Xi Liu, xl3504, Problem Set 10

Problem 1
step 1:
sort the data set in ascending order

```
#include <stdio.h>
#include <stdlib.h>

int cmp(const void * a, const void * b)
{
        return *(int *)a - *(int *)b;
}

int main()
{
        int vals[] = {3, 7, 8, 5, 12, 14, 21, 15, 35, 18, 14};
        int n = sizeof(vals) / sizeof(*vals);
        qsort(vals, n, sizeof(int), cmp);
        printf("n = %d\n", n);
        for(int i = 0; i < n; ++i)
                printf("%d, ", vals[i]);
}
```

sorted data set is:
3, 5, 7, 8, 12, 14,
14, 15, 18, 21, 35

step 2:
calculate $q_{0.25}$, $q_{0.5}$, $q_{0.75}$, min, max, last point within $q_{0.25} - 1.5IQR$, and last point within $q_{0.75} + 1.5IQR$

for a quantile q_p, $p(n+1) = k + \alpha$
$k = \lfloor p(n+1) \rfloor; \quad \alpha = p(n+1) - k$
$q_n(p) = x_k + \alpha(x_{k+1} - x_k)$

n = 11 for the above data set

$q_{0.25}:$
$p(n+1) = 0.25(11+1) = 3$
$k = \lfloor 3 \rfloor = 3$
$\alpha = p(n+1) - k = 3 - 3 = 0$
$q_{0.25} = x_3 = 7$

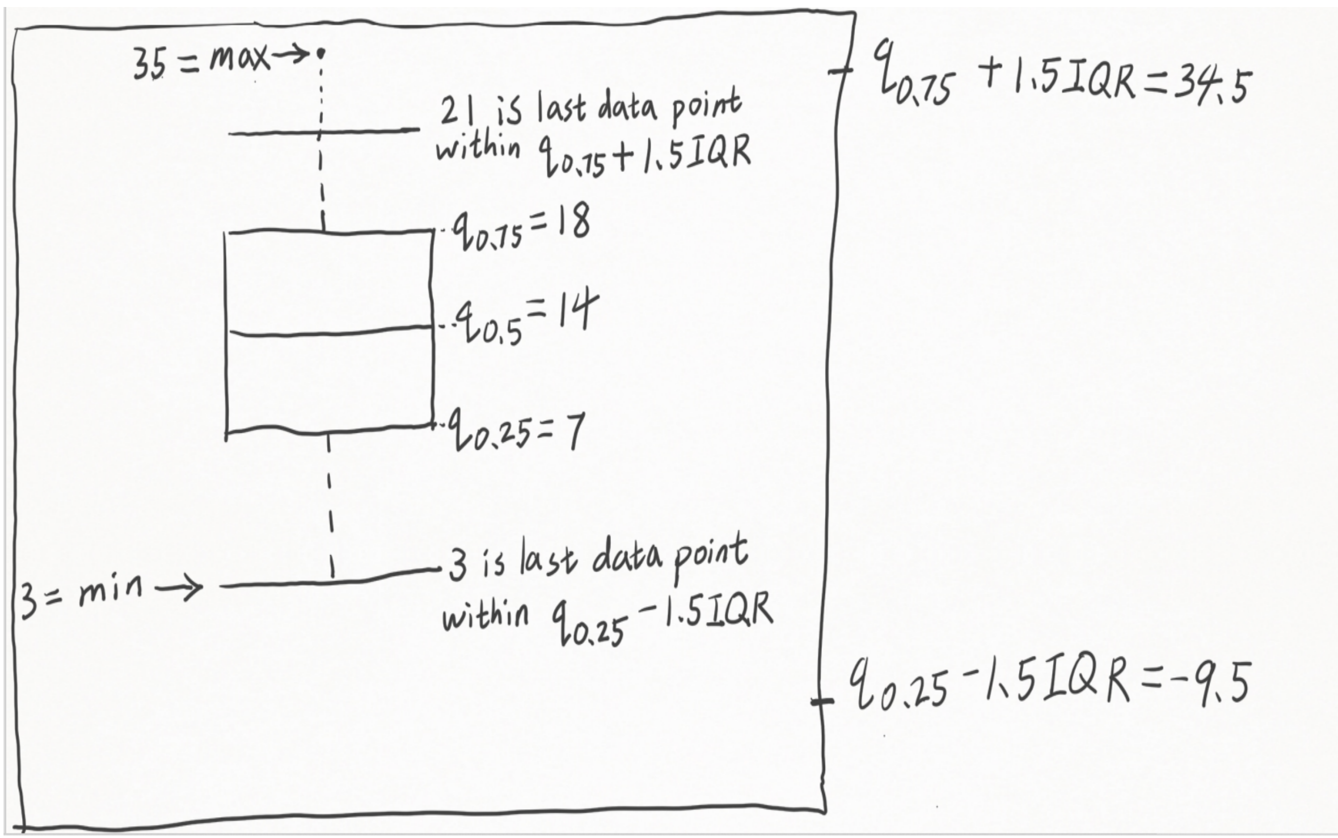
$q_{0.5}:$
$p(n+1) = 0.5(11+1) = 6$
$k = \lfloor 6 \rfloor = 6$
$\alpha = p(n+1) - k = 6 - 6 = 0$
$q_{0.5} = x_6 = 14$

