Homework 8 Solutions

Due: Wednesday, April 28th by 11:59 PM ET

- Collaboration with other students is highly encouraged, but you must **write up your own solutions independently**.
- Please make sure your submission is **well-written and legible**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Section 5.3 Exercises

Problem 1 (3 points each) In many of the statements and proofs involving integration we have proven, we have typically looked at $\int_a^b f$ for $b > a$. In this problem we will show most results directly generalize to the case where $b \leq a$. Recall the meaning of the integral in this case is defined in Section 5.1.3.

(a) Let $f \in \mathscr{R}[a, b]$. Let α, β, γ be arbitrary numbers in $[a, b]$ (not necessarily distinct or ordered in any way). Prove

$$\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$$

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $c \in [a, b]$ be arbitrary. Define

$$F(x) := \int_c^x f$$

Prove that F is differentiable and that $F'(x) = f(x)$ for all $x \in [a, b]$.

(a) We prove this by cases.

- First, if $\alpha = \beta = \gamma$, then all of the integrals are zero by definition, so the desired equality holds.
- Now, we check the case where two are distinct.
  - If $\alpha = \gamma \neq \beta$, then we have $\int_\alpha^\gamma f = 0$ and $\int_\alpha^\beta f = -\int_\beta^\gamma f$, so the desired equality holds.
  - If $\beta = \alpha$ or $\beta = \gamma$, then either the integral $\int_\alpha^\beta f = 0$ or $\int_\beta^\gamma f = 0$ respectively. Likewise, $\int_\beta^\gamma f = \int_\alpha^\gamma f$ or $\int_\alpha^\beta f = \int_\alpha^\gamma f$ respectively. Then, the desired equality holds.
- Finally, we check the case where all three are distinct.

- If $\alpha < \gamma$, we consider three cases
  * If $\alpha < \beta < \gamma$, then we apply the original result of Proposition 5.2.2 for $a = \alpha$, $b = \beta$, $c = \gamma$.
  * If $\beta < \alpha < \gamma$, then we use Proposition 5.2.2 to get

$$\int_\beta^\gamma f = \int_\beta^\alpha f + \int_\alpha^\gamma f$$
$$\implies \int_\alpha^\gamma f = -\int_\beta^\alpha f + \int_\beta^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$$

  * If $\alpha < \gamma < \beta$, then we use Proposition 5.2.2 to get

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$$
$$\implies \int_\alpha^\gamma f = \int_\alpha^\beta f - \int_\gamma^\beta f = \int_\alpha^\beta f + \int_\beta^\gamma f$$

- If $\gamma < \alpha$, then we multiply the equality by -1 to get

$$\int_\gamma^\alpha f = \int_\beta^\alpha f + \int_\gamma^\beta f = \int_\gamma^\beta f + \int_\beta^\alpha f$$

  so the roles of α and γ are switched. We apply the result in the case where $\alpha < \gamma$, except the values of the two symbols are switched, which shows the desired equality.

This exhausts all of the cases, showing the desired result.

(b) Define

$$G(x) := \int_a^x f$$

By the result of the second form of the fundamental theorem of calculus, G is differentiable at all $x \in [a, b]$ with

$$G'(x) = f(x)$$

From the result of part (a), we have that for any $c, x \in [a, b]$

$$F(x) = \int_c^a f + \int_a^x f = C + G(x)$$

Note that $\int_c^a f$ does not depend on x. By linearity of the derivative, of F' exists for all $x \in [a, b]$ and is given by

$$F'(x) = G'(x) = f(x)$$

as desired.

Problem 2 (4 points) Suppose F and G are continuously differentiable functions defined on $[a, b]$ such that $F'(x) = G'(x)$ for all $x \in [a, b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$

(Remark: This is justifying the "rule" of adding a constant $\int f + C$ to indefinite integration when you are computing an antiderivative. Make sure to use the right form of the fundamental theorem of calculus.)

