Taylor's Theorem

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Def. If $f: I \to \mathbb{R}$ is differentiable, we call $f': I \to \mathbb{R}$ the <u>first derivative</u>

" $f': I \to \mathbb{R}$ " " $f'': I \to \mathbb{R}$ the <u>second derivative</u>

$f''', \dots, f^{(n)}$ is the <u>nth derivative</u>

If f possesses n derivatives, we say f is <u>n times differentiable</u>

Taylor's theorem

Def. For an n times differentiable function $f: I \to \mathbb{R}$ with x_0 in the interior of the interval I, define the <u>nth order Taylor polynomial</u> for f at x_0 as:

$$P_n^{x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2}\cdot(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Thrm. (Taylor)

Suppose $f: [a,b] \to \mathbb{R}$ is a function with n continuous derivatives on $[a,b]$ and such that $f^{(n+1)}$ exists on (a,b). Given distinct points $x_0, x \in [a,b]$, we can find $\underbrace{c \text{ strictly between } x_0, x}_{c \in (x, x_0) \text{ or } c \in (x_0, x)}$ such that

$$f(x) = P_n^{x_0}(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}$$

"Lagrange form" of the remainder term

Remarks: Reading 1: $f(x) =$ taylor polynomial + "$O((x-x_0)^{n+1})$"

Reading 2: there exists a solution c_{x,x_0} to the equation

$$\frac{f^{(n+1)}(c)}{(n+1)!} = \frac{f(x) - P_n^{x_0}(x)}{(x-x_0)^{n+1}}$$

Different from Taylor series!

$$\frac{\quad}{(n+1)!} = \quad (x-x_0)^{n+1}$$

<u>Note</u>: Taylor polynomial defined for $n \in \mathbb{N}$. Different from Taylor series! (see section 2.6 on power series)

<u>Idea</u>: MVT is a "special case" of Taylor's theorem:

Take $f: I \to \mathbb{R}$ continuous + differentiable, and $x, x_0 \in I$, $x < x_0$. Then,

$$f'(c) = \frac{f(x)-f(x_0)}{x-x_0} \Rightarrow \underbrace{f(x)}_{f(x)} \overset{\text{actual equality}}{=} \underbrace{f(x_0)}_{\substack{\text{0th order}\\ \text{Taylor polynomial}}} + f'(c) \cdot (x-x_0)$$

$c \in (x, x_0)$

$c = c(x, x_0; f, f')$

<u>Pf.</u> (By solving an equation). Define:

$$M := \boxed{\frac{f(x) - P_n^{x_0}(x)}{(x-x_0)^{n+1}}}$$

all of these are "given" quantities!

- We want to show that there exists some c satisfying

$$\frac{f^{(n+1)}(c)}{(n+1)!} = M$$

- Define a function (note: x, x_0 are given, hence fixed. Thus, s will be our variable)

$$g(s) := f(s) - P_n^{x_0}(s) - M \cdot (s-x_0)^{n+1}$$

- Computing derivatives of $g(s)$,

$$g^{(k)}(s) = f^{(k)}(s) - \left(P_n^{x_0}\right)^{(k)}(s) - \underbrace{M \cdot k! (s-x_0)^{n+1-k}}_{\text{power rule } \frac{d}{dx}x^n = nx^{n-1}}$$

  - Plug in $s = x_0$: Since $\left(P_n^{x_0}\right)^{(k)}(x_0) = f^{(k)}(x_0)$,

$$g(x_0) = g'(x_0) = g''(x_0) = \cdots = g^{(n)}(x_0) = 0$$

  - Plug in $s = x$:

$$g(x) = f(x) - P_n^{x_0}(x) - M \cdot (x-x_0)^{n+1}$$

- Plug in $s = x$.

$$g(x) = f(x) - P_n^{x_0}(x) - M\cdot(x-x_0)^{n+1}$$
$$= 0 \qquad \text{(by def. of } M\text{)}$$

- In particular, $g(x_0) = g(x) = 0$. By MVT, there exists x_1 strictly between x_0, x satisfying

$$g'(x_1) = \frac{g(x_0)-g(x)}{x_0-x} = 0$$

- $g'(x_0) = g'(x_1) = 0$. By MVT, $\exists x_2$ strictly between x_0, x_1 (hence strictly between x_0, x) satisfying

$$g''(x_2) = \frac{g'(x_1)-g'(x_0)}{x_1-x_0} = 0$$

- Repeat this process: $\exists\, x_{n+1}$ strictly between x_0, x_n (hence strictly between x_0, x) satisfying

$$g^{(n+1)}(x_{n+1}) = \frac{g^{(n)}(x_n)-g^{(n)}(x_0)}{x_n-x} = 0$$

- We compute from the def. of $g(s)$,

$$g^{(n+1)}(s) = f^{(n+1)}(s) - (n+1)!\cdot M$$

- Plug in $s = x_{n+1}$. Then,

$$0 = f^{(n+1)}(x_{n+1}) - (n+1)!\cdot M \Rightarrow M = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}$$

- Take $c := x_{n+1}$. This shows the desired claim. $\square$

Ex. From calculus: $\sin(x) \approx x - \frac{x^3}{3!} + O(x^4)$

Now: $\sin(x) = x - \frac{x^3}{3!} + \frac{\sin(c)}{4!}x^4 \quad \left(\frac{d^4}{dx^4}\sin(x) = \sin(x)\right)$

In particular, $\left|\sin(x) - x + \frac{x^3}{3!}\right| = \left|\frac{\sin(c)}{4!}x^4\right| = |\sin(c)|\cdot\frac{|x|^4}{4!} \leq \frac{|x|^4}{4!}$

In particular, $\left|\sin(x) - x + \frac{x^3}{3!}\right| = \left|\frac{\sin(c)}{4!}x^4\right| = |\sin(c)|\cdot\frac{|x|^4}{4!} \underbrace{\leq}_{|\sin(c)|\leq 1} \frac{|x|^4}{4!}$

Prop. (Second derivative test)

Suppose $f:(a,b)\to\mathbb{R}$ is twice continuously differentiable, and there exists $x_0\in(a,b)$ satisfying $f'(x_0)=0$ and $f''(x_0)>0$. Then f has a strict relative minimum at x_0.

Pf. As f'' is continuous and $f''(x_0)>0$, there exists $\delta>0$ such that

$$f''(s)>0 \quad \forall s\in(x_0-\delta, x_0+\delta) \qquad \text{(HW exercise)}$$

Take $x\in(x_0-\delta, x_0+\delta)$ with $x\neq x_0$. By Taylor's theorem, there exists some c between x_0, x satisfying

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(c)}{2}(x-x_0)^2 = f(x_0) + \frac{f''(c)}{2}(x-x_0)^2$$

Since $c\in(x_0-\delta, x_0+\delta)$, $f''(c)>0$. Also $(x-x_0)^2>0$, so

$$f(x)>f(x_0)$$

$\square$