Sets, Set Algebra

Monday, September 5, 2022 11:36 AM

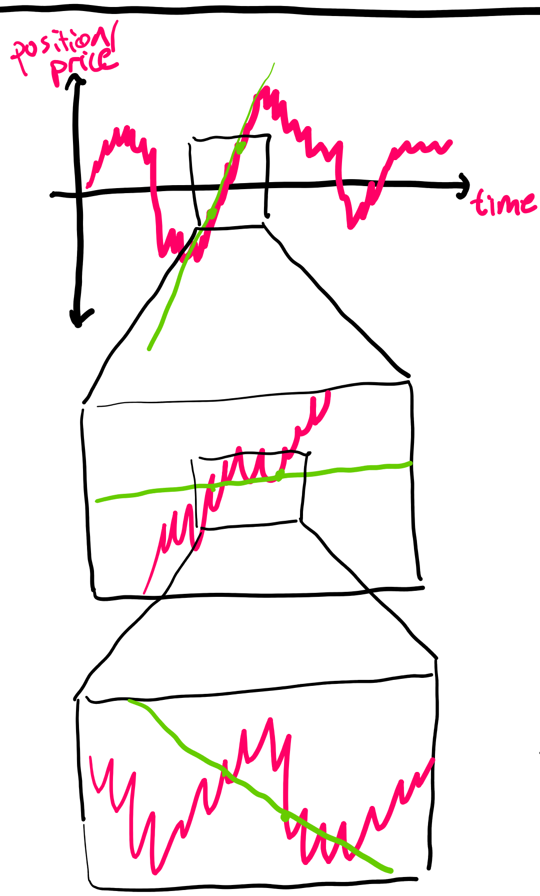
What is Analysis?

$$f(x) := \frac{e^{7x} + \ln(\pi \cdot x)}{\sqrt{3} \cdot (x^2 + 6)}$$

- is $f(x)$ differentiable?
- does $\int_0^1 f(x)\,dx$ have a well-defined value?
- can we define $F(x) := \int_0^x f(s)\,ds$?

- what if we don't have a formula for $f(x)$?

Brownian Motion (Wiener Process)



$f(t) =$ no formula

$f'(t) = ?$

From Calculus:
derivative = limit of secant lines

Secant lines do not converge:
$f(t)$ does not have a "classical" (calculus 1) derivative! (Fractals)

Remark: $f(t)$ needs to be modeled using stochastic calculus. ~~ODE~~ SDE

Roadmap:
- Real numbers, sequences, continuity
- Differentiation + integration
- Modern Analysis

Sets

Def (Naive set theory) A set is a collection of objects called elements or

Sets

Def. (Naive set theory) A *set* is a collection of objects called *elements* or *members*

Ex.

$$S := \{\underbrace{1, 2, 3}_{\text{members}}\}$$

set

definition/assignment

- $\in$ is used to denote set membership

$$1 \in S \qquad 7 \notin S$$

Def. The set with no members is called the *empty* or *null set*, denoted

$$\emptyset := \{\}$$

(Informal) Def. The set of all possible elements under consideration is called the *universe*

Remark: Usually the universe is implicitly the real numbers

Set Builder Notation

- somewhat informal

$$A := \{\underbrace{E(x)}_{\text{"expression": determines form of set elements}} : \underbrace{P(x)}_{\text{"predicate": determines membership.}}\}$$

predicate = a function which evaluates to true or false given an input x

Ex. $\{x : x \leq 3\}$ $\quad P(x) = (x \leq 3)$ $\quad P(1) = \text{true}$, $P(4) = \text{false}$

Ex. (Even numbers) $\{2n : n \in \mathbb{N}\} = \{2, 4, 6, \cdots\}$

- Occasionally,

$$A := \{x \in \underbrace{S}_{\text{"domain": specify universe}} : P(x)\}$$

"domain": specify universe

- Some key sets:
  - natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$
  - integers $\mathbb{Z} := \{0, -1, 1, -2, 2, \dots\}$
  - rational numbers $\mathbb{Q} := \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
  - real numbers $\mathbb{R} := ???$ (Chapter 1)