$q_{0.75}:$
$p(n+1) = 0.75(11+1) = 9$
$k = \lfloor 9 \rfloor = 9$
$\alpha = p(n+1) - k = 9 - 9 = 0$
$q_{0.75} = x_9 = 18$

$min = 3$
$max = 35$

$IQR = q_{0.75} - q_{0.25} = 18 - 7 = 11$
$q_{0.25} - 1.5IQR = 7 - 1.5(11) = -9.5$
$q_{0.75} + 1.5IQR = 18 + 1.5(11) = 34.5$
last point within $q_{0.25} - 1.5IQR = -9.5$ is 3
last point within $q_{0.75} + 1.5IQR = 34.5$ is 21

step 3:
box-and-whisker plot:



Problem 2

1.

T_1 is unbiased, so $E[T_1] = \theta$

$E[W] = 0$

$T_2 = T_1 + W$

$E[T_2] = E[T_1 + W] = E[T_1] + E[W] = \theta + 0 = \theta$

$E[T_2] = \theta$, so T_2 is an unbiased estimator for θ

2.

$$\begin{aligned}
E[T_2] &= E\left[\frac{T_1 - b}{a}\right] \\
&= \frac{1}{a}E[T_1 - b] \\
&= \frac{1}{a}(E[T_1] - E[b]) \\
&= \frac{1}{a}(E[T_1] - b) \quad \text{/* since } b \text{ is a constant */} \\
&= \frac{1}{a}(a\theta + b - b) \\
&= \frac{1}{a}(a\theta) \\
&= \theta
\end{aligned}$$

$E[T_2] = \theta$, so T_2 is an unbiased estimator for θ

Problem 3

uniform distribution probability density function f is

$f(x) = \begin{cases} 0 & \text{if } x \notin [a, b] \\ \frac{1}{b-a} & \text{if } x \in [a, b] \end{cases}$

$f_X(x) = \begin{cases} 0 & \text{if } x \notin [0, \theta] \\ 1/\theta & \text{if } x \in [0, \theta] \end{cases}$

if $x \in [0, \theta],\ F_X(x) = \int_0^x f_X(x)dx = \int_0^x \frac{1}{\theta}dx = \frac{1}{\theta}\int_0^x dx = \frac{x}{\theta}$

