Sequences, Convergence

Tuesday, September 20, 2022 2:25 PM

Sequences

Def. Let $f: D \to \mathbb{R}$ be a function. f is <u>bounded</u> if there exists $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in D$.

- If $f: D \to \mathbb{R}$ is bounded, we define

$$\sup_{x \in D} f(x) := \sup f(D)$$

$$\inf_{x \in D} f(x) := \inf f(D)$$

(note: $\forall f(x) \in f(D)$, $-B \leq f(x) \leq B$)

Def. A <u>sequence</u> (of real numbers) is a function $x: \mathbb{N} \to \mathbb{R}$.

- we write $x_n := x(n)$
- we denote the sequence as a whole by

$$\{x_n\} \quad \text{or} \quad \{x_n\}_{n=1}^{\infty}$$

- A sequence is <u>bounded</u> if there exists $B \in \mathbb{R}$ such that

$$|x_n| \leq B \quad \text{for all } n \in \mathbb{N}$$

Equivalently, $\{x_n : n \in \mathbb{N}\}$ is bounded as a set

$x: \mathbb{N} \to \mathbb{R}$ " as a function

Ex. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

$\{1\}_{n=1}^{\infty} = 1, 1, 1, \ldots$

★★★

Def. A sequence $\{x_n\}$ is said to <u>converge</u> to $\underbrace{\text{a number } L \in \mathbb{R}}_{(0)}$ if, $\underbrace{\text{for every } \varepsilon > 0}_{(1)}$, $\underbrace{\text{there exists } M \in \mathbb{N}}_{(2)}$ such that

$$\underbrace{|x_n - L| < \varepsilon}_{(4)} \quad \underbrace{\text{for all } n \geq M}_{(3)}$$

symbolically: $\underset{(1)}{\forall \varepsilon > 0}, \ \underset{(2)}{\exists M \in \mathbb{N}} : \underset{(3)}{\forall n \geq M}, \ \underset{(4)}{|x_n - L| < \varepsilon}$

- A sequence that converges is called <u>convergent</u>
- We call L the limit of $\{x_n\}$ as $n \to \infty$, and write

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- We call L the <u>limit</u> of $\{x_n\}$ as $n \to \infty$, and write

$$\lim_{n\to\infty} x_n = L$$

- A sequence which does not converge is called <u>divergent</u>

<u>Ex.</u> (constant sequence)

<u>Claim</u>: $\{x_n\} = \{1\}$, $\lim_{n\to\infty} x_n = 1$

<u>Pf.</u> For $\underbrace{\text{any given } \varepsilon > 0}_{(1)}$, let $\underbrace{M = 1 \in \mathbb{N}}_{(2)}$. Then, $\underbrace{\text{for all } n \geq M = 1}_{(3)}$,

$$|x_n - 1| = |1 - 1| = 0 \underbrace{<}_{(4)} \varepsilon$$

Thus, $\lim_{n\to\infty} x_n = 1$ ✓

<u>Claim</u>: $\{\frac{1}{n}\}$ is convergent, and $\lim_{n\to\infty} \frac{1}{n} = 0$

<u>Pf.</u> $\underbrace{\text{Given } \varepsilon > 0}_{(1)}$, by the Archimedean prop. $\underbrace{\text{there exists } M \in \mathbb{N}}_{(2)}$ such that $M \cdot \varepsilon > 1 \Rightarrow \frac{1}{M} < \varepsilon$. Then, $\underbrace{\text{for all } n \geq M}_{(3)}$, we have

$$|x_n - x| = |\tfrac{1}{n} - 0| = \tfrac{1}{n} \leq \tfrac{1}{M} \underbrace{<}_{(4)} \varepsilon$$

✓

<u>Claim</u>: $\{(-1)^n\}_{n=1}^{\infty} = -1, 1, -1, 1, \cdots$ diverges.

<u>Pf.</u> Suppose $\{x_n\}$ converges to some $L \in \mathbb{R}$.

Then for $\underbrace{\varepsilon = \tfrac{1}{2} > 0}_{(1)}$, there exists $\underbrace{M \in \mathbb{N}}_{(2)}$ s.t. $\underbrace{\forall n \geq M}_{(3)}$, $|x_n - L| < \varepsilon$

- For even $n \geq M$, this implies

$$\left.\begin{aligned} |x_n - L| &= |1 - L| < \varepsilon = \tfrac{1}{2} \\ |x_{n+1} - L| &= |-1 - L| < \varepsilon = \tfrac{1}{2} \end{aligned}\right\} (4)$$

- Then we would have

$$\begin{aligned} 2 &= |(1-L) + (1+L)| \\ &\leq |1-L| + |1+L| < 2\varepsilon = 1 \end{aligned}$$

Contradiction!

