Midterm Exam 1 Solutions

Examination Date: Thursday, March 11

1. For each $n \in \mathbb{N}$, let $E_n \subset \mathbb{R}$ be a non-empty bounded set. Define $a_n := \sup E_n$.

   (a) Prove that
   $$\sup(E_1 \cup E_2) = \max\{a_1, a_2\}$$

   (b) Suppose there exists some $B \in \mathbb{R}$ such that $a_n \le B$ for all $n \in \mathbb{N}$. Define the set
   $$D := \bigcup_{n=1}^{\infty} E_n = \{x : x \in E_n \text{ for some } n \in \mathbb{N}\}$$
   Show that $\sup D$ exists, and that
   $$\sup D = \sup\{a_n : n \in \mathbb{N}\}$$

(a) Let $a := \max\{a_1, a_2\}$. Then, for any $x \in E_1 \cup E_2$, either $x \in E_1$ so $x \le a_1 \le a$, or $x \in E_2$ so $x \le a_2 \le a$. Thus, $E_1 \cup E_2$ is bounded above by a, so the least upper bound exists and satisfies $\sup(E_1 \cup E_2) \le a$.

To prove the other inequality, note that $E_1 \subset E_1 \cup E_2$ so $a_1 = \sup(E_1) \le \sup(E_1 \cup E_2)$ (this was proved in HW1). Similarly, $E_2 \subset E_1 \cup E_2$ so $a_2 = \sup(E_2) \le \sup(E_1 \cup E_2)$. Thus, $\sup(E_1 \cup E_2) \ge \max\{a_1, a_2\} = a$.

Combining the two inequalities, $\sup(E_1 \cup E_2) = a = \max\{a_1, a_2\}$

(b) Here are two possible proofs, one without using induction, and one using induction.

Proof without induction (Recommended solution):

The set $\{a_n : n \in \mathbb{N}\}$ is non-empty and bounded above by B, so it has a least upper bound. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. For any $x \in D$, $x \in E_n$ for some $n \in \mathbb{N}$. Therefore, $x \le a_n \le a$ for all $n \in \mathbb{N}$. Thus, D is bounded above by a. D is also non-empty, as each E_n is non-empty. Thus, the least upper bound $\sup D$ exists and satisfies $\sup D \le a$

To prove the other inequality, note that $E_n \subset D$ for all $n \in \mathbb{N}$, so $x \le \sup D$ for all $x \in E_n$. Thus, $\sup D$ is an upper bound for the set E_n, and hence the least upper bound satisfies $\sup(E_n) = a_n \le \sup D$ for all $n \in \mathbb{N}$. Thus, $\sup D$ is an upper bound for the set $A := \{a_n : n \in \mathbb{N}\}$. Thus, the least upper bound of A satisfies $\sup\{a_n : n \in \mathbb{N}\} = a \le \sup D$.

Combining the two inequalities, $\sup D = a = \sup\{a_n : n \in \mathbb{N}\}$

Proof with induction:

Before we proceed, please note the following:

Remark 1: ∞ is *not* a real number. Hence, showing that something holds "for all $n \in \mathbb{N}$" is *not* the same as showing it holds "in the limit as $n \to \infty$", because $\infty \notin \mathbb{N}$. You typically cannot just replace n by ∞ to finish a proof!

Remark 2: We have not covered sequences of sets. While it is possible to define the limit of a sequence of sets $\{D_n\}$ (look up the "set-theoretic limit"), we have not done so in class, and it is not trivial to do so. Thus, we will avoid using limits of sequences of sets in this proof, and stick only to using limits of sequences of real numbers.

Remark 3: Induction is in fact completely superfluous to this proof. I am only writing this solution because I would like to show what a rigorous and complete argument by induction that would have been worth full points would look like on the exam.

