Lecture 12

Tuesday, October 18, 2022 7:04 PM

Today:

- → Finish up alt. characterizations of continuity
- → Facts about continuity
- → Min/max theorem

Prop. (Limit characteritation)
(sequential characterization)

Let $S \subset \mathbb{R}$, $c \in S$, $f: S \to \mathbb{R}$.

(iii) f is continuous at c if and only if, for every sequence $\{x_n\}$ satisfying $x_n \in S$ $\forall n \in \mathbb{N}$ and $x_n \to c$ as $n \to \infty$, the sequence $\{f(x_n)\}$ converges to $f(c)$

"f is cont. at c. $\iff \underbrace{\lim_{n\to\infty} f(x_n)} = \underbrace{f(\lim_{n\to\infty} x_n)}$" $x_n \to c$ as $n \to \infty$

Pf. ($\Rightarrow$) Assume f is cont. at c.

- Let $\{x_n\}$ be any sequence satisfying $x_n \in S$ $\forall n \in \mathbb{N}$ and $x_n \to c$ as $n \to \infty$
- Let $\underbrace{\varepsilon > 0}_{(1)}$ be arbitrary.
  - By continuity of f, $\exists \delta > 0: \forall x \in S \cap \overbrace{(c-\delta, c+\delta)}^{|x-c|<\delta}$, $|f(x) - f(c)| < \varepsilon$
  - Since $x_n \to c$, $\underbrace{\exists M \in \mathbb{N}}_{(2)}: \forall n \geq M$, $|x_n - c| < \delta$

    $\Rightarrow x_n \in S \cap (c-\delta, c+\delta)$ $\forall n \geq M$
  - $\underbrace{\forall n \geq M}_{(3)}$, $|f(x_n) - f(c)| \underbrace{<}_{(4)} \varepsilon$.

  $\Rightarrow \{f(x_n)\}$ converges to $f(c)$ ✓

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($\Leftarrow$) Proof by contrapositive.

WTS: f is discontinuous at c $\Rightarrow$ there exists $\{x_n\}$ satisfying $\underline{x_n \in S\ \forall n \in \mathbb{N} \text{ and } x_n \to c \text{ as } n \to \infty}$ such that $\underline{\{f(x_n)\}}$ does not converge to $\underline{f(c)}$.

- Assume f is discontinuous at c:

$$\Rightarrow \underbrace{\exists \varepsilon > 0}_{(1)} : \underbrace{\forall \delta > 0}_{(2)}, \underbrace{\exists x \in S \cap (c-\delta, c+\delta)}_{(3)} : \underbrace{|f(x) - f(c)| \geq \varepsilon}_{(4)}$$

- For this $\underline{\varepsilon}$, we can construct a sequence by taking

$$x_n \in S \cap (c - \underbrace{1/n}_{\delta = 1/n > 0}, c + 1/n) \quad \forall n \in \mathbb{N}$$

- We have $|x_n - c| < \frac{1}{n}\ \forall n \in \mathbb{N} \Rightarrow x_n \to c$ as $n \to \infty$
- But, we have

$$|f(x_n) - f(c)| \underbrace{\geq}_{\text{cannot be } < \varepsilon' \text{ for all } \varepsilon' > 0} \varepsilon \quad \forall n \in \mathbb{N}$$

$\Rightarrow \{f(x_n)\}$ does not converge to $f(c)$. $\square$

__Cor.__ "Upgrade facts about sequences to continuity"

__Ex.__ __Prop.__ 13.2.5 ($\underline{\text{continuity of alg. op. \#3}}$, for cont.)

Let $\underline{f, g}: S \to \mathbb{R}$ be functions $\underline{\text{continuous at}}$ $c \in S \subset \mathbb{R}$. Then,

(i) $h: S \to \mathbb{R}$ defined by $\underline{h(x) := f(x) + g(x)}$ is continuous at c

