Final

5.(a) G is continuously differentiable:

$\forall c \in (0,\infty)$

$$\lim_{y\to c}\frac{f(y)-f(c)}{y-c} = \lim_{y\to c}\frac{\frac{1}{2}(y+\frac{a}{y})-\frac{1}{2}(c+\frac{a}{c})}{y-c}$$ (use $y'=1$, $(\frac{1}{y})' = -\frac{1}{y^2}$ instead)

$$= \lim_{y\to c}\frac{1}{2}-\frac{a}{2yc} = \frac{1}{2}-\frac{a}{2c^2}$$

so $G'(y)=\frac{1}{2}-\frac{a}{2y^2}$ is continuous

G is continuously differentiable.

When $y<\sqrt{a}$,

$G'(y) < \frac{1}{2}-\frac{a}{2(\sqrt{a})^2} = 0$ ①

When $y>\sqrt{a}$,

$G'(y) > \frac{1}{2}-\frac{a}{2(\sqrt{a})^2} = 0$ ②

By ①, $y<\sqrt{a}$, $G(y)>G(\sqrt{a})$

By ②, $y>\sqrt{a}$, $G(y)>G(\sqrt{a})$

so G achieves an abs min at $y=\sqrt{a}$

(b) $L=\frac{1}{2}$

$\forall x,y\in I$,

$|f(x)-f(y)| = |\frac{1}{2}(x+\frac{a}{x})-\frac{1}{2}(y+\frac{a}{y})| = \frac{1}{2}|(x-y)+(\frac{a}{x}-\frac{a}{y})|$

If $x\geq y$, $x-y\geq 0$, $\frac{a}{x}-\frac{a}{y}\leq 0 \Rightarrow \frac{1}{2}|(x-y)+(\frac{a}{x}-\frac{a}{y})| \leq \frac{1}{2}|x-y|$

$<$ $<$ $>$

(c) By (b), $\forall x, y \in I$, $|f(x) - f(y)| \leq L|x-y|$

so $|y_k - y_n| = |G(y_{k-1}) - G(y_{n-1})|$

$\leq L|y_{k-1} - y_{n-1}| \leq L^2|y_{k-2} - y_{n-2}|$

$\vdots$

$\leq L^n|y_{k-n} - y_0|$ ① Similarly, $|y_i - y_{i-1}|$ $\leq L^{i-1}|y_1 - y_0|$ ③

$|y_{k-n} - y_0| = |\sum_{i=1}^{k-n}(y_i - y_{i-1})|$

$\leq \sum_{i=1}^{k-1}|y_i - y_{i-1}|$ by triangular inequality ②

Therefore,

$|y_k - y_n| \leq L^n|y_{k-n} - y_0|$ by ①

$\leq L^n \sum_{i=1}^{k-1}|y_i - y_{i-1}|$ by ②

$\leq L^n|y_1 - y_0|\sum_{i=1}^{k-n}L^{i-1}$ by ③

$= \frac{L^n(1-L^{k-n})}{1-L}|y_1 - y_0|$ by algebra

$\leq \frac{L^n}{1-L}|y_1 - y_0|$ since $L<1$

Since $0<L<1$, $\frac{1^n}{1-L}|y_1 - y_0| \to 0$ as $n \to \infty$

so $\forall \varepsilon > 0, \exists M \in \mathbb{N}: \forall n, k \geq M, |y_k - y_n| < \varepsilon$

so $\{y_n\}$ is Cauchy.

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(d) $y_{n+1} = G(y_n)$, by (c) $y_{n+1} - y_n \to 0$ as $n \to \infty$

$G(y) - y = \lim_{n\to\infty} (y_{n+1} - y_n) = 0$

so $G(y) = y$

$G(\sqrt{a}) = \frac{1}{2}(\sqrt{a} + \frac{a}{\sqrt{a}}) = \sqrt{a}$

By (b) G is Lipschitz, $\forall x, y \in I$, $|G(x) - G(y)| \le L|x-y|$

$\Rightarrow |G(y) - G(\sqrt{a})| = |y - \sqrt{a}| \le L|y - \sqrt{a}|$

since $L<1$ $\Rightarrow |y - \sqrt{a}| = 0$ $\Rightarrow y = \sqrt{a}$ is the unique solution

