Calculus II Math 152 Learning Objectives and Questions

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Introduction

This document is a list of TILT-ed (Transparency In Teaching and Learning) learning objectives and several corresponding questions for Math 152 Calculus II challenges, based on Hoffman OER textbook content [1]. Many questions or ideas for questions were taken from the resources available at MathQUEST [4], the Cornell GoodQuestions project [5], and the Calculus Video Project [3]. The purpose is for both instructors and students to be transparent on the exact goals of the course. CORE Learning Objectives are especially important for student success, and they will be tested on the Final Part 1.

Learning Objectives

1. **Riemann Sum Computations [CORE].** I can estimate the signed area between a function and the x-axis ever more accurately using Riemann sums. This is perhaps the most important idea in all of Calculus 2 since in later applications we use the same process to estimate various quantities using their rates of change.

   Criteria for Success: I can

   - calculate a left, right, or midpoint Riemann sum for a given function and partition
   - go back and forth between an expanded sum and its sigma notation
   - use the sigma notation to compute Riemann sums for an arbitrary number of rectangles
   - use Riemann sums to estimate the displacement of some moving object, or total accumulation of some quantity

   Resources:

   - Fall 2020 Challenges question number 1 in each challenge (see our Class Notebook on Canvas)
   - Week 1 InClassActivity01: Questions 3, 4, 5, 6, 11
   - Week 1 Online Homework: 4.1: Question 5
   - Week 1 Post-Monday-Class-Activities
   - Week 1 Reading and Videos: 4.0, 4.1, 4.2
   - Week 2 Pre-Monday-Class-Activities except InClassActivity02

Sample questions:

- Consider the graph of $f(x) = x/2$ on the interval $[1, 4]$.
  (a) Approximate the signed area between $y = x/2$ and the x-axis on the interval $[1, 4]$ using any Riemann Sum you wish with 3 rectangles of equal base lengths. Make sure to draw a sketch of the function and the rectangles clearly indicating your choice of left, right, or midpoint Riemann sum. Use desmos `https://www.desmos.com/calculator/oceoomwdiy` to help with visualization.

  Sketch of function with rectangles: Riemann sum computation:

  (b) Express the above Riemann sum computation using sigma notation, where the expression inside the sigma symbo is suppose to be an explicit function of k.

  $$\sum_{k=}$$

- Find the value V of the Riemann sum $V = \sum_{k=1}^{n} f(c_k)\triangle x_k$ for the function $f(x) = x^2 - 5$ using the partition $P = \{1, 4, 5, 8\}$, where the c_k are the midpoints of the partition.
- Scientists have mapped out a 100-km path on the surface of Mars for a rover to follow, and have collected satellite data about the composition of the Martian surface at various points along the route using a LiDAR Spectrometer. Find left and right Riemann sums for dust accumulation. Report answers using sigma notation, where the expression inside the sigma symbo is suppose to be an explicit function of the index k of summation .

Composition	p, the position along path (km)	R(p), the amount of dust per distance traveled (mg/km)
Very sandy	0	6
Moderately sandy	20	3.5
Slightly sandy	40	2.5
Slightly rocky	60	2
Moderately rocky	80	1.5
Very rocky	100	1

$$\sum_{k=}$$

- Express the signed area between $y = x/2$ and the x-axis on the interval $[1, 4]$ in sigma notation using any Riemann Sum you wish with n rectangles of equal base lengths for some arbitrary whole number n. Note that the answer is supposed to contain the variables k, n in it since we would like a general formula we can easily modify for different values of n, and the summation index is k.

$$\sum_{k=}$$

2. **Definite Integrals [CORE].** I can switch between different ways of thinking about the definite integral. The definite integral is the main mathematical concept we will analyze in this course to compute all sorts of quantities depending on the context.

Criteria for Success: I can

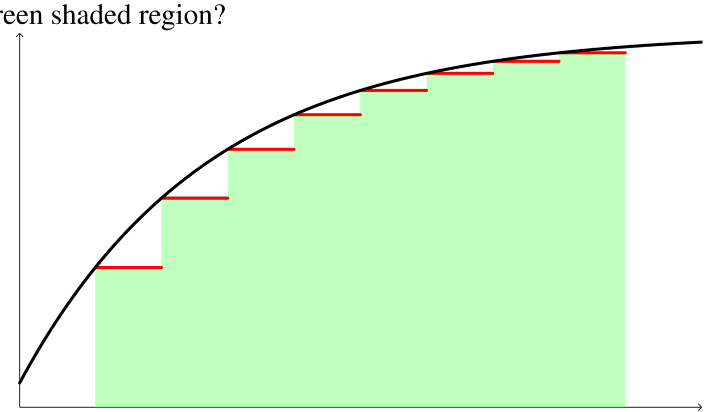
- analyze the Riemann sum definition of a definite integral
- use area formulas from geometry to compute a definite integral, and vice verse, use a definite integral to compute areas
- use properties of definite integrals as needed without relying on the Fundamental Theorem of Calculus.

Resources:

- Fall 2020 Challenges question number 2 in each challenge (see our Class Notebook on Canvas)
- Week 1 InClassActivity01: Questions 11, 15, 16, 17
- Week 1 Online Homework: 4.0 Questions 4, 9, 10; 4.2 All Questions; 4.3 All Questions
- Week 1 Reading and Videos: 4.2, 4.3
- Week 2 Pre-Monday-Class-Activities All items with Definite Integrals in title.
- Week 2 InClassActivity02: Questions 2
- Week 2 Online Homework: 4.4 Question 1; 4.5 Questions 8, 9

Sample questions:

- Answer the following questions:
  (a) What is the formal Riemann sum definition of a definite integral $\int_a^b f(x)dx$?
  (b) Which expression from the Riemann Sum definition of the definite integral represents the green shaded region?



