Feb 28 Homework Solutions
Math 151, Winter 2012

Chapter 6 Problems (pages 287-291)

Problem 6

A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by N_1 the number of test made until the first defective is identified and by N_2 the number of additional tests until the second defective is identified. Find the joint probability mass function of N_1 and N_2.

Note N_1 and N_2 take positive integer values such that $N_1 + N_2 \leq 5$, i.e. $N_1 = 1, \ldots, 4$ and $N_2 = 1, \ldots, 5 - N_1$. Each pair of values for N_1 and N_2 are equally likely, and there are 10 such pairs. So

$$P\{N_1 = i, N_2 = j\} = \frac{1}{10} \text{ for } i = 1, \ldots, 4 \text{ and } j = 1, \ldots, 5 - i$$

Problem 8

The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y}, \quad -y \leq x \leq y, \, 0 < y < \infty.$$

(a) *Find c.*

We want

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^{\infty} \int_{-y}^{y} c(y^2 - x^2)e^{-y} dx dy.$$

We compute

$$\begin{aligned}
\int_0^{\infty} \int_{-y}^{y} (y^2 - x^2)e^{-y} dx dy &= \int_0^{\infty} \int_{-y}^{y} (y^2 - x^2)e^{-y} dx dy \\
&= \int_0^{\infty} \left[\left(xy^2 - \frac{x^3}{3} \right) e^{-y} \right]_{x=-y}^{x=y} dy \\
&= \frac{4}{3} \int_0^{\infty} y^3 e^{-y} dy \\
&= 8 \int_0^{\infty} e^{-y} dy \quad \text{using integration by parts} \\
&= 8.
\end{aligned}$$

Thus $c = 1/8$.

(b) *Find the marginal densities of X and Y.*

The marginal probability density function of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{|x|}^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y}dy \\ &= \int_{|x|}^{\infty} \frac{1}{4} y e^{-y}dy \quad \text{using integration by parts} \\ &= \frac{1}{4}|x|e^{-|x|} + \int_{|x|}^{\infty} \frac{1}{4}e^{-y}dy \quad \text{using integration by parts} \\ &= \frac{1}{4}|x|e^{-|x|} + \frac{1}{4}e^{-|x|} \\ &= \frac{1}{4}e^{-|x|}\big(|x|+1\big) \end{aligned}$$

Let f_Y be the marginal probability density function of Y. For $y < 0$ we have $f_Y(y) = 0$, and for $y \geq 0$ we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \frac{1}{8}\int_{-y}^{y} (y^2 - x^2)e^{-y}dx \\ &= \frac{1}{8}\left[\left(xy^2 - \frac{x^3}{3}\right)e^{-y}\right]_{x=-y}^{x=y} \\ &= \frac{1}{6}y^3 e^{-y} \end{aligned}$$

(c) *Find $E[X]$.*

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = \frac{1}{8}\int_{0}^{\infty}\int_{-y}^{y} x(y^2 - x^2)e^{-y}dxdy = 0,$$

since $x(y^2 - x^2)e^{-y}$ is an odd function of x.

Problem 10

The joint probability density function of X and Y is given by

$$f(x,y) = e^{-x-y}, \quad 0 \leq x < \infty, 0 \leq y < \infty$$

(a) *Find $P\{X < Y\}$.*

$$\begin{aligned} P\{X < Y\} &= \int_0^{\infty}\int_x^{\infty} f(x,y)dydx = \int_0^{\infty}\int_x^{\infty} e^{-x-y}dydx = \int_0^{\infty}\left[-e^{-x-y}\right]_{y=x}^{y=\infty} dx \\ &= \int_0^{\infty} e^{-2x}dx = \left[\tfrac{-1}{2}e^{-2x}\right]_{x=0}^{x=\infty} = \frac{1}{2}. \end{aligned}$$

