Refined Forms of Continuity

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Motivation: Is there something "between" f being continuous and f being differentiable?

→ Yes! In fact, many possible things.

We will cover just a few (sec. 3.4)

Def. Let $S \subset \mathbb{R}$, $f: S \to \mathbb{R}$.

- We say f is <u>uniformly continuous</u> if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in S$ with $|x-y| < \delta$,

$$|f(x) - f(y)| < \varepsilon$$

- We say f is <u>Lipschitz continuous</u> if there exists $K \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq K \cdot |x-y| \quad \forall x, y \in S$$

Ideas: (uniform continuity vs. continuity)

δ depends on ε, c

f is continuous at c $\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

f is continuous $\Leftrightarrow$ $\forall c \in S, \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

f is uniformly cts. $\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in S$ with $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$

δ can only depend on ε, not c!

("hierarchy" of continuity)

- For $c \in S$,

  f differentiable at c $\Rightarrow$ f is continuous at c (proved earlier)

- For an interval $I \subset \mathbb{R}$, $f: I \to \mathbb{R}$

  differentiable + $\Rightarrow$ Lipschitz cts $\Rightarrow$ uniformly cts $\Rightarrow$ continuous

- For an interval $I \subset \mathbb{R}$, $f: I \to \mathbb{R}$

differentiable + bounded derivative $\Rightarrow$ Lipschitz cts. $\Rightarrow$ uniformly cts. $\Rightarrow$ continuous

- For a closed and bounded interval $f:[a,b] \to \mathbb{R}$

cts. derivative $\Rightarrow$ bounded derivative uniformly cts. $\Leftrightarrow$ continuous

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Uniform Continuity

Prop. Let $S \subset \mathbb{R}$, $f: S \to \mathbb{R}$. If f is uniformly continuous, then it is continuous.

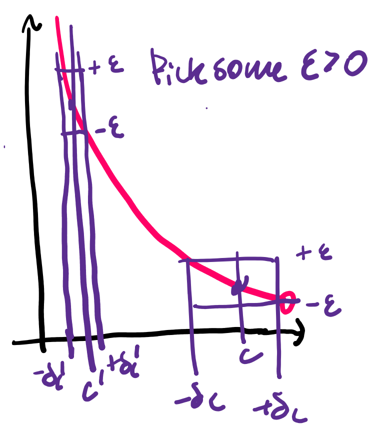
Pf. Let $c \in S$, $\varepsilon > 0$[1] be arbitrary.

f is unif. cts. $\Rightarrow$ $\exists \delta > 0$[2] $: \forall x, y \in S$ with $|x-y|$, $|f(x) - f(y)| < \varepsilon$

Take $y = c$. $\forall x \in S$[3] with $|x-c| < \delta$, $|f(x) - f(c)|$[4] $< \varepsilon$ $\square$

Claim. Let $f: (0,1) \to \mathbb{R}$ $f(x) := 1/x$. Then f is continuous but not uniformly continuous. (i.e. cts $\not\Rightarrow$ uniformly cts. for general S)

Idea:



$\delta_{c'} < \delta_c$

Pf. f is continuous, e.g. by continuity of alg. op.

$\left(c \neq 0, \ \lim_{x \to c} \frac{1}{x} = \frac{1}{c} = f(c) \right)$

- To show f is not uniformly cts., need to prove

$\exists \varepsilon > 0$[1] $: \forall \delta > 0$[2], $\exists x, y \in S$[3] with $|x-y| < \delta : |f(x) - f(y)|$[4] $\geq \varepsilon$

- Pick $\varepsilon = 1 > 0$[1]. Let $\delta > 0$[2] be arbitrary. Define $\eta := \min\{\delta, \frac{1}{2}\}$, and pick $x := \eta$, $y := \eta/2$ (so $x, y \in S$ with $|x-y| < \delta$). Then

• Pick $\varepsilon = 1 > 0$. Let $\delta > 0$ be arbitrary. Define $\eta = \min\{\delta, \frac{1}{2}\}$, and pick $\underbrace{x = \eta,\ y = \eta/2 \text{ (so } x, y \in S \text{ with } |x-y| < \delta)}_{(3)}$. Then,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y-x|}{x \cdot y} = \frac{2|y-x|}{\eta^2} && (x = \eta,\ y = \eta/2) \\
&\geq \frac{2 \cdot \big||y| - |x|\big|}{\eta^2} && (\text{reverse triangle ineq.}) \\
&= \frac{\eta}{\eta^2} = \frac{1}{\eta} && (x = \eta,\ y = \eta/2) \\
&\underbrace{\geq 2 > \varepsilon}_{(4)} && (0 < \eta \leq \tfrac{1}{2})
\end{aligned}$$

• Thus, f is not uniformly continuous. $\square$

<u>Thm.</u> (3.1.4)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

<u>Pf.</u> Will prove by contrapositive.

