Xi Liu

1
(a)
definition 3.5.1
for $f(x)$ to converge to L as $x \to -\infty$
$\forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall x \leq M, |f(x) - L| < \varepsilon$

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0$$
$$\left|\frac{1}{1+x^2} - 0\right| < \varepsilon$$
$$\frac{1}{1+x^2} < \varepsilon$$
$$1 + x^2 > \frac{1}{\varepsilon}$$
$$x^2 > \frac{1}{\varepsilon} - 1$$
$$x > \sqrt{\frac{1}{\varepsilon} - 1}$$
$$\forall \varepsilon > 0, \exists M := \sqrt{\frac{1}{\varepsilon} - 1} \in \mathbb{R}, \forall x \geq M = \sqrt{\frac{1}{\varepsilon} - 1}$$
$$|f(x) - L| = \left|\frac{1}{1+x^2} - 0\right| < \varepsilon$$

$$\lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$
$$\left|\frac{1}{1+x^2} - 0\right| < \varepsilon$$
$$\frac{1}{1+x^2} < \varepsilon$$
$$1 + x^2 > \frac{1}{\varepsilon}$$
$$x^2 > \frac{1}{\varepsilon} - 1$$
$$x < -\sqrt{\frac{1}{\varepsilon} - 1}$$
$$\forall \varepsilon > 0, \exists M := -\sqrt{\frac{1}{\varepsilon} - 1} \in \mathbb{R}, \forall x \leq M = -\sqrt{\frac{1}{\varepsilon} - 1}$$
$$|f(x) - L| = \left|\frac{1}{1+x^2} - 0\right| < \varepsilon$$

(b)
definition 3.2.1, $S \subset \mathbb{R}, c \in S, f : S \to \mathbb{R}$ is continuous at c if
$\forall \varepsilon > 0, \exists \delta > 0, (x \in S \land |x - c| < \delta \to |f(x) - f(c)| < \varepsilon)$

since $\lim_{x\to\infty} f(x) = L$
$\forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall x \geq M, |f(x) - L| < \varepsilon$

since $\lim_{x\to-\infty} f(x) = L$
$\forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall x \leq M, |f(x) - L| < \varepsilon$

need to show $\forall \varepsilon > 0, \exists \delta > 0, (y \in S \land |y - 0| < \delta \to |g(y) - g(0)| < \varepsilon)$
① $\because \lim_{x\to\infty} f(x) = L, \exists M_0 \in \mathbb{R}, \forall 1/y \geq M_0, |f(1/y) - L| < \varepsilon$
② $\because \lim_{x\to-\infty} f(x) = L, \exists M_1 \in \mathbb{R}, \forall 1/y \leq M_1, |f(1/y) - L| < \varepsilon$
①, if $y > 0, 1/y \geq M_0, 0 < y \leq 1/M_0$
②, if $y < 0, 1/y \leq M_1, 0 > y \geq 1/M_1$
③, if $y = 0, |g(y) - g(0)| = |L - L| = 0 < \varepsilon$
combine ①, ②, $1/M_1 \leq y \leq 1/M_0$
let $M := \max\{|M_0|, |M_1|\}$
$-1/M \leq y \leq 1/M$
$|y| \leq 1/M$
$\therefore \exists \delta := 1/M$
$|y - 0| < \delta = 1/M \to |g(y) - g(0)| = |f(1/y) - L| < \varepsilon$

(c)
if f is continuous at 0
$\forall \varepsilon > 0, \exists \delta > 0, (x \in S \land |x - 0| < \delta \to |f(x) - f(0)| < \varepsilon)$
$1/y \in S \land |1/y - 0| < \delta \to |f(1/y) - f(0)| < \varepsilon$
$|1/y - 0| < \delta$
$|1/y| < \delta$
$-\delta < 1/y < \delta$
①, $-1/\delta > y$
②, $y > 1/\delta$

$\forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall y \geq M := 1/\delta, |g(y) - f(0)| = |f(1/y) - f(0)| < \varepsilon$
$\therefore \lim_{y\to\infty} g(y) = f(0)$

