HW8

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1. Let $P = \{x_0, x_1, \cdots x_n\}$, $\tilde{P} = \{\tilde{x}_0, \tilde{x}_1, \cdots \tilde{x}_l\}$

Then $x_0 = \tilde{x}_0$, $x_n = \tilde{x}_l$, $x_j = \tilde{x}_{k_j}$ for $j = 0, 1, \cdots n$

Let $\Delta \tilde{x}_p = \tilde{x}_p - \tilde{x}_{p-1}$, $\tilde{m}_j = \sup\{f(x) : \tilde{x}_{j-1} \le x \le \tilde{x}_j\}$, so

$$m_j \Delta x_j = m_j \sum_{p=k_{j-1}+1}^{k_j} \Delta \tilde{x}_p = \sum_{p=k_{j-1}+1}^{k_j} m_j \Delta \tilde{x}_p \geq \sum_{p=k_{j-1}+1}^{k_j} \tilde{m}_p \Delta \tilde{x}_p$$

$$U(P,f) = \sum_{j=1}^{n} m_j \Delta x_j \geq \sum_{j=1}^{n} \sum_{p=k_{j-1}+1}^{k_j} \tilde{m}_p \Delta \tilde{x}_p = \sum_{j=1}^{l} \tilde{m}_j \Delta \tilde{x}_j = U(\tilde{P}, f)$$

2.(a) Since $B \subset A$, $\forall x \in B \Rightarrow x \in A \Rightarrow x \geq \inf A$

$\Rightarrow \inf A$ is a lower bound of B

$\Rightarrow \inf A \le \inf B$

By Prop. 1.2.8, $\forall \varepsilon > 0$, $\exists x \in A$ s.t. $\inf A \le x < \inf A + \varepsilon$

by hypothesis, $\exists y \in B$ s.t. $x \geq y$

since $y \in B$, $y \geq \inf B$

$\Rightarrow \inf B \le y \le x < \inf A + \varepsilon$

$\Rightarrow \inf B \le \inf A$ since $\varepsilon > 0$ is chosen arbitrarily

Therefore, $\inf B = \inf A$.

(b) $\forall x \in C$, $x = a + b$ for some $a \in A$, $b \in B$

$\Rightarrow x = a + b \le \sup A + \sup B$

$\Rightarrow \sup A + \sup B$ is an upper bound of C

$\Rightarrow \sup C \le \sup A + \sup B$

Let $\varepsilon > 0$ be given, then $\exists a \in A$, $b \in B$ s.t.

$\sup A - \frac{\varepsilon}{2} < a \le \sup A$

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$\sup B - \frac{\varepsilon}{2} < b \le \sup B$

$\Rightarrow \sup A + \sup B - \varepsilon < a + b \le \sup A + \sup B$

Since $a+b \in C$, and $\varepsilon > 0$ is chosen arbitrarily

$\Rightarrow \sup C \ge \sup A + \sup B$

Therefore, $\sup C = \sup A + \sup B$, similarly $\inf C = \inf A + \inf B$

(c) $\forall x \in D$, $f(x) \le \sup\limits_{x \in D} f(x)$, $g(x) \le \sup\limits_{x \in D} g(x)$

$\Rightarrow f(x) + g(x) \le \sup\limits_{x \in D} f(x) + \sup\limits_{x \in D} g(x)$

$\Rightarrow \sup\limits_{x \in D} f(x) + \sup\limits_{x \in D} g(x)$ is an upper bound of $f(x) + g(x)$

$\Rightarrow \sup\limits_{x \in D} (f(x) + g(x)) \le \sup\limits_{x \in D} f(x) + \sup\limits_{x \in D} g(x)$

Similarly, $\inf\limits_{x \in D} (f(x) + g(x)) \le \inf\limits_{x \in D} f(x) + \inf\limits_{x \in D} g(x)$

3. $\overline{\int_a^c} f = \inf\{U(P, f) : P \text{ is a partition of } [a,c]\}$

$= \inf\{U(P, f) : P \text{ is a partition of } [a,c], b \in P\}$

$= \inf\{U(P_1, f) + U(P_2, f) : P_1 \text{ is a partition of } [a,b],$
$P_2 \text{ is a partition of } [b,c]\}$

$= \inf\{U(P_1, f) : P_1 \text{ is a partition of } [a,b]\}$
$+ \inf\{U(P_2, f) : P_2 \text{ is a partition of } [b,c]\}$

