's in RREF.

1.3 [CORE] find and explain the meaning of the solution(s) in the context of the question situation given from the Reduced Row Echelon Form (RREF) of the corresponding augmented matrix or the graphs of the equations in the system of linear equations.

1.4 compute the inverse of a matrix of numbers or variables through row reduction, and use it to solve a matrix equation.

Sample questions:

(1) Suppose we're astronauts in outer space ($\mathbb{R}^3$) and are given the 2 modes of transportation $\vec{v_1} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. We're at the origin and trying to reach $\vec{b} = \begin{pmatrix} 8 \\ 7.5 \\ 9 \end{pmatrix}$. Let x and y be the amount of time in hours we travel on each mode of transportation $\vec{v_1}$ and $\vec{v_2}$ respectively, where negative values mean that we travel in the opposite direction for that amount of time. For example, if $x = -2$, then we're using $\vec{v_1}$ to travel 2 hours in the direction of $-\vec{v_1}$, and therefore move by $-2\vec{v_1} = \begin{pmatrix} -8 \\ -16 \\ -4 \end{pmatrix}$.

(a) Represent this situation with a linear system of equations, a vector equation, a matrix equation $A\vec{x} = \vec{b}$, and an augmented matrix $[A|\vec{b}]$. Rephrase the question accordingly from these perspectives.

(b) The augmented matrix $[A|\vec{b}]$ that you formed in the previous part is not in Reduced Row Echelon Form (RREF). Explain why.

(c) If it is not possible to reach $\vec{b}$, then we should try to get to at least anywhere on the vertical line from $\vec{b}$, that is $\vec{c} = \begin{pmatrix} 8 \\ 7.5 \\ d \end{pmatrix}$, for some real number d. Note that $\vec{c}$ has the same entries as $\vec{b}$, except for the third entry of 9 is replaced by the variable d in order to allow for vertical movement. Show all steps of row reducing matrix $[A|\vec{c}]$ to RREF.

(d) From the RREF of the augmented matrix $[A|\vec{c}]$ obtained in previous part, determine for which values of d is the system is consistent or inconsistent. When consistent, explain the meaning of the solution(s) in the context of the transportation scenario. Are we able to reach $\vec{b}$?

(e) Confirm the above findings for reaching $\vec{b}$ by graphing the 3 equations of lines you found in part (a) within one coordinate system. Describe in your own words what do the graphs look like when the solution is consistent.

(f) Compute the inverse of matrix A (if possible) through row reduction. If the inverse exists, use it to solve the matrix equation $A\vec{x} = \vec{b}$. If it doesn't exist, explain why, and then slightly modify the matrix A to a 2 by 2 matrix A' and vector $\vec{b}$ to a 2 dimensional vector $\vec{b'}$ so that A' has an inverse and find the inverse and solution to the question.

Solution.

(a) In this situation, we're trying to figure out how long we need to travel on each mode of transportation to reach a certain destination, so it's best to start from the vector equation perspective.

Vector Equation:

$$x\begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7.5 \\ 9 \end{bmatrix}$$

Linear System of Equations: We can get this by looking at each component of the above vector equation (i.e., by rows).

Equation 1: $4x + 2y = 8$
Equation 2: $8x + 3y = 7.5$
Equation 3: $2x + y = 9$

Each of the above equations is geometrically a line in 2D space ($\mathbb{R}^2$). The solution then is the common intersection (x, y) of these lines.

Matrix Equation: $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 4 & 2 \\ 8 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 7.5 \\ 9 \end{bmatrix}$$

The left-hand side is a compact way to write the linear combinations of the columns of the matrix A, but it now allows us to think of the question from a function perspective $f(\vec{x}) = \vec{y}$: The matrix A takes the vector $\vec{x}$ as input and spits out a vector $\vec{y}$ through matrix multiplication. The question then is for what input $\vec{x}$ do we get the total revenue $\vec{y} = \vec{b}$ as an output?

Augmented Matrix:

$$\left[\begin{array}{cc|c} 4 & 2 & 8 \\ 8 & 3 & 7.5 \\ 2 & 1 & 9 \end{array}\right]$$

(b) The augmented matrix is not in RREF form since the entries below the main diagonal are not 0. (there's many other correct responses here).

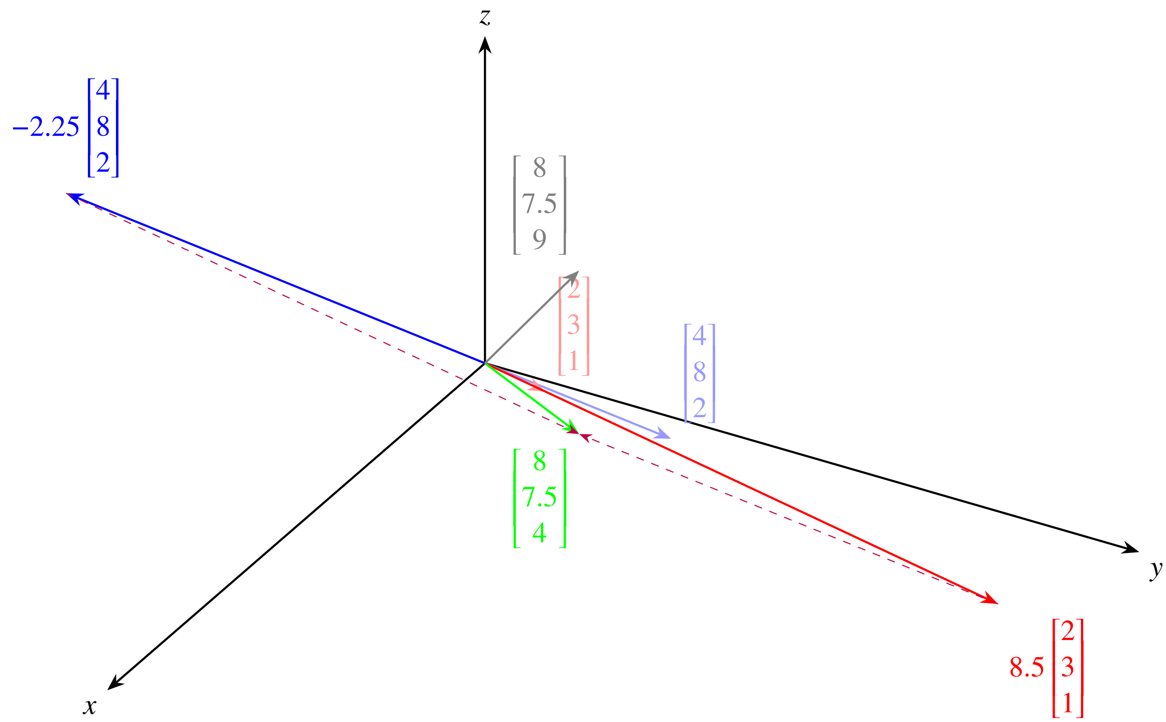
(c) **RREF Steps for Augmented Matrix:**

$$\left[\begin{array}{cc|c} 4 & 2 & 8 \\ 8 & 3 & 7.5 \\ 2 & 1 & d \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 4 & 2 & 8 \\ 0 & -1 & -8.5 \\ 2 & 1 & d \end{array}\right] \xrightarrow{\frac{1}{4}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 1/2 & 2 \\ 0 & -1 & -8.5 \\ 2 & 1 & d \end{array}\right]$$

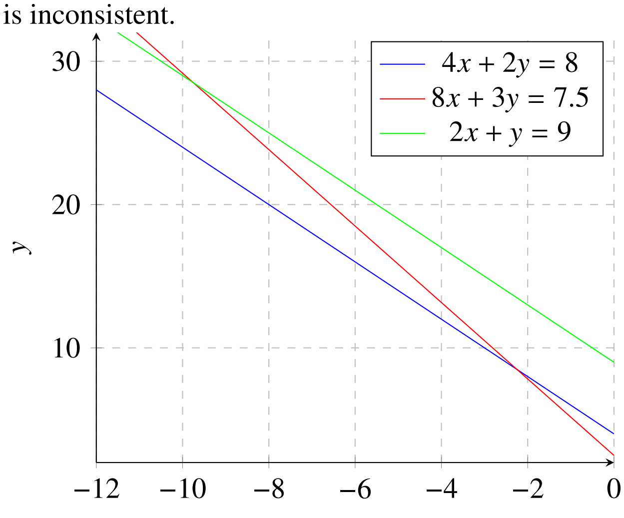
$$\xrightarrow[-R_2 \to R_2]{R_3 - 2R_1 \to R_3} \left[\begin{array}{cc|c} 1 & 1/2 & 2 \\ 0 & 1 & 8.5 \\ 0 & 0 & d-4 \end{array}\right] \xrightarrow{R_1 - (1/2)R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -2.25 \\ 0 & 1 & 8.5 \\ 0 & 0 & d-4 \end{array}\right]$$

(d) With the augmented matrix in RREF, we have $x = -2.25$, $y = 8.5$, and $0 = d - 4$. This tells us that the system is consistent with unique solution $x = -2.25$, $y = 8.5$ precisely when $d = 4$, and otherwise it's inconsistent. When $d = 4$, we have that, we can reach $\vec{c}$ if we travel for a quarter past 2 hours in the opposite direction of $\vec{v_1}$ and 8 hours and a half in the direction of $\vec{v_2}$. When $d \neq 4$ (including $d = 9$), we can never reach $\vec{c}$. In particular, we cannot reach $\vec{b}$ as it lies outside of the plane where $\vec{v_1}$ and $\vec{v_2}$ can travel on (in other words $\vec{b}$ is not in the span of vectors $\vec{v_1}$ and $\vec{v_2}$). This is verified by looking below at the graphical representation of the vector equation of the unique solution:

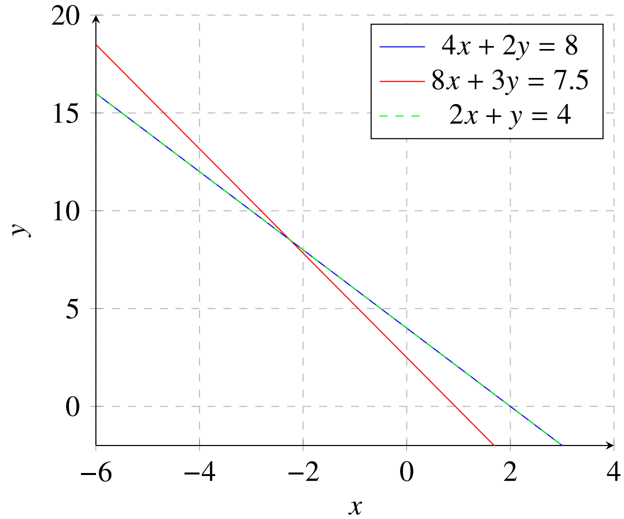
$$-2.25 \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} + 8.5 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7.5 \\ 4 \end{bmatrix}$$



(e) From the following graph, we can see the lines of the first (blue) and third (green) equations are parallel, which means there's no possible intersection point of the 3 lines, and therefore the system is inconsistent.



