Homework 5 Solutions

Due: Monday, October 24th by 11:59 PM ET

Sections 2.3-2.5 Exercises

Problem 1 (4 points each) Exercises on cluster points:

(a) Let $S = (a, b)$ be an open interval with $a, b \in \mathbb{R}$ and $a < b$. Show that $[a, b]$ is the set of all cluster points of S.

(b) Let $S = \mathbb{Z}$. Show that S has no cluster points in $\mathbb{R}$.

(c) Let $S = \mathbb{Q}$. Show that $\mathbb{R}$ is the set of all cluster points of S.

(a) We prove this by cases. Let $c \in \mathbb{R}$ be the candidate cluster point.

$a < c < b$: Let $\varepsilon > 0$ be given. Let $\delta := \min\{\varepsilon/2, (c-a)/2\} > 0$. Then, $a < c - \delta < c < b$ and $c - \varepsilon < c - \delta < c$, so $c - \delta \in (a, b) \setminus \{c\} \cap (c - \varepsilon, c + \varepsilon)$. Thus, $(a, b) \setminus \{c\} \cap (c - \varepsilon, c + \varepsilon)$ is non-empty for all $\varepsilon > 0$, hence c is a cluster point.

$c = a$: Let $\varepsilon > 0$ be given. Then, since $a - \varepsilon < a$, and $a < a + \varepsilon$, we have that $(a, b) \cap (a - \varepsilon, a + \varepsilon) = (a, \min\{a + \varepsilon, b\}) \neq \emptyset$. Thus, a is a cluster point.

$c = b$: Let $\varepsilon > 0$ be given. $b - \varepsilon < b$ and $b < b + \varepsilon$, so $(a, b) \cap (b - \varepsilon, b + \varepsilon) = (\max\{a, b - \varepsilon\}, b) \neq \emptyset$, so b is a cluster point.

$c > a$: Choose $\varepsilon = (c - a)/2$, then $c + \varepsilon < a$ so $(a, b) \cap (c - \varepsilon, c + \varepsilon) = \emptyset$. Thus c is not a cluster point.

$c < b$: Choose $\varepsilon = (b - c)/2$, then $c - \varepsilon > b$ so $(a, b) \cap (c - \varepsilon, c + \varepsilon) = \emptyset$. Thus c is not a cluster point.

(b) Let $m \in \mathbb{Z}$ be any integer. Then, choose $\varepsilon = 1/2$. Since $|m - n| \geq 1$ for any other integer $n \in \mathbb{Z}$ with $n \neq m$, we have that $Z \setminus \{m\} \cap (m - 1/2, m + 1/2) = \emptyset$, so no $m \in \mathbb{Z}$ is a cluster point.

Let $x \in \mathbb{R} \setminus \mathbb{Z}$. Let $\delta := \min\{|n - x| : n \in \mathbb{Z}\}$, which is the distance between x and the closest point in $\mathbb{Z}$. $\delta > 0$, since $x \notin \mathbb{Z}$. Let $\varepsilon = \delta/2$, then

$$\mathbb{Z} \setminus \{x\} \cap (x - \epsilon, x + \epsilon) = \mathbb{Z} \cap (x - \epsilon, x + \epsilon) = \{n \in \mathbb{Z} \colon |n - x| < \delta/2\} = \emptyset$$

thus no number $m \in \mathbb{Z}$ or $x \in \mathbb{R} \setminus \mathbb{Z}$ is a cluster point of $\mathbb{Z}$, so $\mathbb{Z}$ has no cluster points.

(c) Let $c \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then, by the density of rationals in the real numbers, there exists $r \in \mathbb{Q}$ with $c - \varepsilon < r < c$. Thus, $\mathbb{Q} \setminus \{c\} \cap (c - \varepsilon, c + \varepsilon)$ is non-empty. Thus, c is a cluster point.

Problem 2 (4 points each) Prove the following, using the ε-δ definition of the limit of a function:

(a) Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) := \sqrt{x}$. Show that $\lim\limits_{x\to c} f(x) = \sqrt{c}$ for all $c \in [0, \infty)$. Is f a continuous function?

