Homework 10 Solutions

Due: Monday, December 12th by 11:59 PM ET

The Exponential Function

Given a positive real number $a > 0$, it is straightforward to define integer exponents a^n in terms of basic arithmetic operations. We saw how we can also define n-th roots using the tools we've learned in the course (see HW6). However, how exactly do irrational exponents work, if we can't define them in terms of basic arithmetic operations?

In the following problems, we will see how Picard's theorem allows us to define the exponential function, and hence define irrational exponents a^x for positive $a > 0$ and $x \in \mathbb{R}$.

Remark: If you're curious, you can see section 5.4 for how to define irrational exponents starting from integrals and the logarithm instead. Other ways include starting the power series definition of the exponential, or via continuous extension of rational exponents. It is noteworthy that all of these different methods produce the same definition for irrational exponents!

Problem 1 (3 points each) Given any $x_0, y_0 \in \mathbb{R}$, consider the equation and initial conditions

$$f'(x) = f(x) \qquad f(x_0) = y_0$$

(a) Given any positive $h < \frac{1}{2}$, show that we can pick $\alpha > 0$ large enough that the proof of Picard's theorem guarantees a solution for f in the interval $[x_0 - h, x_0 + h]$.

(*Hint*: Read through the statement of Theorem 6.3.2 carefully. Note that Picard's theorem guarantees the existence of at least one $h > 0$ which makes the conclusion of the theorem true. This question is asking you to show that you can "upgrade" Picard's theorem to explicitly show that all $h < 1/2$ makes the conclusion of the theorem hold for this particular ODE.

To do this, you'll need to go through the proof of Picard's theorem and show you can explicitly give values for the "picked" variables (such as M and α) in terms of h, rather than defining h in terms of the picked variables.)

(b) Show that (a) can be used to iteratively extend f to a unique function on all $x \in \mathbb{R}$.

(c) Given $\alpha \in \mathbb{R}$, show the unique solution to the equation $g'(x) = g(x)$ with initial conditions $g(x_0) = \alpha y_0$ is given by $g(x) = \alpha f(x)$ for all $x \in \mathbb{R}$.

(d) Show that if there exists some $c \in \mathbb{R}$ such that $f(c) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$. Conclude that if $y_0 > 0$, then $f(x) > 0$ and f is strictly increasing for all $x \in \mathbb{R}$.

(*Hint*: Recall that $f(x) = f'(x)$. Is it possible for there to exist $a, b \in \mathbb{R}$ such that $f'(a) > 0 > f'(b)$ but $f'(x) \neq 0$ for all $x \in \mathbb{R}$?)

(a) First, the F in Picard's theorem is $F(x,y)=y$. For any convergent sequences of real numbers $\{x_n\}$ and $\{y_n\}$, we have that

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} y_n = y = F(x,y)$$

Thus, F is continuous on any subset $I\times J\subset\mathbb{R}^2$. Next, we also have that for all $x,y,z\in\mathbb{R}$

$$|F(x,y)-F(x,z)| = |y-z|$$

so F is Lipschitz in the second variable with constant $L=1$ for any intervals I,J.

Now, suppose $\alpha>0$ is some positive real number. Choose $I=[x_0-\alpha,x_0+\alpha]$ and $J=[y_0-\alpha,y_0+\alpha]$. $(x_0,y_0)\in I^\circ\times J^\circ$, and furthermore the intervals $[x_0-\alpha,x_0+\alpha]$ and $[y_0-\alpha,y_0+\alpha]$ are contained in I and J respectively. We see that for all $(x,y)\in I\times J$

$$|F(x,y)| = |y| \le \alpha + |y_0| = M$$

Then, the h defined in the proof of Picard's theorem is given by

$$h = \min\left\{\alpha, \frac{\alpha}{M+L\alpha}\right\} = \min\left\{\alpha, \frac{1}{2+|y_0|/\alpha}\right\}$$

Notice that

$$h = \begin{cases} \alpha & \alpha < \frac{1-|y_0|}{2} \\ \frac{1}{2+|y_0|/\alpha} & \alpha \ge \frac{1-|y_0|}{2} \end{cases}$$

which is strictly increasing as a function of α. So, given any positive $h<1/2$, we can pick

$$\alpha = \begin{cases} h & h < \frac{1-|y_0|}{2} \\ \frac{|y_0|}{1/h-2} & h \ge \frac{1-|y_0|}{2} \end{cases} = \max\left\{h, \frac{|y_0|}{1/h-2}\right\}$$

which will guarantee a solution for f in the interval $[x_0-h,x_0+h]$.

