Homework 8

Due: Monday, November 14th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 5.1-5.3

Sections 5.1-5.2 Exercises

Problem 1 (5 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that if P is a partition of $[a, b]$ and $\widetilde{P}$ is a refinement of P, then $U(\widetilde{P}, f) \leq U(P, f)$

Problem 2 (3 points each) In this problem we will review some useful properties of sup/inf.

(a) (Exercise 1.1.9) Let $A, B \subset \mathbb{R}$ be non-empty bounded sets such that $B \subset A$. Suppose that for all $x \in A$, there exists a $y \in B$ such that $x \geq y$. Show that $\inf B = \inf A$.

(Hint: You may find the following variant of Proposition 1.2.8 helpful: If $S \subset \mathbb{R}$ is a nonempty bounded below set, then for every $\varepsilon > 0$ there exists $x \in S$ such that $\inf S \leq x < \inf S + \varepsilon$)

(b) (Exercise 1.2.9) Let $A, B \subset \mathbb{R}$ be non-empty bounded sets. Let $C := \{a + b : a \in A, b \in B\}$. Show that $\inf C$ and $\sup C$ exist, and that

$$\sup C = \sup A + \sup B \qquad \text{and} \qquad \inf C = \inf A + \inf B$$

(c) (Exercise 1.3.7) Let D be a nonempty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions. Then,

$$\sup_{x\in D}(f(x)+g(x)) \leq \sup_{x\in D} f(x)+\sup_{x\in D} g(x) \qquad \text{and} \qquad \inf_{x\in D}(f(x)+g(x)) \geq \inf_{x\in D} f(x)+\inf_{x\in D} g(x)$$

Problem 3 (6 points) Let $a < b < c$ and assume $f : [a, b] \to \mathbb{R}$ is bounded. Show that

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f$$

Problem 4 (6 points) Directly using the definition of Riemann integrable (the upper integral equals the lower integral), show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then so is $-f$ and

$$\int_a^b (-f) = -\int_a^b f$$

(Remark: It is important to prove this statement by definition, and not to use any other properties of the Riemann integral proved in section 5.2. The statement in this problem is used in the proof of linearity, and we do not want to use circular logic.)

Problem 5 (6 points each) In this problem we will prove linearity of the Riemann integral.

(a) Prove Proposition 5.2.5 in the textbook: Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be bounded functions. Then,

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g \qquad \text{and} \qquad \underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$$

(Hint: Try to get an inequality of the form $U(P, f+g) \leq U(P, f)+U(P, g) \leq U(P_1, f)+U(P_2, g)$. You can't use the result of Problem 2b on the middle term, but you can use it on the right-most term (why?).)

(b) Now, suppose $f, g \in \mathscr{R}[a, b]$ (recall that Riemann integrable functions are also bounded). Using your result in (a), prove that $f + g \in \mathscr{R}[a, b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Problem 6 (6 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}$ of $[a, b]$ such that

$$\lim_{k\to\infty} (U(P_k, f) - L(P_k, f)) = 0$$

Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f)$$

Problem 7 (6 points) Let P_n denote the partition of $[0, 1]$ using $n + 1$ uniformly spaced points, that is, $P_n := \{k/n\}_{k=0}^n$. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) := x$. Compute $U(P_n, f)$ and $L(P_n, f)$ for each $n \in \mathbb{N}$.

Then, prove that f is Riemann integrable on $[0, 1]$ and compute $\int_0^1 f$.