Midterm 2

1. (a) Given $f: \mathbb{R} \to \mathbb{R}$, $L \in \mathbb{R}$, $f(x)$ converges to L as $x \to -\infty$ if $\forall \varepsilon > 0, \exists M \in \mathbb{R} : \forall x \le M, |f(x) - L| < \varepsilon$

For all $\varepsilon > 0$, $\exists M = \frac{1}{\sqrt{\varepsilon}} \in \mathbb{R}$ s.t.
$\forall x \ge M, \left|\frac{1}{1+x^2} - 0\right| = \frac{1}{1+x^2} < \frac{1}{x^2} \le \frac{1}{M^2} = \varepsilon$
$\exists M' = -\frac{1}{\sqrt{\varepsilon}} \in \mathbb{R}$ s.t.
$\forall x \le M, \left|\frac{1}{1+x^2} - 0\right| = \frac{1}{1+x^2} < \frac{1}{x^2} \le \frac{1}{M^2} = \varepsilon$
By definition, $\lim_{x\to\infty} \frac{1}{1+x^2} = \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$

(b) g is continuous at 0 iff $\forall \varepsilon > 0, \exists \delta > 0 :$
$\forall y \in \mathbb{R}, |y| \le \delta, |g(y) - g(0)| < \varepsilon$
$\forall \varepsilon > 0$, $\exists \overset{+}{M_1}, \overset{-}{M_2}$ for f, let $\delta = \min\{\frac{1}{M_1}, -\frac{1}{M_2}\}$
$\forall y \in \mathbb{R}, |y| < \delta$ $\Rightarrow \frac{1}{\delta} = \max\{M_1, -M_2\}$
1° $y = 0$, $|g(y) - g(0)| = 0 < \varepsilon$
2° $0 < y < \delta \Rightarrow \frac{1}{y} > \frac{1}{\delta} \ge M_1$
$|g(y) - g(0)| = |f(\frac{1}{y}) - L| < \varepsilon$
3° $-\delta < y < 0 \Rightarrow -\frac{1}{y} > \frac{1}{\delta} \ge -M_2 \Rightarrow \frac{1}{y} < M_2$
$|g(y) - g(0)| = |f(\frac{1}{y}) - L| < \varepsilon$
Therefore, g is continuous at 0.

(c) f is continuous at 0

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0:$

$\forall x \in \mathbb{R}, |x| < \delta, |f(x) - f(0)| < \varepsilon$

so $\exists M_1 = \frac{1}{\delta} \in \mathbb{R}, \forall y \geq M_1 = \frac{1}{\delta},$

$\frac{1}{y} < \delta \Rightarrow |\frac{1}{y}| < \delta \Rightarrow |f(\frac{1}{y}) - f(0)| < \varepsilon$

$\Rightarrow |g(y) - f(0)| < \varepsilon$

$\exists M_2 = -\frac{1}{\delta} \in \mathbb{R}, \forall y \leq M_2 = -\frac{1}{\delta}$

$0 > \frac{1}{y} > -\delta \Rightarrow |\frac{1}{y}| < \delta \Rightarrow |f(\frac{1}{y}) - f(0)| < \varepsilon$

$\Rightarrow |g(y) - f(0)| < \varepsilon$

By definition,

$\lim_{y \to \infty} g(y) = \lim_{y \to -\infty} g(y) = f(0)$

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2. (a) For $[a,b]$,

every sequence $\{x_n\}$ in $[a,b]$ is bounded since $a \le x_n \le b \ \forall n$

& thm 2.3.4. By Bolzano-Weierstrass thm, there exists a convergent subsequence $\{x_{n_i}\}$ s.t. $\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$

Since $x_{n_i} \in [a,b]$,

$$\begin{aligned} \lim_{i\to\infty} x_{n_i} &= \inf\{a_n : n \in \mathbb{N}\} \quad a_n = \sup\{x_k : k \ge n\} \\ &\le \sup\{x_n : n \ge 1\} \\ &\le b \end{aligned}$$

$\lim_{i\to\infty} x_{n_i} \in [a,b]$

so $[a,b]$ is sequentially compact

For (a,b),

let $x_n = b - \frac{b-a}{2^n}$

every subsequence $\{x_{n_k}\}$ converges to b since

$\forall \varepsilon > 0, \exists M = \lceil \log_2(\frac{b-a}{\varepsilon}) \rceil \in \mathbb{N}$ s.t. $\forall k \ge M, |x_{n_k} - b| < \varepsilon$

But $b \notin (a,b)$

so (a,b) is not sequentially compact

(b) Suppose K is not bounded, then $\exists \{x_n\} \subset K$ s.t. $x_n \ge n$

Since K is sequentially compact,

then $\exists \{x_{n_k}\}$ s.t. $\lim_{k\to\infty} x_{n_k} = x$ where $x \in K$

But for all $n_k > x$, $x_{n_k} \ge n_k > x \Rightarrow$ contradiction

Therefore, K is bounded.

