sd4175

P1

$E[XY] = 100 \cdot 0.1 \cdot 100 + 100 \cdot 0.15 \cdot 250 + 200 \cdot 0.2 \cdot 100 + 200 \cdot 0.3 \cdot 250 = 23750$

$E[Y] = 100 \cdot (0.2 + 0.1 + 0.2) + 250 \cdot (0.05 + 0.15 + 0.3) = 175$

$E[X] = 100 \cdot (0.1 + 0.15) + 200 \cdot (0.2 + 0.3) = 125$

$Cov(X, Y) = E[XY] - E[X]E[Y] = 1875$

$E[X^2] = 100^2 \cdot (0.1 + 0.15) + 200^2 \cdot (0.2 + 0.3) = 22500$

$E[Y^2] = 100^2 \cdot (0.2 + 0.1 + 0.2) + 250^2 \cdot (0.05 + 0.15 + 0.3) = 36250$

$Var(X) = 22500 - 125^2 = 6875$

$Var(Y) = 36250 - 175^2 = 5625$

$$\rho(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{1875}{\sqrt{6875 \cdot 5625}} = 0.3015$$

P2

$Y = -X + 1$

1. Area of triangle $= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

$f_{X,Y}(X, Y) = \frac{1}{\text{area}} = 2$

$$\therefore f_{X,Y}(X, Y) = \begin{cases} 2, & 0 < X < 1 - Y, 0 < Y < 1 \\ 0, & \text{otherwise} \end{cases}$$

2. $f_X(x) = \int_0^{1-x} 2\,dy = [2y]_0^{1-x} = \begin{cases} 2-2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$f_Y(y) = \int_0^{1-y} 2\,dx = [2x]_0^{1-y} = \begin{cases} 2-2y, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

3. $f_{X,Y}(x,y) = 2 \neq f_X(x)f_Y(y) = (2-2x)(2-2y)$

$\therefore$ Not Independent

4. $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \int_0^1\int_0^{1-y} 2xy\,dx\,dy = \int_0^1 [x^2y]_0^{1-y}\,dy$$

$$= \int_0^1 (1-y)^2 y\,dy = \int_0^1 y - 2y^2 + y^3\,dy$$

$$= \left[\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right]_0^1$$

$$= \frac{1}{12}$$

$$E[X] = \int_0^1 x(2-2x)\,dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}$$

$$E[Y] = \int_0^1 y(2-2y)\,dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1 = \frac{1}{3}$$

$$\therefore Cov(X,Y) = \frac{1}{12} - \frac{1}{3}\cdot\frac{1}{3} = -\frac{1}{36}$$

P3

1. $Z = |X-Y|$, $Z = \{1, \dots, n-1\}$

The probability for two drawns is $\frac{1}{n(n-1)}$

For $Z=k$, there are $2 \cdot (n-k)$ combinactions

we could exchange 1st and 2nd drawns

$$\therefore P(Z=k) = \frac{2(n-k)}{n(n-1)}$$

$$\text{p.m.f.}: P(Z=z) = \frac{2(n-z)}{n(n-1)}, \quad 1 \leq Z \leq n-1$$

2. $$E[Z] = \sum_{1}^{n-1} z P_Z(z) = \sum_{1}^{n-1} \frac{2z(n-z)}{n(n-1)} = \frac{2}{n^2-n} \sum_{1}^{n-1} z(n-z)$$

$$= \frac{2}{n^2-n} \left[(n-1) + 2(n-2) + \cdots (n-1) \right]$$

$$= \frac{2}{n^2-n} \left[\frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6} \right]$$

$$= \frac{n+1}{3}$$

P4

1. $$P_{N_t}(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$P_{N(3,8)}(0) = \frac{(0.5 \cdot 5)^0}{0!} e^{-0.5 \cdot 5}$$

$= e^{-2.5} = 0.082$

2. $P(N(0,1]=1, N(1,2]=1, N(2,3]=1, N(3,4)=1]$

$= \left[(0.5\cdot 1)\cdot e^{-0.5}\right]^4$

$= 0.5^4 \cdot e^{-2} = 0.00846$

P5

$Cov(N_{t_1}, N_{t_2}) = E(N_{t_1} \cdot N_{t_2}) - E(N_{t_1})E(N_{t_2})$

$= E[N_{t_1} \cdot (N_{t_1} + N_{(t_2-t_1)})] - E(N_{t_1}) \cdot E(N_{t_1} + N_{t_2-t_1})$

$= E[N_{t_1}^2 + N_{t_1} \cdot N_{t_2-t_1}] - E(N_{t_1})^2 - E(N_{t_1}) \cdot E(N_{t_2-t_1})$

$= E(N_{t_1}^2) + E(N_{t_1}) \cdot E(N_{t_2-t_1}) - E(N_{t_1})^2 - E(N_{t_1}) \cdot E(N_{t_2-t_1})$

$= E(N_{t_1}^2) - E(N_{t_1})^2$

By definition $= Var(N_{t_1}) = E((N_{t_1})(N_{t_1}-1)) + \lambda - \lambda^2$

$= \lambda t_1$