THE ESSENTIALS OF

# Computer Organization and Architecture

THIRD EDITION

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# Chapter 3 Special Section

## Focus on Karnaugh Maps

# 3A.1 Introduction

- Simplification of Boolean functions leads to simpler (and usually faster) digital circuits.
- Simplifying Boolean functions using identities is time-consuming and error-prone.
- This special section presents an easy, systematic method for reducing Boolean expressions.

# 3A.1 Introduction

- In 1953, Maurice Karnaugh was a telecommunications engineer at Bell Labs.
- While exploring the new field of digital logic and its application to the design of telephone circuits, he invented a graphical way of visualizing and then simplifying Boolean expressions.
- This graphical representation, now known as a Karnaugh map, or Kmap, is named in his honor.

# 3A.2 Description of Kmaps and Terminology

- A Kmap is a matrix consisting of rows and columns that represent the output values of a Boolean function.
- The output values placed in each cell are derived from the minterms of a Boolean function.
- A *minterm* is a product term that contains all of the function's variables exactly once, either complemented or not complemented.

# 3A.2 Description of Kmaps and Terminology

- For example, the minterms for a function having the inputs *x* and *y* are: $\bar{x}\bar{y}, \bar{x}y, x\bar{y},$ and $xy$
- Consider the Boolean function, $F(x,y) = xy + x\bar{y}$
- Its minterms are:

| Minterm | X | Y |
|---|---|---|
| $\bar{X}\bar{Y}$ | 0 | 0 |
| $\bar{X}Y$ | 0 | 1 |
| $X\bar{Y}$ | 1 | 0 |
| $XY$ | 1 | 1 |

# 3A.2 Description of Kmaps and Terminology

- Similarly, a function having three inputs, has the minterms that are shown in this diagram.

| Minterm | X | Y | Z |
|---|---|---|---|
| $\overline{X}\overline{Y}\overline{Z}$ | 0 | 0 | 0 |
| $\overline{X}\overline{Y}Z$ | 0 | 0 | 1 |
| $\overline{X}Y\overline{Z}$ | 0 | 1 | 0 |
| $\overline{X}YZ$ | 0 | 1 | 1 |
| $X\overline{Y}\overline{Z}$ | 1 | 0 | 0 |
| $X\overline{Y}Z$ | 1 | 0 | 1 |
| $XY\overline{Z}$ | 1 | 1 | 0 |
| $XYZ$ | 1 | 1 | 1 |

# 3A.2 Description of Kmaps and Terminology

- A Kmap has a cell for each minterm.
- This means that it has a cell for each line for the truth table of a function.
- The truth table for the function *F(x,y) = xy* is shown at the right along with its corresponding Kmap.

F(X,Y) = XY

| X | Y | XY |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

# 3A.2 Description of Kmaps and Terminology

- As another example, we give the truth table and KMap for the function, $F(x,y) = x + y$ at the right.
- This function is equivalent to the OR of all of the minterms that have a value of 1. Thus:

$$F(x,y) = x + y = \bar{x}y + x\bar{y} + xy$$

F(X,Y) = X+Y

| X | Y | X+Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

# 3A.3 Kmap Simplification for Two Variables

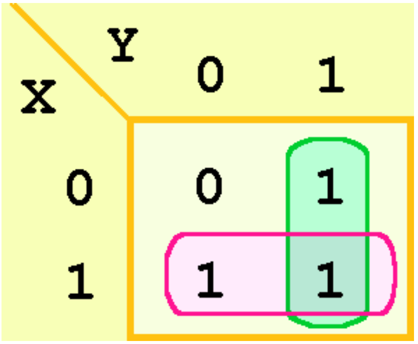
- Of course, the minterm function that we derived from our Kmap was not in simplest terms.
  - That's what we started with in this example.
- We can, however, reduce our complicated expression to its simplest terms by finding adjacent 1s in the Kmap that can be collected into groups that are powers of two.

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

- In our example, we have two such groups.
  - Can you find them?

# 3A.3 Kmap Simplification for Two Variables

- The best way of selecting two groups of 1s form our simple Kmap is shown below.
- We see that both groups are powers of two and that the groups overlap.
- The next slide gives guidance for selecting Kmap groups.