We can see the the unique solution when $d = 4$ is reached when the first (blue) and third (green) equations match (algebraically we can also see that dividing the first equation by 2 gives the third



one).

(f) To find the inverse, we start with the coefficient matrix and the identity matrix:

$$\left[\begin{array}{cc|ccc} 4 & 2 & 1 & 0 & 0 \\ 8 & 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right].$$

Right away we notice that the coefficient matrix A is not even a square matrix, and therefore it does not have an inverse. Row reducing this augmented matrix (following similar steps as in the above row reduction), we would get:

$$\left[\begin{array}{cc|ccc} 1 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 1 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{31} & x_{32} & x_{33} \end{array}\right],$$

for some numbers x_{ij}. The matrix on the right would have been our inverse if the left side were the identity matrix. We could have used this inverse to multiply with the original destination vector $\vec{b}$ or $\vec{c}$ to get the solution. (Food for thought: what would multiplying on the left of the matrix equation $A\vec{x} = \vec{c}$ by the right matrix with x_{ij} entries do to it?)

Here's the modified matrix A' and the row reduction process that yields its inverse:

$$\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 8 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_2-2R_1]{} \left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[\frac{1}{4}R_1]{} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/4 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[-R_2\to R_2]{R_1+\frac{1}{2}R_2\to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -3/4 & 1/2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Therefore we have:

$$A'\vec{x'} = \vec{b'} \iff A'^{-1}A'\vec{x'} = A'^{-1}\vec{b'} \iff \vec{x'} = A'^{-1}\vec{b'} = \begin{bmatrix} -3/4 & 1/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 7.5 \end{bmatrix} = \begin{bmatrix} -2.25 \\ 8.5 \end{bmatrix}.$$

(2) You recently started your own small company that sells three types of products: A, B, and C. Over the course of the first few days, the following sales data are recorded:

- Day 1: 4 units of A, 3 units of B, and 2 units of C were sold for a total revenue of \$200.
- Day 2: 2 units of A, 4 units of B, and 3 units of C were sold for a total revenue of \$170.

- Day 3: 8 units of A, 11 unit of B, and 8 unit of C were sold for a total revenue of \$540.

Let a, b, and c be the prices of the products A, B, and C respectively. You are pretty sure that you picked prices that are multiples of 10, but somehow misplaced the spreadsheet with the exact prices of the products sold. Could you figure out the exact prices based on the above information?

(a) Represent this situation using a linear system of equations, vector equation, matrix equation, and an augmented matrix. Rephrase the question accordingly from these perspectives.

(b) Row reduce the augmented matrix to Reduced Row Echelon Form (RREF). Show all your steps.

(c) What is the cost of each unit of product A, B, and C? Interpret this solution in the context of the question situation.

(d) Compute the inverse of the matrix obtained from the coefficients of products A, B, and C using row reduction, and use it to confirm the prices of each unit of product A, B, and C. If the inverse doesn't exist, then explain why.

Solution.

(a) **Linear system of equations**:

$$\begin{aligned} 4a + 3b + 2c &= 200 \\ 2a + 4b + 3c &= 170 \\ 8a + 11b + 8c &= 540 \end{aligned}$$

Each of the above equations represents the total revenue in two different ways, and it's geometrically a plane in 3D space ($\mathbb{R}^3$). The solution then is the common intersection of these planes.

Vector equation:

$$\begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} a + \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix} b + \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} c = \begin{bmatrix} 200 \\ 170 \\ 540 \end{bmatrix}$$

The 4 vectors above represent the number of units sold for products A, B, and C each day respectively, as well as the total revenues for their sale. From this perspective, we're asking what linear combinations of the product vectors gives the total revenue. Thinking of each of the first 3 vectors as modes of transportation in a 3D space, and the constants a, b, and c as the amount we use each mode of transportation, the question translates to finding a way to use these 3 modes of transportation to reach the total revenue vector.

Matrix equation: $M\vec{x} = \vec{y}$

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 3 \\ 8 & 11 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 200 \\ 170 \\ 540 \end{bmatrix}$$

The left-hand side is a compact way to write the linear combinations of the columns of the matrix M, but it now allows us to think of the question from a function perspective $f(\vec{x}) = \vec{(y)}$: The matrix M takes the vector $\vec{x}$ as input and spits out a vector $\vec{y}$ through matrix multiplication. The question then is for what input $\vec{x}$ do we get the total revenue $\vec{y}$ as an output?

Augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 200 \\ 2 & 4 & 3 & 170 \\ 8 & 11 & 8 & 540 \end{array}\right]$$

This is a compact way to express all of the above perspectives, and go through the row reduction process to compute a solution. It's interesting to think through what each row operation does to the previous perspectives, and in particular why they don't change the solution set of the question.

(b) We start by row reducing the augmented matrix into Reduced Row Echelon Form (RREF). The sequence of operations and corresponding matrices are:

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 200 \\ 2 & 4 & 3 & 170 \\ 8 & 11 & 8 & 540 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 4 & 3 & 170 \\ 4 & 3 & 2 & 200 \\ 8 & 11 & 8 & 540 \end{array}\right] \xrightarrow[R_3-4R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 2 & 4 & 3 & 170 \\ 0 & -5 & -4 & -140 \\ 0 & -5 & -4 & 140 \end{array}\right] \xrightarrow{R_3-R_2\to R_3}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 170 \\ 0 & -5 & -4 & -140 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{5}R_2\to R_2]{\frac{1}{2}R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 85 \\ 0 & 1 & \frac{4}{5} & 28 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{10} & 29 \\ 0 & 1 & \frac{4}{5} & 28 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) The RREF matrix shows that we have two leading variables, a and b, and one free variable, c. Our variables can be represented as:

$$a = 29 + \frac{1}{10}c, \quad b = 28 - \frac{4}{5}c.$$

This means the cost of each unit of product A, B, and C is dependent on the free variable c. In the context of the question situation, this implies that there is an infinite number of possible cost combinations for products A, B, and C, depending on the value of c. We're unable to tell the exact costs for A, B, and C, but we know the relations between them, which can be used to estimate a reasonable cost. Since they're all positive for instance, we know that $0 < c < 28$, $0 < b < 28$, and $29 < a < 31.8$. Since we know the answers are multiples of 10, we can deduce that $a = 30$, as that's the only multiple of 10 between 29 and 31.8. Knowing a allows us to find $b = 20$, and $c = 10$ through the above equations.

It may be helpful to also write the solution set in vector form in other occasions:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 29 \\ 28 \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{1}{10} \\ -\frac{4}{5} \\ 1 \end{bmatrix}.$$

In this vector equation, the first vector represents a specific solution to the system, and the second vector scaled by c represents the "direction" in which we can move to get all possible solutions (it is also a solution to the homogeneous equation $A\vec{x} = \vec{0}$, i.e. it's in the null space of the matrix A). In the context of the question, this means there are infinitely many combinations of prices for products A, B, and C that would satisfy the given conditions, depending on the value of c, but they all lie on the same line that passes through the point $(29, 28, 0)$ and stretch in the direction $\begin{pmatrix} \frac{1}{10} \\ -\frac{4}{5} \\ 1 \end{pmatrix}$. The fact that a, b, and c are positive means that only the part of the line within the first octant (region of space where $x > 0$, $y > 0$, and $z > 0$) is part of the possible solutions.

This line is also the intersection of the 3 planes described by the equations in the system of equations we created in part (a).

(d) To find the inverse, we start with the coefficient matrix and the identity matrix:

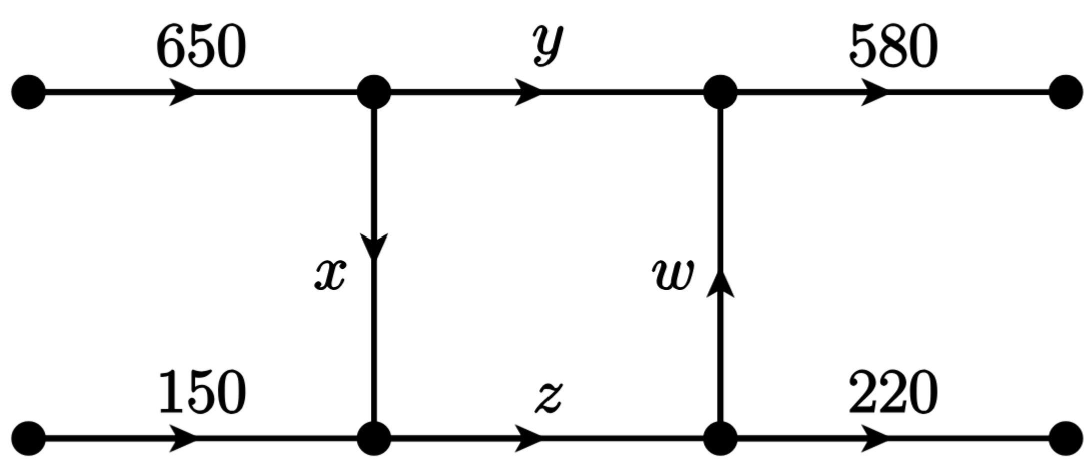
$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 8 & 11 & 8 & 0 & 0 & 1 \end{array}\right].$$

Row reducing this augmented matrix (following similar steps as in the above row reduction), we would get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{10} & x_{11} & x_{12} & x_{13} \\ 0 & 1 & \frac{4}{5} & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 0 & x_{31} & x_{32} & x_{33} \end{array}\right],$$

for some numbers x_{ij}. The matrix on the right would have been our inverse if the left side were the identity matrix. We could have used this inverse to multiply with the original revenue vector to confirm the prices of each unit of product A, B, and C. However, given that the original system is dependent, the inverse does not exist, and indeed the RREF process didn't produce the identity matrix on the left side as needed. (Food for thought: what would multiplying on the left of the matrix equation in part (a) by the right matrix with x_{ij} entries do to it?)

