Sequences of Functions

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Def. For every $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$ be a function. We say the sequence (of functions) $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f : S \to \mathbb{R}$ if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in S$$

- Limit of $\{f_n(x)\}$ is unique $\Rightarrow$ f (if it exists) will be unique
- If $f_n : S \to \mathbb{R}$ converges on some set $T \subset S$ to $f : T \to \mathbb{R}$, then we say $\{f_n\}$ converges on T pointwise to f.

Ex.

$$f_n(x) := \sum_{k=0}^{n} x^k = \frac{1-x^n}{1-x}$$

$$\lim_{n\to\infty} f_n(x) = \frac{1}{1-x} \quad \text{for all } x \in (-1,1)$$

$\Rightarrow$ $\{f_n\}$ converges ptwise to $f : (-1,1) \to \mathbb{R}$, where $f(x) := \frac{1}{1-x}$

Def. (Uniform convergence)

Let $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$. We say $\{f_n\}$ converges uniformly to f if for all $\varepsilon > 0$, there exists $\underbrace{M \in \mathbb{N}}_{\text{cannot depend on } x}$ such that for all $n \geq M$ and $x \in S$,

$$|f_n(x) - f(x)| < \varepsilon$$

Ex. $f_n, f : [0,1] \to \mathbb{R}$

$$f_n(x) := x^n, \quad f(x) := \begin{cases} 1 & x = 1 \\ 0 & 0 \leq x < 1 \end{cases}$$

Claim: $\{f_n\}$ converges ptwise to f but not uniformly.

Pf.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 1 & x = 1 \\ 0 & |x| < 1 \end{cases}$$

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so $\{f_n\}$ converges ptwise to f.

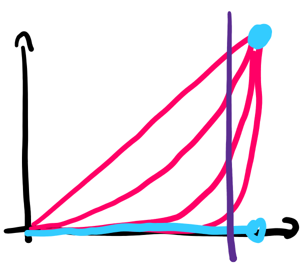
- Now, suppose $\{f_n\}$ converges uniformly to f.
  - Take $\varepsilon = \frac{1}{2} > 0$. Then, $\exists M \in \mathbb{N} : \forall x \in [0,1]$,
    $$|x^M - 0| < \varepsilon = \frac{1}{2} \quad (*)$$
  - Take $\{x_k\} := \{1 - \frac{1}{k}\}$. Note $x_k \in [0,1)$ $\forall k \in \mathbb{N}$, $x_k \to 1$ as $k \to \infty$.
    - Then, by $(*)$
      $$|x_k^M| < \frac{1}{2}$$

    but
    $$\lim_{k\to\infty} |x_k^M| = 1 \nleq \frac{1}{2}$$

  This is a contradiction, so $\{f_n\}$ cannot converge uniformly to f. $\square$

Remarks:

Picture:



at fixed x, $f_n(x) \to f(x)$ as $n \to \infty$

but as $x \to 1$, this convergence becomes arbitrarily slow

"Interchange of limits":

$$\lim_{x\to 1^-} \lim_{n\to\infty} f_n(x) = \lim_{x\to 1^-} f(x) = 0$$

$$\lim_{n\to\infty} \lim_{x\to 1^-} f_n(x) = \lim_{n\to\infty} 1^n = 1$$

Dependence on domain:

$\{f_n\}$ will converge uniformly to f on $[-a, a]$, $0 < a < 1$

Prop. Let $f_n, f : S \to \mathbb{R}$. If $\{f_n\}$ converges uniformly to f, then it also converges pointwise.

Pf. uniform: $\forall \varepsilon > 0, \overbrace{\exists M \in \mathbb{N}}^{\text{cannot depend on } x} : \forall x \in S, \forall n \geq M, |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow$ ptwise: $\forall x \in S, \forall \varepsilon > 0, \underbrace{\exists M \in \mathbb{N}}_{\text{can depend on } x} : \forall n \geq M, |f_n(x) - f(x)| < \varepsilon$

Convergence in Uniform Norm

Motivation: what is a norm?

→ norms assign "magnitude" to objects (usually vectors)

Ex. $\vec{x} = (x, y) \in \mathbb{R}^2$ $\quad \|\vec{x}\|_2 := \sqrt{x^2 + y^2}$ (Euclidean norm, ℓ^2)

$\|\vec{x}\|_1 := |x| + |y|$ (Manhattan norm, ℓ^1)

$\|\vec{x}\|_\infty := \max\{|x|, |y|\}$ (sup norm, ℓ^∞)

$x \in \mathbb{R}$ $\quad |x|$ (Abs. value)

Def. Let $f : S \to \mathbb{R}$ be bounded. Define

$$\|f\|_u := \sup\{|f(x)| : x \in S\}$$

$\|\cdot\|_u$ is called the uniform norm

Prop. A sequence of bounded functions $\{f_n\}$, $f_n : S \to \mathbb{R}$, converges uniformly to $f : S \to \mathbb{R}$ if and only if

$$\lim_{n \to \infty} \|f_n - f\|_u = 0$$

we also say $\{f_n\}$ converges in uniform norm to f.

