Xi Liu, xl3504, Problem Set 8

Problem 1

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\begin{aligned}
E[g(X,Y)] &= \sum_i \sum_j g(a_i, b_j) P(X = a_i, Y = b_j) \\
g(x,y) &= xy \\
E[XY] &= \sum_i \sum_j a_i b_j P(X = a_i, Y = b_j) \\
&= (0)(100)(0.2) + (100)(100)(0.1) + (200)(100)(0.2) \\
&\quad + (0)(250)(0.05) + (100)(250)(0.15) + (200)(250)(0.3) \\
&= 23750
\end{aligned}$$

$$\begin{aligned}
E[X] &= \sum_i a_i P(X = a_i) \\
&= (0)(0.2 + 0.05) + (100)(0.1 + 0.15) + (200)(0.2 + 0.3) \\
&= 125
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \sum_j b_j P(Y = b_j) \\
&= (100)(0.2 + 0.1 + 0.2) + (250)(0.05 + 0.15 + 0.3) \\
&= 175
\end{aligned}$$

$$\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= 23750 - (125)(175) \\
&= \boxed{1875}
\end{aligned}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$\begin{aligned}
E[g(X)] &= \sum_i g(a_i)P(X = a_i) \\
g(x) &= x^2 \\
E[X^2] &= \sum_i a_i^2 P(X = a_i) \\
&= (0)^2(0.2 + 0.05) + (100)^2(0.1 + 0.15) + (200)^2(0.2 + 0.3) \\
&= 22500
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= 22500 - (125)^2 \\
&= 6875
\end{aligned}$$

$$Var(Y) = E[Y^2] - E[Y]^2$$

$$\begin{aligned}
E[g(Y)] &= \sum_j g(b_j)P(Y = b_j) \\
g(y) &= y^2 \\
E[Y^2] &= \sum_j b_j^2 P(Y = b_j) \\
&= (100)^2(0.2 + 0.1 + 0.2) + (250)^2(0.05 + 0.15 + 0.3) \\
&= 36250
\end{aligned}$$

$$
\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= 36250 - (175)^2 \\
&= 5625
\end{aligned}
$$

$$
\begin{aligned}
\rho(X, Y) &= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{1875}{\sqrt{(6875)(5625)}} \\
&= \frac{1875}{1875\sqrt{11}} \\
&= \frac{1}{\sqrt{11}} \\
&= \boxed{\frac{\sqrt{11}}{11}}
\end{aligned}
$$

Problem 2

1.

let $area$ be the area of triangle

$$\exists a \in \mathbb{R}$$
$$f_{X,Y}(x,y) = \begin{cases} a & \text{if } (x,y) \in T \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
1 &= \int_0^1 \int_0^{1-x} f_{X,Y}(x,y) dy dx \\
&= \int_0^1 \int_0^{1-x} a \; dy dx \\
&= \int_0^1 [ay]_{y=0}^{y=1-x} dx \\
&= \int_0^1 a(1-x) dx \\
&= a \left[x - \frac{x^2}{2} \right]_0^1 \\
&= a \left[1 - \frac{1}{2} \right] \\
&= \frac{a}{2}
\end{aligned}$$

$$\begin{aligned}
1 &= \frac{a}{2} \\
a &= 2
\end{aligned}$$

alternatively

$$\begin{aligned}
area &= \frac{(1-0)(1-0)}{2} = \frac{1}{2} \\
a &= \frac{1}{area} = \frac{1}{1/2} = 2
\end{aligned}$$

$$\boxed{f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

2.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_0^{1-x} 2dy \\ &= [2y]_0^{1-x} \\ &= 2(1-x) \\ &= 2-2x \end{aligned}$$

$$\boxed{f_X(x) = \begin{cases} 2-2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{1-y} 2dx \\ &= [2x]_0^{1-y} \\ &= 2(1-y) \\ &= 2-2y \end{aligned}$$

$$\boxed{f_Y(y) = \begin{cases} 2-2y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

3.

$$\begin{aligned} f_X(x)f_Y(y) &= (2-2x)(2-2y) \\ &= 4-4y-4x+4xy \\ f_{X,Y}(x,y) = 2 &\neq f_X(x)f_Y(y) = 4-4y-4x+4xy \end{aligned}$$

for example if $x=0$, $y=0$, $f_X(x)f_Y(y) = 4 \neq f_{X,Y}(x,y) = 2$
so X and Y are not independent random variables

4.

