Homework 3 Solutions

Due: Monday, October 3rd by 11:59 PM ET

Sections 2.1-2.2 Exercises

Problem 1 (4 points) Sometimes we want a little more flexibility with our limit definitions. Given a sequence $\{x_n\}$, a real number $x \in \mathbb{R}$, and another real number $\alpha > 0$, prove that the following two statements are equivalent:

(i) "for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq M$"

(ii) "for every $\varepsilon > 0$, there exists an $M \in \mathbb{R}$ such that $|x_n - x| \leq \alpha\varepsilon$ for all $n \geq M$"

Advice: to avoid confusion in your proof, since the variables ε and M appear in both statements, it may help to rename some of the variables.

We can rename the variables in (ii) to say "for every $\eta > 0$, there exists an $K \in \mathbb{R}$ such that $|x_n - x| \leq \alpha\eta$ for all $n \geq K$"

(i) $\implies$ (ii): Assume that (i) holds. Then, for any $\eta > 0$, take $\varepsilon = \alpha\eta > 0$. Then, by (i), there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$|x_n - x| < \varepsilon \leq a\eta$$

Therefore, by setting $K = M$, (ii) is satisfied.

(ii) $\implies$ (i): Assume that (ii) holds. Then for any $\varepsilon > 0$, take $\eta = \varepsilon/(2\alpha)$. Then, by (ii), there exists $K \in \mathbb{R}$ such that for all $n \geq K$,

$$|x_n - x| \leq \alpha\eta < \varepsilon$$

By the Archimedean property, there exists $M \in \mathbb{N}$ such that $M > K$. Thus, the above inequality holds for all $n \geq M > K$. Thus (i) is satisfied.

Problem 2 (5 points each) Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent sequences with limits $x, y \in \mathbb{R}$, and let $a, b \in \mathbb{R}$. Directly prove the following properties about limits without using the results of Proposition 2.2.5 from the textbook.

You can look at the proofs of Proposition 2.2.5 for inspiration, but for your solutions you should write out your own proofs involving ε and inequalities.

(a) Prove that $\{ax_n + by_n\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} (ax_n + by_n) = a\left(\lim_{n\to\infty} x_n\right) + b\left(\lim_{n\to\infty} y_n\right)$$

(b) Prove that $\{x_n^2\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} \left(x_n^2\right) = \left(\lim_{n\to\infty} x_n\right)^2$$

(c) Assume that $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n \neq 0$. Prove that $\{\frac{1}{x_n}\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$$

(a) First, if $a = b = 0$, then $\{ax_n + by_n\} = \{0\}$ is a constant sequence with limit $0 = ax + by$, which is the desired statement.

Now, assume either $a \neq 0$ or $b \neq 0$, so $|a| + |b| > 0$. Let $\varepsilon > 0$ be given. Then, for $\eta := \frac{\varepsilon}{|a|+|b|} > 0$, there exists M_1 such that $|x_n - x| < \eta$ for $n \geq M_1$, and there exists $M_2 \in \mathbb{N}$ such that $|y_n - y| < \eta$ for $n \geq M_2$. Thus, for all $n \geq M := \max\{M_1, M_2\}$,

$$\begin{aligned}
|ax_n + by_n - (ax + by)| &= |a(x_n - x) + b(y_n - y)| \\
&\leq |a||x_n - x| + |b||y_n - y| \\
&< |a|\frac{\varepsilon}{|a| + |b|} + |b|\frac{\varepsilon}{|a| + |b|} = \varepsilon
\end{aligned}$$

which shows that the sequence $\{ax_n + by_n\}$ converges to the limit $ax + by$.

Remark. Here is an alternative proof, using the result of Problem 1, which might be easier to do.

Again, consider the case where either $a \neq 0$ or $b \neq 0$. Let $\varepsilon > 0$ be given. Then, there exists M_1 such that $|x_n - x| < \varepsilon$ for $n \geq M_1$, and there exists $M_2 \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$ for $n \geq M_2$. Thus, for all $n \geq M := \max\{M_1, M_2\}$,

$$|ax_n + by_n - (ax + by)| \leq |a||x_n - x| + |b||y_n - y| < (|a| + |b|)\varepsilon$$

Since $|a| + |b| > 0$ and does not depend on ε, by the results of Problem 1 this shows that $\{ax_n + by_n\}$ converges to $ax + by$

(b) Since $\{x_n\}$ is convergent, it will be bounded with some bound $B \in \mathbb{R}$. We can take $B > 0$. Let $\varepsilon > 0$ be given. Then for $\frac{\varepsilon}{|B|+|x|} > 0$, there exists $M \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{|B|+|x|}$ for all $n \geq M$. Then, for all $n \geq M$, we also have

$$\left|x_n^2 - x^2\right| = |(x_n + x)(x_n - x)| = |x_n + x||x_n - x| \leq (|x_n|+|x|)\frac{\varepsilon}{|B|+|x|} \leq (|B|+|x|)\frac{\varepsilon}{|B|+|x|} = \varepsilon$$

which shows $\{x_n^2\}$ converges to x^2 as desired.

