Homework 7 Solutions

Due: Monday, November 7th by 11:59 PM ET

Sections 4.1-4.3 Exercises

Problem 1 (4 points each) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \sin(x)$. This problem will walk you through proving that f is differentiable, and that $f'(x) = \cos(x)$.

You may use basic trigonometric identities and inequalities[1], and may find this particular inequality helpful:

$$\sin(x) < x < \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ for } x \in (0, \pi/2)$$

You may also assume that $\sin(x)$ and $\cos(x)$ are continuous functions for $x \in (-\pi/2, \pi/2)$. Recall on HW5 you showed that $\cos(x)$ is continuous, and a very similar proof would show that $\sin(x)$ is continuous.

(a) Prove that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(b) Show that $f(x) = \sin(x)$ is differentiable for all $x \in \mathbb{R}$, and that $f'(x) = \cos(x)$.

(Hint: Try using the sum-to-product identity on $\sin(x) - \sin(c)$.)

(a) For $x \in (0, \pi/2)$, using the inequality $\sin(x) < x$, we have

$$\frac{\sin(x)}{x} < 1$$

and using the inequalities $x < \frac{\sin(x)}{\cos(x)}$ and $0 < \cos(x)$ for $x \in (0, \pi/2)$, we have

$$\cos(x) < \frac{\sin(x)}{x}$$

Then, using the fact that $\sin(-x) = -\sin(x)$, and $\cos(-x) = \cos(x)$, we have for $x \in (-\pi/2, \pi/2) \setminus \{0\}$

$$\cos(x) < \frac{\sin(x)}{x} < 1$$

Since $\cos(x)$ and 1 are continuous functions, we have that their limits exist as $x \to 0$,

$$\lim_{x\to 0} \cos(x) = \cos(0) = 1 = \lim_{x\to 0} 1$$

Thus by the squeeze lemma, the limit of $\frac{\sin(x)}{x}$ exists and is equal to

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

[1] Most trigonometric identities and inequalities have “geometric” proofs, so it doesn’t count as “cheating” to use them to prove facts about calculus. See `https://en.wikipedia.org/wiki/Proofs_of_trigonometric_identities` for example.

(b) Using the sum-to-product identity, we rewrite the difference quotient as

$$\frac{\sin(x)-\sin(c)}{x-c} = \frac{2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)}{x-c} = \frac{\sin\left(\frac{x-c}{2}\right)}{\frac{x-c}{2}}\cos\left(\frac{x+c}{2}\right)$$

By replacing x with $\frac{x-c}{2}$ in the proof of (a), we see that

$$\lim_{x\to c}\frac{\sin\left(\frac{x-c}{2}\right)}{\frac{x-c}{2}} = 1$$

And note that $\cos\left(\frac{x+c}{2}\right)$ is the composition of the continuous functions $\cos(x)$ and $\frac{x+c}{2}$ (both $\mathbb{R}\to\mathbb{R}$), so the limit of $\cos\left(\frac{x+c}{2}\right)$ as $x\to c$ exists. Thus, by the continuity of algebraic operations,

$$f'(c) = \lim_{x\to c}\left[\frac{\sin\left(\frac{x-c}{2}\right)}{\frac{x-c}{2}}\cos\left(\frac{x+c}{2}\right)\right] = \lim_{x\to c}\left(\frac{\sin\left(\frac{x-c}{2}\right)}{\frac{x-c}{2}}\right)\lim_{x\to c}\left(\cos\left(\frac{x+c}{2}\right)\right) = \cos(c)$$

Problem 2 (5 points each) In this problem, we will prove a special case of L'Hôpital's rule.

(a) Let $h : S \to \mathbb{R}$ and c be a cluster point of S. Show that if $\lim_{x \to c} h(x) = L \neq 0$, then there exists some $\delta > 0$ such that for all $x \in (S \setminus \{c\}) \cap (c - \delta, c + \delta)$, $h(x) \neq 0$.

(b) Let $h : S \to \mathbb{R}$ be continuous and c be a cluster point of S. Show that if $h(c) \neq 0$, then there exists some $A \subset S$ such that c is a cluster point of A, $h\big|_A (x) \neq 0$ for all $x \in A$, and

$$\lim_{x \to c} \left(\frac{1}{h|_A(x)} \right) = \frac{1}{\lim_{x \to c} (h|_A(x))} = \frac{1}{h(c)}$$

Note: This result allows us to "abuse notation". We get a slightly more general notion of Corollary 3.1.12.iv and write

$$\lim_{x \to c} \left(\frac{1}{h(x)} \right) = \frac{1}{\lim_{x \to c} h(x)}$$

even though strictly speaking, $1/h(x)$ might not be defined for all $x \in S$.

