Midterm Exam 1 Problem Bank

Examination Date: Tuesday, October 11th

Instructions – please read carefully

- 2 out of 4 problems from this problem bank will appear as-is on the exam
- You may discuss the problems with other students. Furthermore, you can ask the instructor and TA clarifying questions. If you are really stuck, you can ask the instructor and TA for hints to get you started. However, **do not ask for, or share, either partial or full solutions**. You are expected to abide by the NYU CAS Honor Code.
- **You will not be permitted any reference material during the exam**. I advise against rote memorization: as the saying goes, *the easiest way to memorize something is to understand it.*
- You can use any result proved in the course text, in class, or on a previous homework question provided you mention that you are using a result. **You do not need to mention the exact name or reference for the result if you forget it** – the purpose is to demonstrate that you are aware you are using a non-trivial result in your proof.

1. Fix some set X. In lecture, we remarked how $\subset$ is 'kind of like' $\leq$. It is actually possible to take the analogy a little further, and talk about "upper bounds" and "lower bounds" of collections of subsets of X.

   Recall that the powerset $\mathscr{P}(X)$ is the set of all subsets of X. $C \in \mathscr{P}(X)$ means that $C \subset X$. Furthermore, a subset $\mathcal{A} \subset \mathscr{P}(X)$ has elements of the form $B \in \mathcal{A}$, where $B \subset X$. In other words, $\mathcal{A}$ is a collection of sets.

   $P(\{1,2\}) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

   (a) Let $C \in \mathscr{P}(X)$, and let $\mathcal{A} \subset \mathscr{P}(X)$. We say C is an *upper bound* of $\mathcal{A}$ if $B \subset C$ for all $B \in \mathcal{A}$.

   Given $B_1, B_2 \in \mathscr{P}(X)$, show that $B_1 \cup B_2 \in \mathscr{P}(X)$ is an upper bound of $\{B_1, B_2\} \subset \mathscr{P}(X)$.

   $(\Rightarrow)$ $x \in B_1$, $x \in B_2$, $B_1 \cup B_2$ bounded above by $\underbrace{\max\{b_1, b_2\}}_{}$ $= a$ $\leftarrow \sup B_n$, LUB exists and $\sup(B_1, B_2) \leq a$

   $(\Leftarrow)$ $B_1 \subset B_1 \cup B_2$, $b_1 = \sup B_1 \leq \sup(B_1 \cup B_2)$, $b_2 = \sup B_2$, thus $\sup(B_1 \cup B_2) \geq \max\{b_1, b_2\} = a$ (thus "=")

   (b) We say an upper bound S of $\mathcal{A} \subset \mathscr{P}(X)$ is a *least upper bound* if for all upper bounds C of $\mathcal{A}$, $S \subset C$.

   Given $B_1, B_2 \in \mathscr{P}(X)$, show that if C is an upper bound of $\{B_1, B_2\}$, then $B_1 \cup B_2 \subset C$. Use this with your result in (a) to show that $B_1 \cup B_2$ is the unique least upper bound of $\{B_1, B_2\}$.

   (*Hint*: For the second part, first show that $B_1 \cup B_2$ is a least upper bound. Then, show that if S is another least upper bound, $S = B_1 \cup B_2$) let $\sup B_1 \cup B_2 = L$

   $B_1 \cup B_2$ is LUB shown ✓. If $\exists S$ another LUB and $S \neq B_1 \cup B_2$, since S is an upper bound, so $L' = \sup S$, get $L' \leq L$. Because L' is an upper bound of S and $L = \sup S$, $L \leq L'$, so $L = L'$.

   (c) Write down a corresponding definition for some $C \in \mathscr{P}(X)$ to be a *lower bound* of $\mathcal{A} \subset \mathscr{P}(X)$. Use it to prove that $B_1 \cap B_2$ is a lower bound of $\{B_1, B_2\}$.

   ", C is a lower bound of $\mathcal{A}$ if $B \supset C$."

   $(\Rightarrow)$ Problem (a), $\min\{b_1, b_2\}$, $b_n = \inf B_n$ $(\Leftarrow)$ $b_1 = \inf B_1 \geq \inf(B_1 \cap B_2)$, $b_2 = \inf B_2$, thus $\inf(B_1 \cap B_2) \leq \min\{b_1, b_2\}$ thus "="

   (d) Show that $\mathscr{P}(X)$ an analogue to the least upper bound property: every non-empty collection $\mathcal{A} \subset \mathscr{P}(X)$ has a least upper bound S.

