Xi Liu, xl3504, Problem Set 7

Problem 1
1.

$$P(X=1, Y=1, Z=1-2(1)=-1)=\frac{1}{3}$$
$$P(X=1, Y=2, Z=1-2(2)=-3)=\frac{1}{12}$$
$$P(X=2, Y=1, Z=2-2(1)=0)=\frac{1}{6}$$
$$P(X=2, Y=2, Z=2-2(2)=-2)=0$$
$$P(X=4, Y=1, Z=4-2(1)=2)=\frac{1}{12}$$
$$P(X=4, Y=2, Z=4-2(2)=0)=\frac{1}{3}$$

$$P(Z=0)=\frac{1}{6}+\frac{1}{3}=\frac{1+2}{6}=\frac{1}{2}$$

a	-3	-2	-1	0	2
$p(Z=a)$	$\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{12}$

2.

$$g(x, y) = x - 2y$$

$$\begin{aligned}
E[Z] = E[g(X, Y)] &= \sum_j \sum_i g(a_i, b_j) p_{X,Y}(a_i, b_j) \\
&= \sum_j \sum_i g(a_i, b_j) p(X = a_i, Y = b_j) \\
&= g(1,1)p(1,1) + g(1,2)p(1,2) + g(2,1)p(2,1) \\
&\quad + g(2,2)p(2,2) + g(4,1)p(4,1) + g(4,2)p(4,2) \\
&= -1 \cdot \frac{1}{3} + -3 \cdot \frac{1}{12} + 0 \cdot \frac{1}{6} + -2 \cdot 0 + 2 \cdot \frac{1}{12} + 0 \cdot \frac{1}{3} \\
&= \boxed{-\frac{5}{12}}
\end{aligned}$$

3.

$$\begin{aligned}
P(X = 2 | Z = 0) &= \frac{P(X = 2, Z = 0)}{P(Z = 0)} \\
&= \frac{P(X = 2, Y = 1, Z = 2 - 2(1) = 0)}{P(Z = 0)} \\
&= \frac{1/6}{1/2} \\
&= \frac{2}{6} \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

Problem 2

$$p_N(n) = P(N = n) = \frac{\lambda^n}{n!}e^{-\lambda}$$

$P(X = x|N = n) = binomial(n, p) =$
$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x \in [0, n] \cap \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} P(X = x|N = n) &= \frac{P(X = x, N = n)}{P(N = n)} \\ P(X = x, N = n) &= P(X = x|N = n)P(N = n) \\ &= \binom{n}{x} p^x (1-p)^{n-x} P(N = n) \\ &= \boxed{\binom{n}{x} p^x (1-p)^{n-x} \frac{\lambda^n}{n!} e^{-\lambda}} \end{aligned}$$

Problem 3

Y \ X	1	2	3
1	p(1,1) = 1/9 W = 2 Z = 0	p(1,2) = 1/9 W = 3 Z = -1	p(1,3) = 1/9 W = 4 Z = -2
2	p(2,1) = 1/9 W = 3 Z = 1	p(2,2) = 1/9 W = 4 Z = 0	p(2,3) = 1/9 W = 5 Z = -1
3	p(3,1) = 1/9 W = 4 Z = 2	p(3,2) = 1/9 W = 5 Z = 1	p(3,3) = 1/9 W = 6 Z = 0

1.
joint probability mass function of W and Z:

Z \ W	-2	-1	0	1	2
2	0	0	1/9	0	0
3	0	1/9	0	1/9	0
4	1/9	0	1/9	0	1/9
5	0	1/9	0	1/9	0
6	0	0	1/9	0	0

2.

2 discrete random variables X and Y are independent if

$$P(X = a, Y = b) = P(X = a)P(Y = b)$$

W and Z are not independent, since for example, $P(W = 2) = 1/9$, $P(Z = 0) = 3/9$,