Set Algebra

Def. (Set operations) For two sets A, B we define

- The union

$$\underbrace{A \cup B}_{\text{"symbolic"}} := \{x : \underbrace{x \in A \text{ or } x \in B}_{\text{"logical"}}\}$$

- The intersection

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

  - We say A, B are disjoint if $A \cap B = \emptyset$

- The complement of B relative to A

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}$$

  - If A is the universe or an implied set containing B

$$B^c := A \setminus B \quad (\text{e.g. often } B^c = \mathbb{R} \setminus B)$$

Def. (Set relations) For sets A, B

- A is a subset of B if $x \in A$ implies $x \in B$. We write $A \subset B$

Remarks: • $\subset$ is "like" $\leq$ $\qquad 1 \leq 2$

$\{0,1\} \subset \{0,1,2\}$

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- $A \subset B$ ~ $x \in A \Rightarrow x \in B$ ($\Rightarrow$ means "implies")
  "symbolic" "logical"

- empty set: $x \notin \emptyset$ for all x
  $\emptyset \subset A$ "vacuously true"

- A, B are <u>equal</u> if $A \subset B$ and $B \subset A$. We write $A = B$.

<u>Remarks:</u>
- $x \leq y$ and $y \leq x$ $\Leftrightarrow$ $x = y$ ($\Leftrightarrow$ means "if and only if")
- $A \subset B$ and $B \subset A$ ~ $x \in A \Rightarrow x \in B$
  and $x \in B \Rightarrow x \in A$
  ~ $x \in A \Leftrightarrow x \in B$
- $A \neq B$ ~ it is not true that $A = B$

- A is a <u>proper subset</u> of B if $A \subset B$ and $A \neq B$. We write

$A \subsetneq B$

$\subset$ "~" $\leq$ $\subsetneq$ "~" $<$

<u>Thm.</u> (DeMorgan's Laws) symbolic manipulation of sets

Let A, B, C be sets. Then,

$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ "complements distribute
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ by flipping $\cup$ and $\cap$"

<u>Cor.</u>

$(B \cup C)^c = B^c \cap C^c$
$(B \cap C)^c = B^c \cup C^c$

<u>Pf.</u> Let's prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (remainder left as exercise)

- By def of set equality, need to prove (strategy: apply definitions)

① $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$ and

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- ① Assume $x \in A \setminus (B \cup C)$ (strategy: def. of subset, want to show $x \in A \setminus (B \cup C) \Rightarrow x \in (A \setminus B) \cap (A \setminus C)$)

  chain of statements using logic + defs.

  $\Rightarrow x \in A$ and $x \notin (B \cup C)$
  - $x \in B$ or $x \in C$ would imply $x \in (B \cup C)$, so $x \notin B$ and $x \notin C$
  - $x \in A$ and $x \notin B$, so $x \in (A \setminus B)$ (def. of complement)
  - $x \in A$ and $x \notin C$, so $x \in (A \setminus C)$
  - $x \in (A \setminus B)$ and $x \in (A \setminus C) \Rightarrow x \in (A \setminus B) \cap (A \setminus C)$ ✓

- ② Assume $x \in (A \setminus B) \cap (A \setminus C)$

  $\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

  $\quad (A \setminus B) \qquad \cap \qquad (A \setminus C)$
  - $x \notin B$ and $x \notin C \Rightarrow x \notin (B \cup C)$ ($x \in (B \cup C) \Rightarrow x \in B$ or $x \in C$, contradiction)
  - $x \in A$ and $x \notin (B \cup C) \Rightarrow x \in A \setminus (B \cup C)$ ☐

Additional Assigned Readings:

- Union/intersection of a collection of sets
- Induction