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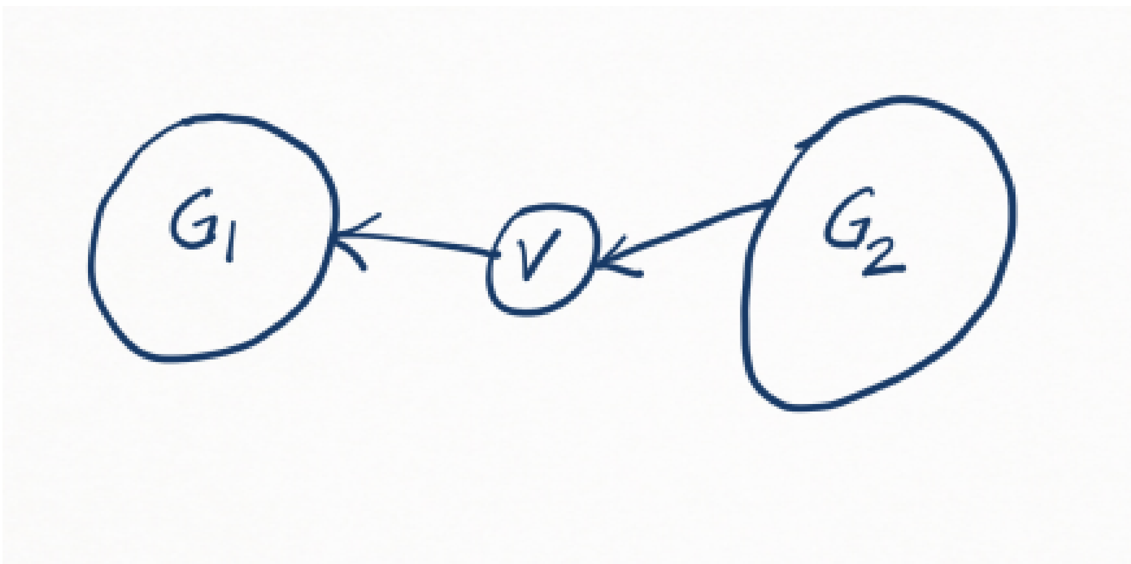
Problem 1
let G_1 be a set containing some vertices of G
G_2 be a set containing some vertices of G
let there be an edge from a vertex in set G_2 to vertex v and an edge from vertex v to a vertex in set G_1 as shown below, then if all of the vertices in set G_1 are visited before visiting v, then there is no outgoing edge of v remaining to visit, so if v is chosen to be visited next, it will be the only vertex in the DFS tree



Problem 2

has_cycle() is an algorithm that determines whether G has a cycle, has_cycle()'s time complexity is $O(|V| + |E|)$ since has_cycle() calls the recursive function has_back_edge() only when $!visited[i]$ which means the vertex has not been visited before. it finds a back edge when there is a vertex i that is adjacent to v and is visited but i is the descendant of the current vertex v (equivalently i is not the parent of v), descendant.discovery_time < ancestor.discovery_time

```
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <list>
using namespace std;

struct graph
{
    int n_v;
    bool * visited;
    list<int> * adj;
    graph(int n_v);
    void clear_visit();
    void add_edge(int v1, int v2);
    bool has_back_edge(int v, int parent);
    bool has_cycle();
};

graph::graph(int n_v)
{
    this->n_v = n_v;
    adj = new list<int>[n_v];
    visited = (bool *)malloc(n_v * sizeof(bool));
}

void graph::clear_visit()
{
    memset(visited, false, n_v * sizeof(bool));
}
```

```
void graph::add_edge(int v1, int v2)
{
    adj[v1].push_back(v2);
    adj[v2].push_back(v1);
}

bool graph::has_back_edge(int v, int parent)
{
    visited[v] = true;
    for(list<int>::iterator i = adj[v].begin(); i != adj[v].end(); ++i)
    {
        if(!visited[*i])
        {
            if(has_back_edge(*i, v))
                return true;
        }
        else if(visited[*i])
        {
            if(*i != parent) /* for an edge between
            {ancestor, descendant},
            descendant.discovery_time < ancestor.discovery_time */
                return true;
        }
    }
    return false;
}

bool graph::has_cycle()
{
    clear_visit();
    for(int i = 0; i < n_v; ++i)
    {
        if(!visited[i])
            if(has_back_edge(i, -1))
                return true;
    }
    return false;
```

