# Properties of R, Absolute Value

Saturday, September 17, 2022 2:32 PM

## Density of $\mathbb{Q}$

<u>Thm.</u> (Archimedean Property of $\mathbb{R}$)

If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that

$$nx > y$$

<u>Pf.</u> Divide inequality by $x$:

- For any $t := y/x$, there exists $n \in \mathbb{N}$ s.t. $n > \frac{y}{x} = t$
  - Equivalently, any $t \in \mathbb{R}$ cannot be an upper bound of $\mathbb{N}$
  - Suppose that $\mathbb{N}$ is bounded above. By LUB of $\mathbb{R}$, there exists

    $$b := \sup \mathbb{N}$$

  - $b - 1$ is not an upper bound, so there exists $m \in \mathbb{N}$ s.t. $m \overset{(+1)}{>} b - 1 \overset{(+1)}{}$.
  - $m + 1 \in \mathbb{N}$ but $m + 1 > b = \sup \mathbb{N}$. Contradiction!
  - $\mathbb{N}$ must be bounded above $\Rightarrow$ there always exists $n \in \mathbb{N}$ s.t.

    $$n > \frac{y}{x} \quad \Leftrightarrow \quad nx > y$$

$\square$

<u>Cor.</u> 1) "$\mathbb{Q}$ is dense in $\mathbb{R}$" (on the HW)

2) "infinity" $\infty$ and "infinitesimals" $dx$ are not real numbers!

Cannot automatically replace variables by $\infty, dx$

($\infty > n \quad \forall n \in \mathbb{N}$ would imply $\mathbb{N}$ was bounded above)

<u>Remark</u>: there exist "models" of Calculus which do include $\infty, dx$

e.g. hyperreal numbers

<u>Cor.</u> $\inf\{1/n : n \in \mathbb{N}\} = 0$

<u>Pf.</u> Let $A := \{1/n : n \in \mathbb{N}\}$

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$\Rightarrow$ $b := \inf A$ exists. As the greatest LB, $b \geq \underbrace{0}_{\text{lower bound}}$

- By the Archimedean Property, for all $a > 0$, there exists $n \in \mathbb{N}$ s.t. $n \cdot a > 1$. Thus $\exists \frac{1}{n} \in A$ with $\frac{1}{n} < a$.
  - Thus any $a > 0$ cannot be a lower bound $\Rightarrow$ $b \leq 0$
- Combining inequalities, $b = 0$ $\square$

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Assigned Readings: 1.2.3 Properties of sup/inf

Remark: For this course we will not use the extended reals (no $\pm\infty$)

Def. If $\sup A \in A$, we call this the maximum of $A$ and write $\max A$

" $\inf A \in A$ " minimum " $\min A$

Ex. $\max\{1,2,3\} = 3$

$\sup(0,3) = 3 \notin (0,3)$

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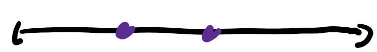
## Absolute Value

Idea: The absolute value quantifies distance



Def. The absolute value of $x \in \mathbb{R}$ is defined

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

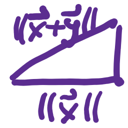
Prop. (1.3.1, basic properties of $|\cdot|$)

$\left.\begin{array}{l} \text{(v) } |x| \leq y \text{ iff } -y \leq x \leq y \\ \text{(vi) } -|x| \leq x \leq |x| \text{ for all } x \in \mathbb{R} \end{array}\right\}$ particularly useful for proofs

Prop. (Triangle inequality)

**Prop.** (Triangle inequality)

$$|x+y| \leq |x|+|y| \quad \text{for all } x,y \in \mathbb{R}$$

(Linear Algebra: $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|$ 

)

**Pf.** For all $x,y \in \mathbb{R}$ we have

$$-|x| \leq x \leq -|x| \quad \text{and} \quad -|y| \leq y \leq |y|$$

Add to get

$$-(|x|+|y|) \leq x+y \leq |x|+|y|$$

Combine

$$|x+y| \leq |x|+|y| \qquad \square$$

**Cor.** (Reverse triangle inequality)

$$\big||x|-|y|\big| \leq |x-y| \quad \left(\leq |x|+|-y| = |x|+|y|\right)$$

**Pf.**

(ii) $|x| = |x-y+y| \leq |x-y|+|y| \Rightarrow \underline{|x|-|y| \leq |x-y|}$

$|y| = |y-x+x| \leq |y-x|+|x| = |x-y|+|x| \Rightarrow \underline{|x|-|y| \geq -|x-y|}$

Combine:

$$-|x-y| \leq |x|-|y| \leq |x-y| \Rightarrow \big||x|-|y|\big| \leq |x-y| \qquad \square$$