MATH-UA.0235 Probability and Statistics – Worksheet # 4

Problem 1

A rich friend proposes the following game to you. She will roll two fair dice; if you get two six, she gives you a million dollars, otherwise you pay her 10000 dollars. What do you choose to do?

Let A be the event "The dice show two six". $P(A) = \frac{1}{36}$, $P(A^c) = \frac{35}{36}$. Your expected earnings in this game thus is

$$10^6 \times \frac{1}{36} - 10^4 \times \frac{35}{36} \approx 18055.56 \ \$$$

This is a paradox of the expected value: it is what one can expect to earn on average per game, in the limit of many games. Here, for each game, the probability of losing is very large, and much money is lost on such occasion. I would therefore not play, unless my friend gives me the promise that we will play a large number of rounds.

Problem 2

One rolls an unfair die such that the faces 1 to 5 have equal probability, and the probability of face 6 is twice the probability of face 5. Let X be the random variable corresponding to the number on the face that comes up.

1. What is the probability mass function of X?

   Let p be the probability of each face 1 to 5. One can write

   $$5p + 2p = 1 \quad \Leftrightarrow \quad p = \frac{1}{7}$$

   The probability mass function can thus be written as

   $$p_X(a) = \begin{cases} \frac{1}{7} & \text{if } \ a = 1, 2, 3, 4, 5 \\ \frac{2}{7} & \text{if } \ a = 6 \\ 0 & \text{otherwise} \end{cases}$$

2. What is the expected value of X?

   The expected value of X is

   $$E[X] = \frac{1}{7} + 2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 5 \cdot \frac{1}{7} + 6 \cdot \frac{2}{7} = \frac{27}{7}$$

Problem 3

One rolls two dice, and considers the random variable X corresponding to the absolute value of the difference between the results of the two dice.

1. What is the probability mass function for X?

   Using a table to consider all possible solutions, one easily finds:

   $$p_X(a) = \begin{cases} \frac{6}{36} & \text{if } \ a = 0 \\ \frac{10}{36} & \text{if } \ a = 1 \\ \frac{8}{36} & \text{if } \ a = 2 \\ \frac{6}{36} & \text{if } \ a = 3 \\ \frac{4}{36} & \text{if } \ a = 4 \\ \frac{2}{36} & \text{if } \ a = 5 \end{cases}$$

2. What is the expected value of X?

We can write

$$E[X] = 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36} = \frac{35}{18}$$

3. What is the variance of X?

We have

$$E[X^2] = 0^2 \cdot \frac{6}{36} + 1^2 \cdot \frac{10}{36} + 2^2 \cdot \frac{8}{36} + 3^2 \cdot \frac{6}{36} + 4^2 \cdot \frac{4}{36} + 5^2 \cdot \frac{2}{36} = \frac{105}{18} = \frac{35}{6}$$

Hence

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{105}{18} - \frac{35^2}{18^2} = \frac{105 \times 18 - 35^2}{324} = \frac{665}{324}$$

Problem 4

Consider a random variable X such that $E[X] = 1$ and $\mathrm{Var}(X) = 5$.

1. Compute $E[(2 + X)^2]$.

$\mathrm{Var}(X) = 5$, so we can write

$$E[X^2] - (E[X])^2 = 5 \quad \Leftrightarrow \quad E[X^2] = 5 + 1 = 6$$

Therefore,

$$E[(2 + X)^2] = E[4 + 4X + X^2] = 4 + 4E[X] + E[X^2] = 14$$

2. Compute $\mathrm{Var}(4 + 3X)$

For any real numbers a and b, we have

$$\begin{aligned}\mathrm{Var}(aX + b) &= E[(aX + b)^2] - (E[aX + b])^2 = E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\ &= a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2 = a^2\left[E[X^2] - (E[X])^2\right] \\ &= a^2\mathrm{Var}(X)\end{aligned}$$

Hence $\mathrm{Var}(4 + 3X) = 9\mathrm{Var}(X) = 45$

Problem 5

Let X be a random variable with expected value μ and variance σ^2. Consider the random variable Y defined by

$$Y = \frac{X - \mu}{\sigma}$$

What is the expected value and variance of Y?

Let us start with the expected value:

$$E[Y] = E[\frac{X}{\sigma} - \frac{\mu}{\sigma}] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma} = 0$$

For the variance of Y, we can use the result derived in Problem 4:

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X) = 1$$

We have such covered the important process of normalization: this operation on the random variable X converts it into a random variable with zero mean and variance equals to 1.