(3) Shown below are some traffic patterns in the downtown area of a large city. The figures give the number of cars per hour traveling along each road. Any car that drives into an intersection must also leave the intersection. This means that the number of cars entering an intersection in an hour is equal to the number of cars leaving the intersection.



(a) How many cars per hour enter the upper left intersection? How many cars per hour leave this intersection? Use this to form a linear equation in the variables x, y, z, w. Form three more linear equations from the other three intersections to form a linear system having four equations in four variables.

(b) Using the previous part, represent this situation with a vector equation, a matrix equation $A\vec{x} = \vec{b}$, and an augmented matrix $[A|\vec{b}]$. Rephrase the question accordingly from these perspectives.

(c) The augmented matrix $[A|\vec{b}]$ that you formed in the previous part is not in Reduced Row Echelon Form (RREF). Explain why.

(d) Row reduce the augmented matrix to Reduced Row Echelon Form (RREF). Show all your steps.

(e) What can you say about the solution of this linear system? Is there exactly one solution or infinitely many solutions? Explain why you would expect this given the information provided. In case there's infinitely many solutions, parameterize them in vector form.

(f) What is the smallest possible amount of traffic flowing through x?

(4) Chemists denote a molecule of water as H_2O, which means it is composed of two atoms of hydrogen (H) and one atom of oxygen (O). The process by which hydrogen burns is described by the chemical reaction

$$xH_2 + yO_2 \to zH_2O.$$

This means that x molecules of hydrogen H_2 combine with y molecules of oxygen O_2 to produce z water molecules. The number of hydrogen atoms is the same before and after the reaction; the same is true of the oxygen atoms.

(a) In terms of x, y, and z, how many hydrogen atoms are there before the reaction? How many hydrogen atoms are there after the reaction? Find a linear equation in x, y, and z, by equating these quantities. Similarly find a second linear equation by equating the number of oxygen atoms before and after the reaction.

(b) Using the previous part, represent this situation with a vector equation, a matrix equation $A\vec{x} = \vec{b}$, and an augmented matrix $[A|\vec{b}]$. Rephrase the question accordingly from these perspectives.

(c) Why can we tell that there will be infinitely many solutions without even solving the system of equations?

(d) The augmented matrix $[A|\vec{b}]$ that you formed in the previous part is not in Reduced Row Echelon Form (RREF). Explain why.

(e) Row reduce the augmented matrix to Reduced Row Echelon Form (RREF). Show all your steps, and parameterize the solution set in vector form.

(f) In this chemical setting, x, y, and z should be positive integers. Find the solution where x, y, and z are the smallest possible positive integers.

Challenge 2: Vectors Exploration

Describe and solve problems using the ideas, properties, and vocabulary of vector spaces (e.g linear combinations, span, subspace, linear independence, basis and a change of basis, dimension). Rephrasing real world questions in terms of vectors and linear algebra terminology allows us to use the intuitive geometrical understanding of vectors as arrows in $\mathbb{R}^n$ to investigate totally unrelated questions from a geometrical perspective.

Learning Objectives: I can

2.1 represent a vector as a linear combination of other vectors both algebraically and geometrically or determine when that's not possible.

2.2 analyze the dot and cross products between two vectors both algebraically and geometrically.

2.3 [CORE] explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property or not and determine if it's a vector space or vector subspace.

2.4 determine if a set of vectors spans a vector space, are linearly dependent or independent, and if they form a basis or not.

2.5 [CORE] find a basis and the dimension of a vector space or vector subspace, and perform a change of basis.

Sample questions:

(1) You may use technology if needed to row reduce matrices. Consider a 2D plane $\mathbb{R}^2$. You are given three modes of transportation (vectors):

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{w} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Part A LO 2.2:

(a) Calculate the dot product of $\vec{u}$ and $\vec{v}$. Show the computation steps.

(b) Geometrically interpret the result of the dot product in terms of the angle between $\vec{u}$ and $\vec{v}$.

Part B LO 2.1, 2.4, and 2.5:

(a) Express the vector $\vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ as a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$, if possible or explain why that's not possible. If possible, draw its geometrical representation using arrows.

(b) Determine if the set $\{\vec{u}, \vec{v}, \vec{w}\}$ spans $\mathbb{R}^2$. Justify your answer algebraically.

(c) Are the vectors in the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent or independent? If they are linearly dependent express $\vec{w}$ as a linear combination of the other two vectors.

(d) Create a basis B for the span of $\{\vec{u}, \vec{v}, \vec{w}\}$ using a subset of $\{\vec{u}, \vec{v}, \vec{w}\}$, and then check they satisfy the conditions for being a basis.

(e) Express the vector $\vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ in the B-basis.

Solution.

Part A LO 2.2:

(a) **Calculate the dot product of $\vec{u}$ and $\vec{v}$:**
The dot product of two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$ is given by:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Using this formula for $\vec{u}$ and $\vec{v}$:

$$\vec{u} \cdot \vec{v} = 2(-1) + 3 \cdot 1 = -2 + 3 = 1$$

So, the dot product of $\vec{u}$ and $\vec{v}$ is 1.

(b) **Geometric interpretation of the dot product:**
The dot product formula can also be expressed in terms of the magnitude of the vectors and the angle between them:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

where θ is the angle between the vectors.
Given that the dot product $\vec{u} \cdot \vec{v} = 1$, we see that the cosine of the angle between them is positive and so we know that the two vectors point in the same quarter of the plane containing them, i.e., the angle between them is less than a right angle. Using the magnitudes of the vectors:

$$|\vec{u}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$|\vec{v}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

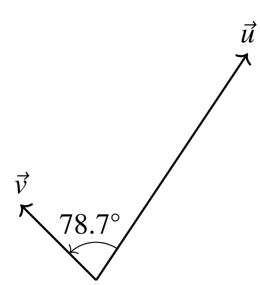
Plugging these values into the dot product formula:

$$1 = \sqrt{13}\sqrt{2}\cos(\theta)$$

From this, we can solve for $\cos(\theta)$:

$$\cos(\theta) = \frac{1}{\sqrt{26}} \iff \theta = 78.7^\circ$$

and get the following picture:



Part B LO 2.1:

(a) To express the vector $\vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ as a linear combination of $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, we set up the equation:

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$$

Recall that in our modes of transportation analogy, this is the same as asking how long should I ride each of the modes of transportation in order to reach at the tip of $\vec{x}$. Substituting the vectors, we get:

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Simplifying, we get the following system of equations:

$$\begin{aligned} 2a - b + 4c &= 3, \\ 3a + b - 2c &= 6. \end{aligned}$$

We can represent this system as an augmented matrix:

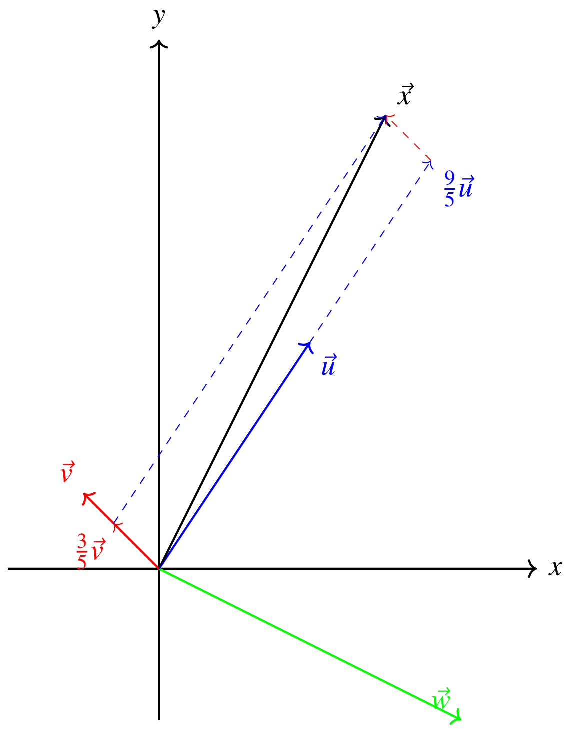
$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 3 \\ 3 & 1 & -2 & 6 \end{array}\right]$$

Row reducing this matrix, we get its RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{9}{5} \\ 0 & 1 & -\frac{16}{5} & \frac{3}{5} \end{array}\right]$$

Here we can let the free variable c take any value and then solve for a and b. Perhaps the easiest thing to do is set $c = 0$, and then $a = \frac{9}{5}$, and $b = \frac{3}{5}$ can be directly read off the last column of the augmented matrix. Therefore we have:

$$\vec{x} = \frac{9}{5}\vec{u} + \frac{3}{5}\vec{v} + 0 \cdot \vec{w} \iff \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{9}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



In our modes of transportation analogy, we can say that it's enough to ride $\vec{u}$ for 1.8 hours and $\vec{v}$ for 0.6 hours in order to reach location $\vec{x}$.

It's perhaps useful to express the full solution set in parametric form as a line in $\mathbb{R}^3$ (the intersection of the two planes determined by the original equations in the system of equations). We therefore have:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{9}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{2}{5} \\ \frac{16}{5} \\ 1 \end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 16 \\ 5 \end{bmatrix}\right)$$

Note that when $c = 2$, we can get the fractions to go away and get $a = 1$ and $b = 7$ as perhaps a nicer looking solution. Can you represent that solution graphically as well? What does this say in terms of our modes of transportation analogy?

