Homework 4 Solutions

Due: Monday, October 17th by 11:59 PM ET

Sections 2.3-2.5 Exercises

Problem 1 (4 points each) For each of the following sequences $\{x_n\}$, find the lim sup and lim inf, and use them to determine if the original sequence $\{x_n\}$ converges or diverges.

(a) $\{(-c)^n\}$ for $0 < c < 1$

(b) $\left\{(-1)^n + \dfrac{1}{n}\right\}$

(a) By induction, since $0 < c < 1$ we have $0 < c^n < c$ for all $n \in \mathbb{N}$ such that $n \geq 2$. Hence, we have

$$a_n := \sup\{(-c)^k : k \geq n\} = \begin{cases} c^{n+1} & n \text{ odd} \\ c^n & n \text{ even} \end{cases}$$

Then, since $0 < a_n \leq c^n$ for all $n \in \mathbb{N}$, and since $\lim\limits_{n\to\infty} c^n = 0$, by the squeeze lemma we have that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n = 0$$

Similarly,

$$b_n := \inf\{(-c)^k : k \geq n\} = \begin{cases} -c^n & n \text{ odd} \\ -c^{n+1} & n \text{ even} \end{cases}$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n = 0$$

Since $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = 0$, the sequence $\{(-c)^n\}$ converges to 0.

(b) Note that

$$\sup\{1 + \tfrac{1}{k} : k \geq n\} = 1 + \sup\{\tfrac{1}{k} : k \geq n\} = 1 + \tfrac{1}{n} > 0$$

and for odd n, $x_n \leq 0$. Thus,

$$a_n := \sup\{(-1)^k + \tfrac{1}{k} : k \geq n\} = \begin{cases} 1 + \frac{1}{n} & n \text{ even} \\ 1 + \frac{1}{n+1} & n \text{ odd} \end{cases}$$

$\lim_{n\to\infty} 1 + \frac{1}{n} = 1$, and since $\{1 + \frac{1}{n+1}\}$ is a subsequence of $\{1 + \frac{1}{n}\}$ it also converges to 1. Since $1 + \frac{1}{n+1} \leq a_n \leq 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, by the squeeze lemma

$$\limsup_{n\to\infty} x_n = 1$$

Now, note that

$$\inf\{-1 + \tfrac{1}{k} : k \geq n\} = -1 + \inf\{\tfrac{1}{k} : k \geq n\} = -1$$

and for even n, $x_n \geq 0$. Thus,

$$b_n := \inf\{(-1)^k + \tfrac{1}{k} : k \geq n\} = -1$$

Thus,

$$\liminf_{n\to\infty} x_n = -1$$

Since $\liminf_{n\to\infty} x_n \neq \limsup_{n\to\infty} x_n$, the sequence $\{x_n\}$ does not converge.

Problem 2 (4 points) Prove that $\{1/n^2\}$ is Cauchy using directly the definition of a Cauchy sequence.

(Hint: How would you prove $\{1/n^2\}$ converges using the original limit definition? Then, try looking at example 2.4.2 in the textbook.)

Let $\varepsilon > 0$ be given. Then, there exists $M \in \mathbb{N}$ such that $M\varepsilon > 2$. Since $M \geq 1$, this implies $\frac{1}{M^2} \leq \frac{1}{M} < \frac{\varepsilon}{2}$. Thus for all $n, k \geq M$,

$$\left|\frac{1}{n^2} - \frac{1}{k^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{k^2}\right| \leq \frac{1}{M^2} + \frac{1}{M^2} < \varepsilon$$

Thus, $\{1/n^2\}$ is Cauchy.

Problem 3 (4 points each) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences.

(a) Show that there exists a subsequence $\{x_{n_i}+y_{n_i}\}_{i=1}^{\infty}$ of the sequence $\{x_n+y_n\}_{n=1}^{\infty}$ which converges to $\liminf_{n\to\infty}(x_n+y_n)$

(b) Show that

$$\liminf_{n\to\infty}(x_n+y_n) \geq \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$$

(Hint: The subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ from (a) does not necessarily converge, but try considering a convergent subsequence of $\{x_{n_i}\}_{i=1}^{\infty}$, then using what you know about adding/subtracting limits.)

(c) Give a explicit example of bounded sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\liminf_{n\to\infty}(x_n+y_n) > \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$$

(Hint: The left-hand side and right-hand side are equal if $\{x_n\}$ and $\{y_n\}$ are convergent, so consider non-convergent sequences)

(a) If $B, B' > 0$ are bounds for $\{x_n\}$ and $\{y_n\}$ respectively, then $|x_n+y_n| \leq |x_n|+|y_n| = B+B'$, so $\{x_n+y_n\}_{n=1}^{\infty}$ is bounded with a bound $B+B'$.