Problem 6

Consider a continuous random variable X with probability density function

$$f_X(x) = \begin{cases} a\cos x & \text{if} \quad x \in [0, \frac{\pi}{2}] \\ 0 & \text{otherwise} \end{cases}$$

1. For which value a is f_X indeed a probability density function?

   We have the condition

   $$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \Leftrightarrow \quad a\int_0^{\frac{\pi}{2}} \cos x dx = 1 \quad \Leftrightarrow \quad a\left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

   We must therefore have $a = 1$.

2. Consider the functions g and G with domain $[0, \frac{\pi}{2}]$, given by

   $$g(x) = x\cos x \qquad G(x) = cx\sin x + d\cos x$$

   For which real numbers c and d is G an antiderivative of g?

   We must have

   $$\frac{dG}{dx} = g(x) \quad \Leftrightarrow \quad c\sin x + cx\cos x - d\sin x = x\cos x$$

   We conclude that $c = 1$ and $d = c = 1$ satisfy the conditions.

3. What is the expected value of X?

   We have

   $$E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^{\frac{\pi}{2}} x\cos x dx$$

   Using the previous question, we then have

   $$E[X] = \left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

Problem 7

Consider a uniform random variable X on the interval $[a, b]$, with expected value μ and variance σ^2. Compute $P(\mu - \sigma \leq X \leq \mu + \sigma)$.

Let us start by computing the expressions for μ and σ^2.

$$\mu = E[X] = \frac{1}{b-a}\int_a^b x dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E[X^2] = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + b^2 + ab}{3}$$

We thus have

$$\sigma^2 = \mathrm{Var}(X) = \frac{a^2 + b^2 + ab}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

and

$$\sigma = \frac{b-a}{2\sqrt{3}}$$

We observe that

$$\mu + \sigma = \frac{\sqrt{3} - 1}{2\sqrt{3}}a + \frac{1 + \sqrt{3}}{2\sqrt{3}}b$$

so $\mu + \sigma \in [a, b]$. Likewise,

$$\mu - \sigma = \frac{1 + \sqrt{3}}{2\sqrt{3}}a + \frac{\sqrt{3} - 1}{2\sqrt{3}}b$$

so $\mu - \sigma \in [a, b]$.

Hence,

$$P(\mu - \sigma \leq X \leq \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f_X(x)dx = \frac{1}{b-a}\int_{\mu-\sigma}^{\mu+\sigma} dx = \frac{1}{b-a}\left[x\right]_{\mu-\sigma}^{\mu+\sigma} = 2\frac{\sigma}{b-a} = \frac{1}{\sqrt{3}}$$

We observe that the result does not depend on the values of a and b.

Problem 8

Let X be a continuous random variable that is uniformly distributed on the interval $[1,3]$. We define the random variable $Y = X^2$.

1. What is the cumulative distribution function of Y?

   Let $c \in \mathbb{R}$.

   If $c < 0$, then $F_Y(c) = P(Y \leq c) = 0$.

   If $c > 0$, then

$$F_Y(c) = P(Y \leq c) = P(-\sqrt{c} \leq X \leq \sqrt{c}) = \int_{-\sqrt{c}}^{\sqrt{c}} f_X(x)dx = \int_{-\infty}^{\sqrt{c}} f_X(x)dx = F_X(\sqrt{c})$$

   where we used the fact that X takes values in $[1,3]$ for the penultimate equality.

   Hence, we may write

$$F_Y(c) = \begin{cases} 0 \quad \text{if} \quad c \leq 1 \\ \frac{\sqrt{c}-1}{2} \quad \text{if} \quad c \in [1,9] \\ 1 \quad \text{otherwise} \end{cases}$$

2. What is the probability density function of Y?

   We have

$$f_Y(c) = \frac{d}{dc}[F_Y(c)]$$

   Hence

$$F_Y(c) = \begin{cases} 0 \quad \text{if} \quad c \leq 1 \\ \frac{1}{4\sqrt{c}} \quad \text{if} \quad c \in [1,9] \\ 0 \quad \text{otherwise} \end{cases}$$

3. What is the expected value of Y?

$$E[Y] = E[X^2] = \int_1^3 \frac{x^2}{2}dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_1^3 = \frac{13}{3}$$

4. What is the variance of Y?

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2 = E[X^4] - \frac{169}{9}$$

   We have

$$E[X^4] = \int_1^3 \frac{x^4}{2}dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_1^3 = \frac{121}{5}$$

   Hence,

$$\text{Var}(Y) = \frac{121}{5} - \frac{169}{9} = \frac{244}{45}$$