Homework 1 Solutions

Due: Monday, September 19th by 11:59 PM ET

Chapter 0 Exercises

Problem 1 (5 points) Let A, B, C be sets. Prove the following set relation properties:

(i) (*Transitivity of set inclusion*) If $A \supset B$ and $B \supset C$, then $A \supset C$

(ii) (*Transitivity of set equality*) If $A = B$ and $B = C$, then $A = C$

(i) Assume $A \subset B$ and $B \subset C$. Then, $x \in A \implies x \in B \implies x \in C$, so $x \in A \implies x \in C$. Thus, $A \subset C$.

(ii) Assume $A = B$ and $B = C$. Then, by definition of set equality, $A \subset B$ and $B \subset C$, and $C \subset B$ and $B \subset A$. By transitivity of set inclusion (i), $A \subset C$ and $C \subset A$. Thus, $A = C$.

Problem 2 (5 points) For each function, determine if it is (i) injective and (ii) surjective. Don't forget to justify your answer with a proof.

(a) $f : (0, 1) \to (1, \infty)$ where $f(x) := 1/x$

(b) $g : \mathbb{R} \to \mathbb{Z}$ given by $g(x) := \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the *floor* function which 'rounds down', i.e. returns the largest integer less than or equal to x.

(a) f is **injective** because for all $y \in (1, \infty)$, $1/x = y$ has at most one solution $x = 1/y$. Thus $f^{-1}(\{y\})$ always consists of one or zero elements.

Furthermore, if $y > 1$, then $x = 1/y < 1$ and $x > 0$. Thus, for all $y \in (1, \infty)$ in the target space, there is at least one element $x = 1/y$ is always in the domain $(0, 1)$ that maps to it. Thus, f is **surjective**.

(b) g is **not injective** because $\lfloor 1.5 \rfloor = \lfloor 1 \rfloor = 1$, so $g^{-1}(1)$ has at least two elements.

g is **surjective**, because $\mathbb{Z} \subset \mathbb{R}$, so every integer $n \in \mathbb{Z}$ can be written as $n = g(n)$, where $n \in \mathbb{R}$ is in the domain.

Problem 3 (6 points) For $p \in \mathbb{N}$, define $\mathbb{N}^p := \mathbb{N} \times ... \times \mathbb{N}$ (p times) to be the set of p-tuples of natural numbers, i.e. $(n_1, n_2, ..., n_p) \in \mathbb{N}^p$.

(a) Let $f_2 : \mathbb{N}^2 \to \mathbb{N}$ be the bijection defined in example 0.3.31, so $f_2(1,1) = 1$, $f_2(1,2) = 2$, etc...

Define a function $f_3 : \mathbb{N}^3 \to \mathbb{N}^2$ by $f_3(n_1, n_2, n_3) := (n_1, f_2(n_2, n_3))$. Show that f_3 is a bijection.

(b) Show using induction that $\mathbb{N}^p$ is countable for any $p \in \mathbb{N}$

(*Note*: It is possible prove this without induction, but you should practice using induction for this problem.)

(a) Since f_2 is bijective, for every $p \in \mathbb{N}$ there is exactly one $(n_2, n_3) \in \mathbb{N}^2$ such that $f_2(n_2, n_3) = p$. Thus for every pair $(n_1, p) \in \mathbb{N}^2$, there is exactly one triplet $(n_1, n_2, n_3) \in \mathbb{N}^3$ such that $f_3(n_1, n_2, n_3) = (n_1, p)$. Thus, f_3 is bijective.

(b) (Basis statement, $p = 2$) The textbook has already proven that $\mathbb{N}^2$ is countable.

(Induction step) Assume $\mathbb{N}^{p-1}$ is countable, so there there exists a function $g : \mathbb{N}^{p-1} \to \mathbb{N}$ which is bijective. Define a function $f_p : \mathbb{N}^p \to \mathbb{N}^{p-1}$ by $f_p(n_1, ..., n_{p-2}, n_{p-1}, n_p) := (n_1, ..., n_{p-2}, f_2(n_{p-1}, n_p))$.

