Xi Liu, xl3504, Problem Set 9

Problem 1

let $N_{(t_1,t_2]}$ be the total number of arrivals in the time interval $(t_1, t_2]$ for the Poisson process

$$P(N_{(t_1,t_2]} = i) = \frac{(\lambda(t_2 - t_1))^i}{i!} e^{-\lambda(t_2 - t_1)}$$

$$P(N_{(0,2]} = 2, \; N_{(1,4]} = 3)$$
$$= P(N_{(0,1]} = 0, \; N_{(1,2]} = 2, \; N_{(2,4]} = 1)$$
$$+ P(N_{(0,1]} = 1, \; N_{(1,2]} = 1, \; N_{(2,4]} = 2)$$
$$+ P(N_{(0,1]} = 2, \; N_{(1,2]} = 0, \; N_{(2,4]} = 3)$$
/* since $(0, 1]$, $(1, 2]$, and $(2, 4]$ are disjoint time intervals,
$N_{(0,1]}$, $N_{(1,2]}$, and $N_{(2,4]}$ are independent random variables */
$$= P(N_{(0,1]} = 0)P(N_{(1,2]} = 2)P(N_{(2,4]} = 1)$$
$$+ P(N_{(0,1]} = 1)P(N_{(1,2]} = 1)P(N_{(2,4]} = 2)$$
$$+ P(N_{(0,1]} = 2)P(N_{(1,2]} = 0)P(N_{(2,4]} = 3)$$
$$= \frac{(\lambda(1 - 0))^0}{0!} e^{-\lambda(1-0)} \frac{(\lambda(2 - 1))^2}{2!} e^{-\lambda(2-1)} \frac{(\lambda(4 - 2))^1}{1!} e^{-\lambda(4-2)}$$
$$+ \frac{(\lambda(1 - 0))^1}{1!} e^{-\lambda(1-0)} \frac{(\lambda(2 - 1))^1}{1!} e^{-\lambda(2-1)} \frac{(\lambda(4 - 2))^2}{2!} e^{-\lambda(4-2)}$$
$$+ \frac{(\lambda(1 - 0))^2}{2!} e^{-\lambda(1-0)} \frac{(\lambda(2 - 1))^0}{0!} e^{-\lambda(2-1)} \frac{(\lambda(4 - 2))^3}{3!} e^{-\lambda(4-2)}$$
$$= e^{-\lambda} \lambda^2 e^{-\lambda} \lambda e^{-2\lambda} + 2\lambda e^{-\lambda} \lambda e^{-\lambda} \lambda^2 e^{-2\lambda} + \frac{\lambda^2}{2} e^{-\lambda} e^{-\lambda} \frac{4\lambda^3}{3} e^{-2\lambda}$$
$$= \lambda^3 e^{-4\lambda} + 2\lambda^4 e^{-4\lambda} + \frac{2\lambda^5}{3} e^{-4\lambda}$$
$$= \boxed{\lambda^3 e^{-4\lambda} \left(1 + 2\lambda + \frac{2\lambda^2}{3}\right)}$$

Problem 2
part 1
if X follows a Poisson distribution with parameter λ, then

$$\forall i \in \mathbb{N}, \ p_X(i) = P(X = i) = \frac{\lambda^i}{i!}e^{-\lambda}$$

if Y follows a Poisson distribution with parameter μ, then

$$\forall i \in \mathbb{N}, \ p_Y(i) = P(Y = i) = \frac{\lambda^i}{i!}e^{-\mu}$$

$$\forall(x, y) \in \mathbb{R}^2, \ p_{X,Y}(x, y) = p_X(x)p_Y(y)$$
$$\forall z \in \mathbb{R}$$
$$p_Z(z) = P(Z = z)$$
$$= \sum_x P(X = x, \ Y = z - x)$$
$$= \sum_x p_{X,Y}(x, z - x)$$
$$= \sum_x p_X(x)p_Y(z - x)$$
$$= \sum_{x=0}^{z} \left(\frac{\lambda^x}{x!}e^{-\lambda}\right)\left(\frac{\mu^{z-x}}{(z-x)!}e^{-\mu}\right)$$
$$= \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!}e^{-(\lambda+\mu)}$$
$$= e^{-(\lambda+\mu)}\sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!}$$
$$= \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z} \frac{z!\lambda^x \mu^{z-x}}{x!(z-x)!}$$
$$= \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z} \frac{z!}{x!(z-x)!}\lambda^x \mu^{z-x}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x}$$

since the binomial theorem says

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

so

$$p_Z(z) = \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x}$$

$$= \boxed{\frac{e^{-(\lambda+\mu)}}{z!} (\lambda + \mu)^z}$$

part 2
application:

$$P_Z(z) = \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} e^{-\lambda-\mu}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!}(\lambda+\mu)^z$$

$$p_Z(10) = \sum_{x=0}^{10} \frac{(8.392)^x (7.854)^{10-x}}{x!(10-x)!} e^{-8.392-7.854}$$

$$= \boxed{\frac{e^{-(8.392+7.854)}}{10!}(8.392+7.854)^{10}}$$

$$\approx \boxed{0.0310566}$$

Problem 3

a binomial distribution with parameters n, p, and k has a probability mass function

$$\forall k \in \mathbb{N}$$

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np$$

$$Var(X) = np(1-p)$$

1.

