Homework 6 Solutions

Due: Monday, October 31th by 11:59 PM ET

Sections 3.2-3.3 Exercises

Problem 1 (3 points each) Let us see how continuity interacts with restrictions.

(a) Let $A \subset S \subset \mathbb{R}$, and $c \in A$ be a point. Suppose $f : S \to \mathbb{R}$ is continuous at c. Prove that the restriction $f\big|_A$ is continuous at c.

(b) Find an example of a function $f : S \to \mathbb{R}$ and a subset $A \subset S$ such that $f\big|_A$ is continuous at some $c \in A$ but f is not continuous at c.

(c) Suppose $S \subset \mathbb{R}$ such that $(c - \alpha, c + \alpha) \subset S$ for some $c \in \mathbb{R}$ and $\alpha > 0$. Let $f : S \to \mathbb{R}$ be a function and $A := (c - \alpha, c + \alpha)$. Prove that if $f\big|_A$ is continuous at c, then f is continuous at c.

(a) Let A, S, c, f be as given in the problem. Let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that for all $x \in S$ with $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \varepsilon$$

Since $A \subset S$, this inequality also holds for all $x \in A$ with $|x - c| < \delta$. Thus, $f\big|_A$ is continuous at c.

(b) Let $f : [-1, 1]$ be given by

$$f(x) := \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

This function is not continuous at $c = 0$ (e.g. since the two-sided limits disagree). However, if we let $A = \{0\}$, then $f\big|_A = 1$. This function is continuous at $c = 0$ (e.g. by the limit characterization, 0 is not a cluster point of A so $f\big|_A$ is continuous there).

(c) Let A, S, c, α, f be as given in the problem. Suppose $f\big|_A$ is continuous at c. Then for all $\varepsilon > 0$, we have that there exists some $\delta_A > 0$ such that for all $x \in A$ with $|x - c| < \delta_A$, we have

$$|f(x) - f(c)| < \varepsilon$$

Take $\delta := \min\{\delta_A, \alpha\}$. Then, for all $x \in S$ with $|x - c| < \delta$, we have that $x \in (c-\alpha, c+\alpha) = A$ and $|x - c| < \delta_A$. Thus, the above inequality holds. Therefore, f is continuous at c.

Problem 2 (5 points) Prove Corollary 3.3.12 in the textbook: If $f : [a, b] \to \mathbb{R}$ is continuous, then the direct image $f([a, b])$ is either a closed and bounded interval, or a single number.

Assume $f : [a, b] \to \mathbb{R}$ is continuous. Then, by the min-max theorem, there exists $c_1, c_2 \in [a, b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a, b]$.

If $f(c_1) = f(c_2)$, then $f(x) = f(c_1) = f(c_2)$ for all $x \in [a, b]$, so the direct image $f([a, b]) = \{f(c_1)\}$ is a single number.

If $f(c_1) < f(c_2)$, then let $a' := \min\{c_1, c_2\}$ and $b' := \max\{c_1, c_2\}$, and consider f restricted to the interval $[a', b'] \subset [a, b]$. f will be continuous on this subset, so by Bolzano's intermediate value theorem, for all $y \in \mathbb{R}$ such that $f(c_1) < y < f(c_2)$, there exists $c \in [a', b']$ such that $f(c) = y$. Thus, $y \in f([a, b])$. Since $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a, b]$, the direct image $f([a, b])$ is then the closed interval $[f(c_1), f(c_2)]$.

Problem 3 (4 points each) Recall early on in the course, we very laboriously showed the existence and uniqueness of $\sqrt{2}$. We will show how the tools we have learned will allow us to show the existence and uniqueness of non-negative nth roots $\sqrt[n]{a}$ for any $n \in \mathbb{N}$ and any non-negative real number $a \in [0, \infty)$.

In the following, let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) := x^n$.

(a) Show that f is strictly increasing for all $n \in \mathbb{N}$, and use it to conclude that f is injective.

We say f is *strictly increasing* if $f(x) < f(y)$ for all $x, y \in [0, \infty)$ with $x < y$.

(b) Show that f is continuous for all $n \in \mathbb{N}$. Then, given $M \in \mathbb{N}$, use part (a) and what you know about continuous functions to show that the restriction $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is both surjective and injective, and hence bijective.

(c) Use the results of part (b) to conclude that for any $a \in [0, \infty)$, there exists a unique non-negative x such that $x^n = a$.

