Homework 9 Solutions

Due: Friday, May 7th by 11:59 PM ET

- Collaboration with other students is highly encouraged, but you must **write up your own solutions independently**.
- Please make sure your submission is **well-written and legible**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Chapter 7 Exercises

Problem 1 (3 points each) For $u \in \mathbb{R}^n$, we define

$$\|u\|_1 := \sum_{i=1}^{n} |u_i| \qquad \|u\|_\infty := \max_{1 \le k \le n} |u_k|$$

and use it to define functions $d_1, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_1(u,v) := \|u-v\|_1 \qquad d_\infty(u,v) := \|u-v\|_\infty$$

(a) Show that $(\mathbb{R}^n, d_1)$ is a metric space.

(b) Show that $(\mathbb{R}^n, d_\infty)$ is a metric space.

(c) Recall d_2 is the standard Euclidean metric on $\mathbb{R}^n$. Given $r > 0$, define the open balls

$$B_1(0,r) := \{u \in \mathbb{R}^n : d_1(u,0) < r\}$$
$$B_2(0,r) := \{u \in \mathbb{R}^n : d_2(u,0) < r\}$$
$$B_\infty(0,r) := \{u \in \mathbb{R}^n : d_\infty(u,0) < r\}$$

For dimension $n = 2$, sketch $B_1(0,1), B_2(0,1)$, and $B_\infty(0,1)$.

(a) For points $u, v, w \in \mathbb{R}^n$ we verify the metric space axioms

(i) (Non-negativity) Since each $|u_i - v_i| \ge 0$, we have

$$d_1(u,v) = \sum_{i=1}^{n} |u_i - v_i| \ge 0$$

(ii) Since $|u_i - v_i| = 0$ iff $u_i = v_i$, and since the sum of non-negative terms is zero iff each of the terms in the sum is zero, then $d_1(u,v) = 0$ iff $u_i = v_i$ for all i, i.e. $u = v$.

(iii) (Symmetry) Since $|u_i - v_i| = |v_i - u_i|$, we compute

$$d_1(u,v) = \sum_{i=1}^{n} |u_i - v_i| = \sum_{i=1}^{n} |v_i - u_i| = d_1(v,u)$$

(iv) (Triangle inequality) By the triangle inequality for absolute value, we have $|u_i - w_i| = |u_i - v_i + v_i - w_i| \leq |u_i - v_i| + |v_i - w_i|$. Thus, we compute

$$d_1(u,w) = \sum_{i=1}^{n} |u_i - w_i| \leq \sum_{i=1}^{n} |u_i - v_i| + \sum_{i=1}^{n} |v_i - w_i| = d_1(u,v) + d_1(v,w)$$

Thus $(\mathbb{R}^n, d_1)$ is a metric space.

(b) For points $u, v, w \in \mathbb{R}^n$ we verify the metric space axioms

(i) (Non-negativity) Since each $|u_i - v_i| \geq 0$, we have

$$d_\infty(u,v) = \max_{1 \leq i \leq n} |u_i - v_i| \geq 0$$

(ii) If $u = v$, then $|u_i - v_i| = 0$ for all i, so $d_\infty(u,v) = 0$.

Conversely, if $d_\infty(u,v) = 0$, then $|u_i - v_i| \leq 0$ for all i. Since the absolute value satisfies $|u_i - v_i| \geq 0$, we have $|u_i - v_i| = 0$ for all i, so $u = v$.

Thus, $d_\infty(u,v) = 0$ iff $u = v$.

(iii) (Symmetry) Since $|u_i - v_i| = |v_i - u_i|$, we compute

$$d_\infty(u,v) = \max_{1 \leq i \leq n} |u_i - v_i| = \max_{1 \leq i \leq n} |v_i - u_i| = d_\infty(v,u)$$

(iv) (Triangle inequality) Let us treat each point as a function $u : S \to \mathbb{R}$ where $S = \{1, 2, ..., n\} \subset \mathbb{R}$. Then, we have that

$$\|u\|_\infty = \max_{1 \leq i \leq n} |u_i| = \sup_{i \in S} |u_i|$$

Since $|u_i + v_i| \leq |u_i| + |v_i|$, we use Proposition 1.3.7 and HW7 Problem 3c to get

