Min/Max, Bolzano IVT

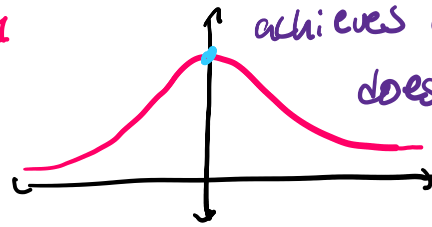
Monday, October 24, 2022 11:17 AM

Min/Max Therorem

Def. We say a function $f: S \to \mathbb{R}$ achieves an absolute maximum at $c \in S$ if $f(x) \leq f(c)$ for all $x \in S$.

Similarly for absolute minimum: $f(x) \geq f(c)$ for all $x \in S$.

Ex. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) := \frac{1}{x^2+1}$



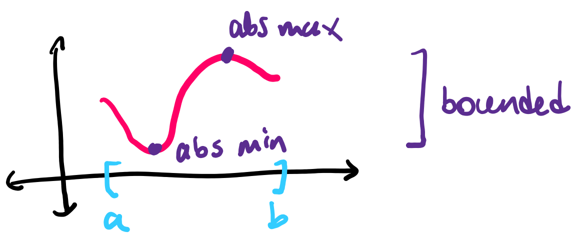
achieves abs. max. at $c = 0$

does not achieve abs. min

Thm. (min/max theorem)

A continuous function $f: [a,b] \to \mathbb{R}$ achieves both an abs. max. and abs. min. on the closed and bounded interval $[a,b]$

Idea:



PR. We've already proven continuous $f: [a,b] \to \mathbb{R}$ is bounded.

- $f([a,b]) = \{f(x) : x \in [a,b]\}$ is bounded, hence has a sup/inf
- There exist sequences $\{f(x_n)\}$, $\{f(y_n)\}$ which approach the sup/inf of $f([a,b])$ $\quad (\exists f(x_n) \in f([a,b]) : \sup f([a,b]) - \frac{1}{n} < f(x_n) \leq \sup f([a,b]))$

  That is,

  $$\lim_{n\to\infty} f(x_n) = \sup f([a,b]) \qquad \lim_{n\to\infty} f(y_n) = \inf f([a,b])$$

- Since $a \leq x_n \leq b$ and $a \leq y_n \leq b$ $\forall n \in \mathbb{N}$, by B-W there exist convergent subsequences $\{x_{n_k}\}$, $\{y_{m_k}\}$.
  - Let $x := \lim x_{n_k}$, $y := \lim y_{m_k}$

convergent subsequences $\{x_{n_k}\}, \{y_{m_k}\}$.

- Let $x := \lim_{k\to\infty} x_{n_k}$, $y := \lim_{k\to\infty} y_{m_k}$.
- Limits preserve non-strict inequalities, so $a \le x \le b$ and $a \le y \le b$

- Now, we can apply seq. char. of continuity:

$$\inf f([a,b]) \underbrace{=}_{\text{by construction}} \lim_{n\to\infty} f(y_n) \underbrace{=}_{\text{subseq.}} \lim_{k\to\infty} f(y_{m_k}) \underbrace{=}_{\text{continuity}} f\left(\lim_{k\to\infty} y_{m_k}\right) = \underbrace{f(y)}_{\text{abs. min}}$$

$$\sup f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x)$$

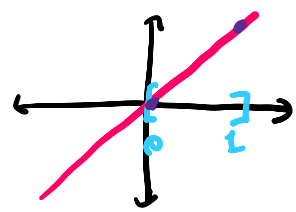
- Thus, f achieves an abs. max at x and an abs. min. at y $\square$

Remarks:

- Domain of definition is important:

$f:\mathbb{R}\to\mathbb{R}$ $\quad f(x):=x$

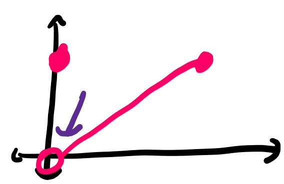
vs. $f|_{[0,1]}$



f is unbounded
$f|_{[0,1]}$ achieves abs min and max

- continuity is important:

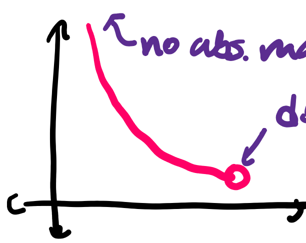
$f:[0,1]\to\mathbb{R}$ $\quad f(x) := \begin{cases} 1 & x=0 \\ x & x>0 \end{cases}$



$\inf_{x\in[0,1]} f(x) = 0$ not achieved

- closed + bounded is important:

$f:(0,1)\to\mathbb{R}$ $\quad f(x) := 1/x$



no abs. max. ($\infty \notin \mathbb{R}$!)
does not achieve abs. min.

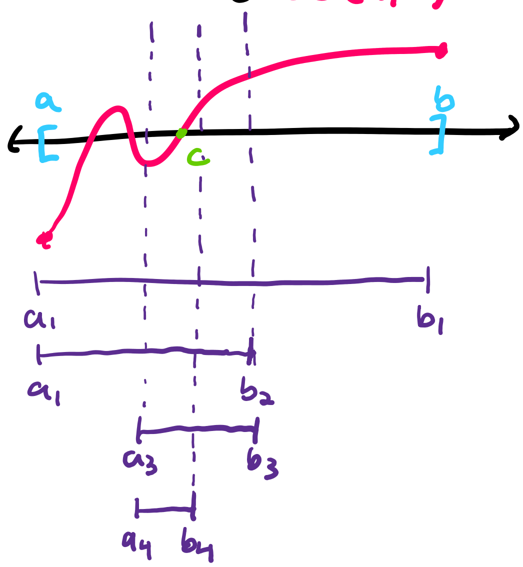
Bolzano's Intermediate Value Theorem

Lemma. (Bisection Method for finding roots)

Let $f:[a,b]\to\mathbb{R}$ be continuous. Suppose $f(a)<0$ and $f(b)>0$.

Let $f:[a,b]\to\mathbb{R}$ be continuous. Suppose $f(a)<0$ and $f(b)>0$.
Then there exists $c\in(a,b)$ such that $f(c)=0$.

Idea:



"bisect" the search interval
find a sequence of intervals
$[a_n, b_n]$ which "narrow in" on c

Pf. We define two sequences $\{a_n\}, \{b_n\}$ inductively.

(Base case) Let $a_1 := a$, $b_1 := b$. Note $a_1 < b_1$ and $b_1 - a_1 = \frac{1}{2^0}(b-a)$

(Induction step). Suppose a_n, b_n are defined, $a_n < b_n$, and $b_n - a_n = \frac{1}{2^{n-1}}(b-a)$

- If $f\left(\frac{a_n+b_n}{2}\right)\geq 0$, let $a_{n+1} := a_n$, $b_{n+1} := \frac{a_n+b_n}{2}$
- If $f\left(\frac{a_n+b_n}{2}\right) < 0$, let $a_{n+1} := \frac{a_n+b_n}{2}$, $b_{n+1} := b_n$

We can also show:

- $a_n < b_n \Rightarrow a_n < \frac{a_n+b_n}{2} < b_n$, so $a_n \leq a_{n+1} < b_{n+1} \leq b_n$
- $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{1}{2^n}(b-a)$

Now, we can see:

- $a \leq a_n \leq a_{n+1} < b_{n+1} \leq b_n \leq b \quad \forall n\in\mathbb{N}$

  $\Rightarrow$ $\{a_n\}, \{b_n\}$ are bounded monotone sequences, hence convergent.

  - Let $c := \lim\limits_{n\to\infty} a_n$, $d := \lim\limits_{n\to\infty} b_n$. Note $c, d \in [a,b]$

- Since $b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \forall n\in\mathbb{N}$

  $d - c = \lim(b_n - a_n) = \lim\left(\frac{1}{2^{n-1}}(b-a)\right) = 0$

• Since $b_n - a_n = \frac{1}{2^{n-1}}(b-a)$ then

$$d - c = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}}(b-a)\right) = 0$$

$\Rightarrow c = d$.