   You may find the following to be helpful: given a collection of sets $\mathcal{A} \subset \mathscr{P}(X)$, you can define a set which is the union over this collection,

$$\bigcup_{B \in \mathcal{A}} B := \{x \in X : x \in B \text{ for some } B \in \mathcal{A}\} \in \mathscr{P}(X)$$

$a_b := \sup B$

$(\Rightarrow)$

$\{a_b\}$ is non-empty and bounded above by L, so it has a LUB.

Let $a := \sup\{a_b : B \in A\}$.

$\forall x \in K$, $\Rightarrow x \in B$.

Then, $\forall B \in A$, $x \leq a_b \leq a$

$\Rightarrow$. K is bounded above by a.

Since B not empty, K is non-empty.

Thus LUB exists, $\sup K \leq a$.

$(\Leftarrow)$

$\forall B \in A$, $B \subset K$.

Then $\forall x \in B$, $x \leq \sup K$.

Thus $\sup K$ is an UB.

Thus $\sup B = a_b \leq \sup K$ for all $B \in A$

Thus $\sup K$ is an UB for set $\{a_b\}$

Thus $\sup\{a_b\} = a \leq \sup K$

Thus, $\sup K = a = \sup\{a_b ; B \in A\}$

Proved $P(X)$ an analogus to LUBP.

2. In the following, let $S \subset \mathbb{R}$ be non-empty and bounded above. In this problem, we will see an interesting connection between the supremum and infinite sets.

   (a) Show that there exists a sequence $\{x_n\}$ with $x_n \in S$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = \sup S$. let $c = \sup S$, From Q8 on HW1, $\forall \varepsilon > 0, \exists x \in S$ s.t. $c - \varepsilon < x \leq c$.

   (*Hint*: Look at Q8 on HW1 and HW2) From Q8 on HW2, $|x_n - x| < \varepsilon_n < \varepsilon$ $(\forall n \geq N)$

   $\Rightarrow \lim_{n\to\infty} x_n = c = \sup S$

   (b) Suppose $\sup S \notin S$. Show that there exists a strictly monotone increasing sequence $\{y_n\}$ with $y_n \in S$ for all $n \in \mathbb{N}$.

   (*Hint*: You may want to approach this problem inductively. Take some $y_1 \in S$, and note that $\sup S - y_1 > 0$ (why?). Use this with HW1 Q8 to find y_2, etc... When doing induction, you may want to prove that $\sup S - y_p > 0$ for all $p \in \mathbb{N}$.)

   $c = \sup S \notin S$, $\forall r > 0$, $\exists s \in S$ with $s \in (c - r, c)$, Take any $x_1 \in S$. choose $x_{n+1} \in S$ s.t. $x_{n+1} \in (c - r_n, c)$ where $r_n = \frac{1}{2}(c - x_n)$. As $r_n > 0$, get $x_{n+1} > c - r_n > c - 2r_n = c_n$

   By induction $n > 0$, $0 < c - x_{n+1} < 2^{-n}(c - x_1)$ so $\lim_{n\to\infty} x_n = c$

   (c) Suppose $\sup S \notin S$. Show that there exists a countably infinite subset $E \subset S$.

   (*Hint*: Consider the set $\{y_n : n \in \mathbb{N}\}$ defined for the sequence $\{y_n\}$ in (b). Can you find some bijection between this set and $\mathbb{N}$?)

let any $x_1 \in S$, assume S_n has been defined an element of S. If S_n were the largest element of S, then it is $\sup S$, and $\sup S \in S$, contradiction! So there must be elements of S larger than S_n. Choose one S_{n+1}. The subset $\{S_n : n \in \mathbb{Z}\}$ is a countably infinite subset of S.

   (d) Suppose $\sup S \in S$. Will there always be a countably infinite subset $E \subset S$? Either prove the statement, or give a counterexample.

   (*Note*: finite sets are never countably infinite)

   No. $S = \{1, 2\}$.

3. So far in the course, we have only talked about sequences which converge to some real number. In this problem, we will use the following definition:

   A sequence (of real numbers) $\{x_n\}$ is said to *diverge to (positive) infinity* if for all $K \in \mathbb{R}$, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$, $x_n > K$. In this case, we abuse notation and write

$$\lim_{n\to\infty} x_n := +\infty$$

$\forall K \in \mathbb{R}$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$ $x_n < K$.

   (a) Write down a corresponding definition for a sequence $\{x_n\}$ which *diverges to negative infinity*, and use it to show that $\lim_{n\to\infty} -n^3 = -\infty$

   $\{x_n\}$ is decreasing

   so when $n \geq M$, $x_n \leq x_m \leq K$.

