Probability 2017 Homework 3

March 20, 2017

1 Q1 (10 points)

Suppose that X is nonnegative continuous random variable. Prove that $E[X] = \int_0^{+\infty} P(X > x)dx$.

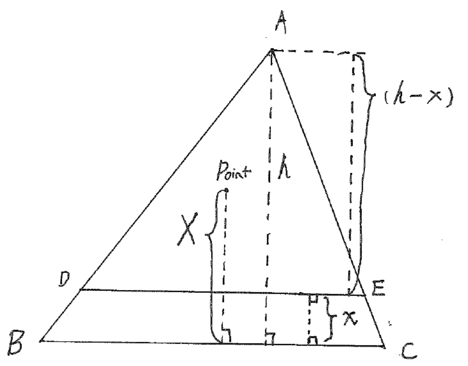
Solution:
Suppose $f(x)$ is the density function of X.

$$\begin{aligned}
\int_0^{+\infty} P(X > x)dx &= \int_0^{+\infty} \int_x^{+\infty} f(y)dydx \qquad (by\ P(X > x) = \int_x^{+\infty} f(y)dy) \\
&= \int_0^{+\infty} \int_0^{y} f(y)dxdy \qquad (by\ changing\ the\ order\ of\ integral) \\
&= \int_0^{+\infty} f(y) \int_0^{y} dxdy \\
&= \int_0^{+\infty} f(y)ydy \\
&= \int_{-\infty}^{+\infty} f(y)ydy \qquad (by\ the\ nonnegtivity\ of\ X) \\
&= E[X]
\end{aligned} \tag{1}$$

2 Q2 (10 points)

Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X.

Solution:
Suppose $\triangle ABC$ is the considered triangle. Let h be the height of $\triangle ABC$, S be the area of $\triangle ABC$. Draw a line parallel to the base within the triangle, whose distance to the base is x. The intersect of this line and the waist of the triangle is D and E respectively.



Then for $0 < x < h$:

$$\begin{aligned}P(X > x) &= P(the\ point\ falls\ in\ \triangle ADE) \\ &= S_{\triangle ADE}/S \quad (by\ uniform\ probability\ law) \\ &= (h-x)^2/h^2 \quad (by\ the\ property\ of\ similar\ triangles)\end{aligned} \tag{2}$$

Thus the CDF of X is:

$$\begin{aligned}P(X \le x) &= 1 - P(X > x) \\ &= \begin{cases} 1-(h-x)^2/h^2, & 0 < x < h \\ 0, & x \le 0 \\ 1, & x \ge h \end{cases}\end{aligned} \tag{3}$$

The PDF is obtained by differentiating CDF:

$$\begin{aligned}f(x) &= \frac{d}{dx}[P(X \le x)] \\ &= \begin{cases} 2(h-x)/h^2, & 0 < x < h \\ 0, & otherwise \end{cases}\end{aligned} \tag{4}$$

3 Q3 (10 points)

Please write down the PDF of a normal random variable with mean μ and standard deviation σ.

Solution:
Suppose X is a normal random variable with mean μ and standard deviation σ. Then the PDF of X is:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (x \in \mathbb{R})$$

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4 Q4 (10 points)

A point is chosen at random (according to a uniform PDF) within a semicircle of the form $\{(x,y) : x^2+y^2 < r^2, y > 0\}$, for some given r>0.
a) Find the joint PDF of the coordinates X and Y of the chosen point.
b) Find the marginal PDF of Y and use it to find $E[Y]$.

Solution:
a) Let $B = \{(x,y) : x^2+y^2 < r^2, y > 0\}$. the joint PDF of the coordinates X and Y is:

$$f(x,y) = \begin{cases} 2/\pi r^2, & (x,y) \in B \\ 0, & otherwise \end{cases} \tag{5}$$

b) The marginal density of Y on $(0,r)$ is:

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{+\infty} f(x,y)dx \\ &= \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} \frac{2}{\pi r^2} dx \\ &= \frac{4}{\pi r^2}\sqrt{r^2-y^2}\end{aligned} \tag{6}$$

Thus, the marginal density of Y is:

$$f_Y(y) = \begin{cases} \frac{4}{\pi r^2}\sqrt{r^2-y^2}, & y \in (0,r) \\ 0, & otherwise \end{cases} \tag{7}$$

The expectation of Y is:

$$\begin{aligned} E[Y] &= \int_{-\infty}^{+\infty} y f_Y(y) dy \\ &= \int_0^r y \frac{4}{\pi r^2} \sqrt{r^2 - y^2} dy \\ &= \frac{4r}{3\pi} \end{aligned} \tag{8}$$

5 Q5 (10 points)

Let $X_1, X_2, ..., X_n$ be independent random variables. We know that variance has linearity for independent random variables, namely

$$Var[X_1 + X_2 + ... + X_n] = Var[X_1] + Var[X_2] + ... + Var[X_n]$$

How about the variance of the product? Can you express

$$Var[X_1 X_2 ... X_n]$$

in terms of $Var[X_i]$ and $E[X_i^2]$?

Solution:
First note two facts:
Fact 1: if $X_1, X_2, ..., X_n$ are independent, then $X_1^2, X_2^2, ..., X_n^2$ are also independent.
Fact 2: if X and Y are independent, then $E[XY] = E[X]E[Y]$.
Then,

$$\begin{aligned} Var[X_1 X_2 ... X_n] &\equiv E[(X_1 X_2 ... X_n - E[X_1 X_2 ... X_n])^2] \\ &= E[(X_1 X_2 ... X_n)^2 - 2E[X_1 X_2 ... X_n](X_1 X_2 ... X_n) + E(X_1 X_2 ... X_n)^2] \\ &= E[(X_1 X_2 ... X_n)^2] - 2E[X_1 X_2 ... X_n]^2 + E[X_1 X_2 ... X_n]^2 \\ &= E[(X_1 X_2 ... X_n)^2] - E[X_1 X_2 ... X_n]^2 \\ &= E[(X_1^2 X_2^2 ... X_n^2)] - E[X_1 X_2 ... X_n]^2 \\ &= E[X_1^2]E[X_2^2]...E[X_n^2] - E[X_1 X_2 ... X_n]^2 \quad (by\ Fact\ 1\ and\ Fact\ 2) \\ &= E[X_1^2]E[X_2^2]...E[X_n^2] - E[X_1]^2 E[X_2]^2 ... E[X_n]^2 \quad (by\ Fact\ 2) \\ &= E[X_1^2]E[X_2^2]...E[X_n^2] - [E[X_1^2] - Var(X_1)][E[X_2^2] - Var(X_2)]...[E[X_n^2] - Var(X_n)] \\ &= \prod_{i=1}^{n} E[X_i^2] - \prod_{i=1}^{n} [E[X_i^2] - Var(X_i)] \end{aligned} \tag{9}$$