Midterm Exam 2 Problem Bank

Examination Date: Monday, November 21st

Instructions – please read carefully

- 2 out of 4 problems from this problem bank will appear as-is on the exam
- You may discuss the problems with other students. Furthermore, you can ask the instructor and TA clarifying questions, and for hints if you are stuck. However, **do not ask for, or share, either partial or full solutions**. You are expected to abide by the NYU CAS Honor Code.
- **You will not be permitted any reference material during the exam**. I advise against rote memorization: as the saying goes, *the easiest way to memorize something is to understand it.*
- You can use any result proved in the course text, in class, or on a previous homework question provided you mention that you are using a result. You do not need to mention an exact name or proposition number – **you only need to demonstrate that you are aware you are using a non-trivial result in your proof.**
- Remember that the exams will be graded to a stricter standard than the homeworks. A good rule of thumb: **if a problem 'covers' chapter N material, then you should put more detail into your proofs of statements involving chapter N**. When in doubt whether or not you need to prove something, it's usually safer to prove it.

1. Let's take a look at limits of functions at infinity.

   Given a function $f : \mathbb{R} \to \mathbb{R}$ and some $L \in \mathbb{R}$, we say $f(x)$ *converges to* L as $x \to \infty$ if for all $\varepsilon > 0$, there exists some $M \in \mathbb{R}$ such that for all $x \geq M$,

$$|f(x) - L| < \varepsilon$$

   In this case, we write $f(x) \to L$ as $x \to \infty$, or

$$\lim_{x\to\infty} f(x) := L$$

   If f does not converge to any $L \in \mathbb{R}$ as $x \to \infty$, we say f *diverges* as $x \to \infty$.

   (a) Write down a corresponding definition for $f(x)$ to converge to L as $x \to -\infty$. Then, use the above definition and the definition you wrote to prove that

$$\lim_{x\to\infty} \frac{1}{1+x^2} = \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = L$$

   for some $L \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) := \begin{cases} f(1/y) & y \neq 0 \\ L & y = 0 \end{cases}$$

   Show that g is continuous at 0.

   (*Hint*: For the ε-δ definition of continuity, show that you can break $|y| < \delta$ into three cases: $y = 0$, $1/y > 1/\delta$, or $-1/y > 1/\delta$. This might help you find the right value of δ.)

   (c) Continuing from (b), show that if f is continuous at 0, then

$$\lim_{y\to\infty} g(y) = \lim_{y\to-\infty} g(y) = f(0)$$

2. In this problem, we will see how to generalize some of the results in section 3.3 to more general domains.

   We say a set $K \subset \mathbb{R}$ is *sequentially compact* if every sequence in K has a convergent subsequence converging to a point in K. In other words, if $\{x_n\}$ is a sequence satisfying $x_n \in K$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ and point $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.

   (a) Prove that $[a, b]$ is sequentially compact, but (a, b) is not sequentially compact.

   (*Hint*: For $[a, b]$, Bolzano-Weierstrass might be helpful. For (a, b), try constructing a convergent sequence which 'escapes' the open interval.)

   (b) Prove that if a set K is sequentially compact, then it is bounded.

   (*Hint*: The contrapositive might be easier to prove, look at the proof of Lemma 3.3.1.)

   (c) Prove that if a set K is sequentially compact, then $\sup(K) \in K$ and $\inf(K) \in K$.

   (*Hint*: Try constructing sequences which converge to $\sup(K)$ and $\inf(K)$.)

   (d) Let K be sequentially compact. Prove that if a function $f : K \to \mathbb{R}$ is continuous, then the direct image $f(K)$ is sequentially compact.

   Use this to prove that f achieves an absolute minimum and absolute maximum on K.

   (*Hint*: For any sequence $\{f(x_n)\}$, observe that $\{x_n\}$ is a sequence in the sequentially compact set K.)

3. In the following, assume $f : \mathbb{R} \to \mathbb{R}$ has 3 continuous derivatives. A common usage of Taylor's theorem is to construct *finite difference schemes* for approximating the derivatives of f. Let's take a look at some basic problems in this vein:

   (a) Let $x \in \mathbb{R}$ and $h > 0$. Show that there exists constants M_1, M_2, possibly depending on x, h, and f and its derivatives, such that

   $$M_1 h \leq \left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) \leq M_2 h$$

   This shows the "first-order accuracy" of the forward-difference approximation. (*Hint*: Use Taylor's theorem for $x, x+h$)

   (b) Let $x \in \mathbb{R}$ and $h > 0$. Show that there exists constants M_3, M_4, possibly depending on x, h, and f and its derivatives, such that

   $$M_3 h^2 \leq \left(\frac{f(x+h) - f(x-h)}{2h} - f'(x) \right) \leq M_4 h^2$$

   This shows the "second-order accuracy" of the centered-difference approximation. (*Hint*: Use Taylor's theorem for $x, x+h$ and $x, x-h$)

   (c) Let $k \in \mathbb{R}$, and let $f(x) := \sin(kx)$. Show that for all $x \in \mathbb{R}$ and $h > 0$,

   $$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \frac{k^2 h}{2}$$
   $$\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| \leq \frac{k^3 h^2}{6}$$

   Remark: Problem (c) shows that while generally "second-order accuracy" is better, the centered-difference scheme can actually do worse if your function has oscillations on the scale of your stepsize, i.e. $kh > 3/2$. This might happen if your function f has "high-frequency noise".

4. In lecture, we've mentioned how the lower and upper Darboux sums 'bound' possible Riemann sums. In this problem, let's make that statement more rigorous.

   (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{P} = \{x_0, ..., x_n\}$ be a partition of $[a, b]$. We say a set of points $\tau := \{c_1, ..., c_n\}$ is a *tagging* of $\mathcal{P}$ if $x_{i-1} \leq c_i \leq x_i$ for all $i = 1, ..., n$.

   Given any partition $\mathcal{P}$ and tagging τ of $\mathcal{P}$, show that

   $$L(\mathcal{P}, f) \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq U(\mathcal{P}, f)$$

   (b) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that for any tagging τ,

   $$\left| \int_a^b f - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$$

   (c) Given a bounded function $f : [a, b] \to \mathbb{R}$ and $n \in \mathbb{N}$, define the *right uniform Riemann sum* $R_n(f, [a, b])$ as

   $$R_n(f, [a, b]) := \sum_{i=1}^{n} f(x_i)\Delta x$$

   where $\Delta x = (b - a)/n$ and $x_i = a + (b - a)(i/n)$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function, which satisfies

   $$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   Show that the sequences $\{R_n(f, [0, 1])\}_{n=1}^{\infty}$, $\{R_n(f, [1, 1+\sqrt{2}])\}_{n=1}^{\infty}$, and $\{R_n(f, [0, 1+\sqrt{2}])\}_{n=1}^{\infty}$ converge, but

   $$\lim_{n\to\infty} (R_n(f, [0, 1]) + R_n(f, [1, 1 + \sqrt{2}])) \neq \lim_{n\to\infty} R_n(f, [0, 1 + \sqrt{2}])$$

   (*Remark*: You may use the fact that if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$, and $rx \notin \mathbb{Q}$ if $r \neq 0$.)

   (d) Prove or disprove (i.e. by counterexample): A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the sequence of right uniform Riemann sums $\{R_n(f, [a, b])\}$ converges as $n \to \infty$.