$F_X(x) = \begin{cases} 0 & \text{if } x \notin [0, \theta] \\ x/\theta & \text{if } x \in [0, \theta] \end{cases}$

$$\begin{aligned} F_T(t) = P(T \leq t) &= P(X_1 \leq t, X_2 \leq t, ..., X_n \leq t) \\ &= \prod_{i=1}^{n} P(X_i \leq t) \\ &= \prod_{i=1}^{n} F_{X_i}(t) \\ &= \prod_{i=1}^{n} \left(\frac{t}{\theta}\right) \\ &= \left(\frac{t}{\theta}\right)^n \end{aligned}$$

$$\begin{aligned} f_T(t) &= \frac{d}{dt}(F_T(t)) \\ &= \frac{d}{dt}\left(\left(\frac{t}{\theta}\right)^n\right) \\ &= n\left(\frac{t}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) \end{aligned}$$

$$f_T(t) = \begin{cases} 0 & \text{if } t \notin [0, \theta] \\ n\left(\frac{t}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) & \text{if } t \in [0, \theta] \end{cases}$$

$$
\begin{aligned}
E[T] &= \int_0^\theta t f_T(t) dt \\
&= \int_0^\theta t \left(n \left(\frac{t}{\theta} \right)^{n-1} \left(\frac{1}{\theta} \right) \right) dt \\
&= \frac{n}{\theta^n} \int_0^\theta t^n dt \\
&= \frac{n}{\theta^n} \left[\frac{t^{n+1}}{n+1} \right]_0^\theta \\
&= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} \\
&= \frac{n}{n+1} \theta
\end{aligned}
$$

$$
\begin{aligned}
B(T) &= E[T] - \theta \\
&= \frac{n}{n+1}\theta - \theta \\
&= \frac{n\theta - (n+1)\theta}{n+1} \\
&= \frac{n\theta - n\theta - \theta}{n+1} \\
&= \boxed{-\frac{\theta}{n+1}}
\end{aligned}
$$

Problem 4

1.

probability to respond yes = probability of responding yes without lying + probability of responding yes with lying = $\frac{1}{6}\mu + \frac{5}{6}(1-\mu) = \frac{5}{6} - \frac{4}{6}\mu$

probability to respond no = probability of responding no without lying + probability of responding no with lying = $\frac{1}{6}(1-\mu) + \frac{5}{6}\mu = \frac{1}{6} + \frac{4}{6}\mu$

$$X_i = \begin{cases} 1 & \text{if the response is yes} \\ 0 & \text{if the response is no} \end{cases}$$

$$\boxed{P(X_i = x) = \begin{cases} \frac{5}{6} - \frac{4}{6}\mu & \text{if } x = 1 \\ \frac{1}{6} + \frac{4}{6}\mu & \text{if } x = 0 \end{cases}}$$

$$P(X_i = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

$$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p = \frac{5}{6} - \frac{4}{6}\mu$$

2.

$$
\begin{aligned}
E[T_n] &= E\left[\frac{5}{4} - \frac{3}{2}\overline{X_n}\right] \\
&= \frac{5}{4} - \frac{3}{2}E[\overline{X_n}] \\
&= \frac{5}{4} - \frac{3}{2}E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] \\
&= \frac{5}{4} - \frac{3}{2}\frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] \\
&= \frac{5}{4} - \frac{3}{2}\frac{n}{n}E[X_i] \\
&= \frac{5}{4} - \frac{3}{2}E[X_i] \\
&= \frac{5}{4} - \frac{3}{2}p \\
&= \frac{5}{4} - \frac{3}{2}\left(\frac{5}{6} - \frac{4}{6}\mu\right) \\
&= \frac{5}{4} - \frac{5}{4} + \mu \\
&= \mu
\end{aligned}
$$

$E[T_n] = \mu$, so T_n is an unbiased estimator for μ

3.

