Xi Liu, xl3504, Problem Set 9

Problem 1
let $N_{(t_1,t_2]}$ be the total number of arrivals in the time interval $(t_1, t_2]$ for the Poisson process

$$P(N_{(t_1,t_2]} = i) = \frac{(\lambda(t_2 - t_1))^i}{i!} e^{-\lambda(t_2-t_1)}$$

$$\begin{aligned}
&P(N_{(0,2]} = 2,\ N_{(1,4]} = 3) \\
&= P(N_{(0,1]} = 0,\ N_{(1,2]} = 2,\ N_{(2,4]} = 1) \\
&+ P(N_{(0,1]} = 1,\ N_{(1,2]} = 1,\ N_{(2,4]} = 2) \\
&+ P(N_{(0,1]} = 2,\ N_{(1,2]} = 0,\ N_{(2,4]} = 3) \\
&\text{/* since } (0,1], (1,2], \text{ and } (2,4] \text{ are disjoint time intervals,} \\
&N_{(0,1]}, N_{(1,2]}, \text{ and } N_{(2,4]} \text{ are independent random variables */} \\
&= P(N_{(0,1]} = 0)P(N_{(1,2]} = 2)P(N_{(2,4]} = 1) \\
&+ P(N_{(0,1]} = 1)P(N_{(1,2]} = 1)P(N_{(2,4]} = 2) \\
&+ P(N_{(0,1]} = 2)P(N_{(1,2]} = 0)P(N_{(2,4]} = 3) \\
&= \frac{(\lambda(1-0))^0}{0!} e^{-\lambda(1-0)} \frac{(\lambda(2-1))^2}{2!} e^{-\lambda(2-1)} \frac{(\lambda(4-2))^1}{1!} e^{-\lambda(4-2)} \\
&+ \frac{(\lambda(1-0))^1}{1!} e^{-\lambda(1-0)} \frac{(\lambda(2-1))^1}{1!} e^{-\lambda(2-1)} \frac{(\lambda(4-2))^2}{2!} e^{-\lambda(4-2)} \\
&+ \frac{(\lambda(1-0))^2}{2!} e^{-\lambda(1-0)} \frac{(\lambda(2-1))^0}{0!} e^{-\lambda(2-1)} \frac{(\lambda(4-2))^3}{3!} e^{-\lambda(4-2)} \\
&= e^{-\lambda}\lambda^2 e^{-\lambda} 2\lambda e^{-2\lambda} + \lambda e^{-\lambda} 2\lambda e^{-\lambda} 2\lambda^2 e^{-2\lambda} + \frac{\lambda}{2} e^{-\lambda} e^{-\lambda} \frac{4\lambda^3}{3} e^{-2\lambda} \\
&= 2\lambda^3 e^{-4\lambda} + 4\lambda^4 e^{-4\lambda} + \frac{2\lambda^4}{3} e^{-4\lambda} \\
&= \boxed{2\lambda^3 e^{-4\lambda}\left(1 + 2\lambda + \frac{\lambda}{3}\right)}
\end{aligned}$$

Problem 2
part 1
if X follows a Poisson distribution with parameter λ, then

$$\forall i \in \mathbb{N},\ p_X(i) = P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}$$

if Y follows a Poisson distribution with parameter μ, then

$$\forall i \in \mathbb{N},\ p_Y(i) = P(Y = i) = \frac{\lambda^i}{i!} e^{-\mu}$$

$$\begin{aligned}
&\forall (x, y) \in \mathbb{R}^2,\ p_{X,Y}(x, y) = p_X(x) p_Y(y) \\
&\forall z \in \mathbb{R} \\
p_Z(z) &= P(Z = z) \\
&= \sum_x P(X = x,\ Y = z - x) \\
&= \sum_x p_{X,Y}(x, z - x) \\
&= \sum_x p_X(x) p_Y(z - x) \\
&= \sum_{x=0}^{z} \left(\frac{\lambda^x}{x!} e^{-\lambda} \right) \left(\frac{\mu^{z-x}}{(z-x)!} e^{-\mu} \right) \\
&= \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} e^{-(\lambda+\mu)} \\
&= e^{-(\lambda+\mu)} \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \frac{z! \lambda^x \mu^{z-x}}{x!(z-x)!} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda^x \mu^{z-x}
\end{aligned}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x}$$

since the binomial theorem says

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

so

$$\begin{aligned} p_Z(z) &= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x} \\ &= \boxed{\frac{e^{-(\lambda+\mu)}}{z!} (\lambda+\mu)^z} \end{aligned}$$

part 2
application:

$$
\begin{aligned}
P_Z(z) &= \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} e^{-\lambda-\mu} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} (\lambda+\mu)^z \\
p_Z(10) &= \sum_{x=0}^{10} \frac{(8.392)^x (7.854)^{10-x}}{x!(10-x)!} e^{-8.392-7.854} \\
&= \boxed{\frac{e^{-(8.392+7.854)}}{10!} (8.392+7.854)^{10}} \\
&\approx \boxed{0.0310566}
\end{aligned}
$$

Problem 3

a binomial distribution with parameters n, p, and k has a probability mass function

$$\begin{aligned}
&\forall k \in \mathbb{N} \\
&p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \\
&E[X] = np \\
&Var(X) = np(1-p)
\end{aligned}$$

1.

