Probability and Statistics – Problem Set 5

February 24, 2022
Due March 3, 2022 in class

Problem 1

Let X be a discrete random variable which takes on values in $\{0, 1, \ldots, N\}$. Show that in that case, the expected value of X can be written as

$$E(X) = \sum_{n=0}^{N-1} P(X > n)$$

Problem 2

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 2Kxe^{-Kx^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where K is a real number such that $K > 0$.

1. Compute the cumulative distribution function F of X.
2. Compute the mean and variance of X.
3. Let Y be a continuous random variable defined by $Y = KX^2$. Compute the probability density function and cumulative distribution function of Y.

Problem 3 (Needed for Problem 4)

A continuous random variable X is said to have an exponential distribution, written $\text{Exp}(\lambda)$, if its probability density function f is such that

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $\lambda > 0$ is a real number.

1. Compute the mean of X
2. Compute the variance of X
3. Compute the cumulative distribution function F of X. Use this to show that for any real numbers s and t such that $s > 0$ and $t > 0$,

   $$P(X > s + t | X > t) = e^{-\lambda s} = P(X > s)$$

   This last property says that the random variable X is memoryless.

Problem 4 (Needs results from Problem 3)

One often models the lifetime of galaxies with exponential laws (see problem 3). Astronomers have found a galaxy for which they estimate that the probability of collapse within the next 1 million years is equal to 0.000002%.

1. Determine the parameter λ for the exponential law of the random variable X corresponding to the lifetime of the galaxy.
2. What is the expected lifetime of the galaxy?
3. What is the probability that the galaxy collapses within the next 3 billion years?
4. What is the probability that the galaxy is still present in 10 billion years?

Problem 5

A factory manager verifies the state of the machines in her factory. Checking her statistics, she finds that for each machine, the probability of having a failure within the first 5 years of operation is 30%. Among the machines which had a failure in the first five years, the probability of having a more significant failure subsequently and permanently going out of order is 75%. Among the machines which did not have a failure in the first 5 years, that probability is only 40%.

1. What is the probability for a machine to go permanently out of order?
2. What is the probability that a machine which is permanently out of order did not have a failure in the first 5 years?
3. Let X be the random variable corresponding to the number of machines which have a failure within the first 5 years among 10 machines chosen at random. Write the probability mass function for X.
4. What is the expected value of X, and what is its variance?

Remember to justify your answers!