Homework 2

Due: Monday, September 26th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 1.3, 2.1-2.2 (optional: 1.4)

Sections 1.2-1.3 Exercises

Problem 1 (4 points) Prove that for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, there is a rational $r \in \mathbb{Q}$ such that $|x - r| < \epsilon$

(*Remark*: This is an alternate way of stating that $\mathbb{Q}$ is dense in $\mathbb{R}$. It says that any real number x can be approximated by a rational number r to an arbitrarily good accuracy ε.

You may use the results of Theorem 1.2.4.ii here, although you should try to understand the proof.)

Problem 2 (4 points) Let $x, y \in \mathbb{R}$. Suppose that for all $\varepsilon > 0$, we have $|x - y| \leq \varepsilon$. Prove that $x = y$.

(*Remark*: This is another way to show $x = y$, and can be easier than proving $x \leq y$ and $x \geq y$ in some cases.)

Problem 3 (2 points each) Suppose $f, g : D \to \mathbb{R}$ are bounded functions.

(a) Show that $f + g : D \to \mathbb{R}$ defined by $(f + g)(x) := f(x) + g(x)$ is bounded.

(b) Show that $fg : D \to \mathbb{R}$ defined by $(fg)(x) := f(x)g(x)$ is bounded.

(c) Suppose $h : D \to \mathbb{R}$ is a function (not necessarily bounded) with the property that there exists $c > 0$ such that $|h(x)| \geq c$ for all $x \in D$. Show that $f/h : D \to \mathbb{R}$ defined as $(f/h)(x) := f(x)/h(x)$ is bounded.

(For example, $f(x) = 1$ and $h(x) = 1 + x^2$ are both defined on $\mathbb{R}$, and the function $f(x)/h(x) = 1/(1 + x^2)$ is bounded.)

(d) Suppose $h : E \to D$ is a function (not necessarily bounded). Show that the composition $f \circ h : E \to \mathbb{R}$ is bounded.

(For example, given $f(x) = \cos(x)$ and $h(x) = x^2$, both functions $\mathbb{R} \to \mathbb{R}$, the composition $f(h(x)) = \cos(x^2)$ is bounded)

Problem 4 (5 points) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded functions where D is non-empty and $f(x) \leq g(x)$ for all $x \in D$. Prove that

$$\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x) \qquad \text{and} \qquad \inf_{x\in D} f(x) \leq \inf_{x\in D} g(x)$$

Use this result to show that if M is a bound for the function $|f| : D \to \mathbb{R}$ defined as $|f|(x) := |f(x)|$, then

$$-M \leq \inf_{x\in D} -|f(x)| \leq \inf_{x\in D} f(x) \leq \sup_{x\in D} f(x) \leq \sup_{x\in D} |f(x)| \leq M$$

Section 2.1 Exercises

Problem 5 (3 points each) In this problem, we will see another helpful way of interpreting the definition of the limit of a sequence.

Definition. Given a sequence $\{x_n\}$ and a predicate $P(x)$, we say $P(x_n)$ *holds at most finitely often* if there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $P(x_n)$ is false.

If $P(x_n)$ does not hold finitely often, we say that $P(x_n)$ *holds infinitely often.*

Example. For the sequence $\{x_n\} := \{1/n\}$, $x_n \geq 1/2$ at most finitely often since for all $n \geq 3$, $x_n < 1/2$.

(a) Let $\{x_n\} := \{(-1)^n\}$. For what values of $\varepsilon \in \mathbb{R}$ does the inequality $|x_n - 1| \geq \varepsilon$ hold at most finitely often?

(b) Let $\{x_n\} := \{(n+1)/n\}$. For what values of $\varepsilon \in \mathbb{R}$ does the inequality $|x_n - 1| \geq \varepsilon$ hold at most finitely often?

(c) Let $\{x_n\}$ be a sequence. Show that $\{x_n\}$ converges to some $L \in \mathbb{R}$ if and only if for all $\varepsilon > 0$, $|x_n - L| \geq \varepsilon$ at most finitely often.

Problem 6 (2.5 points each) Determine if the following sequences converge, and if they do, find the limit. Feel free to use what you know from Calculus to guess the limit, but remember to prove that it is the correct limit.

(a) $\{n^2\}$

(b) $\{x_n\}$ where $x_n := \begin{cases} \sin(n^2) & n < 1000000 \\ 0 & n \geq 1000000 \end{cases}$

(c) $\left\{\frac{4n}{4n+1}\right\}$

(d) $\left\{\frac{2n}{n^2+1}\right\}$

Problem 7 (5 points) Given a sequence $\{x_n\}$ and a number $L \in \mathbb{R}$, show that the statement

(i) for every $\varepsilon' > 0$, there exists an $M' \in \mathbb{N}$ such that $|x_n - L| \leq \varepsilon'$ for all $n \geq M'$

is true if and only if the statement

(ii) for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq M$

is true. In other words, show that it does not matter whether we use a strict inequality or non-strict inequality in the definition of the limit.

Problem 8 (5 points) Show that every real number $x \in \mathbb{R}$ is the limit of some sequence of rational numbers. In other words, for any given $x \in \mathbb{R}$, show there exists a sequence $\{r_n\}$ such that (i) $r_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, and (ii) $\lim_{n\to\infty} r_n = x$.

(*Hint*: Try using the results of problem 1 with a sequence of "approximation errors" $\{\epsilon_n\}$ which converges to a limit of 0.)