Remark: Notice that the problem does not specifically require the given $h<1/2$ to be exactly the same h in the proof of Picard's theorem, only that the h in Picard's theorem should be larger than the h given in the problem.

An alternate solution to the problem is to note that the sequence $\{h_n\}$ given by

$$h_n = \min\left\{n, \frac{1}{2+|y_0|/n}\right\} = \frac{1}{2+|y_0|/n}$$

is a monotone increasing sequence with limit equal to $1/2$. Then, given any $h<1/2$, we have that there exists some $N\in\mathbb{N}$ such that $|h_N-1/2|<(1/2-h) \implies h_N>h$. Pick $\alpha=N$ and I,J the same as above, then Picard's theorem guarantees the existence and uniqueness of f in the interval $[x_0-h,x_0+h]\subset[x_0-h_N,x_0+h_N]$.

(b) Let $h := 1/4$ and $\varepsilon := 1/8$, then by (a) Picard's theorem guarantees a solution for f in the interval $[x_0 - (h+\varepsilon), x_0 + h + \varepsilon]$ since $h + \varepsilon < 1/2$.

(*Remark*: The book simply uses the closed interval $[-h, h]$ instead of $[-h-\varepsilon, h+\varepsilon]$. I am a little confused by the argument in the book since the initial condition should lie inside of the interior of the interval rather than on the boundary for Picard's theorem to work. However, since the book seems to skip this detail, full points will be awarded even if the extra ε is not included.)

We now proceed inductively. Suppose f exists and is unique on the interval $[x_0 - nh - \varepsilon, x_0 + nh + \varepsilon]$ for $n \in \mathbb{N}$. Then, by (a) Picard's theorem guarantees the existence and uniqueness of f_+ satisfying

$$f_+'(x) = f_+(x) \qquad f_+(x_0 + nh) = f(x_0 + nh)$$

on the interval $[x_0 + (n-1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$. Since f also solves this set of equations, by uniqueness f_+ and f must agree on the interval $[x_0 + (n-1)h, x_0 + nh]$. Thus, f can be extended to the domain $[x_0 - nh - \varepsilon, x_0 + (n+1)h + \varepsilon]$

We repeat the argument for f_- satisfying

$$f_-'(x) = f_-(x) \qquad f_-(x_0 - nh) = f(x_0 - nh)$$

on the interval $[x_0 - (n+1)h - \varepsilon, x_0 - (n-1)h + \varepsilon]$, and conclude that f can be extended to a unique function on the interval $[x_0 - (n+1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$. This completes the inductive argument.

Since for any $x \in \mathbb{R}$ we can find n large enough such that $x \in [x_0 - (n+1)h - \varepsilon, x_0 + (n+1)h + \varepsilon]$, we conclude that f can be extended to a unique function on all $x \in \mathbb{R}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the unique solution to $f'(x) = f(x)$ with $f(x_0) = y_0$. Then given $\alpha \in \mathbb{R}$, let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := \alpha f(x)$. By linearity of the derivative, $g'(x) = \alpha f'(x) = \alpha f(x) = g(x)$, and also $g(x_0) = \alpha f(x_0) = \alpha y_0$. Hence by uniqueness, this is the only solution that satisfies the given equation.

(d) Given $c \in \mathbb{R}$, note that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ satisfies the ODE $f'(x) = f(x)$ with initial conditions $f(c) = 0$. Thus, by uniqueness this is the only such solution. Hence, if $f(c) = 0$ for any $c \in \mathbb{R}$, $f(x) = 0$ for all $x \in \mathbb{R}$.

