Solutions Manual for: Understanding Analysis, Second Edition

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Author's note

What began as a desire to sketch out a simple answer key for the problems in *Understanding Analysis* inevitably evolved into something a bit more ambitious. As I was generating solutions for the nearly 200 odd-numbered exercises in the text, I found myself adding regular commentary on common pitfalls and strategies that frequently arise. My sense is that this manual should be a useful supplement to instructors teaching a course or to individuals engaged in an independent study. As with the textbook itself, I tried to write with the introductory student firmly in mind. In my teaching of analysis, I have come to understand the strong correlation between how students learn analysis and how they write it. A final goal I have for these notes is to illustrate by example how the form and grammar of a written argument are intimately connected to the clarity of a proof and, ultimately, to its validity.

The decision to include only the odd-numbered exercises was a compromise between those who view access to the solutions as integral to their educational needs, and those who strongly prefer that no solutions be available because of the potential for misuse. The total number of exercises was significantly increased for the second edition, and almost every even-numbered problem (in the regular sections of the text) is one that did not appear in the first edition. My hope is that this arrangement will provide ample resources to meet the distinct needs of these different audiences.

I would like to thank former students Carrick Detweiller, Katherine Ott, Yared Gurmu, and Yuqiu Jiang for their considerable help with a preliminary draft. I would also like to thank the readers of *Understanding Analysis* for the many comments I have received about the text. Especially appreciated are the constructive suggestions as well as the pointers to errors, and I welcome more of the same.

Middlebury, Vermont — Stephen Abbott
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Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

1.2 Some Preliminaries

Exercise 1.2.1. (a) Assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 3. \tag{1}$$

Let us also assume that p and q have no common factor. Now, equation (1) implies

$$p^2 = 3q^2. \tag{2}$$

From this, we can see that p^2 is a multiple of 3 and hence p must also be a multiple of 3. This allows us to write $p = 3r$, where r is an integer. After substituting $3r$ for p in equation (2), we get $(3r)^2 = 3q^2$, which can be simplified to $3r^2 = q^2$. This implies q^2 is a multiple of 3 and hence q is also a multiple of 3. Thus we have shown p and q have a common factor, namely 3, when they were originally assumed to have no common factor.

A similar argument will work for $\sqrt{6}$ as well because we get $p^2 = 6q^2$ which implies p is a multiple of 2 and 3. After making the necessary substitutions, we can conclude q is a multiple of 6, and therefore $\sqrt{6}$ must be irrational.

(b) In this case, the fact that p^2 is a multiple of 4 does not imply p is also a multiple of 4. Thus, our proof breaks down at this point.

Exercise 1.2.2.

Exercise 1.2.3. (a) False, as seen in Example 1.2.2.

(b) True. This will follow from upcoming results about compactness in Chapter 3.

(c) False. Consider sets $A = \{1, 2, 3\}, B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) True.

(e) True.

Exercise 1.2.4.

Exercise 1.2.5. (a) If $x \in (A \cap B)^c$ then $x \notin (A \cap B)$. But this implies $x \notin A$ or $x \notin B$. From this we know $x \in A^c$ or $x \in B^c$. Thus, $x \in A^c \cup B^c$ by the definition of union.

(b) To show $A^c \cup B^c \subseteq (A \cap B)^c$, let $x \in A^c \cup B^c$ and show $x \in (A \cap B)^c$. So, if $x \in A^c \cup B^c$ then $x \in A^c$ or $x \in B^c$. From this, we know that $x \notin A$ or $x \notin B$, which implies $x \notin (A \cap B)$. This means $x \in (A \cap B)^c$, which is precisely what we wanted to show.

(c) In order to prove $(A \cup B)^c = A^c \cap B^c$ we have to show,

$$(A \cup B)^c \subseteq A^c \cap B^c \text{ and,} \tag{1}$$

$$A^c \cap B^c \subseteq (A \cup B)^c. \tag{2}$$

To demonstrate part (1) take $x \in (A \cup B)^c$ and show that $x \in (A^c \cap B^c)$. So, if $x \in (A \cup B)^c$ then $x \notin (A \cup B)$. From this, we know that $x \notin$A and $x \notin B$ which implies $x \in A^c$ and $x \in B^c$. This means $x \in (A^c \cap B^c)$.

Similarly, part (2) can be shown by taking $x \in (A^c \cap B^c)$ and showing that $x \in (A \cup B)^c$. So, if $x \in (A^c \cap B^c)$ then $x \in A^c$ and $x \in B^c$. From this, we know that $x \notin A$ and $x \notin B$ which implies $x \notin (A \cup B)$. This means $x \in (A \cup B)^c$. Since we have shown inclusion both ways, we conclude that $(A \cup B)^c = A^c \cap B^c$.

Exercise 1.2.6.

Exercise 1.2.7. (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$. In this case, $f(A \cap B) = f(A) \cap f(B) = [1, 4]$ and $f(A \cup B) = f(A) \cup f(B) = [0, 16]$.

(b) Take $A = [0, 2]$ and $B = [-2, 0]$ and note that $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0, 4]$.

(c) We have to show $y \in g(A \cap B)$ implies $y \in g(A) \cap g(B)$. If $y \in g(A \cap B)$ then there exists an $x \in A \cap B$ with $g(x) = y$. But this means $x \in A$ and $x \in B$ and hence $g(x) \in g(A)$ and $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cap g(B)$.

(d) Our claim is $g(A \cup B) = g(A) \cup g(B)$. In order to prove it, we have to show,

$$g(A \cup B) \subseteq g(A) \cup g(B) \text{ and,} \tag{1}$$

$$g(A) \cup g(B) \subseteq g(A \cup B). \tag{2}$$

To demonstrate part (1), we let $y \in g(A \cup B)$ and show $y \in g(A) \cup g(B)$. If $y \in g(A \cup B)$ then there exists $x \in A \cup B$ with $g(x) = y$. But this means

$x \in A$ or $x \in B$, and hence $g(x) \in g(A)$ or $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cup g(B)$.

To demonstrate the reverse inclusion, we let $y \in g(A) \cup g(B)$ and show $y \in g(A \cup B)$. If $y \in g(A) \cup g(B)$ then $y \in g(A)$ or $y \in g(B)$. This means we have an $x \in A$ or $x \in B$ such that $g(x) = y$. This implies, $x \in A \cup B$, and hence $g(x) \in g(A \cup B)$. Since we have shown parts (1) and (2), we can conclude $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8.

Exercise 1.2.9. (a) $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. In this case, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = [-1, 1]$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = [-2, 2]$.

(b) In order to prove $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show,

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cap B)$ and show $x \in g^{-1}(A) \cap g^{-1}(B)$. So, if $x \in g^{-1}(A \cap B)$ then $g(x) \in (A \cap B)$. But this means $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This implies, $x \in g^{-1}(A) \cap g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cap g^{-1}(B)$ and show $x \in g^{-1}(A \cap B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. This implies $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This means, $x \in g^{-1}(A \cap B)$.

Similarly, in order to prove $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$, we have to show,

$$g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cup B)$ and show $x \in g^{-1}(A) \cup g^{-1}(B)$. So, if $x \in g^{-1}(A \cup B)$ then $g(x) \in (A \cup B)$. But this means $g(x) \in A$ or $g(x) \in B$, which implies $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. From this we know $x \in g^{-1}(A) \cup g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cup g^{-1}(B)$ and show $x \in g^{-1}(A \cup B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. This implies $g(x) \in A$ or $g(x) \in B$, and hence $g(x) \in A \cup B$. This means, $x \in g^{-1}(A \cup B)$.

Exercise 1.2.10.

Exercise 1.2.11. (a) There exist two real numbers a and b satisfying $a < b$ such that for all $n \in \mathbf{N}$ we have $a + 1/n \geq b$.

(b) For all real numbers $x > 0$, there exists $n \in \mathbf{N}$ satisfying $x \geq 1/n$.

(c) There exist two distinct rational numbers with the property that every number in between them is irrational.

Exercise 1.2.12.

Exercise 1.2.13. (a) From Exercise 1.2.5 we know $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ which proves the base case. Now we want to show that

if we have $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$, then it follows that

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = A_1^c \cap A_2^c \cap \cdots \cap A_{n+1}^c.$$

Since the union of sets obey the associative law,

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = ((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})^c$$

which is equal to

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c.$$

Now from our induction hypothesis we know that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

which implies that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.$$

By induction, the claim is proved for all $n \in \mathbf{N}$.

(b) Example 1.2.2 illustrates this phenomenon.

(c) In order to prove $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ we have to show,

(1) $$\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c \text{ and,}$$

(2) $$\bigcap_{n=1}^{\infty} A_n^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right)^c.$$

To demonstrate part (1), we let $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ and show $x \in \bigcap_{n=1}^{\infty} A_n^c$. So, if $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ then $x \notin A_n$ for all $n \in \mathbf{N}$. This implies x is in the complement of each A_n and by the definition of intersection $x \in \bigcap_{n=1}^{\infty} A_n^c$.

To demonstrate the reverse inclusion, we let $x \in \bigcap_{n=1}^{\infty} A_n^c$ and show $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$. So, if $x \in \bigcap_{n=1}^{\infty} A_n^c$ then $x \in A_n^c$ for all $n \in \mathbf{N}$ which means $x \notin A_n$ for all $n \in \mathbf{N}$. This implies $x \notin \left(\bigcup_{n=1}^{\infty} A_n\right)$ and we can now conclude $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1. (a) A real number i is the greatest lower bound, or the infimum, for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) i is a lower bound for A; i.e., $i \leq a$ for all $a \in A$, and

(ii) if l is any lower bound for A, then $l \leq i$.

(b) Lemma: Assume $i \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then, $i = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $i + \epsilon > a$.

(i) To prove this in the forward direction, assume $i = \inf A$ and consider $i + \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $i + \epsilon > i$, statement (ii) implies $i + \epsilon$ is not a lower bound for A. Since this is the case, there must be some element $a \in A$ for which $i + \epsilon > a$ because otherwise $i + \epsilon$ would be a lower bound.

(ii) For the backward direction, assume i is a lower bound with the property that no matter how $\epsilon > 0$ is chosen, $i + \epsilon$ is no longer a lower bound for A. This implies that if l is any number greater than i then l is no longer a lower bound for A. Because any number greater than i cannot be a lower bound, it follows that if l is some other lower bound for A, then $l \leq i$. This completes the proof of the lemma.

Exercise 1.3.2.

Exercise 1.3.3. (a) Because A is bounded below, B is not empty. Also, for all $a \in A$ and $b \in B$, we have $b \leq a$. The first thing this tells us is that B is bounded above and thus $\alpha = \sup B$ exists by the Axiom of Completeness. It remains to show that $\alpha = \inf A$. The second thing we see is that every element of A is an upper bound for B. By part (ii) of the definition of supremum, $\alpha \leq a$ for all $a \in A$ and we conclude that α is a lower bound for A.

Is it the greatest lower bound? Sure it is. If l is an arbitrary lower bound for A then $l \in B$, and part (i) of the definition of supremum implies $l \leq \alpha$. This completes the proof.

(b) We do not need to assume that greatest lower bounds exist as part of the Axiom of Completeness because we now have a proof that they exist. By demonstrating that the infimum of a set A is always equal to the supremum of a different set, we can use the existence of least upper bounds to assert the existence of greatest lower bounds.

Another way to achieve the same goal is to consider the set $-A = \{-a : a \in A\}$. If A is bounded below it follows that $-A$ is bounded above and it is not too hard to prove $\inf A = \sup(-A)$.

Exercise 1.3.4.

Exercise 1.3.5. (a) In the case $c = 0$, $cA = \{0\}$ and without too much difficulty we can argue that $\sup(cA) = 0 = c \sup A$. So let's focus on the case where $c > 0$. Observe that $c \sup A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \sup A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $a \leq d/c$ for all $a \in A$. This means d/c is an upper bound for A, and by Definition 1.3.2 $\sup A \leq d/c$. But this implies $c \sup A \leq c(d/c) = d$, which is precisely what we wanted to show.

(b) Assuming the set A is bounded below, we claim $\sup(cA) = c\inf A$ for the case $c < 0$. In order to prove our claim we first show $c\inf A$ is an upper bound for cA. Since $\inf A \le a$ for all $a \in A$, we multiply both sides of the equation to get $c\inf A \ge ca$ for all $a \in A$. This shows that $c\inf A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c\inf A \le d$. We know $ca \le d$ for all $a \in A$, and thus $d/c \le a$ for all $a \in A$. This means d/c is a lower bound for A and from Exercise 1.3.1, $d/c \le \inf A$. But this implies $c\inf A \le c(d/c) \le d$, which is precisely what we wanted to show.

Exercise 1.3.6.

Exercise 1.3.7. Since a is an upper bound for A, we just need to verify the second part of the definition of supremum and show that if d is any upper bound then $a \le d$. By the definition of upper bound $a \le d$ because a is an element of A. Hence, by Definition 1.3.2, a is the supremum of A.

Exercise 1.3.8.

Exercise 1.3.9. (a) Set $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

(b) Take $A = [0, 1]$ and $B = (0, 1)$.

Exercise 1.3.10.

Exercise 1.3.11. (a) True. Observe that all elements of B are contained in A and hence $\sup A \ge b$ for all $b \in B$. By Definition 1.3.2 part (ii), $\sup B$ is less than or equal to any other upper bounds of B. Because $\sup A$ is an upper bound for B, it follows that $\sup B \le \sup A$.

(b) True. Let $c = (\sup A + \inf B)/2$ from which it follows that

$$a \le \sup A < c < \inf B \le b.$$

(c) False. Consider, the open sets $A = (d, c)$ and $B = (c, f)$. Then $a < c < b$ for every $a \in A$ and $b \in B$, but $\sup A = c = \inf B$.

1.4 Consequences of Completeness

Exercise 1.4.1. (a) We have to show if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$. By definition, $\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \ne 0\}$. So take $a = p/q$ and $b = c/d$ where $p, q, c, d \in \mathbf{Z}$ and $q, d \ne 0$. Then, $ab = \frac{pc}{qd}$ where $pc, qd \in \mathbf{Z}$ because $\mathbf{Z}$ is closed under multiplication. This implies $ab \in \mathbf{Q}$. To see that $a + b$ is rational, write

$$\frac{p}{q} + \frac{c}{d} = \frac{pd + qc}{qd},$$

and observe that both $pd + qc$ and qd are integers with $qd \ne 0$.

(b) Assume, for contradiction, that $a + t \in \mathbf{Q}$. Then $t = (t + a) - a$ is the difference of two rational numbers, and by part (a) t must be rational as well. This contradiction implies $a + t \in \mathbf{I}$.

Likewise, if we assume $at \in \mathbf{Q}$, then $t = (at)(1/a)$ would again be rational by the result in (a). This implies $at \in \mathbf{I}$.

(c) The set of irrationals is not closed under addition and multiplication. Given two irrationals s and t, $s + t$ can be either irrational or rational. For instance, if $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s + t = 0$ which is an element of $\mathbf{Q}$. However, if $s = \sqrt{2}$ and $t = 2\sqrt{2}$ then $s + t = \sqrt{2} + 2\sqrt{2} = 3\sqrt{3}$ which is an element of $\mathbf{I}$. Similarly, st can be either irrational or rational. If $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $st = -1$ which is a rational number. However,if $s = \sqrt{2}$ and $t = \sqrt{3}$ then $st = \sqrt{2}\,\sqrt{3} = \sqrt{6}$ which is an irrational number.

Exercise 1.4.2.

Exercise 1.4.3. Let $x \in \mathbf{R}$ be arbitrary. To prove $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$ it is enough to show that $x \notin (0, 1/n)$ for some $n \in \mathbf{N}$. If $x \leq 0$ then we can take $n = 1$ and observe $x \notin (0, 1)$. If $x > 0$ then by Theorem 1.4.2 we know there exists an $n_0 \in \mathbf{N}$ such that $1/n_0 < x$. This implies $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$, and our proof is complete.

Exercise 1.4.4.

Exercise 1.4.5. We have to show the existence of an irrational number between any two real numbers a and b. By applying Theorem 1.4.3 on the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$ we can find a rational number r satisfying $a-\sqrt{2} < r < b-\sqrt{2}$. This implies $a < r + \sqrt{2} < b$. From Exercise 1.4.1(b) we know $r + \sqrt{2}$ is an irrational number between a and b.

Exercise 1.4.6.

Exercise 1.4.7. Now, we need to pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha} \quad \text{or} \quad \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

With this choice of n_0, we have

$$(\alpha - 1/n_0)^2 > \alpha^2 - 2\alpha/n_0 = \alpha^2 - (\alpha^2 - 2) = 2.$$

This means $(\alpha - 1/n_0)$ is an upper bound for T. But $(\alpha - 1/n_0) < \alpha$ and $\alpha = \sup T$ is supposed to be the least upper bound. This contradiction means that the case $\alpha^2 > 2$ can be ruled out. Because we have already ruled out $\alpha^2 < 2$, we are left with $\alpha^2 = 2$ which implies $\alpha = \sqrt{2}$ exists in $\mathbf{R}$.

Exercise 1.4.8.

1.5 Cardinality

Exercise 1.5.1. Next let $n_2 = \min\{n \in \mathbf{N} : f(n) \in A \backslash \{f(n_1)\}\}$ and set $g(2) = f(n_2)$. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbf{N} : f(n) \in A \backslash \{f(n_1) \ldots f(n_{k-1})\}\}$.

To show that $g : N \to A$ is 1–1, observe that $m \neq m'$ implies $n_m \neq n_{m'}$ and it follows that $f(n_m) = g(m) \neq g(m') = f(n_{m'})$ because f is assumed to be 1–1. To show that g is onto, let $a \in A$ be arbitrary. Because f is onto, $a = f(n')$ for some $n' \in \mathbf{N}$. This means $n' \in \{n : f(n) \in A\}$ and as we inductively remove the minimal element, n' must eventually be the minimum by at least the $n' - 1$st step.

Exercise 1.5.2.

Exercise 1.5.3. (a) Because A_1 is countable, there exists a 1–1 and onto function $f : \mathbf{N} \to A_1$.

If $B_2 = \emptyset$, then $A_1 \cup A_2 = A_1$ which we already know to be countable.

If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is a 1–1 and onto follows immediately from the same properties of f.

