Sequences, Limits, and Continuity

Monday, October 17, 2022 10:43 AM

Sequences and Limits

Lemma. (sequential limits lemma)

Let $S \subset \mathbb{R}$ and c be a cluster pt. of S. Let $f: S \to \mathbb{R}$.

Then, $f(x) \to L$ as $x \to c$ if and only if, for every sequence $\{x_n\}$ satisfying $x_n \in S \setminus \{c\}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L

Idea: $\lim_{x\to c} f(x) = L \iff \forall \{x_n\}$ s.t. $\begin{matrix} x_n \in S\setminus\{c\} \\ x_n \to c \end{matrix}$, $\lim_{n\to\infty} f(x_n) = L$

Pf. ($\Rightarrow$) Suppose $f(x) \to L$ as $x \to c$. Let $\{x_n\}$ be a sequence satisfying $x_n \in S\setminus\{c\}$ $\forall n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = c$

- Let $\underbrace{\varepsilon > 0}_{(1)}$ be given. $\exists \delta > 0 : \forall \underbrace{x \in (c-\delta, c+\delta)}\cap S\setminus\{c\}$, $|f(x) - L| < \varepsilon$
  - $\underbrace{\exists M \in \mathbb{N}}_{(2)} : \forall n \geq M$, $\underbrace{|x_n - c| < \delta}$.
  - Thus, $\underbrace{\text{for all } n \geq M}_{(3)}$ since $x_n \in (c-\delta, c+\delta) \cap S\setminus\{c\}$,

    $|f(x_n) - L| \underbrace{< \varepsilon}_{(4)}$

($\Leftarrow$) Prove by contrapositive. Want to show:

$$f(x) \not\to L \text{ as } x \to c \Rightarrow \exists \{x_n\} \text{ s.t. } \overset{(*)}{\boxed{\begin{matrix} x_n \in S\setminus\{c\} \\ x_n \to c \end{matrix}}},\ f(x_n) \not\to L \text{ as } n \to \infty$$

$f(x) \to L$ as $x \to c$: $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in (c-\delta, c+\delta) \cap S\setminus\{c\}$, $|f(x_n) - L| < \varepsilon$

$f(x) \not\to L$ as $x \to c$: $\exists \varepsilon > 0 : \forall \delta > 0, \exists x \in (c-\delta, c+\delta) \cap S\setminus\{c\} : |f(x_n) - L| \geq \varepsilon$

- Assume $f(x) \not\to L$ as $x \to c$. Then, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n \in (c - \underbrace{1/n}_{=\delta>0}, c + 1/n) \cap S\setminus\{c\}$ s.t.

all $n\in\mathbb{N}$, there exists $x_n\in(c-\underbrace{1/n}_{=\delta>0},c+1/n)\cap S\setminus\{c\}$ s.t.

$$|f(x_n)-L|\geq\varepsilon$$

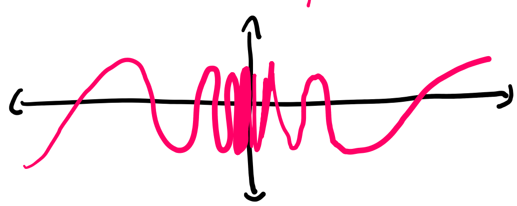
- By construction, $x_n\in S\setminus\{c\}$ $\forall n\in\mathbb{N}$, and $|x_n-c|<1/n$ $\forall n\in\mathbb{N}$ so $\lim_{n\to\infty}x_n=c$. So $\{x_n\}$ satisfies $(*)$
- However, $|f(x_n)-L|\geq\varepsilon>0$ $\forall n\in\mathbb{N}$. Thus, $\{f(x_n)\}$ does not converge to L. $\square$

__Remark:__ why consider $S\setminus\{c\}$ for cluster pts. and limits?

sequence limits: Can't plug in $n=\infty$
function limits: Can't plug in $x=c$

Ex. $\frac{f(x+h)-f(x)}{h}$ $h\neq 0$!

__Ex.__ $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x):=\sin(1/x)$



__Claim__ $f(x)$ diverges as $x\to 0$

__Pf.__ Let $x_n:=\frac{1}{\pi n+\pi/2}$. Then, $x_n\in\mathbb{R}\setminus\{0\}$ $\forall n\in\mathbb{N}$, $\lim_{n\to\infty}x_n=0$

$$f(x_n)=\sin(1/x_n)=\sin(n\pi+\pi/2)=(-1)^n$$

- So $\{f(x_n)\}$ does not converge to any $L\in\mathbb{R}$
- Thus, $f(x)$ cannot converge to any $L\in\mathbb{R}$ as $x\to 0$ by the sequential limits lemma.

__Remark:__ If we had chosen $x_n:=\frac{1}{\pi n}$,

$$f(x_n)=\sin(n\pi)=0 \quad\Rightarrow\quad f(x_n)\to 0 \text{ as } n\to\infty$$

- Sequential limits lemma says something about "all" sequences!