$$P(W=2, Z=0) = 1/9 \neq P(W=2)P(Z=0) = (1/9)(3/9) = 1/27$$

3.
$$g(x,y) := x + y$$

$$\begin{aligned}
E[W] &= E[g(X,Y)] \\
&= \sum_j \sum_i g(a_i, b_j) p_{X,Y}(a_i, b_j) \\
&= \sum_j \sum_i g(a_i, b_j) p(X = a_i, Y = b_j) \\
&= g(1,1)p(1,1) + g(1,2)p(1,2) + g(1,3)p(1,3) \\
&\quad + g(2,1)p(2,1) + g(2,2)p(2,2) + g(2,3)p(2,3) \\
&\quad + g(3,1)p(3,1) + g(3,2)p(3,2) + g(3,3)p(3,3) \\
&= 2(1/9) + 3(1/9) + 4(1/9) \\
&\quad + 3(1/9) + 4(1/9) + 5(1/9) \\
&\quad + 4(1/9) + 5(1/9) + 6(1/9) \\
&= (2+3+4+3+4+5+4+5+6)(1/9) \\
&= \boxed{4}
\end{aligned}$$

$$h(x,y) := x - y$$

$$\begin{aligned}
E[Z] &= E[h(X,Y)] \\
&= \sum_j \sum_i h(a_i, b_j) p_{X,Y}(a_i, b_j) \\
&= \sum_j \sum_i h(a_i, b_j) p(X = a_i, Y = b_j) \\
&= h(1,1)p(1,1) + h(1,2)p(1,2) + h(1,3)p(1,3) \\
&\quad + h(2,1)p(2,1) + h(2,2)p(2,2) + h(2,3)p(2,3) \\
&\quad + h(3,1)p(3,1) + h(3,2)p(3,2) + h(3,3)p(3,3) \\
&= 0(1/9) + -1(1/9) + -2(1/9) \\
&\quad + 1(1/9) + 0(1/9) + -1(1/9) \\
&\quad + 2(1/9) + 1(1/9) + 0(1/9) \\
&= (-1 - 2 + 1 - 1 + 2 + 1)(1/9) \\
&= \boxed{0}
\end{aligned}$$

Problem 4

1.

$$
\begin{aligned}
f_X(i) &= \int_{-\infty}^{\infty} f_{X,Y}(i,y)dy \\
&= \int_{0}^{\infty} abe^{-ai-by}dy \\
&= ab\int_{0}^{\infty} e^{-ai-by}dy \\
&/*\ u := -ai - by; \quad du = -bdy; \quad dy = -\frac{du}{b}\ */ \\
&= -a\left[e^{-ai-by}\right]_0^{y=\infty} \\
&= -a(0 - e^{-ai-0}) \\
&= \boxed{ae^{-ai}}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(j) &= \int_{-\infty}^{\infty} f_{X,Y}(x,j)dx \\
&= \int_{0}^{\infty} abe^{-ax-bj}dx \\
&= ab\int_{0}^{\infty} e^{-ax-bj}dx \\
&/*\ u := -ax - bj; \quad du = -adx; \quad dx = -\frac{du}{a}\ */ \\
&= -b\left[e^{-ax-bj}\right]_0^{\infty} \\
&= -b(0 - (e^{0-bj})) \\
&= \boxed{be^{-bj}}
\end{aligned}
$$

2.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{0}^{\infty} x(ae^{-ax}) dx \\
&= a \int_{0}^{\infty} xe^{-ax} dx \\
&/*\ u := x; \quad dv := e^{-ax}; \\
&du = dx; \quad v = -\frac{1}{a}e^{-ax}\ */ \\
&= a \left[-\frac{x}{a}e^{-ax} - \frac{1}{a^2}e^{-ax} \right]_0^{\infty} \\
&= a \left(0 - \left(-\frac{1}{a^2} \right) \right) \\
&= \boxed{\frac{1}{a}}
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\
&= \int_{0}^{\infty} y(be^{-by}) dy \\
&= b \int_{0}^{\infty} ye^{-by} dy \\
&/*\ u := y; \quad dv := e^{-by}dy; \\
&du = dy; \quad v = -\frac{1}{b}e^{-by}\ */ \\
&= b \left[-\frac{y}{b}e^{-by} - \frac{1}{b^2}e^{-by} \right]_0^{\infty} \\
&= b \left(0 - \left(\frac{1}{b^2} \right) \right) \\
&= \boxed{\frac{1}{b}}
\end{aligned}
$$

3.

