Homework 2 Solutions

Due: Monday, September 26th by 11:59 PM ET

Sections 1.2-1.3 Exercises

Problem 1 (4 points) Prove that for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, there is a rational $r \in \mathbb{Q}$ such that $|x - r| < \epsilon$

(*Remark*: This is an alternate way of stating that $\mathbb{Q}$ is dense in $\mathbb{R}$. It says that any real number x can be approximated by a rational number r to an arbitrarily good accuracy ε.

You may use the results of Theorem 1.2.4.ii here, although you should try to understand the proof.)

By Prop. 1.3.1 from the text, $|x - r| < \epsilon$ if and only if $-\epsilon < x - r < \epsilon$, which is equivalent to $x - \epsilon < r < x + \epsilon$.

Since $x - \epsilon < x + \epsilon$ for any $x \in \mathbb{R}$ and $\varepsilon > 0$, by Theorem 1.2.4 ($\mathbb{Q}$ is dense in $\mathbb{R}$), there exists a rational number $r \in \mathbb{Q}$ which satisfies the inequality $x - \epsilon < r < x + \epsilon$, which is equivalent to the original requirement that $|x - r| < \epsilon$.

Problem 2 (4 points) Let $x, y \in \mathbb{R}$. Suppose that for all $\varepsilon > 0$, we have $|x - y| \leq \varepsilon$. Prove that $x = y$.

(*Remark*: This is another way to show $x = y$, and can be easier than proving $x \leq y$ and $x \geq y$ in some cases.)

We will prove the contrapositive. Suppose $x \neq y$, and therefore $x - y \neq 0$. Then, by Prop. 1.3.1 from the text, $|x - y| > 0$. However, if we set $\varepsilon = \frac{1}{2}|x - y|$, we have found an $\varepsilon > 0$ such that $|x - y| > \frac{1}{2}|x - y| = \varepsilon$.

In other words, $x \neq y$ implies there exists $\varepsilon > 0$ such that $|x - y| > \varepsilon$. The contrapositive of this statement is that if for all $\varepsilon > 0$ we have $|x - y| \leq \varepsilon$, then this implies $x = y$.

Problem 3 (2 points each) Suppose $f, g : D \to \mathbb{R}$ are bounded functions.

(a) Show that $f + g : D \to \mathbb{R}$ defined by $(f + g)(x) := f(x) + g(x)$ is bounded.

(b) Show that $fg : D \to \mathbb{R}$ defined by $(fg)(x) := f(x)g(x)$ is bounded.

(c) Suppose $h : D \to \mathbb{R}$ is a function (not necessarily bounded) with the property that there exists $c > 0$ such that $|h(x)| \geq c$ for all $x \in D$. Show that $f/h : D \to \mathbb{R}$ defined as $(f/h)(x) := f(x)/h(x)$ is bounded.

(For example, $f(x) = 1$ and $h(x) = 1 + x^2$ are both defined on $\mathbb{R}$, and the function $f(x)/h(x) = 1/(1 + x^2)$ is bounded.)

(d) Suppose $h : E \to D$ is a function (not necessarily bounded). Show that the composition $f \circ h : E \to \mathbb{R}$ is bounded.

(For example, given $f(x) = \cos(x)$ and $h(x) = x^2$, both functions $\mathbb{R} \to \mathbb{R}$, the composition $f(h(x)) = \cos(x^2)$ is bounded)

(a) Since f and g are bounded, let $M, M' \in \mathbb{R}$ be numbers such that $|f(x)| \leq M$ and $|g(x)| \leq M'$ for all $x \in D$. Then, for all $x \in D$,

$$|(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + |g(x)| \leq M + M'$$

Thus, $f + g$ is bounded by $M + M'$.

(b) Again, let $M, M' \in \mathbb{R}$ be bounds for f, g respectively. Then, for all $x \in D$,

$$|(fg)(x)| = |f(x)g(x)| = |f(x)||g(x)| \leq M|g(x)| \leq MM'$$

Thus, fg is bounded by MM'

(c) Let $M \in \mathbb{R}$ be a bound for f, and let h and c be as given. The function f/h is well-defined, since $h(x) \neq 0$ for all $x \in D$. Then,

$$\left|\frac{f(x)}{h(x)}\right| = \frac{|f(x)|}{|h(x)|} \leq \frac{M}{|h(x)|} \leq \frac{M}{c}$$

Thus, f/h is bounded by M/c

(d) Let $M \in \mathbb{R}$ be a bound for f, and let h be as given. Then, for all $x \in E$, we have

$$|(f \circ h)(x)| = |f(h(x))| \leq M$$

Thus, $f \circ h$ is bounded by M.

