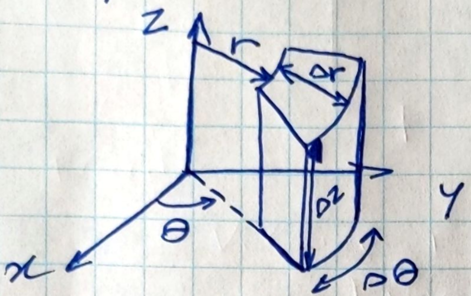
S3 Coordinate Systems & Shell Balance

Cylindrical Coordinates



for small $\Delta r \rightsquigarrow V = r\Delta\theta\,\Delta r\,\Delta z$

$$A_r = r\Delta\theta\,\Delta z$$

$$A_\theta = \Delta r\,\Delta z$$

$$A_z = r\Delta\theta\,\Delta r$$

$$\rightsquigarrow \frac{\partial}{\partial t}(r\Delta\theta\Delta r\Delta z\,\rho) = (\rho u_r\, r\Delta\theta\Delta z)\big|_r - (\rho u_r\, r\Delta\theta\Delta z)\big|_{r+\Delta r} + (\rho u_\theta\,\Delta r\Delta z)\big|_\theta - (\rho u_\theta\,\Delta r\Delta z)\big|_{\theta+\Delta\theta} + (\rho u_z\, r\Delta\theta\Delta r)\big|_z - (\rho u_z\, r\Delta\theta\Delta r)\big|_{z+\Delta z}$$

divide by $\Delta r\Delta\theta\Delta z \Rightarrow \frac{\partial}{\partial t}(r\rho) = \frac{(\rho u_r r)|_r - (\rho u_r r)_{r+\Delta r}}{\Delta r}$
let $\Delta r, \Delta\theta, \Delta z \to 0$

$$+ \frac{(\rho u_\theta)|_\theta - (\rho u_\theta)|_{\theta+\Delta\theta}}{\Delta\theta} + r\,\frac{(\rho u_z)|_z - (\rho u_z)|_{z+\Delta z}}{\Delta z}$$

$$\rightsquigarrow r\frac{\partial\rho}{\partial t} + \frac{\partial(r\rho u_r)}{\partial r} + \frac{\partial(\rho u_\theta)}{\partial\theta} + r\frac{\partial(\rho u_z)}{\partial z} = 0$$

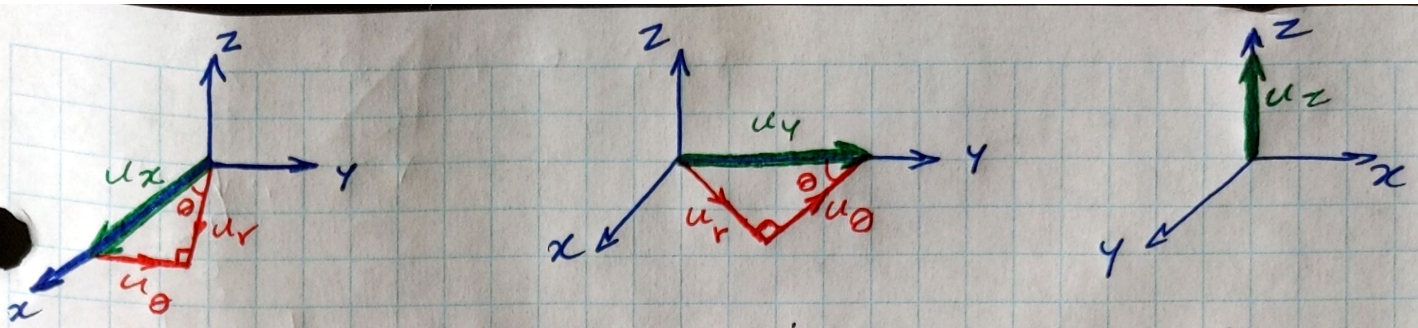
$$\rightsquigarrow \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial\theta} + \frac{\partial(\rho u_z)}{\partial z} = 0$$

Continuity eq in cylindrical coordinates

Could we find this from the eqn of the cartesian coordinates?

Cartesian: $\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_x}{\partial x} + \frac{\partial\rho u_y}{\partial y} + \frac{\partial\rho u_z}{\partial z} = 0$

$\rightsquigarrow$ • express u_x, u_y, u_z in terms of the scalar velocities in the cylindrical coordinates.



$u_x = u_r\cos\theta - u_\theta\sin\theta$ $\qquad$ $u_y = u_r\sin\theta + u_\theta\cos\theta$ $\qquad$ $u_z = u_z$

