HW4

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1\. (a) $x_n = (-c)^n,\ 0 < c < 1$

$$a_n = \sup\{x_k : k \geq n\} = \begin{cases} (-c)^n, & n \text{ is even} \\ (-c)^{n+1}, & n \text{ is odd} \end{cases}$$

$$b_n = \inf\{x_k : k \geq n\} = \begin{cases} (-c)^{n+1}, & n \text{ is even} \\ (-c)^n, & n \text{ is odd} \end{cases}$$

$\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\} = 0$

$\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\} = 0$

Since $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$, $\{x_n\}$ converges

(b) $x_n = (-1)^n + \frac{1}{n}$

$$a_n = \sup\{x_k : k \geq n\} = \begin{cases} 1 + \frac{1}{n}, & n \text{ is even} \\ 1 + \frac{1}{n+1}, & n \text{ is odd} \end{cases}$$

$$b_n = \inf\{x_k : k \geq n\} = \begin{cases} -1 + \frac{1}{n+1}, & n \text{ is even} \\ -1 + \frac{1}{n}, & n \text{ is odd} \end{cases}$$

$\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\} = 1$

$\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\} = -1$

Since $\limsup_{n\to\infty} x_n \neq \liminf_{n\to\infty} x_n$, $\{x_n\}$ diverges.

2\. Given $\varepsilon > 0$, find $M$ such that $M > \sqrt{\frac{2}{\varepsilon}}$.

Then for $n, k \geq M$, we have $\frac{1}{n^2} \leq \frac{1}{M^2} < \frac{\varepsilon}{2}$, $\frac{1}{k^2} \leq \frac{1}{M} < \frac{\varepsilon}{2}$.

Therefore, for $n, k \geq M$, we have

$$\left|\frac{1}{n^2} - \frac{1}{k^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{k^2}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \left\{\frac{1}{n^2}\right\}$ is Cauchy

3.(a) Define $z_n := x_n + y_n$

$\{x_n\}$ is bounded $\Rightarrow \exists\, b_1 \in \mathbb{R}: |x_n| \le b_1 \;\forall n \in \mathbb{N}$

$\{y_n\}$ is bounded $\Rightarrow \exists\, b_2 \in \mathbb{R}: |y_n| \le b_2 \;\forall n \in \mathbb{N}$

$\Rightarrow |z_n| = |x_n + y_n| \le |x_n| + |y_n| \le b_1 + b_2,\ \forall n \in \mathbb{N}$

$\Rightarrow \{z_n\}$ is bounded

By Thm 2.3.4, there exists a subsequence $\{z_{n_i}\} = \{x_{n_i} + y_{n_i}\}$ which converges to $\liminf\limits_{n\to\infty} z_n = \liminf\limits_{n\to\infty}(x_n + y_n)$.

(b) $\forall n \in \mathbb{N},\ i \ge n,$

$x_i + y_i \ge \inf\{x_k, k \ge n\} + \inf\{y_k : k \ge n\}$

So $\forall n \in \mathbb{N}$,

$\inf\{x_k + y_k : k \ge n\} \ge \inf\{x_k : k \ge n\} + \inf\{y_k : k \ge n\}$

By taking the limit as $n \to \infty$ of both sides,

$\liminf\limits_{n\to\infty}(x_n + y_n) \ge (\liminf\limits_{n\to\infty} x_n) + (\liminf\limits_{n\to\infty} y_n)$

(c) $\{x_n\} = \{(-1)^n\}$, $\{y_n\} = \{(-1)^{n+1}\}$, then $\{x_n + y_n\} = \{0\}$

$\liminf\limits_{n\to\infty}(x_n + y_n) = 0 > (-1) + (-1) = \liminf\limits_{n\to\infty} x_n + \liminf\limits_{n\to\infty} y_n$

4.(a) $\{x_n\} = \{n\}$ is unbounded

For every subsequence $\{x_{n_k}\}$,

since $n_k \geq k \Rightarrow x_{n_k} \geq k \quad \forall k$

$\forall L \in \mathbb{R}, \exists \varepsilon = 1 > 0 : \forall M \in \mathbb{N}, \exists k = \max\{M, [L+2]\} \geq M$

s.t. $|x_{n_k} - L| \geq L+1-L = 1 = \varepsilon$

$\Rightarrow \{x_n\}$ diverges

(b) From HW2, we have $\exists r_n \in \mathbb{Q}$ s.t. $\sqrt{2} - \frac{1}{n} < r_n < \sqrt{2} \quad \forall n$

For every subsequence $\{r_{n_k}\}$

$\forall \varepsilon > 0, \exists M > \frac{1}{\varepsilon} : \forall k \geq M$

$|r_{n_k} - \sqrt{2}| < |\frac{1}{n_k}| \leq |\frac{1}{k}| \leq |\frac{1}{M}| < \varepsilon$

$\Rightarrow$ every subsequence converges to $\sqrt{2} \notin \mathbb{Q}$

$\Rightarrow$ no subsequence converges to a rational number.

5.(a) By Prop. 2.4.4, a Cauchy sequence is bouned.

So a Cauchy sequence has some convergent subsequence by Thm. 2.3.8.

Suppose $\{x_n\}$ is Cauchy and $\{x_{n_k}\}$ converges to $L$.

$\forall \varepsilon > 0$

$\exists m \in \mathbb{N}, M_1 = n_m$ s.t. $\forall k \geq m, n_k \geq M_1, |x_{n_k} - L| < \frac{\varepsilon}{2}$

$\exists M_2 \in \mathbb{N}$ s.t. $\forall n, l \geq M_2, |x_n - x_l| < \frac{\varepsilon}{2}$

let $M = \max\{M_1, M_2\}$

$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Therefore, every Cauchy sequence of real numbers is convergent.

(b) We use $\{r_n\}$ from problem 4b that converges to $\sqrt{2} \notin \mathbb{Q}$,

$\sqrt{2} - \frac{1}{n} < r_n < \sqrt{2} \quad \forall n$

So $\forall \varepsilon > 0, \exists M > \frac{2}{\varepsilon}$ s.t. $\forall n, k \geq M$

$$|x_n - x_k| \leq |x_n - \sqrt{2}| + |x_k - \sqrt{2}| < \frac{1}{n} + \frac{1}{k}$$

$$\leq \frac{1}{M} + \frac{1}{M} = \frac{2}{M} < \varepsilon \Rightarrow \text{Cauchy}$$

So there exists a Cauchy sequence $\{r_n\}$ which does not converge to some $r \in \mathbb{Q}$.

6. Since $\forall n, x_n \leq y_n$,

$x_1 = y_1$

if $\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n \Rightarrow \sum_{n=1}^{k+1} x_n = \sum_{n=1}^{k} x_n + x_{n+1}$

$$< \sum_{n=1}^{k} y_n + y_{n+1} = \sum_{n=1}^{k+1} y_n$$

By induction, $\forall N, \sum_{n=1}^{N} x_n \leq \sum_{n=1}^{N} y_n$

$$\Rightarrow \lim_{N\to\infty} \sum_{n=1}^{N} x_n \leq \lim_{N\to\infty} \sum_{n=1}^{N} y_n$$

which is $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$

7. $\forall n, d_n$, $\frac{d_n}{10^n} < \frac{9}{10^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges by comparison

$$0 \leq \sum_{n=1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=1}^{\infty} \frac{9}{10^n}$$

$$= \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = 1$$

Therefore, it is absolutely convergent to a number $x \in [0,1]$.