Problem 4 (5 points) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded functions where D is non-empty and $f(x) \le g(x)$ for all $x \in D$. Prove that

$$\sup_{x\in D} f(x) \le \sup_{x\in D} g(x) \qquad \text{and} \qquad \inf_{x\in D} f(x) \le \inf_{x\in D} g(x)$$

Use this result to show that if M is a bound for the function $|f| : D \to \mathbb{R}$ defined as $|f|(x) := |f(x)|$, then

$$-M \le \inf_{x\in D} -|f(x)| \le \inf_{x\in D} f(x) \le \sup_{x\in D} f(x) \le \sup_{x\in D} |f(x)| \le M$$

If b is any upper bound for $g(D)$, then $f(x) \le g(x) \le b$ for all $x \in D$, so b is also an upper bound for $f(D)$. Since $\sup_{x\in D} g(x)$ is an upper bound of $g(D)$, and $\sup_{x\in D} f(x)$ is the least upper bound of $f(D)$

$$\sup_{x\in D} f(x) = \sup f(D) \le \sup_{x\in D} g(x)$$

Similarly, if c is any lower bound for $f(D)$, then $g(x) \ge f(x) \ge c$ for all $x \in D$, so c is also a lower bound for $g(D)$. Since $\inf_{x\in D} f(x)$ is a lower bound of $f(D)$, and $\inf_{x\in D} f(x)$ is the greatest lower bound of $f(D)$

$$\inf_{x\in D} g(x) = \inf g(D) \ge \inf_{x\in D} f(x)$$

Now, suppose M is a bound for $|f|$. Then, by the properties of the absolute value and the definition of a bound, we have that for all $x \in D$

$$-M \le -|f(x)| \le f(x) \le |f(x)| \le M$$

Thus, we can apply the conclusion of the first part of the problem to the inf of the functions $g_1(x) = -M$, $g_2(x) = -|f(x)|$, and $f(x)$ to get

$$-M = \inf_{x\in D} g_1(x) \le \inf_{x\in D} g_2(x) \le \inf_{x\in D} f(x)$$

We apply it again to the sup of the functions $g_3(x) = |f(x)|$, and $g_4(x) = M$,

$$\sup_{x\in D} f(x) \le \sup_{x\in D} g_3(x) \le \sup_{x\in D} g_4(x) = M$$

Finally, we use the fact (proved in HW1) that $\inf f(D) \le \sup f(D)$ to get

$$\inf_{x\in D} f(x) \le \sup_{x\in D} f(x)$$

which gives all of the desired inequalities.

Section 2.1 Exercises

Problem 5 (3 points each) In this problem, we will see another helpful way of interpreting the definition of the limit of a sequence.

Definition. Given a sequence $\{x_n\}$ and a predicate $P(x)$, we say $P(x_n)$ *holds at most finitely often* if there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $P(x_n)$ is false.

If $P(x_n)$ does not hold finitely often, we say that $P(x_n)$ *holds infinitely often.*

Example. For the sequence $\{x_n\} := \{1/n\}$, $x_n \geq 1/2$ at most finitely often since for all $n \geq 3$, $x_n < 1/2$.

(a) Let $\{x_n\} := \{(-1)^n\}$. For what values of $\varepsilon \in \mathbb{R}$ does the inequality $|x_n - 1| \geq \varepsilon$ hold at most finitely often?

(b) Let $\{x_n\} := \{(n+1)/n\}$. For what values of $\varepsilon \in \mathbb{R}$ does the inequality $|x_n - 1| \geq \varepsilon$ hold at most finitely often?

(c) Let $\{x_n\}$ be a sequence. Show that $\{x_n\}$ converges to some $L \in \mathbb{R}$ if and only if for all $\varepsilon > 0$, $|x_n - L| \geq \varepsilon$ at most finitely often.

(a) Note that

$$|x_n - 1| = |(-1)^n - 1| = \begin{cases} 2 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Thus, for any $\varepsilon \geq 2$, we have that $|x_n - 1| \geq \varepsilon$ at most finitely often (i.e. take $M = 1$).

On the contrary, for any $\varepsilon < 2$ and for any $M \in \mathbb{N}$, we can find some $n \geq M$ (e.g. $n = 2M + 1$) such that $|x_n - 1| = 2 \geq \varepsilon$. Thus, $P(x_n)$ holds infinitely often for $\varepsilon < 2$.

(b) Note that

$$|x_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \frac{1}{n}$$

By the Archimedean property, for any $\varepsilon > 0$ there exists some $M \in \mathbb{N}$ such that $M\varepsilon > 1$, i.e. $1/M < \varepsilon$. Then, for all $n \geq M$,

$$\frac{1}{n} \leq \frac{1}{M} < \varepsilon$$

Thus, $|x_n - 1| \geq \varepsilon$ at most finitely often for any $\varepsilon > 0$.

On the other hand, if $\varepsilon \leq 0$, since $|x_n - 1| = 1/n \geq 0 \geq \varepsilon$ for all $n \in \mathbb{N}$, we have that $|x_n - 1| \geq \varepsilon$ infinitely often.

