# Recitation 1

1. Permutation: How many different letter arrangements can be made from the letters: Mississippi?

   **Solution.**

   $$\frac{11!}{4!2!}$$

2. Combination: From a standard 52 cards deck (without jokers), randomly pick 5 cards. What is the probability to get four of a kind? (Four of a kind means 4 cards with same number plus another random card. eg: heart A + diamond A + spade A + club A + club 2)

   **Solution.**

   Randomly selecting 5 cards from 52, there are: $\binom{52}{5}$ combinations.If 4 Aces are selected first, there are 52 - 4 = 48 cards can be selected as the fifth card. Since there are 13 different cards in poker deck, there are $13 \times 48 = 624$ ways of combination. Therefore, probability to form four of a kind is:

$$\frac{624}{\binom{52}{5}}$$

3. Probability: A 3-person basketball team consists of a guard, a forward, and a center. (a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team? (b) What is the probability that all 3 players selected play the same position?

**Solution.**

Approach 1: Through Counting

(a) $\dfrac{3 \times 2 \times 1}{3^3} = \dfrac{2}{9}$

(b) $\dfrac{3}{3^3} = \dfrac{1}{9}$

Approach 2: Probability Multiplication Rule

(a) $1 \times \dfrac{2}{3} \times \dfrac{1}{3} = \dfrac{2}{9}$

(b) $1 \times \dfrac{1}{3} \times \dfrac{1}{3} = \dfrac{1}{9}$

4. Probability: There are two players A and B. They play a simple game: Both of them roll a fair die, and the one with more dots wins. If they have same dots, it is a tie. What is the probability for player A to win?

   **Solution.**

   Total outcomes $\Omega = |\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}| = 36$ different outcomes. Cases where player A win:

   - $A = 1; B = \phi$
   - $A = 2; B \in \{1\}$
   - $A = 3; B \in \{1, 2\}$
   - $A = 4; B \in \{1, 2, 3\}$
   - $A = 5; B \in \{1, 2, 3, 4\}$
   - $A = 6; B \in \{1, 2, 3, 4, 5\}$

   There are total of 1 + 2 + 3 + 4 + 5 = 15 cases; therefore, probability for player A to win is $\frac{15}{36}$

5. Exclusion-Inclusion: Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing (a) a ring or a necklace? (b) a ring and a necklace?

   **Solution.**

   Visualization (Venn Diagram) is the best way to solve this problem. What we currently know:

   - P(Ring) = 0.20
   - P(Necklace) = 0.30
   - P(None) = 0.60

   Therefore:
   $P(R \cup N) = 1 - P(None) = 1 - 0.60 = 0.40$
   $P(R \cap N) = P(Ring) + P(Nacklace) - P(R \cup N) == 0.30 + 0.20 - 0.40 = 0.10$

6. Bayesian: Imagining a game with 2 players, A and B. Player A goes first and rolls a fair die. If A gets point 1 or 2 he wins; otherwise, player A gives the die to player B. Same rule applies to player B after receiving the die. He rolls the die, if he gets 1 or 2, he wins the game; otherwise he gives it back to player A. What is the probability that player A wins the game / are you willing to be player A or B?

   **Solution.**

   Let the probability for player A to win be $P(A) = p$. When A first throw the die, he has 2 outcomes to win the game, aka, he has $\frac{1}{3}$ probability to win directly. If player A does not win the game in one throw, then Player B basically becomes the first player, and player B then has a probability of $p$ to win the game. In other words, if player A does not win the game in first throw, player A then has a probability of $1 - p$ to win the game. In this case:

$$
\begin{aligned}
P(A) &= P(A \cap \{1,2\}) + P(A \cap \{3,4,5,6\}) \\
&= P(A|\{1,2\})P(\{1,2\}) + P(A|\{3,4,5,6\})P(\{3,4,5,6\}) \\
&= \frac{1}{3} \times 1 + \frac{2}{3}(1 - P(A)) \\
\implies P(A) &= \frac{3}{5}
\end{aligned}
$$

7. Bayesian: An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

**Solution.**

For the sake of convenience, let $A_1, A_2, A_3, A_4$ be the 4 different Aces. Define 4 events.

- $E_1$: $A_1$ in any of the pile
- $E_2$: $A_2$ in any pile that is different from $A_1$
- $E_3$: $A_3$ in any pile that is different from $A_2$ and $A_1$
- $E_4$: $A_4$ in any pile that is different from $A_3$, $A_2$ and $A_1$

In this case, we are trying to calculate:
$P(E_1E_2E_3E_4) = P(E_4|E_3E_2E_1)P(E_3|E_2E_1)P(E_2|E_1)P(E_1)$
And

$$\begin{aligned} P(E_1) &= 1 \\ P(E_2|E_1) &= \frac{39}{51} \\ P(E_3|E_2E_1) &= \frac{26}{50} \\ P(E_4|E_3E_2E_1) &= \frac{13}{49} \end{aligned}$$

Therefore:

$$P(E_1E_2E_3E_4) = 0.105$$

8. The probability of getting a head on a single toss of a coin is $p$. Suppose that $A$ starts and continues to flip the coin until a tail shows up, at which point $B$ starts flipping. Then $B$ continues to flip until a tail comes up, at which point $A$ takes over, and so on. Let $P_{n,m}$ denote the probability that $A$ accumulates a total of $n$ heads before $B$ accumulates $m$.

   Show that: $P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{m,n})$

   **Solution.**

   First, it is important to point out that this is NOT a fair coin. This coin has $p$ probability to get head. Secondly, instead of saying $P_{n,m}$ "denotes the probability that A accumulates a total of n heads before B accumulates m", it is more like "the probability that the one who holds the coin accumulates n heads before its counterparty". Let event $A_{win} = \{$A gets n heads before B gets m heads$\}$, $B_{win} = \{$B gets m heads before A gets n heads$\}$, and $X$ be the initial coin toss. Therefore:

$$P(X = H) = p, P(X = T) = 1 - p$$

   Imagining a scenario, A and B just start this game, and A wins the game if A receives n heads before B receives m heads. Because of total probability and conditional probability:

$$P_{n,m} = P(A_{win}) = P(A_{win}|X = H)P(X = H) + P(A_{win}|X = H)P(X = T)$$

   As the first coin gets head, A needs n-1 heads to finish the game, therefore:

$$P(A_{win}|X = H) = P_{n-1,m}$$

   As the first coin gets tail, A gives the coin to B, and B holds the coin; therefore, $P(B_{win}|X = T) = P_{m,n}$. Thus, $P(A_{win}|X = T) = 1 - P(B_{win}|X = T) = 1 - Pm, n$. Substituting everything back to the original probability we have:

$$P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{m,n})$$