6.(a) Since $|f(k)+g(k)| \le |f(k)|+|g(k)|$ for all $k \in \mathbb{N}$, we have

$$\|f+g\|_\alpha = \sup\{k^\alpha |f(k)+g(k)| : k \in \mathbb{N}\}$$
$$\le \sup\{k^\alpha (|f(k)|+|g(k)|) : k \in \mathbb{N}\}$$
$$\le \sup\{k^\alpha |f(k)| : k \in \mathbb{N}\} + \sup\{k^\alpha |g(k)| : k \in \mathbb{N}\}$$
$$= \|f\|_\alpha + \|g\|_\alpha$$

(b) $\forall f \in l^\infty_m$,

$\{k^m |f(k)| : k \in \mathbb{N}\}$ is bounded, so $\exists B \in \mathbb{R}$

$||k^m |f(k)|| = k^m |f(k)| \le B, \forall k \in \mathbb{N}$

since $k \ge 1$, $|f(k)| \ge 0$

$\Rightarrow k^{m-1} |f(k)| \le k^m |f(k)| \le B, \forall k \in \mathbb{N}$

$\Rightarrow f \in l^\infty_{m-1}$

Therefore, $l^\infty_m \subset l^\infty_{m-1}$.

(c) When $\alpha = 0$, $\{k^\alpha |g_n(k)| : k \in \mathbb{N}\} = \{|g_n(k)| : k \in \mathbb{N}\}$

$\forall g_n \in S$, $\forall k \in \mathbb{N}$, $|g_n(k)| \le 1 \Rightarrow g_n$ is bounded in l^∞_0 norm

$\Rightarrow S$ is bounded in l^∞_0 norm

Assume S is bounded in l^∞_m norm for any $m \in \mathbb{N}$

$\Rightarrow \forall m \in \mathbb{N}, \exists B \in \mathbb{R} : \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k^m |g_n(k)| \le B$

$k = n \Rightarrow n^m \le B$

$m = 2$

let $n = B+1 \Rightarrow n^m = (B+1)^m \ge B$ contradiction

$\Rightarrow S$ is not bounded in l^∞_m norm for any $m \in \mathbb{N}$

(d) Since K is bounded in l_m^∞ norm,

$\exists B \in \mathbb{R}, \forall f \in K, \|f\|_m = \sup\{k^m |f(k)|, k \in \mathbb{N}\} < B$

Take $\varepsilon > 0$ arbitrarily,

Take $C = \sqrt[m]{\frac{B}{\varepsilon}} + 1$ so $\frac{B}{k^m} < \varepsilon$ for all $k \geq C$.

Then for all $k \geq C$, and all $f \in K$,

since B is an upper bound of $\{k^m |f(k)|, k \in \mathbb{N}\}$,

so $k^m |f(k)| \leq B$

$\Rightarrow |f(k)| \leq \frac{B}{k^m}$

(e) $\{f_n\}$ converges pointwise to f

$\Rightarrow$ For all $k \in \mathbb{N}$, $\lim_{n \to \infty} f_n(k) = f(k)$ ①

$\{f_n : n \in \mathbb{N}\}$ is $\underline{USI}$

$\Rightarrow \forall \varepsilon > 0, \exists C \in \mathbb{N} : \forall k \geq C, \forall n \geq N, |f_n(k)| < \varepsilon$ ②

Let $\varepsilon > 0$ be arbitrary.

Let $k \geq C$ be fixed. By ① ②,

$\lim_{n \to \infty} |f_n(k)| = |f(k)| < \frac{\varepsilon}{2}$

$\Rightarrow \forall k \geq C, \forall n \in \mathbb{N}$,

$|f_n(k) - f(k)| \leq |f_n(k)| + |f(k)| < \varepsilon$

Consider $k < C$,

fix some $k < C$, by pointwise convergence,

$\exists M_k \in \mathbb{N} : \forall n \geq M_k, |f_n(k) - f(k)| < \varepsilon$

So $\forall \varepsilon > 0$,

take $M = \max\{M_1, \cdots, M_{k-1}\}$

$\forall k \in \mathbb{N}$,

Case 1. $k < C$, $\forall n \geq M$, $|f_n(k) - f(k)| < \varepsilon$

Case 2: $k > C$, $\forall n \geq M$, $|f_n(k) - f(k)| < \varepsilon$

so f_n converges to f uniformly.