MATH-UA.0235 Probability and Statistics – Worksheet # 7

Problem 1 – Review

Let X and Y be two independent discrete random variables with joint probability mass function $p_{X,Y}$. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $p_{X,Y}(x, y) = p_X(x)p_Y(y)$.

We are interested in the probability mass function p_Z of the discrete random variable $Z = X + Y$. This is readily constructed as follows. For all $z \in \mathbb{R}$,

$$p_Z(z) = P(Z = z) = \sum_x P(X = x, Y = z - x) = \sum_x p_{X,Y}(x, z - x) = \sum_x p_X(x)p_Y(z - x)$$

We thus found the following result:

$$p_Z(z) = \sum_x p_X(x)p_Y(z - x)$$

The probability distribution of the sum of two independent random variables is the convolution of their individual distributions.

Problem 2 – Extension of Problem 1 to continuous random variables

Let X and Y be two independent continuous random variables, with joint probability density function $f_{X,Y}$. Following our usual intuition from class, in which we convert formulas for discrete random variables into formulas for continuous random variables by replacing probability mass functions with probability density functions, and sums with integrals, we may guess that the probability density function of $Z = X + Y$ is given by:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(z - x)dx$$

Let us now prove this formula rigorously.
The most natural method to prove this result is to use an object we have not seen in the lectures but which is quite intuitive: the *conditional probability density function*. Specifically, we will consider the conditional probability density function of Z given that X takes the value x:

$$f_{Z|X}(z|x) = f_Y(z - x)$$

Now, recalling the definition of a conditional probability, we may write, for all $x \in \mathbb{R}$ and $z \in \mathbb{R}$:

$$f_{X,Z}(x, z) = f_{Z|X}(z|x)f_X(x) = f_X(x)f_Y(z - x)$$

Now, we are interested in the probability density function of Z, which is just the marginal probability density function for Z obtained from $f_{X,Z}$. From Lecture 7, we thus know that we can write:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(z - x)dx$$

This is indeed the result we had guessed. The probability density function of the sum of two independent random variables is the convolution of their individual probability density function.

Problem 3

Let T_1 and T_2 be two independent and identically distributed (i.i.d.) random variables, which have an exponential distribution with parameter λ.

1. Characterize the distribution of the random variable $X = T_1 + T_2$.

   Here, we may directly apply the convolution result derived above.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{T_1}(t) f_{T_2}(x-t) dt \\ &= \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt \quad \text{if } x \geq 0 \quad , 0 \quad \text{otherwise} \\ &= \lambda^2 \int_0^x e^{-\lambda x} dt \quad \text{if } x \geq 0 \quad , 0 \quad \text{otherwise} \\ &= \lambda^2 x e^{-\lambda x} \quad \text{if } x \geq 0 \quad , 0 \quad \text{otherwise} \end{aligned}$$

2. Let T_3 be a third random variable with the same distribution as T_1 and T_2, and independent from T_1 and T_2. Let $Y = X + T_3 = T_1 + T_2 + T_3$. Characterize the distribution of Y.

   To answer this question, we may once again apply the convolution result:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f_X(x) f_{T_3}(y-x) dx \quad \text{if } y \geq 0 \quad , 0 \quad \text{otherwise} \\ &= \lambda^3 \int_0^y x e^{-\lambda x} e^{-\lambda(y-x)} dx \quad \text{if } y \geq 0 \quad , 0 \quad \text{otherwise} \\ &= \lambda^3 e^{-\lambda y} \int_0^y x dx \quad \text{if } y \geq 0 \quad , 0 \quad \text{otherwise} \\ &= \frac{\lambda^3}{2} y^2 e^{-\lambda y} \quad \text{if } y \geq 0 \quad , 0 \quad \text{otherwise} \end{aligned}$$

   Following the same process, it is then easy to show, by induction, that the sum Z of n such random variables has a $\text{Gam}(i, \lambda)$ distribution, given by

$$f_Z(z) = \lambda \frac{(\lambda z)^{n-1} e^{-\lambda z}}{(n-1)!} \quad \text{if } z \geq 0 \quad , 0 \quad \text{otherwise}$$

Problem 4

Let X_1 and X_2 be two independent random variables that are uniformly distributed on $[0, l]$. Let $M = \min(X_1, X_2)$ and $T = \max(X_1, X_2)$. What is the joint cumulative distribution function of M and T?

$$\begin{aligned} F_{M,T}(m,t) &= P(M \leq m, T \leq t) = P(((X_1 \leq m) \cup (X_2 \leq m)) \cap ((X_1 \leq t) \cap (X_2 \leq t))) \\ &= P(((X_1 \leq t) \cap (X_2 \leq t)) \setminus ((m < X_1 \leq t) \cap (m < X_2 \leq t))) \\ &= P((X_1 \leq t) \cap (X_2 \leq t)) - P((m < X_1 \leq t) \cap (m < X_2 \leq t)) \\ &= \begin{cases} F_{X_1}(t) F_{X_2}(t) - (F_{X_1}(t) - F_{X_1}(m))(F_{X_1}(t) - F_{X_2}(m)) & \text{if } mleqt \\ F_{X_1}(t) F_{X_2}(t) & \text{if } t < m \end{cases} \end{aligned}$$