(b) To determine if the vectors $\vec{u}, \vec{v}, \vec{w}$ span $\mathbb{R}^2$, we consider a general linear combination of these vectors:

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{x}$$

In terms of our modes of transportation analogy, we ask if it's possible to reach any location $\vec{x}$ in $\mathbb{R}^2$ using the three modes of transportation $\vec{u}, \vec{v}, \vec{w}$? Substituting the given vectors, we get:

$$a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Simplifying, we obtain:

$$\begin{bmatrix} 2a - b + 4c \\ 3a + b - 2c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This can be rewritten as the following system of equations:

$$\begin{aligned} 2a - b + 4c &= x \\ 3a + b - 2c &= y \end{aligned}$$

From the row reduction performed in Part B (a), we know that this system will have at least one solution (and in fact, infinitely many solutions) for any $\vec{x}$ in $\mathbb{R}^2$. Therefore, the vectors $\vec{u}, \vec{v}, \vec{w}$ span $\mathbb{R}^2$.

Without any computations though, we also know that two vectors pointing in different directions is enough to span $\mathbb{R}^2$, so the 3rd vector doesn't really help much. In terms of our analogy, two modes of transportation traveling in different directions is enough to reach any location in $\mathbb{R}^2$.

(c) To check if the vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent or independent, we set up the following equation:

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

From our analogy, this is asking if there's a non-trivial way to get home $\vec{0}$ using these modes of transportation. The trivial way is of course staying home and not traveling at all ($a = b = c = 0$), but if there's redundancy between the modes of transportation, it will lead to multiple ways to get there. The aim of linear independence is to see if there's only one way to get to a certain destination though, so even though in real life we may prefer multiple ways to get home, in linear algebra, we prefer to get there in a unique way. Okay, so the above vector equation leads to the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 0 \\ 3 & 1 & -2 & 0 \end{array}\right]$$

We already row reduced this same matrix with a different last column in part (a), so we can reuse that same row reduced form to get:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{16}{5} & 0 \end{array}\right]$$

Since there is a free variable(c), this indicates that the vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.
To write $\vec{w}$ in terms of the other two, we can let $c = -1$ in the vector equation and get:

$$a\vec{u} + b\vec{v} - \vec{w} = \vec{0} \iff \vec{w} = a\vec{u} + b\vec{v}.$$

Solving for a and b, the dependency relation can be written as:

$$\vec{w} = \frac{2}{5}\vec{u} - \frac{16}{5}\vec{v}$$

From our transportation analogy, we know that in a 2 dimensional space, we will be able to travel everywhere without redundancy with just 2 linearly independent vectors, so the above computation is in line with our expectations.

(d) From part (b) we know that the span of $\vec{u}, \vec{v}, \vec{w}$ is all of $\mathbb{R}^2$. Therefore, we may use any 2 of the 3 vectors $\vec{u}, \vec{v}, \vec{w}$ to form a basis since none of them is a scalar multiple of the other two, and we just need to be able to span a 2 dimensional space. To give an explicit solution, we will show that $B = \{\vec{u}, \vec{v}\}$ is a basis of $\mathbb{R}^2$. To do that, we need to show two things:

I. The vectors in B span $\mathbb{R}^2$ (i.e., we're able to reach any location in $\mathbb{R}^2$): There's many ways to argue this point from what we have done such as for example using row reduction, but perhaps the easiest way is to use the fact that we can reach any location $\vec{x}$ with all three vectors (i.e., $\vec{x}$ is a linear combination of the three vectors) and then substitute $\vec{w} = \frac{2}{5}\vec{u} - \frac{16}{5}\vec{v}$ to get a linear combination of only the vectors in the set B.

II. The vectors in B are linearly independent (i.e., there's no redundancy between these vectors so there is a unique way to reach a location in their span): One can argue this point by our row reduction work earlier, but another way to do so is to note that $\mathbb{R}^2$ is 2 dimensional ($\{\vec{i}, \vec{j}\}$ is a basis and contains 2 vectors), and since we know that our two vectors in B span $\mathbb{R}^2$, it's impossible that one of them is redundant,i.e., a linear combination of the other one).

(e) From item (a), we have that $\vec{x} = \frac{9}{5}\vec{u} + \frac{3}{5}\vec{v}$, and so

$$[\vec{x}]_B = \begin{bmatrix} 9/5 \\ 3/5 \end{bmatrix}_B .$$

(2) Consider the set $L = \{a(2 + 3x) + b(-1 + x) + c(4 - 2x) : a, b, c \in \mathbb{R}\}$.

(a) Show that L is a vector space by checking that all the vector space properties hold.

(b) One easy way to argue that L is a vector space is by representing each of its generating elements as a vector in $\mathbb{R}^2$, and then using the fact that the span of a set of vectors is a vector space. For example, with the convention $1 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $2+3x \to 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}+3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $-1+x \to \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $4 - 2x \to \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. This means that L can be thought as the span of $\vec{u}, \vec{v}, \vec{w}$ from the previous question (the technical math term which we will learn later is "isomorphic"). With that in mind, determine the subset B of $\{2+3x, -1+x, 4-2x\}$ that forms a basis of L, and express the standard basis vectors $\{1, x\}$ in B-coordinates.

(c) Does L contain all possible lines in R^2? Describe the lines contained in L in your own words.

Solution.

(a) To show that L is a vector space, we need to check that it satisfies the following vector space properties:

i. **Closure under addition**: Given $f(x) = a_1(2 + 3x) + b_1(-1 + x) + c_1(4 - 2x)$ and $g(x) = a_2(2 + 3x) + b_2(-1 + x) + c_2(4 - 2x)$ in L, their sum $f(x) + g(x)$ is also in L.

$$f(x) + g(x) = (a_1 + a_2)(2 + 3x) + (b_1 + b_2)(-1 + x) + (c_1 + c_2)(4 - 2x) \in L.$$

ii. **Closure under scalar multiplication**: Given $f(x) = a(2 + 3x) + b(-1 + x) + c(4 - 2x)$ in L and a scalar $k \in \mathbb{R}$, $kf(x)$ is also in L.

$$kf(x) = k[a(2 + 3x) + b(-1 + x) + c(4 - 2x)] = ka(2 + 3x) + kb(-1 + x) + kc(4 - 2x) \in L$$

iii. **Commutativity and associativity of addition**: These properties are inherited from the real numbers.
iv. **Existence of zero vector**: The zero vector 0 is in L when $a = b = c = 0$.
v. **Existence of additive inverse**: For each $f(x)$ in L, the function $-f(x)$ is also in L, since we can take $a' = -a, b' = -b, c' = -c$ to be the coefficients of $(2 + 3x), (-1 + x), (4 - 2x)$ in $-f(x)$.
vi. **Compatibility of scalar multiplication**: This is inherent from multiplication of real numbers.
vii. **Identity element of scalar multiplication**: $1 \cdot f(x) = f(x)$.
viii. **Distributive laws on vectors and scalars**: These properties are also inherited from the real numbers.

In simpler terms, the elements of L are linear combinations of the functions $2 + 3x$, $-1 + x$, and $4 - 2x$ with real coefficients a, b, c, i.e., L is the span of $(2 + 3x), (-1 + x), (4 - 2x)$.

(b) Since L corresponds to the span of $\vec{u}, \vec{v}, \vec{w}$ from the previous question, we can use the results from that question to find a basis and the dimension of L.

From the previous question, we found that the vectors $\vec{u}$ and $\vec{v}$ are linearly independent and span $\mathbb{R}^2$. Therefore, they form a basis for $\mathbb{R}^2$, which means $2 + 3x$ and $-1 + x$ form a basis for L.

The dimension of L is the same as the dimension of the span of $\vec{u}, \vec{v}, \vec{w}$, which is 2.

One can also check directly that $B = \{2 + 3x, -1 + x\}$ forms a basis of L:

I. The vectors in B span L (i.e., we're able to reach any location in L through linear combinations of the linear polynomials in B): There's many ways to argue this point from what we have done such as for example using row reduction, but perhaps the easiest way is to show that $4 - 2x$ is in the span of B, i.e., $4 - 2x = a(2 + 3x) + b(-1 + x)$ for some real numbers a and b. One way to do so is by matching coefficients on both sides like in partial fractions decomposition in calculus 2: Note that $4 - 2x = a(2 + 3x) + b(-1 + x) = (2a - b) + (3a + b)x$, and so we get $2a-b = 4$ and $3a+b = -2$. The corresponding augmented matrix is the same as the left side of the augmented matrix in Part B LO 2.1 (a). That same computation (ignoring the last column) gives $a = 2/5$ and $b = -16/5$. Therefore

$$4 - 2x = \frac{2}{5}(2 + 3x) - \frac{16}{5}(-1 + x),$$

Which is analogous to $\vec{w} = \frac{2}{5}\vec{u} - \frac{16}{5}\vec{v}$ we got in the previous question.

II. The vectors in B are linearly independent (i.e., there's no redundancy between these vectors so there is a unique way to reach a location in their span): $a(2 + 3x) + b(-1 + x) = 0$ has the unique solution $a = b = 0$ by row reduction as in the previous item (the augmented matrix will be the same except the rightmost column will be all zeros). Alternatively, we can see that $2 + 3x$ is not a real multiple of $-1 + x$, because then $2 + 3x = k(-1 + x) = -k + kx$ and so k needs to both be -2 and 3 which is a contradiction.

To find the B-coordinates of 1 and x, we need to find a, b, c, d such that $a(2 + 3x) + b(-1 + x) = 1$, and $c(2 + 3x) + d(-1 + x) = x$. Matching coefficients as above, we get two systems of equations: $2a-b = 1, 3a+b = 0$ and $2c-d = 0, 3c+d = 1$. Putting them in an augmented matrix, we notice that the two augmented matrices are identical on the left side, so in order to reduce the computations, we can solve them at the same time by row reducing the following matrix:

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & -3/5 & 2/5 \end{array}\right]$$

Note that this is the same as finding the inverse of the left side matrix. This means that $a = 1/5, b = -3/5, c = 1/4, d = 2/5$, and therefore

$$1 = \frac{1}{5}(2 + 3x) - \frac{3}{5}(-1 + x), \text{ and } x = \frac{1}{5}(2 + 3x) + \frac{2}{5}(-1 + x).$$

Hence the B-coordinates of 1 and x are respectively $\begin{bmatrix} 1/5 \\ -3/5 \end{bmatrix}$ and $\begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$.

This whole computation is the same as what you get if you try to write the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (corresponding to 1 and x respectively) as a linear combination of the vectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (corresponding to $2 + 3x$ and $-1 + x$ respectively), which is much easier conceptually.

