Properties of Integrals

Wednesday, November 9, 2022 10:11 AM

Notational Stuff

- For an interval $[a,b]$, $b>a$

Def. For any f,

$$\int_a^a f := 0$$

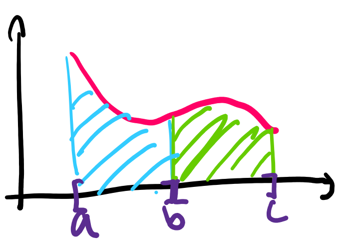
If $b<a$, $\int_a^b f := -\int_b^a f$ (integrate over $[b,a]$)

"Dummy variables" $\int_a^b f(s)\,ds := \int_a^b f(x)\,dx$

Main Properties:

- Additivity: $\int_a^b f + \int_b^c f = \int_a^c f$
- Linearity: $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$, $\alpha, \beta \in \mathbb{R}$
- Monotonicity: $f(x) \le g(x)\ \forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$

Additivity



Lemma. (Additivity of Darboux Integrals)

Suppose $a<b<c$ and $f:[a,c]\to\mathbb{R}$ is bounded. Then,

Suppose [illegible] $[a,c]$ [illegible],

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f \quad \text{and} \quad \overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f$$

<u>Pf.</u> (Additivity of Lower sums)

Take partitions: $P_1 := \{\overset{a}{\overset{\shortparallel}{x_0}}, x_1, \dots, \overset{b}{\overset{\shortparallel}{x_k}}\}$ of $[a,b]$

$P_2 := \{\underset{b}{\underset{\shortparallel}{x_k}}, x_{k+1}, \dots, \underset{c}{\underset{\shortparallel}{x_n}}\}$ of $[b,c]$

- Then, $P := P_1 \cup P_2 = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,c]$

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{k} m_j \Delta x_j + \sum_{j=k+1}^{n} m_j \Delta x_j = L(P_1,f) + L(P_2,f)$$

(upgrade to integrals)

- Let Q be an arbitrary partition of $[a,c]$. Take $P := Q \cup \{b\}$
  - $P = \{x_0, \dots, b, \dots, x_n\} =: P_1 \cup P_2$
  - $Q \subset P$, so P is a refinement of Q

    $\Rightarrow L(P,f) \geq L(Q,f)$

- Now, note the following facts for $A, B \subset \mathbb{R}$ (on HW):

  (i) If $B \subset A$ and $\forall a \in A, \exists b \in B : b \geq a$, then $\sup A = \sup B$

  $(\sup B \leq \sup A)$ $\quad$ $(b \geq a)$

  (ii) $A + B := \{a + b : a \in A, b \in B\}$, $\sup(A+B) = \sup A + \sup B$

- We can then compute

$$\underline{\int_a^c} f = \sup\{L(Q,f) : \forall Q \text{ of } [a,c]\}$$

$$\overset{(i)}{=} \sup\{L(P,f) : \forall P \text{ of } [a,c] \text{ with } b \in P\}$$

$$\overset{(\text{add of } L)}{=} \sup\{L(P_1,f) + L(P_2,f) : \forall P_1 \text{ of } [a,b], \forall P_2 \text{ of } [b,c]\}$$

$$\overset{(ii)}{=} \sup\{L(P_1,f) : \forall P_1 \text{ of } [a,b]\} + \sup\{L(P_2,f) : \forall P_2 \text{ of } [b,c]\}$$

$= \sup\{L(P_1,f)+L(P_2,f) : \forall P_1 \text{ of } [a,b],\ \forall P_2 \text{ of } [b,c]\}$

$$\overset{(ii)}{=} \sup\{L(P_1,f) : \forall P_1 \text{ of } [a,b]\} + \sup\{L(P_2,f) : \forall P_2 \text{ of } [b,c]\}$$

$$\overset{(\text{def of } \underline{\int})}{=} \underline{\int_a^b} f + \underline{\int_b^c} f$$

- Upper sums/integral on HW.

Prop. (Additivity of the Riemann Integral)

Let $a<b<c$. A bounded function $f:[a,c]\to\mathbb{R}$ is Riemann integrable if and only if f is Riemann integrable on $[a,b]$ and $[b,c]$.

Furthermore, if $f\in\mathcal{R}[a,c]$ then

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Pf. $(\Rightarrow)$ Suppose $f\in\mathcal{R}[a,c]$. Then, $\int_a^c f = \overline{\int_a^c} f = \underline{\int_a^c} f$.

- Noting that $\underline{\int_a^b} f \le \overline{\int_a^b} f$ and $\underline{\int_b^c} f \le \overline{\int_b^c} f$ (i), and using add. of Darboux integral:

$$\int_a^c f = \underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f \le \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f = \int_a^c f$$

$$\Rightarrow \underline{\int_a^b} f + \underline{\int_b^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f \quad (ii)$$

- $\overset{(i)}{-\underline{\int_a^b} f \ge -\overline{\int_a^b} f} \overset{(+ii)}{\Rightarrow} \underline{\int_b^c} f \ge \overline{\int_b^c} f \overset{(i)}{\Rightarrow} \underline{\int_b^c} f = \overline{\int_b^c} f \Rightarrow f\in\mathcal{R}[b,c]$

- Similarly can show $f\in\mathcal{R}[a,b]$

$(\Leftarrow)$ Assume $f\in\mathcal{R}[a,b]$, $f\in\mathcal{R}[b,c]$

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f = \int_a^b f + \int_b^c f = \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f$$

$\Rightarrow f\in\mathcal{R}[a,c]$. Furthermore, $\int_a^c f = \underline{\int_a^c} f = \int_a^b f + \int_b^c f$. $\square$

$\Rightarrow f \in \mathcal{R}[a,c]$. Furthermore, $\int_a^c f = \underline{\int}_a^c f = \int_a^b f + \int_b^c f$. $\square$

Cor. If $f \in \mathcal{R}[a,b]$, $[c,d] \subset [a,b]$, then $f|_{[c,d]} \in \mathcal{R}[c,d]$.