- express x, y, z in terms of r, θ, z and vice versa.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \longrightarrow r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$z = z$

$$\rightsquigarrow \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta; \quad \frac{\partial r}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}; \quad \frac{\partial \theta}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u_x)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(\rho u_x)\frac{\partial \theta}{\partial x} + \cancelto{0}{\frac{\partial}{\partial z}(\rho u_x)\frac{\partial z}{\partial x}}$$
$$+ \frac{\partial}{\partial r}(\rho u_y)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}(\rho u_y)\frac{\partial \theta}{\partial y} + \cancelto{0}{\frac{\partial}{\partial z}(\rho u_y)\frac{\partial z}{\partial y}}$$
$$+ \cancelto{0}{\frac{\partial}{\partial r}(\rho u_z)\frac{\partial r}{\partial z}} + \cancelto{0}{\frac{\partial}{\partial \theta}(\rho u_z)\frac{\partial \theta}{\partial z}} + \frac{\partial}{\partial z}(\rho u_z)\cancelto{1}{\frac{\partial z}{\partial z}}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u_r\cos\theta - \rho u_\theta\sin\theta)(\cos\theta) + \frac{\partial}{\partial \theta}(\rho u_r\cos\theta - \rho u_\theta\sin\theta)\left(\frac{-\sin\theta}{r}\right)$$
$$+ \frac{\partial}{\partial r}(\rho u_r\sin\theta + \rho u_\theta\cos\theta)(\sin\theta) + \frac{\partial}{\partial \theta}(\rho u_r\sin\theta + \rho u_\theta\cos\theta)\left(\frac{\cos\theta}{r}\right)$$
$$+ \frac{\partial}{\partial z}(\rho u_z)$$

$$= \frac{\partial \rho}{\partial t} + \cos\theta\left[\cancel{\cos\theta\frac{\partial \rho u_r}{\partial r}} - \cancel{\sin\theta\frac{\partial \rho u_\theta}{\partial r}}\right] - \frac{\sin\theta}{r}\left[\cancel{\cos\theta\frac{\partial \rho u_r}{\partial \theta}} - \sin\theta\,\rho u_r - \cancel{\sin\theta\frac{\partial \rho u_\theta}{\partial \theta}} - \cancel{\cos\theta\,\rho u_\theta}\right]$$
$$+ \sin\theta\left[\cancel{\sin\theta\frac{\partial \rho u_r}{\partial r}} + \cancel{\cos\theta\frac{\partial \rho u_\theta}{\partial r}}\right] + \frac{\cos\theta}{r}\left[\cancel{\sin\theta\frac{\partial \rho u_r}{\partial \theta}} + \cos\theta\,\rho u_r + \cancel{\cos\theta\frac{\partial \rho u_\theta}{\partial \theta}} - \cancel{\sin\theta\,\rho u_\theta}\right]$$
$$+ \frac{\partial}{\partial z}(\rho u_z) = 0$$

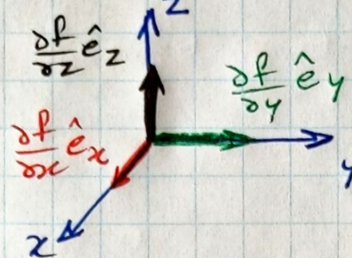
$$\rightsquigarrow \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_r}{\partial r}(\cos^2\theta + \sin^2\theta) + \frac{\partial \rho u_\theta}{\partial \theta}\left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right) + \cancel{\frac{\partial \rho u_r}{\partial \theta}(0)} + \cancel{\frac{\partial \rho u_\theta}{\partial r}(0)}$$
$$+ \rho u_r\left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right) + \cancel{\rho u_\theta(0)} + \frac{\partial \rho u_z}{\partial z} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial \rho u_\theta}{\partial \theta} + \frac{\partial \rho u_z}{\partial z} = 0 \quad ✓$$

Gradient of a scalar ∇f

Cartesian coordinates $\Rightarrow \nabla f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$

$\uparrow$ a vector



$\hat{e}_x, \hat{e}_y, \hat{e}_z$: unit vectors in x, y, z directions

Cylindrical system? $\hat{e}_x = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$

$\hat{e}_y = \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta$

$\hat{e}_z = \hat{e}_z$

$$\rightsquigarrow \nabla f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\left(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta\right) + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\left(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta\right)$$

$$+ \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\left(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta\right) + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\left(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta\right)$$

$$+ \frac{\partial f}{\partial z}\hat{e}_z$$

$$= \frac{\partial f}{\partial r}(\cos\theta)\left(\hat{e}_r\cancel{\cos\theta - \hat{e}_\theta}\sin\theta\right) + \frac{\partial f}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)\left(\cancel{\hat{e}_r}\cos\theta - \cancel{\hat{e}_\theta}\sin\theta\right)$$

$$+ \frac{\partial f}{\partial r}(\sin\theta)\left(\cancel{\hat{e}_r}\sin\theta + \cancel{\hat{e}_\theta}\cos\theta\right) + \frac{\partial f}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\left(\cancel{\hat{e}_r}\sin\theta + \hat{e}_\theta\cos\theta\right) + \frac{\partial f}{\partial z}\hat{e}_z$$

$$= \frac{\partial f}{\partial r}\left[\cos^2\theta\,\hat{e}_r - \cancel{\sin\theta\cos\theta\,\hat{e}_\theta} + \sin^2\theta\,\hat{e}_r + \cancel{\sin\theta\cos\theta\,\hat{e}_\theta}\right]$$

$$+ \frac{\partial f}{\partial \theta}\left[\cancel{\frac{-\sin\theta\cos\theta}{r}\hat{e}_r} + \frac{\sin^2\theta}{r}\hat{e}_\theta + \cancel{\frac{\sin\theta\cos\theta}{r}\hat{e}_r} + \frac{\cos^2\theta}{r}\hat{e}_\theta\right] + \frac{\partial f}{\partial z}\hat{e}_z$$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$