If B_2 is infinite, then by Theorem 1.5.7 it is countable, and so there exists a 1–1 onto function $g : \mathbf{N} \to B_2$. In this case we define $h : A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is 1–1 and onto is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{rccccccc} \mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in Theorem 1.5.8, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m+1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof.

(b) Induction cannot be used when we have an infinite number of sets. It can only be used to prove facts that hold true for each value of $n \in \mathbf{N}$. See the discussion in Exercise 1.2.13 for more on this.

(c) Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve 1-1 correspondence between the set $\mathbf{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbf{N}$ given in the exercise:

$$\begin{array}{llllll} A_1 = a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2 = a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\ A_3 = a_{31} & a_{32} & a_{33} & \cdots & & \\ A_4 = a_{41} & a_{42} & \cdots & & & \\ A_5 = a_{51} & \cdots & & & & \\ \quad\ \vdots & & & & & \end{array}$$

This establishes a 1–1 and onto mapping $g : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j, k) is the row and column location of n in the array for $\mathbf{N}$ given in the exercise.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be 1–1. In this case we could again replace A_n with $B_n = A_n \backslash \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a 1–1 correspondence between $\bigcup_{n=1}^{\infty} A_n$ and an infinite *subset* of $\mathbf{N}$, and then appeal to Theorem 1.5.7.

Exercise 1.5.4.

Exercise 1.5.5. (a) The identity function $f(a) = a$ for all $a \in A$ shows that $A \sim A$.

(b) Since $A \sim B$ we know there is 1-1, onto function from A onto B. This means we can define another function $g : B \to A$ that is also 1-1 and onto. More specifically, if $f : A \to B$ is 1-1 and onto then $f^{-1} : B \to A$ exists and is also 1-1 and onto.

(c) We will show there exists a 1-1, onto function $h : A \to C$. Because $A \sim B$, there exists $g : A \to B$ that is 1–1 and onto. Likewise, $B \sim C$ implies that there exists $f : B \to C$ that is also 1-1 and onto. So let's define $h : A \to C$ by the composition $h = f \circ g$.

In order to show $f \circ g$ is 1-1, take $a_1, a_2 \in A$ where $a_1 \neq a_2$ and show $f(g(a_1)) \neq f(g(a_2))$. Well, $a_1 \neq a_2$ implies that $g(a_1) \neq g(a_2)$ because g is 1–1. And $g(a_1) \neq g(a_2)$ implies that $f(g(a_1)) \neq (f(g(a-2))$ because f is 1–1. This shows $f \circ g$ is 1–1.

In order to show $f \circ g$ is onto, we take $c \in C$ and show that there exists an $a \in A$ with $f(g(a)) = c$. If $c \in C$ then there exists $b \in B$ such that $f(b) = c$ because f is onto. But for this same $b \in B$ we have an $a \in A$ such that $g(a) = b$ since g is onto. This implies $f(b) = f(g(a)) = c$ and therefore $f \circ g$ is onto.

Exercise 1.5.6.

Exercise 1.5.7. (a). The function $f(x) = (x, \frac{1}{3})$ is 1–1 from $(0,1)$ to S.

(b) Given $(x, y) \in S$, let's write x and y in their decimal expansions

$$x = .x_1x_2x_3\dots \quad \text{and} \quad y = .y_1y_2y_3\dots$$

where we make the convention that we always use the terminating form (or repeated 0s) over the repeating 9s form when the situation arises.

Now define $f : S \to (0,1)$ by

$$f(x, y) = .x_1y_1x_2y_2x_3y_3\dots$$

In order to show f is 1–1, assume we have two distinct points $(x, y) \neq (w, z)$ from S. Then it must be that either $x \neq w$ or $y \neq z$, and this implies that in at least one decimal place we have $x_i \neq w_i$ or $y_i \neq z_i$. But this is enough to conclude $f(x, y) \neq f(w, z)$.

The function f is not onto. For instance the point $t = .555959595\dots$ is not in the range of f because the ordered pair (x, y) with $x = .555\dots$ and $y = .5999\dots$ would not be allowed due to our convention of using terminating decimals instead of repeated 9s.

Exercise 1.5.8.

Exercise 1.5.9. (a) $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, $\sqrt[3]{2}$ is a root of the polynomials $x^3 - 2$, and $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$. Since all of these numbers are roots of polynomials with integer coefficients, they are all algebraic.

(b) Fix $n, m \in \mathbf{N}$. The set of polynomials of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

satisfying $|a_n| + |a_{n-1}| + \cdots + |a_0| \leq m$ is *finite* because there are only a finite number of choices for each of the coefficients (given that they must be integers.) If we let A_{nm} be the set of all the roots of polynomials of this form, then because each one of these polynomials has at most n roots, the set A_{nm} is finite. Thus A_n, the set of algebraic numbers obtained as roots of any polynomial (with integer coefficients) of degree n, can be written as a countable union of finite sets

$$A_n = \bigcup_{m=1}^{\infty} A_{nm}.$$

It follows that A_n is countable.

(c) If A is the set of all algebraic numbers, then $A = \bigcup_{n=1}^{\infty} A_n$. Because each A_n is countable, we may use Theorem 1.5.8 to conclude that A is countable as well.

If T is the set transcendental numbers, then $A \cup T = \mathbf{R}$. Now if T were countable, then $\mathbf{R} = A \cup T$ would also be countable. But this is a contradiction because we know $\mathbf{R}$ is uncountable, and hence the collection of transcendental numbers must also be uncountable.

Exercise 1.5.10.

Exercise 1.5.11. (a) For all $x \in A'$, there exists a unique $y \in B'$ satisfying $g(y) = x$. This means that there is a well-defined inverse function $g^{-1}(x) = y$ that maps A' onto B'. Setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}$$

gives the desired 1–1, onto function from X to Y.

(b) To see that the sets $A_1, A_2, A_3, \ldots$ are pairwise disjoint, note that $A_1 \cap A_n = \emptyset$ for all $n \geq 2$ because $A_1 = X \backslash g(Y)$ and $A_n \subseteq g(Y)$ for all $n \geq 2$.

In the general case of $A_n \cap A_m$ where $1 < n < m$, note that if $x \in A_n \cap A_m$ then $f^{-1}(g^{-1}(x)) \in A_{n-1} \cap A_{m-1}$. Continuing in this way, we can show $A_1 \cap A_{m-n+1}$ is not empty, which is a contradiction. Thus $A_n \cap A_m = \emptyset$. Just to be clear, the disjointness of the A_n sets is not crucial to the overall proof, but it does help paint a clearer picture of how the sets A and A' are constructed.

(c) This is very straightforward. Each $x \in A$ comes from some A_n and so $f(x) \in f(A_n) \subseteq B$. Likewise, each $y \in B$ is an element of some $f(A_n)$ and thus $y = f(x)$ for some $x \in A_n \subseteq A$. Thus $f : A \to B$ is onto.

(d) Let $y \in B'$. Then $y \notin f(A_n))$ for all n, and because g is 1–1, $g(y) \notin A_{n+1}$ for all n. Clearly, $g(y) \notin A_1$ either and so g maps B' into A'.

To see that g maps B' onto A', let $x \in A'$ be arbitrary. Because $A' \subseteq g(Y)$, there exists $y \in Y$ with $g(y) = x$. Could y be an element of B? No, because $g(B) \subseteq A$. So $y \in B'$ and we have $g : B' \to A'$ is onto.

1.6 Cantor's Theorem

Exercise 1.6.1. The function $f(x) = (x - 1/2)/(x - x^2)$ is a 1–1, onto mapping from $(0, 1)$ to $\mathbf{R}$. This shows $(0, 1) \sim \mathbf{R}$, and the result follows using the ideas in Exercise 1.5.5.

Exercise 1.6.2.

Exercise 1.6.3. (a) If we imitate the proof to try and show that $\mathbf{Q}$ is uncountable, we can construct a real number x in the same way. This x will again fail to be in the range of our function f, *but there is no reason to expect x to be rational.* The decimal expansions for rational numbers either terminate or repeat, and this will not be true of the constructed x.

(b) By using the digits 2 and 3 in our definition of b_n we eliminate the possibility that the point $x = .b_1 b_2 b_3 \ldots$ has some other possible decimal representation (and thus it cannot exist somewhere in the range of f in a different form.)

Exercise 1.6.4.

Exercise 1.6.5. (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) First, list the n elements of A. To construct a subset of A, we consider each element and associate either a 'Y' if we decide to include it in our subset or an 'N' if we decide not to include it. Thus, to each subset of A there is an associated sequence of length n of Ys and Ns. This correspondence is 1–1, and the proof is done by observing there are 2^n such sequences.

A second way to prove this result is with induction.

Exercise 1.6.6.

Exercise 1.6.7. This solution depends on the mappings chosen in the previous exercise. The key point is that no matter how this is done, the resulting set B should not be in the range of the particular 1–1 mapping used to create it.

Exercise 1.6.8.

Exercise 1.6.9. It is unlikely that there is a reasonably simple way to explicitly define a 1–1, onto mapping from $P(\mathbf{N})$ to $\mathbf{R}$. A more fruitful strategy is to make use of the ideas in Exercise 1.5.5 and 1.5.11. In particular, we have seen earlier in this section that $\mathbf{R} \sim (0,1)$. It is also true that $P(\mathbf{N}) \sim S$ where S is the set of all sequences consisting of 0s and 1s from Exercise 1.6.4. To see why, let $A \in P(\mathbf{N})$ be an arbitrary subset of $\mathbf{N}$. Corresponding to this set A is the sequence (a_n) where $a_n = 1$ if $n \in A$ and $a_n = 0$ otherwise. It is straightforward to show that this correspondence is both 1–1 and onto, and thus $P(\mathbf{N}) \sim S$.

With a nod to Exercise 1.5.5, we can conclude that $P(\mathbf{N}) \sim \mathbf{R}$ if we can demonstrate that $S \sim (0,1)$. Proving this latter fact is easier, but it is still not easy by any means. One way to avoid some technical details, is to use the Schröder–Bernstein Theorem (Exercise 1.5.11). Rather than finding a 1–1, onto function, the punchline of this result is that we will be done if we can find two 1–1 functions, one mapping $(0,1)$ into S, and the other mapping S into $(0,1)$. There are a number of creative ways to produce each of these functions.

Let's focus first on mapping $(0,1)$ into S. A fairly natural idea is to think in terms of binary representations. Given $x \in (0,1)$ let's inductively define a sequence (x_n) in the following way. First, bisect $(0,1)$ into the two parts $(0,1/2)$ and $[1/2,1)$. Then set $x_1 = 0$ if x is in the left half, and $x_1 = 1$ if x is in the right half. Now let I be whichever of these two intervals contains x, and bisect it using the same convention of including the midpoint in the right half. As before we set $x_2 = 0$ if x is in the left half of I and $x_2 = 1$ if x is in the right half. Continuing this process inductively, we get a sequence $(x_n) \in S$ that is uniquely determined by the given $x \in (0,1)$, and thus the mapping is 1–1.

It may seem like this mapping is onto S but it falls just short. Because of our convention about including the midpoint in the right half of each interval, we never get a sequence that is eventually all 1s, nor do we get the sequence of all 0s. This is fixable. The collection of all sequences in S that are NOT in the range of this mapping form a countable set, and it is not too hard to show that the cardinality of S with a countable set removed is the same as the cardinality of S. The other option is to use the Schröder–Bernstein Theorem mentioned previously. Having found a 1–1 function from $(0,1)$ into S, we just need to produce a 1–1 function that goes the other direction. An example of such a function would be the one that takes $(x_n) \in S$ and maps it to the real number with decimal expansion $.x_1x_2x_3x_4\ldots$. Because the only decimal expansions that aren't unique involve 9s, we can be confident that this mapping is 1–1.

The Schröder–Bernstein Theorem now implies $S \sim (0,1)$, and it follows that $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10.

Chapter 2

Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

2.2 The Limit of a Sequence

Exercise 2.2.1. Consider the sequence $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \cdots)$ This sequence verconges to $x = 0$. To see this, note that we only have to produce a *single* $\epsilon > 0$ where the prescribed condition follows, and in this case we can take $\epsilon = 2$. This ϵ works because for all $N \in \mathbf{N}$, it is true that $n \geq N$ implies $|x_n - \frac{1}{2}| < 1$.

This is also an example of a vercongent sequence that is divergent. Notice that the "limit" $x = 0$ is not unique. We could also show this same sequence verconges to $x = 1$ by choosing $\epsilon = 3$.

In general, a vercongent sequence is a bounded sequence. By a bounded sequence, we mean that there exists an $M \geq 0$ satisfying $|x_n| \leq M$ for all $n \in \mathbf{N}$. In this case we can always take $x = 0$ and $\epsilon = M + 1$. Then $|x_n - x| = |x_n| < \epsilon$, and the sequence (x_n) verconges to 0.

Exercise 2.2.2.

Exercise 2.2.3. a) There exists at least one college in the United States where all students are less than seven feet tall.

b) There exists a college in the United States where all professors gave at least one student a grade of C or less.

c) At every college in the United States, there is a student less than six feet tall.

Exercise 2.2.4.

Exercise 2.2.5. a) The limit of (a_n) is zero. To show this let $\epsilon > 0$ be arbitrary. We must show that there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $|[[5/n]] - 0| <$

ϵ. Well, pick $N = 6$. If $n \geq N$ we then have;

$$\left| \left[\left[\frac{5}{n}\right]\right] - 0 \right| = |0 - 0| < \epsilon,$$

because $[[5/n]] = 0$ for all $n > 5$.

b) Here the limit of a_n is 1. Let $\epsilon > 0$ be arbitrary. By picking $N = 7$ we have that for $n \geq N$,

$$\left| \left[\left[\frac{12 + 4n}{3n}\right]\right] - 1 \right| = |1 - 1| < \epsilon,$$

because $[[(12 + 4n)/3n]] = 1$ for all $n \geq 7$.

In these exercises, the choice of N does not depend on ϵ in the usual way. In exercise (b) for instance, setting $N = 7$ is a suitable response for every choice of $\epsilon > 0$. Thus, this is a rare example where a smaller $\epsilon > 0$ does not require a larger N in response.

Exercise 2.2.6.

Exercise 2.2.7. (a) The sequence $(-1)^n$ is *frequently* in the set 1.

(b) Definition (i) is stronger. "Frequently" does not imply "eventually", but "eventually" implies "frequently".

(c) A sequence (a_n) converges to a real number a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is *eventually* in $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2, then (x_n) is *frequently* in the interval (1.9, 2.1). However, (x_n) is not necessarily *eventually* in the interval (1.9, 2.1). Consider the sequence $(2, 0, 2, 0, 2, \cdots)$, for instance.

Exercise 2.2.8.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1. (a) Let $\epsilon > 0$ be arbitrary. We must find an N such that $n \geq N$ implies $|\sqrt{x_n} - 0| < \epsilon$. Because $(x_n) \to 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - 0| = x_n < \epsilon^2$. Using this N, we have $\sqrt{(x_n)^2} < \epsilon^2$, which gives $|\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$, as desired.

(b) Part (a) handles the case $x = 0$, so we may assume $x > 0$. Let $\epsilon > 0$. This time we must find an N such that $n \geq N$ implies $|\sqrt{x_n} - \sqrt{x}| < \epsilon$, for all $n \geq N$. Well,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= |\sqrt{x_n} - \sqrt{x}| \left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right) \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \end{aligned}$$

Now because $(x_n) \to x$ and $x > 0$, we can choose N such that $|x_n - x| < \epsilon\sqrt{x}$ whenever $n \geq N$. And this implies that for all $n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

as desired.

Exercise 2.3.2.

Exercise 2.3.3. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that $n \geq N$ implies $|y_n - l| < \epsilon$. In terms of ϵ-neighborhoods (which are a bit easier to use in this case), we must equivalently show $y_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N$.

Because $(x_n) \to l$, we can pick an N_1 such that $x_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N_1$. Similarly, because $(z_n) \to l$ we can pick an N_2 such that $z_n \in (l-\epsilon, l+\epsilon)$ whenever $n \geq N_2$. Now, because $x_n \leq y_n \leq z_n$, if we let $N = \max\{N_1, N_2\}$, then it follows that $y_n \in (l - \epsilon, l + \epsilon)$, for all $n \geq N$. This completes the proof.

Exercise 2.3.4.

Exercise 2.3.5. $(\Rightarrow)$ Let $\epsilon > 0$ be arbitrary. Let's call the limit that (z_n) converges to L. Then we need to show that there exists an N such that when $n \geq N$, it follows that $|y_n - L| < \epsilon$. Because $(z_n) \to L$, we can pick N so that $|z_n - L| < \epsilon$ for all $n \geq N$. Because $y_n = z_{2N}$ it certainly follows that $|y_n - L| < \epsilon$ whenever $n \geq N$. A similar argument holds for the (x_n) sequence.

$(\Leftarrow)$ Let $\epsilon > 0$ be arbitrary. Again, let L be the common limit of (x_n) and (y_n). We need to show that there exists an N such that when $n \geq N$ it follows that $|z_n - L| < \epsilon$. Choose N_1 so that $|x_n - L| < \epsilon$ for all $n \geq N_1$, and choose N_2 such that $|y_n - L| < \epsilon$ for all $n \geq N_2$. Finally, let $N = \max\{2N_1, 2N_2\}$, and it follows from the construction of the sequence (z_n) that $|z_n - L| < \epsilon$ whenever $n \geq N$.

Exercise 2.3.6.

Exercise 2.3.7. (a) Consider $(x_n) = (-1, 1, -1, 1, \cdots)$, and $(y_n) = (1, -1, 1, -1, \cdots)$. Both sequences diverge but $(x_n + y_n)$ converges.

(b) Such a request is impossible because by the Algebraic Limit Theorem, if $(x_n + y_n)$ converges to l and (x_n) converges to x, then

$$\lim(y_n) = \lim(y_n + x_n - x_n) = \lim(x_n + y_n) - \lim(x_n) = l - x.$$

So (y_n) must also converge.

(c) Consider the sequence $(b_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$. Then $(1/b_n) = (1, 2, 3, 4, \ldots)$ which is unbounded and diverges. It may be tempting to say something like $(1/b_n)$ "converges to infinity." Although it is possible to give a rigorous meaning to the phrase *converges to infinity*, we can avoid the whole issue by adding alternating negative signs to the original sequence.