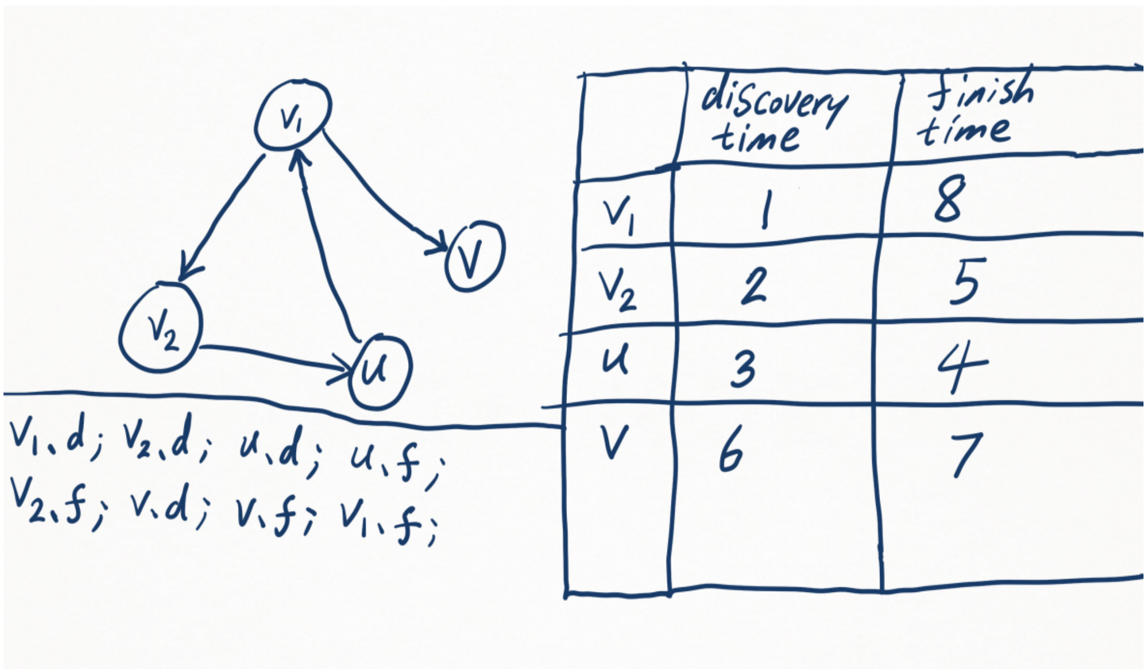
}

Problem 3

(a)

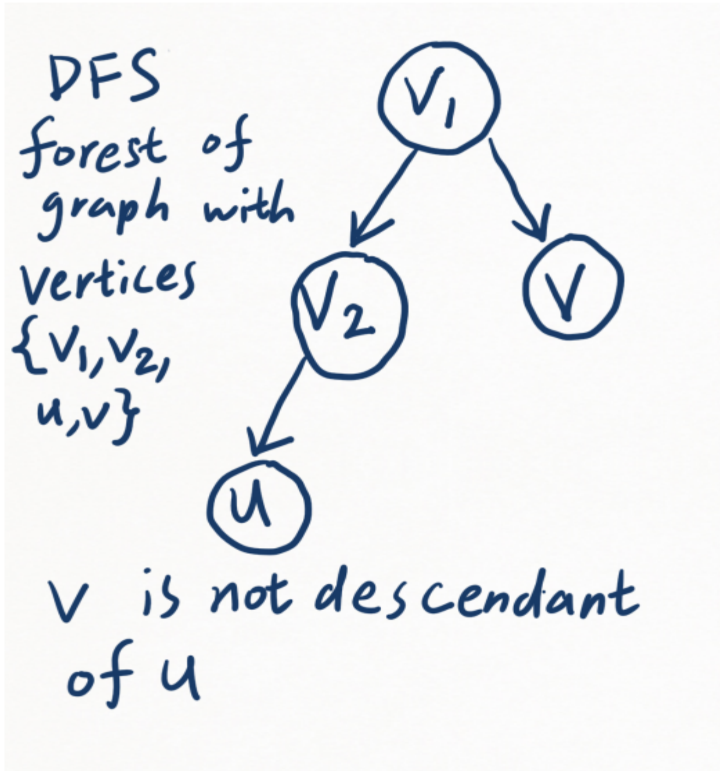
graph with vertices $\{v_1, v_2, u, v\}$ have edges $(v_1, v_2), (v_2, u), (u, v_1), (v_1, v)$ as shown below, there is a path from u to v through $u \rightarrow v_1 \rightarrow v$