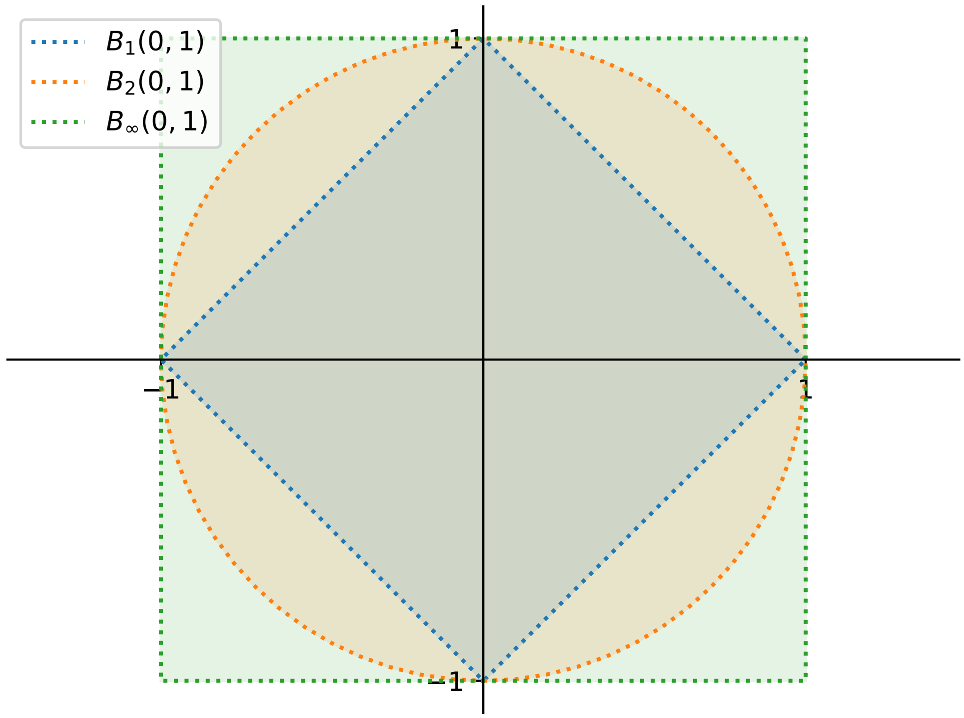
$$\|u + v\|_\infty = \sup_{i \in S} |u_i + v_i| \leq \sup_{i \in S} (|u_i| + |v_i|) \leq \sup_{i \in S} |u_i| + \sup_{i \in S} |v_i| = \|u\|_\infty + \|v\|_\infty$$

So, we show the triangle inequality

$$\begin{aligned} d_\infty(u,w) &= \|u - w\|_\infty = \|u - v + v - w\|_\infty \\ &\leq \|u - v\|_\infty + \|v - w\|_\infty = d_\infty(u,v) + d_\infty(v,w) \end{aligned}$$

Thus $(\mathbb{R}^n, d_\infty)$ is a metric space.

(c)



Problem 2 (8 points) Let $X = C([0,1], \mathbb{R})$ be the set of continuous functions on $[0,1]$, and define the L^1 norm as

$$\|f\|_1 := \int_0^1 |f(x)| \, \mathrm{d}x$$

and use it to define a function $d : X \times X \to \mathbb{R}$ by

$$d(f,g) := \|f-g\|_1$$

Show that (X,d) is a metric space. Note this metric shows up frequently in machine learning.

(Remark: In order to prove that d is a metric, you will also need to show that if the integral over an interval of a continuous non-negative function is zero, then the function on that interval is zero. Proving this statement is also part of the problem.)

Given $f, g, h \in X$ we verify the metric space axioms:

(i) (Non-negativity) Since f, g are continuous, $|f-g|$ is also continuous and hence Riemann integrable. Since $|f(x) - g(x)| \geq 0$ for all $x \in [0,1]$, by monotonicity of the integral we have