Markov's inequality says $\forall a \in \mathbb{R},\ a \geq 0$

$$\begin{aligned}
E[X] &\geq aP(X \geq a) \\
a &:= kn \\
E[X] &\geq knP(X \geq kn) \\
P(X \geq kn) &\leq \frac{E[X]}{kn} = \frac{np}{kn} = \boxed{\frac{p}{k}}
\end{aligned}$$

2.

Chebyshev's inequality says

$$P(|X - E[X]| \geq |a|) \leq \frac{1}{a^2} Var(X)$$

$$
\begin{aligned}
P(X \geq kn) &= P(X - np \geq kn - np) \\
&\leq P(|X - np| \geq kn - np) \\
&\leq \frac{1}{(kn - np)^2} Var(X) \\
&= \frac{n(1-p)}{(n(k-p))^2} \\
&= \frac{n(1-p)}{n^2(k-p)^2} \\
&= \boxed{\frac{1-p}{n(k-p)^2}}
\end{aligned}
$$

3.
Markov's inequality:

$$P(X \geq kn) \leq \frac{p}{k} = \frac{1/2}{3/4} = \frac{2}{3}$$

Chebyshev's inequality:

$$P(X \geq kn) \leq \frac{1-p}{n(k-p)^2} = \frac{1-1/2}{n(3/4-1/2)^2} = \frac{1/2}{n(1/16)} = \frac{8}{n}$$

the smaller upper bound is the tighter upper bound

$$\lim_{n\to\infty} \frac{2/3}{8/n} = \lim_{n\to\infty} \frac{n}{12} = \infty$$

so the Chebyshev's inequality's upper bound ($8/n$) is the tighter bound

Problem 4
apply Chebyshev's inequality:

$$\forall \epsilon > 0$$
$$P(|\overline{X_n} - \mu| > \epsilon) \leq \frac{1}{\epsilon^2} Var(\overline{X_n}) = \frac{\sigma^2}{n\epsilon^2}$$

90% sure that the average of the measurements is within half a degree Kelvin of T:

$$E[U_i] = \mu = 0$$

$$P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \geq 0.9$$
$$-P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \leq -0.9$$
$$1 - P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \leq 1 - 0.9$$
$$P\left(|\overline{X_n} - 0| > \frac{1}{2}\right) \leq 0.1$$

Chebyshev's inequality gives

$$P\left(|\overline{X_n} - 0| > \frac{1}{2}\right) \leq \frac{3}{n(1/2)^2} = \frac{12}{n}$$

so, to be 90% sure that the average of the measurements is within half a degree Kelvin of T, number of measurements n should be such that

$$\frac{12}{n} \leq 0.1$$
$$\boxed{n \geq 120}$$

Problem 5
apply Chebyshev's inequality:

$$\forall \epsilon > 0$$
$$P(|\overline{X_n} - \mu| > \epsilon) \leq \frac{1}{\epsilon^2} Var(\overline{X_n}) = \frac{\sigma^2}{n\epsilon^2}$$

let X correspond to the result of rolling a fair die

$$n := 100$$
$$E[X] = \sum_{a_i} a_i p_X(a_i) = \sum_{a_i=1}^{6} a_i \frac{1}{6} = 3.5 = \frac{7}{2}$$
$$E[X^2] = \sum_{a_i} a_i^2 p_X(a_i) = \sum_{a_i=1}^{6} a_i^2 \frac{1}{6} = \frac{91}{6}$$
$$\sigma^2 = Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\begin{aligned}
P(3.2 \leq \overline{X_{100}} \leq 3.8) &= P(-0.3 \leq \overline{X_{100}} - 3.5 \leq 0.3) \\
&= P(|\overline{X_{100}} - 3.5| \leq 0.3) \\
&= 1 - P(|\overline{X_{100}} - 3.5| > 0.3) \\
& /* \text{ since } P(|\overline{X_{100}} - 3.5| > 0.3) \leq \frac{\sigma^2}{n\epsilon^2} \ */ \\
&\geq 1 - \frac{\sigma^2}{n\epsilon^2} \\
&= 1 - \frac{35/12}{(100)(0.3)^2} \\
&= 1 - \frac{35}{108} \\
&= \boxed{\frac{73}{108}}
\end{aligned}$$