MATH-UA.0235 Probability and Statistics – Worksheet # 5

Problem 1 - Practicing double integrals

Evaluate

$$\iint_R x\cos y\,dxdy$$

where $R = [-2,3] \times [0, \frac{\pi}{2}]$.

$$\int_{-2}^{3}\int_{0}^{\frac{\pi}{2}} x\cos y\,dxdy = \int_{-2}^{3} x\int_{0}^{\frac{\pi}{2}}\cos y\,dydx = \int_{-2}^{3} x\,[\sin y]_0^{\frac{\pi}{2}}\,dx = \int_{-2}^{3} xdx = \left[\frac{x^2}{2}\right]_{-2}^{3} = \frac{5}{2}$$

Now, observe that

$$\left(\int_{-2}^{3} xdx\right)\left(\int_{0}^{\frac{\pi}{2}}\cos y\,dy\right) = \left[\frac{x^2}{2}\right]_{-2}^{3}[\sin y]_0^{\frac{\pi}{2}} = \frac{5}{2}$$

This is a general result: if $R = [a,b] \times [c,d]$, and $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y)dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Problem 2

Let X and Y be a couple of discrete random variables that are uniformly distributed on the set $\{0,1,2,3,4,5,6,7\}^2$.

1. What is the probability mass function of X? What is the probability mass function of Y?

   X takes values in $\{0,1,2,3,4,5,6,7\}$. For $a = 0,1,2,3,4,5,6,7$, we may write

   $$p_X(a) = \sum_{j=0}^{7} p_{X,Y}(a,j) = \sum_{j=0}^{7}\frac{1}{64} = \frac{1}{8}$$

   Y obviously has the same distribution.

2. Are X and Y independent random variables?

   For all $x \in \{0,1,2,3,4,5,6,7\}$ and $y \in \{0,1,2,3,4,5,6,7\}$, we have

   $$p_X(x)p_Y(y) = \frac{1}{8}\cdot\frac{1}{8} = \frac{1}{64} = p_{X,Y}(x,y)$$

   The random variables X and Y are independent.

3. What is the probability mass function of $X+Y$?

   $Z = X+Y$ takes values in $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$. For any a in this set, we can write

   $$p_Z(a) = P(Z=a) = \sum_{j=0}^{a} P((X=j)\cap(Y=a-j))$$

   If $a \le 7$,

   $$p_Z(a) = \sum_{j=0}^{a} P((X=j)\cap(Y=a-j)) = \frac{a+1}{64}$$

   If $14 \ge a > 7$,

   $$p_Z(a) = \sum_{j=0}^{a} P((X=j)\cap(Y=a-j)) = \sum_{j=0}^{7} P((X=j)\cap(Y=a-j)) = \sum_{j=a-7}^{7} P((X=j)\cap(Y=a-j)) = \frac{15-a}{64}$$

Problem 3

Consider two discrete random variables X and Y, which both take values in $\mathbb{N}^*$, and are such that

$$P((X=i)\cap(Y=j)) = \frac{a}{2^{i+j}}$$

for all $i \in \mathbb{N}^*$ and $j \in \mathbb{N}^*$.

1. What is the value of a?

   We first observe that we must have $a \geq 0$. We also have the condition

$$\sum_{i,j\geq 1} \frac{a}{2^{i+j}} = 1 \;\Leftrightarrow\; a\sum_{i,j} \frac{1}{2^i}\frac{1}{2^j} = 1 \;\Leftrightarrow\; a\left(\sum_{i=1}^{+\infty}\frac{1}{2^i}\right)\left(\sum_{j=1}^{+\infty}\frac{1}{2^j}\right) = 1 \;\Leftrightarrow\; a = 1$$

2. What is the marginal distribution p_X of X and p_Y of Y?

   We can write

$$p_X(i) = P(X=i) = \sum_{j=1}^{+\infty} P((X=i)\cap(Y=j)) = \sum_{j=1}^{+\infty}\frac{1}{2^{i+j}} = \frac{1}{2^i}\sum_{j=1}^{+\infty}\frac{1}{2^j} = \frac{1}{2^i}$$

   By symmetry, we immediately conclude that we also have

$$p_Y(j) = \frac{1}{2^j}$$

3. Characterize the distributions of X and Y

   We have

$$p_X(i) = \frac{1}{2}\frac{1}{2^{i-1}}$$

   so X has a geometric distribution with parameter $\frac{1}{2}$. So does Y.

