Cauchy Sequences, Series

Tuesday, October 4, 2022 5:13 PM

Cauchy Sequences

Def. A sequence $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have

$$|x_n - x_k| < \varepsilon$$

Idea: $\{x_n\}$ is convergent $\iff$ $\exists L \in \mathbb{R} : \forall \varepsilon > 0,\ \exists M \in \mathbb{N} : \forall n \geq M,\ |x_n - L| < \varepsilon$

$\{x_n\}$ is Cauchy $\iff$ $\forall \varepsilon > 0,\ \exists M \in \mathbb{N} : \forall n, k \geq M,\ |x_n - x_k| < \varepsilon$

Thrm. ($\mathbb{R}$ is Cauchy-complete)

A sequence of real numbers is Cauchy if and only if it converges.

Prop. A Cauchy sequence is bounded.

Pf Suppose $\{x_n\}$ is Cauchy.

$\exists M \in \mathbb{N} : \forall n, k \geq M,\ |x_n - x_k| < 1$

In particular, $\forall n \geq M$, $|x_n| - |x_M| \leq \underbrace{\big|\,|x_n| - |x_M|\,\big| \leq}_{\text{reverse triangle ineq.}} |x_n - \overbrace{x_M}^{k=M}| < 1$

$\Rightarrow |x_n| < 1 + |x_M|$

Let $B := \max\{|x_1|, |x_2|, \dots, |x_{M-1}|, 1 + |x_M|\}$

then $|x_n| < B \quad \forall n \in \mathbb{N}$

$\Rightarrow \{x_n\}$ is bounded. ✓

Pf. (Cauchy $\iff$ Convergent)

($\Leftarrow$) Suppose $\{x_n\}$ converges to some $L \in \mathbb{R}$. Let $\underset{\sim}{\varepsilon > 0}$ be arbitrary,

($\Leftarrow$) Suppose $\{x_n\}$ converges to some $L \in \mathbb{R}$. Let $\underbrace{\varepsilon > 0}_{(1)}$ be arbitrary,

- $\underbrace{\exists M \in \mathbb{N}}_{(2)}: \forall n \geq M, \ |x_n - L| < \frac{\varepsilon}{2}$
- Then, $\underbrace{\forall n, k \geq M}_{(3)}$ we have

$$\underbrace{|x_n - x_k|} = |x_n - L + L - x_k| \leq |x_n - L| + |x_k - L| \underbrace{<}_{(4)} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \underline{\varepsilon}$$

$\Rightarrow \{x_n\}$ is Cauchy.

($\Rightarrow$) Suppose $\{x_n\}$ is Cauchy. $\{x_n\}$ is bounded, so

$a := \limsup_{n\to\infty} x_n \quad b := \liminf_{n\to\infty} x_n$ (Note: LUB of $\mathbb{R}$ used)

both exist.

- Then, by Thm 2.3.4, there exist subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ with

$$\lim_{k\to\infty} x_{n_k} = a \qquad \lim_{k\to\infty} x_{m_k} = b$$

- Let $\varepsilon > 0$ be arbitrary.
  - $\exists M_1 \in \mathbb{N}: \forall k \geq M_1, \ |x_{n_k} - a| < \varepsilon/3$
  - $\exists M_2 \in \mathbb{N}: \forall k \geq M_2, \ |x_{m_k} - b| < \varepsilon/3$
  - $\underbrace{\exists M_3 \in \mathbb{N}}_{(2)}: \underbrace{\forall n, m \geq M_3}_{(3)}, \ |x_n - x_m| \underbrace{<}_{(4)} \underbrace{\varepsilon/3}_{(1)}$
- Take $M := \max\{M_1, M_2, M_3\}$. Note $n_k, m_k \geq k$. Thus, for all $k \geq M$,

$$\begin{aligned} |a-b| &= |a - x_{n_k} + x_{n_k} - x_{m_k} + x_{m_k} - b| \\ &\leq |a - x_{n_k}| + |x_{n_k} - x_{m_k}| + |x_{m_k} - b| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

- $|a-b| < \varepsilon \ \forall \varepsilon > 0 \Rightarrow |a-b| = 0 \Rightarrow a = \limsup x_n = \liminf x_n = b$
- Then by lim sup/inf convergence test, $\{x_n\}$ is convergent. $\square$

then by the supinf convergence test, $\{x_n\}$ is convergent. $\square$

Ex. (Showing Cauchy from definition)

Claim $\{\frac{1}{n}\}$ is Cauchy.

