Midterm Exam 2 Problem Bank Solutions

Examination Date: Monday, November 21st

1. In this problem, we will see how to generalize some of the results in section 3.3 to more general domains.

   We say a set $K \subset \mathbb{R}$ is *sequentially compact* if every sequence in K has a convergent subsequence converging to a point in K. In other words, if $\{x_n\}$ is a sequence satisfying $x_n \in K$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ and point $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.

   (a) Prove that $[a, b]$ is sequentially compact, but (a, b) is not sequentially compact.

   (*Hint*: For $[a, b]$, Bolzano-Weierstrass might be helpful. For (a, b), try constructing a convergent sequence which 'escapes' the open interval.)

   (b) Prove that if a set K is sequentially compact, then it is bounded.

   (*Hint*: The contrapositive might be easier to prove, look at the proof of Lemma 3.3.1.)

   (c) Prove that if a set K is sequentially compact, then $\sup(K) \in K$ and $\inf(K) \in K$.

   (*Hint*: Try constructing sequences which converge to $\sup(K)$ and $\inf(K)$.)

   (d) Let K be sequentially compact. Prove that if a function $f : K \to \mathbb{R}$ is continuous, then the direct image $f(K)$ is sequentially compact.

   Use this to prove that f achieves an absolute minimum and absolute maximum on K.

   (*Hint*: For any sequence $\{f(x_n)\}$, observe that $\{x_n\}$ is a sequence in the sequentially compact set K.)

(a) Suppose $\{x_n\}$ is a sequence in $[a, b]$. Since $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, it is bounded. By Bolzano-Weierstrass, it has some convergent subsequence $\{x_{n_k}\}$. Since limits preserve non-strict inequalities,

$$a \leq \lim_{k\to\infty} x_{n_k} \leq b$$

which shows that every sequence $\{x_n\}$ in $[a, b]$ has some subsequence which converges to some $x \in [a, b]$.

Now, to show (a, b) is not sequentially compact, consider the sequence

$$x_n := a + \frac{b-a}{2n}$$

Note that $x_n \in (a, b)$ for all $n \in \mathbb{N}$, and $x_n \to a \notin (a, b)$ as $n \to \infty$. Since every subsequence of a convergent sequence converges to the same limit, x_n does not have any subsequence which converges to some $x \in (a, b)$. Thus, (a, b) is not sequentially compact.

(b) We will prove the contrapositive. Suppose K is not bounded. Then, for all $n \in \mathbb{N}$, there exists some $x_n \in K$ satisfying $|x_n| > n$ (i.e. otherwise $|x_n|$ would be a bound for K).

Every subsequence of $\{x_n\}$ is unbounded, e.g. since $|x_{n_k}| \geq n_k \geq k$ for all $k \in \mathbb{N}$. Thus, $\{x_n\}$ cannot have a convergent subsequence, so K is not sequentially compact.

(c) By the properties of the sup/inf, for all $n \in \mathbb{N}$ there exists elements $x_n, y_n \in K$ satisfying

$$\sup(K) - \frac{1}{n} < x_n \leq \sup(K) \qquad\qquad \inf(K) \leq y_n < \inf(K) + \frac{1}{n}$$

Using the squeeze lemma, we have that $x_n \to \sup(K)$ and $y_n \to \inf(K)$ as $n \to \infty$. Since K is sequentially compact, there exist convergent subsequences $\{x_{n_k}\}$ and $\{y_{m_k}\}$ of $\{x_n\}$ and $\{y_n\}$ which converge to some $x, y \in K$. However, since subsequences of a convergent sequence converge to the same limit, $x = \sup(K) \in K$ and $y = \inf(K) \in K$.

(d) Let $\{f(x_n)\}$ be a sequence in $f(K)$. $\{x_n\}$ is a sequence in the sequentially compact set K, so it has some convergent subsequence $\{x_{n_k}\}$ converging to some $x \in K$. Then, by the continuity of f, we have $f(x_{n_k})$ is convergent, with limit

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x) \in f(K)$$

Thus, every sequence $\{f(x_n)\}$ in $f(K)$ has some subsequence converging to some $f(x) \in f(K)$, showing that $f(K)$ is sequentially compact.

