Functions, Cardinality

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# Functions

**Def.** Let $A, B$ be sets. The <u>Cartesian product</u> is the set of tuples

$$A \times B := \{(a,b) : a \in A,\ b \in B\}$$

**Ex.** $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ 2d plane

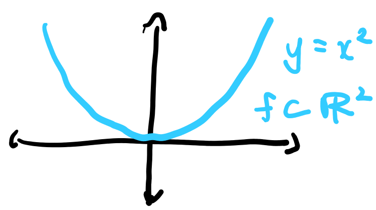
$(0,1) \in \mathbb{R}^2$

**Def.** A <u>function</u> $f : A \to B$ is a subset $f \subset A \times B$ such that for all $x \in A$, there is a unique $(x,y) \in f$

- We write $f(x) = y$
- The set $f$ is sometimes called the <u>graph</u> of the function

**Ex.** $f : \mathbb{R} \to \mathbb{R}$ $f(x) := x^2$

$f = \{(x, x^2) : x \in \mathbb{R}\}$




Other examples:

- Predicate: $P : A \to \{\text{false}, \text{true}\}$
- Derivative: $\underset{\text{input}}{\frac{d}{dx}[x^2]} = \underset{\text{output}}{2x}$ $\frac{d}{dx}$: (certain) functions $\to$ functions
- Definite Integration: $\int_0^1 \underbrace{f(x)}_{\text{input}} dx = \underbrace{I}_{\text{output} \in \mathbb{R}}$ $\int_0^1$: (certain) functions $\to \mathbb{R}$

**Def.** (Sets and functions) Given $f : A \to B$

- $A$ is the <u>domain</u> of $f$, also denoted $\mathrm{Dom}(f)$, $D(f)$
- $B$ is the <u>target space</u> or <u>codomain</u>
- The set

$$R(f) = \mathrm{Ran}(f) := \{y \in B : \overbrace{\exists}^{\text{there exists}} x \in A \underbrace{\text{s.t.}}_{\text{such that}} f(x) = y\}$$

Remark: $R(f)$ may be a proper subset of $B$

such that

**Remark:** $R(f)$ may be a proper subset of $B$

- Given $C \subset A$, the (direct) image of $C$ is
  $$f(C) := \{f(x) \in B : x \in C\} \quad (\subset B)$$
- Given $D \subset B$, the inverse image of $D$ is
  $$f^{-1}(D) := \{x \in A : f(x) \in D\} \quad (\subset A)$$
  Note: $f^{-1}$ is not necessarily a function!

Ex. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) := x^2$

$$\mathrm{Ran}(f) = [0, \infty) \subsetneq \mathbb{R}$$
$$f([0,2]) = \{x^2 : x \in [0,2]\} = [0,4]$$
$$f^{-1}(\{4\}) = \{x \in \mathbb{R} : x^2 \in \{4\}\} = \{-2, 2\}$$

Prop. 0.3.15, 0.3.16 (Properties of direct/inverse images)
Assigned reading / on HW

## Invertibility

Def. Let $f: A \to B$ be a function.

- $f$ is injective or one-to-one, i.e. an injection, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
  - Equivalently, $\forall y \in B$, $f^{-1}(\{y\})$ has at most one element
    "for all"
- $f$ is surjective or onto, i.e. a surjection, if $f(A) = B$
  - Equivalently, $\forall y \in B$, $f^{-1}(\{y\})$ has at least one element
- $f$ is bijective, i.e. a bijection, if $f$ is injective and surjective.

Ex. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) := x^2$ $\quad$ Claim: $f$ is neither injective or surjective

Pf. • $f(2) = f(-2) = 4$ $\quad 2 \neq -2$. Thus, $f$ is not injective.

Pf. • $f(2) = f(-2) = 4$, $2 \neq -2$. Thus, $f$ is not injective.