(c) The above computation shows that 1 and x are a basis of L and so any element of L is a linear combination of them, i.e., has the form $a + bx$, which is the set of all possible lines in $\mathbb{R}^2$ except for those that are vertical (i.e., those that are not functions $x = k$ for some real number k).

(3) Consider the vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$ given by:

$$\vec{a} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -8 \\ 0 \\ 6 \end{bmatrix}$$

(a) What are the angles between the vectors $\vec{a}, \vec{b}, \vec{c}$?

(b) How is the vector cross product vector $\vec{a} \times \vec{b}$ related to the vectors $\vec{a}, \vec{b}, \vec{c}$ geometrically?

(c) Express the vector $\vec{d} = \begin{bmatrix} -12 \\ 30 \\ -16 \end{bmatrix}$ as a linear combination of $\vec{a}, \vec{b}, \vec{c}$, if possible.

(d) Compute the dot products of $\vec{d}$ with $\vec{a}, \vec{b}, \vec{c}$ in two different ways: (1) directly from the coordinates of $\vec{d}$, and (2) by replacing $\vec{d}$ with the linear combination of $\vec{a}, \vec{b}, \vec{c}$ you found in the previous part.

(e) What is the relationship between the coefficients in the linear combination and the dot products computed in the previous item. Think of the geometric meaning of the dot product.

(f) Determine if the set $\{\vec{a}, \vec{b}\}$ spans $\mathbb{R}^3$. If not, what does it look like geometrically (line, plane, 3D space)?

(g) Are the vectors $\vec{a}, \vec{b}, \vec{c}$ linearly dependent or independent? How about the vectors $\vec{a}, \vec{b}, \vec{d}$?

(h) Show that the span of $\{\vec{a}, \vec{b}\}$ satisfies the properties of a vector space.

(i) What is the dimension of the vector space spanned by $\{\vec{a}, \vec{b}\}$? How about the vector space spanned by $\{\vec{a}, \vec{b}, \vec{d}\}$?

Challenge 3: Linear Transformations Exploration

Identify linear transformations using properties of linearity, convert linear transformations to matrix form, and describe the related transformation spaces (including null space and range). Viewing a matrix as a transformation of space is a game changer which allows us to create animations, or predict long term behavior of populations among many other applications.

Learning Objectives: I can

3.1 find and describe the related transformation spaces of a given linear transformation and its transpose, including domain, codomain, null space/kernel, range/image, and their relationships with each other.

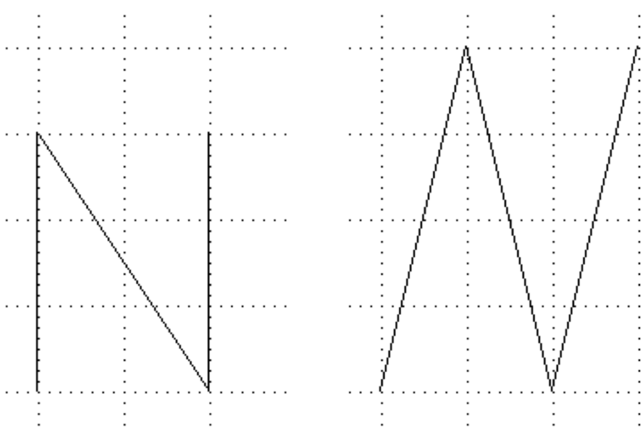
3.2 [CORE] distinguish between linear and non-linear transformations of a vector space, and sketch/describe the effect of a linear transformation.

3.3 convert linear transformations and operations on them to matrix form and operations between matrices, and vice verse.

3.4 analyze a linear transformation through its determinant, and determine when it is injective/one-to-one, onto/surjective, bijective, or an isomorphism.

Sample questions:

(1) Suppose the "N" on the left is written in regular 12-point font. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that transforms "N" into that letter on the right, which is written in 'italics' in 16-point font.



(a) The linear transformation T can be represented by multiplication by a matrix A, which transforms input vectors to output vectors: $T(\vec{x}) = A\vec{x}$. What are the domain and codomain for these input and output spaces?

Domain: **Codomain:**

(b) As a result of the dimensions of the domain and codomain, what is the size of the matrix A?

Size: $m \times n =$

(c) Determining a way to "coordinatize" the original "N" and the italicized "N." In the above pictures, label the origin, x and y axis so as the "N" on the left is 12 units high.

(d) What are some input-output vectors as a result of multiplication by A? Use that information to determine the matrix A of the transformation T.

(e) What would this linear transformation do to other letters written in regular 12-point font? Is that useful in real life?

(f) Find the determinant and interpret its geometric meaning.

(g) What are the rank, null space/kernel, range/image of this linear transformation?

(h) Is the linear transformation T one-to-one/injective? Is it onto/surjective? Is it bijective? Is it an isomorphism?

(i) After class two linear algebra students start talking about linear transformations and the letter "N." One of the students suggested translation (shifting up) as another linear transformation that could be done to the letter "N." The other student disagreed, stating that shifting the "N" up is not an example of a linear transformation. Which student is right? Why? Explain your reasoning by considering the definition of a linear transformation.

Solution.

(a) Both the domain and codomain are $\mathbb{R}^2$ since we're taking a picture in $\mathbb{R}^2$ and transforming it to another picture in $\mathbb{R}^2$.

(b) A is 2×2. Remember the number of columns is the dimension of the domain, and the number of rows is the dimension of the codomain.

(c) There's many possible answers here, but I would label the origin $(0, 0)$ to be the bottom left corners of the letters "N," the positive x-axis horizontally to the right, the positive y-axis vertically up, such that each little square is 4 by 4 units.

(d) $\begin{bmatrix}0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}$ (any linear transformation must send the origin to the origin), $\begin{bmatrix}8\\0\end{bmatrix} \to \begin{bmatrix}8\\0\end{bmatrix}$, $\begin{bmatrix}0\\12\end{bmatrix} \to \begin{bmatrix}4\\16\end{bmatrix}$.
One strategy is to setup two matrix equations from the last two transformations and turn them into a system of 4 linear equations in terms of the 4 entries of the matrix. Then form the corresponding augmented matrix and RREF to get the values of the entries. This strategy will work for any linear transformation as long as you can determine enough input-output vectors. Try to carry it out.

A better strategy is to remember that the first and second columns of A are respectively $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = A\begin{bmatrix}1\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = A\begin{bmatrix}1\\0\end{bmatrix}$. The question is then how to express the basis vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of the known vectors in the domain $\begin{bmatrix}8\\0\end{bmatrix}$ and $\begin{bmatrix}0\\12\end{bmatrix}$, which is quite straightforward: $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{8}\begin{bmatrix}8\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}0\\12\end{bmatrix}$, and by linearity we get

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{8}T\left(\begin{bmatrix}8\\0\end{bmatrix}\right) = \frac{1}{8}\begin{bmatrix}8\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{12}T\left(\begin{bmatrix}0\\12\end{bmatrix}\right) = \frac{1}{12}\begin{bmatrix}4\\16\end{bmatrix} = \begin{bmatrix}1/3\\3/4\end{bmatrix}$$

or equivalently if you prefer matrix multiplication notation instead:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{8}A\begin{bmatrix}8\\0\end{bmatrix} = \frac{1}{8}\begin{bmatrix}8\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } A\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{12}A\begin{bmatrix}0\\12\end{bmatrix} = \frac{1}{12}\begin{bmatrix}4\\16\end{bmatrix} = \begin{bmatrix}1/3\\4/3\end{bmatrix}$$

Therefore we immediately get

$$A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}.$$

(e) It will also change them from 12-point font to 16-point font and italicize them by sheering the top part, which is quite useful if we want to write a paper or document in those fonts. Essentially MS Word could have this built in button that transforms any highlighted text according to this linear transformation.

(f) The determinant is given by $1 \cdot \frac{4}{3} - 0 \cdot \frac{1}{3} = \frac{4}{3}$, which means that any area on the input plane gets scaled by $\frac{4}{3}$, i.e., becomes 33% bigger. This makes sense considering that going from a 12-point font to a 16-point font is a third increase in height, but the length stayed the same.

(g) i. Rank: 2, since the two columns of A are not multiples of one another or since the determinant is not 0.

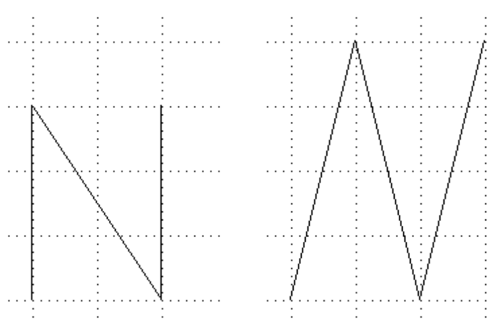
ii. Null Space/Kernel: $\{\vec{0}\}$, since the equation $T(\vec{x}) = A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$ since A is invertible and therefore we can multiply both sides by A^{-1}.

iii. Range/Image: $\mathbb{R}^2$, since the $T(\vec{x}) = A\vec{x}$ is the subspace of $\mathbb{R}^2$ spanned by the columns of A. The columns of A though are linearly independent, so their span is all of $\mathbb{R}^2$.

(h) Yes to all the questions: the kernel is trivial implies the transformation is injective or one-to-one; the image is the full codomain implies surjective or onto; being both injective and surjective implies bijective; and being a linear transformation and bijective implies it's an isomorphism.

(i) No, translation up is not a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ because the origin moves up. However, if we think of the plane with the letter N as say the plane $z = 1$ within $\mathbb{R}^3$, translating the N up corresponds to a sheer transformation in $\mathbb{R}^3$ within the $z = 1$ plane, which is a legitimate linear transformation. Translation in $\mathbb{R}^3$ can also be thought of as a linear transformation if we think of it from the higher dimension $\mathbb{R}^4$. This is in fact how most robot movements are encoded. It's interesting to think how moving up a dimension can solve problems. Look up homogeneous coordinates if you're interested in this.

(2) Suppose the "N" on the left is written in regular 12-point font. In the previous question we found that the matrix $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ will transform N into that letter on the right, which is written in 'italics' in 16-point font.



