HW5

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1.(a) For $c \in [a, b)$, $\forall \delta > 0$,

$\exists x = c + \frac{1}{2}\min\{\delta, b-c\}$ s.t. $x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

$\Rightarrow c$ is a cluster point

For $c = b$, $\forall \delta > 0$,

$\exists x = b - \frac{1}{2}\min\{\delta, b-a\}$ s.t. $x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

$\Rightarrow c$ is a cluster point

For $c < a$,

let $\delta = \frac{a-c}{2}$, then $(c-\delta, c+\delta) \cap S \setminus \{c\} = \emptyset$

$\Rightarrow c$ is not a cluster point

For $c > b$,

let $\delta = \frac{c-b}{2}$, then $(c-\delta, c+\delta) \cap S \setminus \{c\} = \emptyset$

$\Rightarrow c$ is not a cluster point

Therefore, $[a, b]$ is the set of all cluster points of $S$.

(b) $S = \mathbb{Z}$

$\forall c \in \mathbb{Z}$, let $\delta = \frac{1}{2}$, then $(c-\delta, c+\delta) \cap S \setminus \{c\} = \emptyset$

$\forall c \in \mathbb{R} \setminus \mathbb{Z}$, let $\delta = \frac{1}{2}\min\{c - \lfloor c \rfloor, \lceil c \rceil - c\}$,

then $(c-\delta, c+\delta) \cap S \setminus \{c\} = \emptyset$

Therefore, $S$ has no cluster points in $\mathbb{R}$.

(c) $S = \mathbb{Q}$

$\forall c \in \mathbb{R}$, $\forall \delta > 0$,

By HW2 P1, $\exists r \in S$ s.t. $|c - r| < \delta \Rightarrow (c-\delta, c+\delta) \cap S \setminus \{c\} \neq \emptyset$

Therefore, $\mathbb{R}$ is the set of all cluster points of $S$.

2.(a) $\forall c \in (0, \infty)$

$\forall \varepsilon > 0, \exists \delta = \sqrt{c}\,\varepsilon > 0 : \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

$|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{1}{\sqrt{c}}|x-c|$

$< \frac{1}{\sqrt{c}}\delta \leq \frac{1}{\sqrt{c}}\sqrt{c}\,\varepsilon = \varepsilon$

When $c = 0$,

$\forall \varepsilon > 0, \exists \delta = \varepsilon^2 > 0 : \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

$|f(x) - \sqrt{c}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

Therefore, $\lim_{x \to c} f(x) = \sqrt{c}$ for all $c \in [0, \infty)$.

So $f$ is a continuous function.

(b) $\forall c \in \mathbb{R}$

$\forall \varepsilon > 0, \exists \delta = \varepsilon > 0 : \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

$|f(x) - \cos(c)| = |\cos(x) - \cos(c)|$

$= 2\left|\sin\left(\frac{x+c}{2}\right)\right| \cdot \left|\sin\left(\frac{x-c}{2}\right)\right|$

$\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right|$ since $|\sin(x')| \leq 1$

$\leq 2\left|\frac{x-c}{2}\right|$ since $|\sin(x')| \leq |x'|$

$= |x-c|$

$< \delta = \varepsilon$

So $\lim_{x \to c} f(x) = \cos(c)$ for all $c \in \mathbb{R}$

$f$ is a continuous function.

3.(a) Suppose $\lim_{x\to c} f(x) = L_1$, $\lim_{x\to c} g(x) = L_2$

By sequential limits lemma, $\forall \{x_n\}$ s.t. $x_n \in S\setminus\{c\}$ and $\lim_{n\to\infty} x_n = c$,

$\lim_{n\to\infty} f(x_n) = L_1$, $\lim_{n\to\infty} g(x_n) = L_2$

By Prop. 2.2.5, $\lim_{n\to\infty}(f(x_n)+g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L_1 + L_2$

$\Rightarrow \lim_{x\to c}(f(x)+g(x)) = (\lim_{x\to c} f(x)) + (\lim_{x\to c} g(x))$,

By Prop. 2.2.5, $\lim_{n\to\infty}(f(x_n)g(x_n)) = (\lim_{n\to\infty} f(x_n))(\lim_{n\to\infty} g(x_n)) = L_1 L_2$

$\Rightarrow \lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$

If $\lim_{x\to c} g(x) = L_2 \neq 0$,

By Prop 2.2.5, $\lim_{n\to\infty} \dfrac{f(x_n)}{g(x_n)} = \dfrac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \dfrac{L_1}{L_2}$

$\Rightarrow \lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

(b) By sequential limits lemma, $\forall \{x_n\}$ s.t. $x_n \in S\setminus\{c\}$ and $\lim_{n\to\infty} x_n = c$,

$\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x)$, $\lim_{n\to\infty} h(x_n) = \lim_{x\to c} h(x)$

$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n)$

By 2.2.1 Squeeze Lemma, since $f(x) \leq g(x) \leq h(x)$

$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n)$

By sequential limits lemma,

the limit of $g(x)$ as $x \to c$ exists and

$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$

4. If $c$ is a cluster point of $S \cap (-\infty, c)$ and $S \cap (c, \infty)$

$\Rightarrow \forall \delta > 0, \exists x \in S \cap (-\infty, c)$ s.t. $|x - c| < \delta$

