S4 Heat eqn $\equiv$ species continuity eqn

Recall the continuity eqn in the Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0$$

More generally: $\phi x, \phi y, \phi z$ Can be mass, heat ......

$$\frac{\partial \Psi}{\partial t} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} = S$$

S: generation source (quantity per unit volume per unit time)

Ψ: quantity of interest per unit volume

e.g., $\text{Mass}/\text{Volume}$ = Density (ρ)

$\phi_{x,y,z}$: flux of the quantity in x,y,z directions

Ex. For total mass (quantity):

$S=0$, $\Psi=\rho$, $\phi_x=\rho u_x$, $\phi_y=\rho u_y$, $\phi_z=\rho u_z$ $\rightarrow$ Continuity eqn

Cylindrical coordinates? Easy!

$$\frac{\partial \Psi}{\partial t} + \frac{1}{r}\frac{\partial \overset{\rho u_r}{\cancel{\phi_r}}}{\partial r} + \frac{1}{r}\frac{\partial \overset{\rho u_\theta}{\phi_\theta}}{\partial \theta} + \frac{\partial \overset{\rho u_z}{\phi_z}}{\partial z} = S$$

$\nabla f = \frac{d}{dx}\hat{e}_x + \frac{d}{dy}\hat{e}_y + \frac{d}{dz}\hat{e}_z$

$\phi = [\phi_x, \phi_y, \phi_z]$

In vector form $\longrightarrow$ $\boxed{\frac{\partial \Psi}{\partial t} + \underline{\nabla} \cdot \underline{\phi} = S}$

Heat eqn in Cartesian Coordinates:

previously, $\Psi = \frac{\text{mass}}{\text{volume}}$

$$\Psi = \frac{\text{Energy}}{\text{Volume}} = \frac{m C_p T}{V} = \rho C_p T \qquad \frac{d\Psi}{dt} = \frac{d\rho C_p T}{dt}$$

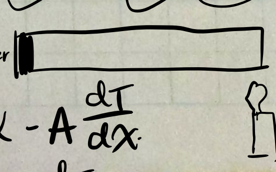
vector form

$$\phi_x = -k\frac{\partial T}{\partial x} + \rho u_x C_p T \qquad (\underline{\phi} = -k\nabla T + \rho \underline{u} C_p T)$$

y, z y, z y, z

Fourier's law (conduction)   flux of energy carried by fluid (Convection)

heat transfer



$$\dot{Q} \propto -A\frac{dT}{dx}$$

$q \propto \frac{dT}{dx} \Rightarrow q = -k\frac{dT}{dx}$

k: conductivity

$\Rightarrow q = -k\nabla T$.

$$\rightarrow \frac{\partial \rho C_p T}{\partial t} + \frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x} + \rho u_x C_p T\right) + \frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y} + \rho u_y C_p T\right) + \frac{\partial}{\partial z}\left(-k\frac{\partial T}{\partial z} + \rho u_z C_p T\right) = S$$

$$\rightsquigarrow \frac{\partial \rho C_p T}{\partial t} + \underbrace{\frac{\partial \rho C_p u_x T}{\partial x} + \frac{\partial \rho C_p u_y T}{\partial y} + \frac{\partial \rho C_p u_z T}{\partial z}}_{\text{convection terms}}$$

$$= \underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}_{\text{conduction terms.}} + S$$

For constant k, ρ, C_p:

$$\rho C_p\left(\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

factor out the constant terms

Heat eqn

In previous:

$$\cancel{\frac{d\rho}{dt}} + \frac{d\rho u_x}{dx} + \frac{d\rho u_y}{dy} + \frac{d\rho u_z}{dz} = 0$$

if ρ is constant:

$$\frac{du_x}{dx} + \frac{du_y}{dy} + \frac{du_z}{dz} = 0.$$

Gradient form:

$$\rho c_p\left[\frac{dT}{dt} + u\cdot\nabla T\right] = k\cdot\nabla^2 T + S$$