(a) Given $0 \le x < y$, we show $x^n < y^n$ by induction.

(Basis statement) $0 \le x < y \implies 0 \le x^1 < y^1$

(Induction step) Assume $0 \le x^{n-1} < y^{n-1}$. Since $y > 0$, we multiply both sides of the inequality to get $yx^{n-1} < y^n$, and since $x \ge 0$ we multiply both sides of the inequality to get $x^n \le yx^{n-1}$, with equality when $x = 0$. Chaining these inequalities together, we get $0 \le x^n < y^n$.

(*Remark*: Be careful, you need to use the fact that $0 \le x < y$. If you only try to use $x < y$, $x^2 < y^2$ is not true in general (e.g. take $x = -2, y = 1$))

Thus, by induction, if $x, y \in [0, \infty)$ and $x < y$, then $f(x) = x^n < y^n = f(y)$, so f is strictly increasing.

Since f is strictly increasing, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, since $x_1 \not< x_2$ and $x_2 \not< x_1$. Thus, f is injective.

(b) We can directly use proposition 3.2.4 (polynomials are continuous) to conclude that x^n, which is a polynomial, is continuous.

Now given $M \in \mathbb{N}$, since the restriction of a continuous function will also be continuous, $f\big|_{[0,M]}$ will map the closed and bounded interval $[0, M]$ to a closed and bounded interval (or a single point).

Since f is increasing, we have $f(0) \le f(x) \le f(M)$ for any $0 \le x \le M$. Thus $f\big|_{[0,M]}$ achieves a max at $x = M$, and a min at $x = 0$, and so it maps the closed and bounded interval $[0, M]$ to $[f(0), f(M)] = [0, M^n]$.

Thus, $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is surjective. It is also injective by (a), since $f(x_1) = f(x_2)$ still implies $x_1 = x_2$ for $x_1, x_2 \in [0, M]$. Thus, $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is bijective.

(c) Let $a \in [0,\infty)$ be given. By the Archimedean property, there exists some $M \in \mathbb{N}$ such that $a < M$. Furthermore, since $1 \leq M$ we have $M \leq M^n$ by induction, so $a < M \leq M^n$, hence $a \in [0, M^n]$.

Thus, we can take $x := \left(f\big|_{[0,M]} \right)^{-1}(a)$ to satisfy the equation and satisfies $f(x) = x^n = a$. x exists and is unique since $f\big|_{[0,M]}$ is a bijection, as we showed in (b).

Lastly, we need to show x does not depend on the choice of M (for example, we want no other solutions appear if we were to increase M). With $x \in [0,\infty)$ as defined above, since f is injective, for any $y \in [0,\infty)$ we have that $f(x) = f(y) = a$ implies $x = y$. Thus we conclude the existence and uniqueness of non-negative nth roots $\sqrt[n]{a} = x$.

Problem 4 (6 points) Suppose $g(x)$ is a monic polynomial of even degree d, that is

$$g(x) = x^d + b_{d-1}x^{d-1} + ... + b_1 x + b_0$$

for some real numbers $b_0, b_1, ..., b_{d-1} \in \mathbb{R}$. Suppose $g(0) < 0$. Show that g has at least two distinct roots, that is, there exists at least two real numbers $c_1 \neq c_2$ such that $g(c_1) = g(c_2) = 0$.

(Hint: This proof shares many similarities with the proof of Proposition 3.3.10 in the textbook. Make sure to use closed and bounded intervals when you are using Bolzano's intermediate value theorem.)

Consider the sequence $\{x_n\}$ defined by

$$x_n := \frac{b_{d-1}n^{d-1} + ... + b_1 n + b_0}{n^d}$$

we can compute the limit of this sequence using continuity of algebraic operations as

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{b_{d-1}}{n} + \lim_{n\to\infty} \frac{b_{d-2}}{n^2} + ... + \lim_{n\to\infty} \frac{b_0}{n^d} = 0$$

Thus, for $\varepsilon = 1 > 0$, there exists $M \in \mathbb{N}$ such that

$$\left| \frac{b_{d-1}M^{d-1} + ... + b_1 M + b_0}{M^d} \right| < 1$$

We use the fact that $|x| < 1 \implies -1 < x$ to get

$$-M^d < b_{d-1}M^{d-1} + ... + b_1 M + b_0$$

which shows that $g(M) > 0$.