Pf. Let $\underbrace{\varepsilon > 0}_{(1)}$ be arbitrary. By Archimedean Prop, $\underbrace{\exists M \in \mathbb{N}}_{(2)} : \frac{1}{M} < \frac{\varepsilon}{2}$

Then, $\underbrace{\forall n, k \geq M}_{(3)}$,

$$\underline{|x_n - x_k|} = \left|\frac{1}{n} - \frac{1}{k}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{k}\right| \leq \frac{1}{M} + \frac{1}{M} \underbrace{<}_{(4)} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \underline{\varepsilon} \quad \square$$

Series

a "thing", not necessarily a number

Def. Given a sequence $\{x_n\}$, we write the "formal object"

$$\sum_{n=1}^{\infty} x_n \quad \text{or} \quad \sum x_n$$

and call it a series.

A series converges if the sequence of partial sums $\{s_k\}$

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + \cdots + x_k$$

converges. In this case, we abuse notation and write

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_k$$

If $\{s_k\}$ diverges, then we say the series diverges. Note $\sum x_n$ is not a real number then.

Prop. (convergence of the geometric series)

Suppose $-1 < r < 1$. Then, the geometric series $\sum_{n=0}^{\infty} r^n$ converges, and

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \left(= 1 + r + r^2 + \cdots \text{ (informally)}\right)$$

$$\sum_{n=0} r^n = \frac{1}{1-r} \quad (= 1 + r + r^2 + \cdots \text{ (informally)})$$

Pf. Idea: "okay" to work with partial sums

- $s_k := \sum_{n=0}^{k-1} r^n = 1 + r + \cdots + r^{k-1}$

- $r s_k = r + r^2 + \cdots + r^k$

- $s_k - r s_k = 1 + \cancel{r} + \cancel{r^2} + \cancel{\cdots} + \cancel{r^{k-1}} - (\cancel{r} + \cancel{r^2} + \cancel{\cdots} + r^k) = 1 - r^k$

  - $(1-r) s_k = 1 - r^k \Rightarrow s_k = \dfrac{1-r^k}{1-r}$

- $$\lim_{k\to\infty} s_k = \frac{\lim_{k\to\infty}(1-\underbrace{r^k}_{\to 0})}{\lim_{k\to\infty}(1-r)} = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \qquad \square$$

Def. A series $\sum x_n$ is Cauchy (or a Cauchy series) if the sequence of partial sums is Cauchy.

Remark Cauchy $\Leftrightarrow$ convergent

Also, if $n \geq k$

$$|s_n - s_k| = \left|\sum_{p=1}^{n} s_p - \sum_{p=1}^{k} s_p\right| = \Bigg|\underbrace{\sum_{p=k+1}^{n} s_p}_{x_{k+1} + \cdots + x_n}\Bigg|$$

Def A series $\sum x_n$ converges absolutely if $\sum |x_n|$ converges.

If a series converges, but not absolutely, it is called conditionally convergent.

Prop. If a series converges absolutely, it converges.

Pf. Assume $\sum x_n$ converges absolutely, i.e. $\sum |x_n|$ converges.

- $\sum |x_n|$ is Cauchy. So $\forall \varepsilon > 0,\ \exists M \in \mathbb{N}: \forall n \geq k \geq M$, Pf. Assume $\sum x_n$ converges absolutely, i.e. $\sum |x_n|$ converges.

- $\sum |x_n|$ is Cauchy. So, $\forall \varepsilon > 0, \exists M \in \mathbb{N} : \forall n \geq k \geq M$,

$$\underbrace{\left| \sum_{p=k+1}^{n} x_p \right|}_{\text{partial sums of } \sum x_n} \leq \sum_{p=k+1}^{n} |x_p| = \underbrace{\left| \sum_{p=k+1}^{n} |x_p| \right|}_{\text{partial sums of } \sum |x_n|} < \varepsilon$$

- Thus, $\sum x_n$ is Cauchy, hence convergent. $\square$

Some assigned readings:

Prop. 2.5.12 (Linearity of series)
Prop. 2.5.16 (Comparison test)
Prop. 2.5.17 (p-series)

} Be familiar with the results!

Some interesting optional readings:

Prop. 2.6.2: there are "many" conditionally convergent series

Sec. 2.6.3: Absolutely convergent series "behave like regular addition" (Prop. 2.6.3; can rearrange terms in the "sum")

Conditionally convergent series do not behave like regular addition!