sd4175

P1

$Z = X - 2Y \quad X = \{1,2,4\},\ Y = \{1,2\}$

$Z = \{-1, -3, 0, -2, 2\}$

1. $P(Z=-1) = \frac{1}{3}$

$P(Z=-3) = \frac{1}{12}$

$P(Z=0) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

$P(Z=-2) = 0$

$P(Z=2) = \frac{1}{12}$

Z	-3	-2	-1	0	2
P(Z)	$\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{12}$

2. $E(Z) = \sum z P(z)$

$= -1 \cdot \frac{1}{3} + (-3) \cdot \frac{1}{12} + 0 \cdot \frac{1}{2} + (-2) \cdot 0 + 2 \cdot \frac{1}{12}$

$= -\frac{5}{12}$

3. $P(X=2 \mid Z=0) = \dfrac{P(X=2 \cap Z=0)}{P(Z=0)} = \dfrac{P_{X,Y}(2,1)}{\frac{1}{2}}$

$= \dfrac{\frac{1}{6}}{\frac{1}{2}}$

$= \frac{1}{3}$

P2

$$P(N=n) = \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

X be the number of girls in the family

$$P(X=x, N=n) = P(X=x \mid N=n) \cdot P(N=n)$$

$$= \frac{\binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda} \cdot \lambda^n}{n!}$$

P3

1. $W = X+Y$, $Z = X-Y$

$X = \{1,2,3\}$, $Y = \{1,2,3\}$

$W = \{2,3,4,5,6\}$

$P_W(2) = P_X(1) \cdot P_Y(1) = \frac{1}{9}$

$P_W(3) = P_X(1) \cdot P_Y(2) + P_X(2) \cdot P_Y(1) = \frac{2}{9}$

$P_W(4) = P_X(1) \cdot P_Y(3) + P_X(2) P_Y(2) + P_X(3) P_Y(1) = \frac{1}{3}$

$P_W(5) = P_X(2) \cdot P_Y(3) + P_X(3) \cdot P_Y(2) = \frac{2}{9}$

$P_W(6) = P_X(3) \cdot P_Y(3) = \frac{1}{9}$

$Z = \{-2, -1, 0, 1, 2\}$

$P_Z(-2) = P_X(1) P_Y(3) = \frac{1}{9}$

$P_Z(-1) = P_X(1) P_Y(2) + P_X(2) P_Y(3) = \frac{2}{9}$

$P_Z(0) = P_X(1) P_Y(1) + P_X(2) P_Y(2) + P_X(3) P_Y(3) = \frac{1}{3}$

$P_Z(1) = P_X(2) P_Y(1) + P_X(3) P_Y(2) = \frac{2}{9}$

$P_Z(2) = P_X(3) P_Y(1) = \frac{1}{9}$

Z \ W	2	3	4	5	6
-2	0	0	$\frac{1}{9}$	0	0
-1	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0
0	$\frac{1}{9}$	0	$\frac{1}{9}$	0	$\frac{1}{9}$
1	0	$\frac{1}{9}$	0	$\frac{1}{9}$	0
2	0	0	$\frac{1}{9}$	0	0

2. Suppose $W=2, Z=2$

$P(W=2, Z=2) = 0$

$P(W=2) \cdot P(Z=2) = \frac{1}{9} \cdot \frac{1}{9} \neq 0$

$$\therefore P(W=a, Z=b) \neq P(W=a)\cdot P(Z=b)$$

not necessary

$\therefore$ W and Z aren't independent

3.

$$E[W] = \sum w P(w) = 2\cdot\frac{2}{9} + 3\cdot\frac{3}{9} + 4\cdot\frac{1}{3} + 5\cdot\frac{2}{9} + 6\frac{1}{9}$$
$$= 4$$

$$E[Z] = \sum z P(z) = (-2)\frac{1}{9} + (-1)\cdot\frac{2}{9} + 0\cdot\frac{1}{3} + 1\cdot\frac{2}{9} + 2\cdot\frac{1}{9}$$
$$= 0$$

P4

1. $$f_x(x) = \int_0^\infty abe^{-ax-by}\,dy$$

$$= ab\int_0^\infty e^{-ax-by}\,dy$$
$$= ab\left[\left(-\frac{1}{b}\right)e^{-ax-by}\right]_0^\infty$$
$$= ab\cdot\frac{1}{b}e^{-ax}$$
$$= a\cdot e^{-ax}$$

$$\therefore f_x(x) = \begin{cases} a\cdot e^{-ax}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^{\infty} abe^{-ax-by} dx$$

$$= ab \left[-\frac{1}{a} e^{-ax-by}\right]_0^{\infty}$$

$$= ab \left[\frac{1}{a} \cdot e^{-by}\right]$$

$$= be^{-by}$$

$$\therefore f_Y(y) = \begin{cases} b \cdot e^{-by}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

2. $$E[X] = \int_0^{\infty} x \cdot ae^{-ax} dx$$

$$= a \int_0^{\infty} x e^{-ax} dx$$

Integral by parts

$u = x$, $u' = 1$

$v' = e^{-ax}$ $v = -\frac{e^{-ax}}{a}$

$$= a \left[-x \cdot \frac{e^{-ax}}{a} + \int \frac{e^{-ax}}{a}\right]_0^{\infty}$$

$$= a \left(-x \cdot \frac{e^{-ax}}{a} - \frac{e^{-ax}}{a^2}\right)_0^{\infty}$$

$$= a \cdot \frac{1}{a^2} = \frac{1}{a}$$

$$E[Y] = \int_0^{\infty} y \cdot be^{-by} dy$$

$$= b \int_0^{\infty} ye^{-by} dx$$

Integral by parts

$$= b \left[-y \cdot \frac{e^{-by}}{} + \int \frac{e^{-by}}{}\right]_0^{\infty}$$

$$= b\left(y\cdot\frac{e^{-by}}{b} - \frac{e^{-by}}{b^2}\right)_0^{\infty}$$

$$= b\cdot\frac{1}{b^2} = \frac{1}{b}$$

3. $P(X<Y) = \int_0^{\infty}\int_0^{Y} f_{X,Y}(x,y)\,dx\,dy$

$$= ab\int_0^{\infty}\int_0^{Y} e^{-ax-by}\,dx\,dy$$

$$= ab\cdot\int_0^{\infty}\left[-\frac{e^{-ax-by}}{a}\right]_0^{Y} dy$$

$$= b\cdot\int_0^{\infty}\left(-e^{-(a+b)Y} + e^{-by}\right)dy$$

$$= b\cdot\left[\frac{e^{-(a+b)Y}}{a+b} - \frac{e^{-by}}{b}\right]_0^{\infty}$$

$$= b\cdot\left(-\frac{1}{a+b} + \frac{1}{b}\right)$$

$$= -\frac{b}{a+b} + 1$$

P5

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & \text{if } 0<x<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_x^{\infty} e^{-y}\,dy = [-e^{-y}]_x^{\infty} = e^{-x}$$

$$f_Y(y) = \int^{y} e^{-y}\,dx = [-e^{-y}\cdot x]_0^{y} = -e^{-y}\cdot y$$

$$P(X=x, Y=y) = e^{-y} \neq e^{-x} \cdot e^{-y} \cdot y = P(X=x) \cdot P(Y=y)$$

$\therefore$ X and Y aren't independent