Let $f(x) = F'(x) = G'(x)$. Since F, G are continuously differentiable, f is continuous and hence Riemann integrable on the interval $[a, b]$. Then using the first form of the fundamental theorem of calculus, we have for any $x \in (a, b]$

$$F(x) - F(a) = \int_a^x f = G(x) - G(a)$$

Then, $F(x) - G(x) = F(a) - G(a) = C$ for all $x \in [a, b]$.

Section 6.1-6.3 Exercises

Problem 3 (4 points each) Practice with pointwise and uniform convergence.

(a) Let $f_n : (0, 1) \to \mathbb{R}$ be given by $f_n(x) := \frac{n+1}{nx}$. Show that $\{f_n\}$ converges pointwise to a continuous function f, but the convergence is not uniform.

(Remark: This shows that pointwise convergence to a continuous function does not imply uniform convergence, so the "converse" to Theorem 6.2.2 is not true. It is also possible to find counterexamples using sequences of continuous functions on $[0, 1]$)

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) := \begin{cases} 0 & x = 0 \\ n & 0 < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

Notice that $f_n \in \mathscr{R}[0, 1]$ since it has a finite number of discontinuities. Show that $\{f_n\}$ converges pointwise to a function $f \in \mathscr{R}[0, 1]$, but the convergence is not uniform (without using Theorem 6.2.4). Furthermore, show that

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$$

(c) Let $f_n(x) = \frac{x^n}{n}$. Show that $\{f_n\}$ converges uniformly to a differentiable function f on $[0, 1]$ (find f). However, show that $f'(1) \neq \lim_{n\to\infty} f_n'(1)$.

(a) For fixed $x \in (0,1)$ we have by continuity of algebraic operations that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\frac{1}{x} + \frac{1}{nx}\right) = \frac{1}{x}$$

Thus, $\{f_n\}$ converges pointwise to the continuous function $f : (0,1) \to \mathbb{R}$ given by $f(x) = 1/x$.

To show that the convergence is not uniform, we attempt to compute

$$\sup_{x\in(0,1)}\left|\frac{n+1}{nx} - \frac{1}{x}\right| = \sup_{x\in(0,1)}\frac{1}{nx}$$

in fact, the function $\frac{1}{nx}$ is not bounded for any $n \in \mathbb{N}$. Hence, for any given $\varepsilon > 0$, there does not exist an $M \in \mathbb{N}$ such that $\sup_{x\in(0,1)}\left|\frac{n+1}{nx} - \frac{1}{x}\right| < \varepsilon$ for all $n \geq M$.

(b) First, note that for fixed $0 < x$, there exists $M_x \in \mathbb{N}$ such that $\frac{1}{M_x} < x$. Thus, $f_n(x) = 0$ for all $n \geq M_x$. $f_n(0) = 0$, so we have that $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Thus, $\{f_n\}$ converges pointwise to $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$.

Then, we compute (e.g. by using additivity and the fundamental theorem of calculus)

$$\int_0^1 f_n = \int_0^{1/n} n \,\mathrm{d}x = 1$$

Then,

$$1 = \lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 f = 0$$

(c) For $0 \leq x \leq 1$ we have that $0 \leq x^n \leq 1$ for $n \in \mathbb{N}$. So, given $\varepsilon > 0$ take $M \in \mathbb{N}$ such that $M > 1/\varepsilon$, then for all $n \geq M$ we have

$$\sup_{x\in[0,1]}\left|\frac{x^n}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Hence, $\{f_n\}$ converges uniformly to $f(x) = 0$ on $[0,1]$.

Now, we compute the derivative of f_n using e.g. the power rule (which can be proved inductively using the product rule), $f_n'(x) = x^{n-1}$. We also have $f'(x) = 0$, so

$$1 = \lim_{n\to\infty} f_n'(1) \neq f'(1) = 0$$

Problem 4 (3 points each) Let f and g be bounded functions on $[a,b]$.

