Homework 7

Due: Wednesday, April 7th by 11:59 PM ET

- Collaboration with other students is highly encouraged, but you must **write up your own solutions independently**.
- Please make sure your submission is **well-written and legible**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Section 4.3 Exercises

Problem 1 (3 points each) Here is an extremely useful application of the mean value theorem: Suppose $f : [a, b] \to \mathbb{R}$ satisfies the assumptions of the MVT, and there is a M such that $|f'(x)| \leq M^*$ for all $x \in (a, b)$. Then, for any $x, y \in [a, b]$, we have from the mean value theorem there is a c between x, y such that

$$f(x) - f(y) = f'(c)(x - y)$$

Taking the absolute value of both sides, we can get a convenient upper bound for $|f(x) - f(y)|$, namely

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq M|x - y|$$

Notice that this is a special case of Taylor's theorem.

Prove the following inequalities:

(a) For any $R > 0$, $n \in \mathbb{N}$, and $x, y \in [-R, R]$, we have $|x^n - y^n| \leq nR^{n-1}|x - y|$

(b) For any $x, y \in \mathbb{R}$, we have $\left|\sqrt{x^2 + 1} - \sqrt{y^2 + 1}\right| \leq |x - y|$

(a) x^n is continuous and differentiable on $[-R, R]$ by repeated application of the product rule on x. Furthermore, the product rule also allows us to compute $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$. Then, we have for $x \in [-R, R]$

$$\left|nx^{n-1}\right| = n|x|^{n-1} \leq nR^{n-1}$$

so we conclude using the MVT,

$$|x^n - y^n| \leq nR^{n-1}|x - y|$$

*There was a typo in the HW PDF, there needs to be an absolute value here, so f' is bounded.

(b) By application of the chain rule, we have that $\sqrt{x^2+1}$ is differentiable for all $x \in \mathbb{R}$, and we compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$$

Now, for all $x \in \mathbb{R}$, we have $0 \le x^2 < x^2+1$, hence we have $|x| = \sqrt{x^2} < \sqrt{x^2+1}$. Thus, for all $x \in \mathbb{R}$ we have

$$\left|\frac{x}{\sqrt{x^2+1}}\right| = \frac{|x|}{\sqrt{x^2+1}} < \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$$

Then, for any $x, y \in \mathbb{R}$, if $x = y$, then

$$\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| = 0 \le |x-y| = 0$$

and if $x \ne y$, then we apply the MVT on the interval bounded by x and y to get

$$\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| = \left|\frac{c}{\sqrt{c^2+1}}\right||x-y| \le |x-y|$$

Section 5.1-5.2 Exercises

Problem 2 (5 points) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Show that if P is a partition of $[a,b]$ and $\widetilde{P}$ is a refinement of P, then $U(\widetilde{P}, f) \le U(P, f)$

Let $\widetilde{P} := \{\widetilde{x}_0, \widetilde{x}_1, ..., \widetilde{x}_m\}$ be a refinement of $P := \{x_0, x_1, ..., x_n\}$. Then, there exist integers $0 = k_0 < k_1 < ... < k_n = m$ such that $x_j = \widetilde{x}_{k_j}$ for $j = 0, 1, 2, ..., n$.

Let $\Delta\widetilde{x}_p = \widetilde{x}_p - \widetilde{x}_{p-1}$. Then,

$$\Delta x_j = x_j - x_{j-1} = \widetilde{x}_{k_j} - \widetilde{x}_{k_{j-1}} = \sum_{p=k_{j-1}+1}^{k_j} \widetilde{x}_p - \widetilde{x}_{p-1} = \sum_{p=k_{j-1}+1}^{k_j} \Delta\widetilde{x}_p$$

Let $M_i := \sup\{f(x) : x_{i-1} \le x \le x_i\}$, and $\widetilde{M_j} := \sup\{f(x) : \widetilde{x}_{j-1} \le x \le \widetilde{x}_j\}$. Then, $M_i \ge \widetilde{M}_p$ for $k_{j-1} < p \le k_j$ (since $\{f(x) : \widetilde{x}_{j-1} \le x \le \widetilde{x}_j\} \subset \{f(x) : x_{i-1} \le x \le x_i\}$). Therefore,

$$M_j\Delta x_j = \sum_{p=k_{j-1}+1}^{k_j} M_j\Delta\widetilde{x}_p \ge \sum_{p=k_{j-1}+1}^{k_j} \widetilde{M}_p\Delta\widetilde{x}_p$$

So,

$$U(P,f) = \sum_{j=1}^{n} M_j\Delta x_j \ge \sum_{j=1}^{n}\sum_{p=k_{j-1}+1}^{k_j} \widetilde{M}_p\Delta\widetilde{x}_p = \sum_{j=1}^{m} \widetilde{M}_j\Delta\widetilde{x}_j = U(\widetilde{P}, f)$$

Problem 3 (3 points each) In this problem we will review some useful properties of sup/inf.