(b) *Find $P\{X < a\}$.*

$$
\begin{aligned}
P\{X < a\} &= \int_0^a \int_0^\infty f(x,y)dydx = \int_0^a \int_0^\infty e^{-x-y}dydx = \int_0^a \left[-e^{-x-y}\right]_{y=0}^{y=\infty} dx \\
&= \int_0^a e^{-x}dx = \left[-e^{-x}\right]_{x=0}^{x=a} = 1 - e^{-a}.
\end{aligned}
$$

Problem 15

The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is

$$
f(x,y) = \begin{cases} c & \text{if } (x,y) \in R, \\ 0 & \text{otherwise.} \end{cases}
$$

(a) *Show that $1/c =$ area of region R.*

Let $\mathbb{R}^2$ be the 2-dimensional plane. Observe that for $E \subseteq \mathbb{R}^2$,

$$
P\{(X,Y) \in E\} = \iint_E f(x,y)dxdy = \iint_{E\cap R} c = c \cdot \text{area}(E \cap R).
$$

Since $f(x,y)$ is a joint density function, we have

$$
1 = P\{(X,Y) \in \mathbb{R}^2\} = c \cdot \text{area}(\mathbb{R}^2 \cap R) = c \cdot \text{area}(R).
$$

So the area of R is $1/c$.

(b) *Suppose that (X, Y) is uniformly distributed over the square centered at $(0,0)$ and with sides of length 2. Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.*

Let $\mathbb{R}$ be the real line. For sets $A, B \subseteq \mathbb{R}$, we have

$$
\begin{aligned}
P\{X \in A, Y \in B\} &= \int_B \int_A f(x,y)dxdy = \int_{B\cap[-1,1]} \int_{A\cap[-1,1]} \frac{1}{4}dxdy \\
&= \frac{1}{4}\text{length}(A \cap [-1,1])\text{length}(B \cap [-1,1])
\end{aligned}
$$

and

$$
P\{X \in A\} = \int_{-\infty}^{\infty} \int_A f(x,y)dxdy = \int_{-1}^{1} \int_{A\cap[-1,1]} \frac{1}{4}dxdy = \frac{1}{2}\text{length}(A \cap [-1,1])
$$

and

$$
P\{Y \in B\} = \int_B \int_{-\infty}^{\infty} f(x,y)dxdy = \int_{B\cap[-1,1]} \int_{-1}^{1} \frac{1}{4}dxdy = \frac{1}{2}\text{length}(B \cap [-1,1]).
$$

Thus

$$
P\{X \in A, Y \in B\} = \frac{1}{4}\text{length}(A\cap[-1,1])\text{length}(B\cap[-1,1]) = P\{X \in A\}P\{Y \in B\},
$$

so X and Y are independent. Also, our formulas for $P\{X \in A\}$ and $P\{Y \in B\}$ show that each of X and Y are uniformly distributed over $(-1, 1)$.

(c) *What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \leq 1\}$.*

Let C denote the interior of the circle of radius 1 centered at the origin. We want to compute $P\{(X, Y) \in C\}$. We have

$$P\{(X,Y) \in C\} = \frac{1}{4} \cdot \text{area}(C \cap R) = \frac{1}{4} \cdot \text{area}(C) = \frac{1}{4} \cdot \pi = \frac{\pi}{4}.$$

Problem 16

Suppose that n points are independently chosen at random on the circumference of a circle, and we want the probability that they all lie in some semicircle. That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line. Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for $180°$, $i = 1, \ldots, n$.

(a) *Express A in terms of the A_i.*

It's clear that $A_i \subset A$ for each i. Hence $\bigcup_{i=1}^n A_i \subseteq A$. Now we want to show the reverse inclusion. If A occurs, then we can rotate the semicircle clockwise until it starts at one of the points P_i, showing that A_i and thus $\bigcup_{i=1}^n A_i$ occurs. Hence $A \subseteq \bigcup_{i=1}^n A_i$. So we've shown that $A = \bigcup_{i=1}^n A_i$.