Now, if $t < 0$, $F_{X_1}(t) = F_{X_2}(t) = 0$. If $t \geq l$, $F_{X_1}(t) = F_{X_2}(t) = 1$. And if $t \in [0, l]$, $F_{X_1}(t) = F_{X_2}(t) = \frac{t}{l}$. Likewise, if $m < 0$, $F_{X_1}(m) = F_{X_2}(m) = 0$. If $m \geq l$, $F_{X_1}(m) = F_{X_2}(m) = 1$. And if $m \in [0, l]$, $F_{X_1}(m) = F_{X_2}(m) = \frac{m}{l}$.

Hence,

$$F_{M,T}(m,t) = \begin{cases} 0 \ \text{ if } m < 0 \text{ or } t < 0 \\ 1 - \left(1 - \frac{s}{l}\right)^2 \ \text{ if } m \in [0, l] \text{ and } t > l \\ \frac{t^2 - (t-m)^2}{l^2} \ \text{ if } 0 \leq m \leq t \leq l \\ \frac{t^2}{l^2} \ \text{ if } t \leq \min(m, l) \\ 1 \ \text{ if } m \geq l \text{ and } t \geq l \end{cases}$$

Problem 5

We consider a Poisson process, and the interval of time $[0,t]$ for that Poisson process. We know that the Poisson process will have two arrivals in that interval ($N_t = 2$ in the notation of the notes for Lecture 9), and we would like to know the distribution of X_1 and X_2, corresponding to the time of arrival of the first and second arrival, respectively, in $[0,t]$.

In the lecture notes, we treated the case for which $N_t = 1$, i.e. one arrival in the interval $[0,t]$. Let us now cover the case $N_t = 2$, with first an arrival at X_1, and a second arrival at X_2. We may write, for $0 \le x_1 \le x_2 \le t$,

$$\begin{aligned} P(X_1 \le x_1, X_2 \le x_2, N_t = 2) &= P(X_2 \le x_2, N_t = 2) - P(X_1 > x_1, X_2 \le x_2, N_t = 2) \\ &= P(N_{[0,x_2]} = 2, N_{[x_2,t]=0}) - P(N_{[0,x_1]} = 0, N_{[x_1,x_2]} = 2, N_{[x_2,t]} = 0) \\ &= \frac{\lambda^2 x_2^2}{2} e^{-\lambda x_2} e^{-\lambda(t-x_2)} - e^{-\lambda x_1} \frac{\lambda^2 (x_2 - x_1)^2}{2} e^{-\lambda(x_2 - x_1)} e^{-\lambda(t-x_2)} \\ &= \frac{\lambda^2}{2} e^{-\lambda t} \left[x_2^2 - (x_2 - x_1)^2 \right] \end{aligned}$$

Therefore,

$$P(X_1 \le x_1, X_2 \le x_2 | N_t = 2) = \frac{P(X_1 \le x_1, X_2 \le x_2, N_t = 2)}{P(N_t = 2)} = \frac{\frac{\lambda^2}{2} e^{-\lambda t} \left[x_2^2 - (x_2 - x_1)^2 \right]}{\frac{(\lambda t)^2}{2} e^{-\lambda t}} = \frac{x_2^2 - (x_2 - x_1)^2}{t^2}$$

We recognize the joint cumulative distribution function for the min and the max of two independent random variables that are uniformly distributed on $[0,t]$ by direct comparison with Problem 4. We conclude that X_1 and X_2 are uniformly distributed on $[0,t]$ and independent, as we discussed in the notes for Lecture 9.

Problem 6

Cars cross a certain line on the highway in accordance with a Poisson process with rate $\lambda = 20$ per minute. If a boar attempts to cross the highway, what is the probability that it will survive if the amount of time that it takes it to cross the road is s seconds? (Assume that if it is on the highway when a car passes by, then it will die.)
The problem is asking the probability that no car passes by during a span of s seconds. Thus

$$P(N_s = 0) = \frac{(20\frac{s}{60})^0}{0!} e^{-20\frac{s}{60}} = e^{-\frac{s}{3}}$$

Problem 7

Suppose that for Problem 6, we do not have a boar, but instead a fox, which is agile enough to escape from a single car. However, if it encounters two or more cars while attempting to cross the road, then it dies. What is the probability that it survives if it takes it s seconds to cross?
This time, two events allow the animal to survive: $N_s = 0$ and $N_s = 1$. These events are disjoint, so the probability p of survival is

$$p = P(N_s = 0) + P(N_s = 1) = e^{-\frac{s}{3}} + \frac{(20\frac{s}{60})^1}{1!} e^{-\frac{s}{3}} = e^{-\frac{s}{3}} \left(1 + \frac{s}{3}\right)$$