# 3A.3 Kmap Simplification for Two Variables

The rules of Kmap simplification are:

- Groupings can contain only 1s; no 0s.
- Groups can be formed only at right angles; diagonal groups are not allowed.
- The number of 1s in a group must be a power of 2 – even if it contains a single 1.
- The groups must be made as large as possible.
- Groups can overlap and wrap around the sides of the Kmap.

# 3A.3 Kmap Simplification for Three Variables

- A Kmap for three variables is constructed as shown in the diagram below.
- We have placed each minterm in the cell that will hold its value.
  - Notice that the values for the *yz* combination at the top of the matrix form a pattern that is not a normal binary sequence.

| x \ yz | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $\overline{x}\,\overline{y}\,\overline{z}$ | $\overline{x}\,\overline{y}z$ | $\overline{x}yz$ | $\overline{x}y\overline{z}$ |
| 1 | $x\overline{y}\,\overline{z}$ | $x\overline{y}z$ | $xyz$ | $xy\overline{z}$ |

# 3A.3 Kmap Simplification for Three Variables

- Thus, the first row of the Kmap contains all minterms where *x* has a value of zero.
- The first column contains all minterms where *y* and *z* both have a value of zero.

| x \ yz | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $\overline{x}\,\overline{y}\,\overline{z}$ | $\overline{x}\,\overline{y}z$ | $\overline{x}yz$ | $\overline{x}y\overline{z}$ |
| 1 | $x\overline{y}\,\overline{z}$ | $x\overline{y}z$ | $xyz$ | $xy\overline{z}$ |

# 3A.3 Kmap Simplification for Three Variables

- Consider the function:

$$F(X,Y) = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}Z + XYZ$$

- Its Kmap is given below.
  - What is the largest group of 1s that is a power of 2?

| X \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |

# 3A.3 Kmap Simplification for Three Variables

- This grouping tells us that changes in the variables *x* and *y* have no influence upon the value of the function: They are irrelevant.
- This means that the function,

$$F(X,Y) = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}Z + XYZ$$

reduces to $F(x) = z$.

| X \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |

**You could verify this reduction with identities or a truth table.**

# 3A.3 Kmap Simplification for Three Variables

- Now for a more complicated Kmap. Consider the function:

$$F(X,Y,Z) = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,\overline{Y}Z + \overline{X}YZ + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XY\overline{Z}$$

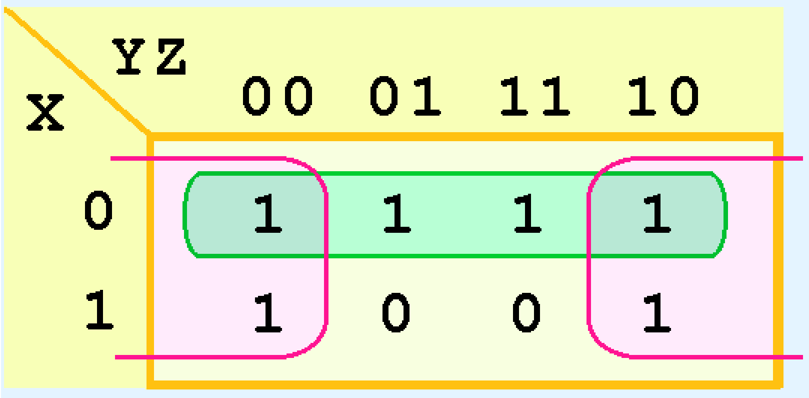
- Its Kmap is shown below. There are (only) two groupings of 1s.
  - Can you find them?

| X \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |

# 3A.3 Kmap Simplification for Three Variables

- In this Kmap, we see an example of a group that wraps around the sides of a Kmap.
- This group tells us that the values of *x* and *y* are not relevant to the term of the function that is encompassed by the group.
  - What does this tell us about this term of the function?

