# Homework 9 Solutions

Due: Monday, December 5th by 11:59 PM ET

## Sections 5.1-5.3 Exercises

**Problem 1** (5 points) Let $A \subset \mathbb{R}$ be a bounded non-empty set, and let $B := \{|x| : x \in A\}$. Prove that

$$\sup B - \inf B \le \sup A - \inf A$$

(*Hint*: Find some $|x|$ close to $\sup B$ and some $|y|$ close to $\inf B$, then use the reverse triangle inequality.)

Let $\varepsilon > 0$ be arbitrary. By the properties of sup/inf, there exists $|x|, |y| \in B$ satisfying

$$\inf B \le |y| < \inf B + \varepsilon/2$$
$$\sup B - \varepsilon/2 < |x| \le \sup B$$

Which we can combine to get

$$\sup B - \inf B - \varepsilon < |x| - |y| \le ||x| - |y|| \le |x - y| \le \sup A - \inf A$$

where the last inequality comes from the fact that

$$x - y \le \sup A - \inf A \quad \text{and} \quad y - x \le \sup A - \inf A \implies |x - y| \le \sup A - \inf A$$

Then, since the inequality is true for all $\varepsilon > 0$, we get that

$$\sup B - \inf B \le \sup A - \inf A$$

**Problem 2** (4 points each) In this problem, we will look at some properties of the Riemann integral of the absolute value of a function.

(a) Suppose $f \in \mathscr{R}[a,b]$. Show that $|f| \in \mathscr{R}[a,b]$, and that

$$0 \leq \left| \int_a^b f \right| \leq \int_a^b |f|$$

(b) Find an example of a function $f : [a,b] \to \mathbb{R}$ such that $|f| \in \mathscr{R}[a,b]$ but $f \notin \mathscr{R}[a,b]$. (*Hint*: Think about the Dirichlet function)

(c) Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that if $f(c) > 0$ for some $c \in [a,b]$, then there exists some $\delta > 0$ such that

$$\int_{c-\delta}^{c+\delta} f > 0$$

(*Remark*: Try to be careful about strict/non-strict inequalities and open/closed intervals in this part.)

(d) Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that

$$\int_a^b |f| = 0$$

if and only if $f(x) = 0$ for all $x \in [a,b]$

**(a)** Using the result from problem 1, we have that for any $S \subset [a,b]$,

$$\sup_{x \in S} |f(x)| - \inf_{x \in S} |f(x)| \leq \sup_{x \in S} f(x) - \inf_{x \in S} f(x)$$

Thus for any partition $\mathcal{P}$ of $[a,b]$, we have that

$$\begin{aligned}
U(\mathcal{P}, |f|) - L(\mathcal{P}, |f|) &= \sum_{i=1}^{n} (M_i^{|f|} - m_i^{|f|}) \Delta x_i \\
&\leq \sum_{i=1}^{n} (M_i^{f} - m_i^{f}) \Delta x_i \\
&= U(\mathcal{P}, f) - L(\mathcal{P}, g)
\end{aligned}$$

Then, let $\varepsilon > 0$ be arbitrary. Since $f \in \mathscr{R}[a,b]$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(\mathcal{P}, |f|) - L(\mathcal{P}, |f|) \leq U(\mathcal{P}, f) - L(\mathcal{P}, g) < \varepsilon$$

which shows that $|f| \in \mathscr{R}[a,b]$ as well.

Then by monotonicity, since $-|f(x)| \le f(x) \le |f(x)|$ for all $x \in [a,b]$, we have

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

which gives the desired inequality

$$0 \le \left|\int_a^b f\right| \le \int_a^b |f|$$

**(b)** Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

Then, $|f|$ is a constant function, so $|f| \in \mathscr{R}[0,1]$. However, $f$ is not Riemann integrable (e.g. since $U(\mathcal{P}, f) = 1$ and $L(\mathcal{P}, f) = -1$ for any partition $\mathcal{P}$ of $[0,1]$).