(b) *Are the A_i mutually exclusive?*

Almost, but not quite. Strictly speaking the A_i and A_j are not mutually exclusive since if $P_i = P_j$, the other P_k can lie in the semicircle beginning at $P_i = P_j$ and going clockwise for 180 degrees in which case both A_i and A_j occur. However, if $P_i \neq P_j$, then A_i and A_j cannot both occur since if P_j is in the semicircle starting at P_i and going clockwise 180 degrees, then P_i is not in the semicircle starting at P_j and going clockwise 180 degrees. So $A_i A_j \subseteq \{P_i = P_j\}$. Moreover $P\{P_i = P_j\} = 0$, so $P(A_i A_j) = 0$.

(c) *Find $P(A)$.*

We argued in part (b) that $P(A_i A_j) = 0$. It follows that $P(A_i A_j A_k) = 0$, and the same is true for all higher order intersections.

We have

$$\begin{aligned}
P(A) &= P(\bigcup_{i=1}^n A_i) \\
&= \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \ldots \\
&\quad \text{by the inclusion-exclusion identity} \\
&= \sum_{i=1}^n P(A_i) \quad \text{since all other terms are zero.}
\end{aligned}$$

Fix $i \in \{1, ..., n\}$ and fix a randomly chosen P_i. For $j \neq i$, the probability that the randomly chosen point P_j is in the semicircle starting at P_i and going clockwise 180 degrees is 1/2. So $P(A_i) = (1/2)^{n-1}$. Therefore

$$P(A) = \sum_{i=1}^{n} P(A_i) = n(1/2)^{n-1}.$$

Problem 18

Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

Note that X and Y are jointly distributed with joint probability density function given by

$$f(x,y) = \begin{cases} \frac{4}{L^2} & 0 \leq x \leq \frac{L}{2} \text{ and } \frac{L}{2} \leq y \leq L \\ 0 & \text{otherwise.} \end{cases}$$

Hence the probability that the distance between the two points is greater than $L/3$ is equal to

$$\begin{aligned}
P(Y - X > L/3) &= \iint_{y-x>L/3} f(x,y)dydx \\
&= \int_0^{L/2} \int_{\max\{L/2, L/3+x\}}^{L} \frac{4}{L^2} dydx \\
&= \int_0^{L/6} \int_{L/2}^{L} \frac{4}{L^2} dydx + \int_{L/6}^{L/2} \int_{L/3+x}^{L} \frac{4}{L^2} dydx \\
&= \int_0^{L/6} \frac{4}{L^2}(L - L/2)dx + \int_{L/6}^{L/2} \frac{4}{L^2}\big(L - (L/3 + x)\big)dx \\
&= \int_0^{L/6} \frac{2}{L} dx + \int_{L/6}^{L/2} \frac{8}{3L} - \frac{4}{L^2}x \; dx \\
&= \frac{2}{L}(L/6 - 0) + \left[\frac{8}{3L}x - \frac{2}{L^2}x^2\right]_{L/6}^{L/2} \\
&= \frac{1}{3} + \left(\frac{4}{3} - \frac{1}{2}\right) - \left(\frac{4}{9} - \frac{1}{18}\right) \\
&= \frac{7}{9}.
\end{aligned}$$

Note: it is much easier to find the limits for the integrals above if you first draw a picture of the region in the plane where $0 \leq x \leq \frac{L}{2}$, where $\frac{L}{2} \leq y \leq L$, and where $y - x > L/3$. Alternatively, after drawing this region in the plane, you can compare the area of this region to the area of the larger rectangle given by $0 \leq x \leq \frac{L}{2}$ and $\frac{L}{2} \leq y \leq L$ to see that the desired probability is 7/9.

Problem 23

The random variables X and Y have joint density function

$$f(x,y) = 12xy(1-x), \quad 0 < x < 1,\ 0 < y < 1.$$

and equal to zero otherwise.

