Metric Spaces

Monday, December 12, 2022 10:35 AM

Motivation:

- Similar looking statements about sequences

→ $\{x_n\}$, $|x_n - x_k|$ cauchy ⟺ convergent "cauchy completeness"

→ $\{f_n\}$, $\|f_n - f_k\|_u$ cauchy ⟺ convergent

key step: triangle inequalities

→ $|x_n - x + x - x_k| \leq |x_n - x| + |x - x_k|$

→ $\|f_n - f + f - f_k\|_u \leq \|f_n - f\|_u + \|f - f_k\|_u$

- Continuity / switching limits

→ $x_n \to x$, $g: \mathbb{R} \to \mathbb{R}$ continuous : $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n)$

→ $f_n \in R[a,b]$, $f_n \to f$ uniformly (so $f \in R[a,b]$)

Note that $\int_a^b : R[a,b] \to \mathbb{R}$ "functional"

$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$ "$\int_a^b : R[a,b] \to \mathbb{R}$ is continuous"

- How much of this course can you generalize?

→ Some of it!

Outline of ch. 7:

- Metric spaces - generalize $|x_n - x + x - x_k| \leq |x_n - x| + |x - x_k|$
- Open/closed sets, "metric topology" - generalize $|x_n - x| < \varepsilon$

(turns out this is enough to generalize (a,b) and $[a,b]$)

- Sequences and convergence - generalize $x_n \to x$

generalize "no gaps" property of $\mathbb{R}$

- Sequences and convergence – generalize $x_n \to x$
- completeness and compactness – generalize "no gaps" property of $\mathbb{R}$, sequential compactness
- cts. functions, Fixed pt. thrm

Metric Spaces

Def. Let X be a set, and $d : X \times X \to \mathbb{R}$ be a function such that for all $x, y, z \in X$,

(i) $d(x,y) \geq 0$ (nonnegativity)

(ii) $d(x,y) = 0$ iff $x = y$ (nondegeneracy)

(iii) $d(x,y) = d(y,x)$ (symmetry)

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

The pair (X,d) is called a metric space, d is called the metric. Sometimes we write X as the metric space if d is clear from context.

Ex. Claim: $(\mathbb{R}, |\cdot|)$ is a metric space

$$d(x,y) := |x - y|$$

Pf. (verify properties)

(i, ii, iii) $|x-y| \geq 0$, $|x-y| = 0 \Leftrightarrow x = y$, $|x-y| = |y-x|$ ✓

(iv) $d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z)$ $\square$

Ex. Let $X := C^0([a,b], \mathbb{R})$ be the set of all continuous real-valued functions on the domain $[a,b]$. Define $d : X \times X \to \mathbb{R}$ by

$$d(f,g) := \|f - g\|_u \qquad f, g \in X$$

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Claim: (X,d) is a metric space

Pf. Let $f,g,h \in X$.

(i) $\|f-g\|_u = \sup\{\underbrace{|f(x)-g(x)|}_{\geq 0} : x \in [a,b]\} \geq 0$ ✓

(ii, iii) Left to reader

(iv) $d(f,h) = \|f-h\|_u = \|f-g+g-h\|_u \underbrace{\leq}_{HW} \|f-g\|_u + \|g-h\|_u = d(f,g) + d(g,h)$ ☑

Remark: norms vs. metrics

Informally:
- A metric measures distance between two points in a set
- A norm assigns a magnitude to one object in a set

- Ex. $|x|$ "norm"

  $d(x,y) = |x-y|$ "metric"

Def. Define the Euclidean norm for $x \in \mathbb{R}^n$

$$\|x\|_2 := \sqrt{\sum_{j=1}^n |x_j|^2}$$

Lemma. (Cauchy-Schwarz Inequality)

If $x,y \in \mathbb{R}^n$, then

$$(x\cdot y)^2 := \left(\sum_{j=1}^n x_j y_j\right)^2 \leq \left(\sum_{j=1}^n x_j^2\right)\left(\sum_{j=1}^n y_j^2\right) = \|x\|_2^2 \, \|y\|_2^2$$

Pf. (Clever calculation; recommended reading)

Claim: Define the Euclidean metric

Claim: Define the Euclidean metric

$$d_2(x,y) := \|x-y\|_2 \qquad x,y \in \mathbb{R}^n$$

then $(\mathbb{R}^n, d_2)$ is a metric space.

Pf. (i, ii, iii) Easy to check.
(iv) uses Cauchy-Schwarz.

Subspaces

Prop. Let (X,d) be a metric space, and $Y \subset X$. Then the restriction
$d' := d|_{Y \times Y}$ is a metric on Y.
(Y, d') is called a subspace of (X,d).

Ex. $[a,b] \subset \mathbb{R}$, can also think of $[a,b]$ as a subspace.

Open/Closed Sets (Briefly)

Def. A sequence in a metric space (X,d) is a function $x : \mathbb{N} \to X$.
As before, we write $x_n := x(n)$ and denote the sequence by $\{x_n\}$, $\{x_n\}_{n=1}^{\infty}$

We say a sequence is bounded if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x_n) \leq B \qquad \forall n \in \mathbb{N}$$

Def. (7.3.2) A sequence $\{x_n\}$ in a metric space (X,d) is said to converge to a point $p \in X$ if for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$d(p, x_n) < \varepsilon$$

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$$d(p, x_n) < \varepsilon$$

we write

$$\lim_{n\to\infty} x_n = p$$

Ex. Uniform convergence of cts. functions on $[a,b]$ is convergence in the metric space $C^0([a,b], \mathbb{R})$ with

$$d(f,g) := \|f-g\|_u$$

Def. Let $A \subset X$ be a subset of a metric space.

- We say $p \in A$ is an interior point of A if there exists $\delta > 0$ such that

$$B(p,\delta) := \{x \in X : d(p,x) < \delta\} \subset A$$

$B(p,\delta)$ is called the open ball of radius δ around p.

- We say $p \in X$ is a limit point of A if there exists a sequence $\{x_n\}$ in A such that $x_n \to p$ as $n \to \infty$.

Informally:

A subset $A \subset X$ is open if and only if every $p \in A$ is an interior point of A

A is closed if and only if it contains all of its limit points.

Ex. A set $K \subset X$ is sequentially compact if every sequence in K has a subsequence converging to some $p \in K$.

Prop. (Informal) Every sequentially compact subset $K \subset X$ is closed and bounded (i.e. $\exists p \in X, B \in \mathbb{R}: \forall x \in K, d(p,x) \leq B$)

Is every closed and bounded set compact?

- $\mathbb{R}^n$: yes! Heine-Borel

Is every closed and bounded set compact?

- $\mathbb{R}^n$: yes! Heine-Borel
- general metric spaces: not necessarily!

$\{\sin(kx) : k \in \mathbb{N}\} \subset C^0([0,2\pi], \mathbb{R})$

closed and bounded, but not compact.