(d) Such a request is impossible. By Theorem 2.3.2, (b_n) is bounded. If $(a_n - b_n)$ were bounded, then we could show that

$$(a_n) = (a_n - b_n) + (b_n)$$

would also have to be bounded, which is not the case. Thus, $(a_n - b_n)$ is unbounded.

(e) Take $(a_n) = 1/n$, and $(b_n) = (-1)^n$. Such a request would be impossible if we were given that $\lim a_n \neq 0$.

Exercise 2.3.8.

Exercise 2.3.9. (a) Because (a_n) is bounded, there exists a K satisfying $|a_n| \leq K$. Let $\epsilon > 0$ be arbitrary. We need to find an N such that when $n \geq N$ it follows that $|a_n b_n - 0| < \epsilon$. Well,

$$|a_n b_n - 0| = |a_n||b_n| \leq K|b_n|.$$

Because $(b_n) \to 0$, we can pick an N such that

$$|b_n| < \frac{\epsilon}{K}.$$

Finally, we conclude that for this choice of N,

$$|a_n b_n - 0| \leq K|b_n| < K\frac{\epsilon}{K} = \epsilon$$

for all $n \geq N$. Therefore, $(a_n b_n) \to 0$.

We may not use the Algebraic Limit Theorem in this case because the hypothesis of that theorem requires that both (a_n) and (b_n) be convergent. (And this may not be so for (a_n).)

(b) No, for instance if $(a_n) = (1, -1, 1, -1, \cdots)$, $(a_n b_n)$ will not converge.

(c) All convergent sequences are bounded. Therefore, if $(a_n) \to a$ and $(b_n) \to 0$, then by part (a), $(a_n b_n) \to 0$.

Exercise 2.3.10.

Exercise 2.3.11. Let $\epsilon > 0$ be arbitrary. Then we need to find an N such that $n \geq N$ implies $|y_n - L| < \epsilon$. Because $(x_n) \to L$, we know that there exists $M > 0$ such that $|x_n - L| < M$ for all n. Also, there exists an $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|x_n - L| < \epsilon/2$. Now for $n \geq N_1$ we can write

$$\begin{aligned}
|y_n - L| &= \left|\frac{x_1 + x_2 + \cdots + x_{N_1} + \cdots + x_n}{n} - \frac{nL}{n}\right| \\
&= \left|\frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} + \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n}\right| \\
&\leq \left|\frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n}\right| + \left|\frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n}\right| \\
&\leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n}.
\end{aligned}$$

Because N_1 and M are fixed constants at this point, we may choose N_2 so that $\frac{(N_1-1)M}{n} < \epsilon/2$ for all $n \geq N_2$. Finally, let $N = \max\{N_1, N_2\}$ be the desired N. To see that this works, keep in mind that $\frac{n-N_1}{n} < 1$ and observe

$$|y_n - L| \leq \frac{(N_1-1)M}{n} + \frac{\epsilon(n-N_1)}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$. This completes the proof.

The sequence $(x_n) = (1, -1, 1, -1, \cdots)$ does not converge, but the averages satisfy $(y_n) \to 0$.

Exercise 2.3.12.

Exercise 2.3.13. (a) First fix $n \in \mathbf{N}$ and let

$$b_n = \lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \frac{1}{1+n/m} = \frac{1}{1+0} = 1.$$

Thus

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1.$$

In the other order, we first fix m and compute the limit along each row of the a_{mn} array to get

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m/n}{(m/n)+1}\right) = \lim_{m\to\infty}\frac{0}{0+1} = 0.$$

From this example we see that it is possible for

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n},$$

and so defining doubly indexed limits in this fashion would be problematic to say the least. Instead, we use the definition given in the exercise. The question to ask is how the definition of the limit relates to the iterated values. Does the existence of one of these imply the existence of the others? Can they all be different? When are they equal?

(b) For the case $a_{mn} = \frac{1}{m+n}$, it straightforward to verify that $\lim_{m,n\to\infty} = 0$. Given $\epsilon > 0$, we choose $N > 1/2\epsilon$. If both n and m are larger than N, the sum $n + m \geq 2N > 1/\epsilon$, and it follows that $1/(m+n) < \epsilon$ as desired.

It is easy to check that both iterated limits from part (a) also yield a value of 0 in this case.

The case $a_{mn} = mn/(m^2+n^2)$ is more subtle. As in the previous example, both iterated limits yield 0. To see this, we can write

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{mn}{m^2+n^2} = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m/n}{(m^2/n^2)+1}\right) = \lim_{m\to\infty}\frac{0}{0+1} = 0.$$

The calculation of the other iterated limit is analogous.

In this case, however, $\lim_{m,n\to\infty} a_{mn}$ does not exist. Perhaps the best way to see this is to note that when $m = n$ we have $a_{mn} = 1/2$. But when $m = 2n$ we get $a_{mn} = 2/5$. Thus, no matter what we choose for our limiting value a, it is going to be impossible to force $|a_{mn} - a|$ to be arbitrarily small for all large values of m and n.

(c) The array $a_{mn} = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}$ has this property. The iterated limits can't be computed but the overall limit is most definitely zero.

(d) As in all convergence proofs, it is important in this argument to be very thoughtful (and explicit) about the order in which we assert the existence of the various variables involved.

Let $\epsilon > 0$. Our job is to produce an $N \in \mathbf{N}$ such that $|b_m - a| < \epsilon$ whenever $m \geq N$. Now for *any* values of m and n the triangle inequality gives us

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a|,$$

and the hypothesis of the problem gives us control over the quantities on the right hand side of this inequality, provided we are careful.

Because $\lim_{m,n\to\infty} a_{mn} = a$, there exists an $N \in \mathbf{N}$ such that $|a_{mn} - a| < \epsilon/2$ for all $m, n \geq N$. We now argue that this same N works for our purposes. To see this, consider an arbitrary $m \geq N$. For this particular m, we are given that $b_m = \lim_{n\to\infty} a_{mn}$. Among other things, this implies that there exists an n_0 such that $|b_m - a_{mn_0}| < \epsilon/2$, and we are on solid ground insisting that $n_0 \geq N$ as well. It follows that

$$\begin{aligned} |b_m - a| &\leq |b_m - a_{mn_0}| + |a_{mn_0} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and the result is proved.

(e) This follows directly from part (d).

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1. (a) We will show that this sequence is decreasing and bounded.

First, let's use induction to show that this sequence is decreasing. Observe that $x_1 = 3 > 1 = x_2$. Now, we need to prove that if $x_n > x_{n+1}$, then $x_{n+1} > x_{n+2}$. Well, $x_n > x_{n+1}$ implies that $-x_n < -x_{n+1}$. Adding 4 to both sides of the inequality gives $4 - x_n < 4 - x_{n+1}$. It follows that

$$\frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}},$$

which is precisely what we need to conclude $x_{n+1} > x_{n+2}$. Thus by induction, (x_n) is decreasing.

The argument above shows that (x_n) is bounded above by 3, so now we'll show that (x_n) is bounded below. Clearly $x_1 > 0$. Now assume $x_n > 0$. Because

(x_n) is decreasing, we know that $x_n \leq x_1 = 3$, which implies that $x_{n+1} = \frac{1}{4-x_n}$ is positive. By induction, (x_n) is bounded below by 0 for all $n \in \mathbf{N}$.

Therefore this sequence converges by the Monotone Convergence Theorem.

(b) Since the sequence (x_{n+1}) is just the sequence (x_n) shifted by 1 (and without the first term), the two sequences have the same limit.

(c) From (b), we can let $x = \lim(x_n) = \lim(x_{n+1})$. Now the Algebraic Limit Theorem tells us that

$$x = \lim x_{n+1} = \lim \frac{1}{4 - x_n} = \frac{1}{4 - x},$$

and it follows that x must satisfy the equation $x^2 - 4x + 1 = 0$. Solving the equation gives

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3},$$

and since $x_1 = 3$ and (x_n) is decreasing, we conclude that $x = 2 - \sqrt{3}$.

Exercise 2.4.2.

Exercise 2.4.3. (a) One way to describe the sequence in this exercise is to set $a_1 = \sqrt{2}$ and let $a_{n+1} = \sqrt{2 + a_n}$.

To prove that (a_n) is increasing we use induction. Clearly $a_1 < a_2$. Now assume $a_n < a_{n+1}$. Adding 2 to both sides and then taking the square root preserves the inequality, and so we have

$$\sqrt{2 + a_n} < \sqrt{2 + a_{n+1}},$$

which is equivalent to asserting $a_{n+1} < a_{n+2}$. By induction, (a_n) is increasing. (Notice that this proof takes as given that the square root function is increasing.) We can also use induction to prove (a_n) is bounded above by 2. It's clear that $a_1 < 2$. Assuming $a_n < 2$, it follows that $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2$.

Because (a_n) is bounded and increasing, it converges to a limit L by MCT. Now, as with the previous exercises, we are justified in taking the limit across the recursive equation to get that L satisfies $L = \sqrt{2 + L}$. A little algebra yields the equation $L^2 - L - 2 = 0$, from which we conclude that $L = 2$.

We should note that the last steps in this problem involved taking the limit inside a square root sign, and this is not a manipulation that is justified by the Algebraic Limit Theorem. Instead we should reference Exercise 2.3.1 to support this part of the argument.

(b) This is remarkably similar to part (a), and in fact has the same answer. First, rewrite the sequence in a recursive way: $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$.

Let's prove that the sequence is increasing by induction. For the base case we observe that

$$x_1 = 2 < \sqrt{2\sqrt{2}} = x_2,$$

so we just need to prove that $x_n < x_{n+1}$ implies $x_{n+1} < x_{n+2}$. But if $x_n < x_{n+1}$ then $\sqrt{x_n} < \sqrt{x_{n+1}}$, and multiplying by $\sqrt{2}$ gives $\sqrt{2x_n} < \sqrt{2x_{n+1}}$. Thus we have $x_{n+1} < x_{n+2}$ and the sequence is increasing.

To show the sequence is bounded above by 2 we first observe that $x_1 < 2$. Now if $x_n < 2$, then $x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$ as well, and (x_n) is bounded.

Therefore this sequence converges by the Monotone Convergence Theorem and we can assert that both (x_n) and (x_{n+1}) converge to some real number l. Taking limits across the recursive equation $x_{n+1} = \sqrt{2x_n}$ yields $l = \sqrt{2l}$, which implies $l = 2$.

Exercise 2.4.4.

Exercise 2.4.5. (a) We first observe that a simple induction argument shows that x_n is positive for all n. We can also write

$$x_{n+1}^2 - 2 = \left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)^2 - 2 = \frac{x_n^2}{4} + \frac{1}{x_n^2} - 1 = \left(\frac{x_n}{2} - \frac{1}{x_n}\right)^2 \geq 0$$

as any number squared is positive. This shows that $x_n^2 \geq 2$ for all choices of n. (It's worth mentioning that this part of the argument, and the next, is not by induction.)

Now let's argue that (x_n) is decreasing. If we write

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n},$$

then we can see that $x_n - x_{n+1}$ is positive because $x_n^2 \geq 2$. Because we have shown that (x_n) is decreasing and bounded below, we may set $x = \lim x_n = \lim x_{n+1}$. Taking limits across the recursive equation we find

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{x}{2} + \frac{1}{x}$$

which implies $x = \sqrt{2}$.

(b) The sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

converges to $\sqrt{c}$ using a similar argument.

Exercise 2.4.6.

Exercise 2.4.7. (a) For each $n \in \mathbf{N}$, set $A_n = \{a_k : k \geq n\}$ so that $y_n = \sup A_n$. Because $A_{n+1} \subseteq A_n$ it follows that $y_{n+1} \leq y_n$ and so (y_n) is decreasing. If L is a lower bound for (a_n), then for all $n \in \mathbf{N}$ it must be that $y_n \geq a_n \geq L$. Thus (y_n) is both decreasing and bounded, and it follows from the Monotone Convergence Theorem that (y_n) converges.

(b)Define the *limit inferior* of (a_n) as

$$\liminf a_n = \lim z_n,$$

where $z_n = \inf\{a_k : k \geq n\}$. The sequence (z_n) is increasing (because we are taking the greatest lower bound of a smaller set each time) and bounded above (because (a_n) is bounded.) Thus (z_n) converges by MCT.

(c) For each $n \in \mathbf{N}$ we have $y_n \geq z_n$, so by the Order Limit Theorem (Theorem 2.3.4) $\lim y_n \geq \lim z_n$. This shows $\limsup a_n \geq \liminf a_n$ for every bounded sequence.

The sequence $(a_n) = (1, 0, 1, 0, 1, 0, \cdots)$ has $\limsup a_n = 1$ and $\liminf a_n = 0$. Notice that this sequence is not convergent.

(d) First let's prove that if $\lim y_n = \lim z_n = l$, then $\lim a_n = l$ as well. Let $\epsilon > 0$. There exists an $N \in \mathbf{N}$ such that $y_n \in V_\epsilon(l)$ and $z_n \in V_\epsilon(l)$ for all $n \geq N$. Because $z_n \leq a_n \leq y_n$, it must also be the case that $a_n \in V_\epsilon(l)$ for all $n \geq N$. Therefore $\lim a_n$ exists and is equal to l.

Next, let's show that if $\lim a_n = l$, then $\lim y_n = l$. (The proof that $\lim z_n = l$ is similar.) Let $\epsilon > 0$ be arbitrary. Because $\lim a_n = l$, there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n \in V_\epsilon(l)$. This means that $l - \epsilon$ and $l + \epsilon$ are lower and upper bounds for the set $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$. It follows that $l - \epsilon \leq y_n \leq l + \epsilon$ for all $n \geq N$. Keeping in mind that we already know $y = \lim y_n$ exists, we can use the Order Limit Theorem to assert that $l - \epsilon \leq y \leq l + \epsilon$, and because ϵ is arbitrary we must have $y = l$. (Theorem 1.2.6 could be referenced in this last step.)

Exercise 2.4.8.

Exercise 2.4.9. We will show that if $\sum_{n=0}^\infty 2^n b_{2n}$ diverges then $\sum_{n=1}^\infty b_n$ diverges by again exploiting a relationship between the partial sums

$$s_m = b_1 + b_2 + \cdots + b_m, \quad \text{and} \quad t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}.$$

Because $\sum_{n=0}^\infty 2^n b_{2n}$ diverges, its monotone sequence of partial sums (t_k) must be unbounded. To show that (s_m) is unbounded it is enough to show that for all $k \in \mathbf{N}$, there is a term s_m satisfying $s_m \geq t_k/2$. This argument is similar to the one for the forward direction, only to get the inequality to go the other way we group the terms in s_m so that the *last* (and hence smallest) term in each group is of the form b_{2^k}.

Given an arbitrary k, we focus our attention on s_{2^k} and observe that

$$\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\
&= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\
&= \frac{1}{2}\left(2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right) \\
&= b_1/2 + t_k/2.
\end{aligned}$$

Because (t_k) is unbounded, the sequence (s_m) must also be unbounded and cannot converge. Therefore, $\sum_{n=1}^\infty b_n$ diverges.

Exercise 2.4.10.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1. (a) Impossible. Theorem 2.5.5 guarantees that all bounded sequences have convergent subsequences. If a subsequence is bounded, then that subsequence has a convergent subsequence which would necessarily be a convergent subsequence of the original sequence.

(b) $(1/2, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 4/5, \cdots, 1/n, (n-1)/n, \cdots)$

(c) The sequence

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \cdots\right)$$

has this property. Notice that there is also a subsequence converging to 0. We shall see that this is unavoidable.

(d) Impossible. This is a slightly technical argument but the idea is straightforward. If there are subsequences converging to every point in the set

$$\{1, 1/2, 1/3, 1/4, \ldots\},$$

then there necessarily has to be a subsequence that converges to 0 as well. To construct it, pick $a_{n_1} \in V_1(1)$. Then choose $n_2 > n_1$ such that $a_{n_2} \in V_{\frac{1}{2}}(1/2)$. The reason such an a_{n_2} must exist is because we know there is a subsequence converging to $1/2$. Continue this process. Having selected $a_{n_k} \in V_{\frac{1}{k}}(1/k)$, select $n_{k+1} > n_k$ so that $a_{n_{k+1}} \in V_{\frac{1}{k+1}}(1/(k+1)$. The resulting subsequence satisfies $0 < a_{n_k} < 2/k$, which is enough to conclude that $(a_{n_k}) \to 0$.

Exercise 2.5.2.

Exercise 2.5.3. (a) Letting $s_n = a_1 + a_2 + \cdots + s_n$, we are given that $\lim s_n = L$. For the regrouped series, let's write

$$\begin{aligned} b_1 &= a_1 + a_2 + \cdots + a_{n_1}, \\ b_2 &= a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}, \\ &\vdots \\ b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}, \end{aligned}$$

and the claim is that the series $\sum_{m=1}^{\infty} b_m$ converges to L as well.

To prove this, just observe that if (t_m) is the sequence of partial sums for the regrouped series, then

$$\begin{aligned} t_m &= b_1 + b_2 + \cdots + b_m \\ &= (a_1 + \cdots + a_{n_1}) + \cdots + (a_{n_{m-1}+1} + \cdots + a_{n_m}) = s_{n_m}. \end{aligned}$$

which means that (t_m) is a subsequence of (s_n) and therefore converges to L by Theorem 2.5.2.

(b) Our proof here does not apply to the example at the end of Section 2.1 because in that case the original series does not converge. The result proved here says that if the series converges, then the associative property holds, but it says nothing about what happens when the original series does not converge.

Exercise 2.5.4.

Exercise 2.5.5. Let's assume, for contradiction, that (a_n) does not converge to a. Paying close attention to the quantifiers in the definition of convergence for a sequence, what this means is that there exists an $\epsilon_0 > 0$ such that for every $N \in \mathbf{N}$ we can find an $n \geq N$ for which $|a - a_n| \geq \epsilon_0$. Using this, we can build a subsequence of (a_n) that never enters the ϵ–neighborhood $V_{\epsilon_0}(a)$. To see how, first pick n_1 so that $|a - a_{n_1}| \geq \epsilon$. Next choose $n_2 > n_1$ so that $|a - a_{n_2}| \geq \epsilon_0$. Because our negated definition says that "...for every $N \in \mathbf{N}$, we can find an $n \geq N$..." we can be sure that having chosen n_j, we may pick $n_{j+1} > n_j$ so that $|a - a_{n_{j+1}}| \geq \epsilon_0$.