$u.d = 3 < v.d = 6$ in depth-first search of G

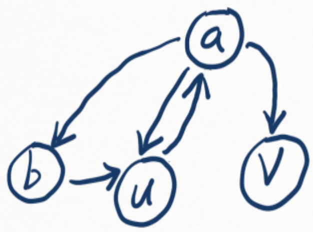


	discovery time	finish time
v1	1	8
v2	2	5
u	3	4
v	6	7

in the depth-first forest generated below for G, v is not a descendant of u

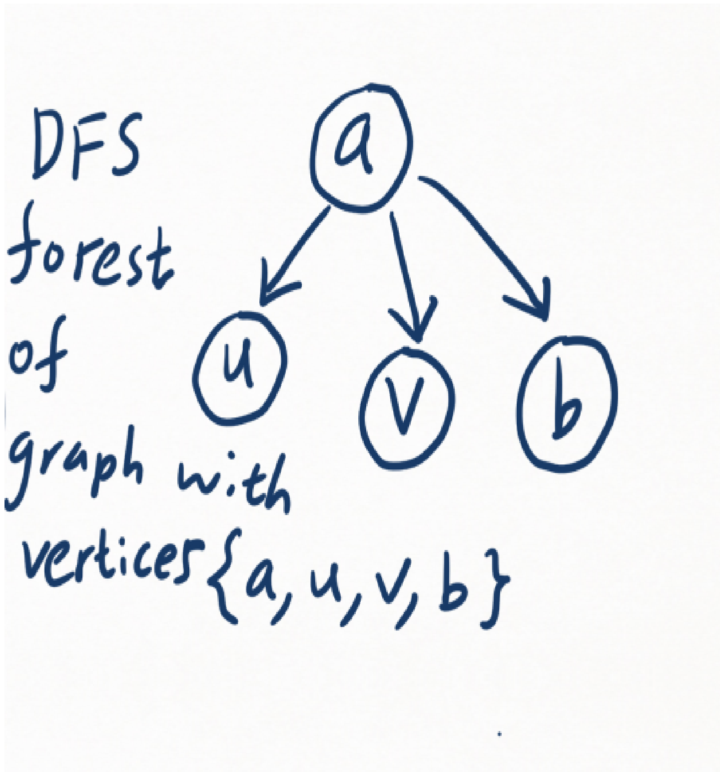


(b)
graph with vertices $\{a, b, u, v\}$ have edges $(a, u), (u, a), (a, v), (a, b)$ as shown below, there is a path from u to v through $u \rightarrow a \rightarrow v$
$v.d \not\leq u.f$ since $v.d = 4 > u.f = 3$ in depth-first search of G



a.d; u.d; u.f
v.d; v.f; b.d
b.f; a.f

	discovery time	finish time
a	1	8
u	2	3
v	4	5
b	6	7


DFS
forest
of
graph with
vertices {a, u, v, b}
a
u
v
b

Problem 4
algorithm:
stage 1:
let G_{red} be a graph containing only red edges
G_{blue} be a graph containing only blue edges
$G_{red} = (V, E_{red})$, in which V is the set containing all vertices in G; E_{red} is the set containing red edges of G
$G_{blue} = (V_{cp}, E_{blue\_cp})$, in which V_{cp} contains the copies of all vertices in V; E_{blue_cp} is the set containing blue edges that connects each vertex in V_{cp} instead of each vertex in V
denote each vertex $v \in V$ and each vertex in $v_{cp} \in V_{cp}$ with an index such that