$\Rightarrow \qquad \exists x \in S \setminus \{c\}$ s.t. $|x - c| < \delta$

$\Rightarrow c$ is a cluster point of $S$

4. (cont.) "$\Rightarrow$" If $\lim_{x\to c} f(x) = L$,

so $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}, |f(x) - L| < \varepsilon$

since $(c-\delta, c+\delta) \cap S \setminus \{c\} = ((c-\delta, c) \cap S) \cup ((c, c+\delta, c) \cap S)$

$\Rightarrow \forall x \in (c-\delta, c) \cap S, |f(x) - L| < \varepsilon \Rightarrow \lim_{x\to c^-} f(x) = L$

$\forall x \in (c, c+\delta) \cap S, |f(x) - L| < \varepsilon \Rightarrow \lim_{x\to c^+} f(x) = L$

$\Rightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

"$\Leftarrow$" If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, $\forall \varepsilon > 0, \exists \varepsilon > 0$

s.t. $\forall x \in (c-\delta, c) \cap S, |f(x) - L| < \varepsilon$

$\forall x \in (c, c+\varepsilon) \cap S, |f(x) - L| < \varepsilon$

$\Rightarrow \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}, |f(x) - L| < \varepsilon$

$\Rightarrow \lim_{x\to c} f(x) = L$

Note:

For $S' = S \cap (-\infty, c)$, $(c-\delta, c+\delta) \cap S' \setminus \{c\}$

$= (c-\delta, c+\delta) \cap S \cap (-\infty, c)$

$= (c-\delta, c) \cap S$

For $S' = S \cap (c, \infty)$, $(c-\delta, c+\delta) \cap S' \setminus \{c\}$

$= (c-\delta, c+\delta) \cap S \cap (c, \infty)$

$= (c, c+\delta) \cap S$

5. (a) Assume $\lim_{x \to x} f(x) = L$, let $\varepsilon = \frac{1}{2}$

then $\exists \delta > 0 : \forall x \in (-\delta, \delta) \setminus \{0\}, \ |f(x) - L| < \frac{1}{2}$

$\exists n \in \mathbb{N}$ st $n > \frac{1}{\pi \delta} \Rightarrow \frac{1}{n\pi} < \delta, \ \frac{1}{(n+1)\pi} < \delta$

$$\left| f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right) \right|$$

$$= |\cos(n\pi) - \cos(n+1)\pi| = |-1-1| = 2$$

$$\Rightarrow 2 = \left| f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right) \right|$$

$$\leq \left| f\left(\frac{1}{n\pi}\right) - L \right| + \left| f\left(\frac{1}{(n+1)\pi}\right) - L \right|$$

$$< \frac{1}{2} + \frac{1}{2} = 1 \quad \Rightarrow \text{contradiction}$$

So $\lim_{x \to 0} f(x)$ does not exist.

(b) $\forall \varepsilon > 0, \ \exists \delta = \sqrt{\varepsilon} > 0$ st. $\forall x \in (-\delta, \delta) \setminus \{0\}$

$$|f(x) - 0| = \left| x^2 \cos\left(\frac{1}{x}\right) \right|$$

$$= |x^2| \cdot \left| \cos\left(\frac{1}{x}\right) \right|$$

$$\leq x^2$$

$$< \delta^2 = \varepsilon$$

So $\lim_{x \to 0} f(0) = 0$

(c) $f: \mathbb{R} \to \mathbb{R}$, so $0$ is a cluster point

By Characterization of Continuity, $f$ is continuous at $0$

$\Leftrightarrow \lim_{x\to 0} f(x) = f(0) = b$

From (b) we know that $\forall \varepsilon > 0$, $\exists \delta = \sqrt{\varepsilon} > 0$ s.t.

$\forall x \in (-\delta, \delta) \setminus \{0\}$, $|f(x) - 0| < \varepsilon$

so $\lim_{x\to 0} f(x) = 0$

Therefore, $b$ is unique and $b = 0$.

6. (a) $\forall c \in \mathbb{R}$

$\exists L = |c|$: $\forall \varepsilon > 0$; $\exists \delta = \varepsilon > 0$: $\forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$,

$|f(x) - L| = \big||x| - |c|\big| \leq |x - c| < \delta = \varepsilon$

So $f$ is continuous at all $c \in \mathbb{R}$.

(b) $\forall a, b \in \mathbb{R}$, if $a \geq b$, $\max\{a, b\} = a = \frac{a+b+a-b}{2} = \frac{a+b+|a-b|}{2}$

if $a < b$, $\max\{a, b\} = b = \frac{a+b+b-a}{2} = \frac{a+b+|a-b|}{2}$

So $h(x) = \max\{f(x), g(x)\} = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$

$\forall c \in S$, since $f, g$ are continuous functions,

by composition of algebraic operations, $f(x) - g(x)$ is continuous at $c$

by (a) and composition of continuous functions, $|f(x) - g(x)|$ is continuous at $c$

by composition of algebraic operations, $h(x)$ is continuous at $c$

Therefore, $h(x)$ is continuous at all $c \in \mathbb{R}$.