Probability and Statistics: Final Exam

Spring 2020

BEFORE STARTING, PLEASE WRITE AND SIGN THE FOLLOWING HONOR PLEDGE:

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

This exam is scheduled for 90 minutes.
Notes and other outside materials are not permitted. Non graphing calculators are allowed; if you do not have any, numerical formulas are enough.
In some of these problems, you may encounter numbers of the form $\frac{4239}{81}$ or $\frac{12!5!}{3!}$. If that is the case, do not worry about simplifying the result.
Show all work to receive full credit, except where specified. The exam is worth 50 points.

Problem Number	Problem Points	Points Earned
MC	10	
TF	5	
FR1	6	
FR2	12	
FR3	12	
FR4	5	
Total	50	

Some possibly useful formulae

- Bernoulli random variable

$$p_X(0) = 1-p \quad \text{and} \quad p_X(1) = p$$
$$E[X] = p$$
$$\text{Var}(X) = p(1-p)$$

- Binomial random variable

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$E[X] = np$$
$$\text{Var}(X) = np(1-p)$$

- Geometric random variable

$$p_X(k) = (1-p)^{k-1} p$$
$$E[X] = \frac{1}{p}$$
$$\text{Var}(X) = \frac{1-p}{p^2}$$

- Poisson random variable

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$
$$E[X] = \lambda$$
$$\text{Var}(X) = \lambda$$

- Uniform random variable, $U(a,b)$

$$f_X(x) = \begin{cases} 0 \text{ if } x \notin [a,b] \\ \dfrac{1}{b-a} \text{ if } x \in [a,b] \end{cases}$$

$$E[X] = \frac{a+b}{2}$$
$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

- Exponential random variable, $Exp(\lambda)$, with $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$
$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Multiple Choice

(2 points each) Circle the correct answer for each question. *You need not justify your answer, but one point of partial credit may be awarded.*

1 A physician records the following measurements for the lengths in centimeters of the femur bones of her patients

$$56, 74, 59, 38, 64$$

What the median of absolute deviations (MAD) for this data set?

(A) 8.96

(B) 7.5

(C) 5

(D) 10

(E) 59

2 The Pareto distribution with parameter a has the following probability density function:

$$f(x) = \frac{a}{x^a} \quad , \quad x \in [1, +\infty)$$

Suppose the data

$$4, 2, 5, 3$$

was drawn independently from such a distribution. What is the maximum likelihood estimate of a?

(A) $\frac{4}{120}$

(B) $\frac{120}{4}$

(C) $\frac{\ln 4}{120}$

(D) $\frac{4}{\ln 120}$

(E) $\frac{\ln 120}{4}$

3 Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a Uniform$(0,\theta)$ distribution, where θ is the unknown parameter of interest. Consider the estimator $\hat{\theta} = 2\overline{X_n}$, where $\overline{X_n} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$ is the sample mean.

What is the mean squared error (MSE) of the estimator $\hat{\theta}$?

(A) $\dfrac{\theta^2}{3n}$

(B) $\dfrac{\theta^4}{9n^2}$

(C) $\dfrac{\theta^2}{3n^2}$

(D) $\dfrac{\theta}{n}$

(E) None of these answers.

4 Consider two random variables X and Y with the following joint probability mass function

		X		
		1	2	3
Y	1	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
	2	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
	3	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
	4	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$

What is $P(X \leq 2 | Y \leq 2)$?

