Homework 1

Due: Monday, September 19th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings 0.2-0.3, 1.1-1.2

Chapter 0 Exercises

Problem 1 (5 points) Let A, B, C be sets. Prove the following set relation properties:

(i) (*Transitivity of set inclusion*) If $A \supset B$ and $B \supset C$, then $A \supset C$

(ii) (*Transitivity of set equality*) If $A = B$ and $B = C$, then $A = C$

Problem 2 (5 points) For each function, determine if it is (i) injective and (ii) surjective. Don't forget to justify your answer with a proof.

(a) $f : (0, 1) \to (1, \infty)$ where $f(x) := 1/x$

(b) $g : \mathbb{R} \to \mathbb{Z}$ given by $g(x) := \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the *floor* function which 'rounds down', i.e. returns the largest integer less than or equal to x.

Problem 3 (6 points) For $p \in \mathbb{N}$, define $\mathbb{N}^p := \mathbb{N} \times ... \times \mathbb{N}$ (p times) to be the set of p-tuples of natural numbers, i.e. $(n_1, n_2, ..., n_p) \in \mathbb{N}^p$.

(a) Let $f_2 : \mathbb{N}^2 \to \mathbb{N}$ be the bijection defined in example 0.3.31, so $f_2(1, 1) = 1$, $f_2(1, 2) = 2$, etc...

Define a function $f_3 : \mathbb{N}^3 \to \mathbb{N}^2$ by $f_3(n_1, n_2, n_3) := (n_1, f_2(n_2, n_3))$. Show that f_3 is a bijection.

(b) Show using induction that $\mathbb{N}^p$ is countable for any $p \in \mathbb{N}$

(*Note*: It is possible prove this without induction, but you should practice using induction for this problem.)

Problem 4 (6 points) Prove Proposition 0.3.16: Consider $f : A \to B$. Let C, D be subsets of A. Then,

$$f(C \cup D) = f(C) \cup f(D)$$
$$f(C \cap D) \subset f(C) \cap f(D)$$

Additionally, find a function $f : A \to B$ and sets C, D such that $f(C \cap D) \not\supset f(C) \cap f(D)$.

Chapter 1 Exercises

Problem 5 (6 points) Let $E = (\infty, b) := \{x \in \mathbb{R} : x < b\}$ where $b \in \mathbb{R}$. Compute $\sup E$ and $\inf E$ if they exist, or prove that E is unbounded above/below if they do not exist. Don't forget to justify your answer by proof.

(*Note*: do not use the extended reals for this problem)

Problem 6 (6 points) Suppose A, B are non-empty sets that are both bounded above and below, and furthermore that $A \subset B$. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

Problem 7 (6 points) Let $B \subset \mathbb{R}$ be bounded above, and let $c = \sup B$. Prove the following statements:

(a) c is unique; that is, if c' is also a supremum of B, then $c = c'$

(b) For any $x \in \mathbb{R}$, if $x > c$ then $x \notin B$

Problem 8 (5 points each) Let $B \subset \mathbb{R}$ be a non-empty subset which is bounded above and below. Let $c = \sup B$ and $d = \inf B$:

(a) For all real numbers $\varepsilon > 0$, there exists $x \in B$ such that $c - \varepsilon < x \leq c$

(b) For every $\varepsilon > 0$, the set $[d, d + \varepsilon) \cap B$ is non-empty.

(*Hint*: The first statement (a) takes the form of a nested quantifier, "$\forall \varepsilon \in (0, \infty), \exists x \in B$ s.t. $P(\varepsilon, x)$ is true". The negation of this double quantifier is "$\exists \varepsilon \in (0, \infty)$ s.t. $\forall x \in B, P(\varepsilon, x)$ is false". This can be seen through 'abstract logic' by negating the statement "$\forall \varepsilon \in (0, \infty), Q(\varepsilon)$ is true" where $Q(\varepsilon)$ is the predicate "$\exists x \in B$ s.t. $P(\varepsilon, x)$ is true".

In plain English, the negation of (a) would be "there exists a real number $\varepsilon > 0$ such that for all $x \in B$, $x \leq c - \varepsilon$ or $x > c$". One way to prove (a) is to assume the negation of (a), then prove a contradiction.

To prove (b), try converting it to a statement similar to (a). Note that (b) provides a "geometric" interpretation of a double quantifier statement.)