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x \geq 0$$

(ii) Suppose $f = g$, so $|f(x) - g(x)| = 0$ for all $x \in [0, 1]$. Then, we integrate the constant function to get

$$d(f, g) = \int_0^1 |f(x) - g(x)| \,\mathrm{d}x = 0$$

Now, suppose $d(f, g) = 0$. Suppose for sake of contradiction there was some $c \in [0, 1]$ such that $M := |f(c) - g(c)| > 0$. Since $|f - g|$ is continuous, there exists some $\delta > 0$ such that $|f(x) - g(x)| > M/2$ for all $x \in (c - \delta, c + \delta)$. Assume without loss of generality we pick δ small enough such that $0 < c - \delta < c + \delta < 1$. Then, by additivity of the integral we have

$$d(f, g) = \int_0^{c-\delta} |f - g| + \int_{c-\delta}^{c+\delta} |f - g| + \int_{c+\delta}^{1} |f - g| := I_1 + I_2 + I_3 = 0$$

By monotonicity of the integral, $I_2 \geq M\delta > 0$. This implies at least one of I_1 or I_3 must be negative. However, since $0 < c - \delta$ and $c + \delta < 1$, and $|f(x) - g(x)| \geq$ for all $x \in [0, 1]$, by monotonicity $I_1 \geq 0$ and $I_3 \geq 0$, which is a contradiction.

Hence, $d(f, g) = 0$ implies $|f(x) - g(x)| = 0$ for all $x \in [0, 1]$, and hence $f = g$.

(iii) (Symmetry) Since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in [0, 1]$, we compute

$$d(f, g) = \int_0^1 |f - g| = \int_0^1 |g - f| = d(g, f)$$

(iv) (Triangle inequality) Since $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$ for all $x \in [0, 1]$, by monotonicity of the integral we have

$$d(f, h) = \int_0^1 |f - h| \leq \int_0^1 |f - g| + \int_0^1 |g - h| = d(f, g) + d(g, h)$$

Thus (X, d) is a metric space.

Problem 3 (8 points) Here is an example of a metric which isn't a norm on a vector space. Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function, and consider the graph of f

$$X = \{(t, f(t)) : t \in [0, 1]\} \subset \mathbb{R}^2$$

Define the function $d : X \times X \to \mathbb{R}$ by

$$d((s, f(s)), (t, f(t))) := \left| \int_s^t \sqrt{1 + |f'(x)|^2} \,\mathrm{d}x \right|$$

Explain why d is well-defined (why does the integral exist) and show that (X, d) is a metric space. Usually d is called the arc-length metric.

Since $1 + |f'(x)|^2 \geq 1$ and is continuous for all $x \in [0, 1]$, we have that $\sqrt{1 + |f'(x)|^2} \geq 1$ and is continuous for all $x \in [0, 1]$. Hence, it is Riemann integrable over any subinterval of $[0, 1]$. Given points $(s, f(s)), (t, f(t)), (u, f(u)) \in X$, we now verify the metric space axioms:

(i) (Non-negativity) Since the integral is always well-defined, d is the absolute value of a real number, hence always satisfies $d((s, f(s)), (t, f(t))) \geq 0$.

(ii) Since $\int_a^a f = 0$ by definition for any $a \in \mathbb{R}$, we have that if $s = t$, then

$$d((s, f(s)), (t, f(t))) = \left| \int_s^t \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| = 0$$

Now, assume that $d((s, f(s)), (t, f(t))) = 0$. Suppose without loss of generality that $s \leq t$ (we can use symmetry, which we will prove next, to switch the two points). Then, since $\sqrt{1 + |f'(x)|^2} \geq 1$ we have that

$$\int_s^t \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \geq (t - s)$$

Hence $d((s, f(s)), (t, f(t))) = 0$ implies $t = s$.

(iii) (Symmetry) Since $\int_a^b f = -\int_b^a f$ by definition for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} d((s, f(s)), (t, f(t))) &= \left| \int_s^t \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| \\ &= \left| \int_t^s \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| = d((t, f(t)), (s, f(s))) \end{aligned}$$

(iv) (Triangle inequality) By the additivity proved in HW8 Problem 1a, we have that

$$\begin{aligned} d((s, f(s)), (u, f(u))) &= \left| \int_s^u \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| \\ &= \left| \int_s^t \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x + \int_t^u \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| \\ &\leq \left| \int_s^t \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| + \left| \int_t^u \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \right| \\ &= d((s, f(s)), (t, f(t))) + d((t, f(t)), (u, f(u))) \end{aligned}$$

Thus (X, d) is a metric space.