By the same logic as in (a), f_p is a bijection. Thus, $g \circ f_p : \mathbb{N}^p \to \mathbb{N}$ is a bijection. Thus, $\mathbb{N}^p$ is countable.

Since $\mathbb{N}^1 = \mathbb{N}$ is countable, this proves that $\mathbb{N}^p$ is countable for any $p \in \mathbb{N}$.

Problem 4 (6 points) Prove Proposition 0.3.16: Consider $f : A \to B$. Let C, D be subsets of A. Then,

$$f(C \cup D) = f(C) \cup f(D)$$
$$f(C \cap D) \subset f(C) \cap f(D)$$

Additionally, find a function $f : A \to B$ and sets C, D such that $f(C \cap D) \not\supset f(C) \cap f(D)$.

Claim : $f(C \cup D) = f(C) \cup f(D)$

Proof : To prove this, we will show $f(C \cup D) \subset f(C) \cup f(D)$ and $f(C \cup D) \supset f(C) \cup f(D)$

First, suppose $y \in f(C \cup D)$. This means there exists $x \in C \cup D$ such that $f(x) = y$. Then, $f(x) = y$ for either $x \in C$ or $x \in D$. Thus, $y \in f(C) \cup f(D)$ so $f(C \cup D) \subset f(C) \cup f(D)$.

Now, suppose $y \in f(C) \cup f(D)$. This means $y = f(x)$ for $x \in C$ or $x \in D$. This means there exists $x \in C \cup D$ such that $y = f(x)$. Thus, $f(C \cup D) \supset f(C) \cup f(D)$. $\square$

Claim : $f(C \cap D) \subset f(C) \cap f(D)$

Proof : Suppose $y \in f(C \cap D)$. Then, there exists some $x \in C \cap D$ such that $f(x) = y$. Since $x \in C \cap D \implies x \in C$ and $x \in D$, then $y \in f(C)$ and $y \in f(D)$, so $y \in f(C) \cap f(D)$. Thus, $f(C \cap D) \subset f(C) \cap f(D)$. $\square$

Counterexample Function : Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, and let $C = (-1, 0)$ and $D = (0, 1)$. Then, $C \cap D = \emptyset$, but $f(C) = f(D) = (0, 1)$ so $f(C) \cap f(D) = (0, 1) \not\subset f(C \cap D) = \emptyset$

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Problem 5 (6 points) Let $E = (-\infty, b) := \{x \in \mathbb{R} : x < b\}$ where $b \in \mathbb{R}$. Compute $\sup E$ and $\inf E$ if they exist, or prove that E is unbounded above/below if they do not exist. Don't forget to justify your answer by proof.

(*Note*: do not use the extended reals for this problem)

Claim : $b = \sup E$

Proof : E is non-empty, since $b - 1 < b$ so $b - 1 \in E$. b is an upper bound for E, since $x < b$ for all $x \in E$. Thus, since E is non-empty and bounded above, it has a least upper bound. In particular, $\sup E \leq b$.

Now, suppose there was another upper bound b' such that $b' < b$. Then, $b' < (b'+b)/2 < b$. However, $(b' + b)/2 \in E$, so b' cannot be an upper bound of E, Thus, any upper bound b' must satisfy $b' \geq b$. In particular, $\sup E \geq b$.

Combining these two inequalities, $\sup E = b$ as desired. □

Claim : $\inf E$ does not exist, since E is unbounded below.

Proof : Suppose a was a lower bound for E. Assume without loss of generality $a < b$. Then, $a - 1 < b$ so $a - 1 \in E$. However, $a - 1 < a$, which contradicts a being a lower bound for E. Thus, E cannot have a lower bound, and is thus unbounded below. □

Problem 6 (6 points) Suppose $A, B \subset \mathbb{R}$ are non-empty sets that are both bounded above and below, and furthermore that $A \subset B$. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

First, since A, B are non-empty and bounded above and below, $\inf B$, $\inf A$, $\sup A$, $\sup B$ all exist.