(Remark: You may use the fact that $0 \le a < b$ if and only if $\sqrt{a} < \sqrt{b}$. As a hint on how to play the ε games, look at the proof of Proposition 2.2.6 in the textbook.)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \cos(x)$. Show that $\lim\limits_{x\to c} f(x) = \cos(c)$ for all $c \in \mathbb{R}$. Is f a continuous function?

(Remark: You may use trigonometric identities here, and the fact that $|\sin(x)| \le |x|$, and $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$. See Example 3.2.6 in the textbook for the necessary algebra; however, you will need explain all of the steps of the proof to receive credit.)

(a) First, consider the case $c = 0$. Let $\varepsilon > 0$ be given. Then, let $\delta := \varepsilon^2 > 0$. Thus, for all $x \in (0, \infty)$ with $|x - c| < \delta$,

$$\left|\sqrt{x} - \sqrt{c}\right| = \left|\sqrt{x}\right| < \left|\sqrt{\varepsilon^2}\right| = \varepsilon$$

Thus $\lim_{x\to 0} \sqrt{x} = 0$

Now, let $c \in (0, \infty)$ and $\varepsilon > 0$ be given. Let $\delta := \varepsilon\sqrt{c}$. Note that $\sqrt{x} + \sqrt{c} \ge \sqrt{c} > 0$. Then for all $x \in [0, \infty) \setminus \{c\}$ with $|x - c| < \delta$,

$$\left|\sqrt{x} - \sqrt{c}\right| = \left|(\sqrt{x} - \sqrt{c})\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}\right| = \frac{|x - c|}{\left|\sqrt{x} + \sqrt{c}\right|} \le \frac{|x - c|}{\sqrt{c}} < \varepsilon$$

Thus $\lim\limits_{x\to c} \sqrt{x} = \sqrt{c}$ for all $c \in [0, \infty)$. This also shows f is a continuous function, since it is continuous for all c in the domain of f.

(b) Let $c \in \mathbb{R}$ and $\varepsilon > 0$ be given. Let $\delta := \varepsilon$. Then, for all $x \in \mathbb{R} \setminus \{c\}$ with $|x - c| < \delta$, we have

$$\begin{aligned}
|\cos(x) - \cos(c)| &= \left|-2\sin\left(\frac{x-c}{2}\right)\sin\left(\frac{x+c}{2}\right)\right| && \text{(Sum-to-product trig identity)} \\
&= 2\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\sin\left(\frac{x+c}{2}\right)\right| \\
&\le 2\left|\sin\left(\frac{x-c}{2}\right)\right| && (\sin(x) \le 1\ \forall x \in \mathbb{R}) \\
&\le 2\left|\frac{x-c}{2}\right| = |x - c| && (|\sin(x)| \le |x|\ \forall x \in \mathbb{R}) \\
&< \varepsilon
\end{aligned}$$

Thus $\lim\limits_{x\to c} \cos(x) = \cos(c)$ for all $c \in \mathbb{R}$. Thus $\cos(x)$ is a continuous function.

Problem 3 (4 points each) Prove the following corollaries to the sequential limits lemma (Lemma 3.1.7 in the textbook):

(a) (Continuity of algebraic operations) Let $S \subset \mathbb{R}$ and c be a cluster point of S. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions. Suppose limits of $f(x)$ and $g(x)$ as x goes to c both exist. Prove that

(i) $\lim\limits_{x\to c} (f(x) + g(x)) = \left(\lim\limits_{x\to c} f(x)\right) + \left(\lim\limits_{x\to c} g(x)\right)$

(ii) $\lim\limits_{x\to c} (f(x)g(x)) = \left(\lim\limits_{x\to c} f(x)\right) \left(\lim\limits_{x\to c} g(x)\right)$

(iii) If $\lim\limits_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

(b) (Squeeze lemma) Let $S \subset \mathbb{R}$ and c be a cluster point of S. Let $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ be functions. Suppose

