Midterm Exam 2 Solutions

Examination Date: Tuesday, April 13th, ± 1 day

1. Recall early on in the course, we very laboriously showed the existence and uniqueness of $\sqrt{2}$. We will show how the tools we have learned will allow us to show the existence and uniqueness of non-negative nth roots $\sqrt[n]{a}$ for any $n \in \mathbb{N}$ and any non-negative real number $a \in [0, \infty)$.

   In the following, let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) := x^n$.

   (a) Show that f is strictly increasing for all $n \in \mathbb{N}$, and use it to conclude that f is injective.

   We say f is strictly increasing if $f(x) < f(y)$ for all $x, y \in [0, \infty)$ with $x < y$.

   (b) Show that f is continuous for all $n \in \mathbb{N}$. Then, given $M \in \mathbb{N}$, use part (a) and what you know about continuous functions to show that the restriction $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is both surjective and injective, and hence bijective.

   (c) Use the results of part (b) to conclude that for any $a \in [0, \infty)$, there exists a unique non-negative x such that $x^n = a$.

(a) Given $0 \leq x < y$, we show $x^n < y^n$ by induction.

(Basis statement) $0 \leq x < y \implies 0 \leq x^1 < y^1$

(Induction step) Assume $0 \leq x^{n-1} < y^{n-1}$. Since $y > 0$, we multiply both sides of the inequality to get $yx^{n-1} < y^n$, and since $x \geq 0$ we multiply both sides of the inequality to get $x^n \leq yx^{n-1}$, with equality when $x = 0$. Chaining these inequalities together, we get $0 \leq x^n < y^n$.

(*Remark*: Be careful, you need to use the fact that $0 \leq x < y$. If you only try to use $x < y$, $x^2 < y^2$ is not true in general (e.g. take $x = -2, y = 1$))

Thus, by induction, if $x, y \in [0, \infty)$ and $x < y$, then $f(x) = x^n < y^n = f(y)$, so f is strictly increasing.

Since f is strictly increasing, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, since $x_1 \not< x_2$ and $x_2 \not< x_1$. Thus, f is injective.

(b) We can directly use proposition 3.2.4 (polynomials are continuous) to conclude that x^n, which is a polynomial, is continuous.

Alternatively, we can also proceed via induction. We showed in lecture that x is continuous. Then, by repeated usage of continuity of algebraic operations, $x^n = x \cdot x^{n-1}$ will be continuous.

Now given $M \in \mathbb{N}$, since the restriction of a continuous function will also be continuous, $f\big|_{[0,M]}$ will map the closed and bounded interval $[0, M]$ to a closed and bounded interval (or a single point).

Since f is increasing, we have $f(0) \leq f(x) \leq f(M)$ for any $0 \leq x \leq M$. Thus $f\big|_{[0,M]}$ achieves a max at $x = M$, and a min at $x = 0$, and so it maps the closed and bounded interval $[0, M]$ to $[f(0), f(M)] = [0, M^n]$.

Thus, $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is surjective. It is also injective by (a), since $f(x_1) = f(x_2)$ still implies $x_1 = x_2$ for $x_1, x_2 \in [0, M]$. Thus, $f\big|_{[0,M]} : [0, M] \to [0, M^n]$ is bijective.

(c) Let $a \in [0, \infty)$ be given. By the Archimedean property, there exists some $M \in \mathbb{N}$ such that $a < M$. Furthermore, since $1 \leq M$ we have $M \leq M^n$ by induction, so $a < M \leq M^n$, hence $a \in [0, M^n]$.

Thus, we can take $x := \left(f\big|_{[0,M]}\right)^{-1}(a)$ to satisfy the equation and satisfies $f(x) = x^n = a$. x exists and is unique since $f\big|_{[0,M]}$ is a bijection, as we showed in (b).

Lastly, we need to show x does not depend on the choice of M. With $x \in [0, \infty)$ as defined above, since f is injective, for any $y \in [0, \infty)$ we have that $f(x) = f(y) = a$ implies $x = y$. Thus we conclude the existence and uniqueness of non-negative nth roots $\sqrt[n]{a} = x$.

2. In this problem, you will prove the nth derivative test, a generalization of Proposition 4.3.3 in the textbook. Suppose $n \in \mathbb{N}$, $x_0 \in (a,b)$, and $f : [a,b] \to \mathbb{R}$ is n times continuously differentiable, with $f^{(k)}(x_0) = 0$ for $k = 1, 2, ..., n-1$ and $f^{(n)}(x_0) > 0$.

   (a) Prove that if $g : [a,b] \to \mathbb{R}$ is continuous and $g(x_0) > 0$, then there exists $\delta > 0$ such that $g(c) > 0$ for all $c \in (x_0 - \delta, x_0 + \delta)$.

   (b) Prove that if n is odd, then f has neither a relative minimum, nor a relative maximum at x_0.

   (c) Prove that if n is even, then f has a strict relative minimum at x_0.

(a) Take g and x_0 as given in the problem statement. Then, since $g(x_0) > 0$, by continuity of g, there exists $\delta_1 > 0$ such that for all $c \in [a,b]$ with $|c - x_0| < \delta_1$, we have

$$|g(c) - g(x_0)| < g(x_0)$$

We use properties of the absolute value and rearrange the inequality to get

$$-g(x_0) < g(c) - g(x_0) \implies g(c) > 0$$

Lastly, since $a < x_0 < b$ there exists some δ_2 such that $a < x_0 - \delta_2 < x_0 + \delta_2 < b$, so we can take $\delta := \min\{\delta_1, \delta_2\}$. Then $g(c) > 0$ for all $c \in (x_0 - \delta, x_0 + \delta)$.