In a previous quarter, two linear algebra students, Pat and Jamie, described their approach to the Italicizing N Task in the following way:

"*In order to find that matrix A, we are going to find a matrix that makes "N" taller (from 12-point to 16-point), find a matrix that italicizes the taller "N," and the combination of those will be the desired matrix A.*"

(a) Try Pat and Jamie's approach. Find a matrix B (for bigger) that makes "N" taller (from 12-point to 16-point), and another matrix S (for sheer) that italicizes the taller "N."

(b) In order to combine them, compute both products BS and SB. Are they equal? Why?

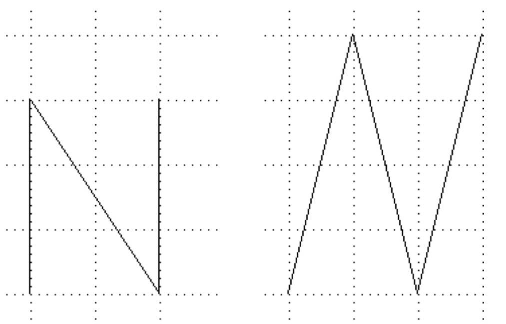
(c) Which of the above two products is the right way to combine the two matrices to get A and why?

(d) Find the determinants of matrices A, B, S, BS, and SB and interpret their geometric meanings.

(e) What are the rank, null space/kernel, range/image of the linear transformations represented by matrices B and S?

(f) Are the linear transformations represented by matrices B and S one-to-one/injective? Are they onto/surjective? Are they bijective? Are they isomorphisms?

(3) Suppose the "N" on the left is written in regular 12-point font. In the previous question we found that the matrix $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ will transform N into that letter on the right, which is written in 'italics' in 16-point font.



The goal of this question is to find a matrix C that will transform the letter on the right back into the letter on the left.

(a) Without computing C and just knowing the matrix A, what is the determinant of C?

(b) Use pairs of input-output vectors to determine the entries of matrix C.

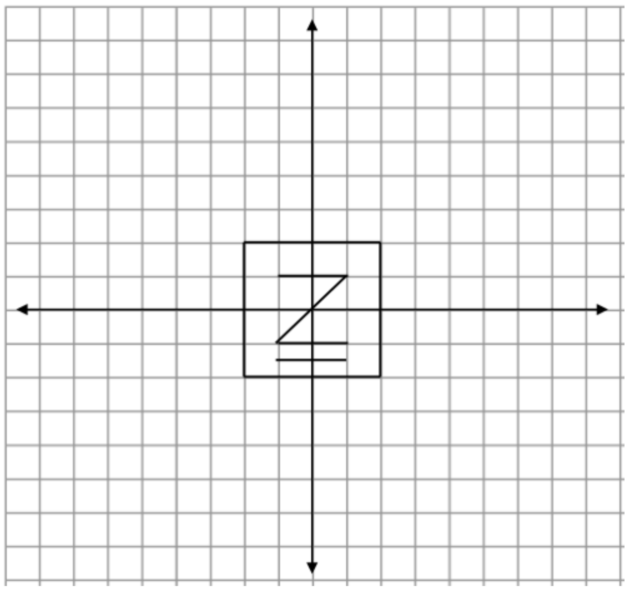
(c) Use both row reductions and the formula for the inverse to compute A^{-1}. You may use technology to compute the RREF form of a matrix. How is it related to the matrix C you computed above.

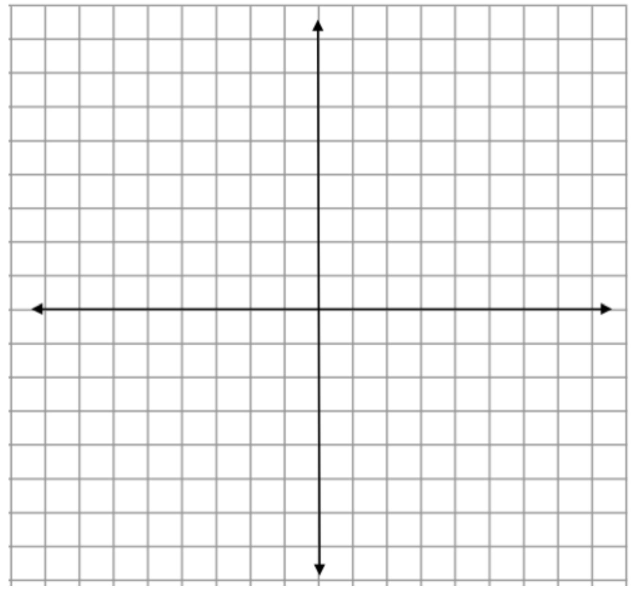
(d) Let B (for bigger) be the linear transformation that makes "N" taller (from 12-point to 16-point), and S (for sheer) the linear transformation that italicizes the taller "N." We know that $A = SB$. Determine S^{-1} and B^{-1}, and use them to determine the matrix C.

(e) What are the rank, null space/kernel, range/image of the linear transformations represented by matrix C?

(f) Is the linear transformations represented by matrix C one-to-one/injective? Is it onto/surjective? Is it bijective? Is it an isomorphism?

(4) Consider the image given below and the transformation matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & -1.5 \end{bmatrix}$.





(a) Sketch on the right coordinate system above what will happen to the image under the tranformation by D.

(b) Describe in words what happened to the image under the transformation.

(c) Describe how you determined what happened. What, if any, calculations did you do? Did you make a prediction? How did you know you were right? etc.

(d) What is the determinant of this linear transformation and what's its geometric significance in terms of the picture being transformed?

(e) What are the rank, null space/kernel, range/image of the linear transformations represented by matrix D?

(f) Is the linear transformations represented by matrix D one-to-one/injective? Is it onto/surjective? Is it bijective? Is it an isomorphism?

(5) After class two linear algebra students started talking about linear transformations. One of the students suggested that squishing down a 2D picture, or more precisely projecting onto the x-axis, is not a linear transformation as it totally destroys the picture, making it impossible to undo. The other student disagreed, stating that projection is a linear transformation even though it's irreversible. Which student is right? Why? Explain your reasoning by considering the definition of a linear transformation.

(a) Find the matrix P describing the above transformation.

(b) What are the rank, kernel, image of the projection transformation discussed above?

(c) Is the projection transformation above one-to-one/injective? Is it onto/surjective? Is it bijective? Is it an isomorphism?

(6) Assume that T is a linear transformation and that $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For each part, answer the following questions:

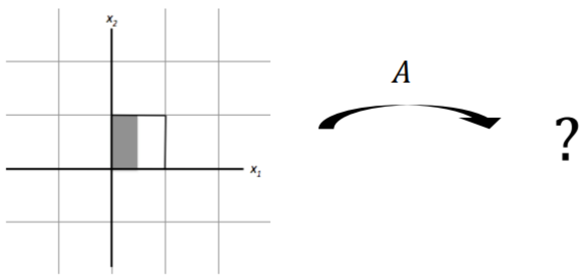
I. Find the standard matrix A for T.

II. Draw the image of the "half-shaded unit square" (shown below) under the given transformation.

III. What is the determinant of this linear transformation and what's its geometric significance in terms of the picture being transformed?

IV. What are the rank, null space/kernel, range/image of the linear transformations?

V. Is the linear transformations one-to-one/injective? Is it onto/surjective? Is it bijective? Is it an isomorphism?



(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotate points about the origin $-\pi/4$ radians, i.e., $\pi/4$ radians clockwise.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical sheer that maps $\vec{e}_1$ to $\vec{e}_1 - \vec{e}_2$, but leaves the vector $\vec{e}_2$ unchanged.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points across the vertical axis, and then rotates points $\pi/2$ radians counterclockwise.

(d) $T : \mathbb{R}^2 \to \mathbb{R}$ projecting onto the line $y = x$. **Hint**: Note that the output will be a single number that represents the length of the projection vector. Recall the dot product.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^3$ taking the $\mathbb{R}^2$ plane and identifying it as the xy-plane within $\mathbb{R}^3$.

(f) $T : \mathbb{R}^3 \to \mathbb{R}^2$ taking $\mathbb{R}^3$ and projecting it onto the xy plane and identifying this plane with $\mathbb{R}^2$.

Challenge 4: Eigen Exploration

Explore the foundational concepts of eigenvalues and eigenvectors, their role in matrix diagonalization, and their real-world applications such as least squares and Markov chains. Dive into orthogonal projections, the creation of orthonormal bases, and their use in diagonalizing symmetric matrices.

Learning Objectives: I can

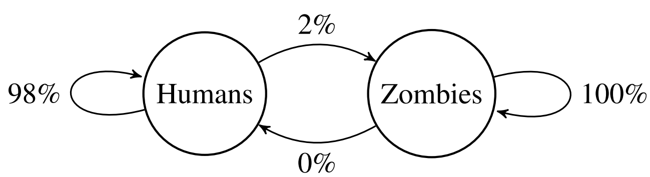
4.1 [CORE] compute eigenvalues and eigenvectors of a linear transformation, diagonalize the corresponding matrix and explain its effects both geometrically and in applied contexts.

4.2 determine orthogonal projections, create an orthogonal or orthonormal basis, and use it to solve least squares questions.

4.3 solve application questions using diagonalization and orthogonal projections.

Sample questions:

(1) **Zombie Apocalypse Model 1**. Suppose a virus that turns Humans to Zombies suddenly appeared and each week 2% of Humans become Zombies. Let's assume that the population of the world initially was $H_0 = 8$ billion Humans and $Z_0 = 0$ billion Zombies. Let H_k and Z_k be the population of Humans and Zombies respectively at the end of Week k. See the following diagram for a visual of the situation and answer the following questions:



(a) What is the population of Humans and Zombies in billions by the end of Week 1 and Week 2?

(b) Create two linear equations that help us compute the population of Humans and Zombies at week $k+1$ provided we know their population at week k by filling out the following boxes with decimals:

$$H_{k+1} = \boxed{}H_k + \boxed{}Z_k$$
$$Z_{k+1} = \boxed{}H_k + \boxed{}Z_k$$

(c) Let $\vec{x}_0 = \begin{bmatrix} H_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, and $\vec{x}_k = \begin{bmatrix} H_k \\ Z_k \end{bmatrix}$ be the vector capturing the population of the world at the end of week k, where the first component is the number of Humans, and the second is the number of Zombies. What is the matrix A that upon multiplying it by $\vec{x}_k$ results in $\vec{x}_{k+1}$, the population at the end of week $k+1$, i.e., $\vec{x}_{k+1} = A\vec{x}_k$?

