HW7

Collaborators: Xi Liu, Cerina Yao

1.(a) For $x \in (0, \frac{\pi}{2})$, $\sin(x) < x < \tan(x)$

$\Rightarrow 1 < \frac{\sin(x)}{x} < \frac{1}{\cos(x)}$

Since $\lim_{x\to 0} \frac{1}{\cos(x)} = 1$,

$1 \leq \lim_{x\to 0} \frac{\sin(x)}{x} \leq 1$

so $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

(b) $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\sin(x)-\sin(c)}{x-c}$

$= \lim_{x\to c} \frac{2\cos(\frac{x+c}{2})\sin(\frac{x-c}{2})}{x-c}$

$= \lim_{x\to c} \cos(\frac{x+c}{2}) \lim_{x\to c} \frac{\sin(\frac{x-c}{2})}{\frac{x-c}{2}}$

$= \cos(\frac{c+c}{2}) \cdot 1$

$= \cos(c)$

so $\forall x \in \mathbb{R}$, $f'(x) = \cos(x)$.

2.(a) $L \neq 0$, let $\varepsilon = |\frac{L}{2}| > 0$, then $\exists \delta > 0 : \forall x \in (c-\delta, c+\delta) \cap S \setminus \{c\}$,

$|h(x) - L| < \varepsilon = |\frac{L}{2}|$

$\Rightarrow |\frac{L}{2}| > |h(x) - L| > ||h(x)| - |L||$

$\Rightarrow -|\frac{L}{2}| < |h(x)| - |L| < |\frac{L}{2}|$

$\Rightarrow |h(x)| > |\frac{L}{2}| \Rightarrow h(x) \neq 0$

(b) Since h is continuous, $h|_A$ is continuous

By part (a), $\exists \delta > 0$; $A = (c-\delta, c+\delta) \cap (S \setminus \{c\})$, $h|_A(x) \neq 0$

since c is a cluster point of S, c is a cluster point of A

$$\lim_{x \to c} \frac{1}{h|_A(x)} = \frac{\lim_{x\to c} 1}{\lim_{x \to c} h|_A(x)} = \frac{1}{\lim_{x \to c} h|_A(x)} = \frac{1}{h(c)}$$

(c) $\displaystyle\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)}$ since $f(c) = g(c) = 0$

$$= \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \quad \text{since } x - c \neq 0$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \lim_{x \to c} \frac{1}{\frac{g(x)-g(c)}{x-c}}$$

$$= \lim_{x \to c} f'(x) \lim_{x \to c} \frac{1}{g'(x)}$$

$$= \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$x > c \Rightarrow f(x) \leq f(c)$

3.(a) Suppose f is decreasing. $\forall x, c \in I$ with $x \neq c$, $x < c \Rightarrow f(x) \geq f(c)$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \leq 0$$

$$\Rightarrow f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leq 0$$

Suppose $f'(x) \leq 0 \ \forall x \in I$. Take $x, y \in I$ with $x < y$.

By MVT, $\exists c \in (a,b)$ s.t. $f(y) - f(x) = \underbrace{f'(c)}_{\geq 0} \underbrace{(y-x)}_{\geq 0} \geq 0$

$\Rightarrow f(y) \geq f(x) \Rightarrow f$ is decreasing

(b) Suppose $f'(x) < 0 \ \forall x \in I$. Take $x, y \in I$ with $x < y$

By MVT, $\exists c \in (a,b)$ s.t. $f(y) - f(x) = \underbrace{f'(c)}_{>0} \underbrace{(y-x)}_{>0} > 0$

$\Rightarrow f(y) > f(x)$

$\Rightarrow f$ is strictly decreasing

4.(a) Let $f(x) = x^n$, then $f'(x) = nx^{n-1}$

$\forall R > 0$, $n \in \mathbb{N}$, $x, y \in [-R, R]$,

by MVT, $\exists c$ between x and y,

$|x^n - y^n| = |f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq nR^{n-1} |x - y|$

(b) Let $f(x) = \sqrt{x^2+1}$, then $f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

$\forall x, y \in \mathbb{R}$, by MVT, $\exists c$ between x and y,

$$\begin{aligned} \left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| &= |f(x) - f(y)| = |f'(c)| \cdot |x - y| \\ &= \left|\frac{c}{\sqrt{c^2+1}}\right| \cdot |x - y| \\ &\leq |x - y| \end{aligned}$$

5. Using Taylor's theorem at $x_0 = a$,
$\forall x \in [a,b]$, $\exists c \in (a,x)$ if $x \neq a$ or $c = a$ if $x = a$

$$f(x) = P_{n-1}^{a}(x) + \frac{f^{n}(c)}{n!}(x-a)^n$$

By Min-Max Theorem, since $f^{n}(x)$ is continuous, it achieves both an abs min M and abs max N on $[a,b]$.

Let $P(x) = P_{n-1}^{a}(x) + \frac{M}{n!}(x-a)^n$

$$Q(x) = P_{n-1}^{a}(x) + \frac{N}{n!}(x-a)^n$$

so $P(x) \leq f(x) \leq Q(x)$ for all $x \in [a,b]$

For $\lambda = \frac{N}{n!} - \frac{M}{n!} \geq 0$.

$Q(x) - P(x) = \lambda(x-a)^n$.