(a) Prove the triangle inequality for the uniform norm,

$$\|f+g\|_u \leq \|f\|_u + \|g\|_u$$

(b) Using your result in (a), prove the reverse triangle inequality for the uniform norm,

$$\left|\|f\|_u - \|g\|_u\right| \leq \|f-g\|_u$$

(Hint: Your proof will look very similar to the proof of the reverse triangle inequality for the absolute value)

(a) Since $|f(x)+g(x)| \leq |f(x)|+|g(x)|$ for all $x \in [a,b]$, we use Proposition 1.3.7 and HW7 Problem 3c to get

$$\begin{aligned}
\|f+g\|_u &= \sup_{x\in[a,b]} (|f(x)+g(x)|) \\
&\leq \sup_{x\in[a,b]} (|f(x)|+|g(x)|) \\
&\leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\|_u + \|g\|_u
\end{aligned}$$

(b) Note that we have

$$\|-f\|_u = \sup_{x\in[a,b]} |-f(x)| = \sup_{x\in[a,b]} |f(x)| = \|f\|_u$$

So, using the triangle inequality, we have that

$$\begin{aligned}
\|f\|_u = \|(f-g)+g\|_u \leq \|f-g\|_u + \|g\|_u &\implies \|f\|_u - \|g\|_u \leq \|f-g\|_u \\
\|g\|_u = \|(g-f)+f\|_u \leq \|g-f\|_u + \|f\|_u &\implies \|f\|_u - \|g\|_u \geq -\|f-g\|_u
\end{aligned}$$

Combining these two inequalities, we have

$$-\|f-g\|_u \leq \|f\|_u - \|g\|_u \leq \|f-g\|_u \implies |\|f\|_u - \|g\|_u| \leq \|f-g\|_u$$

The Exponential Function

In the following problems, we will see how Picard's theorem allows us to define the exponential function, and hence define irrational exponents a^x for positive $a > 0$ and $x \in \mathbb{R}$. If you're curious, see section 5.4 for how to do it starting from integrals and the logarithm instead.

Problem 5 (4 points each) Given any $x_0, y_0 \in \mathbb{R}$, consider the equation and initial conditions

$$f'(x) = f(x) \qquad f(x_0) = y_0$$

(a) Given any positive $h < \frac{1}{2}$, show that we can pick $\alpha > 0$ large enough that the proof of Picard's theorem guarantees a solution for f in the interval $[x_0 - h, x_0 + h]$.

(b) Show that (a) can be used to iteratively extend f to a unique function on all $x \in \mathbb{R}$.

(c) Show that if there exists some $c \in \mathbb{R}$ such that $f(c) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$. Conclude that if $y_0 > 0$, then $f(x) > 0$ and f is strictly increasing for all $x \in \mathbb{R}$.

(d) Given $\alpha \in \mathbb{R}$, show the unique solution to the equation $g'(x) = g(x)$ with initial conditions $g(x_0) = \alpha y_0$ is given by $g(x) = \alpha f(x)$ for all $x \in \mathbb{R}$.

(a) First, the F in Picard's theorem is $F(x,y)=y$. For any convergent sequences of real numbers $\{x_n\}$ and $\{y_n\}$, we have that

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} y_n = y = F(x,y)$$

Thus, F is continuous on any subset $I \times J \subset \mathbb{R}^2$. Next, we also have that for all $x,y,z \in \mathbb{R}$

$$|F(x,y)-F(x,z)| = |y-z|$$

so F is Lipschitz in the second variable with constant $L=1$ for any intervals I,J.

Now, suppose $\alpha > 0$ is some positive real number. Choose $I = [x_0-\alpha, x_0+\alpha]$ and $J=[y_0-\alpha,y_0+\alpha]$. $(x_0,y_0) \in I^\circ \times J^\circ$, and furthermore the intervals $[x_0-\alpha,x_0+\alpha]$ and $[y_0-\alpha,y_0+\alpha]$ are contained in I and J respectively. We see that for all $(x,y)\in I\times J$

$$|F(x,y)| = |y| \le \alpha + |y_0| = M$$

Then, the h defined in the proof of Picard's theorem is given by

$$h = \min\left\{\alpha, \frac{\alpha}{M+L\alpha}\right\} = \min\left\{\alpha, \frac{1}{2+|y_0|/\alpha}\right\}$$

Notice that

$$h = \begin{cases} \alpha & \alpha < \frac{1-|y_0|}{2} \\ \frac{1}{2+|y_0|/\alpha} & \alpha \ge \frac{1-|y_0|}{2} \end{cases}$$

which is strictly increasing as a function of α. So, given any positive $h > 1/2$, we can pick

$$\alpha = \begin{cases} h & h < \frac{1-|y_0|}{2} \\ \frac{|y_0|}{1/h-2} & h \ge \frac{1-|y_0|}{2} \end{cases} = \max\left\{h, \frac{|y_0|}{1/h-2}\right\}$$

which will guarantee a solution for f in the interval $[x_0-h,x_0+h]$.

