HW2

Collaborators: Xi Liu, Cerina Yao

1. For any $x \in \mathbb{R}$ and any $\varepsilon > 0$,
let $y = x + \varepsilon$, so $y \in \mathbb{R}$
By Theorem 1.2.4.ii, $x, y \in \mathbb{R}$ and $x < y$,
then there exists an $r \in \mathbb{Q}$ s.t. $x < r < y \Rightarrow x < r < x + \varepsilon$
$\Rightarrow |x - r| < |x - (x+\varepsilon)| = \varepsilon$

2. Assume $x \neq y$, so $x - y \neq 0$, so $|x-y| > 0$
let $\varepsilon = \frac{|x-y|}{2}$, so $\varepsilon > 0$
Since $|x - y| \leq \varepsilon = \frac{|x-y|}{2}$,
$\Rightarrow |x-y| \leq 0 \Rightarrow$ contradicts with $|x-y| > 0$
Therefore, $x = y$.

3. (a) f, g are bounded $\Rightarrow \exists B_1, B_2 \in \mathbb{R}$ s.t. $|f(x)| \leq B_1$, $|g(x)| \leq B_2$
for all $x \in D$
$$\Rightarrow |(f+g)(x)| = |f(x) + g(x)|$$
$$\leq |f(x)| + |g(x)| \quad \text{(by triangle inequality)}$$
$$\leq B_1 + B_2 \quad \text{for all } x \in D$$
Therefore, $f+g$ is bounded by $B_1 + B_2$.

(b) $$|(fg)(x)| = |f(x)g(x)|$$
$$= |f(x)| \cdot |g(x)|$$
$$\leq B_1 \cdot B_2 \qquad \text{for all } x \in D$$

Therefore, fg is bounded by $B_1 + B_2$.

(c) $|(f/h)(x)| = |f(x)/h(x)|$

$= |f(x)| / |h(x)|$

$\leq B_1 / |h(x)|$ (since $|f(x)| \leq B_1$)

$\leq B_1 / c$ for $\forall x \in D$ (since $|h(x)| \geq c$, $c > 0$)

Therefore, f/h is bounded by B_1/c.

(d) $h: E \to D \Rightarrow \forall x \in E,\ h(x) = y \in D \Rightarrow |f(y)| \leq B_1$

$|f \circ h(x)| = |f(h(x))|$

$= |f(y)|$

$\leq B_1$ for $\forall x \in E$

Therefore, $f \circ h$ is bounded by B_1

4. 1° $\forall x \in D$, $\sup_{x \in D} g(x) \geq g(x)$

$f(x) \leq g(x) \Rightarrow \sup_{x \in D} g(x) \geq f(x)$

$\Rightarrow \sup_{x \in D} g(x)$ is an upper bound for $f(x)$

By definition, $\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x)$

2° $\forall x \in D$, $\inf_{x \in D} f(x) \leq f(x)$

$f(x) \leq g(x) \Rightarrow \inf_{x \in D} f(x) \leq g(x)$

$\Rightarrow \inf_{x \in D} f(x)$ is a lower bound for $g(x)$

By definition, $\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x)$

3° By definition, $\inf_{x\in D} f(x) \le f(x) \le \sup_{x\in D} f(x)$ for all $x\in D$,

Since $f(x) \le |f(x)|$ for all $x\in D$,

by 1°, $\sup_{x\in D} f(x) \le \sup_{x\in D} |f(x)|$

Since $-|f(x)| \le f(x)$ for all $x\in D$,

by 2°, $\inf_{x\in D} -|f(x)| \le \inf_{x\in D} f(x)$

Since M is a bound for $|f|$,

$\Rightarrow$ for all $x\in D$, $|f(x)| \le M \Rightarrow -|f(x)| \ge -M$

$\Rightarrow -M \le \inf_{x\in D} -|f(x)|$, $\sup_{x\in D} |f(x)| \le M$

In conclusion,

$$-M \le \inf_{x\in D} -|f(x)| \le \inf_{x\in D} f(x) \le \sup_{x\in D} f(x) \le \sup_{x\in D} |f(x)| \le M$$

5. (a) When $\varepsilon \le 2$,

$\forall M\in\mathbb{N}$, there exists $n = 2M+1 > M$,

$|x_n - 1| = |(-1)^{2M+1} - 1| = |(-1)-1| = 2 \ge \varepsilon$

so $|x_n - 1| \ge \varepsilon$ holds infinitely often

When $\varepsilon > 2$,

$\exists M = 1 \in \mathbb{N}$ s.t. for all $n \ge M = 1$,

$|x_n - 1| \le |x_n| + |-1|$ (by triangle inequality)

$= 1 + 1$

$= 2 < \varepsilon$

so $|x_n - 1| \ge \varepsilon$ holds at most finitely often

Therefore, $\varepsilon > 2$.

(b) When $\varepsilon \le 0$,
$\forall M \in \mathbb{N}$, there exists $n = M \ge M$,
$|x_n - 1| = |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} = \frac{1}{M} > \varepsilon$
so $|x_n - 1| \ge \varepsilon$ holds infinitely often
When $\varepsilon > 0$,
by Archimedean Property of $\mathbb{R}$,
$\varepsilon, 1 \in \mathbb{R}$ and $\varepsilon > 0$, then $\exists M \in \mathbb{N}$ s.t. $M \cdot \varepsilon > 1 \Rightarrow \varepsilon > \frac{1}{M}$
For all $n \ge M$
$|x_n - 1| = |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} \le \frac{1}{M} < \varepsilon$
so $|x_n - 1| \ge \varepsilon$ holds at most finitely often
Therefore, $\varepsilon > 0$.