• By construction, $f(a_n) < 0 \leq f(b_n)$ $\forall n \in \mathbb{N}$. Since f is continuous,

$$f(c) = \lim_{n\to\infty} f(a_n) \leq 0 \leq \lim_{n\to\infty} f(b_n) = f(c)$$

$\Rightarrow f(c) \leq 0$ and $f(c) \geq 0$ $\Rightarrow$ $f(c) = 0$

• $f(a) < 0 < f(b)$ so $c \neq a$, $c \neq b$

$\Rightarrow$ there exists $c \in (a,b)$ such that $f(c) = 0$. $\square$

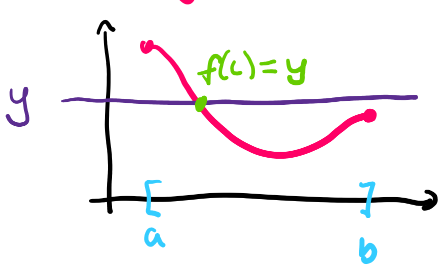
<u>Remark</u>: Overall proof strategy:

1. Construct a sequence
2. Show it converges
3. Compute the limit and show it satisfies the desired properties

<u>Thm.</u> (Bolzano's Intermediate Value Theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Suppose $y \in \mathbb{R}$ satisfies $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Then there exists $c \in (a,b)$ such that $f(c) = y$.

<u>Idea</u>: $f(x) = y$ has a solution $x = c$



<u>Pf.</u> If $f(a) < y < f(b)$, define $g : [a,b] \to \mathbb{R}$, $g(x) := f(x) - y$

• $g(a) = f(a) - y < 0$, $g(b) = f(b) - y > 0$

• Bisection lemma: $\exists c \in (a,b) : g(c) = 0 = f(c) - y$ $\Rightarrow$ $f(c) = y$ ✓

- Bisection lemma: $\exists c \in (a,b): g(c) = 0 = f(c) - y \Rightarrow f(c) = y$ ✓

If $f(b) < y < f(a)$, define $g: [a,b] \to \mathbb{R}$, $g(x) := y - f(x)$

- $g(a) < 0 < g(b)$. Bisection lemma: $\exists c \in (a,b): g(c) = 0$

$\Rightarrow f(c) = y$ ☐

Ex. Prop. Every polynomial of odd degree has a real root.

Pf: Let f be a polynomial of odd degree, so

$$f(x) = \underbrace{a_d}_{\neq 0} x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

- Define $g(x) := \frac{f(x)}{a_d} := x^d + b_{d-1} x^{d-1} + \cdots + b_1 x + b_0$

which has the same roots as f.

- Want to find some closed and bounded interval $[a,b]$ such that $g(a) < 0 < g(b)$ so we can use Bolzano's IVT.
  - $\lim_{n \to \infty} \frac{b_{d-1} n^{d-1} + \cdots + b_0}{n^d} = 0$, so $\exists M \in \mathbb{N}: \forall n \geq M$,

$$\underbrace{-1 < \frac{b_{d-1} n^{d-1} + \cdots + b_0}{n^d} < 1}$$

  - Take $n = M$, so $0 < M^d + b_{d-1} M^{d-1} + \cdots + b_0 = g(M)$
  - $\lim_{k \to \infty} \frac{b_{d-1}(-k)^{d-1} + \cdots + b_0}{(-k)^d} = 0$, similar logic shows $\exists k \in \mathbb{N}: g(-k) < 0$

- Since $g(-k) < 0 < g(M)$, apply BIVT to $g|_{[-k,M]}$

$\Rightarrow \exists c \in (-k, M) \subset \mathbb{R}: g(c) = 0 \Rightarrow f(c) = 0$ ☐