  (a) $f(x_k)\triangle x$
  (b) $\sum_{k=0}^{7} f(x_k)\triangle x$

(c) $\int_a^b f(x)dx$

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\triangle x$

(e) none of the above

- Find the following definite integrals using area formulas from geometry and properties of definite integrals, without using the Fundamental Theorem of Calculus. Sketch a graph representing the integral.

  (a) $\int_{-3}^{10} |9-x| - 5dx =$

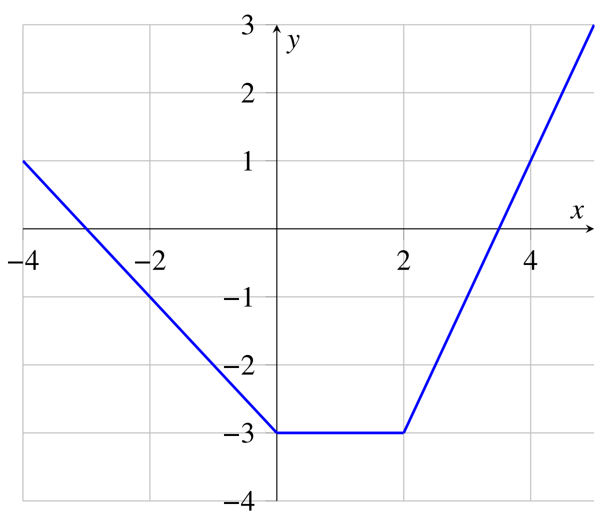
  (b) $\int_{-3}^{3} 5\sqrt{9-x^2}dx =$

- Let $A(x) = \int_{-2}^{x} 2tdt$. Answer the following questions **using geometry**, without using the Fundamental Theorem of Calculus. See `https://www.desmos.com/calculator/ihgs6a6nlv` to help with visualization.

  (a) $A(2) =$

  (b) For any real number x, we have $A(x) =$

- Use the sketch below of the graph of $y = f(x)$ on the interval $[-4, 5]$ to answer the following questions using geometric arguments such as formulas of areas from geometry and properties of definite integrals.



  (a) $\int_{-4}^{3} 2f(x)dx =$

  (b) $\int_{-1}^{1} f(x+1)dx =$

  (c) $\int_{-1}^{1} f(x) + 1dx =$

3. **FTC Applications [CORE].** I can use the Fundamental Theorem of Calculus (FTC) to compute displacement and accumulation. The Fundamental Theorem of Calculus is really the magic behind this class, which lets us compute easily and exactly quantities such as areas of curved shapes that are otherwise impossible to compute. Often times, we are able to measure, or are given information about the rate of change of some quantity. We can use definite integrals to compute the change of the quantity in question by interpreting it as the signed area between the rate of change graph and the x-axis on some desired interval.

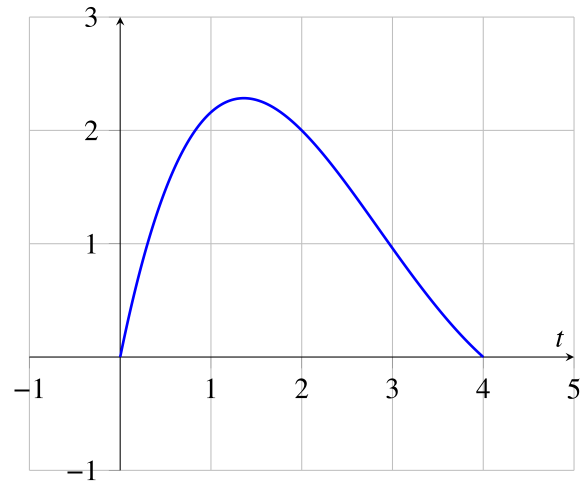
   Criteria for Success: I can

   - compute the distance and displacement of a moving object using a definite integral
   - find the total net change or accumulation of some quantity through a definite integral, and use appropriate units
   - find antiderivatives of functions using the Fundamental Theorem of Calculus
   - analyze definite integrals with functions as bounds

   Sample questions:

   - Water is pouring out of a pipe at the rate of $f(t)$ gallons/minute as graphed below.

t	0					
$W(t)$	0					



   (a) Fill out the table above with relevant times and amount of water $W(t)$ that flows from the pipe starting from $t = 0$, if initially no water had poured out, i.e., $W(0) = 0$. You do not need to fill up the whole table, and values of t could be decimals as well.

   (b) Use the above table and graph to describe the amount of water $W(t)$ that flows from the pipe in your own words. Make sure to include appropriate units in your description.

   (c) Circle all that apply. The amount of water that flows from the pipe between $t = 2$ and $t = 4$ minutes can be represented by:

      (a) $\int_2^4 f(x)dx$
      (b) $f(4) - f(2)$
      (c) $W(4) - W(2)$
      (d) $(4 - 2)f(4)$
      (e) the average of $f(4)$ and $f(2)$ times the amount of time that elapsed

(f) none of the above

(d) Circle all that apply. How is $F(t) = \int_2^t f(x)dx$ related to $W(t)$?

