Interchange of Limits

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Recall: Types of convergence:

$$\underbrace{\text{cauchy in uniform norm} \Leftrightarrow \text{convergent in uniform norm}}_{\text{bounded functions}} \Leftrightarrow \text{uniform convergence} \Leftarrow \text{ptwise convergence}$$

Motivation:

$$\lim_{x \to 1^-} \lim_{n \to \infty} x^n = 0 \neq 1 = \lim_{n \to \infty} \lim_{x \to 1^-} x^n$$

Observe: x^n are continuous but limit $f(x) := \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$ is not!

Continuity of Limits

Thrm. (Uniform convergence preserves continuity)

Let $\{f_n\}$, $f_n : S \to \mathbb{R}$, be a sequence of continuous functions that converges uniformly to some $f : S \to \mathbb{R}$. Then f is continuous.

Remark:
$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{x \to c} f(x) = f(c) = \lim_{n \to \infty} f_n(c) = \lim_{n \to \infty} \lim_{x \to c} f_n(x)$$

"can switch order of limits"

Pf. Let $x \in S$ be fixed, let $\{x_k\}$ be a sequence in S converging to x.

- Let $\varepsilon > 0$ be arbitrary. As $\{f_n\}$ converges uniformly to f, (1)

$$\exists M \in \mathbb{N} : \forall y \in S, \forall n \geq M, \ |f_n(y) - f(y)| < \varepsilon/3$$

- As f_M is continuous, (2) $\exists K \in \mathbb{N} : \forall k \geq K, \ |f_M(x_k) - f_M(x)| < \varepsilon/3$

- Thus, (3) $\forall k \geq K$,

$$|f(x_k) - f(x)| = |f(x_k) - f_M(x_k) + f_M(x_k) - f_M(x) + f_M(x) - f(x)|$$

• Thus, $\forall k \geq K$,

$$|f(x_k) - f(x)| = |f(x_k) - f_M(x_k) + f_M(x_k) - f_M(x) + f_M(x) - f(x)|$$

$$\leq \underbrace{|f(x_k) - f_M(x_k)|}_{\text{unif. conv.}} + \underbrace{|f_M(x_k) - f_M(x)|}_{f_M \text{ cont.}} + \underbrace{|f_M(x) - f(x)|}_{\text{uniform conv.}}$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad (4)$$

$$\Rightarrow \lim_{k\to\infty} f(x_k) = f(x)$$

• Since this holds for all $\{x_k\}$, f is continuous at x

• Since this holds for all $x \in S$, f is continuous. $\square$

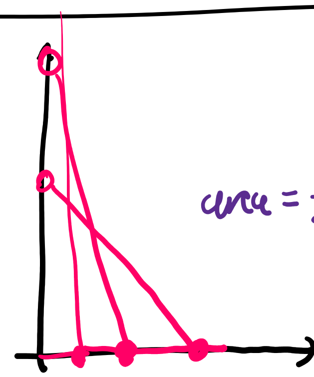
Ex. $f_n : [0,a] \to \mathbb{R}$ $\quad f_n(x) := x^n \quad 0<a<1$

$f(x) := 0$

$\|f_n - f\|_u = a^n \to 0$ as $n \to \infty \Rightarrow f_n \to f$ uniformly.

Observe that f is continuous.

Integrals of Limits



area $= \frac{1}{2} b \cdot h = \frac{1}{2}$

Ex. Let $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) := \begin{cases} 0 & x = 0 \\ n - n^2 x & 0 < x < 1/n \\ 0 & x \geq 1/n \end{cases}$$

• $f_n \in \mathcal{R}[0,1]$ (finite # of discontinuities)

• $f_n \to 0$ pointwise (e.g. $x>0$, $f_n(x) = 0 \;\; \forall n \geq M = 1/x$)

convergence is not uniform

• $\int_0^1 f_n = \frac{1}{2} n \cdot \frac{1}{n} = \frac{1}{2}$ (e.g. by FTC)

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$$\lim_{n\to\infty} \int_0^1 f_n = \frac{1}{2} \neq 0 = \int_0^1 \left(\lim_{n\to\infty} f_n\right) = \int_0^1 0$$

Thrm. Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$ converging uniformly to $f:[a,b]\to\mathbb{R}$. Then, $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \quad \left(= \int_a^b \lim_{n\to\infty} f_n\right)$$

Pf. Let $\overset{(1)}{\varepsilon > 0}$ be arbitrary.

