Xi Liu, xl3504, Problem Set 6

Problem 1

1.

$X \backslash Y$	-2	5	8	$\notin \{-2,\ 5,\ 8\}$		$p_X(x)$
1	$(0.7)(0.3)$ $= 0.21$	$(0.7)(0.5)$ $= 0.35$	$(0.7)(0.2)$ $= 0.14$	$(0.7)(0)$ $= 0$		0.7
2	$(0.3)(0.3)$ $= 0.09$	$(0.3)(0.5)$ $= 0.15$	$(0.3)(0.2)$ $= 0.06$	$(0.3)(0)$ $= 0$		0.3
$\notin \{1,\ 2\}$	$(0)(0.3)$ $= 0$	$(0)(0.5)$ $= 0$	$(0)(0.2)$ $= 0$	$(0.3)(0)$ $= 0$		0
$p_Y(y)$	0.3	0.5	0.2	0		1

2.

$$
\begin{aligned}
P(X \text{ is even},\ Y \text{ is even}) &= p_{X,Y}(2,-2) + p_{X,Y}(2,8) \\
&= 0.09 + 0.06 \\
&= \boxed{0.15}
\end{aligned}
$$

3.

$$
\begin{aligned}
P(X=1 \mid Y>0) &= \frac{P(X=1,\ Y>0)}{P(Y>0)} \\
&= \frac{p_{X,Y}(1,5)+p_{X,Y}(1,8)}{p_Y(5)+p_Y(8)} \\
&= \frac{0.35+0.14}{0.5+0.2} \\
&= \frac{0.49}{0.7} \\
&= \boxed{0.7}
\end{aligned}
$$

Problem 2

1.

X = number of diamonds picked

Y = number of queens picked

in a standard deck of 52 cards

number of diamonds = 13

number of queens = 4

$X \backslash Y$	0	1	2		$p_X(x)$
0	$\frac{105}{221}$	$\frac{18}{221}$	$\frac{1}{442}$		$\frac{19}{34}$
1	$\frac{72}{221}$	$\frac{12}{221}$	$\frac{1}{442}$		$\frac{13}{34}$
2	$\frac{11}{221}$	$\frac{2}{221}$	0		$\frac{1}{17}$
$p_Y(y)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$		1

let $1st$ denote first card picked, $2nd$ denote second card picked, d denote diamond, q denote queen

$$\begin{aligned}
p_{X,Y}(0,0) &= P(1st \neq d,\ 1st \neq q,\ 2nd \neq d,\ 2nd \neq q) \\
&= \frac{52-13-3}{52} \cdot \frac{51-13-3}{51} = \boxed{\frac{105}{221}} \\
p_{X,Y}(0,1) &= P(1st \neq d,\ 1st \neq q,\ 2nd \neq d,\ 2nd = q) \\
&\quad + P(1st \neq d,\ 1st = q,\ 2nd \neq d,\ 2nd \neq q) \\
&= \frac{52-13-3}{52} \cdot \frac{3}{51} + \frac{3}{52} \cdot \frac{51-13-2}{51} = \boxed{\frac{18}{221}} \\
p_{X,Y}(0,2) &= P(1st \neq d,\ 1st = q,\ 2nd \neq d,\ 2nd = q) \\
&= \frac{3}{52} \cdot \frac{2}{51} = \boxed{\frac{1}{442}}
\end{aligned}$$