(a) $F(t) = W(t)$

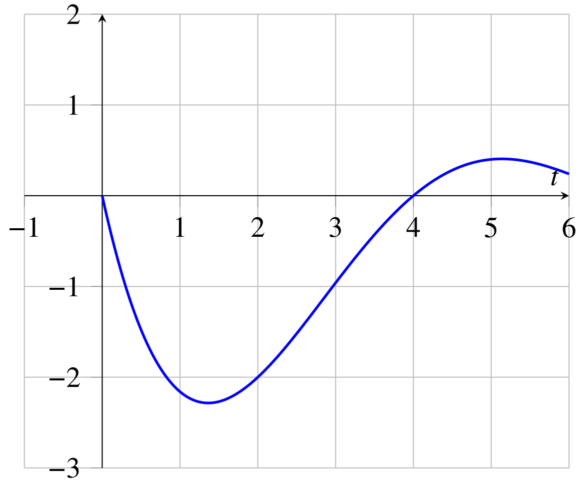
(b) $F(t) = W(t) - W(2)$

(c) $F(t)$ and $W(t)$ are antiderivatives of $f(t)$.

(d) None of the above

- Let the graph below $y = v(t)$ be the velocity of a particle in m/s moving in a straight path on $[0, 6]$. Assume that the position of the particle at time t seconds is given by $S(t)$.

t	0					
$S(t)$	-3					



(a) Fill out the table above with the relevant times and positions of the particle. You do not need to fill up the whole table, and values of t could be decimals as well.

(b) Use the above table and graph to describe the movement of the particle in your own words. Make sure to include appropriate units in your description.

(c) Find the total **distance** (not displacement) the particle moves from $t = 0$ to $t = 6$. Make sure to include appropriate units.

(d) At what time was the particle furthest away from the starting point? Why?

(e) Circle all that apply. How is $F(t) = \int_2^t v(x)dx$ related to $S(t)$?

(a) $F(t) = S(t)$

(b) $F(t) = S(t) + C$ for some constant C.

(c) $F'(t) = S'(t) = v(t)$ for any t in $[0, 6]$.

(d) None of the above

4. **Integrals [CORE].** I can use the antiderivative formulas, u-substitution, and other properties of integrals to compute both definite and indefinite integrals. I'm generally against memorizing formulas, but memorizing properties of integrals, and antiderivatives of common functions are as useful as knowing basic arithmetic, since they are widely used almost every class period in future math classes (most instructors in future math classes will not allow you to use a formula sheet either). The idea behind u-substitution, i.e., of translating your question from the x-world to the u-world so that it's easier to solve there, and then translating the answer back to the x-world is a really common idea in mathematics that will be used in many other forms in your future math classes.

   Criteria for Success: I can

   - compute definite and indefinite integrals of common functions and combinations of them
   - describe the meaning of definite and indefinite integrals
   - perform u-substitution on both definite and indefinite integrals
   - show a good understanding of all the various parts of the u-substitution process

   Sample questions:

   - Find the definite integral $\int_0^1 \frac{1}{2^x} - 4\sec^2(x)dx$, and describe what it means. Clearly state u and du if needed.
   - Find the indefinite integral $\int \frac{1}{2^x} - 4\sec^2(x)dx$, and describe what it means. Clearly state u and du if needed.
   - Find the following integrals, and describe how the answer is related to the integrand function. Clearly state u and du if needed.
     (a) $\int 4x\sec^2(x^2)dx =$
     (b) $\int_0^1 \frac{1}{2^x} - 4\sec^2(x)dx$
   - Circle all that apply and for each option explain why you chose to circle it or not. Clearly state u and du if needed. If $f(x)$ is a continuously differentiable function, then $\int xf'(2x^2)dx$ is
     (a) $\frac{f(2x^2)}{4}$
     (b) $\frac{e^{-x^2}}{-2x}$
     (c) $\frac{F(2x^2)}{4}$
     (d) none of the above
   - Circle all that apply and for each option explain why you chose to circle it or not. Clearly state u and du if needed. The integral $\int_0^{\pi/2} \cos(x)e^{\sin(x)}dx$ is equal to
     (a) $e - 1$
     (b) $e^{\pi/2} - 1$
     (c) $e^{\sin(x)} + C$

(d) $\int_0^{\pi/2} e^u du$

(e) $\int_0^1 e^u du$

(f) $\int_0^{\pi/2} \cos(x) e^u du$

(g) $\int_0^1 \cos(x) e^u dx$

5. **Integrals Challenge.** I deeply understand the concepts behind Riemann sums, definite integrals, and their connection to antiderivatives and indefinite integrals through the Fundamental Theorem of Calculus. It is crucial that we deeply understand the way antiderivatives, definite and indefinite integrals are related, as they're the foundation of calculus. Learning the concepts behind these mathematical objects helps to be able to use them creatively, or generalize to other Calculi. In fact, there's infinitely many calculi that can be develop parallel to the one we're studying in this class, with a huge variety of uses.