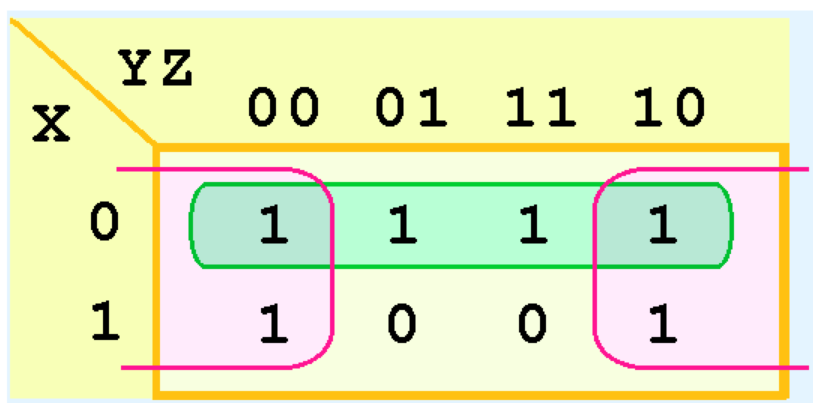
**What about the green group in the top row?**



# 3A.3 Kmap Simplification for Three Variables

- The green group in the top row tells us that only the value of *x* is significant in that group.
- We see that it is complemented in that row, so the other term of the reduced function is $\overline{X}$ .
- Our reduced function is: $F(X,Y,Z) = \overline{X}+\overline{Z}$

**Recall that we had six minterms in our original function!**



# 3A.3 Kmap Simplification for Four Variables

- Our model can be extended to accommodate the 16 minterms that are produced by a four-input function.
- This is the format for a 16-minterm Kmap.

| WX \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | $\overline{W}\overline{X}\overline{Y}\overline{Z}$ | $\overline{W}\overline{X}\overline{Y}Z$ | $\overline{W}\overline{X}YZ$ | $\overline{W}\overline{X}Y\overline{Z}$ |
| 01 | $\overline{W}X\overline{Y}\overline{Z}$ | $\overline{W}X\overline{Y}Z$ | $\overline{W}XYZ$ | $\overline{W}XY\overline{Z}$ |
| 11 | $WX\overline{Y}\overline{Z}$ | $WX\overline{Y}Z$ | $WXYZ$ | $WXY\overline{Z}$ |
| 10 | $W\overline{X}\overline{Y}\overline{Z}$ | $W\overline{X}\overline{Y}Z$ | $W\overline{X}YZ$ | $W\overline{X}Y\overline{Z}$ |

# 3A.3 Kmap Simplification for Four Variables

- We have populated the Kmap shown below with the nonzero minterms from the function:

$$F(W,X,Y,Z) = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z} + \overline{W}\,\overline{X}\,\overline{Y}Z + \overline{W}\,\overline{X}Y\overline{Z} + \overline{W}XY\overline{Z} + W\overline{X}\,\overline{Y}\,\overline{Z} + W\overline{X}\,\overline{Y}Z + W\overline{X}Y\overline{Z}$$

  – Can you identify (only) three groups in this Kmap?

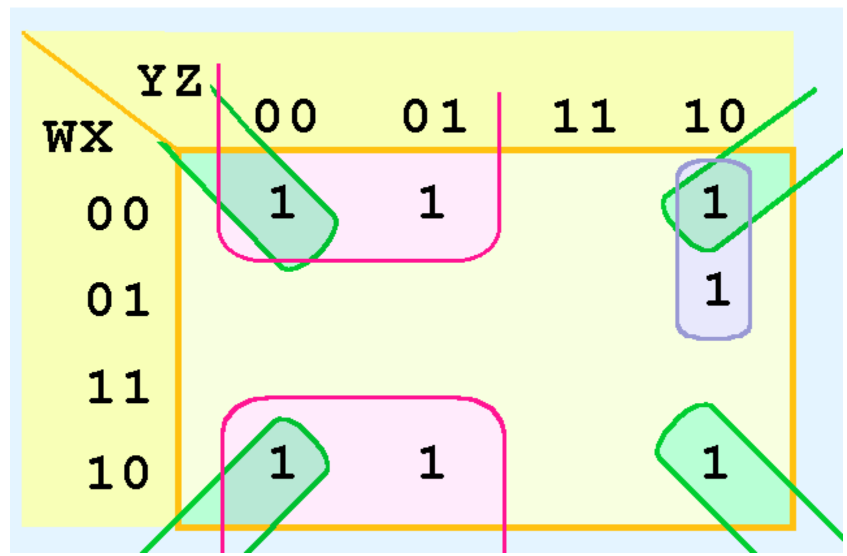
Recall that groups can overlap.

| WX \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | | 1 |
| 01 | | | | 1 |
| 11 | | | | |
| 10 | 1 | 1 | | 1 |