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in S$$

and that the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

Then, the limit of $g(x)$ as x goes to c exists and

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

(a) Let $\{x_n\}$ be any sequence such that $x_n \in S \setminus \{c\}$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} x_n = c$ (at least one exists by Proposition 3.1.2). Then, by the sequential limits lemma, the sequences $\{f(x_n)\}$ and $\{g(x_n)\}$ are convergent, with

$$\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = \lim_{x\to c} g(x)$$

To prove (i) and (ii), apply continuity of algebraic operations on the sequences $\{f(x_n)+g(x_n)\}$ and $\{f(x_n)g(x_n)\}$, which shows that they will be convergent, with

$$\lim_{n\to\infty} (f(x_n) + g(x_n)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right)$$
$$\lim_{n\to\infty} (f(x_n)g(x_n)) = \left(\lim_{x\to c} f(x)\right) \left(\lim_{x\to c} g(x)\right)$$

Since this is true for any sequence $\{x_n\}$ satisfying the hypotheses, by the sequential limits lemma we conclude

$$\lim_{x\to c} (f(x) + g(x)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right)$$
$$\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right) \left(\lim_{x\to c} g(x)\right)$$

We can prove (iii) using the additional assumptions. Since $\lim_{x \to c} g(x) \neq 0$ and $g(x) \neq 0$, then the sequence $\{g(x_n)\}$ as defined above satisfies $g(x_n) \neq 0$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} g(x_n) \neq 0$. Thus, we use continuity of algebraic operations and the sequential limits lemma in the same way to conclude

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

(b) Let $\{x_n\}$ be any sequence such that $x_n \in S \setminus \{c\}$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} x_n = c$. Then, by the sequential limits lemma, the sequences $\{f(x_n)\}$ and $\{h(x_n)\}$ are convergent, with

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} h(x_n) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

Then, we have for the sequence $\{g(x_n)\}$ that $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbb{N}$. Thus, by the squeeze lemma for sequences, we have that $\{g(x_n)\}$ is convergent, with limit

$$\lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} h(x_n)$$

Thus by the sequential limits lemma, $\lim_{x \to c} g(x)$ exists, and is equal to

$$\lim_{x \to c} g(x) = \lim_{n \to \infty} g(x_n) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

Problem 4 (7 points) Two-sided limits are frequently useful. Prove Proposition 3.1.17 in the textbook: Let $S \subset \mathbb{R}$ be a set such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, and let $f : S \to \mathbb{R}$ be a function. Then c is a cluster point of S and

$$\lim_{x\to c} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

First, suppose $\lim_{x\to c} f(x) = L$. Let $\{x_n\}$ be any sequence with $x_n \in S \cap (-\infty, c)$ and $\lim_{n\to\infty} x_n = c$, which exists since c is a cluster point of $S \cap (-\infty, c)$. Then, $\{x_n\}$ is a sequence with $x_n \in S \setminus \{c\}$, so by the sequential limits lemma,

$$\lim_{x\to c^-} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = L$$

The exact same argument holds for sequences $\{x_n\}$ with $x_n \in S \cap (c, \infty)$ and $\lim_{n\to\infty} x_n = c$, so we also conclude

$$\lim_{x\to c^-} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = L$$

To show the other direction, first assume c is a cluster point of $S \cap (-\infty, c)$ (or we could use $S \cap (c, \infty)$, we only need one). Let $\varepsilon > 0$ be given. Then, by the definition of cluster points, there exists some $x \in (S \cap (-\infty, c))$ such that $|x - c| < \varepsilon$. Thus, there exists $x \in S \setminus \{c\}$ such that $|x - c| < \varepsilon$. Thus, the set $(S \setminus \{c\}) \cap (c - \varepsilon, c + \varepsilon)$ is non-empty, and hence c is a cluster point of S.