   Criteria for Success: I can solve conceptual questions related to Riemann sums, definite integrals, and the Fundamental Theorem of Calculus that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

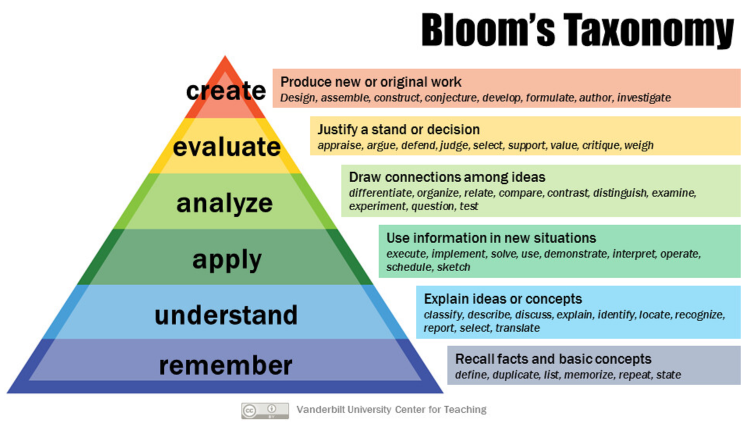
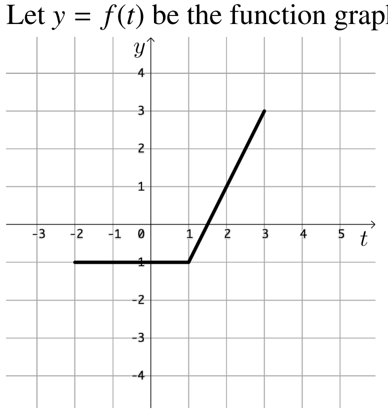


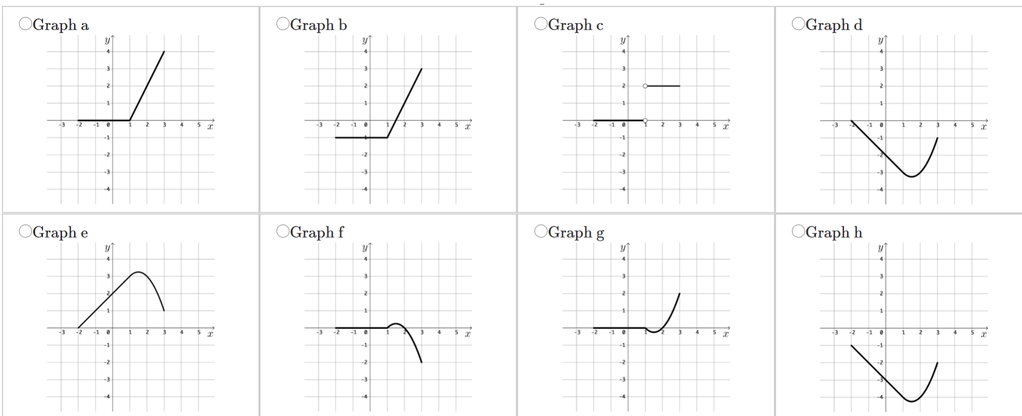
Figure 1: Bloom's Taxonomy. Source: [2]

   Sample questions:

   - True or False. As long as $f(x)$ is defined on $[a,b]$ for some $a < b$, then $\int_a^b f(x)dx$ makes sense, and it's a number.
   - True or False. Let f be continuous on the interval $[a,b]$. The limit of the Riemann Sum with n rectangles of equal base length $\triangle x$,
   $$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\triangle x$$
   may lead to different limits if we choose c_k to be the left-endpoints instead of the midpoints.
   - Let $y = f(t)$ be the function graphed below on the interval $[-2, 3]$.



Which of the following choices could be the corresponding graph of $F(x) = \int_{-2}^{x} f(t)dt$? Explain your thought process.



- Let $F(x) = \int_{-5}^{x} t^2 + \sin(t)dt$. Find the following.
  (a) $F(-5) =$
  (b) $F'(x) =$
  (c) $\dfrac{d}{dx}\left(\int_{\ln(x)}^{\sin(x)} t^2 + \sin(t)dt\right) =$
- Circle all functions that are antiderivatives of $f(x) = e^{-x^2}$, and explain why.
  (a) $e^{-x^3}\dfrac{1}{3}$
  (b) $\dfrac{e^{-x^2}}{-2x}$
  (c) $\int_3^t f(x)dx$
  (d) $\int_3^x e^{-t^2}dx$
  (e) $\int_3^x e^{-t^2}dt$
  (f) $\int_3^{-x} -e^{-t^2}dt$
  (g) none of the above
- True or False. Given any continuous function $f(t)$ defined on the interval $[a, b]$, by the Fundamental Theorem of Calculus we have that the following equality holds. Explain your answer.

$$\int_a^b f(x)dx = \int_a^x f(t)dt + C$$

- Circle and Explain. Let $f(x)$ and $g(x)$ be continuous on $[0, 1]$. If $\int_0^1 f(x)dx = \int_0^1 g(x)dx$ we can deduce
  (a) $f(x) = g(x)$ for x in $[0, 1]$

(b) $\int f(x)dx = \int g(x)dx$ for x in $[0, 1]$

(c) (a) and (b)

(d) none of the above

6. **Area [CORE].** I can use the "divide and conquer strategy"[1] to find areas. In geometry classes so far we have learned formulas for computing areas of shapes that have straight edges, or edges created from circles. In real life, most shapes don't fall into these categories, but with the divide and conquer method of calculus, we can still express their areas as signed areas between a function and the x-axis (i.e., as definite integrals). If these functions happen to have nice antiderivatives, we can use the Fundamental Theorem of Calculus to find exact answers, or otherwise use Riemann sums to get arbitrarily good approximations.