# 3A.3 Kmap Simplification for Four Variables

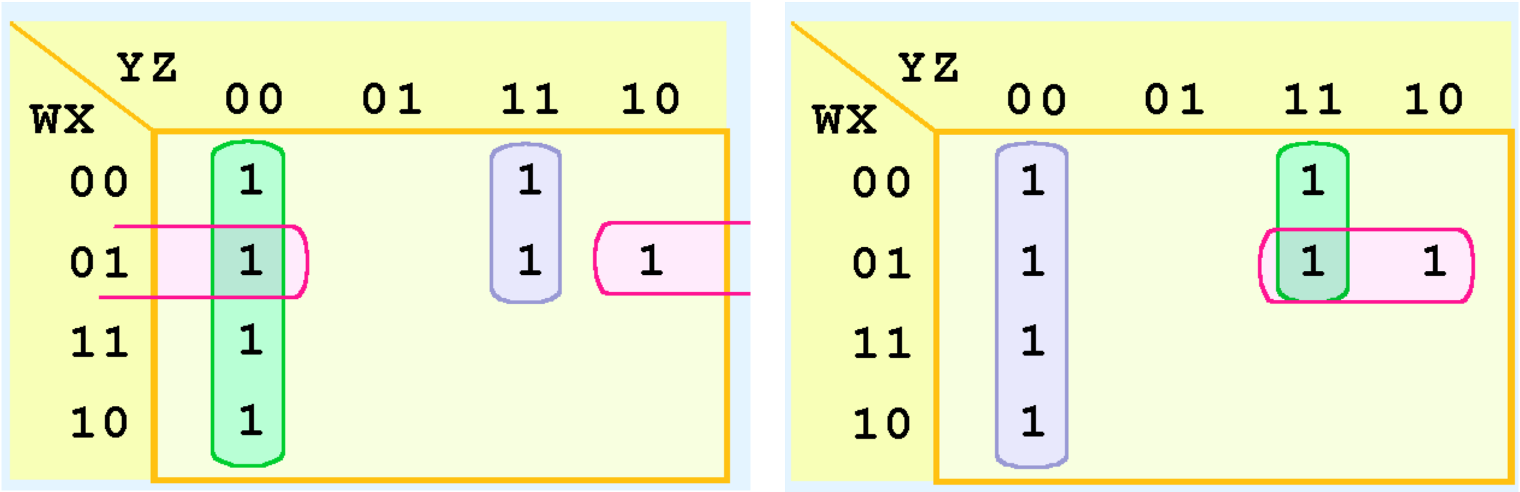
- Our three groups consist of:
  - A purple group entirely within the Kmap at the right.
  - A pink group that wraps the top and bottom.
  - A green group that spans the corners.
- Thus we have three terms in our final function:

$$F(W,X,Y,Z) = \overline{W}\,\overline{Y} + \overline{X}\,\overline{Z} + \overline{W}Y\overline{Z}$$



# 3A.3 Kmap Simplification for Four Variables

- It is possible to have a choice as to how to pick groups within a Kmap, while keeping the groups as large as possible.
- The (different) functions that result from the groupings below are logically equivalent.



# 3A.6 Don’t Care Conditions

- Real circuits don’t always need to have an output defined for every possible input.
  - For example, some calculator displays consist of 7-segment LEDs. These LEDs can display $2^7$ -1 patterns, but only ten of them are useful.
- If a circuit is designed so that a particular set of inputs can never happen, we call this set of inputs a *don’t care* condition.
- They are very helpful to us in Kmap circuit simplification.