$\forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall y \leq M := -1/\delta, |g(y) - f(0)| = |f(1/y) - f(0)| < \varepsilon$
$\therefore \lim_{y\to-\infty} g(y) = f(0)$

$\therefore \lim_{y\to\infty} g(y) = \lim_{y\to-\infty} g(y) = f(0)$

2

(a)

part 1

let $\{x_n\}$ be a sequence satisfying $x_n \in [a,b], \forall n \in \mathbb{N}$

based on bolzano weierstrass theorem, since $\{x_n\}$ is bounded, there exists a convergent subsequence $\{x_{n_i}\}$, $\{x_{n_i}\}$ converge to a number $L \in \mathbb{R}$. since $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$, $\{x_{n_i}\}$ is a sequence satisfying $x_{n_i} \in [a,b], \forall n \in \mathbb{N}$

assume $[a,b]$ is not sequentially compact, then $L \notin [a,b]$, then $L < a$ or $L > b$

if $L > b, L - b > 0$, since $\{x_{n_i}\}$ converges to $L, \forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n \geq M, |x_{n_i} - L| < \varepsilon$

take $\varepsilon < L - b$

$|x_{n_i} - L| < \varepsilon < L - b$

$|x_{n_i} - L| < L - b$

$b - L < x_{n_i} - L < L - b$

$b - L < x_{n_i} - L$

$b < x_{n_i}$

which contradicts with $x_{n_i} \in [a,b]$, so the assumption that $[a,b]$ is not sequentially compact is false, so $\{x_n\}$ is sequentially compact

if $L < a, a - L > 0$, since $\{x_{n_i}\}$ converges to $L, \forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n \geq M, |x_{n_i} - L| < \varepsilon$

take $\varepsilon < a - L$

$|x_{n_i} - L| < \varepsilon < a - L$

$|x_{n_i} - L| < a - L$

$L - a < x_{n_i} - L < a - L$

$x_{n_i} - L < a - L$

$x_{n_i} < a$

which contradicts with $x_{n_i} \in [a,b]$, so the assumption that $[a,b]$ is not sequentially compact is false, so $[a,b]$ is sequentially compact

part 2

$\{x_n\} = \{b - \frac{1}{n}\}$ is a sequence satisfying $x_n \in K = (a,b)$ that converge to b (since $\forall \varepsilon > 0, \exists M \in \mathbb{N}, 0 < 1/M < \varepsilon, \forall n \geq M, |x_n - 0| = |1/n| = 1/n \leq 1/M < \varepsilon$, and based on continuity of algebraic operations 2.2.2), since $x_n \in (a,b)$, so $\{x_n\}$ is bounded, based on proposition 2.3.7, since $\{x_n\}$ is bounded sequence that converge to b, so every convergent subsequence $\{x_{n_k}\}$ converges to b, but $b \notin (a,b)$, so there does not exist a subsequence $\{x_{n_k}\}$ and point $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$, so (a,b) is not sequentially compact

(b)

contrapositive: if set K is not bounded, then the set K is not sequentially compact

for every sequence $x_n \in K$, since K is unbounded, based on definition 1.3.6

$x_n \geq n, x_{n_k} > n_k > k$

/* show for any subsequence $\{x_n\}_{i=1}^{\infty}, \forall i \in \mathbb{N}, n_i \geq i$

base step: $i = 1, 1 \leq n_1, \because n_1 \in \mathbb{N}$

inductive step: assume $i \le n_i$ is true, show $i+1 \le n_{i+1}$ is true

$$\begin{aligned} i &\le n_i \\ i+1 &\le n_i + 1 \\ &\le n_{i+1} \end{aligned}$$

$\because n_i, n_{i+1} \in \mathbb{N}, n_i < n_{i+1}$ implies $n_i \neq n_{i+1}$
least possible n_{i+1} is $n_i + 1$ $\therefore n_i + 1 \le n_{i+1}$ */
$\because x_{n_k} > k$, every subsequence $\{x_{n_k}\}$ diverges, so there is no subsequence $\{x_{n_k}\}$ and point $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$, so $K \subset \mathbb{R}$ is not sequentially compact

