sd4175

P1

3, 5, 7, 8, 12, 14, 14, 15, 18, 21, 35

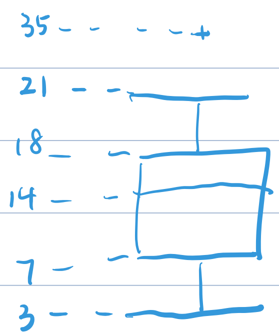
lower quartile: $K = (11+1) \cdot 0.25 = 3 \Rightarrow 7$

upper quartile: $K = (11+1) \cdot 0.75 = 9 \Rightarrow 18$

$IQR = 11$

$UQ + 1.5 IQR = 18 + 1.5 \cdot 11 = 34.5$

$LQ - 1.5 IQR = 7 - 1.5 \cdot 11 = -9.5$



P2

proof.

$E[T_2] = E[T_1 + W] = E[T_1] + \underbrace{E[W]}_{\text{zero mean}} = E[T_1]$

since T_1 is unbiased, $E[T_1] = \theta = E[T_2]$

$\therefore T_2$ is unbiased estimator for θ

2. $E[T_2] = E\left[\frac{T_1 - b}{a}\right] = \frac{1}{a}\left(E[T_1] - E(b)\right]$

$= \frac{1}{a}\ (a\theta + b - b)$

$= \theta$

$\therefore T_2$ is unbiased estimator for θ

P3

$\because X_i \sim U(0, \theta)$

$f(x_i) = \frac{1}{\theta}$

$F_T(t) = P(T < t) = P(\max(X_1, \cdots X_n) < t)$

$= P(X_1 < t)\ P(X_2 < t) \cdots P(X_n < t)$

$= \left(\frac{t}{\theta}\right)^n$

$$f_T(t) = \begin{cases} \frac{n t^{n-1}}{\theta^n}, & 0 < t < \theta \\ 0, & \text{otherwise} \end{cases}$$

$E[T] = \int_0^{\theta} t \cdot \frac{n t^{n-1}}{\theta^n}\, dt$

$= \int_0^{\theta} \frac{n}{\theta^n} \cdot t^n dt = \left[\frac{t^{n+1}}{n+1}\right]_0^{\theta} \frac{n}{\theta^n} = \frac{n}{n+1}\theta$

$\therefore B(T) = E[T] - \theta = -\frac{\theta}{n+1}$

P4

1.

p.m.f.

$$P_X(X_i) = \begin{cases} \frac{1}{6}\mu + \frac{5}{6}(1-\mu) = \frac{5}{6} - \frac{2}{3}\mu & \text{for answer is "yes"} \\ \frac{1}{6}(1-\mu) + \frac{5}{6}\mu = \frac{1}{6} + \frac{2}{3}\mu & \text{for answer is "no"} \end{cases}$$

2. $E[T_n] = E[\frac{5}{4} - \frac{3}{2}\bar{X}_n]$

for "yes", $x = 1$; for "no", $x = 0$

$$E[\bar{X}_n] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = E[X_i] = \frac{5}{6} - \frac{2}{3}\mu$$

$$E[T_n] = \frac{5}{4} - \frac{3}{2}E[\bar{X}_n] = \frac{5}{4} - \frac{3}{2}\left(\frac{5}{6} - \frac{2}{3}\mu\right) = \frac{5}{4} - \frac{5}{4} + \mu = \mu$$

$\therefore$ T_n is unbiased estimator

3.

$$Var(T_n) = Var\left(\frac{5}{4} - \frac{3}{2}\bar{X}_n\right) = Var\left(-\frac{3}{2}\bar{X}_n\right) = \frac{9}{4}Var(\bar{X}_n)$$

$$Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n}Var(X_i)$$

$$= \frac{1}{n}\left[\frac{5}{6} - \frac{2}{3}\mu - \left(\frac{5}{6} - \frac{2}{3}\mu\right)^2\right]$$

$$= \frac{1}{n}\left(\frac{5}{36} + \frac{4}{9}\mu - \frac{4}{9}\mu^2\right)$$

$$= \frac{5 + 16\mu - 16\mu^2}{36n}$$

$\therefore \ Var(T_n) = \frac{7}{4} Var(\bar{x}_n) = \frac{5+16\mu-16\mu^2}{16n}$