Now, let $\{f(x_n)\}$ and $\{f(y_n)\}$ be sequences converging to $\sup(f(K))$ and $\inf(f(K))$ respectively (such sequences exist by the same argument in (c)). Then, $\{x_n\}$ and $\{y_n\}$ have subsequences converging to some $x, y \in K$. Then, again by continuity,

$$\sup(f(K)) = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f(x) \inf(f(K)) = \lim_{n\to\infty} f(y_n) = \lim_{k\to\infty} f(y_{m_k}) = f(y)$$

which shows that f achieves an absolute max at $x \in K$ and an absolute min at $y \in K$.

2. In the following, assume $f : \mathbb{R} \to \mathbb{R}$ has 3 continuous derivatives. A common usage of Taylor's theorem is to construct *finite difference schemes* for approximating the derivatives of f. Let's take a look at some basic problems in this vein:

   (a) Let $x \in \mathbb{R}$ and $h > 0$. Show that there exists constants M_1, M_2, possibly depending on x, h, and f and its derivatives, such that

   $$M_1 h \leq \left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) \leq M_2 h$$

   This shows the "first-order accuracy" of the forward-difference approximation. (*Hint*: Use Taylor's theorem for $x, x+h$)

   (b) Let $x \in \mathbb{R}$ and $h > 0$. Show that there exists constants M_3, M_4, possibly depending on x, h, and f and its derivatives, such that

   $$M_3 h^2 \leq \left(\frac{f(x+h) - f(x-h)}{2h} - f'(x) \right) \leq M_4 h^2$$

   This shows the "second-order accuracy" of the centered-difference approximation. (*Hint*: Use Taylor's theorem for $x, x+h$ and $x, x-h$)

   (c) Let $k \in \mathbb{R}$, and let $f(x) := \sin(kx)$. Show that for all $x \in \mathbb{R}$ and $h > 0$,

   $$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \frac{k^2 h}{2}$$
   $$\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| \leq \frac{k^3 h^2}{6}$$

   Remark: Problem (c) shows that while generally "second-order accuracy" is better, the centered-difference scheme can actually do worse if your function has oscillations on the scale of your stepsize, i.e. $kh > 3/2$. This might happen if your function f has "high-frequency noise".

(a) The restriction of f to $[x, x+h]$ satisfies Taylor's theorem, so there exists some $c \in (x, x+h)$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2$$

which can be rearranged

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{f''(c)}{2}h$$

Since f'' is continuous, it is bounded on the interval $[x, x+h]$. Set

$$M_1 := \inf_{x \in [x, x+h]} \frac{f''(x)}{2} \qquad M_2 := \sup_{x \in [x, x+h]} \frac{f''(x)}{2}$$

then since $h > 0$, we have

$$M_1 h \le f'(x) - \frac{f(x+h) - f(x)}{h} \le M_2 h$$

as desired.

(b) The restriction of f to $[x-h, x]$ and to $[x, x+h]$ both satisfy Taylor's theorem, so there exists some $c_1 \in (x-h, x)$ and some $c_2 \in (x, x+h)$ satisfying

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(c_2)}{6}h^3$$
$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(c_1)}{6}h^3$$

We can subtract the second equation from the first and rearrange to get

$$\frac{f(x+h) - f(x-h)}{2h} - f'(x) = \frac{f'''(c_2) + f'''(c_1)}{12}h^2$$

Since f''' is continuous, it is bounded on the interval $[x-h, x+h]$. Set

$$M_3 := \inf_{x \in [x-h, x+h]} \frac{f'''(x)}{6} \qquad M_4 := \sup_{x \in [x-h, x+h]} \frac{f'''(x)}{6}$$

Then since $h^2 > 0$, we have

$$M_3 h^2 \le f'(x) - \frac{f(x+h) - f(x-h)}{2h} \le M_4 h^2$$

(c) We have that $f'(x) = k\cos(kx)$, $f''(x) = -k^2\sin(kx)$, and $f'''(x) = -k^3\cos(kx)$. Noting that $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for any $x \in \mathbb{R}$, we have that the constants M_i in (a) and (b) satisfy

$$-\frac{k^2}{2} \leq M_1 \leq M_2 \leq \frac{k^2}{2}$$
$$-\frac{k^3}{6} \leq M_3 \leq M_4 \leq \frac{k^3}{6}$$

for any $x \in \mathbb{R}$ and $h > 0$. Then, using the results of (a) and (b), we have

$$-\frac{k^2h}{2} \leq f'(x) - \frac{f(x+h)-h}{h} \leq \frac{k^2h}{2}$$
$$-\frac{k^3h^2}{3} \leq f'(x) - \frac{f(x+h)-f(x-h)}{2h} \leq \frac{k^3h^2}{6}$$

which are equivalent to the desired inequalities.