HW9

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1. By Prop., $\exists x, y \in A$, $|x|, |y| \in B$ s.t $\forall \varepsilon > 0$,

$\sup B - |x| < \frac{\varepsilon}{2}$

$|y| - \inf B < \frac{\varepsilon}{2}$

$$\Rightarrow \sup B - \inf B < |x| + \frac{\varepsilon}{2} - |y| + \frac{\varepsilon}{2}$$
$$= |x| - |y| + \varepsilon$$
$$\leq |x-y| + 2\varepsilon$$
$$\leq \sup A - \inf A + \varepsilon$$

Since ε is arbitrary,

$\sup B - \inf B \leq \sup A - \inf A$

2.(a) Let $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = -\min\{f(x), 0\}$. Then

$|f| = f^+ + f^-$, $f = f^+ + f^-$

$$\text{so } 0 \leq \left|\int_a^b f\right| = \left|\int_a^b f^+ - \int_a^b f^-\right|$$
$$\leq \left|\int_a^b f^+\right| + \left|\int_a^b f^-\right|$$
$$= \int_a^b f^+ + \int_a^b f^-$$
$$= \int_a^b (f^+ + f^-) = \int_a^b |f|$$

Therefore, $0 \leq \left|\int_a^b f\right| \leq \int_a^b |f|$

(b) Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$

so $|f(x)| = 1$ is continuous, thus $|f| \in R[a,b]$

However, for any Partition P, $m_i = -1$, $M_i = 1$,

$L(P,f) = \sum_{i=1}^{n} -1 \cdot (\Delta x_i) = a - b$, $U(P,f) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = b - a$

$\underline{\int_a^b} f = a - b \neq b - a = \overline{\int_a^b} f \Rightarrow f \notin R[a,b]$

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(c) By Prop, $\exists \delta > 0$ s.t. $f(x) > 0$ $\forall x \in (c-2\delta, c+2\delta)$

so $\int_{c-\delta}^{c+\delta} f = \underline{\int}_{c-\delta}^{c+\delta} f \geq L(P, f)$ P is any partition of $[c-\delta, c+\delta]$

$= \sum_{i=1}^{n} m_i \Delta x_i$

> 0 since $m_i > 0$ $\forall i$

(d) "$\Rightarrow$"

If $f(c) \neq 0$ for some $c \in [a, b]$, then $|f(c)| > 0$,

by (c), $\exists \delta > 0$ s.t. $\int_{c-\delta}^{c+\delta} |f| > 0$

by (a), $\int_a^b |f| = \int_a^{c-\delta} |f| + \int_{c-\delta}^{c+\delta} |f| + \int_{c+\delta}^{b} |f|$

$\geq 0 + \int_{c-\delta}^{c+\delta} |f| + 0$

> 0

(If $c-\delta < a$ or $c+\delta > 0$, we can exclude $(c-\delta, a)$ or $(b, c+\delta)$ and other parts are similar)

Therefore, if $\int_a^b |f| = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

"$\Leftarrow$"

If $f(x) = 0$ for all $x \in [a, b]$,

$\int_a^b |f| = \int_a^b 0 = 0$.

then $\int_a^b |f| = 0$.

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3\. By First Form of FTC, since F, G are continuously differentiable

$$\begin{aligned}(F(x)-G(x))-(F(a)-G(a)) &= \int_a^x F'(t)\,dt - \int_a^x G'(t)\,dt \\ &= \int_a^x (F'(t)-G'(t))\,dt \\ &= \int_a^x 0\,dt \\ &= 0\end{aligned}$$

Therefore, there exists $C = F(a)-G(a) \in \mathbb{R}$ s.t. $F(x)-G(x)=C$.

4. (a) $f_n(x) = \frac{n+1}{nx}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n+1}{nx} = \frac{1}{x}$

so $\{f_n\}$ converges to $f: (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ pointwise

$\lim_{n\to\infty} \|f_n - f\|_u = \lim_{n\to\infty} \sup\{\frac{1}{nx} : x \in (0,1)\}$

$= \lim_{n\to\infty} (+\infty) = +\infty$

so $\{f_n\}$ does not converge uniformly

P.S. a more rigorous proof: assume converges uniformly,

$\exists M \in \mathbb{N}: \forall x \in (0,1), \forall n \geq M, |f_n(x) - f(x)| = \frac{1}{nx} < \varepsilon = \frac{1}{2}$

Let $\{x_k\}$ be a sequence s.t. $x_k \in (0,1)$, $x_k \to 0$ as $k \to \infty$

then $\lim_{k\to\infty} \frac{1}{Mx} \geq \frac{1}{M} \cdot \frac{1}{\frac{1}{M}} = 1 > \frac{1}{2}$ contradicts with $\frac{1}{Mx} < \frac{1}{2}$

so $\{f_n\}$ does not converge uniformly.

