Homework 6

Due: Monday, October 31th by 11:59 PM ET

- To fulfill the **collaboration requirement**, clearly write the name(s) of collaborators on the top of your first page. Remember that you must **write up your own solutions independently**.
- Please make sure your submission is **easily readable**. Typed solutions are accepted.
- You can use any result proved in the course text, in class, or on a previous homework question provided you **clearly mention** the result you are using.

Assigned Readings Lebl 3.3, 4.1-4.2

Sections 3.2-3.3 Exercises

Problem 1 (3 points each) Let us see how continuity interacts with restrictions.

(a) Let $A \subset S \subset \mathbb{R}$, and $c \in A$ be a point. Suppose $f : S \to \mathbb{R}$ is continuous at c. Prove that the restriction $f\big|_A$ is continuous at c.

(b) Find an example of a function $f : S \to \mathbb{R}$ and a subset $A \subset S$ such that $f\big|_A$ is continuous at some $c \in A$ but f is not continuous at c.

(c) Suppose $S \subset \mathbb{R}$ such that $(c - \alpha, c + \alpha) \subset S$ for some $c \in \mathbb{R}$ and $\alpha > 0$. Let $f : S \to \mathbb{R}$ be a function and $A := (c - \alpha, c + \alpha)$. Prove that if $f\big|_A$ is continuous at c, then f is continuous at c.

Problem 2 (5 points) Prove Corollary 3.3.12 in the textbook: If $f : [a, b] \to \mathbb{R}$ is continuous, then the direct image $f([a, b])$ is either a closed and bounded interval, or a single number.

Problem 3 (4 points each) Recall early on in the course, we very laboriously showed the existence and uniqueness of $\sqrt{2}$. We will show how the tools we have learned will allow us to show the existence and uniqueness of non-negative nth roots $\sqrt[n]{a}$ for any $n \in \mathbb{N}$ and any non-negative real number $a \in [0, \infty)$.

In the following, let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) := x^n$.

(a) Show that f is strictly increasing for all $n \in \mathbb{N}$, and use it to conclude that f is injective.

We say f is *strictly increasing* if $f(x) < f(y)$ for all $x, y \in [0, \infty)$ with $x < y$.

(b) Show that f is continuous for all $n \in \mathbb{N}$. Then, given $M \in \mathbb{N}$, use part (a) and what you know about continuous functions to show that the restriction $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is both surjective and injective, and hence bijective.

(c) Use the results of part (b) to conclude that for any $a \in [0, \infty)$, there exists a unique non-negative x such that $x^n = a$.

Problem 4 (6 points) Suppose $g(x)$ is a monic polynomial of even degree d, that is

$$g(x) = x^d + b_{d-1}x^{d-1} + ... + b_1 x + b_0$$

for some real numbers $b_0, b_1, ..., b_{d-1} \in \mathbb{R}$. Suppose $g(0) < 0$. Show that g has at least two distinct roots, that is, there exists at least two real numbers $c_1 \neq c_2$ such that $g(c_1) = g(c_2) = 0$.

(Hint: This proof shares many similarities with the proof of Proposition 3.3.10 in the textbook. Make sure to use closed and bounded intervals when you are using Bolzano's intermediate value theorem.)

Problem 5 (4 points) Let $S \subset \mathbb{R}$ be a set, and suppose $E \subset S$ is a subset such that every $x \in S$ is a cluster point of E. Let $f, g : S \to \mathbb{R}$ be continuous functions satisfying $f(x) = g(x)$ for all $x \in E$. Show that $f(x) = g(x)$ for all $x \in S$.

(*Remark*: This problem has to do with "continuous extensions", which we may cover at a later point. The idea here is that E is some set that can "approximate" elements of S, for example, $E = \mathbb{Q}$ and $S = \mathbb{R}$. This problem shows that if $f(x) = g(x)$ on all elements $x \in E$ of the approximating set, then $f(x) = g(x)$ for all elements $x \in S$ of the full set.)

Section 4.1 Exercises

Problem 6 (4 points) Prove linearity for derivatives: Let I be an interval, let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions differentiable at c, and let $\alpha, \beta \in \mathbb{R}$. If $h : I \to \mathbb{R}$ is defined by

$$h(x) := \alpha f(x) + \beta g(x)$$

then h is differentiable at c and

$$h'(c) = \alpha f'(c) + \beta g'(c)$$

Problem 7 (5 points) Prove the product rule for derivatives: Let I be an interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions differentiable at c. If $h : I \to \mathbb{R}$ is defined by

$$h(x) := f(x)g(x)$$

then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c)$$

Problem 8 (5 points) Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) := \sqrt{x}$. Prove, using the limit definition of the derivative, that f is differentiable at all $c \in (0, \infty)$.