- $\{(-1)^n\}$ cannot be convergent. ✓

Graphical Idea of Convergence

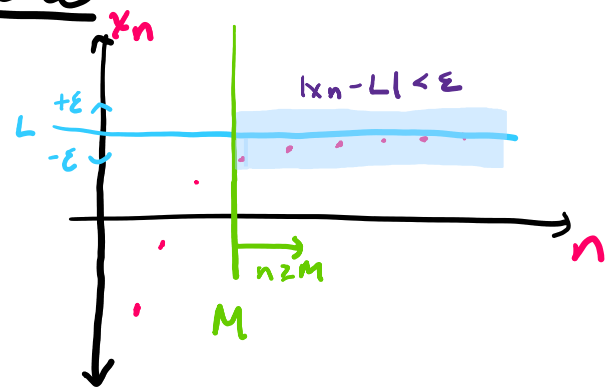
(0) $\lim_{n\to\infty} x_n = L \in \mathbb{R}$ if

(1) for all $\varepsilon > 0$,

(2) there exists $M \in \mathbb{N}$ s.t.

(3) for all $n \geq M$,

(4) $|x_n - L| < \varepsilon$



Prop. A convergent sequence is bounded

Cor. Contrapositive: A sequence which is not bounded is divergent

Pf. Suppose $\{x_n\}$ is convergent to limit L.

- Then, $\underset{(2)}{\exists M \in \mathbb{N}} : \underset{(3)}{\forall n \geq M}, \underset{(4)}{|x_n - L| < 1} \underset{(1)}{= \varepsilon}$
- Let $B_1 := |L| + 1$. Then for $n \geq M$ we have
  $$|x_n| = |x_n - L + L| \overset{\text{triangle ineq.}}{\leq} |x_n - L| + |L| \overset{(4)}{<} 1 + |L| = B_1$$
- $\{|x_1|, |x_2|, \dots, |x_{M-1}|\}$ is a finite set, so take
  $$B_2 := \max\{|x_1|, |x_2|, \dots, |x_{M-1}|\}$$
- Take $B := \max\{B_1, B_2\}$. Then, $\forall n \in \mathbb{N}$, $|x_n| \leq B$
- Thus, $\{x_n\}$ is bounded (by B). $\square$

Warning: convergent $\Rightarrow$ bounded, but bounded $\not\Rightarrow$ convergent!
$\{(-1)^n\}$ is divergent

Prop. A convergent sequence has a unique limit

Remark. $\{(-1)^n\}$ cannot converge to 2 values!

Lemma. "give yourself an ε of room"

Given $x \in \mathbb{R}$, if $x \leq \varepsilon$ for all $\varepsilon > 0$, then $x \leq 0$.

Pf. Let $x \in \mathbb{R}$ be such that for all $\varepsilon > 0$, $x \leq \varepsilon$

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Pf. Let $x \in \mathbb{R}$ be such that for all $\varepsilon > 0$, $x \le \varepsilon$

- Suppose $x > 0$. Then, $0 < \frac{x}{2} < x$, so $\exists\, \varepsilon = \frac{x}{2} > 0 : x > \varepsilon$.
  Contradiction!
- Thus, $x \le 0$.

$\square$

Pf. (unique limits)

- Suppose $\{x_n\}$ has two limits x, y. Take $\varepsilon > 0$ as arbitrary.
  - So, there exists $M_1 \in \mathbb{N}$ s.t. $\forall n \ge M_1$, $|x_n - x| < \varepsilon/2$ (def of $x_n \to x$)
    " $\quad M_2 \overset{(2)}{\in} \mathbb{N}$ s.t. $\forall \overset{(3)}{n} \ge M_2$, $|x_n \overset{(4)}{-} y| < \overset{(1)}{\varepsilon/2}$ (def of $x_n \to y$)

- Now, let $M := \max\{M_1, M_2\}$. Then, for all $n \ge M$,

$$|y - x| = |x_n - x - (x_n - y)| \quad (+x_n - x_n)$$

$$\le |x_n - x| + |x_n - y| \quad \text{(triangle ineq.)}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (\text{def of limit, } n \ge M_1 \text{ and } n \ge M_2)$$

  - $0 \le |y - x| \le \varepsilon$ for all $\varepsilon > 0 \Rightarrow |y - x| = 0 \Rightarrow x = y$

- Thus if $\{x_n\}$ has a limit, it must be unique.

$\square$