Now, assume $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$. Let $\varepsilon > 0$ be given. Then, there exists $\delta_- > 0$ such that for all $x \in S \cap (-\infty, c)$ with $|x - c| < \delta_-$, we have

$$|f(x) - L| < \varepsilon$$

Similarly, there exists $\delta_+ > 0$ such that for all $x \in S \cap (c, \infty)$ with $|x - c| < \delta_+$, we have

$$|f(x) - L| < \varepsilon$$

Let $\delta := \min\{\delta_-, \delta_+\}$. Since $S \setminus \{c\} = (S \cap (-\infty, c)) \cup (S \cap (c, \infty))$, we have that for all $x \in S \setminus \{c\}$ with $|x - c| < \delta$,

$$|f(x) - L| < \varepsilon$$

which is the desired inequality showing $\lim_{x\to c} f(x) = L$.

Problem 5 (3 points each) Let $S = \mathbb{R} \setminus \{0\}$

(a) Let $f : S \to \mathbb{R}$ be defined by $f(x) := \cos(1/x)$. Show that $\lim_{x\to 0} f(x)$ does not exist.

(b) Let $f : S \to \mathbb{R}$ be defined by $f(x) := x^2\cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

(c) Find a value $b \in \mathbb{R}$ for which the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2\cos(1/x) & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{cases}$$

is continuous at 0. Is this b unique?

(a) Let $\{x_n\}$ be defined by $x_n := \frac{1}{\pi n}$. This is a constant multiple of the convergent sequence $\{1/n\}$, and hence converges to 0. Then, the sequence $\{f(x_n)\}$ has $f(x_n) = \cos(n\pi) = (-1)^n$, so it is not convergent. Thus, $\lim_{x\to 0} f(x)$ does not exist.

(b) We have that the polynomials $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ given by $f_1(x) := x^2$ and $f_2(x) := -x^2$ are continuous functions, with $\lim\limits_{x\to c} f_1(x) = \lim\limits_{x\to c} f_2(x) = 0$. Then, since for all $x \in S$ we have

$$-x^2 \leq x^2\cos(1/x) \leq x^2$$

by the squeeze lemma we have that $\lim\limits_{x\to c} x^2\cos(1/x) = 0$.

(c) Let $b = 0$. Then, since $\lim\limits_{x\to c} f(x) = f(0)$, f is continuous at 0. This b is unique, since the limit of a function is unique, so no other choice of b will satisfy $\lim\limits_{x\to c} f(x) = b$.

Problem 6 (3 points each) Practice with continuity.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := |x|$. Show that f is continuous at all $c \in \mathbb{R}$.

(b) Suppose $S \subset \mathbb{R}$ and $f, g : S \to \mathbb{R}$ are continuous functions. Show that $h : S \to \mathbb{R}$ defined by $h(x) := \max\{f(x), g(x)\}$ is continuous at all $c \in \mathbb{R}$.

(Hint: Show that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ for $a, b \in \mathbb{R}$, then use facts about composition of continuous functions, and continuity of algebraic operations.)

(a) Let $c \in \mathbb{R}$ and $\varepsilon > 0$ be given. Let $\delta := \varepsilon$. Then, by the reverse triangle inequality, for all $x \in \mathbb{R}$ with $|x - c| < \delta$,

$$||x| - |c|| \leq |x - c| < \delta = \varepsilon$$

Thus $f(x) = |x|$ is continuous at all $x \in \mathbb{R}$.

(b) First, note that if $a \geq b$, then $a - b \geq 0$, so $|a - b| = a - b$. Then,

$$\max\{a, b\} = a = \frac{a + b + a - b}{2} = \frac{a + b + |a - b|}{2}$$

If $a < b$, then $a - b < 0$ so $|a - b| = -(a - b)$. Then,

$$\max\{a, b\} = b = \frac{a + b - (a - b)}{2} = \frac{a + b + |a - b|}{2}$$

Thus, $\max\{a, b\} = \frac{a+b+|a-b|}{2}$.

Since f, g are continuous functions, $f + g$ will also be a continuous function by continuity of algebraic operations. $|f + g|$ is continuous by composition of continuous functions, since $|x|$ is continuous. Then, by continuity of algebraic operations again, we get that $h : S \to \mathbb{R}$ given by

$$h(x) = \max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) + g(x)|}{2}$$

is also a continuous function.