**(c)** Let $f$ be as given. Since $f$ is continuous and $f(c) > 0$ for some $c \in [a,b]$, there exists some $\delta' > 0$ such that for all $x \in (c-\delta', c+\delta')$

$$-f(c)/2 < f(x) - f(c) < f(c)/2 \implies f(x) > f(c)/2$$

Take $\delta > 0$ such that $\delta < \delta'$ and $[c-\delta, c+\delta] \subset [a,b]$. Since the restriction of $f$ to the interval $[c-\delta, c+\delta]$ is continuous, $f \in \mathscr{R}[c-\delta, c+\delta]$. Then, using monotonicity of the Riemann integral,

$$\int_{c-\delta}^{c+\delta} f \ge (f(c)/2)(2\delta) > 0$$

**(d)** First, suppose $f(x) = 0$ for all $x \in [a,b]$. Then, the integral of the constant function gives

$$\int_a^b |f| = 0$$

Now, we will prove the other direction by contrapositive. Suppose $f(c) \ne 0$ for some $c \in [a,b]$. Since $|f|$ is continuous and $|f(c)| > 0$, by the result of (c) there exists some $\delta > 0$ such that

$$\int_{c-\delta}^{c+\delta} |f| > 0$$

Then, noting that $|f(x)| \ge 0$ for all $x \in [a,b]$, we have

$$\begin{aligned}\int_a^b |f| &= \int_a^{c-\delta} |f| + \int_{c-\delta}^{c+\delta} |f| + \int_{c+\delta}^b |f| \\ &\ge 0 + \int_{c-\delta}^{c+\delta} |f| + 0 > 0\end{aligned}$$

This shows the desired result.

**Problem 3** (5 points) Suppose $F$ and $G$ are continuously differentiable functions defined on $[a,b]$ such that $F'(x) = G'(x)$ for all $x \in [a,b]$. Using the fundamental theorem of calculus, show that $F$ and $G$ differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$

(Remark: This is justifying the "rule" of adding a constant $\int f + C$ to indefinite integration when you are computing an antiderivative. Make sure to use the right form of the fundamental theorem of calculus.)

Let $f(x) = F'(x) = G'(x)$. Since $F, G$ are continuously differentiable, $f$ is continuous and hence Riemann integrable on the interval $[a,b]$. Then using the first form of the fundamental theorem of calculus, we have for any $x \in (a,b]$

$$F(x) - F(a) = \int_a^x f = G(x) - G(a)$$

Then, $F(x) - G(x) = F(a) - G(a) = C$ for all $x \in [a,b]$.

# Sections 6.1-6.2 Exercises

**Problem 4** (4 points each) Practice with pointwise and uniform convergence.

(a) Let $f_n : (0,1) \to \mathbb{R}$ be given by $f_n(x) := \frac{n+1}{nx}$. Show that $\{f_n\}$ converges pointwise to a continuous function $f$, but the convergence is not uniform.

(Remark: This shows that pointwise convergence to a continuous function does not imply uniform convergence, so the "converse" to Theorem 6.2.2 is not true. It is also possible to find counterexamples using sequences of continuous functions on $[0,1]$)

(b) Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) := \begin{cases} 0 & x = 0 \\ n & 0 < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

Notice that $f_n \in \mathscr{R}[0,1]$ since it has a finite number of discontinuities. Show that $\{f_n\}$ converges pointwise to a function $f \in \mathscr{R}[0,1]$, but the convergence is not uniform (without using Theorem 6.2.4). Furthermore, show that

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$$

(c) Let $f_n(x) = \frac{x^n}{n}$. Show that $\{f_n\}$ converges uniformly to a differentiable function $f$ on $[0,1]$ (find $f$). However, show that $f'(1) \neq \lim\limits_{n\to\infty} f_n'(1)$.

**(a)** For fixed $x \in (0,1)$ we have by continuity of algebraic operations that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\frac{1}{x} + \frac{1}{nx}\right) = \frac{1}{x}$$

Thus, $\{f_n\}$ converges pointwise to the continuous function $f : (0,1) \to \mathbb{R}$ given by $f(x) = 1/x$.

To show that the convergence is not uniform, we attempt to compute

$$\sup_{x\in(0,1)} \left|\frac{n+1}{nx} - \frac{1}{x}\right| = \sup_{x\in(0,1)} \frac{1}{nx}$$

in fact, the function $\frac{1}{nx}$ is not bounded for any $n \in \mathbb{N}$. Hence, for any given $\varepsilon > 0$, there does not exist an $M \in \mathbb{N}$ such that $\sup_{x\in(0,1)} \left|\frac{n+1}{nx} - \frac{1}{x}\right| < \varepsilon$ for all $n \geq M$.

**(b)** First, note that for fixed $0 < x$, there exists $M_x \in \mathbb{N}$ such that $\frac{1}{M_x} < x$. Thus, $f_n(x) = 0$ for all $n \geq M_x$. $f_n(0) = 0$, so we have that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Thus, $\{f_n\}$ converges pointwise to $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$.

