Homework 7

Due: Monday, November 7th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 4.2-4.3, 5.1

Sections 4.1-4.3 Exercises

Problem 1 (4 points each) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \sin(x)$. This problem will walk you through proving that f is differentiable, and that $f'(x) = \cos(x)$.

You may use basic trigonometric identities and inequalities[1], and may find this particular inequality helpful:

$$\sin(x) < x < \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ for } x \in (0, \pi/2)$$

You may also assume that $\sin(x)$ and $\cos(x)$ are continuous functions for $x \in (-\pi/2, \pi/2)$. Recall on HW5 you showed that $\cos(x)$ is continuous, and a very similar proof would show that $\sin(x)$ is continuous.

(a) Prove that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(b) Show that $f(x) = \sin(x)$ is differentiable for all $x \in \mathbb{R}$, and that $f'(x) = \cos(x)$.

(Hint: Try using the sum-to-product identity on $\sin(x) - \sin(c)$.)

Problem 2 (5 points each) In this problem, we will prove a special case of L'Hôpital's rule.

(a) Let $h : S \to \mathbb{R}$ and c be a cluster point of S. Show that if $\lim_{x\to c} h(x) = L \neq 0$, then there exists some $\delta > 0$ such that for all $x \in (S \setminus \{c\}) \cap (c - \delta, c + \delta)$, $h(x) \neq 0$.

(b) Let $h : S \to \mathbb{R}$ be continuous and c be a cluster point of S. Show that if $h(c) \neq 0$, then there exists some $A \subset S$ such that c is a cluster point of A, $h\big|_A(x) \neq 0$ for all $x \in A$, and

$$\lim_{x\to c} \left(\frac{1}{h|_A(x)} \right) = \frac{1}{\lim_{x\to c} (h|_A(x))} = \frac{1}{h(c)}$$

[1]Most trigonometric identities and inequalities have "geometric" proofs, so it doesn't count as "cheating" to use them to prove facts about calculus. See https://en.wikipedia.org/wiki/Proofs_of_trigonometric_identities for example.

Note: This result allows us to "abuse notation". We get a slightly more general notion of Corollary 3.1.12.iv and write

$$\lim_{x\to c}\left(\frac{1}{h(x)}\right) = \frac{1}{\lim\limits_{x\to c} h(x)}$$

even though strictly speaking, $1/h(x)$ might not be defined for all $x \in S$.

(c) Suppose $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions. Suppose that at $c \in (a, b)$, $f(c) = g(c) = 0$, and $g'(x) \neq 0$ for all $x \in (a, b)$, and suppose that the limit of $\frac{f'(x)}{g'(x)}$ as $x \to c$ exists. Show that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

(*Hint*: This is similar to the proof that a differentiable function is continuous. Be careful not to divide by 0, and make sure to explain all the steps in your proof.)

Problem 3 (4 points each) Let I be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Prove the following statements:

(a) $f'(x) \leq 0$ for all $x \in I$ if and only f is decreasing.

We say f is decreasing if $f(x) \geq f(y)$ for all $x, y \in I$ with $x < y$

(b) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing.

We say f is strictly decreasing if $f(x) > f(y)$ for all $x, y \in I$ with $x < y$

Problem 4 (4 points each) Here is an extremely useful application of the mean value theorem, which can be thought of as a special case of Taylor's theorem:

Suppose $f : [a, b] \to \mathbb{R}$ satisfies the assumptions of the MVT, and there is a M such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Then, for any $x, y \in [a, b]$, we have from the mean value theorem there is a c between x, y such that

$$f(x) - f(y) = f'(c)(x - y)$$

Taking the absolute value of both sides, we can get a convenient upper bound for $|f(x) - f(y)|$, namely

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq M|x - y|$$

Prove the following inequalities:

(a) For any $R > 0$, $n \in \mathbb{N}$, and $x, y \in [-R, R]$, we have $|x^n - y^n| \leq nR^{n-1}|x - y|$

(b) For any $x, y \in \mathbb{R}$, we have $\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| \leq |x - y|$

Problem 5 (6 points) Here is another way to bound functions using Taylor's theorem:

Suppose $f : \mathbb{R} \to \mathbb{R}$ has n continuous derivatives. Show that for any closed and bounded interval $[a, b] \subset \mathbb{R}$, there exist polynomials P and Q of degree n such that $P(x) \leq f(x) \leq Q(x)$ for all $x \in [a, b]$ and $Q(x) - P(x) = \lambda(x - a)^n$ for some $\lambda \geq 0$.

(*Hint*: Try using Taylor's theorem at $x_0 = a$ with the min/max theorem.)

Problem 6 (5 points) This problem introduces a very reduced version of the inverse function theorem.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Show that if $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then there exists some interval $I = (x_0 - \delta, x_0 + \delta)$ such that $f\big|_I : I \to f(I)$ is bijective.