Probability and Statistics: Midterm Exam 2

Spring 2020

Name: ____________________

This exam is scheduled for 75 minutes. Notes and other outside materials are not permitted. Non graphing calculators are allowed; if you do not have any, numerical formulas are enough.
Show all work to receive full credit, except where specified. The exam is worth 45 points.

Problem Number	Problem Points	Points Earned
MC	10	
TF	5	
FR1	9	
FR2	9	
FR3	12	
Total	45	

Some possibly useful formula

- Bernoulli random variable

$$
\begin{aligned}
&p_X(0) = 1 - p \quad \text{and} \quad p_X(1) = p \\
&E[X] = p \\
&\mathrm{Var}(X) = p(1-p)
\end{aligned}
$$

- Binomial random variable

$$
\begin{aligned}
&p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \\
&E[X] = np \\
&\mathrm{Var}(X) = np(1-p)
\end{aligned}
$$

- Geometric random variable

$$
\begin{aligned}
&p_X(k) = (1-p)^{k-1} p \\
&E[X] = \frac{1}{p} \\
&\mathrm{Var}(X) = \frac{1-p}{p^2}
\end{aligned}
$$

- Poisson random variable

$$
\begin{aligned}
&p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda} \\
&E[X] = \lambda \\
&\mathrm{Var}(X) = \lambda
\end{aligned}
$$

Multiple Choice

(2 points each) Circle the correct answer for each question. *You need not justify your answer, but one point of partial credit may be awarded.*

1 The occurence of goals in a soccer game played by kids is well modeled by a Poisson process with rate $\lambda = 2$ goals per interval of 5 minutes. What is the probability that there are 3 goals scored in the first 5 minutes of the game, and then 1 goal scored in the next 5 minutes of the game?

$\lambda = 2/5$ /min

(A) $\frac{8}{3!}e^{-2}$

(B) $\frac{8}{3!}e^{-2} + 2e^{-2}$

(C) $\frac{16}{3!}e^{-4}$

(D) $\frac{16}{3!}e^{-2}$

(E) $\frac{8}{3!}e^{-4}$

$$\frac{(2)^3 e^{-2}}{3!} \cdot \frac{(2)^1 e^{-2}}{1!} = \frac{16 e^{-4}}{3!}$$

2 Two discrete random variables X and Y have the following joint probability mass function:

		X = 1	X = 2	X = 3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$
	4	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
	5	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{12}$

$$\frac{P(Y=4, X=3)}{P(X=3)} = \frac{1/8}{1/4 + 1/8 + 1/12}$$

What is $P(Y = 4|X = 3)$?

(A) $\frac{4}{15}$

(B) $\frac{3}{11}$

(C) $\frac{3}{8}$

(D) $\frac{1}{8}$

(E) None of these answers.

$$= \frac{1/8}{\frac{6+3+2}{24}} = \frac{3}{11}$$

3 Consider a continuous random variable X with distribution $N(30, 25)$. What is the 25$^{\text{th}}$ percentile $q_{0.25}$ of X?

Note: Tables for the cumulative distribution function Φ of $N(0,1)$ are provided at the end of the exam

(A) $q_{0.25} \approx 21.35$

(B) $q_{0.25} \approx 18.65$

(C) $q_{0.25} \approx 28.175$

(D) $q_{0.25} \approx 26.65$

(E) $q_{0.25} \approx 15.25$

$P(X \leq k) = 0.25$

$\Rightarrow P\left(\frac{X-30}{5} \leq \frac{k-30}{5}\right) = 0.25$

$\frac{k-30}{5} = -0.67$

$\Rightarrow k \approx 26.65$

4 Two continuous random variables X and Y have the following joint probability density function

$$f_{XY}(x,y) = \begin{cases} ax^2 + y & \text{if } 0 \leq x \leq 1 \ , \ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the real number a?

