Derivatives

Wednesday, October 26, 2022 11:25 AM

Derivatives

Def. Let $I \subset \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be a function, and $c \in I$.

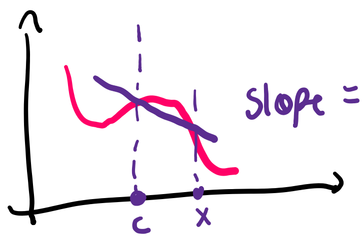
- If the limit

$$L := \lim_{x \to c} \underbrace{\frac{f(x)-f(c)}{x-c}}_{\text{"difference quotient"}}$$

exists, then we say f is <u>differentiable</u> at c, and denote $f'(c) := L$ is the <u>derivative</u> of f at c.

- If f is differentiable at all $c \in I$, we say f is <u>differentiable</u>, and we obtain a function $f': I \to \mathbb{R}$ (also written $\frac{df}{dx}$)

<u>Idea</u>:



slope $= \dfrac{f(x)-f(c)}{x-c}$

limits: $S \setminus \{c\}$, so $\dfrac{f(x)-f(c)}{x-c}$ well defined $\forall x \in I \setminus \{c\}$

<u>Ex.</u> Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) := x^2$. <u>Claim</u>: f is differentiable.

<u>Pf.</u> Let $c \in \mathbb{R}$ be arbitrary.

$$\frac{f(x)-f(c)}{x-c} = \frac{x^2-c^2}{x-c} = \frac{(x+c)(x-c)}{x-c} = x+c$$

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 2c$$

The limit exists, so f is differentiable for all $c \in \mathbb{R}$, $f'(c) = 2c$

<u>Ex.</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) := |x|$ <u>Claim</u>: f is not differentiable at $c = 0$.

<u>Pf.</u> $\dfrac{f(x)-f(0)}{x-0} = \dfrac{|x|-0}{x} = \begin{cases} -1 & x<0 \\ \ldots \end{cases}$ $(x \neq 0)$

Pf. $\frac{f(x)-f(0)}{x-0} = \frac{|x|-0}{x-0} = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$ $(x \neq 0)$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1 \neq \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = -1$$

$\Rightarrow$ limit of $\frac{f(x)-f(0)}{x-0}$ does not exist as $x \to 0$

$\Rightarrow$ f is not differentiable at $c=0$.

Prop. Let $f: I \to \mathbb{R}$ be differentiable at $c \in I$. Then f is continuous at c.

Pf. We know the limits

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{and} \quad 0 = \lim_{x\to c}(x-c)$$

exist. Then, by continuity of alg. op.,

$$\lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\cdot(x-c)\right)$$

$$= \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)\cdot \lim_{x\to c}(x-c)$$

$$= f'(c)\cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

Thus by the limit char. of continuity, f is continuous at c $\square$

Properties of the derivative

Assigned Readings: Linearity $(+,-)$ } HW! Textbook has key identities
Product Rule $(\times)$
Quotient Rule $(\div)$

Quotient Rule ($\div$)

<u>Ex.</u> Let I be an interval, $f, g : I \to \mathbb{R}$ both differentiable at $c \in I$.
Then, $h : I \to \mathbb{R}$, $h(x) := f(x) + g(x)$ is differentiable at $c \in I$.

<u>Prop.</u> (Chain rule, $\circ$)
Let I_1, I_2 be intervals, $g : I_1 \to I_2$ be differentiable at $c \in I_1$,
$f : I_2 \to \mathbb{R}$ be differentiable at $g(c) \in I_2$. Define $h : I_1 \to \mathbb{R}$

$$h(x) := (f \circ g)(x) = f(g(x))$$

Then, h is differentiable at c, and

$$h'(c) = f'(g(c)) \cdot g'(c)$$

<u>Pf.</u> Let $d := g(c)$. Define $u : I_2 \to \mathbb{R}$, $v : I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y) - f(d)}{y - d} & y \neq d \\ f'(d) & y = d \end{cases} \qquad v(x) := \begin{cases} \frac{g(x) - g(c)}{x - c} & x \neq c \\ g'(c) & x = c \end{cases}$$

- f is differentiable at $d \Rightarrow u(y) \to f'(d) = u(d)$ as $y \to d$

  $\Rightarrow u$ is continuous at d.

- Similarly, g is differentiable at $c \Rightarrow v$ is continuous at c.

- For any x, y we have

$$\underbrace{f(y) - f(d) = u(y) \cdot (y - d)}\ (1) \qquad \underbrace{g(x) - g(c) = v(x) \cdot (x - c)}\ (2)$$

- Thus,

$$h(x) - h(c) = f(\underline{g(x)}) - f(\underline{g(c)}) \qquad \text{(apply (1) for } y = g(x),\ d = g(c))$$

$$= u(g(x)) \cdot (\underline{g(x) - g(c)}) \qquad \text{(apply (2))}$$

$$= u(g(x)) \cdot (v(x) \cdot (x - c))$$

$$= u(g(x)) \cdot (v(x) \cdot (x-c)) \qquad \text{(apply (a))}$$

- So, for $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = u(g(x)) \cdot v(x)$$

- Since $u \circ g$, v are continuous at c,

$$\lim_{x \to c} \frac{h(x)-h(c)}{x-c} = \lim_{x \to c} \left(u(g(x)) \cdot v(x) \right)$$

$$= \lim_{x \to c} u(g(x)) \cdot \lim_{x \to c} v(x)$$

$$= u(g(c)) \cdot v(c) = f'(g(c)) \cdot g'(c)$$

Thus, h is differentiable at c, with $h'(c) = f'(g(c)) \cdot g'(c)$. $\square$