A similar computation shows that the sequence $\{y_n\}$ defined by

$$y_n := \frac{b_{d-1}(-n)^{d-1} + ... + b_1(-n) + b_0}{(-n)^d}$$

also converges to zero, and hence there exists $K \in \mathbb{N}$ such that

$$(-K)^d < b_{d-1}(-K)^{d-1} + ... + b_1(-K) + b_0$$

and hence $g(-K) > 0$.

Hence, using Bolzano's IVT, since $g(0) < 0 < g(M)$, there exists some $c_1 \in [0, M]$ such that $g(c_1) = 0$. $c_1 > 0$ since $g(0) \neq 0$. Similarly, since $g(-K) > 0 > g(0)$, there exists some $c_2 \in [-K, 0]$ such that $g(c_2) = 0$, again $c_2 < 0$. Thus, g has at least two distinct roots c_1 and c_2.

Problem 5 (4 points) Let $S \subset \mathbb{R}$ be a set, and suppose $E \subset S$ is a subset such that every $x \in S$ is a cluster point of E. Let $f, g : S \to \mathbb{R}$ be continuous functions satisfying $f(x) = g(x)$ for all $x \in E$. Show that $f(x) = g(x)$ for all $x \in S$.

(*Remark*: This problem has to do with "continuous extensions", which we may cover at a later point. The idea here is that E is some set that can "approximate" elements of S, for example, $E = \mathbb{Q}$ and $S = \mathbb{R}$. This problem shows that if $f(x) = g(x)$ on all elements $x \in E$ of the approximating set, then $f(x) = g(x)$ for all elements $x \in S$ of the full set.)

Let $x \in S$ be arbitrary. Since it is a cluster point of E, there exists a convergent sequence $\{x_n\}$ satisfying $x_n \in E \setminus \{x\}$ for all $n \in \mathbb{N}$, and $x_n \to x$ as $n \to \infty$. Then, since f, g are continuous,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(x)$$

Thus, $f(x) = g(x)$ for all $x \in S$.

Section 4.1 Exercises

Problem 6 (4 points) Prove linearity for derivatives: Let I be an interval, let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions differentiable at c, and let $\alpha, \beta \in \mathbb{R}$. If $h : I \to \mathbb{R}$ is defined by

$$h(x) := \alpha f(x) + \beta g(x)$$

then h is differentiable at c and

$$h'(c) = \alpha f'(c) + \beta g'(c)$$

Suppose f, g are both differentiable at c, and let h be defined as in the problem statement. Then, by continuity of algebraic operations,

$$\begin{aligned}\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c}\left(\frac{\alpha(f(x)-f(c))}{x-c} + \frac{\beta(g(x)-g(c))}{x-c}\right)\\ &= \alpha \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) + \beta \lim_{x\to c}\left(\frac{g(x)-g(c)}{x-c}\right)\\ &= \alpha f'(c) + \beta g'(c) = h'(c)\end{aligned}$$

Thus, h is differentiable at c.

Problem 7 (5 points) Prove the product rule for derivatives: Let I be an interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions differentiable at c. If $h : I \to \mathbb{R}$ is defined by

$$h(x) := f(x)g(x)$$

then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c)$$

Let h be defined as in the problem statement, and suppose f, g are differentiable at c. Then, the difference quotient satisfies

$$\begin{aligned}\frac{h(x)-h(c)}{x-c} &= \frac{f(x)g(x)-f(c)g(c)}{x-c}\\ &= \frac{f(x)(g(x)-g(c)) + (f(x)-f(c))g(c)}{x-c}\end{aligned}$$

Next, note that since f is differentiable at c, it is then continuous at c. Then, by continuity of algebraic operations, we have

$$\begin{aligned}\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{f(x)(g(x)-g(c)) + (f(x)-f(c))g(c)}{x-c}\\ &= \lim_{x\to c} f(x) \lim_{x\to c} \frac{g(x)-g(c)}{x-c} + g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c}\\ &= f(c)g'(c) + f'(c)g(c) = h'(c)\end{aligned}$$

Thus, h is differentiable at c

Problem 8 (5 points) Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) := \sqrt{x}$. Prove, using the limit definition of the derivative, that f is differentiable at all $c \in (0, \infty)$.

For $x, c \in (0, \infty)$ we can compute the limit of the difference quotient directly using continuity of algebraic operations, and the fact that $\sqrt{x}$ is continuous:

$$f'(c) = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}$$

Thus f is differentiable at all $c \in (0, \infty)$