CS473 Machine Learning HW1

September 2022

Question 1

In order for player 1 to win the game, he needs to either succeed in the 1st shot, or he fails the 1st shot, player 2 fails 2nd shot, and player 1 scores again and succeed on his 3rd shot, etc. The probability that player 1 succeeds is 1/5, the probability that he fails is 4/5. The probability that player 2 succeeds is 1/4, the probability that player 2 fails is 3/4. Player 1 wins when all previous shots are failed and he succeeds the 1st, 3rd, 5th, 7th, 9th, ... shots. And the probabilities for each is 1/5, (4/5) * (3/4) * (1/5), (4/5) * (3/4) * (4/5) * (3/4) * (1/5), (4/5) * (3/4) * (4/5) * (3/4) * (4/5) * (3/4) * (1/5), (4/5) * (3/4) * (4/5) * (3/4) * (4/5) * (3/4) * (4/5) * (3/4) * (1/5)...

The probability that player 1 wins is the sum of this geometric sequence, where $a = 1/5, r = (4/5) * (3/4) = 3/5$.

So the probability would be $S = a/(1-r) = (1/5)/(1-(3/5)) = (1/5)/(2/5) = 1/2$.

Question 2

$P(COVID) = 0.01$

$P(notCOVID) = 1 - 0.01 = 0.99$

$P(positive \mid COVID) = 0.90$

$P(positive \mid notCOVID) = 0.10$

$P(COVID \mid positive)$

$= (P(positive \mid COVID) * P(COVID))/(P(positive \mid COVID) * P(COVID) + P(positive \mid notCOVID) * P(notCOVID))$

$= (0.01 * 0.9)/(0.01 * 0.9 + 0.99 * 0.1)$

$= 0.09/(0.09 + 0.099)$

$= 0.09/0.189$

$= 47.62\%$

The probability that you have COVID given that you tested positive is 47.62%.

Question 3

No. By definition, a function is a PDF only if the sum of all the area beneath $f(x)$ and above x is equal to 1 and $f(x)$ is greater or equal to 0 for all values of x. Here, we see that when x = 0, $1/(1+x) = 1/1 = 1$, and when x = 1, $1/(1+x) = 1/2$. Only by looking at the function when x = 0 and 1, the sum of the area beneath these numbers are already $1 + 1/2 = 1.5$, which is greater than 1. Therefore, $\int_{-\infty}^{\infty} 1/(1+x)\, dx > 1$. And the sum of all the area beneath $f(x)$ and above x is not equal to 1. Therefore, this function is not a PDF.

Question 4

The function for X+Y given that X and Y have the same density function is

$$g(x) = X + Y = \begin{cases} 4x & if\ 0 \leq x \leq 0 \\ 0 & otherwise \end{cases}$$

$P(X + Y \leq 1) = totalareaofg(x)below1andabove0/totalareaofg(x)above0$. When x = 0.25, g(x) = X+Y = 4x = 1. When x = 1, g(x) = X+Y = 4x = 4. Therefore $g(x) > 1$ when $x > 0.25$. The total area under $g(x)$ and above the x-axis is 1 * 4 / 2 = 2. The total area under $g(x)$ and above $g(x) = 1$ is $(1 - 0.25) * (4 - 1)/2 = 0.75 * 3/2 = 1.125$. And the total area that $X + Y \leq 1$ is $2 - 1.125 = 0.875$.

Therefore, $P(X + Y \leq 1) = 0.875/2 = 0.4375$.

Question 5 $X \sim Unif(0,1), Y = g(X) = e^X$. $\int_0^1 e^x \, dx$
$= e^x + C \mid_0^1$
$= e^1 - e^0$
$= e - 1$.
The value of $E(y)$ is $e - 1$.

Question 6
$P(X = k) = e^{-\lambda} * \lambda^k/k!$ $P(X_n < 5.5)$
$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$
$= e^{-5} * 5^0/0! + e^{-5} * 5^1/1! + e^{-5} * 5^2/3! + e^{-5} * 5^3/3! + e^{-5} * 5^4/4! + e^{-5} * 5^5/5!$
$= 0.6160$
Therefore the value of $P(X < 5.5) is 0.6160$.

Question 7

Question 8

$Ax = \begin{bmatrix} 50 \\ 17 \\ 35 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix}$

$x^T = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

$x^T A = \begin{bmatrix} 33 & 27 & 36 \end{bmatrix}$

Question 9

(a) Yes. A matrix is invertible if and only if its determinant is not equal to 0.

$det \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$

$= 6 * det \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} - 2 * det \begin{bmatrix} 3 & 1 \\ 10 & 4 \end{bmatrix} + 3 * det \begin{bmatrix} 3 & 1 \\ 10 & 3 \end{bmatrix}$

$= 6 * (1 * 4 - 1 * 3) - 2 * (3 * 4 - 1 * 10) + 3 * (3 * 3 - 1 * 10)$

$= -1 \neq 0$

This matrix is invertible. $A^-1 = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$

(b) No. A matrix is not invertible if its determinant is equal to 0.

$det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

$= 1 * det \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix} - 0 + 1 * det \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

$= 1 * (2 * 5 - 2 * 4) - 0 + 1 * (2 * 2 - 3 * 2)$

$= 0$

This matrix's determinant is equal to 0, so it is not invertible.

Question 10 Eigenvectors of a square matrix are vectors such that the matrix acts on such vectors, they remain in the same direction. An eigenvector of an n*n square matrix A is a nonzero vector $\overrightarrow{x}, such that$A $\overrightarrow{x} = \lambda\overrightarrow{x}$ for some scalar λ, and these λ are the eigenvalue of A.

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ We first find the determinant of $(A - \lambda I)$ and solve for $det(A - \lambda I) = 0$.

$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 0-\lambda & 0 \\ -2 & 2 & 1-\lambda \end{bmatrix}$

$= -\lambda^3 + 2\lambda^2 + \lambda - 2$

$= -(\lambda + 1)(\lambda - 1)(\lambda - 2)$

$\lambda = 1, \lambda = -1, \lambda = 2$

When $\lambda = 1$,

$A - \lambda I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Here, we get the following system of equations:

$$\begin{cases} x_1 - x_2 + 0x_3 = 0 \\ 0x_1 - 0x_2 + x_3 = 0 \end{cases}$$

Simplifying it, we get x$_3 = 0, x_1 = x_2, x_2 = x_2. Therefore, when \lambda = 1$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

When $\lambda = -1$,

$A - \lambda I = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Here, we get the following system of equations:

$$\begin{cases} x_1 + x_2 + 0x_3 = 0 \\ 0x_1 + 2x_2 + x_3 = 0 \end{cases}$$

Simplifying it, we get x$_3 = x_3, x_2 = -1/2x_3, x_1 = 1/2x_3. Therefore, when \lambda = -1$, the eigenvector is $\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

When $\lambda = 2$,

$A - \lambda I = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Here, we get the following system of equations:

$$\begin{cases} x_1 - 2x_2 + 0x_3 = 0 \\ 0x_1 + 2x_2 + x_3 = 0 \end{cases}$$

Simplifying it, we get $x_3 = x_3, x_2 = -1/2x_3, x_1 = x_3. Therefore, when \lambda = -1$, the eigenvector is $\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$.

Therefore, the eigenvalues are 1, -1, and 2. The eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$.