(d) By the above equation we have:

$\vec{x}_1 = A\vec{x}_0$, $\vec{x}_2 = A\vec{x}_1 = A(A\vec{x}_1) = A^2\vec{x}_0$, $\vec{x}_3 = A\vec{x}_2 = A(A^2\vec{x}_0) = A^3\vec{x}_0, \ldots, \vec{x}_k = A\vec{x}_{k-1} = \cdots = A^k\vec{x}_0$.
Use technology to compute $\vec{x}_{100} = A^{100}\vec{x}_0$, and $\vec{x}_{1000} = A^{1000}\vec{x}_0$. Can you explain what's happening to the population of the world over time?

(e) Note that the sum of the entries in each column of the transformation matrix are 1. Why is that so? Does that make sense in the context of the question?

(f) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Explain why the following is true: "If the rows of M add up to the same number k (i.e., if $a + b = k$ and $c + d = k$), then k is an eigenvalue of M corresponding to eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$." Your explanation should include at least 3 sentences. You can use examples or sketches to help if you like.

(g) Use the previous item argument for $k = 1$ to show that the matrix A in your question has an eigenvalue of 1. Note that in A the sum of its columns is 1, whereas M has the sum of its rows equal to 1.

(h) Find the eigenvalues and associated eigenvectors of A by hand, and then use technology to check that you got the correct answer.

(i) Use your own words to describe the effect of the linear transformation represented by A in terms of the "stretch factors" (eigenvalues) and corresponding "stretch directions" (eigenvectors).

(j) Let B be the basis of $\mathbb{R}^2$ formed by the eigenvectors of A, use technology to express $\vec{x}_0 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ as a linear combination of the eigenvectors of A you found in the previous part and then find the B-coordinates of $\vec{x}_0$.

(k) Now write $\vec{x}_k$ as a linear combination of the eigenvectors of A, as well as find its B-coordinates, note that the coefficients will be in terms of k.

(l) Write the vector $\vec{x}_k$ in standard coordinates, where each coordinate is a function of k. Plug in $k = 100$ and $k = 1000$ to confirm the answers you got above using technology for $\vec{x}_{100}$ and $\vec{x}_{1000}$. What happens to the population after a long time? Is it approaching a stable population, exploding, or dying out?

(m) Find a matrix D that captures how multiplication by A transforms the B-coordinates of $\vec{x}_k$ to the B-coordinates of $\vec{x}_{k+1}$, i.e., $[\vec{x}_{k+1}]_B = D\,[\vec{x}_k]_B$.

(n) Let $P = P_{B\to S}$ be the transition matrix from the eigenbasis B to the standard basis S. Then, we have the following commutative diagram:

$$\begin{tikzcd} {[\vec{x}_k]_S} \arrow[r, "A"] \arrow[d, shift right, "P^{-1}"'] & {[\vec{x}_{k+1}]_S} \arrow[d, shift right, "P^{-1}"'] \\ {[\vec{x}_k]_B} \arrow[r, "D"'] \arrow[u, shift right, "P"'] & {[\vec{x}_{k+1}]_B} \arrow[u, shift right, "P"'] \end{tikzcd}$$

Find the matrix P and use it to write the diagonalization of A. Are the matrices A and D similar? Explain why the definition of similar matrices makes sense from this point of view.

(o) Find a formula for A^k where each of its entries is in terms of k, and multiply it by $\vec{x}_0 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ in order to write $\vec{x}_k$ in (standard) coordinates, where each coordinate is in terms of k. (This should match the answer in part (h), and provides an alternate way to computing $\vec{x}_k$).

(p) Does it matter if originally there were a different amount of Humans and Zombies in the world?

Solution.

(a) **End of Week 1:**

$$\text{Humans: } H_1 = 98\% H_0 = 0.98 \cdot 8 = 7.84 \text{ billion.}$$
$$\text{Zombies: } Z_1 = Z_0 + 2\% \cdot H_0 = 0 + 0.02 \cdot 8 = 0.16 \text{ billion.}$$

End of Week 2:

$$\text{Humans: } H_2 = 98\% H_1 = 0.98 \cdot 7.84 = 7.6832 \text{ billion.}$$
$$\text{Zombies: } Z_2 = Z_1 + 2\% H_1 = 0.16 + 0.02 \cdot 7.84 = 0.3168 \text{ billion.}$$

(b) **Linear Equations for Week** $k+1$:

$$\text{For Humans: } H_{k+1} = 0.98 H_k + 0 Z_k$$
$$\text{For Zombies: } Z_{k+1} = 0.02 H_k + 1 Z_k$$

Thus, the filled boxes would be:

$$H_{k+1} = \boxed{0.98} H_k + \boxed{0} Z_k$$
$$Z_{k+1} = \boxed{0.02} H_k + \boxed{1} Z_k$$

(c) The matrix A that transforms $\vec{x}_k$ into $\vec{x}_{k+1}$ is given by:

$$A = \begin{bmatrix} 0.98 & 0 \\ 0.02 & 1 \end{bmatrix}.$$

(d)

$$\vec{x}_{100} = A^{100} \cdot \vec{x}_0 = \begin{bmatrix} 0.13262 & 0 \\ 0.86738 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.061 \\ 6.939 \end{bmatrix} \text{ billion}$$
$$\vec{x}_{1000} = A^{1000} \cdot \vec{x}_0 = \begin{bmatrix} 1.68297 \cdot 10^{-9} & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 13.5 \cdot 10^{-9} \\ 8 \end{bmatrix} \text{ billion}$$

That means after 100 weeks there will be approximately 1 billion Humans and 7 billion Zombies, and after 1000 weeks there will be approximately 13 Humans and almost 8 billion Zombies. The trend being that the population of Humans is turning to Zombies at a frightening rate, and will perhaps continue to do so until there's no humans left.

(e) The sum of the entries in the columns is 1 because that's how 100% of the populations of the Humans and Zombies respectively per column move between each other. Note that in this model the total population is staying fixed at 8 billion. That means no new Humans are born nor die, and that's what makes this model not so realistic.

(f) By matrix vector multiplication we get

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which means $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue k.

(g) Note that $A^T = \begin{bmatrix} 0.98 & 0.02 \\ 0 & 1 \end{bmatrix}$ has the property that its rows add up to 1, and therefore by the previous part, 1 is an eigenvalue of it. Since the eigenvalues of A and A^T are the same (recall that $\det(A-\lambda I) = \det(A-\lambda I)^T = \det(A^T - \lambda I)$), we may conclude that A has an eigenvalue of 1.

(h) We could easily find the characteristic polynomial and solve for the eigenvalues here, but in this case there's a shortcut since we just showed that $\lambda_1 = 1$ is an eigenvalue. Since the trace of a matrix is the sum of the eigenvalues, we know that $\lambda_2 = 1.98 - 1 = 0.98$. Alternatively, the diagonal entries of an upper or lower triangular matrix are the eigenvalues.
For $\lambda_1 = 1$:

$$(A - I)\vec{v}_1 = \vec{0} \iff \left(\begin{bmatrix} 0.98 & 0 \\ 0.02 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{v}_1 = \vec{0} \iff \begin{bmatrix} -0.02 & 0 \\ 0.02 & 0 \end{bmatrix} \vec{v}_1 = \vec{0}.$$

Solving this, we get: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\lambda_1 = 1$.
For $\lambda_2 = 0.98$:

$$(A - 0.98I)\vec{v}_2 = \vec{0} \iff \left(\begin{bmatrix} 0.98 & 0 \\ 0.02 & 1 \end{bmatrix} - \begin{bmatrix} 0.98 & 0 \\ 0 & 0.98 \end{bmatrix} \right) \vec{v}_2 = \vec{0} \iff \begin{bmatrix} 0 & 0 \\ 0.02 & 0.02 \end{bmatrix} \vec{v}_2 = \vec{0}$$

Solving this, we get: $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda_2 = 0.98$.

(i) The linear transformation represented by A does not move all vectors in the direction of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (i.e., y-axis), and shrinks by 2% the vectors in the direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(j) We would like to solve $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ for c_1 and c_2. Translating it to an augmented matrix and using RREF, we get

$$\left[\begin{array}{cc|c} 0 & 1 & 8 \\ 1 & -1 & 0 \end{array}\right] \iff \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 8 \end{array}\right]$$

This means that $\vec{x}_0 = 8\vec{v}_1 + 8\vec{v}_2$, which checks out. That means the B coordinates of $\vec{x}_0$ are $\begin{bmatrix} 8 \\ 8 \end{bmatrix}_B$.

(k) Now that we got the B coordinates of $\vec{x}_0$, applying the linear transformation represented by A simply means scaling each component by the corresponding eigenvalue:

$$\vec{x}_k = A^k\vec{x}_0 = A^k \begin{bmatrix} 8 \\ 8 \end{bmatrix}_B = \begin{bmatrix} \lambda_1^k 8 \\ \lambda_2^k 8 \end{bmatrix}_B = \begin{bmatrix} 8 \\ (0.98)^k 8 \end{bmatrix}_B = 8\vec{v}_1 + (0.98)^k 8\vec{v}_2.$$

Alternatively, you may prefer to think of this whole computation as taking $\vec{x}_0 = 8\vec{v}_1 + 8\vec{v}_2$ and multiplying both sides by A^k in order to get to $\vec{x}_k$:

$$\vec{x}_k = A^k\vec{x}_0 = 8A^k\vec{v}_1 + 8A^k\vec{v}_2 = 8\lambda_1^k\vec{v}_1 + 8\lambda_2^k\vec{v}_2 = 8\vec{v}_1 + (0.98)^k 8\vec{v}_2.$$

(l) Using standard coordinates for $\vec{v}_1$ and $\vec{v}_2$, we get

$$\vec{x}_k = 8\vec{v}_1 + (0.98)^k 8\vec{v}_2 = 8\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (0.98)^k 8 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8(0.98)^k \\ 8\left(1 - (0.98)^k\right) \end{bmatrix}$$

$$\vec{x}_{100} = \begin{bmatrix} 8(0.98)^{100} \\ 8\left(1-(0.98)^{100}\right) \end{bmatrix} = \begin{bmatrix} 1.061 \\ 6.939 \end{bmatrix} \text{ billion and } \vec{x}_{1000} = \begin{bmatrix} 8(0.98)^{1000} \\ 8\left(1-(0.98)^{1000}\right) \end{bmatrix} = \begin{bmatrix} 13.5\cdot 10^{-9} \\ 8 \end{bmatrix} \text{ billion.}$$

As $k \to \infty$, we get

$$\lim_{k\to\infty} \vec{x}_k = \lim_{k\to\infty} \begin{bmatrix} 8(0.98)^k \\ 8\left(1-(0.98)^k\right) \end{bmatrix} = \begin{bmatrix} 8(0) \\ 8\,(1-0) \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix},$$

which confirms that in the long term it seems Humans are completely converted to Zombies and the population stabilizes to 0 Humans and 8 billion Zombies.