Because (a_n) is bounded, the resulting subsequence (a_{n_j}) must be bounded as well. Now apply the Bolzano–Weierstrass Theorem to (a_{n_j}) to say that there exists a convergent subsequence (of (a_{n_j}) and hence also of (a_n)) which we will write as $(a_{n_{j_k}})$. By hypothesis, this convergent subsequence must converge to a, but therein lies the contradiction. Because $(a_{n_{j_k}})$ is a subsequence of (a_{n_j}), it never enters the neighborhood $V_{\epsilon_0}(a)$ and it cannot converge to a. This completes the proof.

Exercise 2.5.6.

Exercise 2.5.7. From Example 2.5.3 we know that this is true for $0 < b < 1$. If $b = 0$ we get the constant sequence $(0, 0, 0, \ldots)$, so let's focus on the case $-1 < b < 0$. Let $\epsilon > 0$ be arbitrary and set $a = |b|$. Because we know $(a^n) \to 0$ (by Example 2.5.3), we may choose N so that $n \geq N$ implies $|a^n - 0| < \epsilon$. But this N will also work for the sequence (b^n) because

$$|b^n - 0| = |b^n| = |a^n| < \epsilon$$

whenever $n \geq N$.

Exercise 2.5.8.

Exercise 2.5.9. Because (a_n) is bounded, the set S is not empty and bounded above. By AoC, we know there exists an $s \in \mathbf{R}$ satisfying $s = \sup S$. For a fixed $k \in \mathbf{N}$ consider $s - 1/k$. Because s is the *least* upper bound, $s - 1/k$ is not an upper bound and there exists a point $s' \in S$ satisfying $s - 1/k < s'$. A quick look at the definition of S then shows that, in fact, $s - 1/k \in S$ and consequently there exist an infinite number of terms a_n satisfying $s - 1/k < a_n$.

Because s is an upper bound for S we can be sure that $s + 1/k \notin S$ from which we can conclude that there are only a finite number of terms a_n satisfying $s + 1/k < a_n$. Taken together, these observations show that for all $k \in \mathbf{N}$, there are an infinite number of terms a_n satisfying

$$s - \frac{1}{k} < a_n \leq s + \frac{1}{k}.$$

To inductively build our convergent subsequence (a_{n_k}) first pick a_{n_1} to satisfy $s-1 < a_{n_1} \leq s+1$. Now given that we have constructed a_{n_k}, choose $n_{k+1} > n_k$ so that

$$s - \frac{1}{k+1} < a_{n_{k+1}} \leq s + \frac{1}{k+1}.$$

(Here we are using the fact that this inequality is satisfied by an infinite number of terms a_n and so there is certainly one where $n > n_k$.) To show $(a_{n_k}) \to s$, we let $\epsilon > 0$ be arbitrary and choose $K > 1/\epsilon$. If $k \geq K$ then $1/k < \epsilon$ which implies $s - \epsilon < a_{n_k} < s + \epsilon$, and the proof is complete.

2.6 The Cauchy Criterion

Exercise 2.6.1. Assume (x_n) converges to x, and let $\epsilon > 0$ be arbitary. Becuase $(x_n) \to x$, there exists $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x| < \epsilon/2$ and $|x_m - x| < \epsilon/2$. By the triangle inequality,

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$, and (x_n) is a Cauchy sequence.

Exercise 2.6.2.

Exercise 2.6.3. (a) Let $\epsilon > 0$ be arbitrary. We need to find an N so that $n, m \geq N$ implies $|(x_n+y_n)-(x_m+y_m)| < \epsilon$. Because (x_n) and (y_n) are Cauchy, we can pick N_1 so that when $n, m \geq N$ it follows that $|x_n - x_m| < \epsilon/2$, and we can pick N_2 so that $n, m \geq N$ implies $|y_n - y_m| < \epsilon/2$. Setting $N = \max\{N_1, N_2\}$, we have

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $n, m \geq N$ as desired.

(b) Let $\epsilon > 0$ be arbitrary. We must produce an N such that $n, m \geq N$ implies $|x_n y_n - x_m y_m| < \epsilon$. Note that

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\ &= |x_n||y_n - y_m| + |y_m||x_n - x_m|. \end{aligned}$$

Because (x_n) and (y_n) are Cauchy, we know by Lemma 2.6.3 that they are bounded. So let $K \geq |x_n|$ and $L \geq |y_m|$ for all m, n. We also know that we can pick N_1 such that $m, n \geq N_1$ implies $|x_n - x_m| < \frac{\epsilon}{2L}$. Similarly, pick N_2 so that $m, n \geq N_2$ implies $|y_n - y_m| < \frac{\epsilon}{2K}$. Now let $N = \max\{N_1, N_2\}$. Then for $m, n \geq N$ it follows that

$$|x_n y_n - x_m y_m| \leq |x_n||y_n - y_m| + |y_m||x_n - x_m| < K\frac{\epsilon}{2K} + L\frac{\epsilon}{2L} = \epsilon.$$

Exercise 2.6.4.

Exercise 2.6.5. Note that the Pseudo-Cauchy definition only requires that the difference between consecutive terms in the sequence become arbitrarily small, whereas the real Cauchy property requires that any two terms beyond a certain point in the sequence differ by an arbitrarily small amount.

(i) Pseudo-Cauchy sequences are not necessarily bounded. A counterexample would be the sequence of partial sums of the harmonic series:

$$(1), (1+1/2), (1+1/2+1/3), (1+1/2+1/3+1/4), \dots$$

Because $s_{n+1} - s_n = 1(n+1)$, it follows that the sequence is Pseudo-Cauchy, and we have seen in a previous example that this sequence is unbounded.

(ii) This is true and can be proved with a straightforward triangle inequality proof.

Let $\epsilon > 0$ be arbitrary. We need to find an N so that $n \geq N$ implies $|(x_n + y_n) - (x_{n+1} + y_{n+1})| < \epsilon$. Because (x_n) and (y_n) are Pseudo-Cauchy we can pick N so that when $n, m \geq N$ it follows that $|x_n - x_{n+1}| < \epsilon/2$ and $|y_n - y_{n+1}| < \epsilon/2$. But in this case we have

$$|(x_n + y_n) - (x_{n+1} + y_{n+1})| \leq |x_n - x_{n+1}| + |y_n - y_{n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

Exercise 2.6.6.

Exercise 2.6.7. (a) Let (a_n) be a bounded increasing sequence. We want to argue that (a_n) converges. Because (a_n) is bounded, we can appeal to the Bolzano-Weierstrass to assert that (a_n) has a convergent subsequence (a_{n_k}). Set $L = \lim a_{n_k}$. The goal is to show that the original sequence converges to this same limit.

Step one is to argue that L is an upper bound for all the terms in (a_n). Assume, for contradiction, that there exists $a_m > L$ and set $\epsilon_0 = a_m - L$. The fact that (a_n) is increasing implies

$$a_n - L \geq a_m - L = \epsilon_0 > 0$$

for all $n \geq m$, which is impossible if $L = \lim a_{n_k}$.

Having established that $a_n \leq L$ for all n, we can show $\lim a_n = L$. Given $\epsilon > 0$, we know there exists a term in the subsequence, call it a_{n_K}, satisfying $L - a_{n_K} < \epsilon$. If $n \geq n_K$ then $L - a_n \leq L - a_{n_K} < \epsilon$, and the result follows.

(b) Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all n. Our goal is to use the Cauchy Criterion to produce a convergent subsequence.

First construct the sequence of closed intervals and the subsequence with $a_{n_k} \in I_k$ according to the method described in the proof of the Bolzano-Weierstrass Theorem in the text. Rather than using NIP to produce a candidate for the limit of this subsequence, we can argue that (a_{n_k}) is convergent by appealing to the Cauchy Criterion.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (Note that this is the place where the Archimedean Property is required. In particular, we need some way to know that $(1/2)^k \to 0$ that doesn't make implicit use of BW or something equivalent to it.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. So for any $s, t \geq N$, because a_{n_s} and a_{n_t} are in I_k, it follows that $|a_{n_s} - a_{n_t}| < \epsilon$. Having shown (a_{n_k}) is a Cauchy sequence, we know it converges.

(c) The rational numbers are an ordered field where the Archimedean Property holds. Since AoC is most definitely not true in $\mathbf{Q}$, it follows that there is no way to prove AoC using only properties possessed by $\mathbf{Q}$.

2.7 Properties of Infinite Series

Exercise 2.7.1. (a) Here we show that the sequence of partial sums (s_n) converges by showing that it is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|s_n - s_m| < \epsilon$. First recall,

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|.$$

Because (a_n) is decreasing and the terms are positive, an induction argument shows that for all $n > m$ we have

$$|a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n| \leq a_{m+1}.$$

So, by virtue of the fact that $(a_n) \to 0$, we can choose N so that $m \geq N$ implies $a_m < \epsilon$. But this implies

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + \cdots \pm a_n| \leq a_{m+1} < \epsilon$$

whenever $n > m \geq N$, as desired.

(b) Let I_1 be the closed interval $[0, s_1]$. Then let I_2 be the closed interval $[s_2, s_1]$, which must be contained in I_1 as (a_n) is decreasing. Continuing in this fashion, we can construct a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$

By the Nested Interval Property there exists at least one point S satisfying $S \in I_n$ for every $n \in \mathbf{N}$. We now have a candidate for the limit, and it remains to show that $(s_n) \to S$.

Let $\epsilon > 0$ be arbitrary. We need to demonstrate that there exists an N such that $|s_n - S| < \epsilon$ whenever $n \geq N$. By construction, the length of I_n is $|s_n - s_{n-1}| = a_n$. Because $(a_n) \to 0$ we can choose N such that $a_n < \epsilon$ whenever $n \geq N$. Thus,

$$|s_n - S| \leq a_n < \epsilon$$

because both $s_n, S \in I_n$.

(c) The subsequence (s_{2n}) is increasing and bounded above (by a_1 for instance.) The Monotone Convergence Theorem allows us to assert that there exists an $S \in \mathbf{R}$ satisfying $S = \lim(s_{2n})$. One way to prove that the other subsequence (s_{2n+1}) converges to the same value is to use the Algebraic Limit Theorem and the fact that $(a_n) \to 0$ to write

$$\lim(s_{2n+1}) = \lim(s_{2n} + a_{2n+1}) = S + \lim(a_{2n+1}) = S + 0 = S.$$

The fact that both (s_{2n}) and (s_{2n+1}) converge to S implies that $(s_n) \to S$ as well. (See Exercise 2.3.5.)

Exercise 2.7.2.

Exercise 2.7.3. (a) (i) Assume $\sum_{k=1}^{\infty} b_k$ converges. Thus, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that $|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$. Since $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < |b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$$

whenever $n > m \geq N$, and $\sum_{k=1}^{\infty} a_k$ converges as well.

(ii) Rather than trying to work with a negated version of the Cauchy Criterion, we can argue by contradiction. This is actually an example of a contrapositive proof. Rather than proving "If P, then Q," we can argue that "Not Q implies not P." In the context of this particular problem, "Not Q implies not P" is just the statement "$\sum_{k=1}^{\infty} b_k$ converges implies that $\sum_{k=1}^{\infty} a_k$ converges." But this is exactly what we showed in (i).

(b)(i) Let $s_n = a_1 + \cdots + a_n$ be the partial sums for $\sum_{k=1}^{\infty} a_k$, and let $t_n = b_1 + \cdots + b_n$ be the partial sums for $\sum_{k=1}^{\infty}$. Because $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, both (s_n) and (t_n) are increasing, and in addition we have $s_n \leq t_n$ for all $n \in \mathbf{N}$. Because $\sum_{k=1}^{\infty} b_k$ converges, (t_n) is bounded and thus (s_n) is also bounded. By MCT, $\sum_{k=1}^{\infty} a_k$ converges.

(ii) As mentioned previously, this is just the contrapositive version of the statement in (i).

Exercise 2.7.4.

Exercise 2.7.5. By the Cauchy Condensation Test (Theorem 2.4.6) $\sum 1/n^p$ converges if and only if $\sum 2^n \left(1/2^n\right)^p$ converges. But notice that

$$\sum 2^n \left(\frac{1}{2^n}\right)^p = \sum \left(\frac{1}{2^n}\right)^{p-1} = \sum \left(\frac{1}{2^{p-1}}\right)^n.$$

By the Geometric Series Test (Example 2.7.5), this series converges if and only if $|\frac{1}{2^{p-1}}| < 1$. Solving for p we find that p must satisfy $p > 1$.

Exercise 2.7.6.

Exercise 2.7.7. (a) The idea here is that eventually the terms a_n "look like" a non-zero constant times $1/n$, and we know that any series of this form diverges. To make this precise, let $\epsilon_0 = l/2 > 0$. Because $(na_n) \to l$, there exists $N \in \mathbf{N}$

such that $na_n \in V_{\epsilon_0}(l)$ for all $n \geq N$. A little algebra shows that this implies we must have $na_n > l/2$, or

$$a_n > (l/2)\,(1/n) \quad \text{ for all } n \geq N.$$

Because this inequality is true for all but some finite number of terms, we may still appeal to the Comparison Test to assert that $\sum a_n$ diverges.

(b) Assume that $\lim(n^2 a_n) \to L \geq 0$. The definition of convergence (with $\epsilon_0 = 1$) tells us that there exists an N such that $n^2 a_n < L + 1$ for all $n \geq N$. This means that eventually $a_n < (L+1)/n^2$. We know that the series $\sum 1/n^2$ converges, and by the Algebraic Limit Theorem for series (Theorem 2.7.1), $\sum (L+1)/n^2$ converges as well. Thus, by the Comparison Test $\sum a_n$ must converge.

Exercise 2.7.8.

Exercise 2.7.9. (a) First, pick an ϵ-neighborhood around r of size $\epsilon_0 = |r - r'|$. Because $\lim \left|\frac{a_{n+1}}{a_n}\right| = r$, there exists an N such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| \in V_{\epsilon_0}(r)$. It follows that $\left|\frac{a_{n+1}}{a_n}\right| \leq r'$ for all $n \geq N$, and this implies the statement in (a)

(b) Having chosen N, $|a_N|$ is now a fixed number. Also, $\sum (r')^n$ is a geometric series with $|r'| < 1$, so it converges. Therefore, by the Algebraic Limit Theorem $|a_N| \sum (r')^n$ converges.

(c) From (a) we know that there exists an N such that $|a_{N+1}| \leq |a_N| r'$. Extending this we find $|a_{N+2}| \leq |a_{N+1}| r' \leq |a_N|(r')^2$, and using induction we can say that

$$|a_k| \leq |a_N|(r')^{k-N} \quad \text{ for all } k \geq N.$$

Thus, $\sum_{k=N}^{\infty} |a_k|$ converges by the Comparison Test and part (b). Because

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k|$$

and $\sum_{k=1}^{N-1} |a_k|$ is just a finite sum, the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Exercise 2.7.10.

Exercise 2.7.11. A preliminary example would be to let

$$(a_n) = (1, 0, 1, 0, 1, \dots) \quad \text{ and } \quad (b_n) = (0, 1, 0, 1, 0, \dots).$$

To handle the more challenging version, we shall construct two positive decreasing sequences (a_n) and (b_n) with $\min\{a_n, b_n\} = 1/n^2$ where $\sum a_n$ and $\sum b_n$ each diverge. First set $a_1 = b_1 = 1$. For $2 \leq n \leq 5$, let $a_n = 1/4$ and let $b_n = 1/n^2$. By holding $a_n = 1/4$ constant over 4 terms, we have added 1 to the partial sums of $\sum a_n$. For $6 \leq n \leq 6 + 24$, let $a_n = 1/n^2$ and hold $b_n = 1/25$ constant. This will add one to the partial sums of $\sum b_n$. Now we switch again and hold $a_n = 1/30^2$ for the next 30^2 terms while letting $b_n = 1/n^2$. Continuing this process will ensure that the partial sums of $\sum a_n$ and $\sum b_n$ are unbounded while $\sum \min\{a_n, b_n\} = \sum 1/n^2$ converges.

Exercise 2.7.12.

Exercise 2.7.13. (a) Let $s_n = \sum_{k=1}^{n} x_k$. By hypothesis, (s_n) converges to a limit L. Among other things, this implies that there exists $M > 0$ satisfying $|s_n| \leq M$ for all $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, Exercise 2.7.12 implies

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}). \tag{1}$$

(b) We would like to take the limit across equation (1) as $n \to \infty$. We know (s_n) and (y_{n+1}) both converge, but what about the sum? Well, using a telescoping argument we can show that it converges absolutely. More precisely, observe that

$$\begin{aligned} \sum_{k=1}^{n} |s_k(y_k - y_{k+1})| &\leq \sum_{k=1}^{n} M(y_k - y_{k+1}) \\ &= M(y_1 - y_{n+1}), \end{aligned}$$

and (y_{n+1}) converges as $n \to \infty$. This proves $\sum_{k=1}^{n} s_k(y_k - y_{k+1})$ converges absolutely. Applying the Algebraic Limit Theorem to equation (1) gives the result.

Exercise 2.7.14.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1. Examining the sum over squares we get $s_{11} = -1$, $s_{22} = -3/2$, $s_{33} = -7/4$, and in general

$$s_{nn} = -2 + \frac{1}{2^{n-1}}.$$

Now taking the limit we find $(s_{nn}) \to -2$. This value corresponds to the value previously computed by fixing j and summing down each column.

Exercise 2.8.2.

Exercise 2.8.3. (a) As we have been doing, let $b_i = \sum_{j=1}^{\infty} |a_{ij}|$ for all $i \in \mathbf{N}$. Our hypothesis tells us that there exists $L \geq 0$ satisfying $\sum_{i=1}^{\infty} b_i = L$. Because we are adding all non-negative terms, it follows that

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Thus, t_{mn} is bounded. We can now conclude that (t_{nn}) converges by the Monotone Convergence Theorem, as it is both increasing and bounded.

(b) Let $\epsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|s_{nn} - s_{mm}| < \epsilon$. Now the expression $s_{nn} - s_{mm}$ is really a sum over a finite collection of a_{ij} terms. If each a_{ij} included in the sum is replaced with $|a_{ij}|$, the sum only gets larger (this is just the triangle inequality), and the result is that

$$|s_{nn} - s_{mm}| = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \right| \leq |t_{nn} - t_{mm}|.$$

We know that (t_{nn}) converges, so pick N so that $n > m \geq N$ implies $|t_{nn} - t_{mm}| < \epsilon$. It follows that (s_{nn}) is Cauchy and must converge.