$$\begin{aligned}
p_{X,Y}(1,0) &= P(1st = d,\ 1st \neq q,\ 2nd \neq d,\ 2nd \neq q) \\
&\quad + P(1st \neq d,\ 1st \neq q,\ 2nd = d,\ 2nd \neq q) \\
&= \frac{12}{52} \cdot \frac{51-12-3}{51} + \frac{52-13-3}{52} \cdot \frac{12}{51} = \boxed{\frac{72}{221}} \\
p_{X,Y}(1,1) &= P(1st = d,\ 1st \neq q,\ 2nd \neq d,\ 2nd = q) \\
&\quad + P(1st \neq d,\ 1st = q,\ 2nd = d,\ 2nd \neq q) \\
&\quad + P(1st = d, 1st = q, 2nd \neq d, 2nd \neq q) \\
&\quad + P(1st \neq d, 1st \neq q, 2nd = d, 2nd = q) \\
&= \frac{12}{52} \cdot \frac{3}{51} + \frac{3}{52} \cdot \frac{12}{51} + \frac{1}{52} \cdot \frac{51-12-3}{51} + \frac{52-13-3}{52} \cdot \frac{1}{51} \\
&= \boxed{\frac{12}{221}}
\end{aligned}$$

$$
\begin{aligned}
p_{X,Y}(1,2) &= P(1st = d,\ 1st = q,\ 2nd \neq d,\ 2nd = q) \\
&\quad + P(1st \neq d,\ 1st = q,\ 2nd = d,\ 2nd = q) \\
&= \frac{1}{52} \cdot \frac{3}{51} + \frac{3}{52} \cdot \frac{1}{51} \\
&= \boxed{\frac{1}{442}} \\
p_{X,Y}(2,0) &= P(1st = d,\ 1st \neq q,\ 2nd = d,\ 2nd \neq q) \\
&= \frac{12}{52} \cdot \frac{11}{51} = \boxed{\frac{11}{221}} \\
p_{X,Y}(2,1) &= P(1st = d,\ 1st = q,\ 2nd = d,\ 2nd \neq q) \\
&\quad + P(1st = d,\ 1st \neq q,\ 2nd = d,\ 2nd = q) \\
&= \frac{1}{52} \cdot \frac{12}{51} + \frac{12}{52} \cdot \frac{1}{51} \\
&= \boxed{\frac{2}{221}} \\
p_{X,Y}(2,2) &= P(1st = d,\ 1st = q,\ 2nd = d,\ 2nd = q) \\
&= \frac{1}{52} \cdot 0 \\
&= \boxed{0}
\end{aligned}
$$

2.

$$
\begin{aligned}
P_X(0) &= \frac{105}{221} + \frac{18}{221} + \frac{1}{442} \\
&= \boxed{\frac{19}{34}} \\
p_X(1) &= \frac{72}{221} + \frac{12}{221} + \frac{1}{442} \\
&= \boxed{\frac{13}{34}} \\
p_X(2) &= \frac{11}{221} + \frac{2}{221} + 0 \\
&= \boxed{\frac{1}{17}} \\
\sum_{i=0}^{3} p_X(i) &= \frac{19}{34} + \frac{13}{34} + \frac{1}{17} = 1
\end{aligned}
$$

$$
\begin{aligned}
p_Y(0) &= \frac{105}{221} + \frac{72}{221} + \frac{11}{221} \\
&= \boxed{\frac{188}{221}} \\
p_Y(1) &= \frac{18}{221} + \frac{12}{221} + \frac{2}{221} \\
&= \boxed{\frac{32}{221}} \\
p_Y(2) &= \frac{1}{442} + \frac{1}{442} \\
&= \boxed{\frac{1}{221}} \\
\sum_{i=1}^{3} p_Y(i) &= \frac{188}{221} + \frac{32}{221} + \frac{1}{221} = 1
\end{aligned}
$$

3.

$$
\begin{aligned}
P(Y \geq 1 \mid X \geq 1) &= \frac{P(Y \geq 1,\ X \geq 1)}{P(X \geq 1)} \\
&= \frac{p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(2,1) + p_{X,Y}(2,2)}{p_X(1) + p_X(2)} \\
&= \frac{\frac{12}{221} + \frac{1}{442} + \frac{2}{221} + 0}{\frac{13}{34} + \frac{1}{17}} \\
&= \frac{29}{195}
\end{aligned}
$$

