MATH-UA.0251 Intro Math Modeling

§1 Dimensional Analysis
Buckingham π theorem

Characteristics of a physical system

- qualitative: length, time, stress, velocity, etc
- quantitative: 2 m, 5 s, 100 Pa, etc.

a number — a standard (unit)

given in terms of "primary" quantities: $L, T, M, \Theta, C, \ldots$

length, temperature, time, mass, charge

(*) primary quantities can then be used to provide qualitative description of any "secondary" quantity.

dimensions

EX. Area $\doteq L^2$, Velocity $\doteq LT^{-1}$, ...

the dimensions of an area is length x length

Dimensional Homogeneity
- dimensions of LHS $\doteq$ dimensions of RHS
- $X + Y \to X$ and Y must have the same dimensions.

(e.g., $L + M$ is meaningless!)

EX. Gravitational force: $F = G\dfrac{m_1 m_2}{r^2}$

m_1: mass of object 1 $\doteq M$

m_2: mass of object 2 $\doteq M$

r: distance between the two objects $\doteq L$

F: force $\doteq MLT^{-2}$

quantitative? $\approx 6.67\times10^{-11}\ \dfrac{m^3}{kg\,s^2}$

$$\rightsquigarrow MLT^{-2} \doteq [G]\,\frac{M\times M}{L^2} \longrightarrow [G] = L^3M^{-1}T^{-2}$$

dimensions of G

Similitude

- Analytical solution to a problem → an explicit formula
- Many laws in physics
- simple models

EX. $F = ma \Rightarrow$ How does "a" depend on "F" and "m"? Easy!

What if we don't have such formula?
Experiments? Numerical solution?

EX. Flow of a fluid through a long, horizontal, circular pipe → our concern: pressure drop per unit length

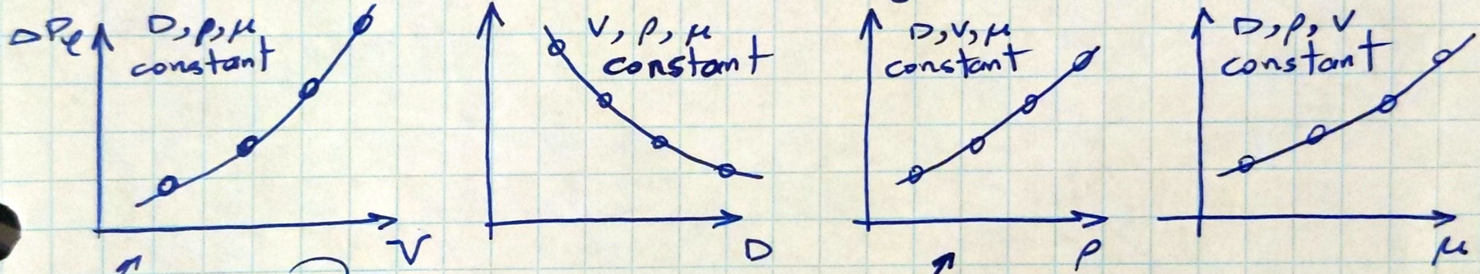
Question: what are the factors that will have an effect on the pressure drop?

$\Delta P_\ell = f(D, \rho, \mu, V)$ → ΔP_ℓ is some function of D, ρ, μ, V.

D: diameter, ρ: density, μ: viscosity, V: velocity

Find the nature of this function by experiments:

change one of the parameters and keep all others constant → what happens to ΔP_ℓ?



valid for the specific typ of fluid used

how possibly can one change ρ and keep μ constant?

* Even if you find all of the fitting curves, how do you combine these data to obtain the desired general functional relationship which woul be valid for any similar pipe system? e.g., water vs oil?

concept of similitude

* too many parameters

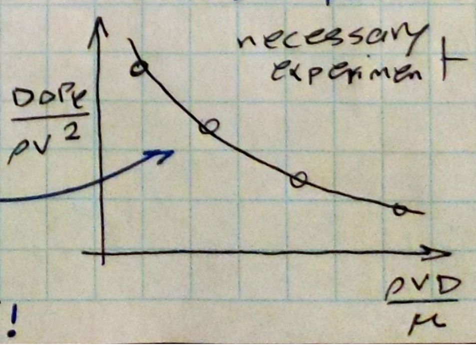
Fortunately, there is a much simpler approach to this problem → 2 dimensionless groups instead of 5 dimensional parameters

$$\frac{D\,\Delta P_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

universal (valid for any similar system)

→ not dependent on the system of units!

EX. If two systems have the same $\frac{\rho V D}{\mu}$, they will have the same $\frac{D\,\Delta P_\ell}{\rho V^2}$ as well.



$\rightsquigarrow$ dimensional analysis $\rightarrow$ Buckingham π theorem

* If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k−r independent dimensionless groups, where r is the minum number of reference dimensions required.

π terms

$$u_1 = f(u_2, u_3, \ldots, u_k) \rightsquigarrow \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Determination of Π terms

- List all the variables that are involved in the problem.
- Express each of the variables in terms of basic dimensions.
- Determine the required number of Π terms: k−r
- select "r" repeating variables (independent) that cover all of the required dimensions together. → (not the dependent one)
- Form a Π term by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent.
- Find the exponents so that each Π term is dimensionless.

EX. $\Delta P_e = f(D, \rho, \mu, V)$

$\Delta P_e \doteq FL^{-3} \doteq ML^{-2}T^{-2}$; $D \doteq L$; $\rho \doteq ML^{-3}$; $\mu \doteq ML^{-1}T^{-1}$; $V \doteq LT^{-1}$

$r = 3 \Rightarrow$ number of Π terms $= 5 - 3 = 2$

repeating variables: take D, ρ, & V

$$\rightsquigarrow \Pi_1 = \Delta P_e D^a V^b \rho^c \doteq ML^{-2}T^{-2} L^a L^b T^{-b} M^c L^{-3c} \doteq M^0 L^0 T^0$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1 \Rightarrow \Pi_1 = \frac{D \Delta P_e}{\rho V^2}$$

$$\Pi_2 = \mu D^a V^b \rho^c \doteq ML^{-1}T^{-1} L^a L^b T^{-b} M^c L^{-3c}$$

$$\Rightarrow a = -1,\ b = -1,\ c = -1 \Rightarrow \Pi_2 = \frac{\mu}{DV\rho}$$

$$\rightsquigarrow \frac{D \Delta P_e}{\rho V^2} = \phi\left(\frac{\mu}{DV\rho}\right)$$

$$= \bar{\phi}\left(\frac{\rho V D}{\mu}\right)$$

Π_1 Π_2