Remark: Notice that the problem does not specifically require the given $h<1/2$ to be exactly the same h in the proof of Picard's theorem, only that the h in Picard's theorem should be larger than the h given in the problem.

An alternate solution to the problem is to note that the sequence $\{h_n\}$ given by

$$h_n = \min\left\{n, \frac{1}{2+|y_0|/n}\right\} = \frac{1}{2+|y_0|/n}$$

is a monotone increasing sequence with limit equal to $1/2$. Then, given any $h<1/2$, we have that there exists some $N\in\mathbb{N}$ such that $|h_N - 1/2| < (1/2-h) \implies h_N > h$. Pick $\alpha = N$ and I,J the same as above, then Picard's theorem guarantees the existence and uniqueness of f in the interval $[x_0-h,x_0+h] \subset [x_0-h_N, x_0+h_N]$.

(b) Let $h := 1/4$ and $\varepsilon := 1/8$, then by (a) Picard's theorem guarantees a solution for f in the interval $[x_0 - (h+\varepsilon), x_0 + h + \varepsilon]$ since $h + \varepsilon < 1/2$.

(*Remark*: The book simply uses the closed interval $[-h, h]$ instead of $[-h-\varepsilon, h+\varepsilon]$. I am a little confused by the argument in the book since the initial condition should lie inside of the interior of the interval rather than on the boundary for Picard's theorem to work. However, since the book seems to skip this detail (there might be some argument I do not see), full points will be awarded even if the extra ε is not included.)

We now proceed inductively. Suppose f exists and is unique on the interval $[x_0 - nh - \varepsilon, x_0 + nh + \varepsilon]$ for $n \in \mathbb{N}$. Then, by (a) Picard's theorem guarantees the existence and uniqueness of f_+ satisfying

$$f_+'(x) = f_+(x) \qquad f_+(x_0 + nh) = f(x_0 + nh)$$

on the interval $[x_0 + (n-1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$. Since f also solves this set of equations, by uniqueness f_+ and f must agree on the interval $[x_0 + (n-1)h, x_0 + nh]$. Thus, f can be extended to the domain $[x_0 - nh - \varepsilon, x_0 + (n+1)h + \varepsilon]$

We repeat the argument for f_- satisfying

$$f_-'(x) = f_-(x) \qquad f_-(x_0 - nh) = f(x_0 - nh)$$

on the interval $[x_0 - (n+1)h - \varepsilon, x_0 - (n-1)h + \varepsilon]$, and conclude that f can be extended to a unique function on the interval $[x_0 - (n+1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$. This completes the inductive argument.

Since for any $x \in \mathbb{R}$ we can find n large enough such that $x \in [x_0 - (n+1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$, we conclude that f can be extended to a unique function on all $x \in \mathbb{R}$.

(c) Given $c \in \mathbb{R}$, note that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ satisfies the ODE $f'(x) = f(x)$ with initial conditions $f(c) = 0$. Thus, by uniqueness this is the only such solution. Hence, if $f(c) = 0$ for any $c \in \mathbb{R}$, $f(x) = 0$ for all $x \in \mathbb{R}$.

Then, if $f(x_0) = y_0 > 0$, suppose for contradiction that $f(c) \leq 0$ for some $c \in \mathbb{R}$. If $f(c) = 0$, then $f(x_0) = 0$ which is a contradiction, so consider $f(c) < 0$. Then, since f is continuous, by Bolzano's IVT, there exists some d in the closed and bounded interval between x_0 and c such that $f(d) = 0$. However, this would also imply that $f(x_0) = f(c) = 0$, which is again a contradiction. Hence, $f(x) > 0$ for all $x \in \mathbb{R}$ if $f(x_0) = y_0 > 0$.