$= \overline{\int_a^b} f + \overline{\int_b^c} f$

4. Since $f: [a,b] \to \mathbb{R}$ is Riemann integrable,

$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

Let $m_i := \inf\{f(x) : x \in [f_{i-1}, f_i]\}$,

$M_i = \sup\{f(x) : x \in [f_{i-1}, f_i]\}$

then $L(P,f) = \sum_{i=1}^n m_i \Delta x_i$

$U(P,f) = \sum_{i=1}^n M_i \Delta x_i$

$\Rightarrow L(P,-f) = \sum_{i=1}^n (-M_i)\Delta x_i = -U(P,f)$

$\Rightarrow \underline{\int_a^b} -f(x)\,dx = \sup\{L(P,-f) : P \text{ is a partition of } [a,b]\}$

$= \sup\{-U(P,f) : P \text{ is a partition of } [a,b]\}$

$= -\inf\{U(P,f) : P \text{ is a partition of } [a,b]\}$

$= -\overline{\int_a^b} f(x)\,dx$

$= -\int_a^b f(x)\,dx$

Similarly, $\overline{\int_a^b} -f(x)\,dx = -\int_a^b f(x)\,dx \Rightarrow \underline{\int_a^b} -f = \overline{\int_a^b} -f = \int_a^b f$

Therefore, $-f$ is Riemann integrable,

$\int_a^b (-f) = -\int_a^b f$.

5. (a) Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$.

By Problem 2(c), $\sup_{x \in [x_{i-1}, x_i]} (f(x)+g(x)) = \sup_{x \in [x_{i-1}, x_i]} f(x) + \sup_{x \in [x_{i-1}, x_i]} g(x)$

$\Rightarrow U(P, f+g) = \sum_{i=0}^n \sup_{x \in [x_{i-1}, x_i]} (f(x)+g(x))$

$\leq \sum_{i=0}^n \sup_{x \in [x_{i-1}, x_i]} f(x) + \sum_{i=0}^n \sup_{x \in [x_{i-1}, x_i]} g(x)$

$= U(P,f) + U(P,g)$

Let P_1, P_2 be partitions of $[a,b]$ s.t. $P = P_1 \cup P_2$

By Prop. 5.1.7, $U(P,f) \leq U(P_1, f)$, $U(P,g) \leq U(P_2, g)$

so $U(P, f+g) \leq U(P,f) + U(P,g) \leq U(P_1, f) + U(P_2, g)$

Thus for any partitions of $[a,b]$, P_1, P_2,

we can let $P = P_1 \cup P_2$ s.t.

$\overline{\int_a^b}(f+g) \leq U(P,f) + U(P,g) \leq U(P_1,f) + U(P_2,g)$

Therefore, $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$

Similarly, $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$

(b) By (a) and Prop 5.1.8, $\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g)$

$\leq \overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$

Since $\underline{\int_a^b} f + \underline{\int_a^b} g = \overline{\int_a^b} f + \overline{\int_a^b} g$, all of them are equal,

$\underline{\int_a^b}(f+g) = \overline{\int_a^b}(f+g)$

so $f+g$ is Riemann integrable.

$\int_a^b f + \int_a^b g = \underline{\int_a^b} f + \underline{\int_a^b} g = \underline{\int_a^b}(f+g) = \int_a^b (f+g)$

Therefore, $\int_a^b f+g = \int_a^b f + \int_a^b g$.

6. $\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$: $\forall k \geq M$, $U(P_k, f) - L(P_k, f) < \varepsilon$

$\Rightarrow 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P_k, f) - L(P_k, f) < \varepsilon$

$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$

$\Rightarrow f$ is Riemann integrable

Since $L(P_k, f) \leq \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \leq U(P_k, f)$,

by squeeze lemma, since $\lim\limits_{k\to\infty} L(P_k, f) = \lim\limits_{k\to\infty} U(P_k, f)$,

$\int_a^b f = \lim\limits_{k\to\infty} U(P_k, f) = \lim\limits_{k\to\infty} L(P_k, f)$

7. $f : [0,1] \to \mathbb{R}$, $f(x) = x$, $P_n = \{\frac{k}{n}\}_{k=0}^{n}$

$$L(P_n, f) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n-1)}{2} = \frac{n-1}{2n}$$

$$U(P_n, f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{(n+1)n}{2} = \frac{n+1}{2n}$$

$$\underline{\int_0^1} f = \sup\left\{ \frac{n-1}{2n} : n \in \mathbb{N} \right\} = \frac{1}{2}$$

$$\overline{\int_0^1} f = \inf\left\{ \frac{n+1}{2n} : n \in \mathbb{N} \right\} = \frac{1}{2}$$

Therefore, f is Riemann integrable and

$$\int_0^1 f = \underline{\int_0^1} f = \overline{\int_0^1} f = \frac{1}{2}.$$