Integrability Tests, FTC

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Lemma. If $f:[a,b]\to\mathbb{R}$ is continuous, then $f\in\mathcal{R}[a,b]$.

Pf. Since f is continuous on a closed and bounded interval, it is uniformly continuous.

- Let $\varepsilon>0$ be arbitrary. $\exists\delta>0:\ \forall x,y\in[a,b]$ with $|x-y|<\delta$, $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$
- Let $P:=\{x_0,\dots,x_n\}$ be a partition of $[a,b]$ satisfying $\Delta x_i<\delta$ for $i=1,\dots,n$.
  - For example, take $n\in\mathbb{N}$ with $\frac{b-a}{n}<\delta$ and let $x_k=\frac{k}{n}(b-a)+a$.
  - Then, $\forall x,y\in[x_{i-1},x_i]$, $|x-y|\le\Delta x_i<\delta$ so

$$f(x)-f(y)\le|f(x)-f(y)|<\frac{\varepsilon}{b-a}$$

- As f is continuous on $[x_{i-1},x_i]$, it achieves a max and min on this interval.
  - Thus, for some $x,y\in[x_{i-1},x_i]$,

$$M_i-m_i=f(x)-f(y)<\frac{\varepsilon}{b-a}$$

- Thus,

$$\begin{aligned}0\le\overline{\int_a^b}f-\underline{\int_a^b}f&\le U(P,f)-L(P,f)\\&=\left(\sum_{i=1}^n M_i\Delta x_i\right)-\left(\sum_{i=1}^n m_i\Delta x_i\right)\\&=\sum_{i=1}^n(M_i-m_i)\Delta x_i\\&<\frac{\varepsilon}{b-a}\sum_{i=1}^n\Delta x_i\\&=\frac{\varepsilon}{b-a}(b-a)=\varepsilon\end{aligned}$$

- As $\varepsilon>0$ was arbitrary, $0\le\overline{\int_a^b}f-\underline{\int_a^b}f\le0$

$$\Rightarrow\ \overline{\int_a^b}f=\underline{\int_a^b}f$$

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Assigned Readings: Be familiar with the results of lemma 5.2.8, thrm. 5.2.9

"Functions with a finite number of discontinuities are Riemann Integrable" covers "most" cases

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Fundamental Theorem of Calculus

"Integrals are antiderivatives"

1st form:	2nd form:
$\int_a^b F'(x)\,dx = F(b) - F(a)$	$\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$
integral of a derivative	derivative of an integral

Thrm. (First form FTC)

- Let $F:[a,b] \to \mathbb{R}$ be a continuous function, differentiable on (a,b)
- Let $f \in \mathcal{R}[a,b]$ be such that $f(x) = F'(x) \ \forall x \in (a,b)$

  Remark: not all derivatives are (proper) Riemann integrable, e.g. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ is not bounded

- Then,

$$\int_a^b f = F(b) - F(a)$$

Remark: Can generalize to finitely many pts. where F is not diff.

Pf. Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. For each interval $[x_{i-1}, x_i]$, we can use MVT on F: $\exists c_i \in (x_{i-1}, x_i)$:

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x_i$$

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$$\Rightarrow \quad m_i \Delta x_i \le f(c_i)\Delta x_i = F(x_i) - F(x_{i-1}) \le M_i \Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \le \underbrace{\sum_{i=1}^{n} F(x_i) - F(x_{i-1})}_{\text{telescoping sum}} \le \sum_{i=1}^{n} M_i \Delta x_i$$

$$F(x_n) - \cancel{F(x_{n-1})} + \left(\cancel{F(x_{n-1})} - F(x_{n-2})\right) + \cdots$$

$$\Rightarrow L(P,f) \underbrace{\le}_{\text{UB}} F(b) - F(a) \underbrace{\le}_{\text{LB}} U(P,f)$$

$$\Rightarrow \overbrace{\underline{\int_a^b} f}^{\sup L} \le F(b) - F(a) \le \overbrace{\overline{\int_a^b} f}^{\inf U}$$

- Since $f \in R[a,b]$,

$$\underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f \quad \Rightarrow \quad \int_a^b f = F(b) - F(a)$$

$\square$

Thm. (Second form FTC)

Let $f \in R[a,b]$. Define

$$F(x) := \int_a^x f$$

Then,

(i) F is Lipschitz continuous on $[a,b]$ (i.e. "bounded rate of change")

(ii) If f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Pf. As f is bounded, $\exists M > 0 : \forall x \in [a,b],\ -M \le f(x) \le M$

- Suppose $x, y \in [a,b]$ with $x > y$

$$|F(x) - F(y)| \underset{\text{by def.}}{=} \left| \int_a^x f - \int_a^y f \right| \underset{\text{additivity}}{=} \left| \int_y^x f \right| \underset{\text{monotonicity}}{\le} M|x-y|$$

  - By symmetry, this also holds for $y > x$

- By symmetry, this also holds for $y > x$

$\Rightarrow |F(x) - F(y)| \leq M|x-y| \quad \forall x, y \in [a,b]$

$\Rightarrow F$ is Lipschitz continuous. ✓

- Suppose f is continuous at $c \in [a,b]$.
  - Let $\varepsilon > 0$ be arbitrary. $\exists \delta > 0 : \forall x \in [a,b]$ with $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon$

    $\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

  - If $x > c$ with $|x-c| < \delta$,

    $$(f(c) - \varepsilon) \cdot (x - c) \leq \int_c^x f \leq (f(c) + \varepsilon) \cdot (x-c) \qquad \text{(by monotonicity)}$$

    If $x < c$ with $|x-c| < \delta$,

    $$-(f(c) - \varepsilon) \cdot (c - x) \geq \int_c^x f = -\int_x^c f \geq -(f(c) + \varepsilon) \cdot (c - x)$$

  - Combining cases: when $x \neq c$ with $|x-c| < \delta$,

    $$f(c) - \varepsilon \leq \frac{\int_c^x f}{x-c} \leq f(c) + \varepsilon$$

  - Using def. of F,

    $$\frac{F(x) - F(c)}{x-c} = \frac{\int_a^x f - \int_a^c f}{x-c} = \frac{\int_c^x f}{x-c}$$

$\Rightarrow \left| \frac{F(x) - F(c)}{x-c} - f(c) \right| \leq \varepsilon \quad \forall x \in [a,b] \setminus \{c\}$ with $|x-c| < \delta$

$\Rightarrow F'(c) = \lim_{x \to c} \frac{F(x) - F(c)}{x - c} = f(c)$ $\square$

Final remarks:

What if we want to integrate unbounded functions or unbounded domains?

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$\rightarrow$ Improper Riemann Integrals.

Ex. $\int_1^\infty \frac{1}{x^p}\,dx := \lim_{c\to\infty} \int_1^c \frac{1}{x^p}\,dx$ (unbounded domain)

$\int_0^1 \frac{1}{x^p}\,dx := \lim_{c\to 0} \int_c^1 \frac{1}{x^p}\,dx$ ($\frac{1}{x^p}$ unbounded as $x \to 0$ for $p>0$)