(c)
let $s := \sup(K)$
since $\mathbb{Q}$ is dense in $\mathbb{R}$
$\exists x_n \in (s - 1/n, s)$
$s - 1/n < x_n < s$
$\because \lim_{n\to\infty} s - 1/n = s$
$\lim_{n\to\infty} s = s$
based on squeeze lemma 2.2.1
$\lim_{n\to\infty}\{x_n\} = s$
based on proposition 2.3.7, since $\{x_n\}$ is a bounded sequence that converge to s, every convergent subsequence $\{x_{n_k}\}$ converges to s, since K is sequentially compact, so every sequence in K has a convergent subsequence converging to a point in K, so s is a point in K, $s = \sup K \in K$

let $i := \inf(K)$
since $\mathbb{Q}$ is dense in $\mathbb{R}$
$\exists x_n \in (i, i + 1/n)$
$i < x_n < i + 1/n$
$\because \lim_{n\to\infty} i = i$
$\lim_{n\to\infty} i + 1/n = i$
based on squeeze lemma 2.2.1
$\lim_{n\to\infty}\{x_n\} = i$
based on proposition 2.3.7, since $\{x_n\}$ is a bounded sequence that converge to i, every convergent subsequence $\{x_{n_k}\}$ converges to i, since K is sequentially compact, so every sequence in K has a convergent subsequence converging to a point in K, so i is a point in K, $i = \inf K \in K$

(d)
definition 3.2.1, $S \subset \mathbb{R}, c \in S, f : S \to \mathbb{R}$ is continuous at c if
$\forall \varepsilon > 0, \exists \delta > 0, (x \in S \land |x - c| < \delta \to |f(x) - f(c)| < \varepsilon)$
definition 0.3.13, direct image $f : K \to \mathbb{R}$, direct image is $f(K) = \{f(x) \in \mathbb{R} : x \in K\}$

part 1
show every sequence in $f(K)$ has a convergent subsequence converging to a point in $f(K)$
pick a sequence $\{y_n\} \subset f(K)$, $y_n \in f(K)$, so $y_n = f(x_n)$ for some $x_n \in K$

since K is sequentially compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ and a point $x \in K$ such that $x_{n_k} \to x \in K$ as $k \to \infty$, bolzano weierstrass theorem 2.3.3

since f is continuous

$$y = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x)$$

$f(x_{n_k}) \to f(x) \in f(K), y \in f(K)$, so $f(K)$ is closed

for contradiction, assume $f(K)$ is not bounded
suppose there is a sequence $\{b_n\} \subset f(K)$ with $\forall n \in \mathbb{N}, n \leq b_n$
suppose there is a sequence $\{a_n\} \subset K$ with $f(a_n) = b_n, \{a_n\}$ is contained in K so have a subsequence a_{n_k} converging to some $a \in K$, but since $f(a) \in \mathbb{R}$

$$\lim_{k\to\infty} n_k \leq \lim_{k\to\infty} f(a_{n_k}) = f(a)$$

which is a contradiction, so $f(K)$ is bounded
$f(x_{n_k})$ is a subsequence of $\{y_n\}$
so $f(K)$ is sequentially compact

part 2
based on what is shown in problem 2.(a), $[a, b]$ is sequentially compact, so K is a closed and bounded interval
since f is a continuous function on a closed and bounded interval K, based on extreme value (min max) theorem 3.3.2, f achieves both an absolute minimum and an absolute maximum on K

3
(a)
let $x_0, c, x \in [a, b], x_0 < c < x$
use taylor's theorem at x, with $n = 0$
$f(x) = P_0^{x_0}(x) + f'(c)(x - x_0) = f(x_0) + f'(c)(x - x_0)$
replace x with $x + h$, replace x_0 with x
$f(x + h) = f(x) + f'(c)(x + h - x) = f(x) + f'(c)h$

$$\frac{f(x+h) - f(x)}{h} = f'(c)$$

let $c \in [x, x + h]$

based on mean value theorem 4.2.3

$$|f'(c) - f'(x)| = |f''(d)(c - x)| \leq |f''(d)|h$$

take $M := \max\{|f''(d)|\}, d \in [x, x + h]$

$$M_1 = -M, \quad M_2 = M$$

$$M_1 \le |f'(c) - f'(x)| \le M_2$$

(b)
let $x_0, c, x \in [a,b], x_0 < c < x$
use taylor's theorem at x, with $n = 2$