(A) $\frac{3}{2}$

(B) $\frac{1}{2}$

(C) 1

(D) 5

(E) 2

$\int_0^1 \int_0^1 (ax^2+y)\, dx\, dy = 1$

$\Rightarrow \int_0^1 \left(\frac{a}{3} + y\right) dy = 1$

$\Rightarrow \frac{a}{3} + \frac{1}{2} = 1 \Rightarrow 3/2$

5 Consider a random variable X with mean 1 and variance 3. What is the *smallest* upper bound (i.e. most accurate estimate) one can get for $P(|X-1| \geq 0.2)$ according to Chebyshev's inequality?

(A) $P(|X-1| \geq 0.2) \leq 25$

(B) $P(|X-1| \geq 0.2) \leq 25.675$

(C) $P(|X-1| \geq 0.2) \leq 75$

(D) $P(|X-1| \geq 0.2) \leq 0.25$

(E) $P(|X-1| \geq 0.2) \leq 250.$

$E[X] = 1$, $Var(X) = 3$

$P(|X-1| \geq 0.2) \leq \frac{Var(X)}{(0.2)^2} = \frac{300}{4}$

$= 75$

True or False

(1 point each) Indicate whether each statement is true or false. No partial credit will be given.

1 Let X and Y be two discrete random variables. Then $\mathrm{Var}(X+Y) \geq \mathrm{Var}(X) + \mathrm{Var}(Y)$

Ⓣ True Ⓕ False

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$

2 Consider a Poisson process, and X_3 the random variable corresponding to the time of the third arrival. X_3 has an exponential distribution.

Ⓣ True Ⓕ False

$X_3 \sim \mathrm{Gamma}(\lambda, 3)$

3 Consider two continuous random variables X and Y with joint cumulative distribution function

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ xy & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ x & \text{if } 0 \leq x \leq 1 \text{ and } y \geq 1 \\ y & \text{if } 0 \leq y \leq 1 \text{ and } x \geq 1 \\ 1 & \text{if } x > 1 \text{ and } y > 1 \end{cases}$$

$$f_{X,Y}(x,y) = \begin{cases} 1 & x \in (0,1),\ y \in (0,1) \\ 0 & \text{otherw.} \end{cases}$$

Then $E[X^3Y] = \dfrac{1}{8}$.

Ⓣ True Ⓕ False

$\int_0^1 \int_0^1 x^3 y \qquad \frac{1}{4} \cdot \frac{1}{2}$

4 Two continuous random variables X and Y are such that their correlation coefficient $\rho(X,Y)$ satisfies $\rho(X,Y) = -1$. The correlation between X and Y is weak.

Ⓣ True Ⓕ False

Strong negative corr.

5 Let X and Y be two random variables such that $E[XY] = E[X]E[Y]$. Then X and Y must be independent random variables.

Ⓣ True Ⓕ False

Y \ X	1	-1	0
-1	0	0	1/4
0	1/4	1/4	0
1	0	0	1/4

$E[XY] = 0$

$E[X] = 1/4 - 1/4 = 0$

$E[Y] = 0$

Free Response

Be sure to show all your work neatly and indicate your final answer where appropriate.

1 (9 Points) A company has 300 employees. For every employee and at any time of the day, the probability for that employee to be on the phone is $\frac{1}{10}$. Furthermore, we assume that internally, employees use emails, Zoom or in-person meetings to communicate, so that all phone calls are with people external to the company.

Use the central limit theorem to estimate the number of phone lines the company should install so that the probability that all the phone lines are used at the same time is at most equal to 0.025.

$$p_{X_i}(x) = \begin{cases} 1/2, & x = 1 \\ 1/2, & x = 0 \end{cases}$$

$$Y = \sum_{i=1}^{300} X_i \longrightarrow \text{No. of calls at a point in time.}$$

$$E[Y] = 300 \cdot 1/2 = 150$$

$$Var(Y) = 300(1/4) = 75$$

$$P(Y \leq s) \geq 0.975$$

$$P\left(\frac{Y - E[Y]}{std(Y)} \leq \frac{s - 150}{\sqrt{75}}\right) \geq 0.975$$

$$\frac{s - 150}{\sqrt{75}} \geq 1.96 \Rightarrow s - 150 \geq 16.97$$

$$\Rightarrow s \geq 166.97 \,/\, 167$$

2 (9 Points) Let X and Y be two discrete random variables with the following joint probability mass function.

