Final Exam Problem Bank

Examination Date: Monday, December 19th

Instructions – please read carefully

- 3 out of 6 problems from this problem bank, totaling 100 points, will appear as-is on the exam
- You may discuss the problems with other students. Furthermore, you can ask the instructor and TA clarifying questions, and for hints if you are stuck. However, **do not ask for, or share, either partial or full solutions**. You are expected to abide by the NYU CAS Honor Code.
- **You will not be permitted any reference material during the exam**. I advise against rote memorization: as the saying goes, *the easiest way to memorize something is to understand it.*
- You can use any result proved in the course text, in class, or on a previous homework question provided you mention that you are using a result. You do not need to mention an exact name or proposition number – **you only need to demonstrate that you are aware you are using a non-trivial result in your proof.**
- Remember that the exams will be graded to a stricter standard than the homeworks. A good rule of thumb: **if a problem 'covers' chapter N material, then you should put more detail into your proofs of statements involving chapter** N. When in doubt whether or not you need to prove something, it's usually safer to prove it.

1. (30 points) In the following, let $S \subset \mathbb{R}$ be non-empty and bounded above. In this problem, we will see an interesting connection between the supremum and infinite sets.

   (a) Show that there exists a sequence $\{x_n\}$ with $x_n \in S$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = \sup S$.

   (*Hint*: Look at Q8 on HW1 and HW2)

   (b) Suppose $\sup S \notin S$. Show that there exists a strictly monotone increasing sequence $\{y_n\}$ with $y_n \in S$ for all $n \in \mathbb{N}$.

   (*Hint*: You may want to approach this problem inductively. Take some $y_1 \in S$, and note that $\sup S - y_1 > 0$ (why?). Use this with HW1 Q8 to find y_2, etc... When doing induction, you may want to prove that $\sup S - y_p > 0$ for all $p \in \mathbb{N}$.)

   (c) Suppose $\sup S \notin S$. Show that there exists a countably infinite subset $E \subset S$.

   (*Hint*: Consider the set $\{y_n : n \in \mathbb{N}\}$ defined for the sequence $\{y_n\}$ in (b). Can you find some bijection between this set and $\mathbb{N}$?)

   (d) Suppose $\sup S \in S$. Will there always be a countably infinite subset $E \subset S$? Either prove the statement, or give a counterexample.

   (*Note*: finite sets are never countably infinite)

2. (30 points) So far in the course, we have only talked about sequences which converge to some real number. In this problem, we will use the following definition:

   A sequence (of real numbers) $\{x_n\}$ is said to *diverge to (positive) infinity* if for all $K \in \mathbb{R}$, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$, $x_n > K$. In this case, we abuse notation and write
$$\lim_{n\to\infty} x_n := +\infty$$

   (a) Write down a corresponding definition for a sequence $\{x_n\}$ which *diverges to negative infinity*, and use it to show that $\lim\limits_{n\to\infty} -n^3 = -\infty$

   (b) Suppose $\{x_n\}$ is a sequence satisfying $x_n > 0$ for all $n \in \mathbb{N}$, and furthermore
$$\lim_{n\to\infty} \frac{1}{x_n} = 0$$
   Show that $\{x_n\}$ diverges to positive infinity.

   (c) Show that a sequence $\{x_n\}$ is unbounded above (i.e. the set $\{x_n : n \in \mathbb{N}\}$ is unbounded above) if and only if $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which diverges to positive infinity.

   (*Hint*: The ($\Longleftarrow$) direction is easier, so you may want to start with that.

   For the ($\Longrightarrow$) direction, I'd recommend splitting the proof into parts:

   (i) prove that if $\{x_n\}$ is unbounded, then every p-tail of $\{x_n\}$ is also unbounded.

   (ii) inductively construct a subsequence $\{x_{n_k}\}$ which satisfies $x_{n_k} > k$ for all $k \in \mathbb{N}$.

   (iii) prove that this subsequence does what you want it to do.)

3. (30 points) Let's take a look at limits of functions at infinity.

   Given a function $f : \mathbb{R} \to \mathbb{R}$ and some $L \in \mathbb{R}$, we say $f(x)$ *converges to* L as $x \to \infty$ if for all $\varepsilon > 0$, there exists some $M \in \mathbb{R}$ such that for all $x \geq M$,

   $$|f(x) - L| < \varepsilon$$

   In this case, we write $f(x) \to L$ as $x \to \infty$, or

   $$\lim_{x\to\infty} f(x) := L$$

   If f does not converge to any $L \in \mathbb{R}$ as $x \to \infty$, we say f *diverges* as $x \to \infty$.