$$\begin{aligned}
①, f(x) &= P_2^{x_0}(x) + f^{(3)}(c)(x - x_0) \\
&= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(c)}{3!}(x - x_0)^3 \\
&\text{from ①, replace } x \text{ with } x+h, \text{ replace } x_0 \text{ with } x, \text{ replace } c \text{ with } c_1 \\
f(x+h) &= f(x) + f'(x)(x+h-x) + \frac{f''(x)}{2}(x+h-x)^2 + \frac{f^{(3)}(c_1)}{3!}(x+h-x)^3 \\
&= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f^{(3)}(c_1)}{3!}h^3 \\
&\text{from ①, replace } x \text{ with } x-h, \text{ replace } x_0 \text{ with } x, \text{ replace } c \text{ with } c_2 \\
f(x-h) &= f(x) + f'(x)(x-h-x) + \frac{f''(x)}{2}(x-h-x)^2 + \frac{f^{(3)}(c_2)}{3!}(x-h-x)^3 \\
&= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f^{(3)}(c_2)}{3!}h^3 \\
\frac{f(x+h) - f(x-h)}{2h} &= \frac{2f'(x)h + \frac{f^{(3)}(c_1)}{3!}h^3 - \frac{f^{(3)}(c_2)}{3!}h^3}{2h} \\
&= \frac{2f'(x)h + \frac{1}{6}(f^{(3)}(c_1)h^3 - f^{(3)}(c_2)h^3)}{2h} \\
&= f'(x) + \frac{1}{12}(f^{(3)}(c_1)h^2 - f^{(3)}(c_2)h^2) \\
&= f'(x) + \frac{f^{(3)}(c_1) - f^{(3)}(c_2)}{12}h^2 \\
f'(x) - \frac{f(x+h) - f(x-h)}{2h} &= f'(x) - \left(f'(x) + \frac{f^{(3)}(c_1) - f^{(3)}(c_2)}{12}h^2\right) \\
&= \frac{f^{(3)}(c_1) - f^{(3)}(c_2)}{12}h^2 \\
&\le \frac{|f^{(3)}(c_1)| + |f^{(3)}(c_2)|}{12}h^2 \\
&\le \frac{2\max\{|f^{(3)}(c_1)|, |f^{(3)}(c_2)|\}}{12}h^2 \\
\text{take } M &:= \frac{2\max\{|f^{(3)}(c_1)|, |f^{(3)}(c_2)|\}}{12}
\end{aligned}$$

$$M_3 = -M, \quad M_4 = M$$

(c)
part 1
let $x_0, c, x \in [a, b], x_0 < c < x$
use taylor's theorem at x, with $n = 1$

$$
\begin{aligned}
① , f(x) &= P_1^{x_0}(x) + f''(c)(x - x_0) \\
&= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2 \\
&\text{from ①, replace } x \text{ with } x + h, \text{ replace } x_0 \text{ with } x, \text{ replace } c \text{ with } c_1 \\
&c_1 \in [x, x + h] \\
f(x + h) &= f(x) + f'(x)(x + h - x) + \frac{f''(c_1)}{2!}(x + h - x)^2 \\
f(x + h) &= f(x) + f'(x)h + \frac{f''(c_1)}{2!}h^2 \\
-\frac{f''(c_1)}{2!}h^2 &= f'(x)h - (f(x + h) - f(x)) \\
\left|-\frac{f''(c_1)}{2!}h\right| &= \left|f'(x) - \frac{f(x + h) - f(x)}{h}\right| \\
\left|\frac{f''(c_1)}{2}h\right| &= \left|f'(x) - \frac{f(x + h) - f(x)}{h}\right| \\
\frac{h}{2}|f''(c_1)| &= \left|f'(x) - \frac{f(x + h) - f(x)}{h}\right|
\end{aligned}
$$

$$
\begin{aligned}
\left|f'(x) - \frac{f(x + h) - f(x)}{h}\right| &= \frac{h}{2}|f''(c_1)| \\
&= \frac{h}{2}| - k^2 \sin(kc_1)| \\
&\leq \frac{h}{2}k^2 \\
&= \frac{k^2 h}{2}
\end{aligned}
$$