Then, for $y_0 > 0$, since $f'(x) = f(x) > 0$ for all $x \in \mathbb{R}$, we have that f is strictly for all $\mathbb{R}$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be the unique solution to $f'(x) = f(x)$ with $f(x_0) = y_0$. Then given $\alpha \in \mathbb{R}$, let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := \alpha f(x)$. By linearity of the derivative, $g'(x) = \alpha f'(x) = \alpha f(x) = g(x)$, and also $g(x_0) = \alpha f(x_0) = \alpha y_0$. Hence by uniqueness, this is the only solution that satisfies the given equation.

Problem 6 (4 points each) Now, we will focus on

$$E'(x) = E(x) \qquad E(0) = 1$$

The solution typically is denoted by $E(x) = e^x$, and is known as the exponential function, and has a unique inverse function $L(x) = \ln(x)$ known as the natural logarithm. However, make sure not to use any properties of either in proving the following.

Recall the following conventions for integer powers and roots: for a positive real number $a > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} a^0 &:= 1 \\ a^n &:= a \cdot a \cdot \ldots \cdot a \quad (n \text{ times}) \\ a^{-n} &:= \frac{1}{a^n} \end{aligned}$$

On midterm exam 2 you also showed the existence and uniqueness of n-th roots $a^{1/n}$ for $n \in \mathbb{N}$, which solve the equation $\left(a^{1/n}\right)^n = a$. Given $m \in \mathbb{Z}$, we define rational powers for $q \in \mathbb{Q}$ with $q = m/n$ as

$$a^q := \left(a^{1/n}\right)^m$$

You will see in the course of this problem that m/n does not need to be a fraction in lowest terms.

(a) Show that $E(a + b) = E(a)E(b)$ for any $a, b \in \mathbb{R}$.

(Hint: Try using 5(d) with $x_0 = 0$ and $y_0 = E(a)$)

(b) Given any $x \in \mathbb{R}$, show that $E(mx) = E(x)^m$ for any $m \in \mathbb{Z}$, and $E(x/n) = E(x)^{1/n}$ for any $n \in \mathbb{N}$. Conclude that $E(x)^q = E(qx)$ for any $q \in \mathbb{Q}$ with $q = m/n$.

(c) Show that $\lim_{n\to\infty} E(-n) = 0$, and that the sequence of real numbers $\{E(n)\}$ is an unbounded monotone increasing sequence.

Use this to conclude that $E : \mathbb{R} \to (0, \infty)$ is bijective, and hence has a unique inverse function $L : (0, \infty) \to \mathbb{R}$ satisfying $E(L(a)) = a$ for all $a \in (0, \infty)$.

(Hint: Refer back to questions 1 and 2 on the midterm 2 problem bank. Don't forget the result of 5(c)!)

(d) Use the fact that E is continuous to show that $a^x := E(xL(a))$ is the unique number satisfying $a^x = \lim_{n\to\infty} a^{q_n}$ for any sequence $\{q_n\}$ of rational numbers with $\lim_{n\to\infty} q_n = x$. Hence, this a natural way to define irrational exponents!

(a) First, let us prove a lemma:

Lemma: Given $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ and $f(x_0) = y_0$ and given $C \in \mathbb{R}$, then $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := f(x + C)$ is the unique solution to the ODE $g'(x) = g(x)$ with $g(x_0) = f(x_0 + C)$.

Proof: Let f, g, C be as given. Then by the chain rule, $g'(x) = f'(x + C) = f(x + C) = g(x)$. Furthermore $g(x_0) = f(x_0 + C)$ by construction. Thus, by Picard's theorem with (b) g is the unique solution to the given ODE. $\square$

Now, we know E exists and is unique by the results of problem 5.

- We have that $f_1 := E$ is the unique solution to the ODE $f_1'(x) = f_1(x)$ with $f_1(b) = E(b)$.
- By the lemma, we have that $f_2(x) := E(x+b)$ is the unique solution to the ODE $f_2'(x) = f_2(x)$ with $f_2(0) = E(b)$ (i.e. take $C = b$, $f = E$, $g = f_2$)
- By 5d, we have that $f_3(x) := E(b)E(x)$ is the unique solution to the ODE $f_3'(x) = f_3(x)$ with $f_3(0) = E(b)$.