On to the actual proof:

First, we need to prove $\sup D$ exists. The set $\{a_n : n \in \mathbb{N}\}$ is non-empty and bounded above by B, so it has a least upper bound. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. For any $x \in D$, $x \in E_n$ for some $n \in \mathbb{N}$. Therefore, $x \leq a_n \leq a$ for all $n \in \mathbb{N}$. Thus, D is bounded above by a. D is also non-empty, as each E_n is non-empty. Thus, the least upper bound $\sup D$ exists and satisfies $\sup D \leq a$

Define the sequence of sets D_n by

$$D_n := \bigcup_{k=1}^{n} E_k$$

And define a sequence of real numbers $\{b_n\}$ by

$$b_n := \max\{a_k : k \leq n\}$$

We will prove by induction that $\sup D_n = b_n$, and that $b_n \leq b_{n+1} \leq B$ so $\{b_n\}$ is a bounded monotone increasing sequence.

(Basis statement) $D_1 = E_1$, so $\sup D_1 = a_1 = b_1 \leq B$.

(Induction step) Let $n \geq 2$. Apply the result of (a) on $D_n = E_n \cup D_{n-1}$. Thus, $b_{n-1} \leq \max\{a_n, b_{n-1}\} = \sup D_n = b_n \leq B$.

Since $b_n = a_k$ for some $k \in \mathbb{N}$, and since $\{b_n\}$ is a bounded monotone increasing sequence, it converges with limit

$$\lim_{n\to\infty} b_n = \sup\{b_n : n \in \mathbb{N}\} = \sup\{a_n : n \in \mathbb{N}\}$$

Since $D_n \subset D$, we have $\sup D_n = b_n \leq \sup D$ for all $n \in \mathbb{N}$. Thus, the limit must satisfy

$$a = \sup\{a_n : n \in \mathbb{N}\} = \lim_{n\to\infty} b_n \leq \sup D$$

Finally, we combine the inequalities $\sup D \geq a$ and $\sup D \leq a$ to conclude $\sup D = a$ as desired.

2. In this problem, you will prove a version of the Bolzano-Weierstrass theorem for points in the plane $\mathbb{R}^2$ using the Euclidean distance metric.

   (a) Define a sequence of points $\{(x_n, y_n)\}$ as a pair of sequences of real numbers $\{x_n\}$ and $\{y_n\}$. We say that $\{(x_n, y_n)\}$ converges to the point (x, y) if for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

   $$\sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$$

   Show that $\{(x_n, y_n)\}$ converges to (x, y) if and only if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y

   (Hint: You may use the fact that if $0 \leq a < b$ if and only if $\sqrt{a} < \sqrt{b}$)

   (b) We say that a sequence of points $\{(x_n, y_n)\}$ is bounded if there exists $B \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

   $$\sqrt{x_n^2 + y_n^2} \leq B$$

   Furthermore, given a strictly increasing sequence of natural numbers $\{n_i\}_{i=1}^{\infty}$, we call the sequence $\{(x_{n_i}, y_{n_i})\}_{i=1}^{\infty}$ a subsequence of $\{(x_n, y_n)\}$.

   Show that if $\{(x_n, y_n)\}$ is bounded, then $\{x_n\}$ and $\{y_n\}$ are bounded. Then, conclude the Bolzano-Weierestrass theorem for $\mathbb{R}^2$: every bounded sequence of points in the plane has a convergent subsequence.

(a) First, assume $\{(x_n, y_n)\}$ converges to (x, y). Let $\varepsilon > 0$ be given. Note $\varepsilon^2 > 0$. Then, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$0 \leq \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon = \sqrt{\varepsilon^2}$$

which is equivalent to

$$(x_n - x)^2 + (y_n - y)^2 < \varepsilon^2$$

Since $a^2 = |a|^2 \geq 0$ for any $a \in \mathbb{R}$, we have that for all $n \geq M$

$$(x_n - x)^2 \leq (x_n - x)^2 + (y_n - y)^2 < \varepsilon^2 \implies |x_n - x| < \varepsilon$$

So $\{x_n\}$ converges to x. Similarly, we have for all $n \geq M$

$$(y_n - y)^2 \leq (x_n - x)^2 + (y_n - y)^2 < \varepsilon^2 \implies |y_n - y| < \varepsilon$$

so $\{y_n\}$ converges to y.