Remark: $x_n \to x$ as $n \to \infty$: $\forall \varepsilon > 0, \exists M \in \mathbb{N} : \forall n \geq M, |x_n - x| < \varepsilon$

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$f_n \to f$ in unif. norm as $n\to\infty$: $\forall \varepsilon>0, \exists M\in\mathbb{N} : \forall n\geq M, \|f_n - f\|_u < \varepsilon$

Pf. $(\Leftarrow)$ Assume $\lim\limits_{n\to\infty} \|f_n - f\|_u = 0$.

- Let $\varepsilon>0$ be given. $\exists \underbrace{M\in\mathbb{N}}_{\text{does not depend on } x} : \forall n\geq M,$

$$\|f_n - f\|_u = \sup\{|f_n(x) - f(x)| : x\in S\} < \varepsilon$$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x\in S$

$\Rightarrow \{f_n\}$ converges uniformly to f.

$(\Rightarrow)$ Assume $\{f_n\}$ converges uniformly to f.

- Let $\varepsilon>0$ be given. $\exists M\in\mathbb{N} : \forall n\geq M,$

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall x\in S$$

$\Rightarrow \|f_n - f\|_u = \sup\{|f_n(x) - f(x)| : x\in S\} \leq \varepsilon/2 < \varepsilon$

$\Rightarrow \{f_n\}$ converges in uniform norm to f. $\square$

Remark: Sometimes we want to test for convergence without knowing what the limit f is.

Def. Let $f_n : S\to\mathbb{R}$ be bounded for all $n\in\mathbb{N}$. We say $\{f_n\}$ is <u>Cauchy in the uniform norm</u> or <u>uniformly Cauchy</u> if, for all $\varepsilon>0$, there exists $M\in\mathbb{N}$ such that for all $n,k\geq M$,

$$\|f_n - f_k\| < \varepsilon$$

Recall: For sequences of real numbers, Cauchy $\Leftrightarrow$ convergent.

Prop. (Cauchy completeness of uniform norm)

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A sequence $\{f_n\}$ of bounded functions $f_n : S \to \mathbb{R}$ is Cauchy in the uniform norm if and only if there exists $f : S \to \mathbb{R}$ such that $\{f_n\}$ converges uniformly to f.

Pf. ($\Rightarrow$) Assume $\{f_n\}$ is Cauchy in uniform norm.

(first define f) Let $x \in S$ be arbitrary. Then, $\{f_n(x)\}$ is Cauchy (as a sequence of real numbers), since

$$|f_n(x) - f_k(x)| \leq \sup\{|f_n(x) - f_k(x)| : x \in S\} = \|f_n - f_k\|_u \quad \forall n, k \in \mathbb{N}$$

- Since $\{f_n(x)\}$ is Cauchy, it is convergent. Define $f : S \to \mathbb{R}$

$$f(x) := \lim_{n \to \infty} f_n(x)$$

(show uniform convergence) Let $\varepsilon > 0$ be arbitrary.

- Take $M \in \mathbb{N}$: $\forall n, k \geq M$, $\|f_n - f_k\|_u < \varepsilon/2$

$$\Rightarrow |f_n(x) - f_k(x)| < \varepsilon/2 \qquad \forall x \in S$$

$$\Rightarrow \lim_{k \to \infty} |f_n(x) - f_k(x)| = |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \qquad \forall x \in S$$

$\Rightarrow \{f_n\}$ converges uniformly to f. ✓

($\Leftarrow$) Suppose $\{f_n\}$ converges uniformly to some f.

- Let $\varepsilon > 0$ be arbitrary. $\exists M \in \mathbb{N}$: $\forall n \geq M$, $|f_n(x) - f(x)| < \varepsilon/4$ $\forall x \in S$
- Then, $\forall n, k \geq M$,

$$\begin{aligned} |f_n(x) - f_k(x)| &= |f_n(x) - f(x) + f(x) - f_k(x)| \\ &\leq |f_n(x) - f(x)| + |f_k(x) - f(x)| \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2 \qquad (\forall x \in S) \end{aligned}$$

$\Rightarrow \|f_n - f_k\|_u = \sup\{|f_n(x) - f_k(x)| : x \in S\} \leq \varepsilon/2 < \varepsilon$

$\Rightarrow$ $\{f_n\}$ is Cauchy in uniform norm. $\square$