$$
\begin{aligned}
Var(T_n) &= Var\left[\frac{5}{4} - \frac{3}{2}\overline{X_n}\right] \\
&= Var\left[-\frac{3}{2}\overline{X_n}\right] \\
&= \left(-\frac{3}{2}\right)^2 Var(\overline{X_n}) \\
&= \frac{9}{4}Var(\overline{X_n}) \\
&= \frac{9}{4}Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) \\
&= \frac{9}{4n^2}Var(\sum_{i=1}^{n} X_i) \\
&= \frac{9}{4n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{9n}{4n^2}Var(X_i) \\
&= \frac{9}{4n}Var(X_i) \\
&= \frac{9}{4n}p(1-p) \\
&= \frac{9}{4n}(\frac{5}{6} - \frac{4}{6}\mu)(1 - (\frac{5}{6} - \frac{4}{6}\mu)) \\
&= \frac{9}{4n}(\frac{5}{6} - \frac{4}{6}\mu)(\frac{1}{6} + \frac{4}{6}\mu) \\
&= \frac{9}{4n}(\frac{5}{36} + \frac{20}{36}\mu - \frac{4}{36}\mu - \frac{16}{36}\mu^2) \\
&= \frac{9}{4n}(\frac{5}{36} + \frac{16}{36}\mu - \frac{16}{36}\mu^2) \\
&= \frac{9}{144n}(5 + 16\mu - 16\mu^2) \\
&= \boxed{\frac{45}{144n} + \frac{\mu}{n} - \frac{\mu^2}{n}}
\end{aligned}
$$

Problem 5

1.

likelihood function $L(\theta)$ is

$$\begin{aligned}
L(\theta) &= f_{\theta,X_1,X_2,...,X_n}(a_1,a_2,...,a_n) \\
&= \prod_{i=1}^{n} f_\theta(a_i) \\
&= \prod_{i=1}^{n} \frac{2}{\sqrt{\pi}\theta^{3/2}} a_i^2 e^{-a_i^2/\theta} \\
&= \left(\frac{2}{\sqrt{\pi}\theta^{3/2}}\right)^n \left(\prod_{i=1}^{n} a_i^2\right) e^{-\sum_{i=1}^{n} a_i^2/\theta}
\end{aligned}$$

loglikelihood function $l(\theta)$ is

$$\begin{aligned}
l(\theta) &= \ln(L(\theta)) \\
&= \ln\left(\left(\frac{2}{\sqrt{\pi}\theta^{3/2}}\right)^n \left(\prod_{i=1}^{n} a_i^2\right) e^{-\sum_{i=1}^{n} a_i^2/\theta}\right) \\
&= \ln\left(\left(\frac{2}{\sqrt{\pi}\theta^{3/2}}\right)^n\right) + \ln\left(\prod_{i=1}^{n} a_i^2\right) + \ln\left(e^{-\sum_{i=1}^{n} a_i^2/\theta}\right) \\
&= n\ln\left(\frac{2}{\sqrt{\pi}\theta^{3/2}}\right) + \left(\sum_{i=1}^{n} \ln\left(a_i^2\right)\right) - \sum_{i=1}^{n} a_i^2/\theta
\end{aligned}$$

$$\begin{aligned}
\frac{dl}{d\theta} &= (-3/2)\frac{2}{\sqrt{\pi}}\theta^{-5/2} n \frac{1}{2/(\sqrt{\pi}\theta^{3/2})} + \sum_{i=1}^{n} a_i^2/\theta^2 \\
&= \frac{-3}{\sqrt{\pi}}\theta^{-5/2} n \frac{\sqrt{\pi}\theta^{3/2}}{2} + \sum_{i=1}^{n} a_i^2/\theta^2 \\
&= \frac{-3n}{2\theta} + \sum_{i=1}^{n} a_i^2/\theta^2
\end{aligned}$$

$$
\begin{aligned}
\frac{dl}{d\theta} &= 0 \\
\frac{-3n}{2\theta} + \sum_{i=1}^{n} a_i^2/\theta^2 &= 0 \\
\frac{-3n}{2\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} a_i^2 &= 0 \\
\frac{1}{\theta^2}\sum_{i=1}^{n} a_i^2 &= \frac{3n}{2\theta} \\
\frac{1}{\theta}\sum_{i=1}^{n} a_i^2 &= \frac{3n}{2} \\
\frac{1}{\theta} &= \frac{3n}{2\sum_{i=1}^{n} a_i^2} \\
\theta &= \frac{2\sum_{i=1}^{n} a_i^2}{3n}
\end{aligned}
$$

so the maximum likelihood estimate of θ is

$$
\hat{\theta} = \frac{2\sum_{i=1}^{n} a_i^2}{3n}
$$

2.

