P1 sd4175

$P(N_{(0,1]}=0,\ N_{(1,2]}=2,\ N_{(2,4]}=1)$

$= e^{-\lambda} \cdot \frac{\lambda^2}{2!} e^{-\lambda} \cdot 2\lambda \cdot e^{-2\lambda} = \lambda^3 e^{-4\lambda}$

$P(N_{(0,1]}=1,\ N_{(1,2]}=1,\ N_{(2,4]}=2)$

$= \lambda e^{-\lambda} \cdot \lambda e^{-\lambda} \cdot \frac{(2\lambda)^2}{2!} e^{-2\lambda} = \frac{2\lambda^4}{2} e^{-4\lambda}$

$P(N_{(0,1]}=2,\ N_{(1,2]}=0,\ N_{(2,4]}=3)$

$= \frac{\lambda^2}{2!} \cdot e^{-\lambda} \cdot e^{-\lambda} \cdot \frac{(2\lambda)^3}{6} e^{-2\lambda} = \frac{2\lambda^5}{3} e^{-4\lambda}$

$\therefore P = (\lambda^3 + 2\lambda^4 + \frac{2}{3}\lambda^5) e^{-4\lambda}$

P2

$Z = X + Y$, $E[X] = \lambda$, $E[Y] = \mu$

$$P_Z(k) = \sum_{i=0}^{k} P(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i)$$

$$= \sum_{i=0}^{k} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \cdot \frac{(\mu t)^{k-i}}{(k-i)!} e^{-\mu t}$$

$$= e^{-(\lambda+\mu)t} \sum_{i=0}^{k} \frac{(\lambda t)^i \cdot (\mu t)^{k-i}}{i! (k-i)!}$$

$$= \frac{e^{-(\lambda+\mu)t} \cdot t^k}{} (\lambda+\mu)^k$$

$$k!$$

$$= \frac{[(\lambda+\mu)t]^k}{k!} e^{-(\lambda+\mu)t}$$

$\lambda+\mu = 8.392 + 7.854 = 16.246$ / 10 second

$$P(Z=10) = \frac{16.246^{10}}{10!} e^{-16.246} = 0.0311$$

P3

$\because$ binomial distribution, $E[X]=np$

1. By Markov's inequality

$$E[X] \geq a \cdot P(X \geq a)$$

$$P(X \geq kn) \leq \frac{E[X]}{kn} = \frac{np}{kn} = \frac{p}{k}$$

$\therefore$ Upper bound $= \frac{p}{k}$

2. By Chebyshev's inequality

$$P(|X-E[X]| \geq a) \leq \frac{Var(X)}{a^2}$$

$$P(X \geq kn) = P(|X-np| \geq kn-np) = P(|X-np| \geq n(k-p))$$

$\because$ binomial distribution, $\sigma = \sqrt{Var(X)} = \sqrt{np(1-p)}$

$$P(|X-np| \geq n(k-p)) \leq \frac{np(1-p)}{n^2(k-p)^2} = \frac{p-p^2}{n(k-p)^2}$$

$\therefore P(X \geq kn) \leq \frac{p-p^2}{n(k-p)^2}$

$\therefore$ Upper bound $= \frac{p-p^2}{n(k-p)^2}$

3.

When $p = \frac{1}{2}, k = \frac{3}{4}$, $\frac{p-p^2}{n(k-p)^2} = \frac{\frac{1}{2}-(\frac{1}{2})^2}{n(\frac{3}{4}-\frac{1}{2})^2} = \frac{4}{n} \rightarrow$ Chebyshev

$\frac{p}{k} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \rightarrow$ Markov

$\frac{4}{n} = \frac{2}{3}$

$n = 6$

For large numbers $n > 6$, Chebyshevs inequality gives a tighter bound ; otherwise, Markov's inequality gives a tigher bound

P4

$E[U_i] = 0$, $Var(U_i) < 3$

$E[T_i] = E[T] + E[U_i] = E[T]$

$$Var(\bar{T_i}) = Var\left(\frac{\sum_{i=1}^{n} T_i}{n}\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(T_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} [Var(T) + Var(U_i)]$$

T is constant

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(U_i)$$

$$= \frac{3}{n}$$

By chebyshev's inequality

$$P(|\bar{T}_i - E[T]| > \tfrac{1}{2}) \le \frac{1}{\frac{1}{4}} Var(\bar{T}_i) = 4 \cdot \frac{3}{n} = \frac{12}{n} \le 0.1$$

$$n \ge 120$$

$\therefore$ at least 120 measurements

P5

$$E[X] = \frac{1}{6}\sum_{i=1}^{6} i = 3.5$$

$$E[X^2] = \frac{1}{6}\sum_{i=1}^{6}(i)^2 = \frac{91}{6}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{35}{12}$$

$$E[\bar{X}_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{100}\sum_{i=1}^{100} 3.5 = 3.5$$

$$Var(\bar{X_i}) = \frac{1}{100^2} Var(\sum_{i=1}^{100} X_i)$$

$$= \frac{1}{100^2} \cdot 100 \cdot \frac{35}{12}$$

$$= \frac{7}{240}$$

$$|3.5 - 3.2| = |3.5 - 3.8| = 0.3$$

By Chebyshev's inequality:

$$P(|X-E[X]| \geq 0.3) \leq \frac{1}{(0.3)^2} Var(X) = \frac{1}{0.09} \cdot \frac{7}{240} = 0.324$$

$$P(3.2 \leq X \leq 3.8) = 1 - P(|X-3.5| \geq 0.3) = 1 - 0.324 = 0.676$$