Then, we compute (e.g. by using additivity and the fundamental theorem of calculus)

$$\int_0^1 f_n = \int_0^{1/n} n \,\mathrm{d}x = 1$$

Then,

$$1 = \lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f = 0$$

**(c)** For $0 \leq x \leq 1$ we have that $0 \leq x^n \leq 1$ for $n \in \mathbb{N}$. So, given $\varepsilon > 0$ take $M \in \mathbb{N}$ such that $M > 1/\varepsilon$, then for all $n \geq M$ we have

$$\sup_{x\in[0,1]} \left| \frac{x^n}{n} - 0 \right| = \frac{1}{n} < \varepsilon$$

Hence, $\{f_n\}$ converges uniformly to $f(x) = 0$ on $[0,1]$.

Now, we compute the derivative of $f_n$ using e.g. the power rule (which can be proved inductively using the product rule), $f_n'(x) = x^{n-1}$. We also have $f'(x) = 0$, so

$$1 = \lim_{n\to\infty} f_n'(1) \neq f'(1) = 0$$

**Problem 5** (3 points each) Let $f$ and $g$ be bounded functions on $[a, b]$.

(a) Prove the triangle inequality for the uniform norm,

$$\|f + g\|_u \leq \|f\|_u + \|g\|_u$$

(b) Using your result in (a), prove the reverse triangle inequality for the uniform norm,

$$|\|f\|_u - \|g\|_u| \leq \|f - g\|_u$$

(Hint: Your proof will look very similar to the proof of the reverse triangle inequality for the absolute value)

**(a)** Since $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all $x \in [a, b]$, we use Proposition 1.3.7 and HW7 Problem 3c to get

$$\begin{aligned} \|f + g\|_u &= \sup_{x \in [a,b]} (|f(x) + g(x)|) \\ &\leq \sup_{x \in [a,b]} (|f(x)| + |g(x)|) \\ &\leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)| = \|f\|_u + \|g\|_u \end{aligned}$$

**(b)** Note that we have

$$\|-f\|_u = \sup_{x \in [a,b]} |-f(x)| = \sup_{x \in [a,b]} |f(x)| = \|f\|_u$$

So, using the triangle inequality, we have that

$$\|f\|_u = \|(f - g) + g\|_u \leq \|f - g\|_u + \|g\|_u \implies \|f\|_u - \|g\|_u \leq \|f - g\|_u$$
$$\|g\|_u = \|(g - f) + f\|_u \leq \|g - f\|_u + \|f\|_u \implies \|f\|_u - \|g\|_u \geq -\|f - g\|_u$$

Combining these two inequalities, we have

$$-\|f - g\|_u \leq \|f\|_u - \|g\|_u \leq \|f - g\|_u \implies |\|f\|_u - \|g\|_u| \leq \|f - g\|_u$$

**Problem 6** (6 points) Consider the sequence of continuous functions $\{f_n\}$ on $[0,1]$ given by

$$f_n(x) := \begin{cases} 1 - nx & 0 \leq x < 1/n \\ 0 & 1/n \leq x \leq 1 \end{cases}$$

Show that $\{f_n\}$ has no subsequence which is convergent in uniform norm.

(*Hint*: Show that every subsequence of $\{f_n\}$ converges pointwise to some function. Can the subsequences converge uniformly?)

(*Remark*: This is an example of a sequence of continuous functions bounded in the uniform norm which has no convergent subsequence.)

For fixed $x \in (0,1]$ there exists $N \in \mathbb{N}$ such that $1/N \leq x$. Then, $f_n(x) = 0$ for all $n \geq N$. Thus, $\{f_n\}$ converges pointwise to a function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$$

Since the sequences $\{f_n(x)\}$ converge to $f(x)$ for fixed $x \in [0,1]$, any subsequence $\{f_{n_j}(x)\}_{j=1}^{\infty}$ will also converge to $f(x)$. Thus, the subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ also converges pointwise to $f$.

Since each of the $\{f_n\}$ are continuous and bounded, if any subsequence $\{f_{n_j}\}$ converged in uniform norm it would be to a continuous function $g : [0,1] \to \mathbb{R}$. Since uniform convergence implies pointwise convergence, and since the pointwise limit is unique (since limits of sequences of real numbers are unique), $g = f$. However, $f$ is not continuous, so this is a contradiction. Thus, no subsequence $\{f_{n_j}\}$ can be convergent in uniform norm.