(a) (Exercise 1.1.9) Let $A, B \subset \mathbb{R}$ be non-empty bounded sets such that $B \subset A$. Suppose that for all $x \in A$, there exists a $y \in B$ such that $x \geq y$. Show that $\inf B = \inf A$.

(Hint: You may find the following variant of Proposition 1.2.8 helpful: If $S \subset \mathbb{R}$ is a nonempty bounded below set, then for every $\varepsilon > 0$ there exists $x \in S$ such that $\inf S \leq x < \inf S + \varepsilon$)

(b) (Exercise 1.2.9) Let $A, B \subset \mathbb{R}$ be non-empty bounded sets. Let $C := \{a + b : a \in A, b \in B\}$. Show that $\inf C$ and $\sup C$ exist, and that

$$\sup C = \sup A + \sup B \qquad \text{and} \qquad \inf C = \inf A + \inf B$$

(c) (Exercise 1.3.7) Let D be a nonempty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions. Then,

$$\sup_{x \in D}(f(x)+g(x)) \leq \sup_{x \in D} f(x)+\sup_{x \in D} g(x) \qquad \text{and} \qquad \inf_{x \in D}(f(x)+g(x)) \geq \inf_{x \in D} f(x)+\inf_{x \in D} g(x)$$

(a) B is non-empty and bounded, so $\inf B$ exists. Since $B \subset A$, we have that $\inf B \geq \inf A$, so $\inf A$ is a lower bound of B.

Now, let $\varepsilon > 0$ be given. By the property of inf, there exists $x \in A$ such that $\inf A \leq x < \inf A + \varepsilon$. Then, by assumption, there exists $y \in B$ such that $y \leq x < \inf A + \varepsilon$. Thus, $\inf A + \varepsilon$ is not a lower bound of B. Hence, $\inf A$ is the greatest lower bound of B, so $\inf B = \inf A$.

(b) Let A, B, C be as given in the problem. Since A, B are non-empty and bounded, their sup and inf exist. C is non-empty since A, B are. For any $c \in C$, we have $x = a + b$ for some $a \in A$ and $b \in B$, so

$$\inf A + \inf B \leq a + b = c \leq \sup A + \sup B$$

Thus, $\inf A + \inf B$ is a lower bound of C, and $\sup A + \sup B$ is an upper bound of C. Hence, C is bounded both above and below, and $\sup C$ and $\inf C$ exist.

Now, let $\varepsilon > 0$ be given. Then, there exists $a \in A$ such that $\sup A - \varepsilon/2 < a \leq \sup A$ and $b \in B$ such that $\sup B - \varepsilon/2 < b \leq \sup B$. Then, there exists $c = a + b \in C$ such that

$$\sup A + \sup B - \varepsilon < a + b = c \leq \sup A + \sup B$$

Thus, $\sup A + \sup B + \varepsilon$ is not an upper bound of C for any $\varepsilon > 0$. Thus $\sup A + \sup B = \sup C$ is the least upper bound of C.

We repeat the argument above, this time with the inequality $\inf A \leq a < \inf A + \varepsilon/2$ (same for B) to get that $\inf A + \inf B = \inf C$ is the greatest lower bound of C.

(c) Let f, g be as given. Since they are bounded, the sup and inf of f, g exist. For all $x \in D$, we have

$$\inf_{x\in D} f(x) + \inf_{x\in D} g(x) \leq f(x) + g(x) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$$

Thus, $f + g$ is bounded, and hence the sup and inf exist. Furthermore, $\sup_{x\in D} f(x) + \sup_{x\in D} g(x)$ is an upper bound of $f + g$, so the least upper bound satisfies $\sup_{x\in D}(f(x) + g(x)) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$. Similarly, the greatest lower bound satisfies $\inf_{x\in D}(f(x) + g(x)) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$.