$$V = \bigcup_{i=0}^{|V|-1} \{v_i\}$$
$$V_{cp} = \bigcup_{i=0}^{|V|-1} \{v_{cp\_i}\}$$
$$v_{cp\_i} \text{ is a copy of } v_i$$

stage 2:
call a vertex a transitional vertex if there exist at least an incoming edge going into the vertex and at least an outgoing edge coming out of the vertex. to find transitional vertices, allocate a 2 dimensional array *trans* that has 2 rows and $|V|$ columns, where

```
bool trans[2][|V|];
memset(trans, false, sizeof(trans));
```

$\forall idx \in [0, |V| - 1] \cap \mathbb{N}$
$$trans[0][idx] = \begin{cases} \text{true} & \text{if vertex } v_{idx} \text{ has an incoming red edge} \\ \text{false} & \text{otherwise} \end{cases}$$
$$trans[1][idx] = \begin{cases} \text{true} & \text{if vertex } v_{idx} \text{ has an outgoing blue edge} \\ \text{false} & \text{otherwise} \end{cases}$$

stage 3:

```
for each directed edge (v_i, v_j) ∈ G.E
{
    if((v_i, v_j) is a red edge)
        trans[0][j] = true;
    if((v_i, v_j) is a blue edge)
        trans[1][i] = true;
}
```

stage 4:

```
vector<edge> E_pink;
for(int j = 0; j < |V|; ++j)
{
    if(trans[0][j] && trans[1][j])
    {
        edge new_pink_edge = new pink edge (v_j, v_cp_j);
        E_pink.push_back(new_pink_edge);
    }
}
E_pink.push_back(new pink edge (s, copy of s));
E_pink.push_back(new pink edge (t, copy of t));
```

stage 5:
let $G_{big} = \{V \cup V_{cp},\ \ E_{red} \cup E_{blue} \cup E_{pink}\}$
use function P with to find the shortest path from vertex s to t_{cp}, then output a sequence of edges or a sequence of vertices, assume the output is a sequence of vertices

stage 6:
traverse through the sequence of vertices, keep track of the vertex immediately before the current vertex, then if the previous vertex and the current vertex forms a pair of v_i and v_{cp_i}, then this is a pink edge that we cre-

ated, so remove the pink edge; and during the traversal, replace every vertex $v_{cp\_i} \in V_{cp}$ with its corresponding vertex $v_i \in V$. then output the modified sequence that is the shortest path from vertex s to t

time complexity:
$T(\text{stage 1}) = |E| + |V|$; traverse through edges and separate them into the red graph and the blue graph takes $|E|$. then $|V|$ is required to create the copies $v_{cp\_i} \in V_{cp}$

$T(\text{stage 2}) = 1$; allocate a 2 dimensional array $trans$ and initialize it requires constant time

$T(\text{stage 3}) = |E|$; traverse all edges and marking them as either red or blue in the 2 dimensional array $trans$, each mark requires constant time

$T(\text{stage 4}) = |V| + 2$; for loop has number of iterations equal to $|V|$ and each iteration requires constant if tests; in addition, 2 pink edges $(s,\ \text{copy of } s), (t,\ \text{copy of } t)$ are added to the E_{pink} set

$T(\text{stage 5}) = O(|E|) + T(P)$; create the new graph G_{big} potentially requires allocating a new adjacency list and then copying all edges in $E_{red} \cup E_{blue} \cup E_{pink}$ to G_{big}'s adjacency list which may take $O(|E|)$, then 1 call to P takes $T(P)$

$T(\text{stage 6}) = O(|E|)$; traversing the the optimal path output of P for which its number of edges is at most $O(|E|)$

$$\begin{aligned}
\text{total time complexity} &= \sum_{i=1}^{6} (\text{stage } i) \\
&= |E| + |V| + 1 + |E| + |V| + 2 + O(|E|) + T(P) + O(|E|) \\
&= O(|E| + |V|) + T(P)
\end{aligned}$$