Since $\{x_n+y_n\}_{n=1}^{\infty}$ is bounded, by Theorem 2.3.4 in the book there exists a subsequence $\{x_{n_i}+y_{n_i}\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty}(x_{n_i}+y_{n_i}) = \liminf_{n\to\infty}(x_n+y_n)$$

(b) Since $\{x_n\}_{n=1}^{\infty}$ is bounded, the subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ (n_i the same as in (a)) is also bounded, and the subsequence has a subsequence $\{x_{n_{i_k}}\}_{k=1}^{\infty}$ which converges with limit equal to $\liminf_{k\to\infty} x_{n_{i_k}}$.

Since $\{x_{n_{i_k}}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, and $\{y_{n_{i_k}}\}_{k=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$, by Proposition 2.3.6 in the book

$$\liminf_{k\to\infty} x_{n_{i_k}} \geq \liminf_{n\to\infty} x_n$$

$$\liminf_{k\to\infty} y_{n_{i_k}} \geq \liminf_{n\to\infty} y_n$$

Since the sequence $\{x_{n_i}+y_{n_i}\}_{i=1}^{\infty}$ is convergent, the subsequence $\{x_{n_{i_k}}+y_{n_{i_k}}\}_{k=1}^{\infty}$ will also be convergent to the same limit. Thus, the difference $\{(x_{n_{i_k}}+y_{n_{i_k}})-x_{n_{i_k}}\}_{k=1}^{\infty} = \{y_{n_{i_k}}\}_{k=1}^{\infty}$ is a convergent sequence with limit equal to $\liminf_{k\to\infty} y_{n_{i_k}}$.

Then, we have that

$$\liminf_{n\to\infty}(x_n+y_n) = \lim_{k\to\infty}\left(x_{n_{i_k}}+y_{n_{i_k}}\right) = \liminf_{k\to\infty} x_{n_{i_k}} + \liminf_{k\to\infty} y_{n_{i_k}} \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$

which is the desired result.

(c) Consider $\{x_n\} := \{(-1)^n\}$ and $\{y_n\} := \{(-1)^{n+1}\}$. Then,

$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = -1$$

and $\{x_n + y_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$ is a constant sequence, which converges to 0, so

$$\limsup_{n\to\infty} (x_n + y_n) = 0 > \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n = -2$$

Problem 4 (4 points each) This question looks at some of the assumptions of the Bolzano-Weierstrass theorem.

(a) The Bolzano-Weierstrass theorem says that any *bounded* sequence has a convergent subsequence.

Find an example of an unbounded sequence $\{x_n\}$ which has the property that every subsequence diverges.

(b) The proof of the Bolzano-Weierstrass theorem relies on the lowest upper bound property of $\mathbb{R}$. We proved in class that $\mathbb{Q}$ does not have the lowest upper bound property, hence we cannot prove a "rational" version of Bolzano-Weierstrass for rational sequences.

Show that there exists a bounded sequence of rational numbers $\{r_n\}$ (that is, $r_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$) such that no subsequence of $\{r_n\}$ converges to a rational number.

(Hint: Use the results of HW2 Problem 8)

(a) Consider the sequence $\{n\}_{n=1}^{\infty}$. It is unbounded, since the set of natural numbers $\mathbb{N}$ is unbounded. Every subsequence is of the form $\{n_i\}_{i=1}^{\infty}$. Since $n_i \geq i$ for all $i \in \mathbb{N}$, the sequence $\{n_i\}_{i=1}^{\infty}$ is unbounded as well (otherwise it would imply the natural numbers are bounded). Hence, every subsequence of $\{n\}_{n=1}^{\infty}$ is unbounded, and hence divergent.

(b) Let $x = \sqrt{2} \in \mathbb{R}$. In the textbook it was proved that $\sqrt{2} \notin \mathbb{Q}$. By the result of HW2 Problem 8, there exists a sequence of rational numbers $\{r_n\}$ which converges to x. Since $\{r_n\}$ is convergent, it is bounded, and every subsequence will also converge to the same limit $x \notin \mathbb{Q}$.

Problem 5 (4 points each) This question looks at the relationship between Bolzano-Weierstrass and the "Cauchy completeness" property of $\mathbb{R}$.

(a) Directly use the Bolzano-Weierstrass theorem (Theorem 2.3.8) to prove that every Cauchy sequence of real numbers is convergent. That is, only make use of the fact that every bounded sequence of real numbers has some convergent subsequence (not necessarily converging to either lim sup/lim inf).

(*Remark*: Proving Cauchy-completeness from Bolzano-Weierstrass without lim sup/inf in this way generalizes more easily to $\mathbb{R}^n$.)