Then, if $f(x_0) = y_0 > 0$, suppose for contradiction that $f(c) \leq 0$ for some $c \in \mathbb{R}$. If $f(c) = 0$, then $f(x_0) = 0$ which is a contradiction, so consider $f(c) < 0$. Then, since f is continuous, by Bolzano's IVT, there exists some d in the closed and bounded interval between x_0 and c such that $f(d) = 0$. However, this would also imply that $f(x_0) = f(c) = 0$, which is again a contradiction. Hence, $f(x) > 0$ for all $x \in \mathbb{R}$ if $f(x_0) = y_0 > 0$.

Then, for $y_0 > 0$, since $f'(x) = f(x) > 0$ for all $x \in \mathbb{R}$, we have that f is strictly increasing for all $\mathbb{R}$.

Problem 2 (3 points each) Now, we will focus on

$$E'(x) = E(x) \qquad E(0) = 1$$

The solution typically is denoted by $E(x) = e^x$, and is known as the exponential function, and has a unique inverse function $L(x) = \ln(x)$ known as the natural logarithm. However, make sure not to use any properties of either in proving the following.

Recall the following conventions for integer powers and roots: for a positive real number $a > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} a^0 &:= 1 \\ a^n &:= a \cdot a \cdot ... \cdot a \qquad (n \text{ times}) \\ a^{-n} &:= \frac{1}{a^n} \end{aligned}$$

On HW6 problem 6 you also showed the existence and uniqueness of n-th roots $a^{1/n}$ for $n \in \mathbb{N}$, which solve the equation $\left(a^{1/n}\right)^n = a$. Given $m \in \mathbb{Z}$, we define rational powers for $q \in \mathbb{Q}$ with $q = m/n$ as

$$a^q := \left(a^{1/n}\right)^m$$

You will see in the course of this problem that m/n does not need to be a fraction in lowest terms.

(a) Given $b \in \mathbb{R}$, define $E_b : \mathbb{R} \to \mathbb{R}$ by $E_b(x) := E(x + b)$. Show that E_b is the unique solution to $E_b'(x) = E_b(x)$ with $E_b(0) = E(b)$.

Use this with your result in 1(c) to show that given $a, b \in \mathbb{R}$, $E(a + b) = E(a)E(b)$.

(*Remark*: This shows the exponential function converts addition into multiplication!)

(b) Given any $x \in \mathbb{R}$, show that $E(mx) = E(x)^m$ for any $m \in \mathbb{Z}$, and $E(x/n) = E(x)^{1/n}$ for any $n \in \mathbb{N}$. Conclude that $E(x)^q = E(qx)$ for any $q \in \mathbb{Q}$ with $q = m/n$.

(c) Show that $\lim_{n\to\infty} E(-n) = 0$, and that the sequence of real numbers $\{E(n)\}$ is an unbounded monotone increasing sequence.

Use this to conclude that $E : \mathbb{R} \to (0, \infty)$ is bijective, and hence has a unique inverse function $L : (0, \infty) \to \mathbb{R}$ satisfying $E(L(a)) = a$ for all $a \in (0, \infty)$.

(Hint: Refer back to HW6. Don't forget the result of 1(d)!)

(d) Use the fact that E is continuous to show that $a^x := E(xL(a))$ is the unique number satisfying $a^x = \lim_{n\to\infty} a^{q_n}$ for any sequence $\{q_n\}$ of rational numbers with $\lim_{n\to\infty} q_n = x$. Hence, this a natural way to define irrational exponents!

(a) First, from the result of problem 3 we know that E exists and is unique.

Let $E_b : \mathbb{R} \to \mathbb{R}$ be as given. By the chain rule we can compute the derivative

$$E_b'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[E(x+b)] = E'(x+b) = E(x+b) = E_b(x)$$

Furthermore, we have that $E_b(0) = E(b)$. By the result of problem 3, E_b is the unique solution satisfying this ODE.