Now, assume $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. Let $\varepsilon > 0$ be given. Then, there exists $M_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/\sqrt{2}$ for $n \geq M_1$. Similarly, there exists $M_2 \in \mathbb{N}$ such that $|y_n - y| < \varepsilon/\sqrt{2}$. Then, for $n \geq M := \max\{M_1, M_2\}$

$$(x_n - x)^2 + (y_n - y)^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2 \implies \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$$

Thus $\{(x_n, y_n)\}$ converges to (x, y).

(b) Suppose $\{(x_n, y_n)\}$ is bounded. Then for all $n \in \mathbb{N}$,

$$\sqrt{x_n^2 + y_n^2} < B \implies x_n^2 + y_n^2 \leq B^2$$

Then, since $a^2 = |a|^2 \geq 0$,

$$x_n^2 \leq x_n^2 + y_n^2 \leq B^2 \implies |x_n| \leq B$$
$$y_n^2 \leq x_n^2 + y_n^2 \leq B^2 \implies |y_n| \leq B$$

Thus, $\{x_n\}$ and $\{y_n\}$ are bounded sequences.

By the Bolzano-Weierstrass theorem for $\mathbb{R}$, since $\{x_n\}$ is bounded, there exists a convergent subsequence $\{x_{m_k}\}$. Then, note the subsequence $\{y_{m_k}\}$ is bounded, hence there exists a convergent subsequence $\{y_{m_{k_i}}\}$. $\{x_{m_{k_i}}\}$ is a subsequence of a convergent sequence, and hence is also convergent. Let $n_i = m_{k_i}$, then $\{x_{n_i}\}$ and $\{y_{n_i}\}$ are both convergent sequences, hence the subsequence $\{(x_{n_i}, y_{n_i})\}$ of the bounded sequence $\{(x_n, y_n)\}$ is convergent.

Remark:

With problem number 2, there were several issues in extending Bolzano-Weierstrass, with the bulk of the problems falling in to one of two categories:

(a) Trying to apply 1-D B-W to both coordinates simultaneously

(b) Trying to apply 1-D B-W to both coordinates independently, and then erroneously combining the results

Claim:

- $\{x_n\}$ is bounded, therefore by Bolzano-Weierstrass there exists a convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x$
- $\{y_n\}$ is bounded, therefore by Bolzano-Weierstrass there exists a convergent subsequence $\{y_{n_j}\}$ such that $y_{n_j} \to y$
- Let $\{n_k\}$ be the monotonically ordered intersection $\{n_i\} \cap \{n_j\}$, then $(x_{n_k}, y_{n_k}) \to (x, y)$, and $\{x_n, y_n\}$ has a convergent subsequence

Problem:

Consider the sequence below. It is certainly bounded, as $\forall n \in \mathbb{N}, \sqrt{x_n^2 + y_n^2} \leq \sqrt{2}$. Now suppose using the above program, as my subsequence for x I choose even x_n (e.g $n_i = 2i$), which converge to 0. As my subsequence for y I choose odd y_n (e.g $n_j = 2j - 1$), which converge to 0. There is no subsequence $(x_{n_k}, y_{n_k}) \to (0, 0)$ and in fact $\{n_i\} \cap \{n_j\} = \emptyset$

$$x_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

$$y_n = \begin{cases} 1 & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases}$$

Claim:

- Ok, well as in that example, just let $n_i = n_j$, we know x_{n_i} is convergent because we selected it that way, and the y_{n_i} will have to "settle down". In the above example we then $\to (1, 0)$ or $\to (0, 1)$

Problem:
There is no such condition on the y_n, and we should always be suspicious when imprecise language enters our proofs. Consider the slightly modified sequences:

$$x_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ \frac{3}{4} + (-\frac{1}{4})^{\frac{n+1}{2}} & n \text{ odd} \end{cases}$$

$$y_n = \begin{cases} \frac{3}{4} + (-\frac{1}{4})^{\frac{n}{2}} & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases}$$

Just as before, the even x_n and odd y_n converge to zero, but the odd x_n and even y_n pop back and forth between 1 and $\frac{1}{2}$. If I choose even n, my x coordinate will converge to 0, but my y coordinate will alternate between $\frac{1}{2}$ and 1.