(b) One interesting thing about the Cauchy sequence definition is that it can be stated without reference to real numbers at all:

We say a sequence of rational numbers $\{r_n\}$ is Cauchy (in the absolute value metric on the rational numbers) if for every $\varepsilon \in \mathbb{Q}$ satisfying $\varepsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $|r_n - r_k| < \varepsilon$.

Prove that $\mathbb{Q}$ is not Cauchy complete, that is, show that there exists a Cauchy sequence $\{r_n\}$ which does not converge to some limit $r \in \mathbb{Q}$.

(*Hint*: Look at problem 4b)

Remark: One way to 'construct' the real numbers (i.e. prove theorem 1.2.1) is to define elements of $\mathbb{R}$ via an equivalence relation on Cauchy sequences of rational numbers.

(a) Suppose $\{x_n\}$ is a Cauchy sequence. Then, it is bounded, so by Bolzano-Weierstrass there exists a convergent subsequence $\{x_{n_k}\}$. Define

$$x := \lim_{k\to\infty} x_{n_k}$$

Let $\varepsilon > 0$ be arbitrary. Then, since $\{x_{n_k}\}$ converges to x, we have that there exists some $M_1 \in \mathbb{N}$ such that for all $k \geq M_1$,

$$|x_{m_k} - x| < \varepsilon/2$$

Note that $m_k \geq k$.

Since $\{x_n\}$ is Cauchy, we have that there exists some $M_2 \in \mathbb{N}$ such that for all $n, k \geq M_2$,

$$|x_n - x_k| < \varepsilon/2$$

Take $M := \max\{M_1, M_2\}$. Then, for all $n \geq M$, we can take some $m_k \geq k \geq M$ such that

$$|x_n - x| = |x_n - x_{m_k} + x_{m_k} - x| \leq |x_n - x_{m_k}| + |x_{m_k} - x| < \varepsilon$$

which shows that $\{x_n\}$ converges (to x), as desired.

(b) Let $x = \sqrt{2} \in \mathbb{R}$. The textbook proves $\sqrt{2} \notin \mathbb{Q}$. By the result of HW2 problem 8, there exists a sequence of rational numbers $\{r_n\}$ which converges to x. Since $\{r_n\}$ is convergent, it is Cauchy (in the sense of the definition given for sequences of real numbers). Since $\mathbb{Q} \subset \mathbb{R}$, this shows that $\{r_n\}$ is Cauchy in the metric topology on the real numbers as well. However, $\{r_n\}$ does not converge to a limit $r \in \mathbb{Q}$, showing that $\mathbb{Q}$ is not Cauchy complete.

Problem 6 (4 points) Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Prove that

$$\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$$

Since $x_n \leq y_n$, by induction the partial sums satisfy for all $k \in \mathbb{N}$

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_k$$

Since limits preserve non-strict inequalities, the limit of the partial sums will satisfy

$$\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_k$$

Problem 7 (6 points) Let $\{d_n\}$ be a sequence of digits, that is $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all $n \in \mathbb{N}$. Let us consider the decimal

$$0.d_1 d_2 d_3 ... := \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Show that the series $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ is absolutely convergent to a number $x \in [0, 1]$.

(*Hint*: Use the comparison test.)

First, we show the convergence of the series

$$\sum_{n=1}^{\infty} \frac{9}{10^n}$$

Following the proof of Proposition 2.5.5 in lecture, we can compute the partial sums

$$s_k := \sum_{n=1}^{k} \frac{9}{10^n} = \frac{9}{10} \sum_{n=0}^{k-1} \frac{1}{10^n} = \frac{9}{10} \frac{1 - (1/10)^{k-1}}{1 - 1/10}$$

Thus, the sequence of partial sums converges, with

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \lim_{n \to \infty} s_k = \frac{9}{10} \frac{1}{1 - 1/10} = 1$$

(*Remark*: Another accepted proof is to use the linearity of series (Proposition 2.5.12) and the convergence of the geometric series (Proposition 2.5.5; covered in lecture). We haven't covered those in lecture, but it is okay to use them, as long as you mention them in your proof.)

Now, proceeding to the problem, for all $n \in \mathbb{N}$ we have the inequality

$$0 \leq \frac{d_n}{10^n} \leq \frac{9}{10^n}$$

Since the series $\sum \frac{9}{10^n}$ converges, by the comparison test (Proposition 2.5.16) we have that $\sum \frac{d_n}{10^n}$ converges as well. Then, we can define

$$x := \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Note this series is absolutely convergent, since $\left|\frac{d_n}{10^n}\right| = \frac{d_n}{10^n}$.

Now, using the fact that

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$$

we can use the result of Problem 6 (twice, to get the two inequalities) to conclude that

$$0 = \sum_{n=1}^{\infty} 0 \leq x \leq \sum_{n=1}^{\infty} \frac{9}{10^n} = 1$$

Hence, $x \in [0, 1]$.