(c) Suppose $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions. Suppose that at $c \in (a, b)$, $f(c) = g(c) = 0$, and $g'(x) \neq 0$ for all $x \in (a, b)$, and suppose that the limit of $\frac{f'(x)}{g'(x)}$ as $x \to c$ exists. Show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

(*Hint*: This is similar to the proof that a differentiable function is continuous. Be careful not to divide by 0, and make sure to explain all the steps in your proof.)

(a) Let h be as given. Let $\varepsilon = |L|/2 > 0$. Then, there exists $\delta > 0$ such that for all $x \in S \setminus \{c\}$ with $|x - c| < \delta$,

$$||h(x)| - |L|| \leq |h(x) - L| < |L|/2 \implies -|L|/2 < |h(x)| - |L| < |L|/2$$

where the first inequality comes from the reverse triangle inequality. Thus, for all $x \in (S \setminus \{c\}) \cap (c - \delta, c + \delta)$ we have

$$|h(x)| > |L|/2 \implies h(x) \neq 0$$

as desired.

(b) Let h be as given. Since h is continuous, $\lim_{x\to c} h(x) = h(c) \neq 0$, thus by the result of (a) there exists some $\delta > 0$ such that for all $x \in A := S \cap (c-\delta, c+\delta)$, we have $h(x) \neq 0$.

To show c is a cluster point of A, let $\delta' > 0$ be arbitrary. Define $\delta'' := \min\{\delta, \delta'\}$. Then, the set $(c-\delta', c+\delta') \cap (A \setminus \{c\}) = (S \setminus \{c\}) \cap (c-\delta'', c+\delta'')$ is non-empty since c is a cluster point of S.

Finally, by Corollary 3.1.12.iv and the limit characterization of continuity, we have

$$\frac{1}{h(c)} = \frac{1}{\lim_{x\to c} (h|_A(x))} = \lim_{x\to c} \left(\frac{1}{h|_A(x)} \right)$$

as desired.

(c) Let f, g be as given, and assume they satisfy the assumptions of the statement. Now, since g' is continuous, we have that $\lim_{x\to c} g'(x) = g'(c) \neq 0$.

$$\begin{aligned}
\lim_{x\to c} \frac{f'(x)}{g'(x)} &= \frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)} && \text{(Corollary 3.1.12.iv)} \\
&= \frac{f'(c)}{g'(c)} && \text{(Continuity of } f', g'\text{)} \\
&= \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} && \text{(Difference quotient definition of derivative)} \\
&= \lim_{x\to c} \left(\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \right) && \text{(Application of (b), since } g'(c) \neq 0\text{)} \\
&= \lim_{x\to c} \frac{f(x)}{g(x)} && \text{(Algebra; } g(c) = f(c) = 0\text{)}
\end{aligned}$$

which shows the desired equality.

Problem 3 (4 points each) Let I be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Prove the following statements:

(a) $f'(x) \leq 0$ for all $x \in I$ if and only f is decreasing.

We say f is decreasing if $f(x) \geq f(y)$ for all $x, y \in I$ with $x < y$

(b) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing.

We say f is strictly decreasing if $f(x) > f(y)$ for all $x, y \in I$ with $x < y$

(a) First, suppose f is decreasing. Then, for all $x, c \in I$ with $x \neq c$, we have

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

Since limits preserve non-strict inequalities, the limit of the difference quotient satisfies $f'(c) \leq 0$.

For the other direction, suppose $f'(x) \leq 0$ for all $x \in I$. Take any $x, y \in I$ with $x < y$. Then, by the mean value theorem (applied to the interval $[x, y]$), there exists some $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x)$$

Since $f'(c) \leq 0$ and $y - x > 0$, we have $f(y) - f(x) \leq 0$, or in other words $f(x) \geq f(y)$, so f is decreasing.

(b) Suppose $f'(x) < 0$ for all $x \in I$. Take any $x, y \in I$ with $x < y$. Then, by the mean value theorem (applied to the interval $[x, y]$), there exists some $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x)$$

Since $f'(c) < 0$ and $y - x > 0$, we have $f(y) - f(x) < 0$, or in other words $f(x) > f(y)$, so f is strictly decreasing.