(ii, iii, iv) $-, \times, \div$

__Pf.__ (i) Take f, g, c as given. Take $h = f + g$. Let $\{x_n\}$ be any sequence satisfying $\underline{x_n \in S\ \forall n \in \mathbb{N} \text{ and } x_n \to c \text{ as } n \to \infty}$, any sequence satisfying $\underline{x_n \in S \ \forall n \in \mathbb{N} \text{ and } x_n \to c \text{ as } n \to \infty}$, (*)

Then,

$$h(c) = f(c) + g(c) = f(\overbrace{\lim_{n\to\infty} x_n}^{x_n \to c}) + g(\lim_{n\to\infty} x_n)$$

$$= \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) \quad \left(\begin{array}{l}\text{seq. characterization of}\\ \text{continuity}, \Rightarrow\end{array}\right)$$

$$= \lim_{n\to\infty} \big(f(x_n) + g(x_n)\big) \quad \text{(cont. of alg. op. for seq.)}$$

$$= \lim_{n\to\infty} h(x_n) \quad \text{(by def. of } h\text{)}$$

$\Rightarrow \lim_{n\to\infty} h(x_n) = h(c) \ \forall \{x_n\}$ satisfying (*)

so, by seq-char. of continuity, h is continuous at c ✓

Ex $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ is continuous

can show first $\boxed{f(x) := x}$ $f: \mathbb{R} \to \mathbb{R}$ is continuous

($\underset{= x_n \to c}{f(x_n)} \to c$ for all sequences $x_n \to c$ as $n \to \infty$)

can then take $p(x) = \underbrace{a_d f(x)^d} + \cdots + \underbrace{a_1 f(x)} + \underbrace{a_0}$

Prop. (compositions preserve continuity)

Let $A, B \subset \mathbb{R}$ and $f: \underline{B} \to \mathbb{R}$, $g: \underline{A \to B}$. If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$, then the composition $f \circ g: A \to \mathbb{R}$ $((f \circ g)(x) = \underline{f(g(\underline{x}))})$ is continuous at c.

Pf. Let f, g be as given.

- Let $\underline{\{x_n\}}$ be a sequence s.t. $x_n \in \underline{A \ \forall n \in \mathbb{N} \text{ and } x_n \to c \text{ as } n \to \infty}$
- Since g is cont. at c, $\underline{\lim_{n\to\infty} g(x_n)} = \underline{g(c)}$

- Since g is cont. at c, $\lim_{n\to\infty} g(x_n) = g(c)$
- Since f is cont. at $g(c)$,

$$(f\circ g)(c) = f(g(c)) = f(\lim_{n\to\infty} g(x_n)) = \lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} (f\circ g)(x_n)$$

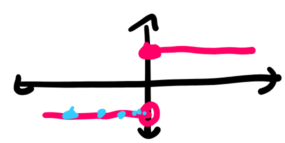
$\Rightarrow$ $f\circ g$ is continuous at c.

Discontinuous functions

Prop. (negation of sequential char. of cont.)
Let $S\subset\mathbb{R}$, $c\in S$, $f: S\to\mathbb{R}$. If there exists $\{x_n\}$ with $x_n\in S$ $\forall n\in\mathbb{N}$ and $x_n\to c$ as $n\to\infty$ s.t. $\{f(x_n)\}$ does not converge to $f(c)$, then f is not continuous at c.

Pf Follows directly from earlier prop.

Ex. $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) := \begin{cases} -1 & x<0 \\ 1 & x\geq 0 \end{cases}$



Claim. f is discont. at 0.

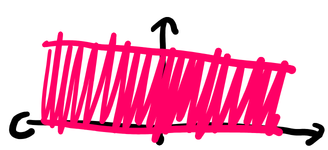
Pf Take $\{-\frac{1}{n}\}$. $-\frac{1}{n}\in\mathbb{R}$ $\forall n\in\mathbb{N}$ and $-\frac{1}{n}\to 0$ as $n\to\infty$

But $f(-\frac{1}{n})\to -1 \neq f(0) = 1$ as $n\to\infty$.