__Remark:__ "Upgrading" statements about sequences to functions

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Ex. Suppose $f(x) \to L_1$ and $g(x) \to L_2$ as $x \to c$. Then,

$\forall \{x_n\}$ with $\lim\limits_{n\to\infty} x_n = c$, $x_n \in S\setminus\{c\}$, $\lim\limits_{n\to\infty}\left(f(x_n)\cdot g(x_n)\right) = \lim\limits_{n\to\infty} f(x_n)\cdot \lim\limits_{n\to\infty} g(x_n) = L_1 \cdot L_2$

$$\Rightarrow \lim_{x\to c}\left(f(x)\cdot g(x)\right) = L_1\cdot L_2 = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right)$$

Problems on HW!

Assigned Reading: Restrictions + One-sided Limits

Continuity

Def. (ε-δ definition of continuity)

Let $S \subset \mathbb{R}$, $c \in S$, $f: S \to \mathbb{R}$. We say f is <u>continuous</u> at c if,
$\underbrace{\text{for all } \varepsilon > 0}_{(1)}$, $\underbrace{\text{there exists } \delta > 0}_{(2)}$ such that $\underbrace{\text{for all } x \in S \text{ with } |x - c| < \delta}_{(3)}$,

$$|f(x) - f(c)| \underbrace{<}_{(4)} \varepsilon.$$

If f is continuous for all $c \in S$, we say f is a <u>continuous function</u>

Prop. (Characterizations of Continuity)

Let $S \subset \mathbb{R}$, $f: S \to \mathbb{R}$, $c \in S$. Then,

(limit characterization)

(i) If c is not a cluster point of S, then f is continuous at c.

(ii) If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and

$$\underbrace{\lim_{x\to c} f(x)}_{} = f(c)$$

$\lim_{x\to c} f(x) = f(c)$

$\underbrace{\lim_{x\to c}}_{\text{does not depend on } f(c)}$

(sequential characterization

(iii) f is continuous at c if and only if, for every sequence $\{x_n\}$ satisfying $x_n \in S\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

In other words, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$

"limits commute with f"

Pf. (Limit characterization)

(i) Suppose c is not a cluster pt. of S

$\Rightarrow \exists \delta > 0 : (c-\delta, c+\delta) \cap S \setminus \{c\}$ is empty

$\Rightarrow (c-\delta, c+\delta) \cap S = \{c\}$

- So, for any $\varepsilon > 0$, $\underbrace{\forall x \in S \text{ with } |x-c| < \delta}_{x \in (c-\delta, c+\delta) \cap S = \{c\}}$, we have

$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$

Thus, f is continuous at c. ✓

(ii) Suppose c is a cluster pt. of S.

- First, suppose $\lim_{x\to c} f(x) = f(c)$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S \setminus \{c\}$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

- Since $|f(c) - f(c)| = 0 < \varepsilon$, this also implies:

$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

so f is continuous at c. ✓

so f is continuous at c.

- Now, suppose f is continuous at c.

  $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

  - $S \setminus \{c\} \subset S$, so this also implies

    $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in S \setminus \{c\}$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

  So $\lim_{x \to c} f(x) = f(c)$

(Sequential Characterization)

(iii, $\Rightarrow$) Suppose f is continuous at c.

- Let $\{x_n\}$ be a sequence satisfying $x_n \in S \ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = c$
- Let $\varepsilon > 0$ be given. $\exists \delta > 0 : \forall x \in S$ with $|x-c| < \delta$
  - Since $\lim_{n \to \infty} x_n = c$, $\exists M \in \mathbb{N} : \forall n \geq M$, $|x_n - c| < \delta$
  - Thus, for all $n \geq M$,

    $$|f(x_n) - f(c)| < \varepsilon$$

  $\Rightarrow \lim_{n \to \infty} f(x_n) = f(c)$

(iii, $\Leftarrow$) Contrapositive: f is not continuous at c

$\Rightarrow \exists \{x_n\}$ with $x_n \in S \ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = c : f(x_n) \not\to f(c)$ as $n \to \infty$

f is not continuous at c: $\exists \varepsilon > 0 : \forall \delta > 0, \exists x \in S$ with $|x-c| < \delta : |f(x) - f(c)| \geq \varepsilon$

- Assume f is not continuous at c,

  $\Rightarrow$ there exists $\varepsilon > 0$ s.t. for all $n \in \mathbb{N}$, $\exists x_n \in S$ with $|x_n - c| < \underbrace{1/n}_{= \delta > 0}$ s.t.

  $$|f(x_n) - f(c)| \geq \varepsilon$$

- Note, $x_n \in S \ \forall n \in \mathbb{N}$, and $|x_n - c| < 1/n \ \forall n \in \mathbb{N} \Rightarrow \lim x_n = c$

- Note $x_n \in S \ \forall n \in \mathbb{N}$, and $|x_n - c| < 1/n \ \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} x_n = c$

However, $|f(x_n) - f(c)| \geq \varepsilon$ so $\{f(x_n)\}$ does not converge to $f(c)$. $\square$