Remark. Here is an alternative proof, similar to the one in the textbook:

Let $\varepsilon > 0$ be given. Let $K := \max\{|x|, \varepsilon/3, 1\}$. Then, there exists $M \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{3K}$ for all $n \geq M$. Thus, for all $n \geq M$ we also have

$$\begin{aligned}\left|x_n^2 - x^2\right| &= \left|((x_n - x) + x)^2 - x^2\right| = \left|(x_n - x)^2 + 2x(x_n - x)\right| \\ &\leq |x_n - x|^2 + 2|x||x_n - x| \\ &< \frac{\varepsilon}{3K}\frac{\varepsilon}{3K} + 2K\frac{\varepsilon}{3K} \\ &\leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon\end{aligned}$$

which shows $\{x_n^2\}$ converges to x^2 as desired.

(c) Let $\varepsilon > 0$ be given. Since $x = \lim_{n\to\infty} x_n \neq 0$, we have that $\min\{|x|^2\frac{\varepsilon}{2}, \frac{|x|}{2}\} > 0$. Then, we can find $M \in \mathbb{N}$ such that for all $n \geq M$ we have

$$|x_n - x| < \min\left\{|x|^2\frac{\varepsilon}{2}, \tfrac{|x|}{2}\right\}$$

Thus, for all $n \geq M$ we have $|x - x_n| < \frac{|x|}{2}$, and so

$$|x| = |x - x_n + x_n| \leq |x - x_n| + |x_n| < \frac{|x|}{2} + |x_n|$$

which we rearrange to get for all $n \geq M$,

$$\frac{1}{|x_n|} < \frac{2}{|x|}$$

We then use this to finish the proof. For all $n \geq M$

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x - x_n|}{|x||x_n|} \leq \frac{|x - x_n|}{|x|}\frac{2}{|x|} < \frac{|x^2|\frac{\varepsilon}{2}}{|x|}\frac{2}{|x|} = \varepsilon$$

which shows the desired statement.

Problem 3 (3 points each) Many of the propositions we have seen so far assume that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. Let's see what happens if we try and alter some of these statements.

For the following, do not assume $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent unless specifically told so. Are the following statements true or false? As usual, you should provide a proof if you believe the statement is true; provide a counterexample otherwise.

(a) If $\{x_n + y_n\}_{n=1}^{\infty}$ converges, then either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ converges.

(b) If $\{x_n y_n\}_{n=1}^{\infty}$ converges, then either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ converges.

(c) If $\{x_n\}_{n=1}^{\infty}$ and $\{x_n + y_n\}_{n=1}^{\infty}$ converge, then $\{y_n\}_{n=1}^{\infty}$ converges.

(d) If $\{x_n\}_{n=1}^{\infty}$ is bounded, and $\{y_n\}_{n=1}^{\infty}$ converges to 0, then $\{x_n y_n\}_{n=1}^{\infty}$ converges to 0.

(e) If $\{x_n\}_{n=1}^{\infty}$ is unbounded, and $\{y_n\}_{n=1}^{\infty}$ converges to 0, then $\{x_n y_n\}_{n=1}^{\infty}$ converges.

(a) False. Let $\{x_n\} := \{n\}$ and $\{y_n\} := \{-n\}$. Then, $\{x_n + y_n\} = \{0\}$ is the constant sequence which converges to 0, but $\{x_n\}$ and $\{y_n\}$ are both unbounded, and hence divergent.

(b) False. Let $\{x_n\} = \{y_n\} = \{(-1)^n\}$. Then, $\{x_n y_n\} = \{1\}$ is the constant sequence which converges to 1. However, we showed in lecture both $\{x_n\}$ and $\{y_n\}$ diverge.

(c) True. Continuity of algebraic operations applies to the difference of the sequences $(x_n + y_n) - (x_n) = y_n$, which shows that $\{y_n\}$ is a convergent sequence.

(d) True. Let $\varepsilon > 0$ be given, and let $B > 0$ be a bound for $\{x_n\}$. Since $\{y_n\}$ converges to 0, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $|y_n| < \frac{\varepsilon}{B}$. Then, for all $n \geq M$ we have

$$|x_n y_n - 0| \leq B|y_n| < \varepsilon$$

Thus $\{x_n y_n\}$ converges to 0.