4. Are X and Y independent?

   We have

$$P((X=i)\cap(Y=j)) = \frac{1}{2^{i+j}} = \frac{1}{2^i}\frac{1}{2^j} = p_X(i)p_Y(j)$$

   so the two random variables are independent.

Problem 4

Consider two continuous random variables X and Y with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & \text{if } x \in \mathbb{R}^+ \text{ and } y \in \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(X < Y)$.

We are looking for the probability of the event $(X<Y)$. For any value y that Y takes, the event occurs if X takes values in $[0, y)$. Hence

$$\begin{aligned} P(X<Y) &= \int_0^{+\infty}\int_0^y f_{X,Y}(x,y)dxdy = \int_0^{+\infty}\int_0^y e^{-(x+y)}dxdy = \int_0^{+\infty} e^{-y}\int_0^y e^{-x}dxdy \\ &= -\int_0^{+\infty} e^{-y}\left[e^{-x}\right]_0^y dy = -\int_0^{+\infty} e^{-y}(e^{-y}-1)dy = \frac{1}{2}\left[e^{-2y}\right]_0^{+\infty} - \left[e^{-y}\right]_0^{+\infty} = \frac{1}{2} \end{aligned}$$

Note that this result could have been obtained in a different, perhaps more intuitive way. The marginal probability density functions f_X and f_Y are

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \int_0^{+\infty} e^{-(x+y)}dy = e^{-x}\int_0^{+\infty} e^{-y}dy = -e^{-x}\left[e^{-y}\right]_0^{+\infty} = e^{-x} \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \int_0^{+\infty} e^{-(x+y)}dx = e^{-y}\int_0^{+\infty} e^{-x}dx = -e^{-y}\left[e^{-x}\right]_0^{+\infty} = e^{-y} \end{aligned}$$

The marginal probability density functions are the same! Hence, $P(X<Y) = P(Y<X)$. Since $P(X=Y)=0$, $P(X<Y)+P(Y<X) = 1 \Rightarrow P(X<Y) = \frac{1}{2}$.

Problem 5

Let X be a discrete random variable with probability mass function

$$p_X(a) = \begin{cases} \frac{1}{3} & \text{if } a = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

We let $Y = X^2$.

1. What is the probability mass function of Y?

   Y can take values 0 and 1. Y takes the value 0 if and only if X takes the value 0. Y takes the value 1 if and only if X takes the value -1 or 1. The latter are mutually exclusive events. Hence

   $$p_Y(a) = \begin{cases} \frac{1}{3} & \text{if } a = 0 \\ \frac{2}{3} & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Are X and Y independent random variables?

   We have $P(X = -1, Y = 0) = 0 \neq p_X(-1)p_Y(0) = \frac{1}{9}$. X and Y are not independent variables, which makes intuitive sense.

3. Evaluate $E[XY]$.

   $$E[XY] = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$$

Problem 6

In this problem, we have N boxes, numbered from 1 to N. Box k contains k balls, numbered from 1 to k. One picks a box randomly, and then a ball randomly in this box. X is the random variable corresponding to the box number, and Y the random variable corresponding to the ball number.

1. Calculate $p_{X,Y}$, the joint probability mass function of X and Y.

   Let $(i, j) \in \{1, 2, \ldots, N\}^2$. If $j > i$, $p_{X,Y}(i, j) = P(X = i, Y = j) = 0$.
   If $j \leq i$,

   $$p_{X,Y}(i, j) = P(X = i, Y = j) = P(Y = j|X = i)P(X = i) = \frac{1}{i} \cdot \frac{1}{N}$$

2. Deduce the probability mass function of Y from the previous question.

   For $j \in \{1, 2, \ldots, N\}$,

   $$p_Y(j) = P(Y = j) = \sum_{i=1}^{N} p_{X,Y}(i, j) = \frac{1}{N} \sum_{i=j}^{N} \frac{1}{i}$$