$\Rightarrow$ f is not cont. at 0.

Ex (Dirichlet Function)

$f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}$



Claim f is discontinuous for all $c\in\mathbb{R}$.

Pf (case where $c\notin\mathbb{Q}$): For any $c\in\mathbb{Q}$, we can find a sequence $\{r_n\}$ with $r_n\in\mathbb{Q}$ $\forall n\in\mathbb{N}$ and $r_n\to c$ as $n\to\infty$

sequence $\{r_n\}$ with $r_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $r_n \to c$ as $n \to \infty$

But $\lim_{n\to\infty} f(r_n) = 1 \neq f(c) = 0$ ✓

(case when $c \notin \mathbb{Q}$) Claim: $\exists \{x_n\}$ s.t. $x_n \notin \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $x_n \to c$ as $n \to \infty$.

(For fun)

Consequences of Continuity

Lemma. A continuous function $f : \underbrace{[a,b]}_{\substack{\text{closed + bdd.} \\ a<b}} \to \mathbb{R}$ is bounded.

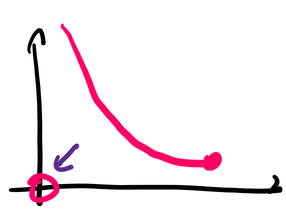
Pf. By contrapositive, WTS: If $f:[a,b] \to \mathbb{R}$ is unbounded, then it is discontinuous for some $c \in [a,b]$.

(recall: f is continuous $\iff$ $\forall c \in [a,b]$, f is continuous at c)

- Suppose f is not bounded. Then, $\forall n \in \mathbb{N}$, $\exists x_n \in [a,b]$: $|f(x_n)| \geq n$ (otherwise n would be a bound for f)
- Since $a \leq x_n \leq b$, so $\{x_n\}$ is bounded.
  - By Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.
  - Since $a \leq x_{n_k} \leq b$ $\forall k \in \mathbb{N}$ $\Rightarrow$ $a \leq \lim_{k\to\infty} x_{n_k} \leq b$
  - $c := \lim_{k\to\infty} x_{n_k} \in [a,b]$
  - We have $\{f(x_{n_k})\}$ is unbounded since $f(x_{n_k}) \geq n_k \geq k$ (subseq.) $\forall k \in \mathbb{N}$. Thus, $\{f(x_{n_k})\}$ is divergent.
  - $\{f(x_{n_k})\}$ does not converge to $f(c)$, so f is not cont. at c $\square$

Ex. $f : [0,1] \to \mathbb{R}$

$$f(x) := \begin{cases} 0 & x = 0 \\ 1/x & x > 0 \end{cases}$$



$\} \; 1/x \quad x>0$

$f(\frac{1}{n})=n \Rightarrow f$ is unbounded.

$f(\lim_{n\to\infty}\frac{1}{n}) = f(0)=0 \neq \cancel{\lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} n}$

$\Rightarrow f$ is discontinuous at 0.

Remarks: Why closed and bounded $[a,b]$?

Bounded: $\boxed{B\text{-}W}$ construct any $x_n\in[a,b]$
extract a convergent subsequence $x_{n_k}\in[a,b]$

Closed: $\lim_{k\to\infty} x_{n_k}\in[a,b]$.

$x_{n_k}\in(a,b) \Rightarrow a<x_{n_k}<b \Rightarrow a\le \lim_{n\to\infty} x_{n_k}\le b \not\Rightarrow \lim x_{n_k}\in(a,b)$

Ex. $f:(0,1)\to\mathbb{R}$

$f(x):=1/x$

Claim: f is continuous and unbounded.

$f(\frac{1}{n})=n \qquad \{f(\frac{1}{n})\}$ is unbounded

$\frac{1}{n}\to 0$ as $n\to\infty \qquad 0\notin(0,1)$

Next time: min/max theorem
Bolzano's intermediate value theorem.