Markov's inequality says $\forall a \in \mathbb{R}, \ a \geq 0$

$$E[X] \geq aP(X \geq a)$$

$$a := kn$$

$$E[X] \geq knP(X \geq kn)$$

$$P(X \geq kn) \leq \frac{E[X]}{kn} = \frac{np}{kn} = \boxed{\frac{p}{k}}$$

2.

Chebyshev's inequality says

$$P(|X - E[X]| \geq |a|) \leq \frac{1}{a^2} Var(X)$$

$$\begin{aligned} P(X \geq kn) &= P(X - np \geq kn - np) \\ &\leq P(|X - np| \geq kn - np) \\ &\leq \frac{1}{(kn - np)^2} Var(X) \\ &= \frac{np(1 - p)}{(n(k - p))^2} \\ &= \frac{np(1 - p)}{n^2(k - p)^2} \\ &= \boxed{\frac{p(1 - p)}{n(k - p)^2}} \end{aligned}$$

3.
Markov's inequality:

$$P(X \geq kn) \leq \frac{p}{k} = \frac{1/2}{3/4} = \frac{2}{3}$$

Chebyshev's inequality:

$$P(X \geq kn) \leq \frac{p(1 - p)}{n(k - p)^2} = \frac{1/2(1 - 1/2)}{n(3/4 - 1/2)^2} = \frac{1/4}{n(1/16)} = \frac{4}{n}$$

the smaller upper bound is the tighter upper bound

$$\lim_{n \to \infty} \frac{2/3}{4/n} = \lim_{n \to \infty} \frac{n}{6} = \infty$$

so the Chebyshev's inequality's upper bound $(4/n)$ is the tighter bound

Problem 4
apply Chebyshev's inequality:

$$\forall \epsilon > 0$$

$$P(|\overline{X_n} - \mu| > \epsilon) \leq \frac{1}{\epsilon^2} Var(\overline{X_n}) = \frac{\sigma^2}{n\epsilon^2}$$

90% sure that the average of the measurements is within half a degree Kelvin of T:

$$E[U_i] = \mu = 0$$

$$P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \geq 0.9$$

$$-P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \leq -0.9$$

$$1 - P\left(|\overline{X_n} - 0| \leq \frac{1}{2}\right) \leq 1 - 0.9$$

$$P\left(|\overline{X_n} - 0| > \frac{1}{2}\right) \leq 0.1$$

Chebyshev's inequality gives

$$P\left(|\overline{X_n} - 0| > \frac{1}{2}\right) \leq \frac{3}{n(1/2)^2} = \frac{12}{n}$$

so, to be 90% sure that the average of the measurements is within half a degree Kelvin of T, number of measurements n should be such that

$$\frac{12}{n} \leq 0.1$$

$$\boxed{n \geq 120}$$

Problem 5
apply Chebyshev's inequality:

$$\forall \epsilon > 0$$

$$P(|\overline{X_n} - \mu| > \epsilon) \leq \frac{1}{\epsilon^2} Var(\overline{X_n}) = \frac{\sigma^2}{n\epsilon^2}$$

let X correspond to the result of rolling a fair die

$$n := 100$$

$$E[X] = \sum_{a_i} a_i p_X(a_i) = \sum_{a_i=1}^{6} a_i \frac{1}{6} = 3.5 = \frac{7}{2}$$

$$E[X^2] = \sum_{a_i} a_i^2 p_X(a_i) = \sum_{a_i=1}^{6} a_i^2 \frac{1}{6} = \frac{91}{6}$$

$$\sigma^2 = Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$P(3.2 \leq \overline{X_{100}} \leq 3.8) = P(-0.3 \leq \overline{X_{100}} - 3.5 \leq 0.3)$$
$$= P(|\overline{X_{100}} - 3.5| \leq 0.3)$$
$$= 1 - P(|\overline{X_{100}} - 3.5| > 0.3)$$
$$/* \text{ since } P(|\overline{X_{100}} - 3.5| > 0.3) \leq \frac{\sigma^2}{n\epsilon^2} \; */$$
$$\geq 1 - \frac{\sigma^2}{n\epsilon^2}$$
$$= 1 - \frac{35/12}{(100)(0.3)^2}$$
$$= 1 - \frac{35}{108}$$
$$= \boxed{\frac{73}{108}}$$