Linearity

Prop. (Sub/super additivity of Darboux integrals; 5.2.5)

Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions. Then,

$$\overline{\int_a^b}(f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g \qquad \underline{\int_a^b}(f+g) \ge \underline{\int_a^b} f + \underline{\int_a^b} g$$

Idea:

$$f(x) + g(x) \le \sup_{x \in S} f(x) + \sup_{x \in S} g(x) \quad \forall x \in S$$

$$\sup_{x \in S}(f(x) + g(x)) \le \sup_{x \in S} f(x) + \sup_{x \in S} g(x)$$

Similarly, $\inf(f+g) \ge \inf f + \inf g$

Pf. (HW exercise)

Main idea:
$$\begin{aligned} M_i^{f+g} &:= \sup\{f(x)+g(x) : x \in [x_{i-1}, x_i]\} \\ &\le \sup\{f(x) : x \in [x_{i-1}, x_i]\} + \sup\{g(x) : x \in [x_{i-1}, x_i]\} \\ &= M_i^f + M_i^g \end{aligned}$$

Prop. (Linearity of Riemann Integral)

Let $f, g \in \mathcal{R}[a,b]$, $\alpha \in \mathbb{R}$. Then,

(i) $\alpha f \in \mathcal{R}[a,b]$ with

$$\int_a^b \alpha f = \alpha \int_a^b f$$

(ii) $f+g \in \mathcal{R}[a,b]$ with

(ii) $f+g \in R[a,b]$ with

$$\int_a^b (f+g) = \int_a^b f + \int_b^c f$$

Pf. (i) Consider case of $\alpha \geq 0$ ($\alpha = -1$ on HW)

- Let P be a partition of $[a,b]$.
  - $m_i^{\alpha f} = \inf\{\alpha f(x) : x \in [x_{i-1}, x_i]\} = \alpha \inf\{f(x) : x \in [x_{i-1}, x_i]\} = \alpha m_i^f$

    $(\inf(\alpha A) = \alpha \inf(A)$ for $\alpha \geq 0)$

$$\Rightarrow L(P, \alpha f) = \sum_{i=1}^{n} m_i^{\alpha f} \Delta x_i = \alpha \sum_{i=1}^{n} m_i^f \Delta x_i = \alpha L(P, f)$$

- Similar proof shows $U(P, \alpha f) = \alpha U(P, f)$

- $$\underline{\int_a^b} \alpha f = \sup\{L(P, \alpha f) : \forall P\}$$
  $$= \sup\{\alpha L(P, f) : \forall P\}$$
  $$= \alpha \sup\{L(P, f) : \forall P\} = \alpha \underline{\int_a^b} f$$

- Similar proof: $\overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f$

$$\Rightarrow \underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f \overset{(f \in R[a,b])}{=} \alpha \overline{\int_a^b} f = \overline{\int_a^b} \alpha f$$

$\Rightarrow \alpha f \in R[a,b]$, and $\int_a^b \alpha f = \alpha \int_a^b f$ ✓

In order to finish (i), need to show $-f \in R[a,b]$

$(\inf(-A) = -\sup(A))$

(ii) HW, follows directly from prop. 5.2.5

Monotonicity

Monotonicity

Prop. (Monotonicity of Darboux + Riemann integrals; 5.2.6)

Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions with $f(x) \leq g(x)\ \forall x \in [a,b]$. Then,

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \quad \text{and} \quad \overline{\int_a^b} f \leq \overline{\int_a^b} g$$

Furthermore, if $f, g \in \mathcal{R}[a,b]$, then

$$\int_a^b f \leq \int_a^b g \qquad \text{"integrals preserve inequalities"}$$

Pf. Let P be a partition of $[a,b]$. Then,

- $f(x) \leq g(x) \quad \forall x \in [a,b]$

$\Rightarrow m_i^f = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \leq \inf\{g(x) : x \in [x_{i-1}, x_i]\} = m_i^g \quad \forall i \leq n$

$\Rightarrow L(P,f) = \sum_{i=1}^n m_i^f \Delta x_i \leq \sum_{i=1}^n m_i^g \Delta x_i = L(P,g) \quad \forall P$

$\Rightarrow \underline{\int_a^b} f = \sup\{L(P,f) : \forall P\} \leq \sup\{L(P,g) : \forall P\} = \underline{\int_a^b} g$ ✓

- Similar proof shows $\overline{\int_a^b} f \leq \overline{\int_a^b} g$

- Finally, if $f, g \in \mathcal{R}[a,b]$, then

$$\int_a^b f = \underline{\int_a^b} f \leq \underline{\int_a^b} g = \int_a^b g$$

$\square$