Problem 4 (6 points) Let $a < b < c$ and assume $f : [a, b] \to \mathbb{R}$ is bounded. Show that

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f$$

Consider partitions $P_1 := \{x_0, x_1, ..., x_k\}$ of $[a, b]$ and $P_2 := \{x_k, x_{k+1}, ..., x_n\}$ of $[b, c]$. Then, the set $P := P_1 \cup P_2 = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, c]$. Then,

$$U(P, f) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{k} M_j \Delta x_j + \sum_{j=k}^{n} M_j \Delta x_j = U(P_1, f) + U(P_2, f)$$

Note that the partition P that occurs in the left-hand side is a partition containing b. If Q is any partition of $[a, c]$, and $P = Q \cup \{b\}$, then P is a refinement of Q and so $U(Q, f) \geq U(P, f)$. Since

$$\{U(P, f) : P \text{ a partition of } [a, c], b \in P\} \subset \{U(P, f) : P \text{ a partition of } [a, c]\}$$

by the result of Problem 4a (i.e. exercise 1.1.9), we have that the two sets have the same inf. Then, we apply the result of Problem 4b (i.e. exercise 1.2.9) using the equality to get

$$\begin{aligned}
\overline{\int_a^c} f &= \inf\{U(P, f) : P \text{ a partition of } [a, c]\} \\
&= \inf\{U(P, f) : P \text{ a partition of } [a, c], b \in P\} \\
&= \inf\{U(P_1, f) + U(P_2, f) : P_1 \text{ a partition of } [a, b], P_2 \text{ a partition of } [b, c]\} \\
&= \inf\{U(P_1, f) : P_1 \text{ a partition of } [a, b]\} + \inf\{U(P_2, f) : P_2 \text{ a partition of } [b, c]\} \\
&= \overline{\int_a^b} f + \overline{\int_b^c} f
\end{aligned}$$

which is the desired equality.

Problem 5 (6 points) Directly using the definition of Riemann integrable (the upper integral equals the lower integral), show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then so is $-f$ and

$$\int_a^b (-f) = -\int_a^b f$$

(Remark: It is important to prove this statement by definition, and not to use any other properties of the Riemann integral proved in section 5.2 because the statement in this problem is used in the proof of linearity, and we do not want to use circular logic.)

Recall that given a bounded set $A \subset \mathbb{R}$, the set $-A := \{-a : a \in A\}$ satisfies $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.

Now, let P be a partition of $[a, b]$, and define $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Note that

$$\inf\{-f(x) : x \in [x_{i-1}, x_i]\} = -\sup\{f(x) : x \in [x_{i-1}, x_i]\} = -M_i$$
$$\sup\{-f(x) : x \in [x_{i-1}, x_i]\} = -\inf\{f(x) : x \in [x_{i-1}, x_i]\} = -m_i$$

So,

$$L(P, -f) = \sum_{i=1}^{n} \inf\{-f(x) : x \in [x_{i-1}, x_i]\}\Delta x_i = -\sum_{i=1}^{n} M_i \Delta x_i = -U(P, f)$$
$$U(P, -f) = \sum_{i=1}^{n} \sup\{-f(x) : x \in [x_{i-1}, x_i]\}\Delta x_i = -\sum_{i=1}^{n} m_i \Delta x_i = -L(P, f)$$

Hence,

$$\begin{aligned}
\underline{\int_a^b}(-f) &= \sup\{L(P, -f) : P \text{ a partition of } [a, b]\} \\
&= \sup\{-U(P, f) : P \text{ a partition of } [a, b]\} \\
&= -\inf\{U(P, f) : P \text{ a partition of } [a, b]\} \\
&= -\overline{\int_a^b} f
\end{aligned}$$

A similar sequence of manipulations gets us

$$\overline{\int_a^b}(-f) = -\underline{\int_a^b} f$$

Thus, since f is Riemann integrable

$$\underline{\int_a^b}(-f) = -\overline{\int_a^b} f = -\underline{\int_a^b} f = \overline{\int_a^b}(-f)$$

So $-f$ is Riemann integrable, with

$$\int_a^b (-f) = \underline{\int_a^b}(-f) = -\overline{\int_a^b} f = -\int_a^b f$$

Problem 6 (6 points each) In this problem we will prove linearity of the Riemann integral.

(a) Prove Proposition 5.2.5 in the textbook: Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be bounded functions. Then,

$$\overline{\int_a^b}(f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g \qquad \text{and} \qquad \underline{\int_a^b}(f+g) \ge \underline{\int_a^b} f + \underline{\int_a^b} g$$

(Hint: Try to get an inequality of the form $U(P, f+g) \le U(P, f)+U(P, g) \le U(P_1, f)+U(P_2, g)$. You can't use the result of Problem 4b on the middle term, but you can use it on the right-most term (why?).)

