Homework 10

Due: Monday, December 12th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 7.1-7.2

The Exponential Function

Given a positive real number $a > 0$, it is straightforward to define integer exponents a^n in terms of basic arithmetic operations. We saw how we can also define n-th roots using the tools we've learned in the course (see HW6). However, how exactly do irrational exponents work, if we can't define them in terms of basic arithmetic operations?

In the following problems, we will see how Picard's theorem allows us to define the exponential function, and hence define irrational exponents a^x for positive $a > 0$ and $x \in \mathbb{R}$.

Remark: If you're curious, you can see section 5.4 for how to define irrational exponents starting from integrals and the logarithm instead. Other ways include starting the power series definition of the exponential, or via continuous extension of rational exponents. It is noteworthy that all of these different methods produce the same definition for irrational exponents!

Problem 1 (3 points each) Given any $x_0, y_0 \in \mathbb{R}$, consider the equation and initial conditions

$$f'(x) = f(x) \qquad f(x_0) = y_0$$

(a) Given any positive $h < \frac{1}{2}$, show that we can pick $\alpha > 0$ large enough that the proof of Picard's theorem guarantees a solution for f in the interval $[x_0 - h, x_0 + h]$.

(*Hint*: Read through the statement of Theorem 6.3.2 carefully. Note that Picard's theorem guarantees the existence of at least one $h > 0$ which makes the conclusion of the theorem true. This question is asking you to show that you can "upgrade" Picard's theorem to explicitly show that all $h < 1/2$ makes the conclusion of the theorem hold for this particular ODE.

To do this, you'll need to go through the proof of Picard's theorem and show you can explicitly give values for the "picked" variables (such as M and α) in terms of h, rather than defining h in terms of the picked variables.)

(b) Show that (a) can be used to iteratively extend f to a unique function on all $x \in \mathbb{R}$.

(c) Given $\alpha \in \mathbb{R}$, show the unique solution to the equation $g'(x) = g(x)$ with initial conditions $g(x_0) = \alpha y_0$ is given by $g(x) = \alpha f(x)$ for all $x \in \mathbb{R}$.

(d) Show that if there exists some $c \in \mathbb{R}$ such that $f(c) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$. Conclude that if $y_0 > 0$, then $f(x) > 0$ and f is strictly increasing for all $x \in \mathbb{R}$.

(*Hint*: Recall that $f(x) = f'(x)$. Is it possible for there to exist $a, b \in \mathbb{R}$ such that $f'(a) > 0 > f'(b)$ but $f'(x) \neq 0$ for all $x \in \mathbb{R}$?)

Problem 2 (3 points each) Now, we will focus on

$$E'(x) = E(x) \qquad E(0) = 1$$

The solution typically is denoted by $E(x) = e^x$, and is known as the exponential function, and has a unique inverse function $L(x) = \ln(x)$ known as the natural logarithm. However, make sure not to use any properties of either in proving the following.

Recall the following conventions for integer powers and roots: for a positive real number $a > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} a^0 &:= 1 \\ a^n &:= a \cdot a \cdot \ldots \cdot a \qquad (n \text{ times}) \\ a^{-n} &:= \frac{1}{a^n} \end{aligned}$$

On HW6 problem 6 you also showed the existence and uniqueness of n-th roots $a^{1/n}$ for $n \in \mathbb{N}$, which solve the equation $\left(a^{1/n}\right)^n = a$. Given $m \in \mathbb{Z}$, we define rational powers for $q \in \mathbb{Q}$ with $q = m/n$ as

$$a^q := \left(a^{1/n}\right)^m$$

You will see in the course of this problem that m/n does not need to be a fraction in lowest terms.

(a) Given $b \in \mathbb{R}$, define $E_b : \mathbb{R} \to \mathbb{R}$ by $E_b(x) := E(x + b)$. Show that E_b is the unique solution to $E_b'(x) = E_b(x)$ with $E_b(0) = E(b)$.

Use this with your result in 1(c) to show that given $a, b \in \mathbb{R}$, $E(a + b) = E(a)E(b)$.

(*Remark*: This shows the exponential function converts addition into multiplication!)

(b) Given any $x \in \mathbb{R}$, show that $E(mx) = E(x)^m$ for any $m \in \mathbb{Z}$, and $E(x/n) = E(x)^{1/n}$ for any $n \in \mathbb{N}$. Conclude that $E(x)^q = E(qx)$ for any $q \in \mathbb{Q}$ with $q = m/n$.

(c) Show that $\lim_{n\to\infty} E(-n) = 0$, and that the sequence of real numbers $\{E(n)\}$ is an unbounded monotone increasing sequence.

Use this to conclude that $E : \mathbb{R} \to (0, \infty)$ is bijective, and hence has a unique inverse function $L : (0, \infty) \to \mathbb{R}$ satisfying $E(L(a)) = a$ for all $a \in (0, \infty)$.

(Hint: Refer back to HW6. Don't forget the result of 1(d)!)

(d) Use the fact that E is continuous to show that $a^x := E(xL(a))$ is the unique number satisfying $a^x = \lim_{n\to\infty} a^{q_n}$ for any sequence $\{q_n\}$ of rational numbers with $\lim_{n\to\infty} q_n = x$. Hence, this a natural way to define irrational exponents!