Midterm 1

1. (a) $\forall b_1 \in B_1$, $b_1 \in B_1 \cup B_2 \Rightarrow B_1 \subset B_1 \cup B_2$

$\forall b_2 \in B_2$, $b_2 \in B_1 \cup B_2 \Rightarrow B_2 \subset B_1 \cup B_2$

so for all $B \in \{B_1, B_2\}$, $B \subset B_1 \cup B_2$

$\Rightarrow$ by def, $B_1 \cup B_2$ is an upper bound of $\{B_1, B_2\}$

(b) If C is an upper bound of $\{B_1, B_2\}$,

$B_1 \in C$, $B_2 \subset C$

$\forall x \in B_1 \cup B_2 \Rightarrow x \in B_1$ or $x \in B_2$

$\Rightarrow x \in C$ in both cases

$\Rightarrow B_1 \cup B_2 \subset C$

so for all upper bounds C, $B_1 \cup B_2 \subset C$

From (a) we know that $B_1 \cup B_2$ is an upper bound.

$\Rightarrow$ by def., $B_1 \cup B_2$ is a least upper bound

If S is another least upper bound,

$B_1 \cup B_2 \subset S$

by def of least upper bound S, $S \subset B_1 \cup B_2$

$\Rightarrow S = B_1 \cup B_2$

(c) Let $C \in P(X)$ and let $A \subset P(X)$. We say C is a lower bound of A if $C \subset B$ for all $B \subset A$.

$\forall b \in B_1 \cap B_2$, $b \in B_1 \Rightarrow B_1 \cap B_2 \subset B_1$

$\forall b \in B_1 \cap B_2$, $b \in B_2 \Rightarrow B_1 \cap B_2 \subset B_2$

so for all $B \in \{B_1, B_2\}$, $B_1 \cap B_2 \subset B$

$\Rightarrow$ by def, $B_1 \cap B_2$ is a lower bound of $\{B_1, B_2\}$

(d) $S = \bigcup_{B \in A} B$

For all $B' \in A$, $\forall x \in B' \Rightarrow x \in B$ for some $B \in A$

$\Rightarrow x \in S$

$\Rightarrow S$ is an upper bound

For all upper bounds C' of A,

$\forall x \in S \Rightarrow x \in B$ for some $B \in A$

C' is an upper bound $\Rightarrow B \subset C' \Rightarrow x \in C'$

$\Rightarrow S \subset C$

By def, S is a least upper bound.

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2. (a) If $\sup S \in S$, let $x_n = \sup S$ for all n, then $\lim_{n\to\infty} x_n = \sup S$

If $\sup S \notin S$, from homework we have

$\exists x_1 \in S$ s.t. $\sup S - 1 < x_1 \le \sup S$

$\vdots$

$\exists x_n \in S$ s.t. $\sup S - \frac{1}{n} < x_n \le \sup S$

Since $\lim_{n\to\infty} (\sup S - \frac{1}{n}) = \sup S$, $\lim_{n\to\infty} \sup S = \sup S$,

by Squeeze Lemma, $\lim_{n\to\infty} x_n = \sup S$

(b) By definition of suprenum, and since $\sup S \notin S$, $y_1 \in S$

take some $y_1 \in S$, then $\sup S - y_1 > 0$

From HW1 Q8 we have

$\exists y_2 \in S$ s.t. $\sup S - (\sup S - y_1) = y_1 < y_2 \le \sup S$

since $\sup S \notin S \Rightarrow \sup S - y_2 > 0$

Suppose we have $y_k \in S$ s.t. $\sup S - y_k > 0$

$\exists y_{k+1} \in S$ s.t. $\sup S - (\sup S - y_k) = y_k < y_{k+1} \le \sup S \Rightarrow$ strictly monotone increasing

since $\sup S \notin S \Rightarrow \sup S - y_{k+1} > 0$

So $\sup S - y_p > 0$, $y_p < y_{p+1}$, $y_p \in S$ for all $p \in \mathbb{N}$

Such a sequence exists.

(c) Consider $\{y_n : n \in \mathbb{N}\}$ defined for $\{y_n\}$ in (b).

If $y_a = y_b$, since $\{y_n\}$ is strictly monotone increasing,

if $a<b$, $y_a < y_b$; if $a>b$, $y_a > y_b$

$\Rightarrow a = b \Rightarrow$ injective

$\forall y_a \in \{y_n\}$, there exists $n = a \in \mathbb{N}$ s.t. $y_n = y_a \Rightarrow$ surjective

So $f: \mathbb{N} \to \{y_n : n \in \mathbb{N}\}$, $f(n) = y_n$ is bijective

$|\{y_n : n \in \mathbb{N}\}| = |\mathbb{N}|$

Since $y_n \in S$ for all $n \in \mathbb{N}$, $\{y_n : n \in \mathbb{N}\}$ is a countably infinite subset of S

(d) No.

Counterexample: $S = \{0\}$

$\sup S = 0 \in S$

all subsets $E = \emptyset$ or $\{0\}$, both are finite

3. (a) A sequence $\{x_n\}$ diverges to negative infinity if for all $K \in \mathbb{R}$, there exists some $M \in \mathbb{N}$ s.t. for all $n \geq M$, $x_n < K$.