Next, since A is non-empty, there exists some $x \in A$ which satisfies $\inf A \leq x \leq \sup A$. Thus, $\inf A \leq \sup A$.

Then, since $A \subset B$, we have $x \in B$ for every $x \in A$. Thus, $\inf B \leq x \leq \sup B$ for every $x \in A$. Thus, $\inf B$ and $\sup B$ are respectively a lower bound and an upper bound for A. Thus, the greatest lower bound of A satisfies $\inf B \leq \inf A$, and the least upper bound of a satisfies $\sup A \leq \sup B$.

Finally, chaining the inequalities together, we get $\inf B \leq \inf A \leq \sup A \leq \sup B$

Problem 7 (6 points) Let $B \subset \mathbb{R}$ be bounded above, and let $c = \sup B$. Prove the following statements:

(a) c is unique; that is, if c' is also a supremum of B, then $c = c'$

(b) For any $x \in \mathbb{R}$, if $x > c$ then $x \notin B$

(a) If c and c' are suprema of B, then $c \leq c'$ and $c' \leq c$ since c and c' are both least upper bounds, so $c' = c$.

(b) Since c is an upper bound of B, $y \leq c$ for all $y \in B$, hence any $x \in \mathbb{R}$ with $x > c$ cannot be in B.

Problem 8 (5 points each) Let $B \subset \mathbb{R}$ be a non-empty subset which is bounded above and below. Let $c = \sup B$ and $d = \inf B$:

(a) For all real numbers $\varepsilon > 0$, there exists $x \in B$ such that $c - \varepsilon < x \leq c$

(b) For every $\varepsilon > 0$, the set $[d, d + \varepsilon) \cap B$ is non-empty.

(*Hint*: The first statement (a) takes the form of a nested quantifier, "$\forall \varepsilon \in (0, \infty), \exists x \in B$ s.t. $P(\varepsilon, x)$ is true". The negation of this double quantifier is "$\exists \varepsilon \in (0, \infty)$ s.t. $\forall x \in B, P(\varepsilon, x)$ is false". This can be seen through 'abstract logic' by negating the statement "$\forall \varepsilon \in (0, \infty), Q(\varepsilon)$ is true" where $Q(\varepsilon)$ is the predicate "$\exists x \in B$ s.t. $P(\varepsilon, x)$ is true".

In plain English, the negation of (a) would be "there exists a real number $\varepsilon > 0$ such that for all $x \in B$, $x \leq c - \varepsilon$ or $x > c$". One way to prove (a) is to assume the negation of (a), then prove a contradiction.

To prove (b), try converting it to a statement similar to (a). Note that (b) provides a "geometric" interpretation of a double quantifier statement.)

(a) We will show this via contradiction. Assume the negation of (a), i.e. assume there exists a real number $\varepsilon > 0$ such that for all $x \in B$, either $x \leq c - \varepsilon$ or $x > c$.

Since c is an upper bound of B, $x > c$ cannot be true. Thus, $x \leq c - \varepsilon$ for all $x \in B$. However, this would imply that $c - \varepsilon$ is an upper bound of B. However, $c - \varepsilon < c$ contradicts the assumption that c is the least upper bound of B. This proves a contradiction.

Thus, for all $\varepsilon > 0$, there must exist an $x \in B$ such that $c - \varepsilon < x \leq c$.

(b) Note that $[d, d + \varepsilon) \cap B$ is non-empty if and only if there exists an $x \in B$ such that $d \leq x < d + \varepsilon$. Thus, (b) is equivalent to the statement "for all $\varepsilon > 0$, there exists $x \in B$ such that $d \leq x < d + \varepsilon$".

Assume for sake of contradiction that there exists $\varepsilon > 0$ such that for all $x \in B$, either $x < d$ or $x \geq d + \varepsilon$. $x < d$ cannot be true since d is a lower bound of B, so $x \geq d + \varepsilon$ for all $x \in B$. This implies $d + \varepsilon$ is a lower bound of B, which contradicts the assumption that d is the greatest lower bound. Thus, the original statement must hold. This implies the desired statement.