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(c) If K is sequentially compact, by (b) it is bounded.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$,
$\exists x_1$ s.t. $\sup K - 1 < x_1 < \sup K$
$\exists x_2$ s.t. $\sup K - \frac{1}{2} < x_2 < \sup K$
$\vdots$
$\exists x_n$ s.t. $\sup K - \frac{1}{n} < x_n < \sup K$
Since $x_n < \sup K$ for all n, $x_n \in K$, $\{x_n\}$ in K.
By cor. of squeeze lemma, $\{x_n\}$ converges to $\sup K$ since $\sup K - \frac{1}{n} \to K$.
Since K is sequentially compack,
$\sup K \in K$, similarly $\inf K \in K$.

(d) For any sequence $\{y_n\} \subset f(K)$, $\{y_n\} = \{f(x_n)\}$ where $\{x_n\} \subset K$
Since K is sequentially compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$
s.t. $\lim\limits_{k\to\infty} x_{n_k} = x$ for $x \in K$.
Then $\lim\limits_{k\to\infty} y_{n_k} = \lim\limits_{k\to\infty} f(x_{n_k}) = f(x)$

since $x \in K$, $f(x) \in f(K)$
Therefore $f(K)$ is sequentially compact.

By (c), $\sup f(K) \in f(K)$, $\inf f(K) \in f(K)$,
since $\forall x \in K$, $f(x) \le \sup f(K)$, $f(x) \ge \inf f(K)$
Therefore, f achieves an abs min and abs max in K.

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3. (a) By Taylor Theorem, given $x, x+h \in \mathbb{R}$,
$\exists c$ strictly between x and $x+h$ such that

$$f(x+h) = P_1^x(x+h) + \frac{f''(c)}{2!}(x+h-x)^2$$

$$= f(x) + f'(x)h + \frac{f''(c)}{2}h^2$$

$$\frac{f(x+h)-f(x)}{h} - f'(x) = \frac{f''(c)}{2}h$$

In $[x, x+h]$, f has 3 continuous derivative $\Rightarrow f''$ bounded
$\Rightarrow m_1 \le f''(c) \le m_2$
Let $M_1 = \frac{m_1}{2}$, $M_2 = \frac{m_2}{2}$
$\Rightarrow M_1 h \le \frac{f''(c)}{2}h \le M_2 h$

(b) By Taylor Theorem,
$\exists c_1 \in (x, x+h)$, $c_2 \in (x-h, x)$ s.t.

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f^{(3)}(c_1)}{6}h^3$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f^{(3)}(c_2)}{6}h^3$$

$$\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \frac{f^{(3)}(c_1) + f^{(3)}(c_2)}{12}h^2$$

In $[x, x+h]$ and $[x-h, x]$,
f has 3 continuous derivative $\Rightarrow f^{(3)}$ bounded
$\Rightarrow m_1 \le f^{(3)}(c_1) \le m_2$, $m_3 \le f^{(3)}(c_2) \le m_3$
Let $M_3 = \frac{m_1+m_3}{12}$, $M_4 = \frac{m_2+m_4}{12}$ $\Rightarrow$ ✓

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(c) $f(x) = \sin(kx)$

$f'(x) = k\cos(kx)$

$f''(x) = -k^2\sin(kx)$

$f^{(3)}(x) = -k^3\cos(kx)$

By part (a),

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| = \left|\frac{f''(c)}{2}h\right|$$

$$= \left|\frac{k^2\sin(kx)h}{2}\right| = \frac{k^2h}{2}\left|\sin(kx)\right| \le \frac{k^2h}{2}$$

By part (b), $k>0$?