Problem 4 (3 points each) In this problem, consider $a, b \in \mathbb{R}$ with $a < b$. Prove the following:

(a) (a, b), $(-\infty, a)$, and (b, ∞) are open in $\mathbb{R}$

(b) $[a, b]$, $(-\infty, a]$, and $[b, \infty)$ are closed in $\mathbb{R}$

(c) Recall that the restriction of the standard metric d on $\mathbb{R}$ to a subset $Y \subset \mathbb{R}$ defines a new metric space $(Y, d|_{Y\times Y})$.

Show that the set $[a, b)$ is:

(i) neither open nor closed in $\mathbb{R}$

(ii) open but not closed in the subspace $[a, \infty)$

(iii) both open and closed in the subspace $[a, b)$

(a) $(a, b) = B(\frac{a+b}{2}, \frac{b-a}{2})$ so it is an open ball, which is open as proved in lecture.

For every $x \in (b, \infty)$, let $\delta = x - b > 0$. Then, the open ball $B(x, \delta) = (b, 2x - b) \subset (b, \infty)$, so (b, ∞) is open.

For every $x \in (-\infty, a)$, let $\delta = a - x > 0$. Then, the open ball $B(x, \delta) = (2x - a, a) \subset (-\infty, a)$, so $(-\infty, a)$ is open.

(b) We have that $[a, b]^c = (-\infty, a) \cup (b, \infty)$ which is the union of two open sets and hence open. Thus $[a, b]$ is closed.

Similarly, $(-\infty, a]^c = (a, \infty)$ and $[b, \infty)^c = (-\infty, b)$ are open sets, hence $(-\infty, a]$ and $[b, \infty)$ are both closed.

(c)

(i) Consider $a \in [a, b)$. For any $\delta > 0$ we have e.g. $a - \frac{\delta}{2} \in B_{\mathbb{R}}(a, \delta) = (a - \delta, a + \delta)$, so $B_{\mathbb{R}}(a, \delta) \not\subset [a, b)$ for any $\delta > 0$, hence $[a, b)$ is not open in $\mathbb{R}$.

Now, consider $b \in [a, b)^c = (-\infty, a) \cup [b, \infty)$. For any $\delta > 0$, we can find $c \in B_{\mathbb{R}}(b, \delta)$ such that $a \leq c$ and $b - \delta < c < b$. Hence, $B_{\mathbb{R}}(b, \delta) \not\subset [a, b)^c$, so $[a, b)$ is not closed in $\mathbb{R}$.

(ii) We have that $[a, b) = \{x \in [a, \infty) : |x - a| < (b - a)\} = B_{[a,\infty)}(x, b - a)$, which is an open ball in the subspace topology, and hence open.

Now, in the subspace we have $[a, b)^c = [a, \infty) \setminus [a, b) = [b, \infty)$. For any $\delta > 0$, we can find $c \in B_{[a,\infty)}(b, \delta)$ such that $b - \delta < c < b$. Hence, $B_{[a,\infty)}(b, \delta) \not\subset [a, b)^c$, so $[a, b)$ is not closed in $[a, \infty)$.

(iii) Using Proposition 7.2.6, we have that $[a, b)$ is open in the subspace $[a, b)$. Furthermore, we have $[a, b)^c = [a, b) \setminus [a, b) = \emptyset$ which is open, hence $[a, b)^c$ is also closed in the subspace $[a, b)$.

Problem 5 (4 points each)

(a) Suppose I is some set (not necessarily finite or countable, though it could be) and for each $\lambda \in I$ we have a set A_λ. Recall the union and intersection of these sets indexed by I is defined as

$$\bigcup_{\lambda \in I} A_\lambda := \{x : x \in A_\lambda \text{ for some } \lambda \in I\}$$
$$\bigcap_{\lambda \in I} A_\lambda := \{x : x \in A_\lambda \text{ for all } \lambda \in I\}$$

Show that the complement is given by

$$\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \bigcap_{\lambda \in I} (A_\lambda)^c$$
$$\left(\bigcap_{\lambda \in I} A_\lambda\right)^c = \bigcup_{\lambda \in I} (A_\lambda)^c$$

(Warning: Induction works only for finite sets I. Try proving the statement directly without using induction.)

(b) Use your result in (a) to prove proposition 7.2.8 using the results of proposition 7.2.6 in the textbook.

