Midterm Exam 1 Solutions

Examination Date: Tuesday, October 11th

1. Fix some set X. In lecture, we remarked how $\subset$ is 'kind of like' $\leq$. It is actually possible to take the analogy a little further, and talk about "upper bounds" and "lower bounds" of collections of subsets of X.

   Recall that the powerset $\mathscr{P}(X)$ is the set of all subsets of X. $C \in \mathscr{P}(X)$ means that $C \subset X$. Furthermore, a subset $\mathcal{A} \subset \mathscr{P}(X)$ has elements of the form $B \in \mathcal{A}$, where $B \subset X$. In other words, $\mathcal{A}$ is a collection of sets.

   (a) Let $C \in \mathscr{P}(X)$, and let $\mathcal{A} \subset \mathscr{P}(X)$. We say C is an *upper bound* of $\mathcal{A}$ if $B \subset C$ for all $B \in \mathcal{A}$.

   Given $B_1, B_2 \in \mathscr{P}(X)$, show that $B_1 \cup B_2 \in \mathscr{P}(X)$ is an upper bound of $\{B_1, B_2\} \subset \mathscr{P}(X)$.

   (b) We say an upper bound S of $\mathcal{A} \subset \mathscr{P}(X)$ is a *least upper bound* if for all upper bounds C of $\mathcal{A}$, $S \subset C$.

   Given $B_1, B_2 \in \mathscr{P}(X)$, show that if C is an upper bound of $\{B_1, B_2\}$, then $B_1 \cup B_2 \subset C$. Use this with your result in (a) to show that $B_1 \cup B_2$ is the unique least upper bound of $\{B_1, B_2\}$.

   (*Hint*: For the second part, first show that $B_1 \cup B_2$ is a least upper bound. Then, show that if S is another least upper bound, $S = B_1 \cup B_2$)

   (c) Write down a corresponding definition for some $C \in \mathscr{P}(X)$ to be a *lower bound* of $\mathcal{A} \subset \mathscr{P}(X)$. Use it to prove that $B_1 \cap B_2$ is a lower bound of $\{B_1, B_2\}$.

   (d) Show that $\mathscr{P}(X)$ an analogue to the least upper bound property: every non-empty collection $\mathcal{A} \subset \mathscr{P}(X)$ has a least upper bound S.

   You may find the following to be helpful: given a collection of sets $\mathcal{A} \subset \mathscr{P}(X)$, you can define a set which is the union over this collection,

$$\bigcup_{B \in \mathcal{A}} B := \{x \in X : x \in B \text{ for some } B \in \mathcal{A}\} \in \mathscr{P}(X)$$

(a) By the definition of the set union, $x \in B_1$ or $x \in B_2$ implies that $x \in B_1 \cup B_2$. Thus, $B_1 \subset B_1 \cup B_2$ and $B_2 \subset B_1 \cup B_2$. Thus, $B_1 \cup B_2$ is an upper bound of $\{B_1, B_2\}$

(b) Suppose C is an upper bound of $\{B_1, B_2\}$, so $B_1 \subset C$ and $B_2 \subset C$. In particular, $x \in B_1$ or $x \in B_2$ implies $x \in C$. Thus, $x \in B_1 \cup B_2$ implies $x \in C$, so $B_1 \cup B_2 \subset C$.

Since $B_1 \cup B_2$ is an upper bound and $B_1 \cup B_2 \subset C$ for every upper bound C, this shows that $B_1 \cup B_2$ must be a least upper bound.

Now, suppose S is a least upper bound of $\{B_1, B_2\}$. In particular, $S \subset B_1 \cup B_2$ since $B_1 \cup B_2$ is an upper bound. Similarly, $B_1 \cup B_2 \subset S$ since $B_1 \cup B_2$ is also a least upper bound. Thus, $S = B_1 \cup B_2$

(c) We say $C \in \mathscr{P}(X)$ is a *lower bound* of $\mathcal{A} \subset \mathscr{P}(X)$ if $C \subset B$ for all $B \in \mathcal{A}$.

By the definition of the set intersection, $x \in B_1 \cap B_2$ implies that $x \in B_1$ and $x \in B_2$. Thus, $B_1 \cap B_2 \subset B_1$ and $B_1 \cap B_2 \subset B_2$. Thus shows that $B_1 \cap B_2$ is an upper bound of $\{B_1, B_2\}$.

