HW3

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1. (i) $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n \geq M, |x_n - x| < \varepsilon$

(ii) $\forall \varepsilon' > 0, \exists M' \in \mathbb{N}$ s.t. $\forall n \geq M', |x_n - x| < \alpha \varepsilon'$

(i) $\Rightarrow$ (ii)

$\forall \varepsilon' > 0$, given (i), let $\varepsilon = \alpha \varepsilon'$,

$\exists M = M'$ s.t. $\forall n \geq M, |x_n - x| < \varepsilon = \alpha \varepsilon'$

(ii) $\Rightarrow$ (i)

$\forall \varepsilon > 0$, given (ii), let $\varepsilon' = \frac{1}{\alpha} \varepsilon$

$\exists M' = M$ s.t. $\forall n \geq M', |x_n - x| < \alpha \varepsilon' = \alpha$

Therefore, the two statements are equivalent.

2. (a) Take $\varepsilon > 0$. Define $z = ax + by$, where $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$.

Since $x_n \to x$: $\exists M_1 \in \mathbb{N}: \forall n \geq M_1, |x_n - x| < \frac{\varepsilon}{2a}$

$y_n \to y$: $\exists M_2 \in \mathbb{N}: \forall n \geq M_2, |y_n - y| < \frac{\varepsilon}{2b}$

Take $M = \max\{M_1, M_2\}$, then for all $n \geq M$, $z_n = ax_n + by_n$

$$\begin{aligned}
|z_n - z| &= |ax_n + by_n - ax - by| \\
&\leq |ax_n - ax| + |by_n - by| \\
&= a|x_n - x| + b|y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\Rightarrow \lim_{n\to\infty}(ax_n + by_n) = a(\lim_{n\to\infty} x_n) + b(\lim_{n\to\infty} y_n)$

(b) Let $x_n \to x$ as $n \to \infty$. Take $z_n := x_n^2$, $z = x^2$.

$$\begin{aligned}
|z_n - z| &= |x_n^2 - x^2| \\
&= |(x_n - x + x)^2 - x^2| \\
&= |2(x_n - x) \cdot x + (x_n - x)^2| \\
&\leq 2|x_n - x| \cdot |x| + |x_n - x|^2
\end{aligned}$$

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Let $\varepsilon > 0$ be given. Take $k := \max\{|x|, 1, \frac{\varepsilon}{3}\}$

$\exists M_1 \in \mathbb{N} : \forall n \geq M_1, |x_n - x| < \frac{\varepsilon}{3k} \quad (\leq 1)$

Take $M := M_1$.

for all $n \geq M$,

$$|z_n - z| \leq 2|x_n - x| \cdot |x| + |x_n - x|^2$$
$$< 2 \cdot \frac{\varepsilon}{3k} \cdot k + \left(\frac{\varepsilon}{3k}\right)^2$$
$$\leq \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$$

$\Rightarrow \lim_{n\to\infty}(x_n^2) = (\lim_{n\to\infty} x_n)^2$

(c) Let $x_n \to x$ as $n \to \infty$. Take $z_n := \frac{1}{x_n}$, $z = \frac{1}{x}$. Let $\varepsilon > 0$.

$$|z_n - z| = \left|\frac{1}{x_n} - \frac{1}{x}\right|$$
$$= \left|\frac{x_n - x}{x_n x}\right|$$
$$= \frac{|x_n - x|}{|x_n| \cdot |x|}$$

$\exists M \in \mathbb{N} : \forall n \geq M, |x_n - x| < \min\{|x|^2 \frac{\varepsilon}{2}, \frac{|x|}{2}\}$

For all $n \geq M$, $|x - x_n| < \frac{|x|}{2}$

$\Rightarrow |x| = |x - x_n + x_n| \leq |x - x_n| + |x_n| < \frac{|x|}{2} + |x_n|$

$\Rightarrow \frac{|x|}{2} < |x_n|$

$\Rightarrow \frac{1}{|x_n|} < \frac{2}{|x|}$

$$|z_n - z| = \frac{|x_n - x|}{|x_n| \cdot |x|} \leq \frac{|x_n - x|}{|x|} \cdot \frac{2}{|x|}$$
$$< \frac{|x|^2 \frac{\varepsilon}{2}}{|x|} \cdot \frac{2}{|x|} = \varepsilon$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim_{n\to\infty} x_n}$

3.(a) False.