Hence by uniqueness, $f_2 = f_3$. Thus, for any $a \in \mathbb{R}$ we have that $E(b+a) = E(b)E(a)$ as desired.

(b) Fix $x \in \mathbb{R}$.
First, we show $E(mx) = E(x)^m$ for $m \in \mathbb{Z}$.

- If $m = 0$, then $E(0x) = 1 = E(x)^0$ as desired.
- If $m \in \mathbb{N}$. We have that $E(1x) = E(x)^1$ by definition. Then, assuming $E((m-1)x) = E(x)^{m-1}$ we have

$$E(mx) = E((m-1)x + x) = E((m-1)x)E(x) = E(x)^{m-1}E(x) = E(x)^m$$

  proving the desired equality for any $m \in \mathbb{N}$.
- If $m < 0$, then $-m \in \mathbb{N}$. Note that

$$1 = E(0) = E(x-x) = E(x)E(-x) \implies E(-x) = \frac{1}{E(x)} = E(x)^{-1}$$

  which is always well defined since $E(x) > 0$ for any $x \in \mathbb{R}$. Hence, we compute

$$E(mx) = \frac{1}{E(-mx)} = \frac{1}{E(-x)^{-m}} = \left(\frac{1}{E(-x)}\right)^{-m} = E(x)^m$$

  which shows the desired equality for any $m \in \mathbb{Z}$.

Next, given $n \in \mathbb{N}$, we note that

$$E(x) = E\left(n\tfrac{x}{n}\right) = E\left(\tfrac{x}{n}\right)^n$$

Since $E(x) > 0$ and $E(\frac{x}{n}) > 0$, by uniqueness of non-negative nth roots we have that

$$E(x/n) = E(x)^{1/n}$$

as desired.
Finally, given $q = m/n \in \mathbb{Q}$, we have that

$$E(qx) = E(\tfrac{m}{n}x) = E(x/n)^m = \left(E(x)^{1/n}\right)^m = E(x)^q$$

showing the desired equality. Notice that m and n do not have to be in reduced form.

(c) Since E is strictly increasing, we have that $E(-1) < E(0) = 1 < E(1)$. Then, $\{E(-n)\} = \{E(-1)^n\}$ converges to 0 by the ratio test. Similarly, $\{E(n)\} = \{E(1)^n\}$ is unbounded by the ratio test. Furthermore, $E(1)^n < E(1)^{n+1}$ for any $n \in \mathbb{N}$, so $\{E(n)\}$ is monotone increasing.

Since E is strictly increasing, we have that for $x, y \in \mathbb{R}$, $x > y \implies E(x) > E(y)$ so $E(x) = E(y) \implies x = y$, hence E is injective. Furthermore, given $a > 0$, we have $\exists N \in \mathbb{N}$ such that $E(-N) < a$ by convergence of $\{E(-n)\}$ to 0, and $\exists K \in \mathbb{N}$ such that $E(K) > a$ by unboundedness of $\{E(n)\}$. Hence, by Bolzano's IVT, there exists some $c \in [-N, K]$ such that $E(c) = a$. Thus, $E : \mathbb{R} \to (0, \infty)$ is surjective and hence bijective.

This allows us to concluded there is a unique function $L : (0, \infty) \to \mathbb{R}$ satisfying $E(L(a)) = a$ for all $a \in (0, \infty)$ given by the inverse function to E.

(d) By continuity of E, we have that for any sequence of rational numbers $\{q_n\}$ with $\lim_{n\to\infty} q_n = x$, we have

$$a^x := E(xL(a)) = E(\lim_{n\to\infty}(q_n L(a))) = \lim_{n\to\infty} E(q_n L(a)) = \lim_{n\to\infty} E(L(a))^{q_n} = \lim_{n\to\infty} a^{q_n}$$

Since limits are unique, this is the only such number a^x which satisfies this property.