Mean Value Theorem

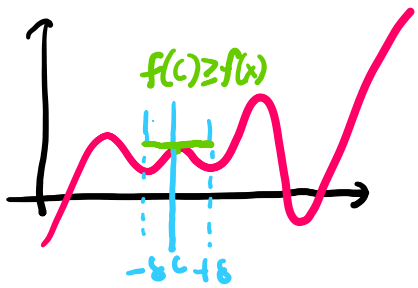
Monday, October 31, 2022 10:27 AM

Derivative as "Rate of Change"

<u>Def.</u> Let $S \subset \mathbb{R}$, $f: S \to \mathbb{R}$. We say f has a <u>relative (local) maximum</u> at $c \in S$ if there exists $\delta > 0$ such that for all $x \in S$ with $|x-c| < \delta$, we have $f(x) \leq f(c)$

- <u>Relative min.</u> defined similarly.

<u>Idea:</u>



<u>Lemma:</u> (rel min/max $f(c) \Rightarrow$ critical pt. $f'(c) = 0$)

Suppose $f: [a,b] \to \mathbb{R}$ is differentiable at $c \in (a,b)$, and f has a relative min/max at c. Then, $f'(c) = 0$

<u>Pf.</u> Consider case of rel. max. Rel. min. proved by considering $-f$.

- Let c be a relative max of f.

  $\exists \delta > 0 : \forall x \in [a,b]$ with $|x-c| < \delta$, $f(x) - f(c) \leq 0$

- Consider difference quotient:

  case $x \in (c, c+\delta)$: $x - c > 0$, so

  $$\frac{f(x)-f(c)}{x-c} \leq 0$$

  case $y \in (c-\delta, c)$: $y - c < 0$, so

  $$\frac{f(y)-f(c)}{y-c} \geq 0$$

$$\frac{f(y)-f(c)}{y-c} \geq 0$$

- Take sequences $\{x_n\}, \{y_n\}$ satisfying $x_n \in (c, c+\delta)$, $y_n \in (c-\delta, c)$ $\forall n \in \mathbb{N}$
  $x_n \to c$, $y_n \to c$ as $n \to \infty$

Then,

$$\underbrace{f'(c) = \lim_{n\to\infty}}_{\text{seq. lim. lemma}} \frac{f(x_n)-f(c)}{x_n - c} \underbrace{\leq}_{\text{limits preserve} \leq} 0$$

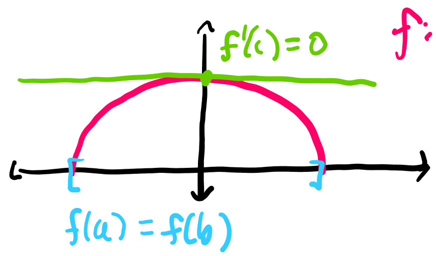
$$f'(c) = \lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n - c} \geq 0$$

$\Rightarrow$ $f'(c) = 0$ as desired. $\square$

Thrm. (Rolle's thrm.)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function, differentiable on (a,b), satisfying $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.

Idea:



$f: [-1,1] \to \mathbb{R}$ $\quad f(x) := \sqrt{1-x^2}$
semicircle func.

$f'(x) = \frac{-x}{\sqrt{1-x^2}}$ $\quad f': (-1,1) \to \mathbb{R}$

cts. on $[-1,1]$, diff. on $(-1,1)$

Pf. f is continuous on a closed + bdd. interval, so by min/max thrm.
f achieves an abs max + abs min on $[a,b]$.

- Proof by cases: Let $K := f(a) = f(b)$.

