Manipulating Limits

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Manipulating Limits

Motivation: How do limits interact with $+, -, \times, \div, <$?

Prop. (Continuity of Algebraic Operations; Prop. 2.2.5)

Let $\{x_n\}, \{y_n\}$ be convergent sequences.

(i) The sequence $\{z_n\}$, $z_n := x_n + y_n$ converges and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

(ii) $z_n := x_n - y_n$, then $\{z_n\}$ converges and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$$

(iii) $z_n := x_n \cdot y_n$, then $\{z_n\}$ converges and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n \cdot y_n) = (\lim_{n\to\infty} x_n)\cdot(\lim_{n\to\infty} y_n)$$

(iv) If $\lim_{n\to\infty} y_n \neq 0$ and $\forall n \in \mathbb{N},\ y_n \neq 0$, then $\{z_n\} := \{\frac{x_n}{y_n}\}$ converges and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

Idea: Continuity ~ "we can switch the order of $+, -, \times, \div$ and limits (when convergent)"

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

add first / lim second $\qquad$ lim first / add second

Pf. (i) Let $\{x_n\}, \{y_n\}, \{z_n\}$ be as given. Let

$$x := \lim_{n\to\infty} x_n \qquad y := \lim_{n\to\infty} y_n \qquad z := x + y$$

- Let $\varepsilon > 0$ be arbitrary. Then,

· Let $\varepsilon > 0$ be arbitrary. Then,

$\exists M_1 \in \mathbb{N} : \forall n \geq M_1, \quad |x_n - x| < \varepsilon/2$

$\exists M_2 \in \mathbb{N} : \forall n \geq M_2, \quad |y_n - y| < \varepsilon/2$

(2) (3) (4) (1)

Take $M := \max\{M_1, M_2\}$. Then $\forall n \geq M$,

$$
\begin{aligned}
|z_n - z| &= |\; x_n + y_n \; - \; x + y \;| && \text{(def of } z_n, z\text{)} \\
&= |(x_n - x) + (y_n - y)| && \text{(rearrange)} \\
&\leq |x_n - x| + |y_n - y| && \text{(triangle ineq.)} \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}
$$

Thus $\{z_n\}$ is convergent with

$$\lim_{n\to\infty} z_n = x + y = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

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(iii) Let $x_n \to x$, $y_n \to y$ as $n \to \infty$. Take $z_n := x_n y_n$, $z := x \cdot y$

Strategy: "ε games"

$$
\begin{aligned}
|z_n - z| &= |x_n \cdot y_n - x \cdot y| && \text{(defs.)} \\
&= |(x_n - x + x) \cdot (y_n - y + y) - x \cdot y| && (-x+x, \; -y+y) \\
&= |(x_n - x) \cdot y + x \cdot (y_n - y) + (x_n - x) \cdot (y_n - y)| \\
&\leq |x_n - x| \cdot |y| + |x| \cdot |y_n - y| + |x_n - x| \cdot |y_n - y| && \text{(triangle ineq)}
\end{aligned}
$$

Intuition: |big·big − big·big| becomes

small·big + big·small + small·small.

Let $\varepsilon > 0$ be given. Take $K := \max\{|x|, |y|, 1, \varepsilon/3\}$ "K is big"

$\exists M_1 \in \mathbb{N} : \forall n \geq M_1, \quad |x_n - x| < \frac{\varepsilon}{3K}$ $(\leq 1$ since $K \geq \varepsilon/3)$

$\exists M_2 \in \mathbb{N} : \forall n \geq M_2, \quad |y_n - y| < \frac{\varepsilon}{3K}$ $\frac{\text{small}}{\text{big}} \sim$ "extrasmall"

Take $M := \max\{M_1, M_2\}$. For all $n \geq M$

$\exists M_2 \in \mathbb{N}: \forall n \geq M_2, \ |y_n - y| < \frac{\varepsilon}{3K}$ $\quad \frac{\text{small}}{\text{big}} \sim$ "extra small"

Take $M := \max\{M_1, M_2\}$. For all $n \geq M$,

$$
\begin{aligned}
|z_n - z| &\leq |x_n - x|\,|y| + |x| \cdot |y_n - y| + |x_n - x| \cdot |y_n - y| \\
&< \frac{\varepsilon}{3K} \cdot K + K \cdot \frac{\varepsilon}{3K} + \frac{\varepsilon}{3K} \cdot \frac{\varepsilon}{3K} \\
&\leq \varepsilon/3 + \varepsilon/3 + \frac{\varepsilon}{3K} \qquad \left(\frac{\varepsilon}{3K} \leq 1\right) \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \qquad (K \geq 1) \\
&= \varepsilon
\end{aligned}
$$

Thus, $\{z_n\}$ converges to $z = x \cdot y$, so

$$\lim_{n\to\infty}(x_n \cdot y_n) = \left(\lim_{n\to\infty} x_n\right) \cdot \left(\lim_{n\to\infty} y_n\right)$$

$\square$

Lemma (Limits preserve non-strict inequalities; lemma 2.2.3)

Let $\{x_n\}, \{y_n\}$ be convergent sequences such that

$$x_n \leq y_n \quad \forall n \in \mathbb{N}$$

Then,

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$

Pf. Let $x := \lim_{n\to\infty} x_n$, $y := \lim_{n\to\infty} y_n$. Let $\varepsilon > 0$ be arbitrary. Then,

$\exists M_1 \in \mathbb{N}: \forall n \geq M_1, \ |x_n - x| < \varepsilon/2 \Rightarrow -\varepsilon/2 < x_n - x < \varepsilon/2$

$\exists M_2 \in \mathbb{N}: \forall n \geq M_2, \ |y_n - y| < \varepsilon/2 \Rightarrow y_n - y < \varepsilon/2$

Then, for $n \geq \max\{M_1, M_2\}$,

$$(x - x_n) - (y - y_n) < \varepsilon \Rightarrow x - y < \varepsilon - \underbrace{(y_n - x_n)}_{y_n \geq x_n} \leq \varepsilon$$

$$-(y_n - x_n)$$

$-(y_n - x_n)$

$\underbrace{\qquad}_{y_n \geq x_n}$

Since $\forall \varepsilon > 0,\ x - y \leq \varepsilon \Rightarrow x - y \leq 0$. Thus,

$$x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n = y.$$

Remark: $\frac{1}{n} > 0 \quad \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{1}{n} = 0 \geq 0$

$$x_n < y_n \Rightarrow x_n \leq y_n \Rightarrow \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$

$$x_n < y_n \not\Rightarrow \lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$$