3. In this problem, you will explore "derivatives" of the Fourier sine series. For the following, let $C > 0$, and $0 < r < 1$ be real numbers.

   (a) Let $\{a_n\}$ be a sequence of real numbers. We define the Fourier sine series for $x \in [0, \pi]$ as the series sum (if it exists)

   $$f(x) := \sum_{n=1}^{\infty} a_n \sin(nx)$$

   Show that if $|a_n| \leq Cr^n$ for all $n \in \mathbb{N}$, then the series sum converges absolutely for all $x \in [0, \pi]$, and hence $f(x)$ is well defined over this interval.

   (b) Let $k \in \mathbb{N}$ be a natural number. Show that the sum

   $$\sum_{n=1}^{\infty} n^k r^n$$

   converges absolutely.

   (c) We could try taking the k-th "derivative" of $f(x)$ by taking the derivative of the sine function,

   $$f^{(k)}(x) := \begin{cases} \sum_{n=1}^{\infty} (-1)^{\frac{k}{2}} a_n n^k \sin(nx) & k \text{ even} \\ \sum_{n=1}^{\infty} (-1)^{\frac{k-1}{2}} a_n n^k \cos(nx) & k \text{ odd} \end{cases}$$

   Show that if $|a_n| \leq Cr^n$ for all $n \in \mathbb{N}$, then the series sum converges absolutely for all k and all $x \in [0, \pi]$

(a) Since $|\sin(nx)| \leq 1$ for all x, we have that for all $n \in \mathbb{N}$ and $x \in [0, \pi]$

$$0 \leq |a_n \sin(nx)| \leq |a_n| \leq Cr^n$$

Since $|r| < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ converges. By linearity, so does $\sum_{n=1}^{\infty} Cr^n$. Thus, by the comparison test, $\sum_{n=1}^{\infty} |a_n \sin(nx)|$ converges, so $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges absolutely.

Remark: Make sure to carefully read the requirements of the comparison test. The inequality required by the test is satisfied by the individual terms x_n and y_n of the series, not the series $\sum x_n$ or $\sum y_n$ as a whole.

The reason for this distinction is because while the following inequalities are true,

$$0 \leq \sum_{n=1}^{\infty} |a_n \sin(nx)| \leq \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} Cr^n$$

we haven't yet established the convergence of the series $\sum_{n=1}^{\infty} |a_n \sin(nx)|$ or $\sum_{n=1}^{\infty} |a_n|$ at the beginning of the proof.

Recall that $\sum x_n$, confusingly, denotes both the series as a formal object (not a real number), and if the series converges, also denotes the value that the series converges to (a real number). It doesn't make sense to write down inequalities that a series $\sum x_n$ satisfies before you prove it converges, since the series might not converge to anything, and hence might not represent a real number. However, the individual terms of the series x_n, as well as the partial sums s_n, are well-defined real numbers, hence we can (usually) write inequalities involving them before we prove that the series converges.

(b) We compute the limit (using continuity of algebraic operations)

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k r = r \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^k = r\left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^k = r < 1$$

Thus, by the ratio test for series, $\sum n^k r^n$ converges absolutely.

(c) Note for even k and all $x \in [0, \pi]$ we have for all $n \in \mathbb{N}$

$$0 \le \left|(-1)^{\frac{k}{2}} a_n n^k \sin(nx)\right| \le \left|a_n n^k\right| \le C n^k r^n$$

and for odd k we have

$$0 \le \left|(-1)^{\frac{k-1}{2}} a_n n^k \cos(nx)\right| \le \left|a_n n^k\right| \le C n^k r^n$$

Thus by the comparison test, the series will converge for all $k \in \mathbb{N}$ and $x \in [0, \pi]$.