(b) Now, suppose $f, g \in \mathscr{R}[a, b]$ (recall that Riemann integrable functions are also bounded). Using your result in (a), prove that $f + g \in \mathscr{R}[a, b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(a) Let P_1 and P_2 be partitions of $[a, b]$. Define $P := P_1 \cup P_2$, which is a refinement of both P_1 and P_2. Let $P = \{x_0, x_1, ..., x_n\}$. Thus, using the results of Problem 4c,

$$\begin{aligned} M_i^{f+g} &:= \sup\{f(x) + g(x) : x \in [x_{i-1}, x_i]\} \\ &\le \sup\{f(x) : x \in [x_{i-1}, x_i]\} + \sup\{g(x) : x \in [x_{i-1}, x_i]\} = M_i^f + M_i^g \end{aligned}$$

which implies

$$U(P, f+g) = \sum_{i=1}^n M_i^{f+g}\Delta x_i \le \sum_{i=1}^n \left(M_i^f + M_i^g\right)\Delta x_i = U(P, f) + U(P, g)$$

then, using Proposition 5.1.7, we have

$$U(P, f+g) \le U(P, f) + U(P, g) \le U(P_1, f) + U(P_2, g)$$

Then, by the results of Problem 4c, we can take the inf over P for the left-hand side of the inequality, and the inf over P_1 and P_2 for the right-hand side of the inequality, to get

$$\begin{aligned} \overline{\int_a^b}(f+g) &= \inf\{U(P, f+g) : P \text{ a partition of } [a, b]\} \\ &\le \inf\{U(P_1, f) + U(P_2, g) : P_1, P_2 \text{ partitions of } [a, b]\} \\ &= \inf\{U(P_1, f) : P_1 \text{ a partition of } [a, b]\} + \inf\{U(P_2, g) : P_2 \text{ a partition of } [a, b]\} \\ &= \overline{\int_a^b} f + \overline{\int_a^b} g \end{aligned}$$

which is one of the desired inequalities.

A similar computation using the properties of inf shows that

$$L(P, f+g) \ge L(P, f) + L(P, g) \ge L(P_1, f) + L(P_2, f)$$

which we then use to conclude

$$\underline{\int_a^b}(f+g) \ge \underline{\int_a^b} f + \underline{\int_a^b} g$$

(b) Suppose f, g are Riemann integrable over $[a, b]$. Then, using the results of (a), we have

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g = \int_a^b f + \int_a^b g = \underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g) \tag{1}$$

Combine this with the inequality given by Proposition 5.1.8

$$\underline{\int_a^b}(f+g) \leq \overline{\int_a^b}(f+g)$$

which shows that

$$\underline{\int_a^b}(f+g) = \overline{\int_a^b}(f+g)$$

and hence $f+g$ is Riemann integrable. Furthermore, the inequalities on the far sides of (1) become equality signs, showing that

$$\int_a^b (f+g) = \overline{\int_a^b}(f+g) = \int_a^b f + \int_a^b g$$

Problem 7 (6 points) Let P_n denote the partition of $[0, 1]$ using $n+1$ uniformly spaced points, that is, $P_n := \{k/n\}_{k=0}^n$. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) := x$. Compute $U(P_n, f)$ and $L(P_n, f)$ for each $n \in \mathbb{N}$.

Then, via Proposition 5.1.13, prove that f is Riemann integrable on $[0, 1]$ and compute $\int_0^1 f$.

Let P_n and f be defined as given, then,

$$m_i := \inf\{x : x \in [(i-1)/n, i/n]\} = \frac{i-1}{n}$$
$$M_i := \sup\{x : x \in [(i-1)/n, i/n]\} = \frac{i}{n}$$

Furthermore, note that $\Delta x_i = 1/n$. Then,

$$L(P_n, f) = \sum_{i=1}^n m_i \Delta x_i = \frac{1}{n^2}\sum_{i=1}^n (i-1) = \frac{(n-1)n}{2n^2} = \frac{1}{2} - \frac{1}{2n}$$
$$U(P_n, f) = \sum_{i=1}^n M_i \Delta x_i = \frac{1}{n^2}\sum_{i=1}^n i = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n}$$

Notice that $U(P_n, f) - L(P_n, f) = \frac{1}{n}$. So, given $\varepsilon > 0$, pick $M \in \mathbb{N}$ such that $M > 1/\varepsilon$. Then, there exists a partition $P := P_M$ such that

$$U(P_M, f) - L(P_M, f) = \frac{1}{M} < \varepsilon$$

Thus, by Proposition 5.1.13, f is Riemann integrable on $[0, 1]$.

Finally, noting that

$$\{L(P_n, f) : n \in \mathbb{N}\} \subset \{L(P, f) : P \text{ a partition of } [0, 1]\}$$

since $\sup\{L(P_n, f) : n \in \mathbb{N}\} = \frac{1}{2}$, and f is Riemann integrable (so the Riemann integral is equal to the lower Darboux integral), we have that

$$\int_0^1 f = \underline{\int_0^1} f \geq \frac{1}{2}$$

The corresponding statement for the inf of $U(P_n, f)$ gets

$$\int_0^1 f = \overline{\int_0^1} f \leq \frac{1}{2}$$

which we combine to conclude

$$\int_0^1 f = \frac{1}{2}$$