(c) If $\{x_n\}$ converges to some $L \in \mathbb{R}$,
$\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$: $\forall n \ge M$, $|x_n - L| < \varepsilon$
$\Rightarrow \forall \varepsilon > 0$, $\exists M \in \mathbb{N}$: $\forall n \ge M$, $|x_n - L| \ge \varepsilon$ is false
$\Rightarrow$ for all $\varepsilon > 0$, $|x_n - L| \ge \varepsilon$ holds at most finitely often.

If for all $\varepsilon > 0$, $|x_n - L| \ge \varepsilon$ holds at most finitely often,
for all $\varepsilon > 0$, $\exists M \in \mathbb{N}$: $\forall n \ge M$, $|x_n - L| \ge \varepsilon$ is false
$\Rightarrow \forall \varepsilon > 0$, $\exists M \in \mathbb{N}$: $\forall n \ge M$, $|x_n - L| < \varepsilon$
$\Rightarrow \{x_n\}$ converges to some $L \in \mathbb{R}$

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6. (a) $\{n^2\}$ does not converge

pf. Assume $\{n^2\}$ converges to some $L \in \mathbb{R}$

Then for $\varepsilon = \frac{1}{2} > 0$, there exists $M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|x_n - L| < \varepsilon$

For $n = M$, $n = M+1$, this implies

$|M^2 - L| < \varepsilon = \frac{1}{2}$

$|(M+1)^2 - L| < \varepsilon = \frac{1}{2}$

Then we would have

$$2M+1 = |[(M+1)^2 - L] - (M^2 - L)|$$
$$\leq |(M+1)^2 - L| + |M^2 - L| < 2\varepsilon = 1 \Rightarrow M < 0 \text{ contradiction!}$$

$\Rightarrow$ Therefore, $\{n^2\}$ does not converge.

(b) $\{x_n\}$ where $x_n := \begin{cases} \sin(n^2) & n < 1000000 \\ 0 & n \geq 1000000 \end{cases}$ converges to 0

For all $\varepsilon > 0$, there exists $M = 1000000 \in \mathbb{N}$ such that

$|x_n - L| = |0 - 0| = 0 < \varepsilon$ for all $n \geq M$

so $\{x_n\}$ converges to 0

(c) $\left\{\frac{4n}{4n+1}\right\}$ converges to 1

For all $\varepsilon > 0$, by Archimedian Property of $\mathbb{R}$,

there exists $M \in \mathbb{N}$ s.t. $M \cdot (4\varepsilon) > 1 \Rightarrow M > \frac{1}{4\varepsilon}$

$$|x_n - L| = \left|\frac{4n}{4n+1} - 1\right| = \frac{1}{4n+1} \leq \frac{1}{4M+1}$$
$$< \frac{1}{4 \cdot \frac{1}{4\varepsilon} + 1} = \frac{1}{\frac{1}{\varepsilon} + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \quad \text{for all } n \geq M$$

so $\left\{\frac{4n}{4n+1}\right\}$ converges to 1

(d) $\left\{\frac{2n}{n^2+1}\right\}$ converges to 0

For all $\varepsilon > 0$, by Archimedian Property of $\mathbb{R}$,

there exists $M \in \mathbb{N}$ s.t. $M \cdot \varepsilon > 2 \Rightarrow \frac{2}{M} < \varepsilon$

$|x_n - L| = \left|\frac{2n}{n^2+1} - 0\right| = \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} \leq \frac{2}{M} < \varepsilon$ for all $n \geq M$

so $\left\{\frac{2n}{n^2+1}\right\}$ converges to 0

7. 1° (ii) $\Rightarrow$ (i)

For every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon$ for all $n \geq M$

$\Rightarrow$ For every $\varepsilon' > 0$, let $\varepsilon = \varepsilon'$, so $\varepsilon > 0$,

there exist $M \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon$ for all $n \geq M$

$\varepsilon = \varepsilon' \Rightarrow |x_n - L| < \varepsilon' \Rightarrow |x_n - L| \leq \varepsilon'$

2° (i) $\Rightarrow$ (ii)

For every $\varepsilon' > 0$, there exists $M \in \mathbb{N}$ s.t. $|x_n - L| \leq \varepsilon'$ for all $n \geq M$

$\Rightarrow$ For every ε, let $\varepsilon' = \frac{\varepsilon}{2}$, so $\varepsilon' > 0$,

there exists $M \in \mathbb{N}$ s.t. $|x_n - L| \leq \varepsilon'$ for all $n \geq M$

$\varepsilon' = \frac{\varepsilon}{2} \Rightarrow |x_n - L| \leq \frac{\varepsilon}{2} < \varepsilon$

8. For any given $x \in \mathbb{R}$, let $\varepsilon_n = \frac{1}{n}$,
from problem 1 we know that for x, ε_n,
there is a $r_n \in \mathbb{Q}$ s.t. $|x - r_n| < \varepsilon_n$
for all $n \in \mathbb{N}$, we choose r_n in this way
so $|r_n - x| = |x - r_n| < \varepsilon_n = \frac{1}{n}$
By Archimedian Property of $\mathbb{R}$, $\forall \varepsilon > 0$,
there exists $M \in \mathbb{N}$ s.t. $M \cdot \varepsilon > 1 \Rightarrow \varepsilon > \frac{1}{M}$
so $|r_n - x| < \frac{1}{n} \leq \frac{1}{M} < \varepsilon$ for all $n \geq M$
so $\lim_{n \to \infty} r_n = x$.

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