1. Player 1: Hitting: $\frac{1}{5}$ Missing: $\frac{4}{5}$

Player 2: Hitting: $\frac{1}{4}$ Missing: $\frac{3}{4}$

- The probability that Player 1 shot the target : $\frac{1}{5} \Rightarrow K$
- The probability that both 1 & 2 miss the target $= (1-\frac{1}{5})\cdot(1-\frac{1}{4})$

$$= \frac{4}{5}\cdot\frac{3}{4} = \frac{3}{5} \Rightarrow R$$

- The solution will then b a geometric progression

$$K\cdot R^0 + K\cdot R^1 + K\cdot R^2 + \cdots\cdots + K\cdot R^n$$

n: the # of games which Player 1 took to win

$$Sum = \frac{K}{1-R} \qquad \cancel{\forall |R|<1} \quad \forall |R|<1$$

$$= \frac{\frac{1}{5}}{1-\frac{3}{5}} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$$

2. 1% population have covid.

↳ 90% of covid people gets positive test result

10% population who don't have covid also test positive

Probability: Given Test Positive Probability of having covid.

e.g: Probability that the person does not have COVID receive positive: 0.1

Probability that the person does have COVID receives positive: 0.9

Probability that a person has covid: 0.01

$$P(D|+) = \frac{P(+|D)P(D)}{P(+)}$$

$P(D) = 0.01$

$P(+|H) = 0.1$

$P(+|D) = 0.9$

$$P(+) = P(+|D)P(D) + P(+|H)P(H)$$

$$= P(+|D)P(D) + P(+|H)P(H)$$

$P(H) = 1 - P(D)$

$= 1 - 0.01$

$= 0.99$

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|H)P(H)}$$

$$= \frac{(0.9)(0.01)}{0.9(0.01) + 0.1(0.99)} = \frac{0.009}{0.009 + 0.099}$$

$$= \frac{0.009}{0.108}$$

$$= 0.083$$

3.

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{1+x} & x \geq 0 \end{cases}$$

In order to be a probability density function, the area underneath the PDF has to be equal to 1.

$$1 \stackrel{?}{=} \int_0^{\infty} \frac{1}{1+x}\, dx$$

→ However, the integral is divergent. Therefore, the function is not a pdf

4.

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$P(X+Y \leq 1)$

$$\boxed{\begin{array}{l} f(X) = \begin{cases} 2X & \text{if} \quad 0 \leq X \leq 1 \\ 0 & \text{otherwise} \end{cases} \\ \\ f(Y) = \begin{cases} 2Y & \text{if} \quad 0 \leq Y \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{array}}$$

$$f(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

$$P(X+Y \leq 1) = \int_0^1 \int_0^{1-x} 4xy\, dy\, dx$$

$$= \int_0^1 2x(1-x)^2\, dx$$

$$= \int_0^1 2x - 4x^2 + 2x^3 dx$$

$$= \left[x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 \right]_0^1$$

$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$

Question 5:

$X \sim Unif(0,1)$. Let $Y = g(x) = e^x$

$E(Y) = ?$

$f(x) = 1 \qquad 0 < x < 1$

$$E(Y) = E(g(x)) = \int_0^1 f(x) \cdot g(x)\, dx$$

$$= \int_0^1 e^x \, dx$$

$$= [e^x]_0^1$$

$$= e^1 - e^0 = e - 1 = 1.718$$

Question 6

Suppose that number of errors per computer program has a Poisson distribution with mean 5. We have 125 program submissions. Let $X_1, X_2, \text{-----}, X_{125}$ denote the number of errors in the program. Find $P(\bar{X}_n < 5.5)$?

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!}$$

Central Limit Theorem:

$\bar{X}_n = n^{-1} \sum_i X_i$ has a distribution which is approximately normal with mean μ and variance $\frac{\sigma^2}{n}$.