(e) False. Let $\{x_n\} := \{n^2\}$ and $\{y_n\} := \{1/n\}$. Then $\{x_n\}$ is unbounded (by the Archimedean property, for all $B \in \mathbb{R}$ there exists n such that $n^2 \geq n > B$), and $\{y_n\}$ converges to 0 as we showed in lecture. However, $\{x_n y_n\} = \{n\}$ is unbounded and hence divergent.

Problem 4 (5 points) Here's a fun and useful fact you can prove using convergence: Suppose $\{x_n\}$ and $\{y_n\}$ are sequences (not necessarily convergent) such that $y_n > 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0$$

Prove that there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $y_n > |x_n|$

Note this says that $|x_n| \geq y_n$ at most finitely often.

Use this to prove that for any polynomial of degree $d \in \mathbb{N}$ given by

$$p(n) := n^d + c_{d-1}n^{d-1} + ... + c_0$$

with coefficients $c_r \in \mathbb{R}$, there exists some $M \in \mathbb{N}$ such that $2n^d > p(n)$ for all $n \in \mathbb{N}$. This can be used to simplify inequalities using polynomials.

(*Hints*: For the first part, try picking a clever value of ε then manipulate the resulting inequality. For the second part, try taking $\{y_n\} = \{n^d\}$ and $\{x_n\} = \{p(n) - n^d\}$)

Suppose $\{x_n\}$ and $\{y_n\}$ are as given. Then for $\varepsilon = 1 > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\left|\frac{x_n}{y_n} - 0\right| = \frac{|x_n|}{y_n} < 1 \implies |x_n| < y_n$$

which is the desired statement we want to prove.

To prove the second part, take $\{y_n\} := \{n^d\}$ and $\{x_n\} := \{p(n) - n^d\}$. Note that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left[\frac{c_{d-1}}{n} + ... + \frac{c_0}{n^d}\right] = 0$$

Then, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $y_n > |x_n|$, or in other words,

$$n^d > \left|p(n) - n^d\right| \geq p(n) - n^d \implies 2n^d > p(n)$$

which is the desired inequality.

Problem 5 (2.5 points each) For the following determine whether the sequence converges, and if it does, find its limit.

(a) $\{x_n\} := \{\frac{n\cos(n)}{n^2+1}\}$. *Note*: You may use the fact that $|\cos(y)| \leq 1$ for all $y \in \mathbb{R}$.

(b) $\{x_n\} := \{\frac{2^n}{n^2}\}$

(a) Note that for any $\varepsilon > 0$, by the Archimedean property there exists $M \in \mathbb{N}$ satisfying $1/M < \varepsilon$ so that for all $n \geq M$,

$$\left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{M} < \varepsilon$$

Thus, $\{\frac{n}{n^2+1}\}$ converges to 0. Multiplying this convergent sequence by -1, we get $\{-\frac{n}{n^2+1}\}$ also converges to 0.

Then since for all $n \in \mathbb{N}$ we have

$$-\frac{n}{n^2+1} \leq \frac{n\cos}{n^2+1} \leq \frac{n}{n^2+1}$$

by the squeeze lemma $\{x_n\}$ converges to 0.

(b) We check the ratio

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \left(\frac{n^2}{(n+1)^2}\frac{2^{n+1}}{2^n}\right) = \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1} \geq \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 > 1$$

Thus by the ratio test, the sequence $\{x_n\}$ does not converge.

Problem 6 (6 points) Prove that a bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit $x \in \mathbb{R}$ if and only if every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to x.

(*Remark*: One thing that may be helpful to prove is that for any subsequence $\{x_{n_i}\}_{i=1}^{\infty}$, $n_i \geq i$ for all $i \in \mathbb{N}$.)

First, assume $\lim_{n\to\infty} x_n = x$. Then, for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $n \geq M$

$$|x_n - x| < \varepsilon$$

Now given any subsequence $\{x_{n_i}\}_{i=1}^{\infty}$, we will show by induction that $n_i \geq i$. Let $P(i)$ be the statement $n_i \geq i$.

(Basis statement) For $i = 1$, since $m \geq 1$ for all $m \in \mathbb{N}$, $n_1 \geq 1$.

(Induction step) Assume $P(i)$ is true, so $n_i \geq i$. Since $n_{i+1} > n_i$, we have that $n_{i+1} \geq n_i + 1 \geq i + 1$. So, $P(i+1)$ is true.

Thus, by induction $n_i \geq i$ for all $i \in \mathbb{N}$.

Therefore, for all $i \geq M$, we have $n_i \geq M$, so

$$|x_{n_i} - x| < \varepsilon$$

which shows $\{x_{n_i}\}_{i=1}^{\infty}$ also converges to x.

Now, assume every subsequence of $\{x_n\}_{n=1}^{\infty}$ also converges to x. The sequence $\{x_n\}_{n=1}^{\infty}$ is a subsequence of itself, so it converges to x.