part 2
from problem 3.(b)

$$
\begin{aligned}
f'(x) - \frac{f(x + h) - f(x - h)}{2h} &= \frac{f^{(3)}(c_1) - f^{(3)}(c_2)}{12}h^2 \\
\left|f'(x) - \frac{f(x + h) - f(x - h)}{2h}\right| &= \left|\frac{f^{(3)}(c_1) - f^{(3)}(c_2)}{12}h^2\right| \\
&= \left|\frac{(-k^3 \cos(kc_1)) - (-k^3 \cos(kc_2))}{12}h^2\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \frac{-k^3\cos(kc_1) + k^3\cos(kc_2)}{12} h^2 \right| \\
&= \left| \frac{k^3(\cos(kc_2) - \cos(kc_1))}{12} h^2 \right| \\
&= |\cos(kc_2) - \cos(kc_1)| \left| \frac{k^3h^2}{12} \right| \\
&\leq ||\cos(kc_2)| + |\cos(kc_1)|| \left| \frac{k^3h^2}{12} \right| \\
&\leq \frac{2k^3h^2}{12} \\
&= \frac{k^3h^2}{6}
\end{aligned}
$$

4
(a)

$$
\begin{aligned}
m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \\
M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\
L(\mathcal{P}, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\
U(\mathcal{P}, f) &= \sum_{i=1}^{n} M_i \Delta x_i \\
\forall c_i &\in [x_{i-1}, x_i] \\
\inf\{f(x) : x_{i-1} \leq x \leq x_i\} &\leq f(c_i) \leq \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\
\forall i \in \{1, 2, ..., n\}, m_i &\leq f(c_i) \leq M_i \\
\sum_{i=1}^{n} m_i &\leq \sum_{i=1}^{n} f(c_i) \leq \sum_{i=1}^{n} M_i \\
\sum_{i=1}^{n} m_i \Delta x_i &\leq \sum_{i=1}^{n} f(c_i) \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \\
L(\mathcal{P}, f) &\leq \sum_{i=1}^{n} f(c_i) \Delta x_i \leq U(\mathcal{P}, f)
\end{aligned}
$$

(b)

definition 5.1.9, riemann integrable

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$
$$\underline{\int_a^b} f(x)dx = \sup\{L(P,f) : P \text{ a partition of } [a,b]\}$$
$$\overline{\int_a^b} f(x)dx = \inf\{U(P,f) : P \text{ a partition of } [a,b]\}$$

$$\int_a^b f = \underline{\int_a^b} f = \sup\{L(P,f)\}$$
$$\text{let } P_1 \text{ be a partition of } [a,b], \int_a^b f - \varepsilon < L(P_1,f) < \underline{\int_a^b} f$$

$$\int_a^b f = \overline{\int_a^b} f = \inf\{U(P,f)\}$$
$$\text{let } P_2 \text{ be a partition of } [a,b], \overline{\int_a^b} f < U(P_2,f) < \int_a^b f + \varepsilon$$

let P_3 be a partition of $[a,b], P_3 := P_1 \cup P_2$

$\because P_1 \subset P_3, P_3$ is a refinement of $P_1, \therefore L(P_1,f) \leq L(P_3,f)$

$\because P_2 \subset P_3, P_3$ is a refinement of $P_2, \therefore U(P_3,f) \leq U(P_1,f)$

$$\int_a^b f - \varepsilon < L(P_1,f) \leq L(P_3,f) \leq \sum_{i=1}^{n} f(x_i)\Delta x_i \leq U(P_3,f) \leq U(P_2,f) < \int_a^b f + \varepsilon$$
$$\int_a^b f - \varepsilon < \sum_{i=1}^{n} f(x_i)\Delta x_i < \int_a^b f + \varepsilon$$
$$-\varepsilon < \sum_{i=1}^{n} f(x_i)\Delta x_i - \int_a^b f < \varepsilon$$
$$\left| \sum_{i=1}^{n} f(x_i)\Delta x_i - \int_a^b f \right| < \varepsilon$$
$$\left| \int_a^b f - \sum_{i=1}^{n} f(x_i)\Delta x_i \right| < \varepsilon$$