(d) Let $\mathcal{A} \subset \mathscr{P}(X)$ be a non-empty collection of sets. Define

$$S := \bigcup_{B \in \mathcal{A}} B$$

Now, for every $B \in \mathcal{A}$, we have that $x \in B$ implies $x \in S$ (i.e. since $x \in B$ for some $B \in \mathcal{A}$), so $B \subset S$. Thus, S is an upper bound of $\mathcal{A}$.

To show it is a least upper bound, suppose C is an upper bound of $\mathcal{A}$. In particular, every $B \in \mathcal{A}$ satisfies $B \subset C$.

For all $x \in S$, by the definition of the union we have $x \in B$ for some $B \in \mathcal{A}$. Since $B \subset C$, we have $x \in B$ implies $x \in C$. Thus, $x \in S$ implies $x \in C$, which shows that $S \subset C$, so S is a least upper bound.

Remark: Even though the proof is short, the logic to show $S \subset C$ can be quite tricky! This is one reason you should fully write out the "simple" proofs in part (a) and (b), e.g. showing that $B_1 \subset B_1 \cup B_2$. Writing out the explicit proofs should help you to generalize them to this more difficult problem.

2. For a sequence $\{x_n\}$, we define the arithmetic mean sequence $\{\bar{x}_n\}$ as

$$\bar{x}_n := \frac{x_1 + x_2 + ... + x_n}{n}$$

We will see how the convergence of $\{\bar{x}_n\}$ relates to the convergence of $\{x_n\}$. Note this question does not require any knowledge of series.

(a) Let $\{x_n\} := \{(-1)^n\}$. Show that $\{\bar{x}_n\}$ converges.

(b) Suppose $\lim_{n\to\infty} x_n = L$. Let $\varepsilon > 0$ be arbitrary, and take $M \in \mathbb{N}$ such that for all $n \geq M$, $|x_n - L| < \varepsilon/2$.
Show that there is some $K \geq 0$ such that for all $n \geq M$, we have

$$|\bar{x}_n - L| \leq \frac{MK}{n} + \frac{(n-M)\varepsilon}{2n}$$

(*Hint*: Use $\bar{x}_n - L = \sum_{k=1}^{n} \frac{x_k - L}{n}$ with the triangle inequality.)

(c) Note the right hand side of the inequality in (b) converges to $\varepsilon/2$ as $n \to \infty$. Prove that there exists some $M' \in \mathbb{N}$ such that for all $n \geq M'$,

$$\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} < \varepsilon$$

and use it to conclude that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} \bar{x}_n = L$.

Remark: As it turns out, not all bounded sequences have convergent mean sequences.

(a) We can compute

$$\bar{x}_n = \begin{cases} -1/n & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Since $-1/n \le \bar{x}_n \le 0$, we have by the squeeze lemma that $\bar{x}_n$ converges to 0.

(b) Let ε, M be as given. Take

$$K := \max\{|x_1 - L|, ..., |x_M - L|\} \ge 0$$

Then for all $n \ge M$, since $|x_n - L| < \varepsilon/2$, we have that

$$\begin{aligned} |\bar{x}_n - L| &= \left| \sum_{k=1}^{n} \frac{x_k - L}{n} \right| && \text{(by definition of } \bar{x}_n\text{)} \\ &\le \sum_{k=1}^{n} \left| \frac{x_k - L}{n} \right| && \text{(by triangle inequality)} \\ &= \sum_{k=1}^{M} \frac{|x_k - L|}{n} + \sum_{k=M+1}^{n} \frac{|x_k - L|}{n} && \text{(splitting a finite sum up)} \\ &\le \frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} \end{aligned}$$

which is the desired inequality

(c) By continuity of algebraic operations, since M, K, ε are constants we can compute

$$\lim_{n\to\infty} \left[\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} \right] = \frac{\varepsilon}{2}$$

In particular, noting that $\varepsilon/2 > 0$, this means that there exists some $M' \in \mathbb{N}$ such that for all $n \ge M'$,

$$\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} - \frac{\varepsilon}{2} \le \left| \frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} - \frac{\varepsilon}{2} \right| < \frac{\varepsilon}{2}$$

which can be manipulated to say

$$\frac{MK}{n} + \frac{(n-M)\varepsilon}{2n} < \varepsilon$$

Take $M'' := \max\{M, M'\}$. Then, for all $n \ge M''$, we have that

$$|\bar{x}_n - L| < \varepsilon$$

which shows that $\{\bar{x}_n\}$ converges to L.