$$bias(\hat{\theta}) = E[\hat{\theta}] - \theta$$

$$\begin{aligned}
E[\hat{\theta}] &= E\left[\frac{2\sum_{i=1}^{n} a_i^2}{3n}\right] \\
&= \frac{2}{3n}E[\sum_{i=1}^{n} a_i^2] \\
&= \frac{2}{3n}\sum_{i=1}^{n} E[a_i^2] \\
&= \frac{2n}{3n}E[a_i^2] \\
&= \frac{2}{3}E[a_i^2]
\end{aligned}$$

$$\begin{aligned}
E[x^2] &= \int_{-\infty}^{\infty} x^2 f_\theta(x)dx \\
&= \int_{-\infty}^{\infty} x^2 \left(\frac{2}{\sqrt{\pi}\theta^{3/2}} x^2 e^{-x^2/\theta}\right) dx \\
&= \frac{2}{\sqrt{\pi}\theta^{3/2}} \int_{-\infty}^{\infty} x^4 e^{-x^2/\theta} dx
\end{aligned}$$

to calculate $\int_{-\infty}^{\infty} x^4 e^{-x^2/\theta} dx$:
a continuous random variable has a normal distribution with parameters μ and $\sigma^2 > 0$ if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
\text{/* set } \mu = 0; \qquad K &= \frac{1}{2\sigma^2}; \qquad \sigma = \frac{1}{\sqrt{2K}} \text{ */} \\
&= \frac{1}{\sqrt{2\pi(1/\sqrt{2K})^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-0}{1/(\sqrt{2K})}\right)^2} dx \\
&= \frac{1}{\sqrt{2\pi}(1/\sqrt{2K})} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\sqrt{2K}x\right)^2} dx \\
&= \frac{\sqrt{2K}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(2Kx^2\right)} dx \\
&= \sqrt{\frac{K}{\pi}} \int_{-\infty}^{\infty} e^{-Kx^2} dx = 1
\end{aligned}
$$

/* since $\int_{-\infty}^{\infty} f(x)dx = 1$ for a f that is the probability density function of a continuous random variable X */

$$
\int_{-\infty}^{\infty} e^{-Kx^2} dx = \sqrt{\frac{\pi}{K}}
$$

$$
\begin{aligned}
I(K) &:= \int_{-\infty}^{\infty} e^{-Kx^2} dx = \sqrt{\frac{\pi}{K}} = \sqrt{\pi} K^{-1/2} \\
\frac{\partial I}{\partial K} &= \int_{-\infty}^{\infty} (-x^2) e^{-Kx^2} dx = -\frac{\sqrt{\pi}}{2} K^{-3/2} \\
\frac{\partial^2 I}{\partial K^2} &= \int_{-\infty}^{\infty} x^4 e^{-Kx^2} dx = \frac{3\sqrt{\pi}}{4} K^{-5/2} \\
& \text{/* set } K = \frac{1}{\theta} \text{ */} \\
& \int_{-\infty}^{\infty} x^4 e^{-x^2/\theta} dx = \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\theta}\right)^{-5/2} = \frac{3\sqrt{\pi}}{4} \theta^{5/2}
\end{aligned}
$$

return to calculate $E[x^2]$

$$
\begin{aligned}
E[x^2] &= \frac{2}{\sqrt{\pi}\theta^{3/2}} \int_{-\infty}^{\infty} x^4 e^{-x^2/\theta} dx \\
&= \frac{2}{\sqrt{\pi}\theta^{3/2}} \left(\frac{3\sqrt{\pi}}{4} \theta^{5/2}\right) \\
&= \frac{3}{2}\theta
\end{aligned}
$$

return to calculate $bias(\hat{\theta})$

$$
\begin{aligned}
bias(\hat{\theta}) &= E[\hat{\theta}] - \theta \\
&= \frac{2}{3} E[a_i^2] - \theta \\
&= \frac{2}{3} \cdot \frac{3}{2}\theta - \theta \\
&= \theta - \theta \\
&= \boxed{0}
\end{aligned}
$$