(c) ($\Longrightarrow$) First, assume $\{x_n\}$ converges to $L \in \mathbb{R}$. Then, for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$|x_n - L| < \varepsilon$$

This is exactly the definition that $|x_n - L| \geq \varepsilon$ at most finitely often.

($\Longleftarrow$) Suppose that for all $\varepsilon > 0$, $|x_n - L| \geq \varepsilon$ at most finitely often.

Let $\varepsilon > 0$ be given, by definition of at most finitely often, there exists some $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$|x_n - L| < \varepsilon$$

thus, $\{x_n\}$ converges to L.

Problem 6 (2.5 points each) Determine if the following sequences converge, and if they do, find the limit. Feel free to use what you know from Calculus to guess the limit, but remember to prove that it is the correct limit.

(a) $\{n^2\}$

(b) $\{x_n\}$ where $x_n := \begin{cases} \sin(n^2) & n < 1000000 \\ 0 & n \geq 1000000 \end{cases}$

(c) $\left\{\frac{4n}{4n+1}\right\}$

(d) $\left\{\frac{2n}{n^2+1}\right\}$

(a) $\{n^2\}$ is unbounded (i.e. since any upper bound t would satisfy $t \geq n^2 \geq n$ for all $n \in \mathbb{N}$, which would imply $\mathbb{N}$ was bounded above), so it is divergent.

(b) Let $\varepsilon > 0$ be given. Then for any $n \geq M = 1000000 \in \mathbb{N}$, we have

$$|x_n - 0| = 0 < \varepsilon$$

thus $\{x_n\}$ converges to 0.

(c) Let $\varepsilon > 0$ be given. By the Archimedean property, there exists $M \in \mathbb{N}$ such that $1/4M < \varepsilon$. Then for any $n \geq M$, we have

$$\left|\frac{4n}{4n+1} - 1\right| = \frac{1}{4n} \leq \frac{1}{4M} < \varepsilon$$

Thus, $\{x_n\}$ converges to 1.

(d) Let $\varepsilon > 0$ be given. By the Archimedean property, there exists $M \in \mathbb{N}$ such that $2/M < \varepsilon$. Then for all $n \geq M$,

$$\left|\frac{2n}{n^2+1}\right| = \frac{2n}{n^2+1} \leq \frac{2n}{n^2} \leq \frac{2}{M} < \varepsilon$$

Thus $\{x_n\}$ converges to 0.

Problem 7 (5 points) Given a sequence $\{x_n\}$ and a number $L \in \mathbb{R}$, show that the statement

(i) for every $\varepsilon' > 0$, there exists an $M' \in \mathbb{N}$ such that $|x_n - L| \leq \varepsilon'$ for all $n \geq M'$

is true if and only if the statement

(ii) for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq M$

is true. In other words, show that it does not matter whether we use a strict inequality or non-strict inequality in the definition of the limit.

(ii) $\implies$ (i): Assume that (ii) holds and let $\varepsilon' = \varepsilon$ and $M' = M$. Then, for every $\varepsilon' = \varepsilon > 0$, there exists $M' = M \in \mathbb{N}$ such that $|x_n - L| < \varepsilon \leq \varepsilon = \varepsilon'$. Thus, (i) holds.

(i) $\implies$ (ii): Assume that (i) holds. For every $\varepsilon > 0$, set $\varepsilon' = \varepsilon/2 > 0$. Then, by (i), there exists $M' \in \mathbb{N}$ such that $|x_n - L| \leq \varepsilon' < 2\varepsilon' = \varepsilon$. Thus, by choosing $M = M'$, (ii) holds.

Problem 8 (5 points) Show that every real number $x \in \mathbb{R}$ is the limit of some sequence of rational numbers. In other words, for any given $x \in \mathbb{R}$, show there exists a sequence $\{r_n\}$ such that (i) $r_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, and (ii) $\lim_{n\to\infty} r_n = x$.

(*Hint*: Try using the results of problem 1 with a sequence of "approximation errors" $\{\epsilon_n\}$ which converges to a limit of 0.)

Let $\{\epsilon_n\}$ be any sequence that satisfies $\epsilon_n > 0$ for all $n \in \mathbb{N}$, and has $\lim_{n\to\infty} \epsilon_n = 0$. For example, $\epsilon_n = \frac{1}{n}$ works.

By the result of problem 1, for each $\epsilon_n > 0$ there exists a rational number $r_n \in \mathbb{Q}$ such that $|r_n - x| < \epsilon_n$. This gives us a sequence of rational numbers $\{r_n\}$.

Since $\lim_{n\to\infty} \epsilon_n = 0$, and $\epsilon_n > 0$, this means that for every $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that $|\epsilon_n| = \epsilon_n < \varepsilon$ for all $n \geq M$. Therefore, $|r_n - x| < \epsilon_n < \varepsilon$ for all $n \geq M$. This shows that $\lim_{n\to\infty} r_n = x$ as desired.