(a) The negation of "$x \in A_\lambda$ for some $\lambda \in I$" is the statement "$x \notin A_\lambda$ for all $\lambda \in I$", which is equivalent to "$x \in A_\lambda^c$ for all $\lambda \in I$". Hence, we have

$$\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \{x : x \in A_\lambda^c \text{ for all } \lambda \in I\} = \bigcap_{\lambda \in I} (A_\lambda)^c$$

(Remark: A different way of notating the union is "$\exists \lambda \in I$ s.t. $x \in A_\lambda$". The negation of this statement is "$\forall \lambda \in I,\ x \notin A_\lambda$")

Similarly, the negation of "$x \in A_\lambda$ for all $\lambda \in I$" is the statement "$x \in A_\lambda^c$ for some $\lambda \in I$", so

$$\left(\bigcap_{\lambda \in I} A_\lambda\right)^c = \{x : x \in A_\lambda^c \text{ for some } \lambda \in I\} = \bigcup_{\lambda \in I} (A_\lambda)^c$$

(b) Given a metric space (X, d), we assume the results of Prop 7.2.6. Then, we prove the items of Prop 7.2.8 using the corresponding statements in Prop 7.2.6:

(i) $\emptyset^c = X$ and $X^c = \emptyset$ are open, so $\emptyset$ and X are closed.

(ii) If $\{E_\lambda\}_{\lambda\in I}$ is an arbitrary collection of closed sets, then $\{E_\lambda^c\}_{\lambda\in I}$ is a collection of open sets. $\left(\bigcap_{\lambda\in I} E_\lambda\right)^c$ is open since it is the union of open sets:

$$\left(\bigcap_{\lambda\in I} E_\lambda\right)^c = \bigcup_{\lambda\in I} E_\lambda^c$$

Thus, $\bigcap_{\lambda\in I} E_\lambda$ is closed.

(iii) If $E_1, E_2, ..., E_k$ are closed, then $E_1^c, E_2^c, ..., E_k^c$ are open. Thus, $\left(\cup_{j=1}^k E_j\right)^c$ is open because it is the finite intersection of open setsL

$$\left(\bigcup_{j=1}^{k} E_\lambda\right)^c = \bigcap_{j=1}^{k} E_\lambda^c$$

Thus $\cup_{j=1}^k E_j$ is closed.

Problem 6 (8 points) Consider the sequence of continuous functions $\{f_n\}$ on $[0,1]$ given by

$$f_n(x) := \begin{cases} 1 - nx & 0 \le x < 1/n \\ 0 & 1/n \le x \le 1 \end{cases}$$

Show that $\{f_n\}$ has no subsequence which is convergent in uniform norm.

(Hint: Show that every subsequence of $\{f_n\}$ converges pointwise to some function. Can the subsequences converge uniformly?)

(Remark: This is an example of a sequence of continuous functions bounded in the uniform norm which has no convergent subsequence. Many of the results proved in this course rely in a crucial way on Bolzano-Weierstrass and Heine-Borel, which are results particular to $\mathbb{R}^n$. Showing in what ways the results in this course we have proved generalize to other metric spaces is a topic you can study in advanced analysis courses.)

For fixed $x \in (0,1]$ there exists $N \in \mathbb{N}$ such that $1/N \le x$. Then, $f_n(x) = 0$ for all $n \ge N$. Thus, $\{f_n\}$ converges pointwise to a function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \le 1 \end{cases}$$

Since the sequences $\{f_n(x)\}$ converge to $f(x)$ for fixed $x \in [0,1]$, any subsequence $\{f_{n_j}(x)\}_{j=1}^\infty$ will also converge to $f(x)$. Thus, the subsequence $\{f_{n_j}\}_{j=1}^\infty$ also converges pointwise to f.

Since each of the $\{f_n\}$ are continuous and bounded, if any subsequence $\{f_{n_j}\}$ converged in uniform norm it would be to a continuous function $g : [0,1] \to \mathbb{R}$. Since uniform convergence implies pointwise convergence, and since the pointwise limit is unique (since limits of sequences of real numbers are unique), $g = f$. However, f is not continuous, so this is a contradiction. Thus, no subsequence $\{f_{n_j}\}$ can be convergent in uniform norm.