Exercise 2.8.4.

Exercise 2.8.5. (a) Thinking of m as fixed and n as the limiting variable, the Algebraic Limit Theorem can be applied to the finite number of components of

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}$$

to conclude that

$$\lim_{n \to \infty} s_{mn} = r_1 + r_2 + \cdots + r_m.$$

If, in addition, we insist that $m \geq N$ (where N is the one constructed in the previous exercise), then we have that

$$-\epsilon < s_{mn} - S < \epsilon$$

is eventually true once n is larger than N. Applying the Order Limit Theorem we find

$$-\epsilon \leq (r_1 + r_2 + \cdots + r_m) - S \leq \epsilon$$

for all $m \geq N$.

This last statement is extremely close to what we need to conclude that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S. Given an arbitrary $\epsilon > 0$, we have produced an N such that

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon \quad \text{for all } m \geq N$$

The only distraction is that our definition of convergence requires a strict inequality, and we have a "less than or equal to ϵ" result. This, however, is not a problem. Because ϵ is arbitrary, we could just as easily have chosen to let $\epsilon' < \epsilon$ at the beginning and constructed our argument using ϵ' throughout the proof. On a more general note, while we strive at the introductory level to adhere to the exact wording of our definitions, there comes a point in epsilon–style arguments where it becomes more convenient to simply make quantities less than something that we know can be made arbitrarily small.

(b) As the exercise explains, the same argument can be used to prove $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converges to S once we show that for each $j \in \mathbf{N}$ the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

To show $\sum_{i=1}^{\infty} a_{ij}$ converges for each $j \in \mathbf{N}$, it suffices to prove that the absolute series $\sum_{i=1}^{\infty} |a_{ij}|$ converges. Recall that $b_i = \sum_{j=1}^{\infty} |a_{ij}|$, so it is certainly the case that $b_i \geq |a_{ij}|$ for all $i, j \in \mathbf{N}$. But our hypothesis says that $\sum_{i=1}^{\infty} b_i$ converges, and so by the Comparison Test, $\sum_{i=1}^{\infty} |a_{ij}|$ converges for all values of j.

Exercise 2.8.6.

Exercise 2.8.7. (a) Let

$$\sum_{i=1}^{\infty} |a_i| = L \quad \text{and} \quad \sum_{j=1}^{\infty} |b_j| = M.$$

For each fixed $i \in \mathbf{N}$, the Algebraic Limit Theorem allows us write $\sum_{j=1}^{\infty} |a_i b_j| = |a_i| \sum_{j=1}^{\infty} |b_j|$. Continuing this process, we see

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| = \sum_{i=1}^{\infty} |a_i| \sum_{j=1}^{\infty} |b_j| = \sum_{i=1}^{\infty} |a_i| M = M \sum_{i=1}^{\infty} |a_i| = ML,$$

and therefore $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges to ML.

(b) Taking advantage of the fact that the sums are both finite, we can write

$$\lim_{n \to \infty} s_{nn} = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j = \lim_{n \to \infty} \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{j=1}^{n} b_j \right).$$

Applying the Algebraic Limit Theorem to the limits of these partial sums we find that $\lim_{n \to \infty} s_{nn} = AB$. From part (a) we know that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges, so we can use Theorem 2.8.1 and Exercise 2.8.6 to conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = \lim_{n \to \infty} s_{nn} = AB.$$

Chapter 3

Basic Topology of R

3.1 Discussion: The Cantor Set

3.2 Open and Closed Sets

Exercise 3.2.1. (a) We cannot always take minimums of infinite sets. Therefore the step where we let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\} > 0$ requires that we are working with a finite collection of open sets. We can, however, take the infimum of an infinite set, but the infimum of the set could be 0.

(b) Let $O_n = (\frac{-1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^{\infty} O_n = \{0\}$.

Exercise 3.2.2.

Exercise 3.2.3. (a) Neither. Given any point in $\mathbf{Q}$, there is no ϵ-neighborhood contained in $\mathbf{Q}$ because the irrational numbers are dense. The set of limit points not contained in $\mathbf{Q}$ is $\mathbf{I}$.

(b) Closed. Given any point in $\mathbf{N}$, there is no ϵ-neighborhood of that point contained in the set.

(c) Open. The limit point 0 is not contained in the set $\{x \in \mathbf{R} : x \neq 0\}$.

(d) Neither. None of the points in this set have ϵ-neighborhoods contained in the set. The set is not closed because this particular sequence converges to some finite limit (i.e., $\sum 1/n^2$ converges) and this limit, whatever it might be, is not an element of the set.

(e) Closed. There is no ϵ-neighborhood of any point in the set contained in $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$. Because the series $\sum 1/n$ diverges, this particular sequence of points does not have a limit point in $\mathbf{R}$, and is thus a closed set.

Exercise 3.2.4.

Exercise 3.2.5. ($\Rightarrow$) Assume that the set $F \subseteq \mathbf{R}$ is closed. Then F contains its limit points. We will show that that every Cauchy sequence (a_n) contained in F has its limit in F by showing that the limit of (a_n) is either a limit point or possibly an isolated point of F. Because (a_n) is Cauchy, we know $x = \lim a_n$ exists. If $a_n \neq x$ for all n, then it follows from Theorem 3.2.5 that x is a limit point of F. Now consider a Cauchy sequence a_n where $a_n = x$ for some n. Because $(a_n) \subseteq F$ it follows that $x \in F$ as well. (Note that if a_n is eventually equal to x, then it may not be true that x is a limit point of F.)

($\Leftarrow$) Assume that every Cauchy sequence contained in F has a limit that is also an element of F. To show that F is closed we want to show that it contains its limit points. Let x be a limit point of F. By Theorem 3.2.5, $x = \lim a_n$ for some sequence (a_n). Because (a_n) converges, it must be a Cauchy sequence. So x is contained in F, and therefore F is closed.

Exercise 3.2.6.

Exercise 3.2.7. (a) Let L be the set of limit points of A, and suppose that x is a limit point of L. We want to show that x is an element of L; in other words, that x is a limit point of A. Let $V_\epsilon(x)$ be arbitrary. By the definition of a limit point, $V_\epsilon(x)$ intersects L at a point $l \in L$, where $l \neq x$. Now choose $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and $x \notin V_{\epsilon'}(l)$. Since $l \in L$, l is a limit point of A and so $V_{\epsilon'}(l)$ intersects A. This implies $V_\epsilon(x)$ intersects A at a point different than x, and therefore x is a limit point of A and thus an element of L.

(b) Assume x is a limit point of $A \cup L$ and consider the ϵ-neighborhood $V_\epsilon(x)$ for an arbitrary $\epsilon > 0$. We know $V_\epsilon(x)$ must intersect $A \cup L$ and we would like to argue that it in fact intersects A. If $V_\epsilon(x)$ intersects A at a point different than x we are done, so let's assume that there exists an $l \in L$ with $l \in V_\epsilon(x)$. Using the same argument employed in (a), we take $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$, and $x \notin V_{\epsilon'}(l)$. Because l is a limit point of A we have that there exists an $a \in V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and thus $V_\epsilon(x)$ intersects A at some point other than x, as desired.

Because any limit point of $A \cup L$ is a limit point of A (and thus an element of L), it follows that $A \cup L$ contains its limit points; i.e., $\overline{A} = A \cup L$ is a closed set. This proves Theorem 3.2.12.

Exercise 3.2.8.

Exercise 3.2.9. (a) Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Then x is not an element of E_λ for all λ. Hence $x \in E_\lambda^c$ for all λ. So $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. We have just shown that $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Now we will show that $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then for all λ, $x \notin E_\lambda$. So $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$, and hence $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Therefore

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

Secondly, we want to show that

$$\left(\bigcap_{\lambda\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

Let $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. Then there exists a $\lambda' \in \Lambda$ for which x is not an element of $E_{\lambda'}$. Therefore $x \in E^c_{\lambda'}$. So $x \in \bigcup_{\lambda\in\Lambda} E^c_\lambda$, and we have $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcup_{\lambda\in\Lambda} E^c_\lambda$. Now assume $x \in \bigcup_{\lambda\in\Lambda} E^c_\lambda$. Then there exists a $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Therefore $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$, so $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. So it is also true that $\bigcup_{\lambda\in\Lambda} E^c_\lambda \subseteq \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$ and we have reached our desired conclusion.

(b) (i) Suppose that E_λ is a finite collection of closed sets. Then their complements, E^c_λ are a finite collection of open sets. We know by Theorem 3.2.3 that the intersection of a finite collection of open sets is open. In symbols,

$$\bigcap_{\lambda\in\Lambda} E^c_\lambda = \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$$

is an open set. Therefore the union of a finite collection of closed sets, $\bigcup_{\lambda\in\lambda} E_\lambda$ is closed.

(ii) Now suppose that E_λ is an arbitrary collection of closed sets. Then $\bigcup_{\lambda\in\Lambda} E^c_\lambda$ is open by Theorem 3.2.3. By De Morgan's Laws,

$$\bigcup_{\lambda\in\Lambda} E^c_\lambda = \left(\bigcap_{\lambda\in\lambda} E_\lambda\right)^c.$$

It then follows from Theorem 3.2.13 that the intersection of an arbitrary collection of closed sets is closed.

Exercise 3.2.10.

Exercise 3.2.11. (a) Clearly $A \subseteq A \cup B$, and any limit point of A will by definition be a limit point of $A \cup B$. Thus $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. It follows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. We also have that $A \cup B \subseteq \overline{A} \cup \overline{B}$, and so $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. But by Theorem 3.2.14, $\overline{A} \cup \overline{B}$ is closed, so $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and so $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) No. Take $A_n = \{1/n\}$. Then $\bigcup_{n=1}^\infty \overline{A_n} = \{1/n : n \in \mathbf{N}\}$. But $\overline{\bigcup_{n=1}^\infty A_n} = \{1/n : n \in \mathbf{N}\} \cup \{0\}$.

Exercise 3.2.12.

Exercise 3.2.13. For contradiction, assume that there exists a nonempty set A, different from $\mathbf{R}$, that is both open and closed. Because $A \neq \mathbf{R}$, $B = A^c$ is also nonempty, and B is open and closed as well. Pick a point $a_1 \in A$ and $b_1 \in B$. We can assume, without loss of generality, that $a_1 < b_1$. Bisect the interval $[a_1, b_1]$ at $c = (b_1 - a_1)/2$. Now $c \in A$ or $c \in B$. If $c \in A$, let $a_2 = c$ and let $b_2 = b_1$. If $c \in B$, let $b_2 = c$ and let $a_2 = a_1$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$ and $b_n \in B$. By the Nested Interval Property, there exists an $x \in \bigcap_{n=1}^\infty I_n$. Because the lengths $(b_n - a_n) \to 0$, we can show $\lim a_n = x$ which implies that $x \in A$ because A is closed. However, it is also true that $\lim b_n = x$ and thus $x \in B$ because B is closed. Thus we have shown $x \in A$ and $x \in A^c$. This contradiction implies that no such A exists, and we conclude that $\mathbf{R}$ and $\emptyset$ are the only two sets that are both open and closed. (This argument is closely related to the discussion of connected sets in the next section.)

Exercise 3.2.14.

Exercise 3.2.15. (a) $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$.

(b) $(a, b] = \bigcap_{n=1}^{\infty}(a, b + 1/n)$; $(a, b] = \bigcup_{n=1}^{\infty}[a + 1/n, b]$

(c) Because $\mathbf{Q}$ is countable, we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$. Note that each singleton set $\{r_n\}$ is closed and the complement $\{r_n\}^c$ is open. Then $\mathbf{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$ shows that $\mathbf{Q}$ is an F_σ set, and $\mathbf{I} = \mathbf{Q}^c = \bigcap_{n=1}^{\infty}\{r_n\}^c$ shows that $\mathbf{I}$ is a G_δ set.

3.3 Compact Sets

Exercise 3.3.1. Let K be compact and nonempty. Then by Theorem 3.3.4, K is closed and bounded. By the Axiom of Completeness, $\sup K$ exists, so let's set $\alpha = \sup K$.

Using the definition of the supremum, we can assert that for each $n \in \mathbf{N}$, there exists an $x_n \in K$ satisfying $\alpha - \frac{1}{n} < x_n \leq \alpha$. It follows that $\lim x_n = \alpha$. Because $(x_n) \subseteq K$ and K is closed, we can conclude $\alpha \in K$.

A similar argument shows $\inf K \in K$.

Exercise 3.3.2.

Exercise 3.3.3. Let $K \subseteq \mathbf{R}$ be closed and bounded. Since K is bounded, the Bolzano-Weierstrass Theorem guarantees that for any sequence (a_n) contained in K, we can find a convergent subsequence (a_{n_k}). Because the set is closed, the limit of this subsequence is also in K. Hence K is compact.

Exercise 3.3.4.

Exercise 3.3.5. (a) True. By Theorem 3.2.14, an arbitrary intersection of closed sets is closed. Boundedness is also preserved by intersections; therefore, the arbitrary intersection of compact sets will be compact.

(b) False. The union of the compact sets $[1/n, 2]$ over all $n \in \mathbf{N}$ is equal to $(0, 2]$ which is not closed and thus not compact. The union of the compact sets $[n, n + 1]$ over all $n \in \mathbf{N}$ is the unbounded set $[1, \infty)$.

(c) False. Take $A = (0, 1)$ and $K = [0, 1]$. Then $A \cap K = (0, 1)$ is not compact.

(d) False. Let $F_n = [n, \infty)$. Then F_n is closed for all n, but the intersection of these sets is empty.

Exercise 3.3.6.

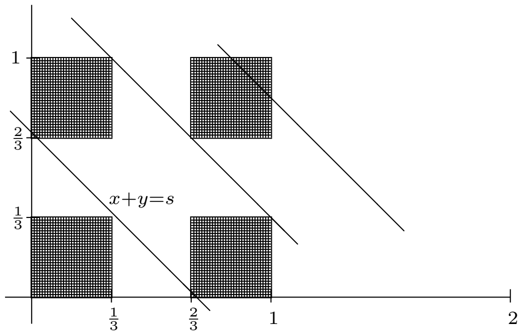

Figure 3.1: $x + y = s$ MUST INTERSECT $C_1 \times C_1$.

Exercise 3.3.7. (a) Fix $s \in [0, 2]$. We want to find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$. We know that $C_1 = [0, 1/3] \cup [2/3, 1]$. Then we have that:

$$[0, 1/3] + [0, 1/3] = [0, 2/3]$$

$$[0, 1/3] + [2/3, 1] = [2/3, 4/3]$$

$$[2/3, 1] + [2/3, 1] = [4/3, 1].$$

Hence $C_1 + C_1 = [0, 2/3] \cup [2/3, 4/3] \cup [4/3, 2] = [0, 2]$, so for any $s \in [0, 2]$, we can find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$.

A convenient way to visualize this result in the (x, y)–plane is to shade in the four squares corresponding to the components of $C_1 \times C_1$ (see Figure 3.1) and observe that, for each $s \in [0, 2]$, the line $x + y = s$ must intersect at least one of the squares. For each n we can draw a similar picture (with increasing numbers of smaller squares), and our job is to argue that the line $x + y = s$ continues to intersect at least one of the smaller squares

To argue by induction, suppose that we can find $x_n, y_n \in C_n$ such that $x_n + y_n = s$. To show that this must hold for $n + 1$, let's focus attention on a square from the nth stage where $x_n + y_n = s$ holds (i.e., where $x + y = s$ intersects an nth stage square). Moving to the $n + 1$th stage means removing the open middle third of this shaded region. But this results in a situation precisely like the one in Figure 3.1, implying that the line $x + y = s$ must intersect a $(n + 1)$st stage square. This shows that there exist $x_{n+1}, y_{n+1} \in C_{n+1}$ where $x_{n+1} + y_{n+1} = s$.

(b) We have (x_n) and (y_n) with $x_n, y_n \in C_n$ and $x_n + y_n = s$ for all n. The sequence (x_n) doesn't converge, but (x_n) is bounded so by the Bolzano-Weierstrass Theorem there exists a convergent subsequence (x_{n_k}). Set $x = \lim x_{n_k}$. Now look at the corresponding subsequence $(y_{n_k}) = s - x_{n_k}$. Using the Algebraic Limit Theorem, we see that this subsequence converges to $y = \lim(x - x_{n_k}) = s - x$. This shows $x + y = s$. We now need to argue that $x, y \in C$.

One temptation is to say that because C is closed, $x = \lim(x_{n_k})$ must be in C. However, we don't know (and it probably isn't true) that (x_{n_k}) is in C. We *can* say that (x_{n_k}) is in C_1, and because C_1 is closed we may conclude $x \in C_1$. In fact, given any fixed n_0, we can argue that $x \in C_{n_0}$ because x_{n_k} is (with the exception of some finite number of terms) contained in C_{n_0}. This implies $x \in \bigcap_{n=1}^{\infty} C_n = C$ as desired, and a similar argument works for y.

Exercise 3.3.8.

Exercise 3.3.9. (a) This is a standard bisection argument. Bisect I_0 into two halves A_1 and B_1. If $A_1 \cap K$ and $B_1 \cap K$ both had finite subcovers consisting of sets from the collection $\{O_\lambda : \lambda \in \Lambda\}$, then there would exist a finite subcover for K. But we assumed that such a finite subcover did not exist for K. Hence either $A_1 \cap K$ or $B_1 \cap K$ (or both) has no finite subcover.

Let I_1 be a half of I_0 whose intersection with K does not have a finite subcover, so that $I_1 \cap K$ cannot be finitely covered and $I_1 \subseteq I_0$. Then bisect I_1 into two closed intervals, A_2 and B_2 and again let $I_2 = A_2$ if $A_2 \cap K$ does not have a finite subcover. Otherwise, let $I_2 = B_2$. Continuing this process of bisecting the interval I_n, we get the desired sequence I_n with $\lim |I_n| = 0$.

(b) Because K is compact, $K \cap I_n$ is also compact for each $n \in \mathbf{N}$. By Theorem 3.3.5, $\bigcap_{n=1}^{\infty} I_n \cap K$ is nonempty, and there exists an $x \in K \cap I_n$ for all n.