X \ Y	0	1	2
0	1/8	1/8	0
1	1/8	1/4	1/8
2	0	1/8	1/8

$Z \in \{0, 1, 4, 2, 5, 8\}$

(A) (2 Points) Construct the marginal probability mass functions of X and Y.

(B) (3 Points) Compute $\text{Cov}(X,Y)$.

(C) (2 Points) Compute the correlation coefficient $\rho(X,Y)$.

(D) (2 Points) Let $Z = X^2 + Y^2$. What is the probability mass function of Z?

(A)

$$p_X(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \\ 0 & \text{other} \end{cases}$$

$$p_Y(y) = \begin{cases} 1/4, & y = 0 \\ 1/2, & y = 1 \\ 1/4, & y = 2 \\ 0, & \text{other} \end{cases}$$

$$\text{Cov}(XY) = E[XY] - E[X]E[Y]$$

$$E[X] = 0(1/4) + 1(1/2) + 2(1/4) = 1$$

$$E[Y] = 1$$

(B) $E[XY] = \frac{1}{4}(1)(1) + \frac{1}{8}(1)(2) + \frac{1}{8}(2)(1) + \frac{1}{8}(2)(2)$

$1 + \frac{1}{4}$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}$

$= \frac{5}{4}$

$\text{Cov} = 1 - \frac{5}{4} = -\frac{1}{4}$

(C) $$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$\text{Var}(X) = E[X^2] - \{E[X]\}^2$

3 (12 Points) **Important note: This problem is constructed in such a way that you can answer questions even if you did not answer previous questions in the problem.**

In this problem, we assume that the odds of giving birth to a girl or to a boy are $\frac{1}{2}$ each.

We consider a country in which, because of tradition and particular socio-economic circumstances, parents give birth to children until they give birth to their first son, at which point they stop having children.

The point of the problem is to evaluate the proportion of children who are boys in a given generation. We will do this in several steps.

A (3 Points) Let X be the random variable corresponding to the number of children of a given couple chosen at random in the population.
What is the probability mass function of X?
Verify that $E[X] = 2$ and $\text{Var}(X) = 2$ with the formulas provided at the beginning of this exam.

B (3 Points) We now assume that in the country, there are N couples who can have children, and for a given couple i, we call X_i the random variable corresponding to the number of children this couple will have.
Let P be the random variable corresponding to the fraction of boys among all children. Express P in terms of $X_1, X_2, \ldots, X_N$ and N. Then write P in terms of the sample mean $\overline{X_N}$ defined by

$$\overline{X_N} = \frac{X_1 + X_2 + \ldots + X_N}{N}$$

C (2 Points) Assume the X_i are independent random variables.
What is the expected value of $\overline{X_N}$ and what is the variance of $\overline{X_N}$?

D (4 Points) Assume N is large enough that the answer to the problem is well approximated by taking the limit $N \to +\infty$.
What does the law of large numbers tell us about $\overline{X_N}$?
What is then the ratio P of boys among all children?

A

$X \in \{1, 2, \ldots\}$

$P(X=k) = (1-p)^{k-1} \cdot p$

$E[X] = 1/p = 2$

$\text{Var}(X) = \frac{1-p}{p^2} = 2$

(B) $$P = \frac{1+1+1+\cdots N \text{ times}}{X_1 + X_2 + \cdots + X_n} = \frac{N}{\sum_{i=1}^{N} X_i}$$

$$= \frac{1}{\bar{X}_N}$$

(C) $$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1} X_i\right]$$

$$= \frac{1}{n} \cdot n\, E[X_i]$$

$$= 1/p$$

$$\mathrm{Var}(\bar{X}_n) = \mathrm{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} \cdot \frac{n(1-p)}{p^2}$$

$$= \frac{(1-p)}{np^2}$$

(D) $$\bar{X}_n \xrightarrow{n\to\infty} E[X]$$

$$p = 1/2$$

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997