   Criteria for Success: I can

   - use the divide and conquer method to slice a region vertically or horizontally, find the area of a general slice, and setup the corresponding Riemann sum and definite integral
   - solve questions related to computing areas or average values of functions

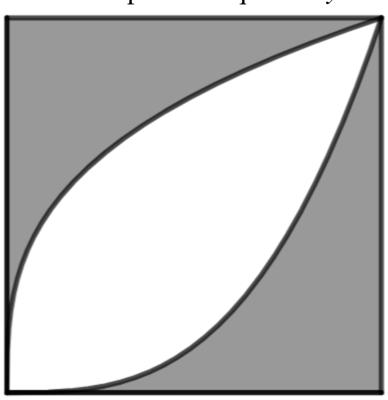
   Sample questions:

   - Sketch and compute the area of the region on the first quadrant enclosed by the graphs $y = 1/x$, $y = x$, and $y = 1/2$. Check out Desmos for help with visualizing: `https://www.desmos.com/calculator/gmdidaeuhy`.

     Area of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

   - What is the area of the white eye in the middle of the following unit square given that one of the curved shapes has equation $y = x^2$, and the other is its reflection about the diagonal of the square?



[1]**Divide and Conquer Strategy:**

Step 1. Divide the larger problem into smaller "*similar*" parts (slices).

Step 2. Solve it for a general representative of those parts (slices).

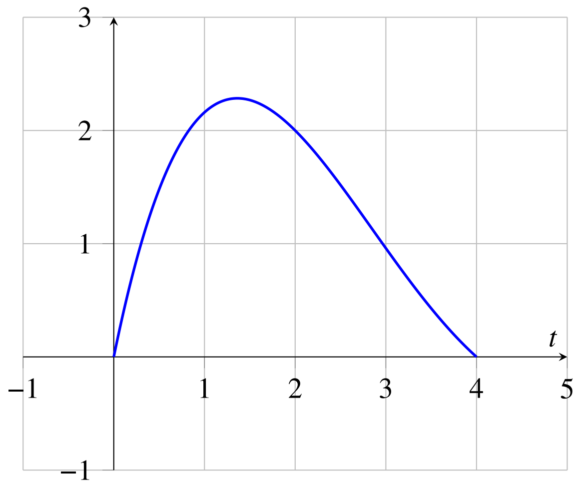
Step 3. Sum up the solution of those parts (slices) using summation notation.

Step 4. Rewrite the sum as a Riemann sum.

Step 5. Take the limit to form the corresponding definite integral.

Step 6. Use various calculus techniques to solve it.

- Suppose you deposit \$4000 at 8% interest compounded continuously. Find the average value of your account during the first 4 years. The amount of money into your account is given by $A(t) = Pe^{rt}$, where P is the initial deposit (principal), r is the interest rate, and t is time in years.
- Water is pouring out of a pipe at the rate of $f(t)$ gallons/minute as graphed below. Estimate the average value of the rate $f(t)$ over the interval $[0, 4]$.



7. **Arclength and Surface Area.** I can use the divide and conquer strategy to find arclengths and surface areas. In geometry classes so far we have learned formulas for computing arclengths of paths formed by a combination of straight edges, or edges created from a piece of a circle; and analogously we have learned formulas for computing surface areas that are mostly formed by flat shapes, or parts of spheres. In real life, most shapes don't fall into these categories, but with the divide and conquer method of calculus, we can still express their arclengths or surface areas as signed areas between a function and the x-axis (i.e., as definite integrals). If these functions happen to have nice antiderivatives, we can use the Fundamental Theorem of Calculus to find exact answers, or otherwise use Riemann sums to get arbitrarily good approximations.

   Criteria for Success: I can

   - use the divide and conquer method to slice a path/shape, find the arclength/surface area of a general slice, and setup the corresponding Riemann sum and definite integral
   - solve questions related to computing arclengths both using Cartesian and parametric equations
   - solve questions related to computing surface areas of revolved graphs about some line

   Sample questions:

   - What is the arc-length of one period of $y = \sin(x)$? **Setup but do not solve the answer involving a definite integral.**

     Length of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

   - Use the parametric formula for arc-length to find the length of the curve given by $x(t) = \cos(t)$ and $y(t) = \sin(t)$ over the interval $[0, 2\pi]$. **Setup but do not solve the answer involving a definite integral.**

     Length of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

   - What is the surface area formed by rotating the function $y = x^3$ on $[0, 2]$ around the x-axis. **Setup but do not solve the answer involving a definite integral.**

     Surface area of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

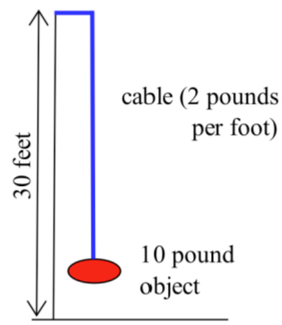
8. **Work.** I can use the divide and conquer strategy to find work. In introductory physics courses we learn how to compute work as force multiplied by distance. If the force varies along the distance, we can use the divide and conquer strategy of Calculus to compute work by expressing it as the area between a function and the x-axis (i.e., a definite integral).

Criteria for Success: I can

- use the divide and conquer method to split the work process into small parts, find the work of a general part, and setup the corresponding Riemann sum and definite integral
- solve questions related to computing work, including questions that require the use of similar figures, and Pythagorean Theorem.