(b+c) Take f and x_0 as given in the problem statement. Then, since f is n times continuously differentiable, $f^{(n)}$ is continuous. Since $f^{(n)}(x_0) > 0$, by the result of problem (a), we can take $\delta_+ > 0$ such that $f^{(n)}(c) > 0$ for all $c \in (x_0 - \delta_+, x_0 + \delta_+)$.

For any $x \in [a,b]$ distinct from x_0, we have that f satisfies the requirements of Taylor's theorem. From the assumptions of the problem, we have that the Taylor polynomial is given by

$$P_{n-1}^{x_0}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x_0)$$

so there exists c between x and x_0 such that

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

(b) Suppose n is odd. To show that f does not have a relative minimum at x_0, let $\delta > 0$ be given. Define $\delta' := \min\{\delta, \delta_+\}$. Then, take $x_1 \in [a,b]$ such that $x_1 < x$ and $|x_1 - x_0| < \delta'$. Then, for any c between x_0 and x_1, we have $c \in (x_0 - \delta_+, c_0 + \delta_+)$, so $f^{(n)}(c) > 0$ in this interval. We also have $x_1 - x_0 < 0$ so $(x_1 - x_0)^n < 0$ since n is odd. Thus, by Taylor's theorem we have

$$f(x_1) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x_1 - x_0)^n < 0 \implies f(x_1) < f(x_0)$$

Since we can find such an x_1 for any given $\delta > 0$, we have that $f(x_0)$ does not have a relative minimum at x_0.

To show f does not have a relative maximum at x_0, we repeat the argument, except this time we choose $x_2 > x$. Then, $(x_2 - x_0)^n > 0$, and

$$f(x_2) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x_2 - x_0)^n > 0 \implies f(x_2) > f(x_0)$$

Thus, f does not have a relative maximum at x_0 either.

(c) Suppose n is even. Then, for all $x \in [a, b]$ distinct from x_0, we have $(x - x_0)^n > 0$. Then, for all $x \in (x_0 - \delta_+, x_0 + \delta_+)$ distinct from x_0, we have that for all c between x and x_0, $f^{(n)}(c) > 0$. Thus, by Taylor's theorem, we have that for all $x \in [a, b]$ distinct from x_0 with $|x - x_0| < \delta_+$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n > 0 \implies f(x) > f(x_0)$$

Thus, f has a strict relative minimum at x_0.

3. In this problem, we will study functions of *bounded variation*, which show up in a few areas of math, including probability theory.

   In the following, let $f : [a, b] \to \mathbb{R}$ be increasing.

   (a) Show that f is bounded, and furthermore that $f \in \mathscr{R}[a, b]$.
   (Hint: Try using a partition P_n of $n + 1$ uniformly spaced points.)

   (b) Use part (a) to show that a decreasing function is Riemann integrable.

   (c) We say a function $h : [a, b] \to \mathbb{R}$ is of *bounded variation* if $h = f - g$ where f, g are increasing functions on $[a, b]$. Show that h is Riemann integrable.

(a) Let f be as given. Then, for all $x \in [a, b]$, since f is increasing, we have

$$f(a) \leq f(x) \leq f(b)$$

thus f is bounded.

Now, to show f is Riemann integrable, let $\varepsilon > 0$ be given. Choose $n \in \mathbb{N}$ such that $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$. Let $P_n := \{x_0, x_1, ..., x_n\}$ be a partition of $n + 1$ uniformly spaced points. Explicitly, $x_i = a + \frac{i}{n}(b - a)$, and $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$.

Since f is increasing, $f(x_{i-1}) \leq f(x) \leq f(x_i)$ for $x_{i-1} \leq x \leq x_i$. Thus,

$$m_i := \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_{i-1})$$
$$M_i := \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_i)$$
$$L(P_n, f) = \sum_{i=1}^{n} m_i \Delta x_i = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$$
$$U(P_n, f) = \sum_{i=1}^{n} M_i \Delta x_i = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

Then we compute

$$U(P_n, f) - L(P_n, f) = \frac{b-a}{n}(f(b) - f(a)) < \varepsilon$$

Thus by proposition 5.1.13 in the text, f is Riemann integrable on $[a, b]$.

Remark: As an alternative to using proposition 5.1.13, one can also use the properties of the Darboux integrals (since they are the sup and inf of the upper and lower Darboux sums)

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P_n, f) - L(P_n, f) = \frac{b-a}{n}(f(b) - f(a))$$

then, taking the limit as $n \to \infty$ gets $\overline{\int_a^b} f - \underline{\int_a^b} f \leq 0$, which combined with $\overline{\int_a^b} f \geq \underline{\int_a^b} f$ implies $\overline{\int_a^b} f = \underline{\int_a^b} f$, and hence f is Riemann integrable on $[a, b]$.

(b) Let $g : [a, b] \to \mathbb{R}$ be a decreasing function. Then, for $x, y \in [a, b]$ with $x < y$, we have $g(x) > g(y)$, which can be rearranged as $-g(x) < -g(y)$. Thus, $-g$ is an increasing function, and so by (a) is Riemann integrable. Then, by linearity of the integral, $-(-g) = g$ is also Riemann integrable. Thus, decreasing functions are also Riemann integrable on $[a, b]$.

(c) By (a), both f and g are Riemann integrable on $[a, b]$. Thus, by linearity of the integral, we have $h = f - g = f + (-g)$ is also Riemann integrable. Thus, functions of bounded variation are Riemann integrable.