• $\forall x \in \mathbb{R}$, $f(x) = x^2 \geq 0$, so $-1 \in B$ but $-1 \notin f(A)$.
Thus, $f(A) \neq B$ (i.e. $-1 \in B \not\Rightarrow -1 \in f(A)$, so $B \not\subseteq f(A)$)
so $f$ is not surjective. $\square$

Remark: $g: [0,\infty) \to [0,\infty)$ $g(x) := x^2$ is both injective and surjective!

Def. Given a bijection $f: A \to B$, the inverse function $f^{-1}: B \to A$ is defined as $f^{-1}(y) := x$, where $x$ is the unique element of $f^{-1}(\{y\})$

Def. Given $f: A \to B$, $g: B \to C$, the composition of $f, g$ is a new function $g \circ f: A \to C$ defined

$$(g \circ f)(x) := g(f(x))$$

Remark: If $f: A \to B$, $g: B \to C$ are bijections, then so is $g \circ f: A \to C$

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## Cardinality

Motivation: there are different sizes of infinity!

Def. Let $A, B$ be sets. We say $A, B$ have the same cardinality if there exists a bijection $f: A \to B$. We write $|A| = |B|$

Remarks: • Informally, cardinality measures the "size" of a set

$A = \{1, 2, 3\}$ $B = \{\text{apple}, \text{bird}, \text{cat}\}$ $C = \{1, 2, 3, 4\}$

$|A| = |B|$ $|A| \neq |C|$

• (optional reading) More precisely, $|A|$ can be concretely defined via equivalence relations, see 0.3.4 and 0.3.5

**Def.** (Countable Cardinalities) Given a set $A$,

- $A$ is <u>finite</u> if $A$ is empty or $|A| = |\{1,2,\dots,n\}|$ for some $n \in \mathbb{N}$.
  Otherwise, $A$ is <u>infinite</u>.
- $A$ is <u>countably infinite</u> if $|A| = |\mathbb{N}|$.
- $A$ is <u>countable</u> if it is finite or countably infinite.

**Ex.** <u>Claim</u>: $E = \{2n : n \in \mathbb{N}\}$ is countably infinite.

<u>Pf.</u> Define $f : \mathbb{N} \to E$ by $f(n) := 2n$

- $2n_1 = 2n_2 \Rightarrow n_1 = n_2$ so $f$ is an injection.
- $f(\mathbb{N}) = \{2n : n \in \mathbb{N}\} = E$ so $f$ is a surjection.

Thus, $f$ is a bijection $\Rightarrow |E| = |\mathbb{N}|$ $\square$

<u>Claim</u>: $|\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

<u>Pf.</u> (Sketch)

$$\begin{array}{rcl} \mathbb{Z} &=& \{0, -1, 1, -2, 2, \dots\} \\ & & \ \ \updownarrow \ \ \ \updownarrow \ \ \updownarrow \ \ \ \updownarrow \ \ \updownarrow \\ \mathbb{N} &=& \{1, 2, 3, 4, 5, \dots\} \end{array}$$

$\mathbb{N} = \{(m,n) : m,n \in \mathbb{N}\}$

$$\begin{array}{cccc} \overset{1}{(1,1)} & \overset{2}{(1,2)} & \overset{4}{(1,3)} & \cdots \\ \overset{3}{(2,1)} & \overset{5}{(2,2)} & \vdots & \\ \overset{6}{(3,1)} & \vdots & & \\ \vdots & & & \end{array}$$

<u>Claim</u>: $|\mathbb{Q}| = |\mathbb{N}|$.

<u>Pf.</u> (Informal)

$$\begin{array}{cccc} {}^{1}\frac{1}{1} & {}^{2}\frac{2}{1} & {}^{4}\frac{3}{1} & \cdots \\ {}^{3}\frac{1}{2} & \cancel{\frac{2}{2}} & \vdots & \\ {}^{5}\frac{1}{3} & \vdots & & \\ \vdots & & & \end{array}$$

⋮

<u>Thm.</u> (Cantor) There exist infinite sets which are not countable
(optional reading 0.3.34)