   Then $\lim_{n\to\infty} x_n = -\infty$

   (b) Suppose $\{x_n\}$ is a sequence satisfying $x_n > 0$ for all $n \in \mathbb{N}$, and furthermore

$$\lim_{n\to\infty} \frac{1}{x_n} = 0$$

   As $\lim_{n\to\infty} \frac{1}{x_n} = 0$, so $\forall k \in \mathbb{R}$, $\exists M \in \mathbb{N}$.

   If $n \geq M$, $\left|\frac{1}{x_n} - 0\right| \leq \frac{1}{k}$. $\Rightarrow x_n \geq K$

   Show that $\{x_n\}$ diverges to positive infinity.

   As k is any number,

   $\lim_{n\to\infty} x_n$ diverges to PI.

   (c) Show that a sequence $\{x_n\}$ is unbounded above (i.e. the set $\{x_n : n \in \mathbb{N}\}$ is unbounded above) if and only if $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which diverges to positive infinity.

   (*Hint*: The ($\Longleftarrow$) direction is easier, so you may want to start with that.

   For the ($\Longrightarrow$) direction, I'd recommend splitting the proof into parts:

   (i) prove that if $\{x_n\}$ is unbounded, then every p-tail of $\{x_n\}$ is also unbounded.

   (ii) inductively construct a subsequence $\{x_{n_k}\}$ which satisfies $x_{n_k} > k$ for all $k \in \mathbb{N}$.

   (iii) prove that this subsequence does what you want it to do.)

   ($\Leftarrow$) contrapositive to if X_n converges, then every subseq X_{nk} converges.

   ($\Rightarrow$) (i) Since $\{X_n\}$ is unbounded, p-tail of $\{X_n\}$ is only taking the first p finite number of elements, so T_p ($p \in \mathbb{N}$) is also unbounded.

   (ii) $x_{n_1} > 1$.

   Inductive: $X_{n_k} > X_{n_{k-1}} > k-1$

   diverge to PI.

   (iii) ?. Proved,

4. For a sequence $\{x_n\}$, we define the arithmetic mean sequence $\{\bar{x}_n\}$ as

$$\bar{x}_n := \frac{x_1 + x_2 + ... + x_n}{n}$$

We will see how the convergence of $\{\bar{x}_n\}$ relates to the convergence of $\{x_n\}$. Note this question does not require any knowledge of series.

(a) Let $\{x_n\} := \{(-1)^n\}$. Show that $\{\bar{x}_n\}$ converges.

$\overline{x_n} = \frac{(-1)+1+(-1)+1}{n}$

As $n \to \infty$, $\overline{x_n} \to 0$.

(b) Suppose $\lim_{n\to\infty} x_n = L$. Let $\varepsilon > 0$ be arbitrary, and take $M \in \mathbb{N}$ such that for all $n \geq M$, $|x_n - L| < \varepsilon/2$.
Show that there is some $K \geq 0$ such that for all $n \geq M$, we have

$$|\bar{x}_n - L| \leq \frac{MK}{n} + \frac{(n-M)\varepsilon}{2n}$$

(*Hint*: Use $\bar{x}_n - L = \sum_{k=1}^{n} \frac{x_k - L}{n}$ with the triangle inequality.)

$|x_n - L| = \left|\sum_{k=1}^{n} \frac{x_k - L}{n}\right|$

$\le \sum_{k=1}^{n} \frac{|x_k - L|}{n}$

$\le \sum_{k=1}^{m} \frac{|x_k - L|}{n} + \sum_{m+1}^{n} \frac{|x_k - L|}{n}$

$< \frac{MK}{n} + \frac{(n-M)}{n}\frac{\varepsilon}{2}$

(c) Note the right hand side of the inequality in (b) converges to $\varepsilon/2$ as $n \to \infty$. Prove that there exists some $M' \in \mathbb{N}$ such that for all $n \geq M'$,

$$\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} < \varepsilon$$

and use it to conclude that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} \bar{x}_n = L$.

$\lim \overline{x_n} = L$.

$\underset{\uparrow 0}{\frac{MK}{n}} + \underset{\uparrow 1}{\frac{(n-M)}{n}}\frac{\varepsilon}{2}$ converges to $\frac{\varepsilon}{2}$

Remark: As it turns out, not all bounded sequences have convergent mean sequences.

$\lim_{n\to\infty} |\overline{x_n} - 2| = 0 \le \underset{\lim}{\frac{MK}{n}} + \frac{n-M}{n} \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$

Thus $\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} < \varepsilon$

$|x_n - L| < \frac{\varepsilon}{2} \Rightarrow |\overline{x_n} - L| < \frac{\varepsilon}{2}$

$\lim_{n\to\infty} x_n = L \Rightarrow \lim_{n\to\infty} \overline{x_n} = L$