   (a) Write down a corresponding definition for $f(x)$ to converge to L as $x \to -\infty$. Then, use the above definition and the definition you wrote to prove that

   $$\lim_{x\to\infty} \frac{1}{1+x^2} = \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

   $$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = L$$

   for some $L \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

   $$g(y) := \begin{cases} f(1/y) & y \neq 0 \\ L & y = 0 \end{cases}$$

   Show that g is continuous at 0.

   (*Hint*: For the ε-δ definition of continuity, show that you can break $|y| < \delta$ into three cases: $y = 0$, $1/y > 1/\delta$, or $-1/y > 1/\delta$. This might help you find the right value of δ.)

   (c) Continuing from (b), show that if f is continuous at 0, then

   $$\lim_{y\to\infty} g(y) = \lim_{y\to-\infty} g(y) = f(0)$$

4. (30 points) In lecture, we've mentioned how the lower and upper Darboux sums 'bound' possible Riemann sums. In this problem, let's make that statement more rigorous.

   (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{P} = \{x_0, ..., x_n\}$ be a partition of $[a, b]$. We say a set of points $\tau := \{c_1, ..., c_n\}$ is a *tagging* of $\mathcal{P}$ if $x_{i-1} \le c_i \le x_i$ for all $i = 1, ..., n$.

   Given any partition $\mathcal{P}$ and tagging τ of $\mathcal{P}$, show that

   $$L(\mathcal{P}, f) \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le U(\mathcal{P}, f)$$

   (b) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that for any tagging τ,

   $$\left| \int_a^b f - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \varepsilon$$

   (c) Given a bounded function $f : [a, b] \to \mathbb{R}$ and $n \in \mathbb{N}$, define the *right uniform Riemann sum* $R_n(f, [a, b])$ as

   $$R_n(f, [a, b]) := \sum_{i=1}^{n} f(x_i)\Delta x$$

   where $\Delta x = (b - a)/n$ and $x_i = a + (b - a)(i/n)$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function, which satisfies

   $$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   Show that the sequences $\{R_n(f, [0, 1])\}_{n=1}^{\infty}$, $\{R_n(f, [1, 1+\sqrt{2}])\}_{n=1}^{\infty}$, and $\{R_n(f, [0, 1+\sqrt{2}])\}_{n=1}^{\infty}$ converge, but

   $$\lim_{n\to\infty} (R_n(f, [0, 1]) + R_n(f, [1, 1 + \sqrt{2}])) \ne \lim_{n\to\infty} R_n(f, [0, 1 + \sqrt{2}])$$

   (*Remark*: You may use the fact that if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$, and $rx \notin \mathbb{Q}$ if $r \ne 0$.)

   (d) Prove or disprove (i.e. by counterexample): A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the sequence of right uniform Riemann sums $\{R_n(f, [a, b])\}$ converges as $n \to \infty$.

5. (40 points) It turns out that Picard's theorem can be seen as an application of something called the *fixed point theorem for contraction mappings*. While the general theorem is covered in chapter 7, in this problem we will see how the proof techniques can be extended to other problems.

   Fix some $a > 0$. Consider the function $G : (0, \infty) \to (0, \infty)$ given by

$$G(y) := \frac{1}{2}\left(y + \frac{a}{y}\right)$$

   (a) Show that G is continuously differentiable, and that $G'(y) < 0$ for $y < \sqrt{a}$ and $G'(y) > 0$ for $y > \sqrt{a}$. Use this to show that G achieves an absolute minimum at $y = \sqrt{a}$.

   (*Remark*: Feel free to use the fact that $y' = 1$ and $(1/y)' = -1/y^2$)

   (b) Let $I := [\sqrt{a}, \infty)$. Show that the restriction $G\big|_I$ is Lipschitz continuous, with some Lipschitz constant $L < 1$.

   (*Hint*: For an easy way to find the Lipschitz constant, look at HW7)

   (c) Define $y_0 := \max\{a, 1\}$. Note that $y_0 \geq \sqrt{a}$. Define iterates

$$y_n := G(y_{n-1})$$

   Show that the sequence of iterates $\{y_n\}$ satisfies $y_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$. Then, show that for all $k > n$, the following chain of inequalities holds (i.e. explain where each line comes from):

$$\begin{aligned}
|y_k - y_n| &\leq L^n |y_{k-n} - y_0| \\
&\leq L^n \sum_{i=1}^{k-n} |y_i - y_{i-1}| \\
&\leq L^n |y_1 - y_0| \sum_{i=1}^{k-n} L^{i-1} \\
&= \frac{L^n(1 - L^{k-n})}{1 - L} |y_1 - y_0| \\
&\leq \frac{L^n}{1 - L} |y_1 - y_0|
\end{aligned}$$

   Use this to show that $\{y_n\}$ is Cauchy.