Problem 3

1

$\mathcal{X}$:= random variable corresponding to number of failures of system X

$\mathcal{Y}$:= random variable corresponding to number of failures of system Y

Y has at least two failures per day:

$$P(\mathcal{Y} \geq 2) = 0.5 + 0.17 + 0.03 = \boxed{0.7}$$

the number of failures of X is strictly less than 2, and the number of failures of Y is greater than or equal to 3:

$$\begin{aligned} P(\mathcal{X} < 2,\ \mathcal{Y} \geq 3) &= p_{\mathcal{X},\mathcal{Y}}(0,3) + p_{\mathcal{X},\mathcal{Y}}(0,4) + p_{\mathcal{X},\mathcal{Y}}(1,3) + p_{\mathcal{X},\mathcal{Y}}(1,4) \\ &= p_{\mathcal{X}}(0) \cdot p_{\mathcal{Y}}(3) + p_{\mathcal{X}}(0) \cdot p_{\mathcal{Y}}(4) + p_{\mathcal{X}}(1) \cdot p_{\mathcal{Y}}(3) + p_{\mathcal{X}}(1) \cdot p_{\mathcal{Y}}(4) \\ &= (0.07)(0.17) + (0.07)(0.03) + (0.35)(0.17) + (0.35)(0.03) \\ &= \boxed{0.084} \end{aligned}$$

there is only one failure in the day

$$\begin{aligned} P(\text{number of failure } = 1) &= p_{\mathcal{X},\mathcal{Y}}(0,1) + p_{\mathcal{X},\mathcal{Y}}(1,0) \\ &= p_{\mathcal{X}}(0) \cdot p_{\mathcal{Y}}(1) + p_{\mathcal{X}}(1) \cdot p_{\mathcal{Y}}(0) \\ &= (0.07)(0.2) + (0.35)(0.1) \\ &= \boxed{0.049} \end{aligned}$$

X has the same number of failures as Y

$$\begin{aligned} P(\mathcal{X} = \mathcal{Y}) &= p_{\mathcal{X},\mathcal{Y}}(0,0) + p_{\mathcal{X},\mathcal{Y}}(1,1) + p_{\mathcal{X},\mathcal{Y}})(2,2) + p_{\mathcal{X},\mathcal{Y}}(3,3) + p_{\mathcal{X},\mathcal{Y}}(4,4) \\ &= p_{\mathcal{X}}(0) \cdot p_{\mathcal{Y}}(0) + p_{\mathcal{X}}(1) \cdot p_{\mathcal{Y}}(1) + p_{\mathcal{X}}(2) \cdot p_{\mathcal{Y}}(2) + p_{\mathcal{X}}(3) \cdot p_{\mathcal{Y}}(3) + p_{\mathcal{X}}(4) \cdot p_{\mathcal{Y}}(4) \\ &= (0.07)(0.1) + (0.35)(0.2) + (0.34)(0.5) + (0.18)(0.17) + (0.06)(0.03) \\ &= \boxed{0.2794} \end{aligned}$$

2.

$\mathcal{X}$ \ $\mathcal{Y}$	0	1	2	3	4		$p_X(x)$
0	$(0.07)(0.1)$ $= 0.007$	$(0.07)(0.2)$ $= 0.014$	$(0.07)(0.5)$ $= 0.035$	$(0.07)(0.17)$ $= 0.0119$	$(0.07)(0.03)$ $= 0.0021$		0.07
1	$(0.35)(0.1)$ $= 0.035$	$(0.35)(0.2)$ $= 0.07$	$(0.35)(0.5)$ $= 0.175$	$(0.35)(0.17)$ $= 0.0595$	$(0.35)(0.03)$ $= 0.0105$		0.35
2	$(0.34)(0.1)$ $= 0.034$	$(0.34)(0.2)$ $= 0.068$	$(0.34)(0.5)$ $= 0.17$	$(0.34)(0.17)$ $= 0.0578$	$(0.34)(0.03)$ $= 0.0102$		0.34
3	$(0.18)(0.1)$ $= 0.018$	$(0.18)(0.2)$ $= 0.036$	$(0.18)(0.5)$ $= 0.09$	$(0.18)(0.17)$ $= 0.0306$	$(0.18)(0.03)$ $= 0.0054$		0.18
4	$(0.06)(0.1)$ $= 0.006$	$(0.06)(0.2)$ $= 0.012$	$(0.06)(0.5)$ $= 0.03$	$(0.06)(0.17)$ $= 0.0102$	$(0.06)(0.03)$ $= 0.0018$		0.06
$p_Y(y)$	0.1	0.2	0.5	0.17	0.03		1