(a) *Are X and Y are independent?*

Let $\mathbb{R}$ be the real line. Let $A, B \subseteq \mathbb{R}$. Without loss of generality, we may assume that $A, B \subseteq (0,1)$ since f is zero outside this range. We have

$$\begin{aligned} P\{X \in A\} &= \int_{-\infty}^{\infty}\int_A f(x,y)dxdy = \int_0^1 \int_A 12xy(1-x)dxdy \\ &= 12\int_A x(1-x)dx \cdot \int_0^1 ydy = 12\int_A x(1-x)dx \cdot \frac{1}{2} \\ &= 6\int_A x(1-x)dx \end{aligned}$$

and

$$\begin{aligned} P\{Y \in B\} &= \int_B \int_{-\infty}^{\infty} f(x,y)dxdy = \int_B \int_0^1 12xy(1-x)dxdy \\ &= 12\int_0^1 x(1-x)dx \cdot \int_B ydy = 12 \cdot \frac{1}{6} \cdot \int_B ydy = 2 \cdot \int_B ydy \end{aligned}$$

and

$$\begin{aligned} P\{X \in A, Y \in B\} &= \int_B \int_A f(x,y)dxdy = \int_B \int_A 12xy(1-x)dxdy \\ &= 6\int_A x(1-x)dx \cdot 2\int_B ydy = P\{X \in A\} \cdot P\{Y \in B\}. \end{aligned}$$

Since $P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\}$, this means that X and Y are independent. Note that we've also shown that $f_X(x) = 6x(1-x)$ and $f_Y(y) = 2y$, and we will use this information in the remaining parts of this problem.

(b) *Find $E[X]$.*

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 6x^2(1-x)dx = \left[2x^3 - 3x^4/2\right]_0^1 = 1/2.$$

(c) *Find $E[Y]$.*

$$E[Y] = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 2y^2dx = \left[2y^3/3\right]_0^1 = 2/3.$$

(d) *Find $Var(X)$.*

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 6x^3(1-x)dx = \left[3x^4/2 - 6x^5/5\right]_0^1 = 3/10,$$

so

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 3/10 - (1/2)^2 = 1/20.$$

(e) *Find $Var(Y)$.*

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_0^1 2y^3 dx = \left[y^4/2\right]_0^1 = 1/2.$$

so

$$\text{Var}(Y) = E[Y^2] - E[Y]^2 = 1/2 - (2/3)^2 = 1/18.$$

Problem 26

Suppose that A, B, C are independent random variables, each being uniformly distributed over $(0,1)$.

(a) *What is the joint cumulative distribution function of A, B, C?*

The joint cumulative distribution function is

$$f(a,b,c) = f_A(a)f_B(b)f_C(c) = \begin{cases} 1 & \text{if } 0 < x, y, z < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) *What is the probability that all the roots of the equation $Ax^2 + Bx + C = 0$ are real?*

Recall that the roots of $Ax^2 + Bx + C = 0$ are real if $B^2 - 4AC \geq 0$, which occurs with probability

$$\begin{aligned}
\int_0^1 \int_0^1 \int_{\min\{1,\sqrt{4ac}\}}^1 1 dbdcda &= \int_0^1 \int_0^{\min\{1,1/(4a)\}} \int_{\sqrt{4ac}}^1 1 dbdcda \\
&= \int_0^{1/4} \int_0^1 \int_{\sqrt{4ac}}^1 1 dbdcda + \int_{1/4}^1 \int_0^{1/(4a)} \int_{\sqrt{4ac}}^1 1 dbdcda \\
&= \int_0^{1/4} \int_0^1 (1 - 2a^{1/2}c^{1/2})dcda + \int_{1/4}^1 \int_0^{1/(4a)} (1 - 2a^{1/2}c^{1/2})dcda \\
&= \int_0^{1/4} \left[c - \tfrac{4}{3}a^{1/2}c^{3/2}\right]_{c=0}^{c=1} da + \int_{1/4}^1 \left[c - \tfrac{4}{3}a^{1/2}c^{3/2}\right]_{c=0}^{c=1/(4a)} da \\
&= \int_0^{1/4} 1 - \tfrac{4}{3}a^{1/2} da + \int_{1/4}^1 \tfrac{1}{12}a^{-1} da \\
&= \left[a - \tfrac{8}{9}a^{3/2}\right]_{a=0}^{a=1/4} + \left[\tfrac{1}{12}\log(a)\right]_{a=1/4}^{a=1} \\
&= \tfrac{5}{36} + \tfrac{1}{12}\log(4).
\end{aligned}$$

Section 6 Theoretical Exercises (page 291-293)

Problem 4

Solve Buffon's needle problem when $L > D$.

