MATH-UA.0235 Probability and Statistics – Worksheet # 3

Continuous Random Variables

Problem 1

Consider a continuous random variable X with probability density function

$$f(x) = \begin{cases} \frac{4}{3}(1-x)^{\frac{1}{3}} & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(0.488 < X \leq 1.2)$.

We are looking for $P(0.488 < X \leq 1.2) = \int_{0.488}^{1.2} f(x)dx = \frac{4}{3}\int_{0.488}^{1}(1-x)^{\frac{1}{3}}dx = -\left[(1-x)^{\frac{4}{3}}\right]_{0.488}^{1} = 0.512^{\frac{4}{3}} \approx 0.4096.$

Problem 2

In a gas station, the weekly demand for gas, in units of thousands of gallons, is a continuous random variable with probability density function

$$f(x) = \begin{cases} a(1-x)^4 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

1. For what value of a is f an appropriate probability density function?

   We know that f must satisfy the condition

   $$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \Leftrightarrow \quad a\int_0^1 (1-x)^4 dx = 1 \quad \Leftrightarrow \quad -\frac{a}{5}\left[(1-x)^5\right]_0^1 = 1 \quad \Leftrightarrow \quad a = 5$$

2. The gas station is refilled every Monday morning at 3am. How big must the gas reservoir of the gas station be for the probability of the gas station to run out of gas in a given week to be less than 10^{-5}?

   We are looking for L such that $P(X \geq L) \leq 10^{-5} \quad \Leftrightarrow \quad P(X \leq L) \geq 0.99999.$

   We thus want to find L such that $F(L) \geq 0.99999$. This can be written as

   $$\int_{-\infty}^{L} f(x)dx \geq 0.99999 \quad \Leftrightarrow 5\int_0^L (1-x)^4 dx \geq 0.99999 \quad \Leftrightarrow \quad -\left[(1-x)^5\right]_0^L \geq 0.99999 \quad \Leftrightarrow \quad (1-L)^5 \leq 0.00001$$

   We thus have $L \geq 0.9$. Since we are measuring in units of thousands of gallons, the reservoir must contain at least 900 gallons.

Problem 3

Consider the function

$$f(x) = \frac{a}{1+x^2}$$

1. For which value of a can f be the probability density function of a continuous random variable?

   For f to be a probability density function, we need the additional condition

   $$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \Leftrightarrow \quad a\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 1 \quad \Leftrightarrow \quad a\left[\text{atan}(x)\right]_{-\infty}^{+\infty} = 1 \quad \Leftrightarrow \Leftrightarrow a(\pi/2 - (-\pi/2)) = 1$$

   We thus need to have $a = \frac{1}{\pi}$.

2. For that value of a, f is called the *standard Cauchy distribution*. Compute the cumulative distribution function F of the standard Cauchy distribution.

   We have
$$F(b) = \int_{-\infty}^{b} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{b} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\operatorname{atan}(b) + \frac{\pi}{2}\right) = \frac{\operatorname{atan}(b)}{\pi} + \frac{1}{2}$$

3. Does the Cauchy distribution have a defined expected value? If yes, what is the expected value?

   The expected value is given by
$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x}{1+x^2}dx = \frac{1}{2}\left[1+x^2\right]_{-\infty}^{+\infty}$$

   We observe that the last expression does not have a finite limit, which means that the integral does not converge, and the random variable does not have a defined expected value.

Problem 4

Consider the function
$$f(x) = \begin{cases} \frac{a}{x\sqrt{x}} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. For which value of a can f be the probability density function of a continuous random variable?

   For f to be a probability density function, we need the additional condition
$$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad\Leftrightarrow\quad a\int_{1}^{+\infty} \frac{dx}{x\sqrt{x}} = 1 \quad\Leftrightarrow\quad -2a\left[\frac{1}{\sqrt{x}}\right]_{1}^{+\infty} = 1$$

   We thus need to have $a = \frac{1}{2}$.

2. For that value of a, compute the corresponding cumulative distribution function.

   For $b < 1$, we have $F(b) = 0$. For $b \geq 1$, we may write
$$F(b) = \int_{-\infty}^{b} f(x)dx = \frac{1}{2}\int_{1}^{b} \frac{dx}{x\sqrt{x}} = -\left[\frac{1}{\sqrt{x}}\right]_{1}^{b} = 1 - \frac{1}{\sqrt{b}}$$

3. Does this random variable have a defined expected value? If yes, what is the expected value?

   The expected value is given by
$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{2}\int_{1}^{+\infty} \frac{1}{\sqrt{x}}dx = \left[\sqrt{x}\right]_{1}^{+\infty}$$

   We observe that the last expression does not have a finite limit, which means that the integral does not converge, and the random variable does not have a defined expected value.

Problem 5

Consider the function
$$f(x) = ae^{-|x|}$$

1. For which value of a can f be the probability density function of a continuous random variable?

   For f to be a probability density function, we need the additional condition
$$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad\Leftrightarrow\quad a\left(\int_{-\infty}^{0} e^{x}dx + \int_{0}^{+\infty} e^{-x}dx\right) = 1 \quad\Leftrightarrow\quad a\left(\left[e^{x}\right]_{-\infty}^{0} - \left[e^{-x}\right]_{0}^{+\infty}\right) = 1$$

   We thus need
$$a = \frac{1}{2}$$

2. What is the expected value of this continuous random variable?

   The expected value of the random variable is given by

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

   Since f is an even function, $g(x) = xf(x)$ is an odd function, so $E[X] = 0$.

Problem 6

Let X be a continuous random variable which takes values in $[-1, 1]$. The graph of its probability density function f is an isosceles triangle (including the x-axis as one of the sides).

1. Give an expression for f.

   We know the area under the curve must be equal to one, so the highest value for f must be 1, obtained at $x = 0$. We may therefore write

$$f(x) = \begin{cases} x+1 & \text{if } x \in [-1, 0] \\ 1-x & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

2. Compute the corresponding cumulative distribution function F.

   Without any calculation, we may readily write

$$F(b) = \begin{cases} 0 & \text{if } b \leq -1 \\ 1 & \text{if } b \geq 1 \end{cases}$$

   Now, if $b \in [-1, 0]$,

$$F(b) = \int_{-\infty}^{b} f(x) dx = \int_{-1}^{b} (x+1) dx = \frac{1}{2}\left[(x+1)^2\right]_{-1}^{b} = \frac{(b+1)^2}{2}$$

   If $b \in [0, 1]$,

$$F(b) = \int_{-\infty}^{b} f(x) dx = \left(\int_{-1}^{0} (x+1) dx + \int_{0}^{b} (1-x) dx\right) = \frac{1}{2}\left(\left[(x+1)^2\right]_{-1}^{0} - \left[(1-x)^2\right]_{0}^{b}\right) = \frac{2-(1-b)^2}{2}$$

3. What is the expected value of this random variable?

   Since f is an even function, $g(x) = xf(x)$ is an odd function, so $E[X] = 0$.