Now, note that problem 3c says that the solution to $f'(x) = f(x)$ with $f(0) = E(b)$ is $E(b)E(x)$. Since the solution is unique, it must agree with the earlier solution, so

$$E(b)E(a) = E_b(a) = E(b+a)$$

for any $a \in \mathbb{R}$ as desired.

(b) Fix $x \in \mathbb{R}$.

First, we show $E(mx) = E(x)^m$ for $m \in \mathbb{Z}$.

- If $m = 0$, then $E(0x) = 1 = E(x)^0$ as desired.
- If $m \in \mathbb{N}$. We have that $E(1x) = E(x)^1$ by definition. Then, assuming $E((m-1)x) = E(x)^{m-1}$ we have

$$E(mx) = E((m-1)x + x) = E((m-1)x)E(x) = E(x)^{m-1}E(x) = E(x)^m$$

  proving the desired equality for any $m \in \mathbb{N}$.
- If $m < 0$, then $-m \in \mathbb{N}$. Note that

$$1 = E(0) = E(x-x) = E(x)E(-x) \implies E(-x) = \frac{1}{E(x)} = E(x)^{-1}$$

  which is always well defined since $E(x) > 0$ for any $x \in \mathbb{R}$. Hence, we compute

$$E(mx) = \frac{1}{E(-mx)} = \frac{1}{E(-x)^{-m}} = \left(\frac{1}{E(-x)}\right)^{-m} = E(x)^m$$

  which shows the desired equality for any $m \in \mathbb{Z}$.

Next, given $n \in \mathbb{N}$, we note that

$$E(x) = E\left(n\tfrac{x}{n}\right) = E\left(\tfrac{x}{n}\right)^n$$

Since $E(x) > 0$ and $E(\frac{x}{n}) > 0$, by uniqueness of non-negative nth roots we have that

$$E(x/n) = E(x)^{1/n}$$

as desired.

Finally, given $q = m/n \in \mathbb{Q}$, we have that

$$E(qx) = E(\tfrac{m}{n}x) = E(x/n)^m = \left(E(x)^{1/n}\right)^m = E(x)^q$$

showing the desired equality. Notice that m and n do not have to be in reduced form.

(c) Since E is strictly increasing, we have that $E(-1) < E(0) = 1 < E(1)$. Then, $\{E(-n)\} = \{E(-1)^n\}$ converges to 0 by the ratio test. Similarly, $\{E(n)\} = \{E(1)^n\}$ is unbounded by the ratio test. Furthermore, $E(1)^n < E(1)^{n+1}$ for any $n \in \mathbb{N}$, so $\{E(n)\}$ is monotone increasing.

Since E is strictly increasing, we have that for $x, y \in \mathbb{R}$, $x > y \implies E(x) > E(y)$ so $E(x) = E(y) \implies x = y$, hence E is injective. Furthermore, given $a > 0$, we have $\exists N \in \mathbb{N}$ such that $E(-N) < a$ by convergence of $\{E(-n)\}$ to 0, and $\exists K \in \mathbb{N}$ such that $E(K) > a$ by unboundedness of $\{E(n)\}$. Hence, by Bolzano's IVT, there exists some $c \in [-N, K]$ such that $E(c) = a$. Thus, $E : \mathbb{R} \to (0, \infty)$ is surjective and hence bijective.

This allows us to conclude there is a unique function $L : (0, \infty) \to \mathbb{R}$ satisfying $E(L(a)) = a$ for all $a \in (0, \infty)$, which is the inverse function to E.

(d) By continuity of E, we have that for any sequence of rational numbers $\{q_n\}$ with $\lim\limits_{n\to\infty} q_n = x$, we have

$$a^x := E(xL(a)) = E(\lim_{n\to\infty}(q_n L(a))) = \lim_{n\to\infty} E(q_n L(a)) = \lim_{n\to\infty} E(L(a))^{q_n} = \lim_{n\to\infty} a^{q_n}$$

Since limits are unique, this is the only such number a^x which satisfies this property.