$$\begin{aligned}
p_{\mathcal{X}}(0) &= 0.007 + 0.014 + 0.035 + 0.0119 + 0.0021 \\
&= 0.07 \\
p_{\mathcal{X}}(1) &= 0.035 + 0.07 + 0.175 + 0.0595 + 0.0105 \\
&= 0.35 \\
p_{\mathcal{X}}(2) &= 0.034 + 0.068 + 0.17 + 0.0578 + 0.0102 \\
&= 0.34 \\
p_{\mathcal{X}}(3) &= 0.018 + 0.036 + 0.09 + 0.0306 + 0.0054 \\
&= 0.18 \\
p_{\mathcal{X}}(4) &= 0.006 + 0.012 + 0.03 + 0.0102 + 0.0018 \\
&= 0.06 \\
\sum_{i=0}^{4} p_{\mathcal{X}}(i) &= 0.07 + 0.35 + 0.34 + 0.18 + 0.06 \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
p_{\mathcal{Y}}(0) &= 0.007 + 0.035 + 0.034 + 0.018 + 0.006 \\
&= 0.1 \\
p_{\mathcal{Y}}(1) &= 0.014 + 0.07 + 0.068 + 0.036 + 0.012 \\
&= 0.2 \\
p_{\mathcal{Y}}(2) &= 0.035 + 0.175 + 0.17 + 0.09 + 0.03 \\
&= 0.5 \\
p_{\mathcal{Y}}(3) &= 0.0119 + 0.0595 + 0.0578 + 0.0306 + 0.0102 \\
&= 0.17 \\
p_{\mathcal{Y}}(4) &= 0.0021 + 0.0105 + 0.0102 + 0.0054 + 0.0018 \\
&= 0.03 \\
\sum_{i=0}^{4} p_{\mathcal{Y}}(i) &= 0.1 + 0.2 + 0.5 + 0.17 + 0.03 \\
&= 1
\end{aligned}
$$

3.

$$
\begin{aligned}
E[\mathcal{X}] &= \sum_{i=0}^{4} x_i p_{\mathcal{X}}(x_i) \\
&= 0 \cdot p_{\mathcal{X}}(0) + 1 \cdot p_{\mathcal{X}}(1) + 2 \cdot p_{\mathcal{X}}(2) + 3 \cdot p_{\mathcal{X}}(3) + 4 \cdot p_{\mathcal{X}}(4) \\
&= (0)(0.07) + (1)(0.35) + (2)(0.34) + (3)(0.18) + (4)(0.06) \\
&= \boxed{1.81} \\
E[\mathcal{Y}] &= \sum_{i=0}^{4} y_i p_{\mathcal{Y}}(y_i) \\
&= 0 \cdot p_{\mathcal{Y}}(0) + 1 \cdot p_{\mathcal{Y}}(1) + 2 \cdot p_{\mathcal{Y}}(2) + 3 \cdot p_{\mathcal{Y}}(3) + 4 \cdot p_{\mathcal{Y}}(4) \\
&= (0)(0.1) + (1)(0.2) + (2)(0.5) + (3)(0.17) + (4)(0.03) \\
&= \boxed{1.83}
\end{aligned}
$$