(A) $\dfrac{1}{4}$

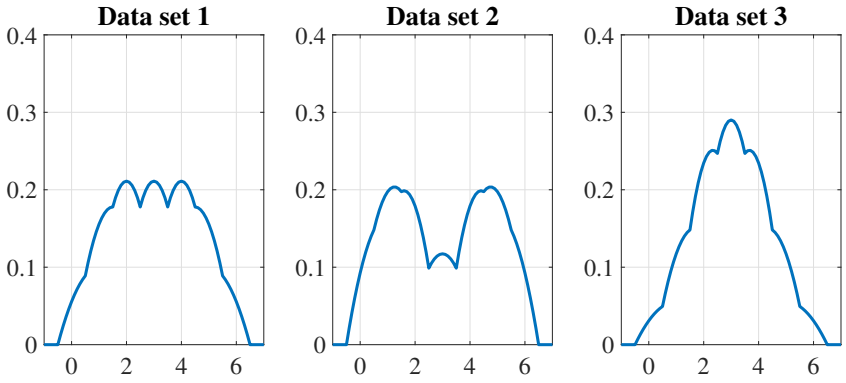
(B) $\dfrac{1}{8}$

(C) $\dfrac{1}{6}$

(D) $\dfrac{3}{12}$

(E) $\dfrac{1}{2}$

5 The three graphs below correspond to the kernel density estimations for three different data sets. Which answer orders the data sets from smallest to biggest standard deviation?



(A) 3 – 2 – 1

(B) 3 – 1 – 2

(C) 1 – 2 – 3

(D) 1 – 3 – 2

(E) 2 – 3 – 1

True or False

(1 point each) Indicate whether each statement is true or false. No partial credit will be given.

1 Consider two continuous random variables X and Y with joint probability density function

$$f_{X,Y} = \begin{cases} 12x^2y^3 & \text{if } 0 \le x \le 1\ ,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The random variables X and Y are independent.

(T) True (F) False

2 Consider two continuous random variables X and Y with joint probability density function

$$f_{X,Y} = \begin{cases} 6e^{-(3x+2y)} & \text{if } x \in [0,+\infty)\ ,\ y \in [0,+\infty) \\ 0 & \text{otherwise} \end{cases}$$

The random variables X and Y are independent.

(T) True (F) False

3 Consider the discrete random variable X with the following cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/6 & \text{if } 0 \le x < 0.5 \\ 3/7 & \text{if } 0.5 \le x < 3 \\ 7/9 & \text{if } 3 \le x < 6.5 \\ 1 & \text{if } x \ge 6.5 \end{cases}$$

Then $p_X(0.5) = \dfrac{3}{7}$.

(T) True (F) False

4 For any data set, the sample mean is greater than or equal to the sample median.

(T) True (F) False

5 For any data set, the median of absolute deviations (MAD) is smaller than or equal to the sample median.

(T) True (F) False

Free Response

Be sure to show all your work neatly and indicate your final answer where appropriate.

1 (6 points) Consider a coin with probability q of landing on heads, and probability $1-q$ of landing on tails.

(A) (2 Points) The coin is tossed N times. What is the probability that the coin lands k times on heads.

(B) (4 Points) The coin is tossed 100 times, and lands on heads 70 times. Using your result from the previous question, what is the maximum likelihood estimate for q?

2 (12 Points) You just bought a new fridge, and plug it in your kitchen. You are interested in how fast the temperature inside the fridge (in Celsius) decreases as a function of time. This is what you find with the thermometer you put in the fridge:

Time	1	3	5
Temperature	8	2	1

(A) (4 Points) Compute the best linear fit in the least squares sense for this data set.

(B) (1 Point) Plot the data points and the line for the best linear fit you found in the previous question to verify that your answer indeed gives a good approximation of the data.

(C) (2 Points) Compute the residuals r_1, r_2, r_3 for this data set, measuring the difference between the data points and the best linear fit.

(D) (5 Points) Looking at your plot more closely, you think that a curve of the form $y = \frac{a}{x}$ would give a better fit to the data points than a linear fit. Use the method of least squares to find the value of a for such a fit.

3 (12 points) A university professor gave a hard quiz to his students, worth 20 points, and obtained the following scores for his students, after grading the quiz:

1, 1, 1, 2, 3, 5, 5, 8, 12, 13, 14, 14, 14, 18

(A) (3 Points) Plot the empirical cumulative distribution function for this data set.

(B) (2 Points) Compute the sample mean and the sample median of the data set.

(C) (3 Points) Compute the lower and upper quartiles of the data set.

(D) (4 Points) Construct the box-and-whisker plot for this data set.
Make sure to explain how you got all the information required to plot the boxplot.

4 (5 points) The average heart rate at rest of infants between 6 and 12 months old is 100 beats per minute (BPM) with standard deviation 15 BPM.

Use the central limit theorem to estimate the probability that a randomly selected group of 100 infants between 6 and 12 months old has an average heart rate at rest above 103 BPM.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997