(c)
part 1

$$R_n(f,[0,1]) = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} = \frac{1}{n} \\
x_i &= a + (b-a)(i/n) = 0 + (1-0)(i/n) = \frac{i}{n} \\
\because i/n \in \mathbb{Q}, f(i/n) = 1 \\
R_n(f,[0,1]) &= \sum_{i=1}^{n} (1)(1/n) \\
&= \sum_{i=1}^{n} \frac{1}{n} \\
&= 1 \\
\therefore \{R_n(f,[0,1])\}_{n=1}^{\infty} &= \{1\}_{n=1}^{\infty} \text{ converges to } 1
\end{aligned}
$$

part 2

$$
\begin{aligned}
R_n(f,[1,1+\sqrt{2}]) &= \sum_{i=1}^{n} f(x_i)\Delta x \\
\Delta x &= \frac{b-a}{n} = \frac{1+\sqrt{2}-1}{n} = \frac{\sqrt{2}}{n} \\
x_i &= a + (b-a)(i/n) = 1 + (1+\sqrt{2}-1)(i/n) = 1 + (\sqrt{2}i/n) \\
\because \sqrt{2}i/n \notin \mathbb{Q}, 1 + (\sqrt{2}i/n) \notin \mathbb{Q}, f(1+(\sqrt{2}i/n)) = 0 \\
R_n(f,[1,1+\sqrt{2}]) &= \sum_{i=1}^{n} 0(\sqrt{2}/n) \\
&= 0 \\
\therefore \{R_n(f,[0,1])\}_{n=1}^{\infty} &= \{0\}_{n=1}^{\infty} \text{ converges to } 0
\end{aligned}
$$

part 3

$$
\begin{aligned}
R_n(f,[0,1+\sqrt{2}]) &= \sum_{i=1}^{n} f(x_i)\Delta x \\
\Delta x &= \frac{b-a}{n} = \frac{1+\sqrt{2}-0}{n} = \frac{1+\sqrt{2}}{n} \\
x_i &= a + (b-a)(i/n) = 0 + (1+\sqrt{2}-0)(i/n) = (1+\sqrt{2})(i/n) \\
&= i/n + (\sqrt{2}i/n) \\
\because \sqrt{2}i/n \notin \mathbb{Q}, i/n + (\sqrt{2}i/n) \notin \mathbb{Q}, f(i/n + (\sqrt{2}i/n)) = 0 \\
R_n(f,[0,1+\sqrt{2}]) &= \sum_{i=1}^{n} (0)\frac{1+\sqrt{2}}{n} \\
&= 0
\end{aligned}
$$

$$\therefore \{R_n(f,[0,1+\sqrt{2}])\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty} \text{ converges to } 0$$

part 4

$$\begin{aligned}
R_n(f,[0,1]) + R_n(f,[1,1+\sqrt{2}]) &= 1+0 = 1 \neq R_n(f,[0,1+\sqrt{2}]) = 0 \\
\lim_{n\to\infty} R_n(f,[0,1]) + R_n(f,[1,1+\sqrt{2}]) &= 1+0 = 1 \neq \lim_{n\to\infty} R_n(f,[0,1+\sqrt{2}]) = 0
\end{aligned}$$

(d)
disprove
$\{R_n(f,[0,1])\}$ converges to 1 as $n \to \infty$ as shown in 4.(c) part 1
but the dirichlet function defined in 4.(c) is not riemann integrable since

$$\begin{aligned}
m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0 \\
M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1 \\
L(P,f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (0)\Delta x_i = 0 \\
U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (1)\Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1 - 0 = 1 \\
\underline{\int_a^b} f(x)dx &= \sup\{L(P,f) : P \text{ a partition of } [a,b]\} = 0 \\
\overline{\int_a^b} f(x)dx &= \inf\{U(P,f) : P \text{ a partition of } [a,b]\} = 1 \\
\underline{\int_a^b} f(x)dx &= 0 \neq \overline{\int_a^b} f(x)dx = 1
\end{aligned}$$

so f is not riemann integrable