Sample questions:

- How much work is done lifting a 10 pound object from the ground to the top of a 30 foot building if the cable used weighs 2 pounds per foot? **Setup but do not solve the definite integral.**



Work for slice: Riemann Sum:

Sketch of slice: Definite Integral:

- How much work do you do when drinking water in a cup using a straw to a point 2 inches above its top edge? Assume that the graph $x = e^y$ for y in $[0, 1]$ models the side edge of this cup with the bottom on the x-axis and the y-axis pointing vertically, and that the weight density of water is 0.5787 ounces/in^3. Furthermore, assume that the glass is filled with water up to $y = 0.5$ inches. See `https://www.desmos.com/calculator/l21bf2ptnf`. **Setup but do not solve the definite integral.**

Work for slice: Riemann Sum:

Sketch of slice: Definite Integral:

- How much work do you do when drinking water in a cup the shape of a half sphere of radius 5 inches using a straw to a point 2 inches above its top edge? Assume that the weight density of water is 0.5787 ounces/in^3. Furthermore, assume that the glass is filled with water up to $y = 0.5$ inches. **Setup but do not solve the definite integral.**

Work for slice: Riemann Sum:

Sketch of slice: Definite Integral:

- A trough is 3 meters long, 2 meters wide, and 1 meters deep. The vertical cross-section of the trough parallel to an end is shaped like an isosceles triangle (with height 1 meters, and base, on top, of length 2 meters). The trough is full of water (density 1000 kg/m^3). Find the amount of work in joules required to empty the trough by pumping the water over the top. (Note: Use g = 9.8 m/s^2 as the acceleration due to gravity.) **Setup but do not solve the definite integral.**

  Work for slice: Riemann Sum:

  Sketch of slice: Definite Integral:

9. **Volume and Mass/Weight [CORE].** I can use the divide and conquer strategy to find the volume, mass or weight of an object. In geometry classes so far we have learned formulas for computing volumes of shapes that have flat sides, or sides created from spheres. In real life, most shapes don't fall into these categories, but with the divide and conquer method of calculus, we can still express their volumes or masses as signed areas between a function and the x-axis (i.e., as definite integrals). If these functions happen to have nice antiderivatives, we can use the Fundamental Theorem of Calculus to find exact answers, or otherwise use Riemann sums to get arbitrarily good approximations. Humans like symmetry, and objects we get by revolving around a line are especially pleasing to look at, so they often make up various parts of the architecture of a building.

   Criteria for Success: I can

   - use the divide and conquer method to slice a shape, find the volume or mass/weight of a general slice using similar figures, or the Pythagorean Theorem, and setup the corresponding Riemann sum and definite integral
   - use the divide and conquer method to slice a shape of revolution, find the volume of a general slice by discs/washers or tubes/shells, and setup the corresponding Riemann sum and definite integral

   Sample questions:

   - A boneless baked turkey breast that is ten inches long from one end to the other is sliced up into very thin slices. Each slice has a cross-sectional area of $(-x^2 + 10x)$ square inches for each x between 0 and 10. What is the volume of the turkey breast? Show work by filling out the information below. **Setup but do not solve the answer involving a definite integral.**

     Volume of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

   - Suppose the density of a conical rod of height 10 m and circular base of radius 0.1 m has a density of material varying height-wise given by $\rho(y) = \frac{y^3}{(y^2+1)^3}$ kg/m^3, where y is the height in meters. See `https://www.desmos.com/calculator/z65c8niznw` for a visual. What is the mass of the rod? Show work by filling out the information below. **Setup but do not solve the answer involving a definite integral.**

     Mass of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

   - Find the volume of a sphere of radius R centered at the origin by using the divide and conquer method with vertical slices perpendicular to the x axis.

     Volume of slice: Riemann Sum:

     Sketch of slice: Definite Integral:

- What is the volume of the solid formed by taking the the area between $y = \sin(x)$ and $y = 2$ on the interval $[0, 2\pi]$, and rotating it about $y = 2$? Check out Desmos for a visual here: `https://www.desmos.com/calculator/s8ou42dgv2`. **Setup but do not solve the definite integral.**

  Volume of slice: Riemann Sum:

  Sketch of slice: Definite Integral:

- What is the volume of the solid formed by taking the the area between $y = \sin(x)$ and $y = 2$ on the interval $[0, 2\pi]$, and rotating it about $x = -1$? **Setup but do not solve the definite integral.**

  Volume of slice: Riemann Sum:

  Sketch of slice: Definite Integral:

10. **Divide and Conquer Applications Challenge.** I can analyze the divide and conquer strategy in a variety of applications including area, average value, arclength, surface area, work, volume, mass/weight and center of mass by the divide and conquer strategy. The heart of Calculus 2 is the divide and conquer strategy, which can be applied to seemingly unrelated questions both in this and future math classes.