Counterexample: $x_n=(-1)^n$, $y_n=(-1)^{n+1}$, then $\{x_n\},\{y_n\}$ diverges

$\forall n\in\mathbb{N}$, $x_n+y_n=0$, then $\{x_n+y_n\}$ converges to 0

(b) False.

Counterexample: $x_n=(-1)^n$, $y_n=(-1)^{n+1}$, then $\{x_n\}.\{y_n\}$ diverges

$\forall n\in\mathbb{N}$, $x_ny_n=-1$, then $\{x_ny_n\}$ converges to -1

(c) True

Pf. Take $\varepsilon>0$. Let $x_n\to x$, $x_n+y_n\to z$, $y=z-x$

$\exists M_1\in\mathbb{N}:\ \forall n\geq M_1,\ |x_n-x|<\frac{\varepsilon}{2}$

$\exists M_2\in\mathbb{N}:\ \forall n\geq M_2,\ |(x_n+y_n)-(x+y)|<\frac{\varepsilon}{2}$

Take $M=\max\{M_1,M_2\}$, for all $n\geq M$,

$$\begin{aligned}|y_n-y|&=|(x_n+y_n)-(x+y)-x_n+x|\\&\leq|(x_n+y_n)-(x+y)|+|x_n-x|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

so $\{y_n\}$ converges.

(d) True.

Pf. Take $\varepsilon>0$.

$\exists B\in\mathbb{R}$ s.t. $|x_n|\leq B$ for all n

$\exists M\in\mathbb{N}$ s.t. $\forall n\geq M$, $|y_n-0|<\frac{\varepsilon}{B}$

So $\forall n\geq M$, $|x_ny_n-0|=|x_n|\cdot|y_n|<B\cdot\frac{\varepsilon}{B}=\varepsilon$

so $\{x_ny_n\}$ converges to 0.

(e) False.

Counterexample: $x_n = n^2$, $y_n = \frac{1}{n}$

then $\{x_n\}$ is unbounded, $\{y_n\}$ converges to 0

$\forall n \in \mathbb{N}$, $x_n y_n = n$, then $\{x_n y_n\}$ diverges

4. 1° $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 0$

$\Rightarrow M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\left|\frac{x_n}{y_n} - 0\right| < 1$

$\frac{|x_n|}{|y_n|} < 1$

$|x_n| < |y_n|$

since $y_n > 0 \Rightarrow y_n > |x_n|$

2° Take $\{y_n\} = \{n^d\}$, $\{x_n\} = \{p(n) - n^d\}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{C_{d-1} n^{d-1} + \cdots + C_0}{n^d}$$

$$= \lim_{n\to\infty}\left(\frac{C_{d-1}}{n} + \cdots + \frac{C_0}{n^d}\right) = 0$$

$\Rightarrow \exists M \in \mathbb{N}$ st. $\forall n \geq M$, $y_n > |x_n|$

$n^d > C_{n-1} n^{d-1} + \cdots + C_0$

$2n^d > n^d + C_{n-1} n^{d-1} + \cdots + C_0 = p(n)$

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5. (a) The sequence converges.

For all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ s.t. $M \cdot \varepsilon > 1$

$$\left| \frac{n\cos n}{n^2+1} - 0 \right| = \frac{|n| \cdot |\cos n|}{|n^2+1|} \leq \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{M} < \varepsilon \text{ for all } n \geq M$$

So $\frac{n\cos n}{n^2+1} \to 0$ as $n \to \infty$.

(b) The sequence diverges.

6. 1° "$\Leftarrow$" If every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to x

Let $n_i = i$, $\{x_{n_i}\}_{i=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$

Then $\{x_n\}_{n=1}^{\infty}$ converges to x.

2° "$\Rightarrow$" If $\{x_n\}_{n=1}^{\infty}$ converges to x,

for any subsequence $\{x_{n_i}\}_{i=1}^{\infty}$,

for $i=1$, $n_i \geq i = 1$

suppose for $i \leq k$, $n_i \geq i$

for $i = k+1$, $n_i = n_{k+1} \geq n_k + 1 \geq k+1 = i$

So $n_i \geq i$ for all $i \in \mathbb{N}$.

For every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ s.t. for all $n \geq M$,

$|x_n - x| < \varepsilon$

Since $n_i \geq i$, $i > M \Rightarrow n_i \geq M$

$\Rightarrow |x_{n_i} - x| < \varepsilon$

$\Rightarrow \{x_{n_i}\}_{i=1}^{\infty}$ converges to x.

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