For $\{x_n\} = \{-n^3\}$,

$\forall K \in \mathbb{R}$, let M to be the least integer that is greater than $\sqrt[3]{|K|}$, so $\forall n \geq M$

if $K \geq 0$, $x_n = -n^3 < 0 < K$

if $K < 0$, $x_n = -n^3 < -M^3 < -|K| = K$

By def., $\lim_{n \to \infty} -n^3 = -\infty$

(b) By def. of lim, $\forall \varepsilon$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$,

$\left|\frac{1}{x_n} - 0\right| < \varepsilon$

since $x_n > 0 \Rightarrow \frac{1}{x_n} < \varepsilon \Rightarrow x_n > \frac{1}{\varepsilon}$

$\forall K \in \mathbb{R}$, let $\varepsilon = \frac{1}{K} \Rightarrow x_n > K \Rightarrow$ by def., $\{x_n\}$ diverges to $+\infty$

(c) "$\Leftarrow$"

If $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which diverges to $+\infty$, assume $\{x_n\}$ has an upper bound b s.t. $x_n \le b$ for all n

For $b \in \mathbb{R}$, there exists $M \in \mathbb{N}$ s.t. $\forall k \ge M$,

$x_{n_k} > b \Rightarrow x_n > b$ for $n = n_k$

$\Rightarrow$ contradicts with $x_n \le b$

So $\{x_n\}$ is unbounded above.

"$\Rightarrow$"

If $\{x_n\}$ is unbounded above,

there doesn't exist $b \in \mathbb{R}$ s.t. $x_n \le b$ for all $n \in \mathbb{N}$

$\Rightarrow \forall b$, $x_n > b$ for some $n \in \mathbb{N}$

For any p-tail $\{x_{n+p}\}_{n=1}^{\infty}$, if it has an upper bound b',

then $x_{n+p} \le b'$ $\forall n \in \mathbb{N}$

then $b = \max\{x_1, x_2, \ldots x_p, b'\} \ge x_n$ $\forall n \in \mathbb{N}$

$\Rightarrow b$ is an upper bound of $\{x_n\}$ $\Rightarrow$ contradiction

So every p-tail of $\{x_n\}$ is also bounded.

We then construct $\{x_{n_k}\}$

For all $K \in \mathbb{R}$,

since $\{x_n\}$ is unbounded, $x_{n_1} > K$ for some $n \in \mathbb{N}$

Inductively, suppose we have $x_{n_1}, \cdots x_{n_k} > K$,

since $\{x_{n+n_k}\}$ is unbounded, $x_{n_{k+1}} > K$ for some $n_{k+1} > n_k$

So $\exists M = 1 \in \mathbb{N}$ s.t. for all $k \ge M$, $x_{n_k} > K$

then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which diverges to $+\infty$.

4. (a) $\{x_n\} = \{(-1)^n\}$

If n is even, $\overline{x}_n = \frac{-1+1\cdots-1+1}{n} = 0$

If n is odd, $\overline{x}_n = \frac{(-1+1-\cdots-1+1)-1}{n} = -\frac{1}{n}$

$\forall \varepsilon > 0,\ \exists M \in \mathbb{N}$ s.t. $\varepsilon M > 1 \Rightarrow M > \frac{1}{\varepsilon}$

$\forall n \geq M,\ |\overline{x}_n - 0| = -\overline{x}_n \leq \frac{1}{n} \leq \frac{1}{M} < \varepsilon$

so $\{\overline{x}_n\}$ converges to 0

(b) $|\overline{x}_n - L| = \left|\sum_{k=1}^{n} \frac{x_k - L}{n}\right| = \frac{1}{n}|x_1 + x_2 + \cdots + x_n - nL|$

$\leq \frac{1}{n}(|x_1 - L| + \cdots + |x_m - L| + |x_{m+1} - L| + \cdots + |x_n - L|)$

$< \frac{1}{n}(|x_1 - L| + \cdots + |x_m - L|) + \frac{1}{n} \times \frac{\varepsilon}{2} \times (n - M)$

Let $k = \frac{|x_1 - L| + \cdots + |x_m - L|}{M}$

$\Rightarrow |\overline{x}_n - L| < \frac{Mk}{n} + \frac{(n-M)\varepsilon}{2n}$

(c) Let $\varepsilon > 0$ be arbitrary.

$\frac{Mk}{n} + \frac{(n-M)\varepsilon}{2n} < \varepsilon \Leftrightarrow \frac{Mk}{n} < \frac{(n+M)}{2n}\varepsilon \Leftrightarrow 2Mk < (n+M)\varepsilon$

$\Leftrightarrow n\varepsilon > 2Mk - M\varepsilon \Leftrightarrow n > \frac{2Mk}{\varepsilon} - M$

Let M' be the smallest integer that is greater than $\frac{2Mk}{\varepsilon} - M$

$\Rightarrow |\overline{x}_n - L| < \varepsilon \Rightarrow \lim_{n\to\infty} \overline{x}_n = L$