3. What is the expected value of Y?

   We have

   $$\begin{aligned} E[Y] &= \sum_{j=1}^{N} j \cdot p_Y(j) = \frac{1}{N} \sum_{j=1}^{N} \sum_{i=j}^{N} \frac{j}{i} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{i} \frac{j}{i} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{i} \sum_{j=1}^{i} j \\ &= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{i} \cdot \frac{i(i+1)}{2} = \frac{1}{N} \sum_{i=1}^{N} \frac{i+1}{2} = \left(\frac{1}{2N} \sum_{i=1}^{N} i \right) + \frac{1}{2N} \sum_{i=1}^{N} 1 = \frac{N+1}{4} + \frac{1}{2} \\ &= \frac{N+3}{4} \end{aligned}$$

Problem 7

Consider two continuous random variables X and Y such that

$$f_{X,Y}(x,y) = \begin{cases} c\left(\frac{1}{x^2}+y^2\right) & \text{if } 1 \leq x \leq 5 \text{ and } -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. For which value of c is $f_{X,Y}$ indeed a joint probability density function?

   The condition can be written as

   $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dydx = 1 \quad \Leftrightarrow \quad c\int_1^5\left[\int_{-1}^1\left(\frac{1}{x^2}+y^2\right)dy\right]dx = 1$$

   We have

   $$\int_1^5\left[\int_{-1}^1\left(\frac{1}{x^2}+y^2\right)dy\right]dx = \int_1^5\left(\left[\frac{y}{x^2}+\frac{y^3}{3}\right]_{-1}^1\right)dx = \int_1^5\left(\frac{2}{x^2}+\frac{2}{3}\right)dx = \left[-\frac{2}{x}+\frac{2}{3}x\right]_1^5 = \frac{64}{15}$$

   Therefore, we must have $c = \frac{15}{64}$.

2. Compute the marginal probability density functions f_X and f_Y.

   If $x \leq 1$ or $x \geq 5$, $f_{X,Y}(x,y) = 0$ for all values of y. Thus $f_X(x) = 0$. If $x \in [1,5]$,

   $$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \frac{15}{64}\int_{-1}^1\left(\frac{1}{x^2}+y^2\right)dy = \frac{15}{64}\left[\frac{y}{x^2}+\frac{y^3}{3}\right]_{-1}^1 = \frac{15}{32}\left(\frac{1}{x^2}+\frac{1}{3}\right)$$

   Hence

   $$f_X(x) = \begin{cases} \frac{15}{32}\left(\frac{1}{x^2}+\frac{1}{3}\right) & \text{if } x \in [1,5] \\ 0 & \text{otherwise} \end{cases}$$

   Likewise, If $y \leq -1$ or $y \geq 1$, $f_{X,Y}(x,y) = 0$ for all values of x. Thus $f_Y(y) = 0$. If $y \in [-1,1]$,

   $$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \frac{15}{64}\int_1^5\left(\frac{1}{x^2}+y^2\right)dx = \frac{15}{64}\left[-\frac{1}{x}+xy^2\right]_1^5 = \frac{15}{64}\left(\frac{4}{5}+4y^2\right) = \frac{15}{16}\left(\frac{1}{5}+y^2\right)$$

   Hence

   $$f_Y(y) = \begin{cases} \frac{15}{16}\left(y^2+\frac{1}{5}\right) & \text{if } y \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

3. Are X and Y independent random variables?

   $$f_{X,Y}(2,0) = \frac{15}{256}$$
   $$f_X(2) = \frac{35}{128} \quad , \quad f_Y(0) = \frac{3}{16} \quad \Rightarrow f_X(2)f_Y(0) = \frac{105}{2048} \neq \frac{15}{256}$$

   X and Y are not independent random variables.

4. Compute $E[XY]$.

   By definition

   $$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_{X,Y}(x,y)dydx = \frac{15}{64}\int_1^5\left[\int_{-1}^1\left(\frac{y}{x}+y^3x\right)dy\right]dx$$
   $$= \frac{15}{64}\int_1^5\left[\frac{y^2}{2x}+\frac{xy^4}{4}\right]_{-1}^1 dx = 0$$