  (i) $\exists x \in (a,b) ; f(x) > K$: then abs. max $f(c) > K \Rightarrow c \in (a,b)$.
  Since abs. max. is also a rel. max, $f'(c) = 0$

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(ii) $\exists x \in (a,b) : f(x) > K$: then abs min $f(c) < K \Rightarrow c \in (a,b)$, $f'(c) = 0$

(iii) $\forall x \in (a,b)$, $f(x) = K$: then for any $c \in (a,b)$, $f'(c) = \lim_{x \to c} \frac{K-K}{x-c} = 0$ $\square$

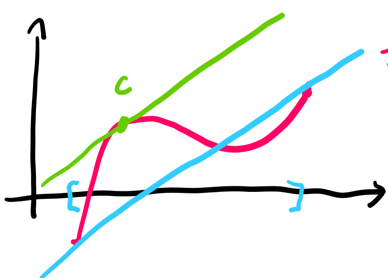
Thm. (Mean Value thm., MVT)

Let $f : [a,b] \to \mathbb{R}$ be a continuous function differentiable on (a,b).

Then, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Idea: Geometric:



$f'(c) = \frac{f(b)-f(a)}{b-a}$

Analytic: $f|_{[x_0,x]}$

"Intuition" $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

MVT: $\exists c \in (x, x_0)$: $f'(c) = \frac{f(x) - f(x_0)}{x - x_0}$

$$\Rightarrow \underset{=}{f(x)} = f(x_0) + f'(c) \cdot (x - x_0)$$ (equality!)

Pf. (Proof by calculation)

- Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) := f(x) - \underbrace{\left[f(b) + \frac{f(b) - f(a)}{b-a} \cdot (x - b) \right]}_{\text{secant line passing}}$$

$\underbrace{\qquad\qquad\qquad}$
secant line passing
through $(a,f(a))$, $(b,f(b))$

- g is continuous on $[a,b]$ (by cont. of alg. op.), and differentiable on (a,b) (by linearity of deriv.)

Furthermore, $g(a)=g(b)=0$.

$\Rightarrow$ By Rolle's thrm., $\exists\, c\in(a,b) : g'(c)=0$

Then,

$$g'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}=0 \Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a} \qquad \square$$

Applications of MVT

Prop. ("solving our first diff. eq.")

Let I be an interval and $f: I\to\mathbb{R}$ be a differentiable function satisfying $f'(x)=0\ \forall x\in I$. Then, f is constant.

Pf: Take arbitrary $x,y\in I$ with $x<y$. Then, $f|_{[x,y]}$ is continuous and differentiable, so it satisfies hypotheses of MVT.

- Therefore, $\exists\, c\in(x,y)$ such that

$$f(y)-f(x)=f'(c)\cdot(y-x)=0 \Rightarrow f(x)=f(y)$$

- This shows $f(x)=f(y)\ \forall x,y\in I$, so f is constant.

Prop. (sign of derivative vs. inc/dec)

Let I be an interval, $f: I\to\mathbb{R}$ be differentiable.

(i) f is increasing iff $f'(x)\geq 0\ \forall x\in I$

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(f is increasing: $x > y \Rightarrow f(x) \geq f(y)$)

(ii) If $f'(x) > 0 \quad \forall x \in I$, then f is strictly increasing

(f is strictly increasing: $x > y \Rightarrow f(x) > f(y)$)

Pf. (i) Suppose f is increasing. $\forall x, c \in I$ with $x \neq c$, $x > c \Rightarrow f(x) \geq f(c)$, $x < c \Rightarrow f(x) \leq f(c)$

$\Rightarrow \forall x, c \in I$ with $x \neq c$, $\dfrac{f(x) - f(c)}{x - c} \geq 0 \Rightarrow f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \geq 0$ ✓

- Suppose $f'(x) \geq 0 \quad \forall x \in I$. Take $x, y \in I$ with $x < y$. By MVT, $\exists c \in (a, b)$ such that

$$f(y) - f(x) = \underbrace{f'(c)}_{\geq 0} \cdot \underbrace{(y - x)}_{> 0} \geq 0 \Rightarrow f(y) \geq f(x)$$

Thus, f is increasing. ✓

Pf. of (ii) is similar. ☑