Homework 5

Due: Monday, October 24th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 3.1-3.3

Sections 2.3-2.5 Exercises

Problem 1 (4 points each) Exercises on cluster points:

(a) Let $S = (a, b)$ be an open interval with $a, b \in \mathbb{R}$ and $a < b$. Show that $[a, b]$ is the set of all cluster points of S.

(b) Let $S = \mathbb{Z}$. Show that S has no cluster points in $\mathbb{R}$.

(c) Let $S = \mathbb{Q}$. Show that $\mathbb{R}$ is the set of all cluster points of S.

Problem 2 (4 points each) Prove the following, using the ε-δ definition of the limit of a function:

(a) Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) := \sqrt{x}$. Show that $\lim_{x\to c} f(x) = \sqrt{c}$ for all $c \in [0, \infty)$. Is f a continuous function?

(Remark: You may use the fact that $0 \le a < b$ if and only if $\sqrt{a} < \sqrt{b}$. As a hint on how to play the ε games, look at the proof of Proposition 2.2.6 in the textbook.)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \cos(x)$. Show that $\lim_{x\to c} f(x) = \cos(c)$ for all $c \in \mathbb{R}$. Is f a continuous function?

(Remark: You may use trigonometric identities here, and the fact that $|\sin(x)| \le |x|$, and $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$. See Example 3.2.6 in the textbook for the necessary algebra; however, you will need explain all of the steps of the proof to receive credit.)

Problem 3 (4 points each) Prove the following corollaries to the sequential limits lemma (Lemma 3.1.7 in the textbook):

(a) (Continuity of algebraic operations) Let $S \subset \mathbb{R}$ and c be a cluster point of S. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions. Suppose limits of $f(x)$ and $g(x)$ as x goes to c both exist. Prove that

(i) $\lim_{x\to c} (f(x) + g(x)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right)$

(ii) $\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

(iii) If $\lim_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

(b) (Squeeze lemma) Let $S \subset \mathbb{R}$ and c be a cluster point of S. Let $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ be functions. Suppose

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in S$$

and that the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

Then, the limit of $g(x)$ as x goes to c exists and

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

Problem 4 (7 points) Two-sided limits are frequently useful. Prove Proposition 3.1.17 in the textbook: Let $S \subset \mathbb{R}$ be a set such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, and let $f : S \to \mathbb{R}$ be a function. Then c is a cluster point of S and

$$\lim_{x\to c} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Problem 5 (3 points each) Let $S = \mathbb{R} \setminus \{0\}$

(a) Let $f : S \to \mathbb{R}$ be defined by $f(x) := \cos(1/x)$. Show that $\lim_{x\to 0} f(x)$ does not exist.

(b) Let $f : S \to \mathbb{R}$ be defined by $f(x) := x^2 \cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

(c) Find a value $b \in \mathbb{R}$ for which the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{cases}$$

is continuous at 0. Is this b unique?

Problem 6 (3 points each) Practice with continuity.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := |x|$. Show that f is continuous at all $c \in \mathbb{R}$.

(b) Suppose $S \subset \mathbb{R}$ and $f, g : S \to \mathbb{R}$ are continuous functions. Show that $h : S \to \mathbb{R}$ defined by $h(x) := \max\{f(x), g(x)\}$ is continuous at all $c \in \mathbb{R}$.

(Hint: Show that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ for $a, b \in \mathbb{R}$, then use facts about composition of continuous functions, and continuity of algebraic operations.)