(b) $\{f_n\}$ converges pointwise to $f: [0,1] \to \mathbb{R}$, $f(x) = 0$

since $\frac{1}{n} \to 0$ as $n \to \infty$

Assume $\{f_n\}$ converges uniformly,

$\exists M \in \mathbb{N}. \forall x \in [0,1], \forall n \geq M, |f_n(x) - f(x)| < \varepsilon = \frac{1}{2}$

$\Rightarrow |f_M(x) - f(x)| < \frac{1}{2}$

let $x = \frac{1}{2M}$, $|f_M(x) - f(x)| = |2M - 0| = 2M > \frac{1}{2} \Rightarrow$ contradiction

so $\{f_n\}$ does not converge uniformly

$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \int_0^{\frac{1}{n}} n = 1$

$\int_0^1 f = \int_0^1 0 = 0$

so $\lim_{n\to\infty} \int_0^1 f \neq \int_0^1 f$

(c) $f_n(x) = \frac{x^n}{n}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{n} = 0 \quad \forall x \in [0,1]$

so $\{f_n\}$ converges pointwise to $f: [0,1] \to \mathbb{R}$, $f(x) = 0$

$\|f_n - f\|_u = \sup_{x\in[0,1]} \left|\frac{x^n}{n} - 0\right| = \frac{1}{n} \to 0$ as $n \to \infty$

so $\{f_n\}$ converges uniformly

$f'(1) = 0$

$\lim_{n\to\infty} f_n'(1) = \lim_{n\to\infty} 1^{n-1} = 1$

so $f'(1) \neq \lim_{n\to\infty} f_n'(1)$.

5.(a) $\forall x \in [a,b]$, $|f(x)| \leq \|f\|_u$, $|g(x)| \leq \|g\|_u$

$\Rightarrow |f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_u + \|g\|_u$

$\Rightarrow \|f\|_u + \|g\|_u$ is an upper bound of $f(x)+g(x)$

Also $\|f+g\|_u = \sup\{|f(x)+g(x)| : x \in [a,b]\}$ is the least upper bound of $f(x)+g(x)$

Therefore, $\|f+g\|_u \leq \|f\|_u + \|g\|_u$

(b) $\|f\|_u = \|f-g+g\|_u$
$\leq \|f-g\|_u + \|g\|_u$ by (a)
$\Rightarrow \|f\|_u - \|g\|_u \leq \|f-g\|_u$
$\|g\|_u = \|g-f+f\|_u$
$\leq \|g-f\|_u + \|f\|_u$ by (a)
$= \|f-g\|_u + \|f\|_u$
$\Rightarrow \|f\|_u - \|g\|_u \geq -\|f-g\|_u$
Combine them $\Rightarrow \big|\|f\|_u - \|g\|_u\big| \leq \|f-g\|_u$

6. $\forall x \in (0,1]$, $\lim_{n\to\infty} f_n(x) = 0$

so $\{f_n\}$ converges pointwise to: $f: [0,1] \to \mathbb{R}$
$f(x) = \begin{cases} 0 & x=0 \\ 1 & 0<x\leq 1 \end{cases}$
every subsequence $\{f_{n_k}\}$ converges pointwise to $f(x)$
Assume there exists $\{f_{n_k}\}$ converges uniformly
$\exists M \in \mathbb{N}: \forall x \in [0,1], \forall k \geq M \Rightarrow n_k \geq k \geq M$
$|f_{n_k}(x) - f(x)| < \varepsilon = \frac{1}{2}$
$\Rightarrow |1 - Mx - 0| < \frac{1}{2}$ $\forall x \in (0,1]$
Take $x = \frac{1}{4M}$, then $|1-Mx-0| = |1-\frac{1}{4}| = \frac{3}{4} > \frac{1}{2} \Rightarrow$ contradiction
so no subsequence converges uniformly.

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