Lim Sup/Inf, Bolzano-Weierstrass

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Lim Sup/Inf

Def. Let $\{x_n\}$ be a bounded (not necessarily convergent) sequence.
Define $\{a_n\}, \{b_n\}$ by:

$$a_n := \sup\{x_k : x \geq n\}$$
$$b_n := \inf\{x_k : x \geq n\}$$

Then we define (if the limits exist)

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n$$

Ex. $\{\frac{1}{n}\}$

$a_n = \sup\{\frac{1}{k} : k \geq n\} = \frac{1}{n}$

$b_n = \inf\{\frac{1}{k} : k \geq n\} = 0$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} 0 = 0$$

Prop. Let $\{x_n\}$ be a bounded sequence, and let $\{a_n\}, \{b_n\}$ be defined as above.

(i) $\{a_n\}$ is a bounded monotone decreasing sequence
$\{b_n\}$ " increasing " (existence)

So $\limsup_{n\to\infty} x_n \left(= \lim_{n\to\infty} a_n\right)$ and $\liminf_{n\to\infty} x_n$ exist.

(ii) $\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\}$ (formula)

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$\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\}$

(iii) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ (inequality)

Pf. (i) Show $\{a_n\}$ is increasing.

- Define $S_n := \{x_k : k \geq n\}$. $S_{n+1} \subset S_n$

  $\Rightarrow a_{n+1} = \sup S_{n+1} \leq \sup S_n = a_n$

  Thus, $\{a_n\}$ is monotone decreasing.

- To show $\{a_n\}$ is bounded, $S_n \subset S_1$

  $\Rightarrow \inf S_1 \leq \inf S_n \leq \sup S_n \leq \sup S_1$

  $\Rightarrow \underbrace{b_1} \leq b_n \underbrace{\leq a_n \leq a_1}$

  Thus, $\{a_n\}$ (and $\{b_n\}$) are bounded.

- Thus, $\{a_n\}$ converges ✓, and

  (ii) $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$ ✓

- Proof for $\{b_n\}$ similar.

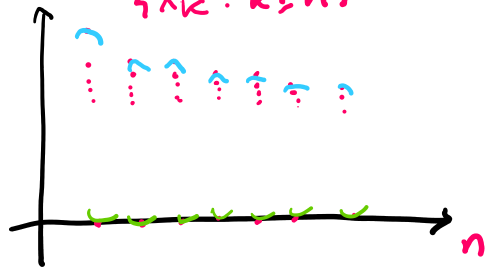
(iii) Since $\{a_n\}, \{b_n\}$ are convergent sequences satisfying $b_n \leq a_n \ \forall n \in \mathbb{N}$,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n \underbrace{\leq}_{\text{limits preserve non-strict ineq.}} \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$$

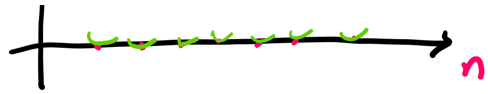
☑

Ex. $x_n := \begin{cases} 1 + \frac{1}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\{x_k : k \geq n\}$



$\sup\{x_n : k \geq n\} = \begin{cases} 1 + \frac{1}{n} & n \text{ odd} \\ 1 + \frac{1}{n+1} & n \text{ even} \end{cases}$

$\sup\{x_k : k \geq n\} = \begin{cases} 1 + \frac{1}{n+1} & n \text{ even} \end{cases}$ 

$= a_n$

$\inf\{x_k : k \geq n\} = 0$

$\liminf_{n\to\infty} x_n = 0 \leq \limsup_{n\to\infty} x_n = 1$

Existence of convergent subsequences (Bolzano-Weierstrass)

Thm. (2.3.4)

If $\{x_n\}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$$

Similarly, there exists a (possibly different) subsequence $\{x_{m_k}\}$ s.t.

$$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$$

Idea: Can extract a convergent sequence from any bounded sequence

Pf Strategy: Inductively define $\{n_k\}$, then show convergence.

- Let $a_n := \sup\{x_k : k \geq n\}$

  $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$

- We inductively construct $\{n_k\}$:

  (basis statement) $n_1 = 1$

  (induction step) Suppose n_p is defined.

  $a_{(n_p+1)} = \sup\{x_k : k \geq n_p + 1\}$

  $\Rightarrow \exists\, m \geq n_p + 1$ s.t. $a_{(n_p+1)} - \frac{1}{p+1} < x_m \leq a_{(n_p+1)}$

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$$\sup S - \varepsilon < x_m \leq \sup S$$

Set $n_{(p+1)} := m > n_p$

By induction, we get a subsequence $\{x_{n_k}\}$ since $\{n_k\}$ is strictly increasing.

(existence) ✓

- To show $\{x_{n_k}\}$ converges, (book: ε games lecture: squeeze lemma)

$$\underbrace{a_{(n_k+1)}}_{\substack{\text{subseq of} \\ \{a_n\} \\ \to x}} - \underbrace{\frac{1}{k+1}}_{\substack{\{\frac{1}{k+1}\} \\ \to 0}} < \underbrace{x_{n_{(k+1)}}}_{\substack{\text{1-tail of} \\ \{x_{n_k}\}}} \leq \underbrace{a_{(n_k+1)}}_{\to x} \quad \forall k \in \mathbb{N}$$

- Since $\lim\limits_{n\to\infty} a_{(n_k+1)} - \frac{1}{k+1} = \lim\limits_{n\to\infty} a_{(n_k+1)} = \limsup\limits_{n\to\infty} x_n$

By the squeeze lemma, 1-tail of $\{x_{n_k}\}$ converges to $\limsup\limits_{n\to\infty} x_n$

$\Rightarrow$ Thus, there exists a subsequence of $\{x_n\}$ which converges to the lim sup. ✓

- Proof for lim inf is similar. □

Consequences of Thm. 2.3.4

Prop. (lim inf/sup convergence test)

Let $\{x_n\}$ be a bounded sequence. Then, $\{x_n\}$ converges iff

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

Furthermore if $\{x_n\}$ converges,

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

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Pf. Let $\{a_n\}$, $\{b_n\}$ be defined as above. Then,

$$b_n \leq x_n \leq a_n \quad \forall n \in \mathbb{N}$$

- Suppose $\liminf x_n = \limsup x_n$. By the squeeze lemma, $\{x_n\}$ converges and

$$\lim x_n = \liminf x_n = \limsup x_n$$

- Suppose $\{x_n\}$ converges to some $x \in \mathbb{R}$. By thm. 2.3.4, there exist subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ which converge to lim sup/inf
  - Then, since all subsequences converge to the same limit,

$$\liminf x_n = \limsup x_n = \lim x_n \qquad \square$$

Prop. (2.3.6; lim inf/sup bound subsequential limits)

★★★ Thm. (2.3.8; Bolzano-Weierstrass)

Suppose $\{x_n\}$ is a bounded sequence of real numbers. Then, there exists a convergent subsequence $\{x_{n_k}\}$

Pf. Follows directly from thm. 2.3.4., as there exists a subsequence $\{x_{n_k}\}$ where

$$\lim x_{n_k} = \limsup x_n \qquad \square$$

Remark: Depends fundamentally on LUB of $\mathbb{R}$!