   (d) Let $y := \lim_{n\to\infty} y_n$. Show that $G(y) = y$, and use it to conclude that $y = \sqrt{a}$.

   (*Hint*: Note that $G(\sqrt{a}) = \sqrt{a}$. Are there any other possible solutions?)

   Remark: The sequence of iterates $\{y_n\}$ gives approximations to $\sqrt{a}$ using *Newton's method*, also sometimes called the *Babylonian method*. It generally converges to $\sqrt{a}$ much faster than the bisection method (covered in chapter 3) does.

6. (40 points) In this problem, we will take a look at different norms on spaces of functions $f : \mathbb{N} \to \mathbb{R}$. Such functions might arise from, e.g. the Fourier sine/cosine series of some periodic function $g : [0, 2\pi] \to \mathbb{R}$.

   Given some $f : \mathbb{N} \to \mathbb{R}$ and $\alpha \geq 0$, we say that $f \in \ell^\infty_\alpha$ if the set $\{k^\alpha |f(k)| : k \in \mathbb{N}\}$ is bounded. This is called the k^α *weighted* ℓ^∞ *space.*

   When $f \in \ell^\infty_\alpha$, we can define the ℓ^∞_α *norm* of f as

   $$\|f\|_\alpha := \sup\{k^\alpha |f(k)| : k \in \mathbb{N}\} = \|k^\alpha f\|_u$$

   where $\|\cdot\|_u$ is the uniform norm. Note that $\|f\|_0 = \|f\|_u$, so these norms generalize the uniform norm.

   (a) Prove the triangle inequality for the ℓ^∞_α norm: if $f, g \in \ell^\infty_\alpha$, then

   $$\|f + g\|_\alpha \leq \|f\|_\alpha + \|g\|_\alpha$$

   (b) Let $m \in \mathbb{N}$. Show that $\ell^\infty_m \subset \ell^\infty_{m-1}$.

   (c) We say a collection of functions $S \subset \ell^\infty_\alpha$ is *bounded in* ℓ^∞_α *norm* if there exists some $B \in \mathbb{R}$ such that for all $f \in S$, $\|f\|_\alpha \leq B$.

   Show that the set of functions $S = \{g_n : n \in \mathbb{N}\}$ given by

   $$g_n(k) := \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

   is bounded in ℓ^∞_0 norm, but not bounded in ℓ^∞_m norm for any $m \in \mathbb{N}$.

   (d) We say a collection of functions $K \subset \ell^\infty_0$ is *uniformly small at infinity (USI)* if for all $\varepsilon > 0$, there exists some $C \in \mathbb{N}$ such that for all $k \geq C$ and for all $f \in K$,

   $$|f(k)| < \varepsilon$$

   Show that if K is bounded in ℓ^∞_m norm for some $m \in \mathbb{N}$, then K is uniformly small at infinity.

   (*Hint*: If B is the bound of K in ℓ^∞_m norm, try picking M such that $B/k^m < \varepsilon$ for all $k \geq M$ (why can you do this?). Then, what happens to $|f(k)|$?)

   (e) Let $\{f_n\}$ be a sequence of functions in ℓ^∞_0 converging pointwise to some $f \in \ell^\infty_0$. Show that if the collection $\{f_n : n \in \mathbb{N}\}$ is uniformly small at infinity, then f_n converges to f uniformly.

   (*Hint*: Use the USI condition to find some cutoff C such that $|f_n(k) - f(k)| < \varepsilon$ for all $n \in \mathbb{N}$ and all $k \geq C$. Then there are only a finite number of sequences $\{f_n(k)\}_{n=1}^\infty$, $k < C$ to deal with.)

   Remark: The 'weights' used in this problem have something to do with the idea of "decay in Fourier coefficients corresponds to regularity in real space". Parts (d) and (e) have something to do with compact subsets of function spaces – notice that for a sequence $\{f_n\}$ in some bounded USI collection K, each of the sequences $\{f_n(k)\}_{n=1}^\infty$ is a bounded sequence of real numbers. Perhaps we could extract pointwise convergent subequences... (check out the Arzelà-Ascoli theorem if you are interested!)