Homework 4

Due: Monday, October 17th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 2.3-2.5, 3.1-3.2

Sections 2.3-2.5 Exercises

Problem 1 (4 points each) For each of the following sequences $\{x_n\}$, find the lim sup and lim inf, and use them to determine if the original sequence $\{x_n\}$ converges or diverges.

(a) $\{(-c)^n\}$ for $0 < c < 1$

(b) $\left\{(-1)^n + \frac{1}{n}\right\}$

Problem 2 (4 points) Prove that $\{1/n^2\}$ is Cauchy using directly the definition of a Cauchy sequence.

(Hint: How would you prove $\{1/n^2\}$ converges using the original limit definition? Then, try looking at example 2.4.2 in the textbook.)

Problem 3 (4 points each) Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be bounded sequences.

(a) Show that there exists a subsequence $\{x_{n_i} + y_{n_i}\}_{i=1}^\infty$ of the sequence $\{x_n + y_n\}_{n=1}^\infty$ which converges to $\liminf_{n\to\infty} (x_n + y_n)$

(b) Show that

$$\liminf_{n\to\infty} (x_n + y_n) \geq \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$$

(Hint: The subsequence $\{x_{n_i}\}_{i=1}^\infty$ from (a) does not necessarily converge, but try considering a convergent subsequence of $\{x_{n_i}\}_{i=1}^\infty$, then using what you know about adding/subtracting limits.)

(c) Give a explicit example of bounded sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\liminf_{n\to\infty} (x_n + y_n) > \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$$

(Hint: The left-hand side and right-hand side are equal if $\{x_n\}$ and $\{y_n\}$ are convergent, so consider non-convergent sequences)

Problem 4 (4 points each) This question looks at some of the assumptions of the Bolzano-Weierstrass theorem.

(a) The Bolzano-Weierstrass theorem says that any *bounded* sequence has a convergent subsequence.

Find an example of an unbounded sequence $\{x_n\}$ which has the property that every subsequence diverges.

(b) The proof of the Bolzano-Weierstrass theorem relies on the lowest upper bound property of $\mathbb{R}$. We proved in class that $\mathbb{Q}$ does not have the lowest upper bound property, hence we cannot prove a "rational" version of Bolzano-Weierstrass for rational sequences.

Show that there exists a bounded sequence of rational numbers $\{r_n\}$ (that is, $r_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$) such that no subsequence of $\{r_n\}$ converges to a rational number.

(Hint: Use the results of HW2 Problem 8)

Problem 5 (4 points each) This question looks at the relationship between Bolzano-Weierstrass and the "Cauchy completeness" property of $\mathbb{R}$.

(a) Directly use the Bolzano-Weierstrass theorem (Theorem 2.3.8) to prove that every Cauchy sequence of real numbers is convergent. That is, only make use of the fact that every bounded sequence of real numbers has some convergent subsequence (not necessarily converging to either lim sup/lim inf).

(*Remark*: Proving Cauchy-completeness from Bolzano-Weierstrass without lim sup/inf in this way generalizes more easily to $\mathbb{R}^n$.)

(b) One interesting thing about the Cauchy sequence definition is that it can be stated without reference to real numbers at all:

We say a sequence of rational numbers $\{r_n\}$ is Cauchy (in the absolute value metric on the rational numbers) if for every $\varepsilon \in \mathbb{Q}$ satisfying $\varepsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $|r_n - r_k| < \varepsilon$.

Prove that $\mathbb{Q}$ is not Cauchy complete, that is, show that there exists a Cauchy sequence $\{r_n\}$ which does not converge to some limit $r \in \mathbb{Q}$.

(*Hint*: Look at problem 4b)

Remark: One way to 'construct' the real numbers (i.e. prove theorem 1.2.1) is to define elements of $\mathbb{R}$ via an equivalence relation on Cauchy sequences of rational numbers.

Problem 6 (4 points) Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Prove that

$$\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$$

Problem 7 (6 points) Let $\{d_n\}$ be a sequence of digits, that is $d_n \in \{0,1,2,3,4,5,6,7,8,9\}$ for all $n \in \mathbb{N}$. Let us consider the decimal

$$0.d_1 d_2 d_3 ... := \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Show that the series $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ is absolutely convergent to a number $x \in [0,1]$.

(*Hint*: Use the comparison test.)