Let $X_1, \text{-----} X_n$ be iid with mean μ and variance σ^2. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n$ Then $Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$ where $Z \sim N(0,1)$

$\mu = 5 \qquad \sigma^2 = Var(X_1) = \lambda = 5$

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{125}(\bar{X}_n - 5)}{\sqrt{5}} = \frac{5\sqrt{5}(\bar{X}_n - 5)}{\sqrt{5}}$$

$$= 5(\bar{X}_n - 5) \approx N(0,1)$$

$$P(\bar{X}_n < 5.5) = P(5(\bar{X}_n - 5) < (5.5 - 5)5)$$

$\approx P(Z_n < 2.5)$

$= 0.9938$

Question 7

Let $X_n = f(W_n, X_{n-1})$ for $n=1, \ldots, P$, for some function $f()$. Let us define the value of variable E as

$$E = \| C - X_P \|^2 \text{ for some constant } C.$$

What is the value of the gradient $\frac{\partial E}{\partial X_0}$?

Norms : size of a vector

Vector : set of ~~the~~ numbers

feature : $X = \{X_1, X_2\}$

$a = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

L2 norm: Euclidean Distance $= \left(\sum_{i=1}^{k} |x_i|^2 \right)^{\frac{1}{2}}$

$$E = (c - f(w_p, x_{p-1})) \cdot (c - f(w_p, x_{p-1}))$$

$$= c^2 - 2cf(w_p, x_{p-1}) + f(w_p, x_{p-1})^2$$

$$= c^2 - 2cf(w_p, f(w_{p-1}, x_{p-2})) + f(w_p, f(w_{p-1}, x_{p-2}))^2$$

$$E = c^2 - 2cf(w_p, f(w_{p-1}, f(w_{p-2}, \ldots f(w_1, x_0)) + f(w_p, f(w_{p-1}, f(w_{p-2}, \ldots f(w_1, x_0))^2$$

$$\frac{\partial E}{\partial X_0} = -2cf'(w_p, x_p) \ldots f'(w_0, x_0) + 2f'(w_p, x_p) \ldots f'(w_0, x_0)$$

Question 8

Let A be the matrix $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$ and let x be the column vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Let A^T and x^T denote the transpose of A and x respective.

Compute Ax, A^T and x^TA

$$x^T = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 50 \\ 17 \\ 35 \end{bmatrix}$$

$$x^TA = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 33 & 27 & 36 \end{bmatrix}$$

Question 9

a) $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$

$$\det A = 6\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 10 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 10 & 3 \end{vmatrix}$$

$$= 6(4-3) - 2(12-10) + 3(9-10)$$

$$= 6 - 4 - 3 = -1$$

$C_{1,1} = (-1)^{1+1}(4-3) = 1$

$C_{1,2} = (-1)^{1+2}(12-10) = -2$

$C_{1,3} = (-1)^{1+3}(9-10) = -1$

$C_{2,1} = (-1)^{1+2}(8-9) = 1$

$C_{2,2} = (-1)^{2+2}(24-30) = -6$

$C_{2,3} = (-1)^{2+3}(18-20) = 2$

$C_{3,1} = (-1)^{3+1}(2-3) = -1$
$C_{3,2} = (-1)^{3+2}(6-9) = 3$
$C_{3,3} = (-1)^{3+3}(6-6) = 0$

$$C = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -6 & 3 \\ -1 & 2 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C = -C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

$\det A = 1(10-8) - 2(0-2) + 3(0-2)$

$= 2 - 2(-2) + 3(-2)$

$= 2 + 4 - 6 = 0$ $\Rightarrow$ which is not invertible

Question 10

Let A be an $n \times n$ matrix and let $X \in C^n$ be a nonzero vector for which
$AX = \lambda X$ for some scalar λ.

Then λ is called an eigenvalue of matrix A and X is called an eigenvector of A associated with λ.

We first find the eigenvalue, and put an identity matrix on the right side of the equation

$AX = \lambda I X$

Bring all to left hand side,

$AX - \lambda I X = 0$ Then we can calculate the eigenvalues using determinant. with the equation $\det|A - \lambda I| = 0$
when X is non-zero

Then put in the λ that we got from previous step into the equation $AX = \lambda X$
Then solve for equations to get eigenvector.

$$\det\left|\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\det\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ -2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\left(-\lambda(1-\lambda)\right) - 1\left(2 - 2\lambda\right) = 0$$

$$(1^2 - 2\lambda + \lambda^2)(-\lambda) - 2 + 2\lambda = 0$$

$$-\lambda + 2\lambda^2 - \lambda^3 - 2 + 2\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$\lambda_1 = 1$

$\lambda_2 = -1$

$\lambda_3 = 2$