(c) Let $x \in K$ and let O_{λ_0} be an open set that contains x. Because O_{λ_0} is open, there exists $\epsilon_0 > 0$ such that $V_{\epsilon_0}(x) \subseteq O_{\lambda_0}$. Now choose n_0 such that $|I_{n_0}| < \epsilon_0$. Then I_{n_0} is contained in the single open set O_{λ_0} and thus it has a finite subcover. This contradiction implies that K must have originally had a finite subcover.

Exercise 3.3.10.

Exercise 3.3.11. (a) Let $O_\lambda = (\lambda - 1, \lambda + 1)$ where $\lambda \in \mathbf{N}$. This open cover has no finite subcover.

(b) Let α be a fixed irrational number in the interval $(0, 1)$. For each $n \in \mathbf{N}$ set $O_n = (-1, \alpha - 1/n) \cup (\alpha + 1/n, 2)$. The union over n of all these sets gives $(-1, \alpha) \cup (\alpha, 2)$ which contains $\mathbf{Q} \cap [0, 1]$. This cover has no finite subcover.

(c) The Cantor set is compact.

(d) For each point $s_n = 1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2$ in the set, let $O_n = (s_n - 1/(n+1)^2, s_n + 1/(1+n)^2)$. This open cover has no finite subcover.

(e) This set is compact

Exercise 3.3.12.

Exercise 3.3.13. If A is a finite set then it is clearly clompact. Conversely, assume A is clompact. Because a singleton set is a closed set, the collection of singleton sets consisting of the elements of A is a closed cover. This cover must have a finite subcover, and it follows that A is a finite set. To summarize, a set is "clompact" if and only if it is finite.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1. Let P be a perfect set and let K be compact. Consider the set $P \cap K$. This set is closed by Theorem 3.2.14. Since K is bounded, $P \cap K$ will be bounded as well, and thus the intersection of the two sets is compact. However, $P \cap K$ is not necessarily perfect. For example, let K be a singleton set contained in P. Then $P \cap K$ is a singleton set and is not perfect.

Exercise 3.4.2.

Exercise 3.4.3. (a) We are given an arbitrary $x \in C$. Because $x \in C_1 \subseteq C$, x must fall in one of the two intervals that make up C_1. The key idea to remember is that C contains at least the endpoints of these two intervals. Thus, if $0 \leq x < 1/3$, let $x_1 = 1/3$. If $x = 1/3$, then take $x = 0$. We can do a similar thing if x falls in the other interval. This is, if $2/3 \leq x < 1$, then let $x_1 = 1$, and if $x = 1$ then set $x_1 = 2/3$. In all of these cases we have $x_1 \in C$ with $|x - x_1| \leq 1/3$.

(b) For each $n \in \mathbf{N}$, the length of each interval that makes up C_n is $1/3^n$. It is also true that the endpoints of these intervals are always elements of C. For every n, let x_n be an endpoint of the interval that contains x. If x happens to be an endpoint of a C_n interval, then let x_n be the opposite endpoint of this interval. Thus we have $x_n \in C$ with $x_n \neq x$ such that $|x - x_n| \leq 1/3^n$. Because $1/3^n \to 0$, it follows that $(x_n) \to x$. This means that $x \in C$ is not an isolated point. Having already seen that C is closed, we conclude that C is perfect.

Exercise 3.4.4.

Exercise 3.4.5. Let U and V be disjoint, open sets with $A \subseteq U$ and $B \subseteq V$. We claim that $\overline{U} \cap V = \emptyset$ and $U \cap \overline{V} = \emptyset$. To see why this is true, note that because U and V are disjoint we have $U \subseteq V^c$. Now V^c is closed (because V is open) and thus $\overline{U}$ must also satisfy $\overline{U} \subseteq V^c$ by Theorem 3.2.12. This proves $\overline{U} \cap V = \emptyset$, and the other statement has a similar proof.

Since $A \subseteq U$, limit points of A will also be limit points of U and we get $\overline{A} \subseteq \overline{U}$. Hence $\overline{A} \cap V = \emptyset$ and therefore $\overline{A} \cap B = \emptyset$. Similarly, $\overline{B} \subseteq \overline{V}$, so $A \cap \overline{B} = \emptyset$. Therefore, A and B are separated.

Exercise 3.4.6.

Exercise 3.4.7. (a) Given any $x, y \in \mathbf{Q}$, choose $z \in \mathbf{I}$ such that $x < z < y$. We know that such a z exists because the irrational numbers are dense. Then let $\mathbf{Q} = A \cup B$, where $A = \mathbf{Q} \cap (-\infty, z)$ and $B = \mathbf{Q} \cap (z, \infty)$. The sets A and B are separated (see Example 3.4.5(ii)), and $x \in A$ and $y \in B$.

(b) The set of irrational numbers is totally disconnected because the rational numbers are also dense in $\mathbf{R}$. Thus we can follow the same argument as in part (a) by letting $x, y \in \mathbf{I}$ and choosing $z \in \mathbf{Q}$.

Exercise 3.4.8.

Exercise 3.4.9. (a) Since $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ is a union of open sets, O is open. Therefore F is closed. Every rational number is contained in O, so F must contain only irrationals. Now we must argue that F is nonempty. To informally see this, look at the "length" of O. Since O is the union of open sets of length $1/2^{n-1}$, the length of O must be no greater than $\sum_{n=1}^{\infty} 1/2^n n - 1 = 2$. Therefore the entire real line cannot be covered by O, and hence F is nonempty.

A way to avoid applying the concept of "length" to sets that are not *finite* unions of intervals would be to assume, for contradiction, that $F = \emptyset$. Then $O = \mathbf{R}$ and, in particular, the compact set $[0,3]$ is covered by $\{V_{\epsilon_n}(r_n) : n \in \mathbf{N}\}$. Now let $\{V_{\epsilon_{n_1}}(r_{n_1}), V_{\epsilon_{n_2}}(r_{n_2}), \ldots V_{\epsilon_{n_m}}(r_{n_m})\}$ be a finite subcover for $[0,3]$. The lengths of this finite collection of open intervals must sum to a total less than 2, and therefore they cannot cover the set $[0,3]$.

(b) No, the set F does not contain any nonempty open intervals. Every non-trivial interval contains a rational number and this rational is not an element of F. Hence F contains no such intervals. This proves that F is totally disconnected. Given arbitrary $a, b \in F$ with $a < b$, we can find a rational number c with $a < c < b$. Then writing $F = A \cup B$ where $A = F \cap (-\infty, c)$ and $B = F \cap (c, \infty)$ finishes the argument.

(c) It is not possible to know whether F is perfect as it is possible for F to contain isolated points.

There does exist a nonempty perfect set of irrational numbers. To modify the construction, we again write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$, but this time we define ϵ_n inductively. Set $\epsilon_1 = \sqrt{2}/2$ and, as a convention, let $V_\epsilon(x) = \emptyset$ whenever $\epsilon = 0$. For $n \geq 2$, let $\epsilon_n = \min\{\sqrt{2}/2^n, d_n/2\}$ where

$$d_n = \inf\{|x - r_n| : x \in \bigcup_{k=1}^{n-1} V_{\epsilon_k}(r_k)\}.$$

Geometrically, d_n is the distance from r_n to the set $O_{n-1} = \bigcup_{k=1}^{n-1} V_{\epsilon_k}(r_k)$. The idea is to inductively build the open set O as a disjoint union of positively spaced neighborhoods of the form $V_{\epsilon_n}(r_n)$. If $d_n = 0$, then because ϵ_n is always irrational whenever it is non-zero, we may conclude $r_n \in O_{n-1}$. If $d_n > 0$, then the definition of ϵ_n ensures that

$$\overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_m}(r_m)} = \emptyset \quad \text{for all } 1 \leq m < n. \tag{1}$$

Now $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ is open and contains $\mathbf{Q}$, so $F = O^c$ is again a closed set inside the irrationals. It remains to show that it contains no isolated points.

Let $x \in F$ be arbitrary and assume, for contradiction, that x is isolated. Thus there exits $\epsilon_0 > 0$ such that $(x - \epsilon_0, x)$ and $(x, x + \epsilon_0)$ are both contained in O. Because of the way we constructed O it now follows that there must exist n' and m' such that

$$(x - \epsilon_0, x) \subseteq V_{\epsilon_{n'}}(r_{n'}) \quad \text{and} \quad (x, x + \epsilon_0) \subseteq V_{\epsilon_{m'}}(r_{m'}).$$

But this contradicts statement (1) above because the point x is a limit point of each of these two neighborhoods. This contradiction proves x is not isolated and the proof is complete.

3.5 Baire's Theorem

Exercise 3.5.1. ($\Rightarrow$) Let A be a G_δ set. We want to show that this implies that A^c is an F_σ set. By the definition of a G_δ set, A can be written as the countable intersection of open sets. In symbols, $A = \bigcap_{n=1}^{\infty} O_n$ where O_n is open for each $n \in \mathbf{N}$. Then by De Morgan's Law, $A^c = \bigcup_{n=1}^{\infty} O_n^c$. Because O_n is open, O_n^c is closed. Hence, A^c is the countable union of closed closed sets, and therefore it is an F_σ set.

($\Leftarrow$) Now let B be an F_σ set. Then we know that $B = \bigcup_{n=1}^{\infty} F_n$, where F_n is closed for each $n \in \mathbf{N}$. It then follows from De Morgan's Law that $B^c = \bigcap_{n=1}^{\infty} F_n^c$. Therefore, B^c is the countable intersection of open sets, which makes it a G_δ set.

Exercise 3.5.2.

Exercise 3.5.3. (a) $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$.

(b) $(a, b] = \bigcap_{n=1}^{\infty}(a, b + 1/n)$; $(a, b] = \bigcup_{n=1}^{\infty}[a + 1/n, b]$

(c) Because $\mathbf{Q}$ is countable, we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$. Note that each singleton set $\{r_n\}$ is closed and the complement $\{r_n\}^c$ is open. Then $\mathbf{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$ shows that $\mathbf{Q}$ is an F_σ set, and $\mathbf{I} = \mathbf{Q}^c = \bigcap_{n=1}^{\infty}\{r_n\}^c$ shows that $\mathbf{I}$ is a G_δ set.

Exercise 3.5.4.

Exercise 3.5.5. Let F be a closed set containing no nonempty open intervals. Then F^c is open and we claim that it must also be *dense* in $\mathbf{R}$. To see why, assume $x, y \in R$ satisfy $x < y$. By hypothesis, the open interval (x, y) is *not* contained in F which means there exists a point $z \in F^c$ satisfying $x < z < y$. This proves F^c is dense.

Turning to the statement in the exercise, assume for contradiction that $\mathbf{R} = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed set containing no nonempty open intervals. Taking complements we get $\emptyset = \bigcap_{n=1}^{\infty} F_n^c$, and we have just seen that each F_n^c is a dense open set in $\mathbf{R}$. But this is a contradiction, because the intersection of dense open sets is not empty, as the previous theorem states.

Exercise 3.5.6.

Exercise 3.5.7. The set $(\mathbf{I} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$ is neither an F_σ set nor a G_δ set.

Exercise 3.5.8.

Exercise 3.5.9. (a) Somewhere in between.

(b) Nowhere dense.

(c) Dense.

(d) Nowhere dense.

Exercise 3.5.10.

Chapter 4

Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

4.2 Functional Limits

Exercise 4.2.1. (a) Showing $\lim_{x \to c}[f(x)+g(x)] = L+M$ is equivalent to showing $f(x_n)+g(x_n) \to L+M$ whenever $(x_n) \to c$. Since we are given $f(x_n) \to L$ and $g(x_n) \to M$, we can use Theorem 2.3.3 part (ii) to conclude $f(x_n) + g(x_n) \to L + M$.

(b) Let $\epsilon > 0$ be arbitrary. We need to show, there exists δ such that $0 < |x - c| < \delta$ implies $|(f(x) + g(x)) - (L + M)| < \epsilon$. Note that,

$$|(f(x)+g(x))-(L+M)| = |(f(x)-L)+(g(x)-M)| \leq |f(x)-L|+|g(x)-M|.$$

Since $\lim_{x\to c} f(x) = L$, there exists δ_1 such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \epsilon/2$. In addition, because $\lim_{x\to c} g(x) = M$, there exists δ_2 such that $0 < |x - c| < \delta_2$ implies $|g(x) - M| < \epsilon/2$. Now if we pick $\delta = \min\{\delta_1, \delta_2\}$ then $0 < |x - c| < \delta$ implies that

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

as desired.

(c) Showing $\lim_{x \to c}[f(x)g(x)] = LM$ is equivalent to showing $f(x_n)g(x_n) \to LM$ whenever $(x_n) \to c$. Since we are given $f(x_n) \to L$ and $g(x_n) \to M$, we can use Theorem 2.3.3 part (iii) to conclude $f(x_n)g(x_n) \to LM$.

Now let's write another proof of the corollary based on Definition 4.2.1. Note that,

$$\begin{aligned} |f(x)g(x) - (LM)| &= |f(x)g(x) - f(x)M + f(x)M - (LM)| \\ &\leq |f(x)(g(x) - M)| + |M(f(x) - L)| \\ &= |f(x)||g(x) - M| + |M||f(x) - L|. \end{aligned}$$

Since $\lim_{x\to c} f(x) = L$, there exists δ_1 such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \epsilon/(2M)$.

Next we need a lemma that says $f(x)$ is bounded. Although this may not be the case over the whole domain A, it is certainly true in some neighborhood around $x = c$. Given $\epsilon_0 = 1$, for instance, we know there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|f(x) - L| < 1$, and in this case we then have $|f(x)| < |L| + 1$.

We now use the fact that $\lim_{x\to c} g(x) = M$ to assert that there exists $\delta_3 > 0$ such that $0 < |x - c| < \delta_3$ implies $|g(x) - M| < \epsilon/(2(|L| + 1))$. Finally, if we pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then

$$\begin{aligned} |f(x)g(x) - (LM)| &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &< (|L| + 1)\left(\frac{\epsilon}{2(|L| + 1)}\right) + M\left(\frac{\epsilon}{2M}\right) = \epsilon \end{aligned}$$

whenever $0 < |x - c| < \delta$.

Finally, we point out that this proof assumes $M \neq 0$. This minor hiccup can be remedied by replacing M with $M + 1$ in the above argument. Another strategy is to handle it as a special case. It also follows as a corollary to Exercise 4.2.7.

Exercise 4.2.2.

Exercise 4.2.3. (a) Let $x_n = (n+1)/n$, $y_n = \sqrt{(n+1)/n}$ and $z_n = (2n+1)/2n$. Note that $\lim(x_n) = \lim(y_n) = \lim(z_n) = 1$.

(b) For (x_n) we get $t(x_n) = 1/n$ which converges to 0.

For (y_n) we get $t(y_n) = 0$ which converges to 0.

For (z_n) we get $t(z_n) = 1/2n$ which converges to 0.

(c) The point to make is that the closer a rational number is to 1, the larger its denominator has to be, and thus the smaller the value of $t(x)$. Because $t(x) = 0$ for all irrational numbers, the conjecture is that $\lim_{x\to 1} t(x) = 0$.

In order to prove our claim, we have to show that given $\epsilon > 0$, there exists a δ neighborhood around 1 such that $x \in V_\delta(1)$ implies $t(x) \in V_\epsilon(0)$. If we set $T = \{x \in R : t(x) \geq \epsilon\}$, then notice that $x \in T$ if and only if x is a rational number of the form $x = m/n$ where $n \leq 1/\epsilon$. If we focus on some finite interval such as $[0, 2]$ then the restriction on the size of n implies that the set $T \cap [0, 2]$ is *finite*. With finite sets, we are allowed to take minimums and so let

$$\delta = \min\{y : y \in T \cap [0, 2]\} > 0.$$

To see that this choice of δ "works", we note that if $x \in V_\delta(1)$ then $x \notin T$ and thus $t(x) \in V_\epsilon(0)$.

Exercise 4.2.4.

Exercise 4.2.5. (a) Let $\epsilon > 0$. Notice that

$$|f(x) - 10| = |(3x + 4) - 10| = |3x - 6| = 3|x - 2|.$$

Choose $\delta = \epsilon/3$. Then $0 < |x - 2| < \delta = \epsilon/3$ implies that

$$|f(x) - 10| = 3|x - 2| < 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

(b) Let $\epsilon > 0$. Choose $\delta = \epsilon^{\frac{1}{3}}$. Then $0 < |x| < \delta = \epsilon^{\frac{1}{3}}$ implies that

$$|f(x) - 0| = |x^3| < (\epsilon^{\frac{1}{3}})^3 = \epsilon.$$

(c) Given an arbitrary $\epsilon > 0$, our goal is to make $|(x^2 + x - 1) - 5| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. Note that

$$|(x^2 + x - 1) - 5| = |x^2 + x - 6| = |x + 3|\,|x - 2|.$$

By insisting that $\delta \leq 1$, we can restrict x to fall in the interval $(1, 3)$. This implies $|x + 3| < 3 + 3 = 6$.

Now choose $\delta = \min\{1, \epsilon/6\}$. If $0 < |x - 2| < \delta$, then it follows that

$$|(x^2 + x - 1) - 5| = |x + 3|\,|x - 2| \leq 6\left(\frac{\epsilon}{6}\right) = \epsilon$$

as desired.

(d) Let $\epsilon > 0$ be arbitrary. First write

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3 - x|}{|3x|}.$$

To bound this expression we need to prevent x from being too close to zero. To this end, let's insist again that $\delta \leq 1$ so that $x \in (2, 4)$. Under this restriction, $1/|3x| \leq 1/6$.

Now choose $\delta = \min\{1, 6\epsilon\}$. If $0 < |x - 3| < \delta$, then it follows that

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3 - x|}{|3x|} < \frac{1}{6}(6\epsilon) = \epsilon,$$

as desired.

Exercise 4.2.6.

Exercise 4.2.7. We are given that there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. Because we know $\lim_{x\to c} g(x) = 0$, there exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|g(x) - 0| = |g(x)| < \epsilon/M$. It follows that

$$|g(x)f(x) - 0| = |g(x)||f(x)| < \left(\frac{\epsilon}{M}\right) M = \epsilon$$

whenever $0 < |x - c| < \delta$. Therefore, $\lim_{x\to c} g(x)f(x) = 0$.