See page 243 for the version of the problem where $L \leq D$. What is new for $L > D$ is that now $X < \min\{D/2, (L/2)\cos\theta\}$. This does not reduce to $X < (L/2)\cos\theta$ as it did before. Let ϕ be the angle such that $D/2 = (L/2)\cos\phi$, that is, $\cos\phi = D/L$ (note we are using slightly different notation from the text). Then

$$\begin{aligned}
P\left\{X < \frac{L}{2}\cos\theta\right\} &= \iint_{x<(L/2)\cos y} f_X(x)f_\theta(y)dxdy \\
&= \frac{4}{\pi D}\int_0^{\pi/2}\int_0^{\min\{D/2,(L/2)\cos y} dxdy \\
&= \frac{4}{\pi D}\int_0^{\phi}\int_0^{D/2} dxdy + \frac{4}{\pi D}\int_\phi^{\pi/2}\int_0^{(L/2)\cos y} dxdy \\
&= \frac{4}{\pi D}\cdot\frac{\phi D}{2} + \frac{4}{\pi D}\int_\phi^{\pi/2}\frac{L}{2}\cos y\ dy \\
&= \frac{2\phi}{\pi} + \frac{2L}{\pi D}(1-\sin\phi),
\end{aligned}$$

where ϕ is the angle such that $\cos\phi = D/L$.

Problem 5

If X and Y are independent continuous positive random variables, express the density function of (a) $Z = X/Y$ and (b) $Z = XY$ in terms of the density functions of X and Y. Evaluate the density functions in the special case where X and Y are both exponential random variables.

(a) $Z = X/Y$.

Note $f_Z(z) = 0$ for $z \le 0$, since X and Y are both positive. For $z > 0$, we compute

$$P\{Z < z\} = P\{X/Y < z\} = P\{X < zY\} = \int_0^\infty\int_{-\infty}^{zy} f_X(x)f_Y(y)dxdy,$$

so

$$f_Z(z) = \frac{d}{dz}P\{Z < z\} = \int_0^\infty \frac{d}{dz}\int_{-\infty}^{zy} f_X(x)f_Y(y)dxdy = \int_0^\infty y f_X(zy)f_Y(y)dy.$$

(b) $Z = XY$.

Note $f_Z(z) = 0$ for $z \le 0$, since X and Y are both positive. For $z > 0$, we compute

$$P\{Z < z\} = P\{XY < z\} = P\{X < z/Y\} = \int_0^\infty\int_{-\infty}^{z/y} f_X(x)f_Y(y)dxdy,$$

so

$$f_Z(z) = \frac{d}{dz}P\{Z < z\} = \int_0^\infty \frac{d}{dz}\int_{-\infty}^{z/y} f_X(x)f_Y(y)dxdy = \int_0^\infty \frac{1}{y}f_X(z/y)f_Y(y)dy.$$

Now, let X and Y be exponential random variables with parameters λ and μ, respectively. We can now evaluate the density functions a bit more explicitly.

(a) $Z = X/Y$.

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y f_X(zy) f_Y(y) dy = \int_0^\infty \lambda\mu y e^{-\lambda z y} e^{-\mu y} dy = \int_0^\infty \lambda\mu y e^{-(\lambda z+\mu)y} dy \\
&= \frac{\lambda\mu}{\lambda z+\mu} \int_0^\infty e^{-(\lambda z+\mu)y} dy \quad \text{after integrating by parts} \\
&= \frac{\lambda\mu}{(\lambda z+\mu)^2}
\end{aligned}$$

(b) $Z = XY$.

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_X(z/y) f_Y(y) dy = \int_0^\infty \frac{\lambda\mu}{y} e^{-\lambda z/y} e^{-\mu y} dy = \int_0^\infty \frac{\lambda\mu}{y} e^{-\lambda z/y-\mu y} dy.$$