Homework 9

Due: Monday, December 5th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 6.1-6.3

Sections 5.1-5.3 Exercises

Problem 1 (5 points) Let $A \subset \mathbb{R}$ be a bounded non-empty set, and let $B := \{|x| : x \in A\}$. Prove that

$$\sup B - \inf B \leq \sup A - \inf A$$

(*Hint*: Find some $|x|$ close to $\sup B$ and some $|y|$ close to $\inf B$, then use the reverse triangle inequality.)

Problem 2 (4 points each) In this problem, we will look at some properties of the Riemann integral of the absolute value of a function.

(a) Suppose $f \in \mathscr{R}[a,b]$. Show that $|f| \in \mathscr{R}[a,b]$, and that

$$0 \leq \left| \int_a^b f \right| \leq \int_a^b |f|$$

(b) Find an example of a function $f : [a,b] \to \mathbb{R}$ such that $|f| \in \mathscr{R}[a,b]$ but $f \notin \mathscr{R}[a,b]$.

(*Hint*: Think about the Dirichlet function)

(c) Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that if $f(c) > 0$ for some $c \in [a,b]$, then there exists some $\delta > 0$ such that

$$\int_{c-\delta}^{c+\delta} f > 0$$

(*Remark*: Try to be careful about strict/non-strict inequalities and open/closed intervals in this part.)

(d) Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that

$$\int_a^b |f| = 0$$

if and only if $f(x) = 0$ for all $x \in [a,b]$

Problem 3 (5 points) Suppose F and G are continuously differentiable functions defined on $[a, b]$ such that $F'(x) = G'(x)$ for all $x \in [a, b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$

(Remark: This is justifying the "rule" of adding a constant $\int f + C$ to indefinite integration when you are computing an antiderivative. Make sure to use the right form of the fundamental theorem of calculus.)

Sections 6.1-6.2 Exercises

Problem 4 (4 points each) Practice with pointwise and uniform convergence.

(a) Let $f_n : (0, 1) \to \mathbb{R}$ be given by $f_n(x) := \frac{n+1}{nx}$. Show that $\{f_n\}$ converges pointwise to a continuous function f, but the convergence is not uniform.

(Remark: This shows that pointwise convergence to a continuous function does not imply uniform convergence, so the "converse" to Theorem 6.2.2 is not true. It is also possible to find counterexamples using sequences of continuous functions on $[0, 1]$)

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) := \begin{cases} 0 & x = 0 \\ n & 0 < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

Notice that $f_n \in \mathscr{R}[0, 1]$ since it has a finite number of discontinuities. Show that $\{f_n\}$ converges pointwise to a function $f \in \mathscr{R}[0, 1]$, but the convergence is not uniform (without using Theorem 6.2.4). Furthermore, show that

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$$

(c) Let $f_n(x) = \frac{x^n}{n}$. Show that $\{f_n\}$ converges uniformly to a differentiable function f on $[0, 1]$ (find f). However, show that $f'(1) \neq \lim\limits_{n\to\infty} f_n'(1)$.

Problem 5 (3 points each) Let f and g be bounded functions on $[a, b]$.

(a) Prove the triangle inequality for the uniform norm,

$$\|f + g\|_u \leq \|f\|_u + \|g\|_u$$

(b) Using your result in (a), prove the reverse triangle inequality for the uniform norm,

$$\left|\|f\|_u - \|g\|_u\right| \leq \|f - g\|_u$$

(Hint: Your proof will look very similar to the proof of the reverse triangle inequality for the absolute value)

Problem 6 (6 points) Consider the sequence of continuous functions $\{f_n\}$ on $[0,1]$ given by

$$f_n(x) := \begin{cases} 1 - nx & 0 \leq x < 1/n \\ 0 & 1/n \leq x \leq 1 \end{cases}$$

Show that $\{f_n\}$ has no subsequence which is convergent in uniform norm.

(*Hint*: Show that every subsequence of $\{f_n\}$ converges pointwise to some function. Can the subsequences converge uniformly?)

(*Remark*: This is an example of a sequence of continuous functions bounded in the uniform norm which has no convergent subsequence.)