3.

$$Var(\hat{\theta}) = E[\hat{\theta}^2] - E[\hat{\theta}]^2$$

$$\begin{aligned}
E[\hat{\theta}^2] &= E\left[\left(\frac{2\sum_{i=1}^n a_i^2}{3n}\right)^2\right] \\
&= \frac{4}{9n^2} E\left[\left(\sum_{i=1}^n a_i^2\right)^2\right] \\
&= \frac{4}{9n^2} E\left[(na_i^2)^2\right] \\
&= \frac{4}{9n^2} E\left[n^2 a_i^4\right] \\
&= \frac{4}{9} E\left[a_i^4\right]
\end{aligned}$$

$$\begin{aligned}
E[x^4] &= \int_{-\infty}^{\infty} x^4 f_\theta(x) dx \\
&= \int_{-\infty}^{\infty} x^4 \left(\frac{2}{\sqrt{\pi}\theta^{3/2}} x^2 e^{-x^2/\theta}\right) dx \\
&= \frac{2}{\sqrt{\pi}\theta^{3/2}} \int_{-\infty}^{\infty} x^6 e^{-x^2/\theta} dx
\end{aligned}$$

to calculate $\int_{-\infty}^{\infty} x^6 e^{-x^2/\theta} dx$:
from Problem 5. 2 it is shown that

$$\begin{aligned}
\frac{\partial^2 I}{\partial K^2} &= \int_{-\infty}^{\infty} x^4 e^{-Kx^2} dx = \frac{3\sqrt{\pi}}{4} K^{-5/2} \\
\frac{\partial^3 I}{\partial K^3} &= \int_{-\infty}^{\infty} -x^6 e^{-Kx^2} dx = -\frac{15\sqrt{\pi}}{8} K^{-7/2}
\end{aligned}$$

$$\int_{-\infty}^{\infty} x^6 e^{-Kx^2} dx = \frac{15\sqrt{\pi}}{8} K^{-7/2}$$

$$/\text{* set } K = \frac{1}{\theta} \text{ */}$$

$$\int_{-\infty}^{\infty} x^6 e^{-x^2/\theta} dx = \frac{15\sqrt{\pi}}{8} \left(\frac{1}{\theta}\right)^{-7/2}$$

$$= \frac{15\sqrt{\pi}}{8} \theta^{7/2}$$

return to calculate $E[x^4]$

$$\begin{aligned} E[x^4] &= \frac{2}{\sqrt{\pi}\theta^{3/2}} \int_{-\infty}^{\infty} x^6 e^{-x^2/\theta} dx \\ &= \frac{2}{\sqrt{\pi}\theta^{3/2}} \left(\frac{15\sqrt{\pi}}{8}\theta^{7/2}\right) \\ &= \frac{15}{4}\theta^2 \end{aligned}$$

return to calculate $Var(\hat{\theta})$

$$\begin{aligned} Var(\hat{\theta}) &= E[\hat{\theta}^2] - E[\hat{\theta}]^2 \\ &= \frac{4}{9} E\left[a_i^4\right] - \theta^2 \\ &= \frac{4}{9} \cdot \frac{15}{4}\theta^2 - \theta^2 \\ &= \frac{15}{9}\theta^2 - \theta^2 \\ &= \frac{15-9}{9}\theta^2 \\ &= \boxed{\frac{2}{3}\theta^2} \end{aligned}$$

as one increases the number n of independent observations, $Var(\hat{\theta})$ does not change since $Var(\hat{\theta})$ does not depend on n