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$
\begin{aligned}
E[g(X,Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy \\
g(x,y) &= xy; \quad 0 \le x \le 1-y; \quad 0 \le y \le 1, \quad f_{X,Y}(x,y) = 2 \\
E[XY] &= 2\int_0^1 \int_0^{1-y} xy \; dxdy \\
&= 2\int_0^1 \left[\frac{x^2y}{2}\right]_{x=0}^{x=1-y} dy \\
&= \int_0^1 (1-y)^2 y dy \\
&= \int_0^1 (1-2y+y^2) y dy \\
&= \int_0^1 (y-2y^2+y^3) dy \\
&= \left[\frac{y}{2} - \frac{1}{3}y^3 + \frac{1}{4}y^4\right]_0^1 \\
&= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\
&= \frac{1}{12}
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\
f_X(x) &= 2-2x; \quad 0 \le x \le 1 \\
E[X] &= \int_0^1 x(2-2x)\; dx \\
&= \int_0^1 (2x-2x^2)\; dx \\
&= \left[x^2 - \frac{2}{3}x^3\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\
f_Y(y) &= 2 - 2y; \quad 0 \le y \le 1 \\
E[Y] &= \int_0^1 y(2-2y)\, dy \\
&= \int_0^1 (2y - 2y^2)\, dy \\
&= \left[y^2 - \frac{2}{3} y^3 \right]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{12} - \frac{1}{3} \cdot \frac{1}{3} \\
&= \boxed{-\frac{1}{36}}
\end{aligned}
$$

Problem 3

1.

since there is no replacement, $x = y$ is not possible

$$\begin{aligned}
p_Z(z) &= P(Z = z) \\
&= \sum_{x<y} P(Z = z) + \sum_{x>y} P(Z = z) \\
&= \sum_{x=1}^{n-z} \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right) + \sum_{y=1}^{n-z} \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right) \\
&= \frac{n-z-1+1}{n(n-1)} + \frac{n-z-1+1}{n(n-1)} \\
&= \frac{2n-2z}{n(n-1)}
\end{aligned}$$

$$\boxed{P(Z = z) = \frac{2n-2z}{n(n-1)} \quad \text{if } z \in [1, n-1] \cap \mathbb{N}}$$

2.

$$
\begin{aligned}
E[Z] &= \sum_{z=1}^{n-1} zP(Z=z) \\
&= \sum_{z=1}^{n-1} z\left(\frac{2n-2z}{n(n-1)}\right) \\
&= \sum_{z=1}^{n-1} \frac{2nz-2z^2}{n(n-1)} \\
&= \left(\frac{2n}{n(n-1)}\sum_{z=1}^{n-1} z\right) - \left(\frac{2}{n(n-1)}\sum_{z=1}^{n-1} z^2\right) \\
&/*\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\ */ \\
&= \frac{2n}{n(n-1)} \cdot \frac{(n-1)((n-1)+1)}{2} \\
&\quad - \frac{2}{n(n-1)} \cdot \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\
&= \frac{2}{n-1} \cdot \frac{n(n-1)}{2} - \frac{2}{n(n-1)} \cdot \frac{n(n-1)(2n-1)}{6} \\
&= \boxed{n - \frac{2n-1}{3}}
\end{aligned}
$$

Problem 4

1.

let $N_{[x,t]}$ be the total number of arrivals in the interval $[x,t]$ for the Poisson process

$$P(N_{[x,t]} = k) = \frac{(\lambda(t-x))^k}{k!} e^{-\lambda(t-x)}$$

$$\begin{aligned} P(N_{(3,8]} = 0) &= \frac{(0.5(8-3))^0}{0!} e^{-0.5(8-3)} \\ &= \boxed{e^{-5/2}} \end{aligned}$$

2.