- Suppose f is not uniformly cts. (want to show f is not cts. at some c)

  $\Rightarrow \exists\, \varepsilon > 0 : \forall\, \delta > 0,\ \exists\, x, y \in [a,b]$ with $|x-y| < \delta : |f(x) - f(y)| \geq \varepsilon$

- Take $\delta_n := 1/n > 0$. Then, $\forall n \in \mathbb{N},\ \exists\, x_n, y_n \in [a,b]$ with $|x_n - y_n| < \delta_n$ such that

$$|f(x_n) - f(y_n)| \geq \varepsilon$$

- By B-W, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $c := \lim_{k \to \infty} x_{n_k}$. $\quad x_n \in [a,b] \Rightarrow c \in [a,b]$.

  - $|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| = \frac{1}{n_k} + |x_{n_k} - c|$
  - (RHS) $\to 0$ as $k \to \infty$, so $y_{n_k} \to c$ as $k \to \infty$ (can you show it?)

- If f were continuous at c,

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$$\lim_{k\to\infty}|f(x_{n_k})-f(y_{n_k})| = |f(\lim x_{n_k}) - f(\lim y_{n_k})| = |f(c)-f(c)|=0$$

but

$$|f(x_n)-f(y_n)| \geq \varepsilon \quad \forall n\in\mathbb{N} \Rightarrow \lim_{k\to\infty}|f(x_{n_k})-f(y_{n_k})| \geq \varepsilon > 0$$

which is a contradiction!

- Thus, f is not continuous at $c\in[a,b] \Rightarrow f$ is not continuous. $\square$

Lipschitz Continuity

Prop. If $f: I\to\mathbb{R}$ is differentiable and $f': I\to\mathbb{R}$ is bounded, then f is Lipschitz continuous.

Pf. (By MVT, on HW)

Claim. Let $f:[-1,1]\to\mathbb{R}$, $f(x):=|x|$. f is Lipschitz continuous but not differentiable (i.e. Lipschitz cts. $\not\Rightarrow$ bdd. deriv)

Pf. $\forall x,y\in[-1,1]$,

$$|f(x)-f(y)| = \big||x|-|y|\big| \leq 1\cdot|x-y|$$

$\Rightarrow f$ is Lipschitz continuous, with Lipschitz constant $K=1$.

- $|x|$ is not differentiable at $x=0$. $\square$

Prop. Let $S\subset\mathbb{R}$, $f:S\to\mathbb{R}$. If f is Lipschitz, then it is uniformly continuous.

Pf. Suppose f is Lipschitz. $\Rightarrow \exists K\in\mathbb{R}: \forall x,y\in S,\ |f(x)-f(y)|\leq K\cdot|x-y|$

Let $\overset{(1)}{\varepsilon>0}$ be arbitrary. Take $\delta \overset{(2)}{:=} \varepsilon/K$. Then, $\overset{(3)}{\forall x,y\in S}$ with $|x-y|<\delta$.

— [illegible] $|f(x)-f(y)| \le K \cdot |x-y|$

Let $\varepsilon \overset{(1)}{>} 0$ be arbitrary. Take $\delta \overset{(2)}{:=} \varepsilon / K$. Then, $\forall x, y \overset{(3)}{\in} S$ with $|x-y| < \delta$,

$$\underline{|f(x)-f(y)|} \le K \cdot |x-y| \underset{(4)}{\underline{<}} K \cdot \delta = K \cdot \frac{\varepsilon}{K} = \underline{\varepsilon}$$

$\Rightarrow$ f is uniformly continuous. $\square$

<u>Claim.</u> Let $f : [0,1] \to \mathbb{R}$, $f(x) := \sqrt{x}$. Then, f is uniformly continuous but not Lipschitz cts. (uniform cts. $\not\Rightarrow$ Lipschitz cts.)

<u>Pf.</u> f is continuous on $[0,1]$ $\Rightarrow$ f is uniformly continuous.

Suppose f were Lipschitz. $\Rightarrow$ $\exists K \in \mathbb{R} : \forall x, y \in [0,1]$

$$|f(x)-f(y)| \le K|x-y| \;\Rightarrow\; \left|\frac{f(x)-f(y)}{x-y}\right| \le K$$

$\Rightarrow$ f' is bounded. But $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded.

Hence, f is not Lipschitz. $\square$