P5

$$f_\theta(x) = \frac{2}{\sqrt{\pi}\,\theta^{3/2}} x^2 e^{-x^2/\theta}, \theta > 0$$

1. $L(\theta) = f_\theta(a_1) f_\theta(a_2) \cdots f_\theta(a_n)$

$$= \prod_{i=1}^{n} f_\theta(a_i) = \prod_{i=1}^{n} \frac{2}{\sqrt{\pi}\,\theta^{3/2}} x_i^2 \cdot e^{-x_i^2/\theta}$$

$$= \frac{2}{\sqrt{\pi}\,\theta^{3/2}} \prod_{i=1}^{n} x_i^2 \cdot e^{-x_i^2/\theta}$$

$$l(\theta) = \ln 2 - \frac{1}{2}\ln(\pi) - \frac{3}{2} n \ln(\theta) + \sum_{i=1}^{n} \left[2\ln(x_i) - \frac{x_i^2}{\theta} \ln(e) \right]$$

$$\frac{dl}{d\theta} = 0 \Rightarrow \quad -\frac{3n}{2\theta} + \sum_{i=1}^{n} \frac{x_i^2}{\theta^2} = 0$$

$$\frac{3}{2} n\theta = \sum_{i=1}^{n} x_i^2$$

$$\hat{\theta} = \frac{2}{3n} \sum_{i=1}^{n} x_i^2$$

$$\theta = \frac{2}{3n} \sum_{i=1}^{n} a_i^2$$

2.

$$E[\hat{\theta}] = E\left[\frac{2}{3n} \sum_{i=1}^{n} a_i^2\right] = \frac{2}{3n} E\left[\sum_{i=1}^{n} x_i^2\right] = \frac{2}{3n} \sum_{i=1}^{n} E[x_i^2]$$

since $a_1, \cdots a_n$ are iid

$$E[X_i^2] = \int_{-\infty}^{+\infty} \frac{2}{\sqrt{\pi}\,\theta^{3/2}} x^2 \cdot x^2 e^{-\frac{x^2}{\theta}} dx$$

integral by part + integral calculator

$$= \frac{3}{2}\theta$$

$$\therefore E[\hat{\theta}] = \frac{2}{3n} \sum_{i=1}^{n} \frac{3}{2}\theta = \theta$$

$\therefore \hat{\theta}$ is unbiased

3.

$$Var(\hat{\theta}) = Var\left(\frac{2}{3n}\sum_{i=1}^{n} X_i^2\right) = \frac{4}{9n^2} Var\left(\sum_{i=1}^{n} X_i^2\right) = \frac{4}{9n^2} \underbrace{\sum_{i=1}^{n}}_{\text{since } a_1 \cdots a_n \text{ is i.i.d}} Var(X_i^2)$$

$$Var(X^2) = E(X^4) - E(X^2)^2 = \int_{-\infty}^{\infty} x^4 f_\theta(x)dx - \left(\frac{3}{2}\theta\right)^2$$

$$= \int_{-\infty}^{\infty} x^4 \left(\frac{2}{\sqrt{\pi}\,\theta^{3/2}} x^2 e^{-x^2/\theta}\right)dx - \frac{9}{4}\theta^2$$

by lecture and online calculator $= \frac{15}{4}\theta^2 - \frac{9}{4}\theta^2$

$$= \frac{3}{2}\theta^2$$

$$\therefore Var(\hat{\theta}) = \frac{4}{9n^2} \sum_{i=1}^{n} \frac{3}{2}\theta^2$$

$$= \frac{4}{9n} \cdot \frac{3}{2}\theta^2 = \frac{2\theta^2}{3n}$$

As n increase, $Var(\hat{\theta})$ will decrease

$$\lim_{n \to \infty} Var(\hat{\theta}) = 0$$