$$\begin{aligned} P(N_{(0,1]} = 1) &= \frac{(0.5(1-0))^1}{1!} e^{-0.5(1-0)} \\ &= \frac{1}{2} e^{-1/2} \end{aligned}$$

$$\begin{aligned} P(N_{(1,2]} = 1) &= \frac{(0.5(2-1))^1}{1!} e^{-0.5(2-1)} \\ &= \frac{1}{2} e^{-1/2} \end{aligned}$$

$$\begin{aligned} P(N_{(2,3]} = 1) &= \frac{(0.5(3-2))^1}{1!} e^{-0.5(3-2)} \\ &= \frac{1}{2} e^{-1/2} \end{aligned}$$

$$
\begin{aligned}
P(N_{(3,4]} = 1) &= \frac{(0.5(4-3))^1}{1!} e^{-0.5(4-3)} \\
&= \frac{1}{2} e^{-1/2}
\end{aligned}
$$

since the 4 time intervals are disjoint

$$
\begin{aligned}
&P(N_{(0,1]} = 1, N_{(1,2]} = 1, N_{(2,3]} = 1, N_{(3,4]} = 1) \\
&= P(N_{(0,1]} = 1) P(N_{(1,2]} = 1) P(N_{(2,3]} = 1) P(N_{(3,4]} = 1) \\
&= \boxed{\left(\frac{1}{2} e^{-1/2}\right)^4} \\
&= \boxed{\frac{1}{16} e^{-2}}
\end{aligned}
$$

Problem 5

$$Cov(N_{t_1}, N_{t_2}) = E[N_{t_1} N_{t_2}] - E[N_{t_1}]E[N_{t_2}]$$

$$\begin{aligned}
P(N_{t_1} = k) &= \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1} \\
P(N_{t_2} = k) &= \frac{(\lambda t_2)^k}{k!} e^{-\lambda t_2} \\
E[N_{t_1}] &= \lambda t_1 \\
E[N_{t_2}] &= \lambda t_2 \\
Var(N_{t_1}) &= \lambda t_1
\end{aligned}$$

$$\begin{aligned}
E[N_{t_1} N_{t_2}] &= E[N_{t_1}(N_{t_2} - N_{t_1} + N_{t_1})] \\
&= E[N_{t_1}(N_{t_2} - N_{t_1}) + N_{t_1}^2] \\
&= E[N_{t_1}(N_{t_2} - N_{t_1})] + E[N_{t_1}^2]
\end{aligned}$$

/* since $(0, t_1]$ and $(t_1, t_2]$ are disjoint time intervals,
N_{t_1} and $N_{t_2} - N_{t_1}$ are independent random variables */

$$= E[N_{t_1}]E[N_{t_2} - N_{t_1}] + E[N_{t_1}^2]$$

$$Var(N_{t_1}) = E[N_{t_1}^2] - E[N_{t_1}]^2$$

$$\begin{aligned}
E[N_{t_1}^2] &= Var(N_{t_1}) + E[N_{t_1}]^2 \\
&= \lambda t_1 + (\lambda t_1)^2
\end{aligned}$$

$$\begin{aligned}
E[N_{t_1} N_{t_2}] &= E[N_{t_1}]E[N_{t_2} - N_{t_1}] + E[N_{t_1}^2] \\
&= (\lambda t_1)(\lambda(t_2 - t_1)) + (\lambda t_1 + (\lambda t_1)^2) \\
&= (\lambda t_1)(\lambda(t_2 - t_1)) + \lambda t_1 + (\lambda t_1)^2 \\
&= \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda t_1 + \lambda^2 t_1^2 \\
&= \lambda^2 t_1 t_2 + \lambda t_1
\end{aligned}$$

$$
\begin{aligned}
Cov(N_{t_1}, N_{t_2}) &= E[N_{t_1} N_{t_2}] - E[N_{t_1}]E[N_{t_2}] \\
&= \lambda^2 t_1 t_2 + \lambda t_1 - (\lambda t_1)(\lambda t_2) \\
&= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2 \\
&= \boxed{\lambda t_1}
\end{aligned}
$$