$$
\begin{aligned}
P(X < Y) &= \int_0^\infty \int_0^y f_{X,Y}(x,y)dxdy \\
&= \int_0^\infty \int_0^y (abe^{-ax-by})dxdy \\
&= ab \int_0^\infty \int_0^y e^{-ax-by}dxdy \\
&/* \ u := -ax-by; \quad du = -adx; \quad dx = -\frac{du}{a} \ */ \\
&= -b \int_0^\infty \left[e^{-ax-by}\right]_{x=0}^{x=y} dy \\
&= -b \int_0^\infty \left(e^{-(a+b)y} - e^{-by}\right) dy \\
&= -b \left[-\frac{1}{a+b}e^{-(a+b)y} + \frac{1}{b}e^{-by}\right]_0^\infty \\
&= -b \left(0 - \left(-\frac{1}{a+b} + \frac{1}{b}\right)\right) \\
&= b \left(-\frac{1}{a+b} + \frac{1}{b}\right) \\
&= \boxed{-\frac{b}{a+b} + 1}
\end{aligned}
$$

Problem 5

$$
\begin{aligned}
f_X(a) &= \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy \\
&= \int_{a}^{\infty} e^{-y}dy \\
&= -\left[e^{-y}\right]_a^{\infty} \\
&= -(0 - e^{-a}) \\
&= e^{-a}
\end{aligned}
$$

$$
\begin{aligned}
F_X(a) &= \int_{-\infty}^{a} f_X(x)dx \\
&= \int_{0}^{a} e^{-x}dx \\
&= -[e^{-x}]_0^a \\
&= -(e^{-a} - e^{0}) \\
&= 1 - e^{-a}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(b) &= \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx \\
&= \int_{0}^{b} e^{-b}dx \\
&= [xe^{-b}]_0^b \\
&= be^{-b}
\end{aligned}
$$

$$
\begin{aligned}
F_Y(b) &= \int_{-\infty}^{b} f_Y(y)dy \\
&= \int_0^b ye^{-y}dy \\
&/* \ u := y; \quad dv := e^{-y}dy; \\
&du = dy; \quad v = -e^{-y} \ */ \\
&= [-ye^{-y}]_0^b - \int_0^b (-e^{-y})dy \\
&= -be^{-b} + \int_0^b e^{-y}dy \\
&= -be^{-b} - [e^{-y}]_0^b \\
&= -be^{-b} - (e^{-b} - e^0) \\
&= -be^{-b} - e^{-b} + 1
\end{aligned}
$$

if $a < b$

$$
\begin{aligned}
F_{X,Y}(a,b) &= \int_0^a \int_x^b f_{X,Y}(x,y)dydx \\
&= \int_0^a \int_x^b e^{-y}dydx \\
&= -\int_0^a [e^{-y}]_{y=x}^{y=b} dx \\
&= -\int_0^a (e^{-b} - e^{-x})dx \\
&= -[xe^{-b} + e^{-x}]_0^a \\
&= -(ae^{-b} + e^{-a} - 1) \\
&= 1 - ae^{-b} - e^{-a}
\end{aligned}
$$

if $a \geq b$

$$
\begin{aligned}
F_{X,Y}(a,b) &= \int_0^b \int_x^b f_{X,Y}(x,y) dy dx \\
&= \int_0^b \int_x^b e^{-y} dy dx \\
&= -\int_0^b [e^{-y}]_{y=x}^{y=b} dx \\
&= -\int_0^b (e^{-b} - e^{-x}) dx \\
&= -[xe^{-b} + e^{-x}]_0^b \\
&= -(be^{-b} + e^{-b} - 1) \\
&= 1 - be^{-b} - e^{-b}
\end{aligned}
$$

$$
P(X \leq a, Y \leq b) = F_{X,Y}(a,b) = \begin{cases} 1 - ae^{-b} - e^{-a} & \text{if } a < b \\ 1 - be^{-b} - e^{-b} & \text{if } a \geq b \end{cases}
$$

$$
\begin{aligned}
P(X \leq a)P(Y \leq b) = F_X(a)F_Y(b) &= (1 - e^{-a})(-be^{-b} - e^{-b} + 1) \\
&= -be^{-b} - e^{-b} + 1 + be^{-a}e^{-b} + e^{-a}e^{-b} - e^{-a}
\end{aligned}
$$

$$
P(X \leq a, Y \leq b) \neq P(X \leq a)P(Y \leq b)
$$

so X and Y are not independent