Problem 4

		x					$p_Y(y)$
		1	2	3	4	5	
y	1	1/14	1/14	1/14	1/14	1/14	5/14
	2	0	1/14	1/14	1/14	1/14	4/14
	3	0	1/14	1/14	0	0	2/14
	4	0	1/14	1/14	0	0	2/14
	5	0	1/14	0	0	0	1/14
$p_X(x)$		1/14	5/14	4/14	2/14	2/14	1

steps:
fill row $y = 1$
fill column $x = 2$
fill row $y = 5$
fill column $x = 1$
fill row $y = 2$
fill column $x = 3$
fill column $x = 4$
fill column $x = 5$

Problem 5

1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx \\
&= \int_{0}^{\infty}\int_{0}^{\infty} Ke^{-3x-2y}\,dy\,dx \\
&= K\int_{0}^{\infty}\int_{0}^{\infty} e^{-3x-2y}\,dy\,dx \\
&/\!*\ u := -3x-2y; \quad \partial u = -2\partial y\ *\!/ \\
&= K\int_{0}^{\infty} \frac{1}{-2}\left[e^{-3x-2y}\right]_{y=0}^{y=\infty}\,dx \\
&= -\frac{K}{2}\int_{0}^{\infty} \left[e^{-3x-2y}\right]_{y=0}^{y=\infty}\,dx \\
&= -\frac{K}{2}\int_{0}^{\infty} (e^{-3x}\cdot 0 - e^{-3x})\,dx \\
&= \frac{K}{2}\int_{0}^{\infty} e^{-3x}\,dx \\
&/\!*\ u_2 := -3x; \quad du_2 = -3dx\ *\!/ \\
&= -\frac{K}{6}[e^{-3x}]_0^{\infty} \\
&= -\frac{K}{6}(0-1) \\
&= \frac{K}{6}
\end{aligned}$$

$$\begin{aligned}
1 &= \frac{K}{6} \\
K &= \boxed{6}
\end{aligned}$$

2.

$$\begin{aligned}
& a := x; \quad b := y \\
F_{X,Y}(a,b) &= P(X \leq a, Y \leq b) \\
&= P(-\infty \leq X \leq a, -\infty \leq Y \leq b) \\
&= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y)\, dy\, dx \\
&= \int_{0}^{a} \int_{0}^{b} 6e^{-3x-2y}\, dy\, dx \\
&= 6 \int_{0}^{a} \int_{0}^{b} e^{-3x-2y}\, dy\, dx \\
& /^{*}\ u := -3x - 2y; \quad \partial u = -2\partial y\ ^{*}/ \\
&= -3 \int_{0}^{a} [e^{-3x-2y}]_{y=0}^{y=b}\, dx \\
&= -3 \int_{0}^{a} (e^{-3x-2b} - e^{-3x})\, dx \\
&= -3 \int_{0}^{a} (e^{-3x})(e^{-2b} - 1)\, dx \\
&= -3(e^{-2b} - 1) \int_{0}^{a} e^{-3x}\, dx \\
& /^{*}\ u_2 := -3x; \quad du_2 = -3dx\ ^{*}/ \\
&= (e^{-2b} - 1)[e^{-3x}]_{0}^{a} \\
&= (e^{-2b} - 1)(e^{-3a} - 1) \\
&= e^{-2b-3a} - e^{-2b} - e^{-3a} + 1
\end{aligned}$$

$$\boxed{F_{X,Y}(x,y) = \begin{cases} e^{-2y-3x} - e^{-2y} - e^{-3x} + 1 & \text{if } x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}}$$

3.

$$\begin{aligned}
P(X \le 1, Y \le 2) &= F_{X,Y}(1,2) \\
&= e^{-2(2)-3(1)} - e^{-2(2)} - e^{-3(1)} + 1 \\
&= \boxed{\frac{1}{e^7} - \frac{1}{e^4} - \frac{1}{e^3} + 1}
\end{aligned}$$