Problem 4 (4 points each) Here is an extremely useful application of the mean value theorem, which can be thought of as a special case of Taylor's theorem:

Suppose $f : [a, b] \to \mathbb{R}$ satisfies the assumptions of the MVT, and there is a M such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Then, for any $x, y \in [a, b]$, we have from the mean value theorem there is a c between x, y such that

$$f(x) - f(y) = f'(c)(x - y)$$

Taking the absolute value of both sides, we can get a convenient upper bound for $|f(x) - f(y)|$, namely

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq M|x - y|$$

Prove the following inequalities:

(a) For any $R > 0$, $n \in \mathbb{N}$, and $x, y \in [-R, R]$, we have $|x^n - y^n| \leq nR^{n-1}|x - y|$

(b) For any $x, y \in \mathbb{R}$, we have $\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| \leq |x - y|$

(a) x^n is continuous and differentiable on $[-R, R]$ by repeated application of the product rule on x. Furthermore, the product rule also allows us to compute $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$. Then, we have for $x \in [-R, R]$

$$\left|nx^{n-1}\right| = n|x|^{n-1} \leq nR^{n-1}$$

so we conclude using the MVT,

$$|x^n - y^n| \leq nR^{n-1}|x - y|$$

(b) By application of the chain rule, we have that $\sqrt{x^2+1}$ is differentiable for all $x \in \mathbb{R}$, and we compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$$

Now, for all $x \in \mathbb{R}$, we have $0 \leq x^2 < x^2 + 1$, hence we have $|x| = \sqrt{x^2} < \sqrt{x^2+1}$. Thus, for all $x \in \mathbb{R}$ we have

$$\left|\frac{x}{\sqrt{x^2+1}}\right| = \frac{|x|}{\sqrt{x^2+1}} < \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$$

Then, for any $x, y \in \mathbb{R}$, if $x = y$, then

$$\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| = 0 \leq |x - y| = 0$$

and if $x \neq y$, then we apply the MVT on the interval bounded by x and y to get

$$\left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| = \left|\frac{c}{\sqrt{c^2+1}}\right||x - y| \leq |x - y|$$

Problem 5 (6 points) Here is another way to bound functions using Taylor's theorem:

Suppose $f : \mathbb{R} \to \mathbb{R}$ has n continuous derivatives. Show that for any closed and bounded interval $[a, b] \subset \mathbb{R}$, there exist polynomials P and Q of degree n such that $P(x) \le f(x) \le Q(x)$ for all $x \in [a, b]$ and $Q(x) - P(x) = \lambda(x - a)^n$ for some $\lambda \ge 0$.

(*Hint*: Try using Taylor's theorem at $x_0 = a$ with the min/max theorem.)

Let $[a, b]$ be a closed and bounded interval. Then, by Taylor's theorem, for $x \in (a, b]$, we can find a point c between a and x such that

$$f(x) = P^a_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x - a)^n$$

Since $f^{(n)}$ is continuous on the closed and bounded interval $[a, b]$, it achieves its infinum and supremum. Write

$$M_1 := \inf_{x\in[a,b]} f^{(n)}(x) \qquad M_2 := \sup_{x\in[a,b]} f^{(n)}(x)$$

Then, define polynomials

$$P(x) := P^a_{n-1}(x) + M_1(x - a)^n \qquad Q(x) := P^a_{n-1}(x) + M_2(x - a)^n$$

These satisfy $Q(x) - P(x) = (M_2 - M_1)(x - a)^n$, where $M_2 - M_1 \ge 0$.

Note that for all $x \in (a, b]$, $M_1 \le f^{(n)}(c) \le M_2$, and $(x - a)^n > 0$. Furthermore, for $x = a$, $P^a_{n-1}(a) = f(a)$. Thus, we have that for all $x \in [a, b]$,

$$P(x) \le f(x) \le Q(x)$$

which is the desired inequality.

Problem 6 (5 points) This problem introduces a very reduced version of the inverse function theorem.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Show that if $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then there exists some interval $I = (x_0 - \delta, x_0 + \delta)$ such that $f\big|_I : I \to f(I)$ is bijective.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and that $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Since $f'(x)$ is continuous, by the results of problem 2a, there exists some $\delta > 0$ such that $f'(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Take $I := (x_0 - \delta, x_0 + \delta)$.

Since $f'(x) > 0$ for all $x \in I$, we have that $f\big|_I$ is strictly monotone increasing on this interval. In particular, for any $x, y \in I$, $x > y \implies f(x) > f(y)$, so $f(x) = f(y) \implies x = y$. Thus $f\big|_I$ is injective. $f\big|_I(I) = f(I)$, so it is also surjective, hence bijective.