Cluster Points, Limits of Functions

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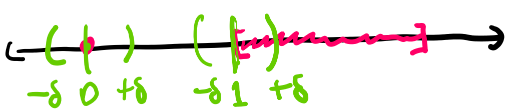
Cluster Points

<u>Def</u> Let $S \subset \mathbb{R}$ be a set. A number $c \in \mathbb{R}$ is called a <u>cluster point</u> of S if for all $\delta > 0$, $\underbrace{\text{there exists } x \in S \setminus \{c\} \text{ such that } |x-c| < \delta}_{\text{"the set } (c-\delta,\, c+\delta) \cap S \setminus \{c\} \text{ is non-empty"}}$

<u>Idea:</u>
c is not a cluster point of $S \Leftrightarrow \exists \delta > 0 : (c-\delta, c+\delta) \cap S \setminus \{c\}$ is empty
"c is isolated from other points of S"

<u>Ex.</u> $S = \{0\} \cup [1,2]$



- 0 is not a cluster point of S
  e.g. take $\delta = 1/2$, then $(c-\delta, c+\delta) \cap S \setminus \{c\} = (-\frac{1}{2}, \frac{1}{2}) \cap [1,2] = \emptyset$
- 1 is a cluster point of S
  e.g. for any $0 < \delta < 1$, $(1-\delta, 1+\delta) \cap S \setminus \{1\} = (1, 1+\delta)$

<u>Prop.</u> (Limit characterization of cluster points)
Let $S \subset \mathbb{R}$. Then $c \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence $\{x_n\}$ such that $x_n \in S \setminus \{c\}$ $\forall n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} x_n = c$.

<u>Pf</u> ($\Rightarrow$) Suppose c is a cluster point of S.
- For any $n \in \mathbb{N}$, pick

$$x_n \in \underbrace{(\overset{c-\delta}{c - \tfrac{1}{n}},\ \overset{c+\delta}{c + \tfrac{1}{n}}) \cap S \setminus \{c\}}_{\text{non-empty since } c \text{ is a cluster pt.}}$$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{non-empty since } c \text{ is a cluster pt.}}$

- Then, $\forall \varepsilon > 0$, $\exists M \in \mathbb{N} : \frac{1}{M} < \varepsilon$. Then, $\forall n \geq M$,

$$|x_n - c| < \frac{1}{n} \leq \frac{1}{M} < \varepsilon$$

So $\lim_{n\to\infty} x_n = c$. ✓

($\Leftarrow$) Suppose $\{x_n\}$ converges to c, with $x_n \in S \setminus \{c\}$ $\forall n \in \mathbb{N}$.

- Let $\delta > 0$ be given. $\exists M \in \mathbb{N} : \forall n \geq M$, $|x_n - c| < \delta$

$\Rightarrow x_M \in (c-\delta, c+\delta) \cap S \setminus \{c\}$

- Thus, $\forall \delta > 0$, $(c-\delta, c+\delta) \cap S \setminus \{c\}$ is non-empty.

$\Rightarrow$ c is a cluster pt. of S $\square$

★★★ **Def** (ε-δ definition of the limit of a function)

Let $f : S \to \mathbb{R}$ be a function, and c be a cluster point of $S \subset \mathbb{R}$. Suppose there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, we have

$$|f(x) - L| < \varepsilon$$

Then, we say $f(x)$ converges to L as x goes to c. We write

$$\lim_{x \to c} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to c$$

If no such L exists, we say f diverges at c.

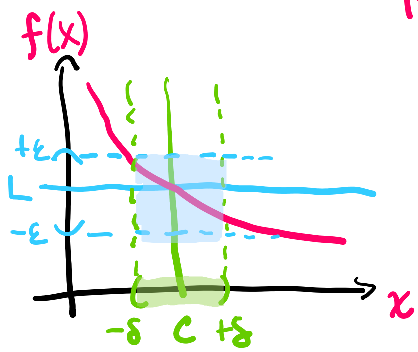
Idea:

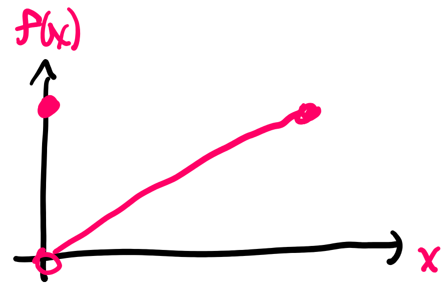
$f(x)$ converges at $c \Leftrightarrow \overset{(0)}{\exists L \in \mathbb{R}} : \overset{(1)}{\forall \varepsilon > 0}, \overset{(2)}{\exists \delta > 0} : \overset{(3)}{\forall x \in} \underbrace{(c-\delta, c+\delta) \cap S \setminus \{c\}}_{\text{non-empty } \forall \delta > 0}$, $\underset{(4)}{|f(x) - L| < \varepsilon}$

$\underset{\wedge}{f(x)}$

$|f(x)-L| < \varepsilon$
(4)

non-empty $\forall \delta > 0$





Ex. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x & 0 < x \le 1 \\ 1 & x = 0 \end{cases}$$

Claim: $\lim_{x \to 0} f(x) = 0$

Pf. Let $\varepsilon > 0$ be given. Take $\delta := \varepsilon$. Then, for all

$$x \in \underbrace{(0-\delta, 0+\delta)}_{x \in (-\delta, \delta)} \cap [0,1] \setminus \{0\},$$

$$|f(x) - L| = |x - 0| < \delta = \varepsilon$$

Prop. Let c be a cluster point of S, and let $f : S \to \mathbb{R}$ be a function that converges as $x \to c$. Then the limit of $f(x)$ as $x \to c$ is unique.

Pf. Let L_1, L_2 be limits for $f(x)$ as $x \to c$. Let $\varepsilon > 0$ be given.

- $\exists \delta_1 > 0 : \forall x \in (c - \delta_1, c + \delta_1) \cap S \setminus \{c\}, \; |f(x) - L_1| < \varepsilon/2$
- $\exists \delta_2 > 0 : \forall x \in (c - \delta_2, c + \delta_2) \cap S \setminus \{c\}, \; |f(x) - L_2| < \varepsilon/2$

Take $\delta := \min\{\delta_1, \delta_2\}$. Then, $\forall x \in (c - \delta, c + \delta) \cap S \setminus \{c\}$,

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |f(x) - L_1| + |f(x) - L_2| \end{aligned}$$

$$\leq |f(x)-L_1| + |f(x)-L_2|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow |L_1 - L_2| < \varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \quad L_1 = L_2$ $\square$