    Criteria for Success: I can solve conceptual questions related to arclength, surface area, volume, work, and center of mass that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

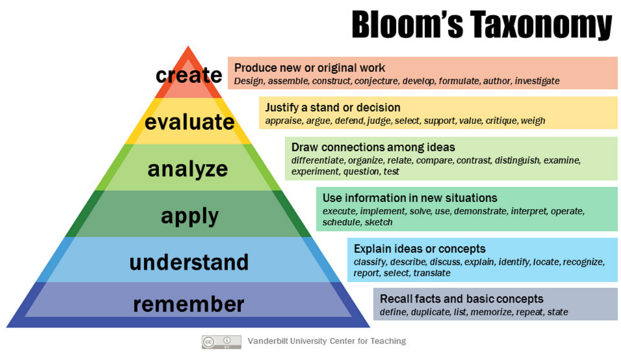


Figure 2: Bloom's Taxonomy. Source: [2]

Sample questions:

- Consider the region under $y = 4 - x^2$ and above the x-axis. Find the number L such that the line $y = L$ cuts the area of this region in half. Consider using Desmos to help.
- Circle all that apply and explain. Suppose we want to find the arclength of a function $f(x)$ on the interval $[a, b]$. We partition $[a, b]$ into n pieces. The arclength of the i-th line segment is
  - (a) $f(x_i)\triangle x_i$
  - (b) $\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$.
  - (c) $\triangle x_i$
  - (d) $\sqrt{1 + f'(c_i)^2}\triangle x_i$ for some c_i in $[x_{i-1}, x_i]$.
  - (e) none of the above
- Circle all that apply and explain. Consider the center of mass of the region between the x-axis and $f(x)$ on $[2, 3]$, assuming uniform weight distribution. Assume also that $f(x)$ is positive, increasing, and such that $f(2) = 0$, and $f(3) = 1$. The the moment $\dfrac{1}{2}\displaystyle\int_2^3 f(x)^2 dx$ is also equal to
  - (a) $\displaystyle\int_0^1 y(3 - f^{-1}(y))dy$
  - (b) $\displaystyle\int_2^3 x(3 - f^{-1}(x))dx$
  - (c) $\displaystyle\int_2^3 xf(x)dx$
  - (d) $\displaystyle\int_0^1 yf(y)dy$
  - (e) none of the above

11. **Integration By Parts.** I can use integration by parts to solve integrals. This method is especially useful in Fourier Analysis for computing Fourier series coefficients, or Fourier Transforms.

Criteria for Success: I can

- identify u and dv for applying integration by parts
- use the integration by parts formula to trade a harder integral with an easier one

Sample questions:

- Use integration by parts to solve the following integrals:
  (a) $\int_0^1 x^2 e^{-x} dx =$
  (b) $\int \arcsin(x) dx =$
- Let $f(x)$ be twice continuously differentiable. Find the integral in terms of $f(x)$ or its derivatives.
  $\int x f''(x) dx =$
- Consider the integral $\int x^2 \sin(x) dx$.
  (a) Let $u = x^2$, $dv = \sin(x)dx$ and apply integration by parts once. Do not try to solve the integral you get.
  $\int x^2 \sin(x) dx =$
  (b) Let $u = \sin(x)$, $dv = x^2 dx$ and apply integration by parts once. Do not try to solve the integral you get.
  $\int x^2 \sin(x) dx =$
  (c) Use one of the above methods to finish solving the integral.
  $\int x^2 \sin(x) dx =$

12. **Partial Fractions.** I can apply partial fraction as needed. Partial Fractions is useful for solving a big array of integrals involving rational functions, but it's also useful to solve recurrence relations in Discrete Structures too.

Criteria for Success: I can

- use long division in combination with partial fractions to solve integrals
- decompose a fraction into a sum of two or more fractions of a specific form.

Sample questions:

- Find A, B, C such that $\dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{(x+1)^2}$. No need to solve any integrals here.
- Solve the following integral using partial fractions: $\displaystyle\int \frac{x^3 - 1}{x(x+1)} dx$

13. **Trigonometric Substitution.** I can apply trigonometric substitution as needed. All the formulas we learned in the table of integrals involving inverse trigonometric functions can be derived using trigonometric substitution.

Criteria for Success: I can

- identify the correct trigonometric substitution
- perform trigonometric substitution as needed

Sample questions:

- Select the correct substitution, and carry it through.

$$\int \frac{\sqrt{16x^2 - 49}}{x^4} dx =$$

- Carry out an appropriate trigonometric substitution for solving the definite integral $\int x\sqrt{9 + 4x^2} dx$. Stop once the integral is written in terms of the angle θ. Clearly state what x and dx are in terms of θ.

14. **Improper Integrals.** I can recognize and use limit notation to solve improper integrals. Improper integrals are crucial to understand several convergence tests for infinite series in Calculus 3. Interesting applications such as the classic overhang problem or Gabriel's Horn depend on computing improper integrals.

Criteria for Success: I can

- solve improper integrals with continuity issues or where at least one of the bounds is $\pm\infty$
- identify improper integrals
- split up an improper integrals so that each of them has only one improper bound
- use appropriate limit notation and properties to solve improper integrals

Sample questions:

- Circle all improper integrals from the list below, and then write each as a sum of limits of integrals. No need to solve the respective pieces.
  (a) $\int \frac{1}{x} dx$
  (b) $\int_0^1 \frac{1}{2x-1} dx$
  (c) $\int_4^5 \frac{1}{5x-1} dx$
  (d) $\int_{-\infty}^{\infty} \frac{\sin(x)}{1+3x^2} dx$
  (e) $\int_1^3 \ln(x-1) dx$
- Solve the following integrals by hand. For improper integrals, I'm testing to see in particular appropriate limits wherever needed. **Hint:** Use (a) to do parts (b) and (c). Then, use those for part (d).
  (a) $\int \frac{1}{\sqrt[3]{x}} dx =$
  (b) $\int_0^1 \frac{1}{\sqrt[3]{x}} dx =$
  (c) $\int_1^{\infty} \frac{1}{\sqrt[3]{x}} dx =$
  (d) $\int_0^{\infty} \frac{1}{\sqrt[3]{x}} dx =$
- Find the error in the following computation, and then solve the improper integral correctly. I'm testing to see in particular appropriate limits wherever needed.
  $\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx = (\infty)^2 \ln(1+(\infty)^2) - (-\infty)^2 \ln(1+(-\infty)^2) = \infty - \infty = 0$

15. **Differential Equations.** I can solve and/or analyze differential equations or initial value problems. Most of the time in the real world we model a construction or process using rates of change, which lead to differential equations.