# 3A.6 Don’t Care Conditions

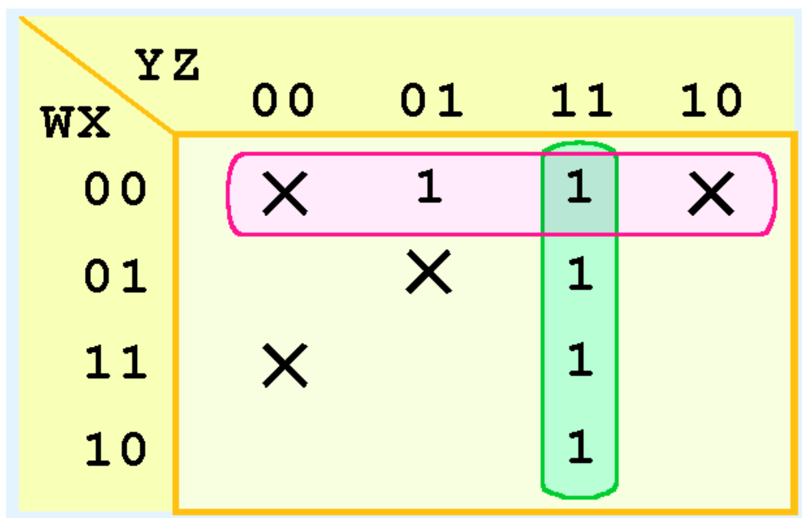
- In a Kmap, a don’t care condition is identified by an *X* in the cell of the minterm(s) for the don’t care inputs, as shown below.
- In performing the simplification, we are free to include or ignore the *X*’s when creating our groups.

| WX \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | X | 1 | 1 | X |
| 01 | | X | 1 | |
| 11 | X | | 1 | |
| 10 | | | 1 | |

# 3A.6 Don't Care Conditions

- In one grouping in the Kmap below, we have the function:

$$F(W,X,Y,Z) = \overline{W}\,\overline{Y} + YZ$$



# 3A.6 Don't Care Conditions

- A different grouping gives us the function:

$$F(W,X,Y,Z) = \bar{W}Z + YZ$$

| WX \ YZ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | × | 1 | 1 | × |
| 01 | | × | 1 | |
| 11 | × | | 1 | |
| 10 | | | 1 | |

# 3A.6 Don’t Care Conditions

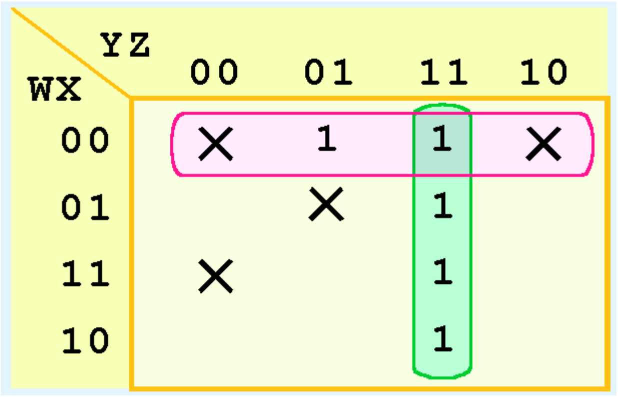
- The truth table of:

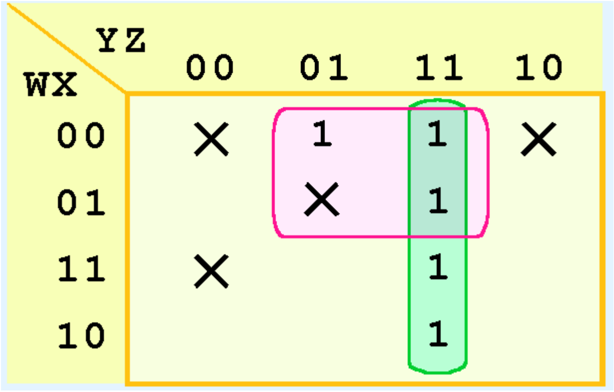
$$F(W,X,Y,Z) = \overline{W}\,\overline{Y} + YZ$$

differs from the truth table of:

$$F(W,X,Y,Z) = \overline{W}Z + YZ$$

- However, the values for which they differ, are the inputs for which we have don’t care conditions.



# 3A Conclusion

- Kmaps provide an easy graphical method of simplifying Boolean expressions.
- A Kmap is a matrix consisting of the outputs of the minterms of a Boolean function.
- In this section, we have discussed 2- 3- and 4-input Kmaps. This method can be extended to any number of inputs through the use of multiple tables.

# 3A Conclusion

Recapping the rules of Kmap simplification:

- Groupings can contain only 1s; no 0s.
- Groups can be formed only at right angles; diagonal groups are not allowed.
- The number of 1s in a group must be a power of 2 – even if it contains a single 1.
- The groups must be made as large as possible.
- Groups can overlap and wrap around the sides of the Kmap.
- Use don’t care conditions when you can.

End of Chapter 3A