$$\left|\frac{f(x+h)-f(x-h)}{2h} - f'(x)\right| = \left|\frac{f^{(3)}(c_1)+f^{(3)}(c_2)}{12}h^2\right|$$

$$= \frac{h^2}{12}\left|-k^3\cos(c_1x) - k^3\cos(c_2x)\right|$$

$$\le \frac{k^3h^2}{12}\left(\left|\cos(c_1x)\right| + \left|\cos(c_2x)\right|\right)$$

$$\le \frac{k^3h^2}{12}(1+1) = \frac{k^3h^2}{6}$$

4. (a) $m_i = \inf\{f(x): x \in [t_{i-1}, t_i]\}$

$M_i = \sup\{f(x): x \in [t_{i-1}, t_i]\}$

$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$

$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$

$\Rightarrow m_i \le f(c_i) \le M_i$ for $i = 1, \ldots n$ $\Rightarrow m_i \Delta x_i \le f(c_i)\Delta x_i \le M_i \Delta x_i$

$\Rightarrow L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i = U(P, f)$

(b) Since f is Riemann integrable, $\int_a^b f = \underline{\int}_a^b f = \overline{\int}_a^b$

$= \sup\{L(P, f)\} = \inf\{U(P, f)\}$

so $\exists$ partition P_1 of $[a, b]$ s.t. $\underline{\int}_a^b f - \varepsilon < L(P_1, f) < \underline{\int}_a^b f$

$\int_a^b f - \varepsilon < L(P_1, f) < \int_a^b f$

P_2 $\quad \int_a^b f < U(P_2, f) < \int_a^b f + \varepsilon$

Let $P = P_1 \cup P_2$ so it is a partition of $[a, b]$

P is a refinement of P_1, so $L(P_1, f) \le L(P, f)$

P_2 $\quad U(P_2, f) \ge U(P, f)$

$\Rightarrow \int_a^b f - \varepsilon < L(P, f) \le U(P, f) < \int_a^b f + \varepsilon$

For any tagging τ,

$\Rightarrow \int_a^b f - \varepsilon < L(P, f) \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le U(P, f) < \int_a^b f + \varepsilon$

$\Rightarrow \left| \int_a^b f - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$

(c) For $[0, 1]$, $x_i = 0 + (1-0)\cdot \frac{i}{n} = \frac{i}{n} \in \mathbb{Q}$ $\forall n \in \mathbb{N}$

For $[1, 1+\sqrt{2}]$, $x_i = 1 + (1+\sqrt{2} - 1)\cdot \frac{i}{n} = 1 + \sqrt{2}\left(\frac{i}{n}\right) \notin \mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$

$\left(\frac{i}{n}\right) \in \mathbb{Q}$

For $[0, 1+\sqrt{2}]$, $x_i = 0 + (1+\sqrt{2} - 0)\cdot \frac{i}{n} = (1+\sqrt{2})\cdot \frac{i}{n} \notin \mathbb{Q}$

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$R_n(f,[0,1]) = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} 1\cdot\frac{1}{n} = 1 \to 1$ as $n\to\infty$

$R_n(f,[1,1+\sqrt{2}]) = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} 0\cdot\Delta x_i = 0 \to 0$ as $n\to\infty$

$R_n(f,[0,1+\sqrt{2}]) = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} 0\cdot\Delta x_i = 0 \to 0$ as $n\to\infty$

so all of them converges

$$\lim_{n\to\infty}(R_n(f,[0,1]) + R_n(f,[1,1+\sqrt{2}])) = \lim_{n\to\infty}(1+0) = 1+0 = 1$$

$\lim_{n\to\infty} R_n(f,[0,1+\sqrt{2}]) = 0$

so they are not equal.

(d) Not true.

For $\{R_n(f,[0,1])\}$ in (c) where $f:[0,1]\to\mathbb{R}$

$$f(x) := \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$$

For any partition P, $m_i = \inf\{f(x): x_{i-1}\le x\le x_i\} = 0$

$M_i = \sup\{f(x): x_{i-1}\le x\le x_i\} = 0$

so $L(P,f) = \sum_{i=1}^{n} m_i\Delta x_i = \sum_{i=1}^{n} 0\cdot\Delta x_i = 0$

$U(P,f) = \sum_{i=1}^{n} M_i\Delta x_i = \sum_{i=1}^{n} 1\cdot\Delta x_i = 1$

so $\underline{\int_a^b} f(x)\,dx = 0$

$\overline{\int_a^b} f(x)\,dx = 1 \neq \underline{\int_a^b} f(x)\,dx$

so f is not Riemann integrable

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