(m) The matrix D such that $[\vec{x}_{k+1}]_B = D\,[\vec{x}_k]_B$ is the diagonal matrix created with the eigenvalues of A on the diagonal, that is similar to A, and geometrically scales the first coordinate by $\lambda_1 = 1$ and the second coordinate by $\lambda_2 = 0.98$. Algebraically, that also makes sense:

$$[\vec{x}_{k+1}]_B = D\,[\vec{x}_k]_B \iff \begin{bmatrix} 8 \\ (0.98)^{k+1}8 \end{bmatrix}_B = D\begin{bmatrix} 8 \\ (0.98)^k 8 \end{bmatrix}_B \iff D = \begin{bmatrix} 1 & 0 \\ 0 & 0.98 \end{bmatrix}.$$

(n) The matrix P is the matrix with columns the eigenvectors of A. Following the commutative diagram

$$\begin{array}{ccc} [\vec{x}_k]_S & \xrightarrow{\quad A \quad} & [\vec{x}_{k+1}]_S \\ {\scriptstyle P^{-1}}\downarrow & & \uparrow{\scriptstyle P} \\ [\vec{x}_k]_B & \xrightarrow[\quad D \quad]{} & [\vec{x}_{k+1}]_B \end{array}$$

from left to right on top we get A, but going down, right, and up, we get PDP^{-1} (remember composition is always read from right to left). Therefore we have

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.98 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.98 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Two matrices M and N are similar if there exists an invertible matrix R such that $M = RNR^{-1}$. The matrices A and D are similar since they are related to each other by a change of basis matrix P, which in this case contains eigenvectors as columns. The definition of similar matrices makes sense from this perspective since it's just saying that two linear transformations act in the same way after a change of coordinates. In our case the matrix D keeps the vectors in the x-axis direction fixed/stationary, and shrinks the vectors in the y-axis by 2%, which is similar to what A is doing; A is simply doing those same things in different directions.

(o) Note the pattern:

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}, A^3 = A^2A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}, \ldots$$

Therefore, we have

$$A^k = PD^kP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & (0.98)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (0.98)^k \\ 1 & -(0.98)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (0.98)^k & 0 \\ 1-(0.98)^k & 1 \end{bmatrix}.$$

This implies that

$$\vec{x}_k = A^k\vec{x}_0 = \begin{bmatrix} (0.98)^k & 0 \\ 1-(0.98)^k & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 8(0.98)^k \\ 8(1-(0.98)^k) \end{bmatrix},$$

matching our previous computations.

(p) If there were say H_0 Humans and Z_0 Zombies to begin with, then after k weeks there would have been:

$$\vec{x}_k = A^k\vec{x}_0 = \begin{bmatrix} (0.98)^k & 0 \\ 1-(0.98)^k & 1 \end{bmatrix}\begin{bmatrix} H_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} (0.98)^k Z_0 \\ (1-(0.98)^k)H_0 + Z_0 \end{bmatrix} \text{ billion}$$

Taking the limit as $k \to \infty$, we get

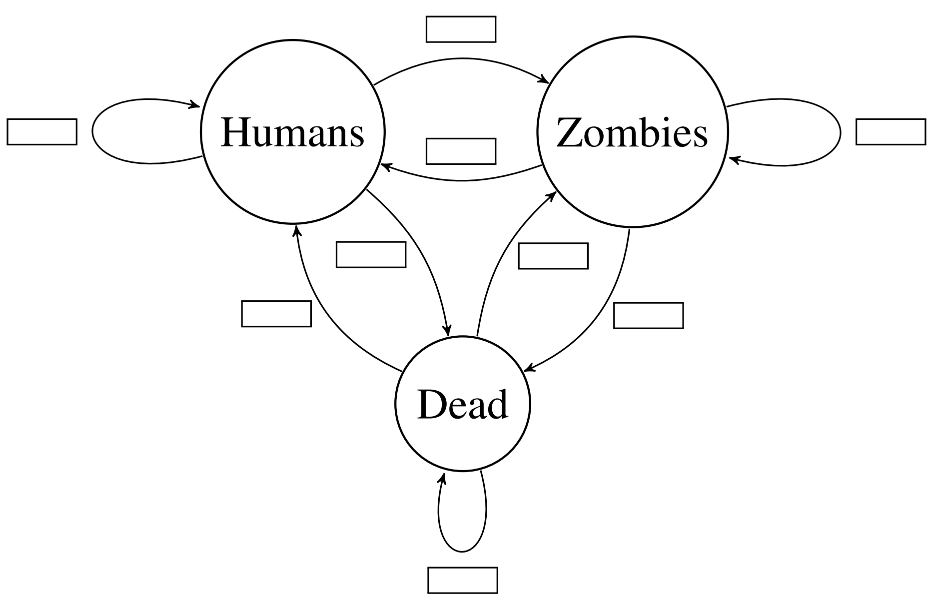
$$\lim_{k\to\infty} \vec{x}_k = \lim_{k\to\infty} A^k\vec{x}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} H_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ H_0 + Z_0 \end{bmatrix} \text{ billion.}$$

So, the entire population of Humans turns to Zombies as expected no matter the starting numbers of Humans and Zombies.

(2) **Zombie Apocalypse Model 2**. Suppose a virus that turns Humans to Zombies suddenly appeared. Let's assume that the population of the world initially was $H_0 = 8$ billion Humans, $Z_0 = 0$ billion Zombies, and $D_0 = 0$ billion Dead. Let H_k, Z_k, and D_k be the population of Humans, Zombies, and Dead respectively at the end of Week k. The transformation each week is give by the following matrix equation:

$$\begin{bmatrix} H_{k+1} \\ Z_{k+1} \\ D_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0.02 & 0.1 & 0 \\ 0.08 & 0.9 & 1 \end{bmatrix}\begin{bmatrix} H_k \\ Z_k \\ D_k \end{bmatrix}$$

Fill out the following diagram with percentages according to this transformation and then describe in your own words what's happening? What percentages of Humans are turning to Zombies for instance? Then answer the analog questions as in the **Zombie Apocalypse Model 1**.



(3) **Google Engine Search.** Suppose the vector $\vec{b} = \begin{bmatrix} 3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$ encodes a certain google search phrase we typed into www.google.com. Further suppose Google's web-page database $\mathfrak{D}$ is described as the span of the

basis $\left\{\begin{bmatrix}1\\2\\-2\\0\end{bmatrix},\begin{bmatrix}-3\\0\\4\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. [1] The goal of this question is to find a best match for the vector $\vec{b}$ in our database by answering the following questions.[2]

(a) Write a vector equation describing the problem, and then translate it to a matrix equation. Rephrase the question from the vector equation perspective.

(b) Generate an orthonormal basis for our database $\mathfrak{D}$ through the Gram-Schmidt process.[3]

(c) Compute $\mathrm{Proj}_{\mathfrak{D}}(\vec{b})$.

(d) Solve $A\vec{x} = \mathrm{Proj}_{\mathfrak{D}}\vec{b}$, where A is the matrix made up by the basis of $\mathfrak{D}$.

(e) What do the components of the solution $\vec{x}$ mean in the context of the Google search? Hint: Change of Basis

References

[1] OpenAI. (2023). *ChatGPT: Language Model*. Retrieved [today's date], from `https://chat.openai.com/`.

[2] Dale Hoffman. *Contemporary Calculus*. Washington State Colleges. Retrieved July 15th, 2024, from `https://www.contemporarycalculus.com/dh/Calculus_all/Calculus_all.html`

[3] Austin, D. (2023). *Understanding Linear Algebra*. Department of Mathematics, Grand Valley State University. Retrieved July 15th, 2024, from `https://understandinglinearalgebra.org/ula.html`.

[4] Clontz, S., & Lewis, D., et al. (2023). *Linear Algebra for Team-Based Inquiry Learning*. Department of Mathematics and Statistics, University of South Alabama. Retrieved July 15th, 2024, from `https://teambasedinquirylearning.github.io/linear-algebra/2023/linear-algebra-for-team-based-inquiry-learning.html`.

[5] Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). *Inquiry Oriented Linear Algebra: Course Materials*. Available at `http://iola.math.vt.edu`. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

[1] In reality these vectors have way more than 4 dimension depending on the length of the google search and the various attributes of each web-page, but the idea is still the same

[2] Note that each element of the database is currently described as a vector in 4D, which needs 4 components based on various characteristics of the web-page it's encoding. However, the database is only 3D due to having a basis of 3 vectors (it's geometrically a hyperplane in 4D space), and so each web-page can thus be encoded using only 3 numbers (the coordinates in this basis), which therefore reduce data storage by 25%. This can be a huge win considering in reality we have billions of web-pages in the database to store and it's an indispensable part of data compression. The 3 coordinates in the new basis are generally meaningless in the real world, so the user will still need their coordinates in the standard basis of 4D, which is why change of basis one would need to find a 4th vector linearly independent to the given 3 that forms a basis of $\mathbb{R}^4$ and keep performing a change of basis.

[3] It is super useful to have an orthonormal basis for a database, because then switching between bases avoids the tedious and expensive calculation of the inverse of the change of basis matrix, which in the case of an orthonormal matrix is simply the transpose. It also makes finding projections super easy as you'll see on the next part.