Exercise 4.2.8.

Exercise 4.2.9. (a) Let $M > 0$ be arbitrary. To prove $\lim_{x\to 0} 1/x^2 = \infty$, we can choose $\delta = \sqrt{\frac{1}{M}}$. Then $0 < |x| < \delta = \sqrt{\frac{1}{M}}$ implies $x^2 < \frac{1}{M}$ from which it follows that $1/x^2 > M$, as desired.

(b) We say $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there exists $K > 0$, such that whenever $x > K$ it follows that $|f(x) - L| < \epsilon$.

Let $\epsilon > 0$. To prove $\lim_{x\to\infty} 1/x = 0$, choose $K = 1/\epsilon$. If $x > K = 1/\epsilon$, then $1/x < \epsilon$ as desired.

(c) We say $\lim_{x\to\infty} f(x) = \infty$ if for every $M > 0$ there exists $K > 0$, such that whenever $x > K$ it follows that $f(x) > M$. An example of function with such a limit would be $f(x) = \sqrt{x}$. Given an arbitrary $M > 0$, choose $K = M^2$. If $x > K = M^2$, then it follows that $\sqrt{x} > M$ as desired.

Exercise 4.2.10.

Exercise 4.2.11. This is another situation where we could use the analogous statement for sequences (Exercise 2.3.3) to prove the functional limit version. Instead, we shall give a proof in terms of the ϵ–δ definition of functional limits.

Let $\epsilon > 0$. Because $\lim_{x\to c} f(x) = L$, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $L - \epsilon < f(x) < L + \epsilon$. Likewise, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $L - \epsilon < h(x) < L + \epsilon$. Choosing $\delta = \min\{\delta_1, \delta_2\}$, we see that

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

whenever $0 < |x - c| < \delta$, which implies $|g(x) - L| < \epsilon$ as desired.

4.3 Continuous Functions

Exercise 4.3.1. (a) Let $\epsilon > 0$. Note that $|g(x) - c| = |\sqrt[3]{x} - 0| = |\sqrt[3]{x}|$ where $c = 0$. Now if we set $\delta = \epsilon^3$, then $|x - 0| < \delta = \epsilon^3$ implies $|\sqrt[3]{x}| < \epsilon$. This shows $g(x)$ is continuous at $c = 0$.

(b) For $c \neq 0$ write,

$$\begin{aligned} |g(x) - g(c)| &= |\sqrt[3]{x} - \sqrt[3]{c}| = |\sqrt[3]{x} - \sqrt[3]{c}| \left(\frac{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \right) \\ &= \frac{|x - c|}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \leq \frac{|x - c|}{\sqrt[3]{c^2}}. \end{aligned}$$

Therefore, if we pick $\delta = \epsilon\sqrt[3]{c^2}$, then $|x - c| < \delta = \epsilon\sqrt[3]{c^2}$ implies

$$|g(x) - g(c)| = |\sqrt[3]{x} - \sqrt[3]{c}| \leq \frac{|x - c|}{\sqrt[3]{c^2}} < \frac{\epsilon\sqrt[3]{c^2}}{\sqrt[3]{c^2}} = \epsilon.$$

Exercise 4.3.2.

Exercise 4.3.3. (a) Let $\epsilon > 0$. Because g is continuous at $f(c) \in B$, for every $\epsilon > 0$, there exists an $\alpha > 0$ such that $|g(y) - g(f(c))| < \epsilon$ whenever y satisfies $|y - f(c)| < \alpha$. Now, because f is continuous at $c \in A$, for this value of α, we can find a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \delta$. Combining the two statements, we see that for $\epsilon > 0$, there exists $\delta > 0$ such that $|x-c| < \delta$ implies $|g(f(x)) - g(f(c))| < \epsilon$. Therefore, $g \circ f$ is continuous at c.

(b) Let's now prove Theorem 4.3.9 using the sequential characterization of continuity in Theorem 4.3.2 (iv).

Assume $(x_n) \to c$ (with $c \in A$). Our goal is to show $g(f(x_n)) \to g(f(c))$. Because f is continuous at c, we know $f(x_n) \to f(c)$. Then, because g is continuous at $f(c)$, we know that $g(f(x_n)) \to g(f(c))$. This completes the proof.

Exercise 4.3.4.

Exercise 4.3.5. Let $\epsilon > 0$. If c is an isolated point of A, then there exists a neighborhood $V_\delta(c)$ that intersects the set A only at c. Because $x \in V_\delta(c) \cap A$ implies that $x = c$, we see $f(x) = f(c) \in V_\epsilon(f(c))$. Thus $f(x)$ is continuous at the isolated point c using the criterion in Theorem 4.3.2 (ii).

Exercise 4.3.6.

Exercise 4.3.7. (a) We are asked to show Dirichlet's function $g(x)$ is nowhere-continuous on $\mathbf{R}$. First consider an arbitrary $r \in \mathbf{Q}$. Because $\mathbf{I}$ is dense in $\mathbf{R}$ there exists a sequence $(x_n) \subseteq \mathbf{I}$ with $(x_n) \to r$. Then, $g(x_n) = 0$ for all $n \in N$ while $g(r) = 1$. Since $\lim g(x_n) = 0 \neq g(r)$ we can use Corollary 4.3.3 to conclude $g(x)$ is not continuous at $r \in \mathbf{Q}$.

Now let's consider an arbitrary $i \in \mathbf{I}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$ we can find a sequence $(y_n) \subseteq \mathbf{Q}$ with $(y_n) \to i$. This time $g(y_n) = 1$ for all $n \in N$ while $g(i) = 0$. Because $\lim g(y_n) = 1 \neq g(i)$ we can conclude that g is not continuous at i. Combining the two results, we can conclude that Dirichlet's function is indeed nowhere continuous on $\mathbf{R}$.

(b) Consider an arbitrary rational number $r \in \mathbf{Q}$ and observe that $t(r) \neq 0$. Because $\mathbf{I}$ is dense, there exists a sequence $(x_n) \subseteq \mathbf{I}$ with $(x_n) \to r$. Then, $t(x_n) = 0$ for all $n \in N$ while $t(r) \neq 0$. Thus, $\lim t(x_n) \neq t(r)$ and $t(x)$ is not continuous at r.

(c) Consider an arbitrary $c \in \mathbf{I}$. Given $\epsilon > 0$, set $T = \{x \in R : t(x) \geq \epsilon\}$. If $x \in T$, then x is a rational number of the form $x = m/n$ with $m, n \in \mathbf{Z}$ where n satisfies $|n| \leq 1/\epsilon$. By focusing our attention on the interval $[c-1, c+1]$ around the point c, we see that the restriction on the size of n implies that the set $T \cap [c-1, c+1]$ is *finite*. In a finite set, all points are isolated so we can pick a neighborhood $V_\delta(c)$ around c such that all $x \in V_\delta(c)$ implies $x \notin T$. But if $x \notin T$ then $t(x) < \epsilon$ or $t(x) \in V_\epsilon(t(c))$. By Theorem 4.3.2 (iii), we conclude $t(x)$ is continuous at c.

Exercise 4.3.8.

Exercise 4.3.9. We will prove the set K is closed by showing that it contains all its limit points. Let c be a limit point of K. By Theorem 3.2.5 there is a sequence

$(x_n) \subseteq K$ with $(x_n) \to c$. Because h is continuous on $\mathbf{R}$, $\lim h(x_n) = h(c)$. But notice $x_n \in K$, implies $h(x_n) = 0$, and thus $\lim h(x_n) = 0$. We conclude $h(c) = 0$, which implies $c \in K$, as desired.

Exercise 4.3.10.

Exercise 4.3.11. Geometrically speaking, the condition on f described in this problem says that if f is applied to any two points x and y, then the image values $f(x)$ and $f(y)$ are closer together (in a uniform way) than x and y. This is the reason for the term "contraction."

(a) Let $\epsilon > 0$ and fix $y \in \mathbf{R}$. To show f is continuous at y, choose $\delta = \epsilon/c$, and observe that $|x - y| < \delta = \epsilon/c$ implies

$$|f(x) - f(y)| \leq c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

Because y is arbitrary, $f(x)$ must be continuous on $\mathbf{R}$.

(b) Observe that for any fixed $n \in \mathbf{N}$,

$$|y_{m+1} - y_{m+2}| = |f(y_m) - f(y_{m+1})| \leq c|y_m - y_{m+1}|.$$

This idea can be extended inductively to conclude that

$$\begin{aligned} |y_{m+1} - y_{m+2}| &\leq c|y_m - y_{m+1}| \\ &\leq c^2|y_{m-1} - y_m| \\ &\leq \cdots \leq c^m|y_1 - y_2|. \end{aligned}$$

The fact that $0 < c < 1$ means $\sum_{n=1}^{\infty} c^n$ converges, and this will enable us to conclude that (y_n) is a Cauchy sequence. To see how, first note that for $m < n$ we have

$$\begin{aligned} |y_m - y_n| &\leq |y_m - y_{m+1}| + |y_{m+1} - y_{m+2}| + \cdots + |y_{n-1} - y_n| \\ &\leq c^{m-1}|y_1 - y_2| + c^m|y_1 - y_2| + \cdots + c^{n-2}|y_1 - y_2| \\ &= c^{m-1}|y_1 - y_2|(1 + c + \cdots + c^{n-m-1}) \\ &< c^{m-1}|y_1 - y_2|\left(\frac{1}{1-c}\right). \end{aligned}$$

Let $\epsilon > 0$, and choose $N \in \mathbf{N}$ large enough so that $c^{N-1} < \epsilon(1-c)/|y_1 - y_2|$. Then the previous calculation shows that $n > m \geq \mathbf{N}$ implies $|y_m - y_n| < \epsilon$. We conclude that (y_n) is Cauchy.

(c) Set $y = \lim y_n$. Because f is continuous, $f(y) = \lim f(y_n)$. But $f(y_n) = y_{n+1}$, and so $f(y) = \lim y_{n+1}$. Because $\lim y_{n+1} = \lim y_n = y$, it follows that $f(y) = y$ and y is a "fixed point."

(d) The argument in (b) and (c) applies to any sequence of iterates. Thus, given an arbitrary x, we may assert that $(x, f(x), f(f(x)), \ldots)$ converges to a limit x' and that x' is a fixed point of f. But y is also a fixed point and so

$$|f(x') - f(y)| = |x' - y|.$$

However,

$$|f(x') - f(y)| \leq c|x' - y|,$$

must also be true, and because $0 < c < 1$ we conclude that $x' = y$.

In summary, if f is a contraction on $\mathbf{R}$, then f has a unique fixed point, and every sequence of iterates converges to this unique point.

Exercise 4.3.12.

Exercise 4.3.13. (a) Note that $f(0) = f(0+0) = f(0) + f(0)$ which implies $f(0) = 0$. For any $x \in \mathbf{R}$, $f(0) = f(x - x) = f(x) + f(-x) = 0$. This implies $f(-x) = -f(x)$.

(b) Fix $c \in \mathbf{R}$ and let $(x_n) \to c$. To prove that f is continuous at c it is enough to show $\lim f(x_n) = f(c)$.

Now $(c - x_n) \to 0$. Because we are given that f is continuous at 0, it follows that

$$\lim f(x_n - c) = f(0) = 0.$$

Combining the additive condition on f with the Algebraic Limit Theorem then gives

$$0 = \lim f(c - x_n) = \lim(f(c) - f(x_n)) = f(c) - \lim f(x_n),$$

and we get $f(c) = \lim f(x_n)$ as desired.

(c) For any $n \in \mathbf{N}$,

$$f(n) = f(1 + 1 + \ldots + 1) = f(1) + f(1) + \ldots + f(1) = nf(1) = nk.$$

For $z \in \mathbf{Z}$, the case $z < 0$ is all that remains to do. In (a) we saw $f(-x) = -f(x)$. Observing that $z = -|z|$ and $|z| \in \mathbf{N}$, we can write

$$f(z) = f(-|z|) = -f(|z|) = -|z|k = zk. \tag{1}$$

Before taking on an arbitrary rational number, let's consider $1/n$ where $n \in \mathbf{N}$. In this case,

$$\begin{aligned} k = f(1) &= f\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \\ &= nf\left(\frac{1}{n}\right), \end{aligned}$$

which gives $f(1/n) = k/n$. For $m, n \in \mathbf{N}$ we then get

$$\begin{aligned} f(m/n) &= f\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \\ &= mf\left(\frac{1}{n}\right) = k(m/n). \end{aligned}$$

Finally, for any $r \in \mathbf{Q}$ satisfying $r < 0$, an argument similar to equation (1) above gives the result.

(d) Fix $x \in \mathbf{R}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a sequence $(r_n) \subseteq \mathbf{Q}$ with $(r_n) \to x$. By our work in part (c) we know that $f(r_n) = kr_n$ for all n. Then, because f is continuous at x, we have

$$f(x) = \lim f(r_n) = \lim kr_n = kx.$$

This completes the proof.

Exercise 4.3.14.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1. (a) Fix $c \in \mathbf{R}$ and write

$$|f(x) - f(c)| = |x^3 - c^3| = |x - c||x^2 + xc + c^2|.$$

Insisting that $\delta \leq 1$ means that x will fall in the interval $(c-1, c+1)$ and thus

$$|x^2 + xc + c^2| < (c+1)^2 + (c+1)^2 + c^2 < 3(c+1)^2.$$

Now pick $\delta = \min\{1, \epsilon/(3(c+1)^2)\}$. Then $|x - c| < \delta$ implies

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3(c+1)^2}\right) 3(c+1)^2 = \epsilon.$$

(b) The dependence of ϵ on the point c is evident in the previous formula with larger choices of c resulting in smaller values of δ. This means that the sequences (x_n) and (y_n) we seek are necessarily going to tend to infinity.

Set $x_n = n$ and $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n$ tends to zero as required, while

$$|f(x_n) - f(y_n)| = |n^3 - \left(n + \frac{1}{n}\right)^3| = 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3,$$

stays $\epsilon_0 = 3$ units apart for all $n \in \mathbf{N}$. This proves f is not uniformly continuous on $\mathbf{R}$.

(c) Let A be bounded by M. If $x, c \in A$ then $|x^2 + xc + c^2| \leq 3M^2$. Given $\epsilon > 0$ we can now choose $\delta = \epsilon/(3M^2)$, which is independent of c. If $|x - c| < \delta$, it follow that

$$|f(x) - f(c)| \leq \left(\frac{\epsilon}{3M^2}\right) 3M^2 = \epsilon,$$

and f is uniformly continuous on A.

Exercise 4.4.2.

Exercise 4.4.3. For $f(x) = 1/x^2$ we see

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = |y - x| \left(\frac{y + x}{x^2y^2}\right).$$

If we restrict our attention to $x, y \geq 1$, then we can estimate

$$\frac{y+x}{x^2y^2} = \frac{1}{x^2y} + \frac{1}{xy^2} \leq 1 + 1 = 2.$$

Given $\epsilon > 0$, we may then choose $\delta = \epsilon/2$ (independent of x and y), and it follows that $|f(x) - f(y)| < (\epsilon/2)2 = \epsilon$ whenever $|x - y| < \delta$. This shows f is uniformly continuous on $[1, \infty)$.

If x and y are allowed to be arbitrarily close to zero, then the expression $(x+y)/(x^2y^2)$ is unbounded and we get into trouble. To see this more explicitly, set $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$. Then $|x_n - y_n| \to 0$ while

$$|f(x_n) - f(y_n)| = |n - (n+1)| = 1.$$

By the criterion in Theorem 4.4.5, we conclude that f is not uniformly continuous on $(0, 1]$.

Exercise 4.4.4.

Exercise 4.4.5. Let $\epsilon > 0$ be arbitrary. Because f is uniformly continuous on $(a, b]$, there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $x, y \in (a, b]$ satisfy $|x - y| < \delta_1$. Likewise, there exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $x, y \in [b, c)$ satisfy $|x - y| < \delta_2$.

Now set $\delta = \min\{\delta_1, \delta_2\}$ and assume we have x and y satisfying $|x-y| < \delta$. If both x and y fall in $(a, b]$, or if they both fall in $[b, c)$, then we get $|f(x)-f(y)| < \epsilon/2 < \epsilon$. In the case where $x < b$ and $y > b$ we may write

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Because δ_1 and δ_2 are both independent of x and y, δ is as well and we conclude that f is uniformly continuous on (a, c).

Exercise 4.4.6.

Exercise 4.4.7. Let's first focus our attention on the domain $[1, \infty)$. If $x, y \geq 1$, it follows that

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq |x - y| \frac{1}{2}.$$

So, given $\epsilon > 0$ we can choose $\delta = 2\epsilon$, and it follows that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

Now the interval $[0, 1]$ is compact, so f is uniformly continuous on this domain as well. Putting these two pieces together and repeating the argument in Exercise 4.4.5 gives the result.

Exercise 4.4.8.

Exercise 4.4.9. (a) First write the Lipschitz condition in the form

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in A.$$

Given $\epsilon > 0$, we choose $\delta = \epsilon/M$. Then $|x - y| < \delta$ implies

$$|f(x) - f(y)| < M\frac{\epsilon}{M} = \epsilon.$$

This proves f is uniformly continuous.

(b) No, all uniformly continuous functions are not Lipschitz. Consider $f(x) = \sqrt{x}$ on $[0, 1]$. A continuous function on a compact set is uniformly continuous. However, if we set $y = 0$ and consider $x > 0$, then we get

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{\sqrt{x}}{x}\right| = \frac{1}{\sqrt{x}},$$

which is *not* bounded for x values arbitrarily close to zero.

Exercise 4.4.10.

Exercise 4.4.11. ($\Rightarrow$) Assume g is continuous on $\mathbf{R}$ and let $O \subseteq \mathbf{R}$ be open. We want to prove $g^{-1}(O)$ is open. To do this, we fix $c \in g^{-1}(O)$ and show that there is a δ–neighborhood of c satisfying $V_\delta(c) \subseteq g^{-1}(O)$.

Because $c \in g^{-1}(O)$, we know $g(c) \in O$. Now O is open, so there exists an $\epsilon > 0$ such that $V_\epsilon(g(c)) \subseteq O$. Given this particular ϵ, the continuity of g at c allows us to assert that there exists a neighborhood $V_\delta(c)$ with the property that $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(g(c)) \subseteq O$. But this implies $V_\delta(c) \subseteq g^{-1}(O)$, which proves that $g^{-1}(O)$ is open.

