HW6

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1.(a) f is continuous at c, so $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\forall x \in S, |x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

so $\forall x' \in A, |x'-c| < \delta \Rightarrow x' \in S$ since $A \subset S$

$\Rightarrow |f(x') - f(c)| < \varepsilon$

so $f|_A$ is continuous at c.

(b) $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

$S = \mathbb{R}, A = \mathbb{Q}, c = 0$

When $x \in A$, $f(x) = 1$, so $\forall \varepsilon > 0, \exists \delta = 1$ s.t.

$\forall x \in A, |x-c| < \delta$, then $|f(x) - f(c)| = 0 < \varepsilon$

so $f|_A$ is continuous at c

We prove in class that f is not continuous everywhere.

(c) $f|_A$ is continuous at c, so $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\forall x \in (c-\alpha, c+\alpha), |x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

$\exists \delta' = \min\{\alpha, \delta\}$ s.t.

$\forall x \in S, |x-c| < \delta' \Rightarrow x \in (c-\alpha, c+\alpha), |x-c| < \delta$.

then $|f(x) - f(c)| < \varepsilon$

so f is continuous at c

2. By Min-Max Theorem, $f: [a,b] \to \mathbb{R}$ achieves both an absolute min M and an absolute max N on $[a,b]$

If $M = N$, then $f[(a,b)]$ is a single number M.

If $M < N$, suppose $f(m) = M$, $f(n) = N$, $m \le n$, by Bolzano's IVT, $\forall y \in \mathbb{R}$ s.t. $M < y < N$, then $\exists c \in (m,n)$ s.t. $f(c) = y$

so $f[(a,b)] = [M, N]$ is a closed and bounded interval

3. (a) $\forall x, y \in [0, \infty)$ with $x < y$

$f(y) - f(x) = y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + x^{n-1}) > 0$

so $f(y) > f(x)$

so f is strictly increasing for all $n \in \mathbb{N}$

so $\forall x < y$, $f(x) \neq f(y)$ since $f(x) < f(y)$

so f is injective

(b) $\forall n \in \mathbb{N}$, $\forall c \in [0, M]$,

$\forall \varepsilon > 0$, $\exists \delta = \min\{M, \frac{\varepsilon}{nM^{n-1}}\}$, $\forall x \in S$, $|x - c| < \delta$

$$
\begin{aligned}
|f(x) - f(c)| = |x^n - c^n| &< |(c+\delta)^n - c^n| \\
&= \delta(c^{n-1} + c^{n-2}\delta + \cdots + \delta^{n-1}) \\
&< \delta(M^{n-1} + \cdots + M^{n-1}) \\
&= n\delta M^{n-1} < n \cdot \frac{\varepsilon}{nM^{n-1}} M^{n-1} = \varepsilon
\end{aligned}
$$

so f is continuous

$f(0) = 0$, $f(M) = M^n$, by Bolzano's IVT, $\forall y \in \mathbb{R}$ s.t. $0 < y < M^n$,

$\exists c \in [0, M]$ s.t. $f(c) = y$ $\Rightarrow$ $f|_{[0,M]}$ is surjective

also f is injective by (a) $\Rightarrow$ $f|_{[0,M]}$ is bijective

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(c) $\forall a \in [0,\infty)$, $\exists M \in \mathbb{N}$ s.t. $M^n > a$

Since $f|_{[0,M]}$ is bijective and $a \in [0, M^n]$,

there exists a unique $x \in [0,M]$ s.t $x^n = a$.

4. $$\left|\frac{b_{d-1}n^{d-1}+\cdots+b_1 n+b_0}{n^d}\right| \le \frac{|b_{d-1}|n^{d-1}+\cdots+|b_1|n+|b_0|}{n^d}$$

$$\le \frac{|b_{d-1}|n^{d-1}+\cdots+|b_1|n^{d-1}+|b_0|n^{d-1}}{n^d}$$

$$= \frac{1}{n}(|b_{d-1}|+\cdots+|b_1|+|b_0|)$$

so $\lim_{n\to\infty} \frac{b_{d-1}n^{d-1}+\cdots+b_1 n+b_0}{n^d} = 0$, $\lim_{n\to-\infty} \frac{b_{d-1}n^{d-1}+\cdots+b_0}{n^d} = 0$

since d is even

so $\exists M \in \mathbb{N}$, $M>0$ s.t. $\left|\frac{b_{d-1}M^{d-1}+\cdots+b_1 M+b_0}{M^d}\right| < 1$

so $g(M) > 0$

similarly, $\exists N \in \mathbb{N}$, $N<0$ s.t. $g(N) > 0$

By Bolzano's IVM, g is continuous on $[N,0]$, $[0,M]$,

$g(0) < 0 < g(N) \Rightarrow \exists c_1 \in (N,0)$ s.t. $g(c_1) = 0$

$g(0) < 0 < g(M) \Rightarrow \exists c_2 \in (0,M)$ s.t. $g(c_2) = 0$

so we find $c_1 \neq c_2$ s.t. $g(c_1) = g(c_2) = 0$

6. $$\lim_{x\to c}\frac{h(x)-h(c)}{x-c} = \lim_{x\to c}\frac{\alpha f(x)+\beta g(x)-\alpha f(c)-\beta g(c)}{x-c}$$

$$= \lim_{x\to c}\left[\alpha\frac{f(x)-f(c)}{x-c}+\beta\frac{g(x)-g(c)}{x-c}\right]$$

$$= \alpha\lim_{x\to c}\frac{f(x)-f(c)}{x-c}+\beta\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$= \alpha f'(c)+\beta g'(c)$$

so h is differentiable at c and $h'(c)=\alpha f'(c)+\beta g'(c)$

7. $$\frac{h(x)-h(c)}{x-c} = \frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$$

$$= g(x)\frac{f(x)-f(c)}{x-c}+f(c)\frac{g(x)-g(c)}{x-c}$$

$$\lim_{x\to c}\frac{h(x)-h(c)}{x-c} = \lim_{x\to c}g(x)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}+f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$= g(c)f'(c)+f(c)g'(c)$$

so h is differentiable at c and

$h'(c)=f(c)g'(c)+f'(c)g(c)$

8. $\frac{f(x)-f(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}+\sqrt{c})(\sqrt{x}-\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{1}{\sqrt{x}+\sqrt{c}}$$

$$= \lim_{x\to c}\frac{1}{\sqrt{c}+\sqrt{c}} = \frac{1}{2\sqrt{c}} \quad \text{for all } c\in(0,\infty)$$

so f is differentiable at all $c\in(0,\infty)$