Criteria for Success: I can

- use the slope field to analyze solutions of differential equations or initial value problems
- solve differential equations or initial value problems using separation of variables
- check if a given function is a solution to a differential equation or initial value problem
- find constants in given functions so that they become solutions of differential equations or initial value problems

Sample questions:

- The solution to a differential equation is $y = e^{x^3}$. Which of the following direction fields would correspond to that differential equation? Explain in words why that's the case?

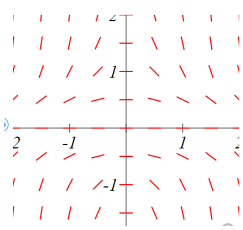


Figure 3: Slope Field

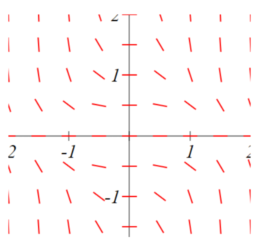


Figure 4: Slope Field

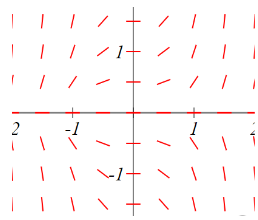


Figure 5: Slope Field

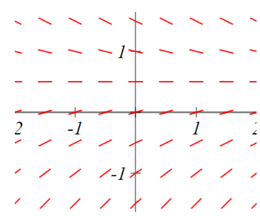
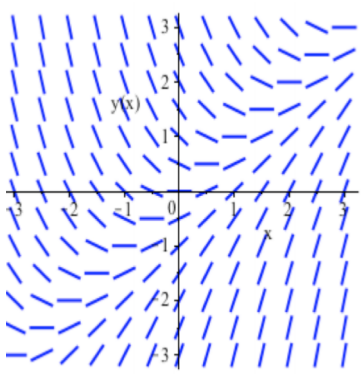


Figure 6: Slope Field

- Given the following slope field of an initial value problem with initial condition $y(0) = 1$, sketch the graph of its solution.



- Consider the differential equation $y'' - 4y' - 4y = e^x \iff \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 4y = e^x$. Is $y = Ae^x + B$ a solution for some constants A and B?
- Solve the following initial value problem: $\dfrac{dy}{dx} = \dfrac{9 + 13x}{xy^2}$ for $x > 0$, with initial condition $y(1) = 4$.

16. **Integration Techniques and Differential Equations Challenge.** I can use improper integrals and differential equations creatively in new situations that require a deep understanding of them. Modeling the spread of a virus such as COVID-19 based on different criteria, and new factors requires a deep understanding of differential equations. The fact that the area under the bell curve in statistics is 1 (or 100%) and other important formulas related to it come from computing improper integrals.

    Criteria for Success: I can solve conceptual questions related to improper integrals and differntial equations that lie on the top half of Bloom's Taxonomy (analyze, evaluate, and create) shown below.

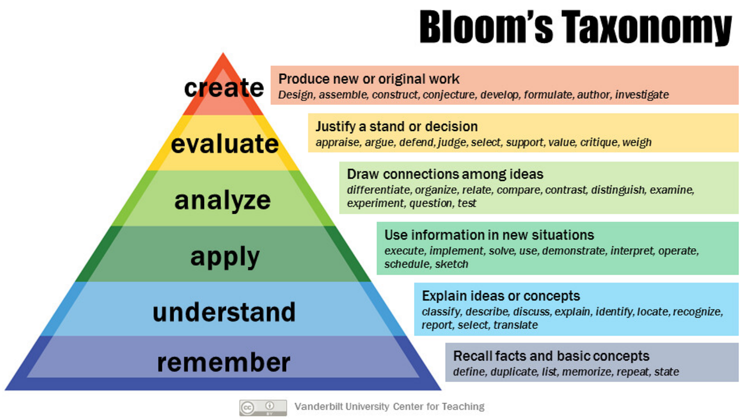


Figure 7: Bloom's Taxonomy. Source: [2]

Sample questions:

- Which integration techniques can be used to find the integral of the function $\frac{f(x)}{x^2+1}$ for some polynomial $f(x)$? Explicitly list the steps to solve this integral.
- Consider the integral $\int \sin(x)f(x)dx$ for some function $f(x)$.
  (a) Pick a function $f(x)$ so that the integration by parts with $u = \sin(x)$ and $dv = f(x)dx$ can help you solve the given integral. Then show that this is the case by solving it.

  $$\int \sin(x)f(x)dx =$$

  (b) Pick a function $f(x)$ so that you don't need to use integration by parts to solve the given integral. Then show that this is the case by solving it.

  $$\int \sin(x)f(x)dx =$$

- Find a function $f(x)$ with a vertical asymptote at $x = 0$ such that the improper definite integral $\int_0^1 f(x)dx$ diverges, and show all work checking that indeed it does diverge.
- Create an initial value problem whose solution is the function $y = \pi$.
- Suppose $f'(x) = 1/x^2$ and $f(1) = 1$. Explain why $f(-1) = 3$ is incorrect.

References

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