# Homework 3

Due: Monday, October 3rd by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

**Assigned Readings** Lebl 2.1-2.2

## Sections 2.1-2.2 Exercises

**Problem 1** (4 points) Sometimes we want a little more flexibility with our limit definitions. Given a sequence $\{x_n\}$, a real number $x \in \mathbb{R}$, and another real number $\alpha > 0$, prove that the following two statements are equivalent:

(i) "for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq M$"

(ii) "for every $\varepsilon > 0$, there exists an $M \in \mathbb{R}$ such that $|x_n - x| \leq \alpha\varepsilon$ for all $n \geq M$"

*Advice*: to avoid confusion in your proof, since the variables $\varepsilon$ and $M$ appear in both statements, it may help to rename some of the variables.

**Problem 2** (5 points each) Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent sequences with limits $x, y \in \mathbb{R}$, and let $a, b \in \mathbb{R}$. Directly prove the following properties about limits without using the results of Proposition 2.2.5 from the textbook.

You can look at the proofs of Proposition 2.2.5 for inspiration, but for your solutions you should write out your own proofs involving $\varepsilon$ and inequalities.

(a) Prove that $\{ax_n + by_n\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} (ax_n + by_n) = a\left(\lim_{n\to\infty} x_n\right) + b\left(\lim_{n\to\infty} y_n\right)$$

(b) Prove that $\{x_n^2\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} (x_n^2) = \left(\lim_{n\to\infty} x_n\right)^2$$

(c) Assume that $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n \neq 0$. Prove that $\{\frac{1}{x_n}\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$$

**Problem 3** (3 points each) Many of the propositions we have seen so far assume that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. Let's see what happens if we try and alter some of these statements.

For the following, do not assume $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent unless specifically told so. Are the following statements true or false? As usual, you should provide a proof if you believe the statement is true; provide a counterexample otherwise.

(a) If $\{x_n + y_n\}_{n=1}^{\infty}$ converges, then either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ converges.

(b) If $\{x_n y_n\}_{n=1}^{\infty}$ converges, then either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ converges.

(c) If $\{x_n\}_{n=1}^{\infty}$ and $\{x_n + y_n\}_{n=1}^{\infty}$ converge, then $\{y_n\}_{n=1}^{\infty}$ converges.

(d) If $\{x_n\}_{n=1}^{\infty}$ is bounded, and $\{y_n\}_{n=1}^{\infty}$ converges to 0, then $\{x_n y_n\}_{n=1}^{\infty}$ converges to 0.

(e) If $\{x_n\}_{n=1}^{\infty}$ is unbounded, and $\{y_n\}_{n=1}^{\infty}$ converges to 0, then $\{x_n y_n\}_{n=1}^{\infty}$ converges.

**Problem 4** (5 points) Here's a fun and useful fact you can prove using convergence: Suppose $\{x_n\}$ and $\{y_n\}$ are sequences (not necessarily convergent) such that $y_n > 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0$$

Prove that there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $y_n > |x_n|$

Note this says that $|x_n| \geq y_n$ at most finitely often.

Use this to prove that for any polynomial of degree $d \in \mathbb{N}$ given by

$$p(n) := n^d + c_{d-1}n^{d-1} + ... + c_0$$

with coefficients $c_r \in \mathbb{R}$, there exists some $M \in \mathbb{N}$ such that $2n^d > p(n)$ for all $n \in \mathbb{N}$. This can be used to simplify inequalities using polynomials.

(*Hints*: For the first part, try picking a clever value of $\varepsilon$ then manipulate the resulting inequality. For the second part, try taking $\{y_n\} = \{n^d\}$ and $\{x_n\} = \{p(n) - n^d\}$)

**Problem 5** (2.5 points each) For the following determine whether the sequence converges, and if it does, find its limit.

(a) $\{x_n\} := \{\frac{n\cos(n)}{n^2+1}\}$. *Note*: You may use the fact that $|\cos(y)| \leq 1$ for all $y \in \mathbb{R}$.

(b) $\{x_n\} := \{\frac{2^n}{n^2}\}$

**Problem 6** (6 points) Prove that a bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit $x \in \mathbb{R}$ if and only if every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to $x$.

(*Remark*: One thing that may be helpful to prove is that for any subsequence $\{x_{n_i}\}_{i=1}^{\infty}$, $n_i \geq i$ for all $i \in \mathbb{N}$.)