$\exists \overset{(2)}{M} \in \mathbb{N} : \forall n \overset{(3)}{\geq} M, \forall x \in [a,b], \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}$

- $|f(x)| \leq \frac{\varepsilon}{2(b-a)} + |f_M(x)| \Rightarrow f$ is bounded since f_M is bounded, so Darboux integrals exist.

$$\overline{\int_a^b} f - \underline{\int_a^b} f = \overline{\int_a^b}(f - f_n + f_n) - \underline{\int_a^b}(f - f_n + f_n)$$

$$\leq \overline{\int_a^b}(f - f_n) + \cancel{\overline{\int_a^b} f_n} - \left(\underline{\int_a^b}(f - f_n) + \cancel{\underline{\int_a^b} f_n}\right) \quad \left(\overline{\int} f+g \leq \overline{\int} f + \overline{\int} g\right)$$

$$= \overline{\int_a^b} f - f_n - \underline{\int_a^b} f - f_n \quad \left(|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}\right)$$

$$\leq \frac{\varepsilon}{2(b-a)} \cdot (b-a) + \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \varepsilon$$

$\Rightarrow \forall \varepsilon > 0, \ \overline{\int_a^b} f - \underline{\int_a^b} f \leq \varepsilon \Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f \leq 0$

$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f \Rightarrow f \in \mathcal{R}[a,b]$

To compute $\int_a^b f$,

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$$\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b f - f_n\right| \leq \frac{\varepsilon}{2(b-a)} \cdot |b-a| \overset{(4)}{<} \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n = \int_a^b f \qquad \square$$

Derivatives of Limits

Ex. $f_n : \mathbb{R} \to \mathbb{R}$, $\quad f_n(x) := \frac{\sin(nx)}{n}$

$\|f_n - 0\|_u = \frac{1}{n} \to 0$ as $n \to \infty$

- $\{f_n\}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$, $f(x) := 0$
- $f_n'(x) = \cos(nx)$ does not converge ptwise to any $g : \mathbb{R} \to \mathbb{R}$
- However, $f'(x) = 0$.

$\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = 0$, $\quad \{f_n'(x)\}$ does not converge for most $x \in \mathbb{R}$!

Thm. Let I be a bounded interval, and let $f_n : I \to \mathbb{R}$ be cts. differentiable functions. Suppose $\{f_n'\}$ converges uniformly to some $g : I \to \mathbb{R}$, and suppose $\{f_n(c)\}$ is convergent for some $c \in I$.

Then, $\{f_n\}$ converges uniformly to some cts. diff. function $f : I \to \mathbb{R}$, and $f' = g$

$$f' = \frac{d}{dx}\left(\lim_{n\to\infty} f(x)\right) = \lim_{n\to\infty}\left(\frac{d}{dx} f(x)\right) = g$$

Remark: Continuity of f_n' not required

Pf. (By FTC)

- Define $f(c) := \lim_{n\to\infty} f_n(c)$
- ... are Riemann integrable.

- Define $f(c) := \lim_{n\to\infty} f_n(c)$
- As f_n' are continuous, they are Riemann integrable.
  - By FTC, $\forall x \in I$,
    $$f_n(x) = f_n(c) + \int_c^x f_n'$$
  - Since $f_n' \to g$ uniformly on I, it will also converge uniformly on $[x,c]$ or $[c,x] \subset I$. Therefore,
    $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(c) + \lim_{n\to\infty} \int_c^x f_n'$$
    $$= f(c) + \int_c^x \lim_{n\to\infty} f_n' = f(c) + \int_c^x g =: f(x)$$
  - Since g is the uniform limit of cts. functions, g is also cts.
  - By 2nd form FTC, f is differentiable, with $f'(x) = g(x)\ \forall x \in I$
- To show $f_n \to f$ uniformly, let $\overset{(1)}{\varepsilon > 0}$ be arbitrary.

  $\underset{(2)}{\exists M \in \mathbb{N}}$: $\underset{(3)}{\forall n \geq M}$, $|f_n(c) - f(c)| < \varepsilon/2$ and $(f_n(c) \to f(c))$

  $|f_n'(x) - g(x)| < \frac{\varepsilon}{2\cdot(b-a)}$ $(f_n' \to g \text{ unif.})$
- Let $\overset{(3)}{x \in I}$ be arbitrary.
  $$|f_n(x) - f(x)| = \left|f_n(c) + \int_c^x f_n' - f(c) - \int_c^x g\right|$$
  $$\leq |f_n(c) - f(c)| + \left|\int_c^x f_n' - g\right|$$
  $$\underline{<}\ \varepsilon/2 + \frac{\varepsilon}{2\cdot(b-a)}\cdot(b-a) = \underset{(4)}{\underline{\varepsilon}}$$
- Thus, $f_n \to f$ uniformly.