($\Leftarrow$) Conversely, we assume $g^{-1}(O)$ is open whenever O is open, and show that g is continuous at an arbitrary point $c \in \mathbf{R}$.

Let $\epsilon > 0$, and set $O = V_\epsilon(g(c))$. Certainly O is open, so our hypothesis gives us that $g^{-1}(O)$ is open. Because $c \in g^{-1}(O)$, there exists a $\delta > 0$ with $V_\delta(c) \subseteq g^{-1}(O)$. But this means that whenever $x \in V_\delta(c)$ we get $g(x) \in O = V_\epsilon(g(c))$, and we conclude that g is continuous at c by the criterion in Theorem 4.3.2 (ii).

Exercise 4.4.12.

Exercise 4.4.13. (a) We want to show that $f(x_n)$ is a Cauchy sequence, so let $\epsilon > 0$ be arbitrary. Because f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Given this δ, we use the fact that (x_n) is a Cauchy sequence to say that there exists an $N \in \mathbf{N}$ such that $|x_n - y_n| < \delta$ whenever $m, n \geq N$. Combining the last two statements we see that $|f(x_n) - f(y_n)| < \epsilon$ whenever $m, n \geq N$, which shows that $f(x_n)$ is Cauchy.

(b) ($\Rightarrow$) Let's first assume f is uniformly continuous on (a, b). Now fix a sequence (x_n) in (a, b) with $(x_n) \to a$. It follows from (a) that $g(x_n)$ converges, so let's *define* the value of $g(a)$ by asserting that $g(a) = \lim g(x_n)$.

Proving that g is continuous at a amounts to showing that if we now take an *arbitrary* sequence (y_n) that converges to a, then it follows that $g(a) = \lim g(y_n)$ as well. This is equivalent to showing that

$$\lim[g(y_n) - g(x_n)] = 0.$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|g(y) - g(x)| < \epsilon$ whenever $|x-y| < \delta$. Because (x_n) and (y_n) each converge to a, we see that $(y_n - x_n) \to 0$. Thus, there exists an $N \in N$ such that $|y_n - x_n| < \delta$ for all $n \geq N$. But this implies

$$|g(y_n) - g(x_n)| < \epsilon \quad \text{for all } n \geq N,$$

and we conclude $\lim[g(y_n) - g(x_n)] = 0$. Because this implies $g(a) = \lim g(y_n)$, we see that g is continuous at a.

A similar argument can be used for the point b.

($\Leftarrow$) Given that g can be continuously extended to the domain $[a, b]$, we immediately get that g is uniformly continuous because $[a, b]$ is a compact set. Thus g is certainly uniformly continuous on the smaller set (a, b).

Exercise 4.4.14.

4.5 The Intermediate Value Theorem

Exercise 4.5.1. The set $[a, b]$ is connected, and so by Theorem 4.5.2, the image set $f([a, b])$ is also connected. Because $f(a)$ and $f(b)$ are both elements of $f([a, b])$, we see that $L \in f([a, b])$ is as well by Theorem 3.4.6. But this implies that there exists a point $c \in (a, b)$ satisfying $L = f(c)$, as desired.

Exercise 4.5.2.

Exercise 4.5.3. Assume $f : [a, b] \to \infty$ is increasing and satisfies the intermediate value property stated in Definition 4.5.3. Let's fix $c \in (a, b)$ (the case where c is an endpoint is similar), and let $\epsilon > 0$. Our task is to produce a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

We know $f(a) \leq f(c)$. If $f(c) - \epsilon/2 < f(a)$, then set $x_1 = a$. If $f(a) \leq f(c) - \epsilon/2$, then the intermediate value property for f implies that there exists $x_1 < c$ where $f(x_1) = f(c) - \epsilon/2$. Because f is increasing, we see that in either case $x \in (x_1, c]$ implies

$$f(c) - \frac{\epsilon}{2} = f(x_1) \leq f(x) \leq f(c).$$

We can follow a similar process on the other side to get that there exists a point $x_2 > c$ with the property that

$$f(c) \leq f(x) \leq f(x_2) = f(c) + \frac{\epsilon}{2},$$

whenever $x \in [c, x_2)$. Finally, we set $\delta = \min\{c - x_1, x_2 - c\}$, and it follows that

$$f(c) - \frac{\epsilon}{2} \leq f(x) \leq f(c) + \frac{\epsilon}{2} \quad \text{provided } |x - c| < \delta.$$

This completes the proof.

Exercise 4.5.4.

Exercise 4.5.5. (a) Assume, for contradiction, that $f(c) > 0$. If we set $\epsilon_0 = f(c)$, then the continuity of f implies that there exists a $\delta_0 > 0$ with the property that $x \in V_{\delta_0}(c)$ implies $f(x) \in V_{\epsilon_0}(f(c))$. But this implies that $f(x) > 0$ and thus $x \notin K$ for all $x \in V_{\delta_0}(c)$. What this means is that if c is an upper bound on K, then $c - \delta_0$ is a smaller upper bound, violating the definition of the supremum. We conclude that $f(x) > 0$ is not allowed.

Now assume that $f(c) < 0$. This time, the continuity of f allows us to produce a neighborhood $V_{\delta_1}(c)$ where $x \in V_{\delta_1}(c)$ implies $f(x) < 0$. But this implies that a point such as $c + \delta_1/2$ is an element of K, violating the fact that c is an upper bound for K.

It follows that $f(c) < 0$ is also impossible, and we conclude that $f(c) = 0$ as desired.

This proves the Intermediate Value Theorem for the special case where $L = 0$. To prove the more general version, we consider the auxiliary function $h(x) = f(x) - L$ which is certainly continuous. From the special case just considered we know $h(c) = 0$ for some point $c \in (a, b)$ from which it follows that $f(c) = L$.

(b) By repeating the construction started in the text, we get a nested sequence of intervals $I_n = [a_n, b_n]$ where $f(a_n) < 0$ and $f(b_n) \geq 0$ for all $n \in \mathbf{N}$. By the Nested Interval Property, there exists a point $c \in \bigcap_{n=1}^{\infty} I_n$. The fact that the lengths of the intervals are tending to zero means that the two sequences (a_n) and (b_n) each converge to c.

Because f is continuous at c, we get $f(c) = \lim f(a_n)$ where $f(a_n) < 0$ for all n. Then the Order Limit Theorem implies $f(c) \leq 0$. Because we also have $f(c) = \lim f(b_n)$ with $f(b_n) \geq 0$, it must be that $f(c) \geq 0$. We conclude that $f(c) = 0$.

Exercise 4.5.6.

Exercise 4.5.7. The trick here is to apply the Intermediate Value Theorem to the function $g(x) = f(x) - x$. Because the range of f is contained in the interval $[0, 1]$ we see that

$$g(0) = f(0) \geq 0 \quad \text{and} \quad g(1) = f(1) - 1 \leq 0.$$

It follows from IVT that we must have $g(c) = 0$ for some point $c \in [0, 1]$, and this is equivalent to $f(c) = c$.

Exercise 4.5.8.

4.6 Sets of Discontinuity

Exercise 4.6.1. Let g be the Dirichlet function (which equals 1 on $\mathbf{Q}$ and 0 otherwise.)

(a) The functions $f(x) = g(x)\sin(\pi x)$ is continuous at each integer and is discontinuous everywhere else.

(b) For $x \in [0, 1]$ set $f(x) = xg(x)$, and let $f(x) = 0$ otherwise.

Exercise 4.6.2.

Exercise 4.6.3. We say that $\lim_{x\to c^-} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta$.

Exercise 4.6.4.

Exercise 4.6.5. This argument is very similar in spirit to the proof of the Monotone Convergence Theorem.

Given $c \in \mathbf{R}$, let's prove that $\lim_{x\to c^-} f(x)$ exists for an increasing function f. Our first task is to produce a candidate for the value of the limit. To this end, set

$$A = \{f(x) : x < c\}.$$

Because f is increasing, A is bounded above by $f(c)$. By AoC, we can set $L = \sup A$. The claim is that $\lim_{x\to c^-} f(x) = L$.

Let $\epsilon > 0$. By the least upper bound property of the supremum, we know that there exists an $x_0 < c$ satisfying

$$L - \epsilon < f(x_0) \leq L.$$

If we set $\delta = c - x_0$, then the fact that f is increasing implies that

$$L - \epsilon < f(x_0) \leq f(x) \leq L$$

whenever $0 < c - x < \delta$. This proves the claim.

For the right-hand limit we can fashion a similar argument to show that

$$\lim_{x\to c^+} f(x) = L',$$

where $L' = \inf\{f(x) : x > c\}$. A final consequence of this argument is that the value of the function at c must satisfy

$$L \leq f(c) \leq L'.$$

If $L = L'$ then f is continuous at c, and if $L < L'$ then we have a jump discontinuity. There are no other possibilities.

Exercise 4.6.6.

Exercise 4.6.7. (a) For Dirichlet's function we see $\mathbf{R}$ is closed.

For the modified Dirichlet function, we set $A_n = (-\infty, -1/n] \cup [1/n, \infty)$ which is closed for each $n \in \mathbf{N}$. Then $\mathbf{R}\backslash\{0\} = \bigcup_{n=1}^{\infty} A_n$ is an F_σ set.

For Thomae's function we observe that $\mathbf{Q}$ is the countable union of singleton sets, and a singleton set is closed.

(b) Let $n \in \mathbf{N}$ satisfy $n \geq 3$ and let $z \in \mathbf{Z}$. Define $U(n,z)$ to be the closed interval $[z + (1/n), (z+1) - (1/n)]$, and note that

$$U(n) = \bigcup_{z\in\mathbf{Z}} U(n,z)$$

is a closed set that doesn't contain any integers. Finally,

$$\mathbf{Z}^c = \bigcup_{n=3}^{\infty} U(n),$$

and therefore $\mathbf{Z}^c$ is an F_σ set.

For the second example we can write $(0,1] = \bigcup_{n=1}^{\infty}[1/n, 1]$.

Exercise 4.6.8.

Exercise 4.6.9. Assume $\alpha_1 < \alpha_2$ and let $c \in D_f^{\alpha_2}$. Given $\delta > 0$, the statement $c \in D_f^{\alpha_2}$ implies that there exist $y, z \in V_\delta(c)$ satisfying

$$|f(y) - f(z)| \geq \alpha_2 > \alpha_1.$$

Thus $c \in D_f^{\alpha_1}$ as well.

Exercise 4.6.10.

Exercise 4.6.11. Assume f is not continuous at x. Negating the ϵ–δ definition of continuity we get that there exists an $\epsilon_0 > 0$ with the property that for all $\delta > 0$ there exists a point $y \in V_\delta(x)$ where $|f(y) - f(x)| \geq \epsilon_0$. Noting simply that both $x, y \in V_\delta(x)$, we conclude that f is not α-continuous for $\alpha = \epsilon_0$ (or anything smaller.)

To prove $D_f = \bigcup_{n=1}^{\infty} D_f^{1/n}$ we argue for inclusion each way. If $x \in D_f$, then we have just shown that $x \in D_f^{\epsilon_0}$ for some $\epsilon_0 > 0$. Choosing $n_0 \in \mathbf{N}$ small enough so that $1/n_0 \leq \epsilon_0$, it follows that $x \in D_f^{1/n_0}$. This proves $D_f \subseteq \bigcup_{n=1}^{\infty} D_f^{1/n}$.

For the reverse inclusion we observe that Exercise 4.6.10 implies $D_f^{1/n} \subseteq D_f$ for all $n \in \mathbf{N}$, and the result follows.

Chapter 5

The Derivative

5.1 Discussion: Are Derivatives Continuous?

5.2 Derivatives and the Intermediate Value Property

Exercise 5.2.1. (i) First we rewrite the difference quotient as

$$\begin{aligned}\frac{(f+g)(x)-(f+g)(c)}{x-c} &= \frac{f(x)+g(x)-f(c)-g(c)}{x-c} \\ &= \frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}.\end{aligned}$$

The fact that f and g are differentiable at c together with the functional-limit version of the Algebraic Limit Theorem (Theorem 4.2.4) justifies the conclusion

$$(f+g)'(c) = f'(c) + g'(c).$$

(ii) This time we rewrite the difference quotient as

$$\begin{aligned}\frac{(kf)(x)-(kf)(c)}{x-c} &= \frac{kf(x)-kf(c)}{x-c} \\ &= k\left(\frac{f(x)-f(c)}{x-c}\right)\end{aligned}$$

Because f is differentiable at c, it follows from the functional-limit version of the Algebraic Limit Theorem that

$$(kf)'(c) = kf'(c).$$

Exercise 5.2.2.

Exercise 5.2.3. (a) For $c \neq 0$, the derivative of h at c is given by the formula

$$h'(c) = \lim_{x \to c} \frac{1/x - 1/c}{x - c} = \lim_{x \to c} \frac{(c-x)/xc}{x-c} = \lim_{x \to c} \frac{-1}{xc} = \frac{-1}{c^2}.$$

(b) By the Chain Rule,

$$\left(\frac{1}{g(x)}\right)' = (h \circ g)'(x) = \frac{-g'(x)}{[g(x)]^2}.$$

Then using the product rule (Theorem 5.2.4 (iii)), we have

$$\begin{aligned} \left(\frac{f}{g}\right)'(x) = [f(x)(h \circ g)(x)]' &= f'(x)(h \circ g)(x) + f(x)(h \circ g)'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

provided that $g(c) \neq 0$.

(c) Rewrite the difference quotient as

$$\begin{aligned} \frac{(f/g)(x) - (f/g)(c)}{x-c} &= \frac{1}{x-c}\left(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right) \\ &= \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right) \\ &= \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)}\right) \\ &= \frac{1}{g(x)g(c)}\left(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c}\right). \end{aligned}$$

Applying the Algebraic Limit Theorem for functional limits gives

$$\left(\frac{f}{g}\right)'(c) = \frac{1}{[g(c)]^2}\left(g(c)f'(c) - f(c)g'(c)\right),$$

which gives the result.

Exercise 5.2.4.

Exercise 5.2.5. (a) From the left side of zero we have $\lim_{x \to 0^-} f(x) = 0$, so we require that $\lim_{x \to 0^+} x^a = 0$ as well. This occurs if and only if $a > 0$.

(b) From (a) we know $f_a(0) = 0$. For $f_a'(0)$ we again begin by considering the limit from the left and see that

$$\lim_{x \to 0^-} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0.$$

Thus, we require that

$$\lim_{x\to 0^+} \frac{x^a}{x} = \lim_{x\to 0^+} x^{a-1} = 0$$

as well. This occurs if and only if $a > 1$. The derivative formula $(x^a)' = ax^{a-1}$ (which we have not justified for $a \notin \mathbf{N}$) shows that $f_a'(x)$ is continuous in this case.

(c) Because we continue to get zero on the left, for the second derivative to exist we must have

$$\lim_{x\to 0^+} \frac{(x^a)' - 0}{x - 0} = \lim_{x\to 0^+} \frac{ax^{a-1}}{x} = \lim_{x\to 0^+} ax^{a-2} = 0.$$

This occurs whenever $a > 2$.

Exercise 5.2.6.

Exercise 5.2.7. (a) With regards to the existence of $g_a'(x)$ at $x = 0$ we see that

$$g_a'(0) = \lim_{x\to 0} \frac{x^a \sin(1/x)}{x} = \lim_{x\to 0} x^{a-1}\sin(1/x) = 0,$$

as long as $a > 1$. For $x \neq 0$, $g_a'(x)$ always exists and using the standard rules of differentiation we get

$$g_a'(x) = -x^{a-2}\cos(1/x) + ax^{a-1}\sin(1/x).$$

Setting $1 < a < 2$ makes $x^{a-2}\cos(1/x)$ unbounded near zero and yields the desired function.

(b) For $g_a'(x)$ to be continuous we need

$$\lim_{x\to 0} g_a'(x) = g_a'(0) = 0$$

and, looking at the above expression for $g_a'(x)$, we see that this happens as long as $a > 2$. For the second derivative $g_a''(0)$ we consider the limit

$$\begin{aligned} g_a''(0) = \lim_{x\to 0} \frac{g_a'(x)}{x} &= \lim_{x\to 0} \left(\frac{1}{x}\right)\left(-x^{a-2}\cos(1/x) + ax^{a-1}\sin(1/x)\right) \\ &= \lim_{x\to 0} \left(-x^{a-3}\cos(1/x) + ax^{a-2}\sin(1/x)\right) \end{aligned}$$

which exists if and only if $a > 3$. Thus setting $2 < a \leq 3$ gives the desired function.

(c) From (b) we see that choosing $a > 3$ makes g_a' differentiable at zero. Away from zero we get

$$g_a''(x) = -x^{a-4}\sin(1/x) - (2a-2)x^{a-3}\cos(1/x) + a(a-1)x^{a-2}\sin(1/x),$$

which fails to be continuous at zero when $a \leq 4$. Setting $3 < a \leq 4$ gives the desired function.

Exercise 5.2.8.

Exercise 5.2.9. (a) True. Although the derivative function need not be continuous, it does satisfy the intermediate value property. Thus, if the derivative of a function takes on two distinct values then it attains every value–rational and irrational–in between these two.

(b) False. Consider

$$f(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

At zero we can show that $f'(0) = 1/2$. Away from zero we get

$$f'(x) = 1/2 - \cos(1/x) + 2x \sin(1/x),$$

which takes on negative values in every δ-neighborhood of zero.

(c)True. Assume, for contradiction, that $L \neq f'(0)$ and choose $\epsilon_0 > 0$ so that $\epsilon_0 < |f'(0) - L|$. From the hypothesis that $\lim_{x\to 0} f'(x) = L$ we know there exists a $\delta > 0$ such that $0 < |x| < \delta$ implies that $|f'(x) - L| < \epsilon_0$. Now our choice of ϵ_0 guarantees that there exists a value α between $f'(0)$ and L but outside $V_{\epsilon_0}(L)$. However, by Darboux's Theorem, there exists a point $x \in V_\delta(0)$ such that $f'(x) = \alpha$. This suggests that $\alpha \in V_{\epsilon_0}(L)$, which is a contradiction. Therefore $L = f'(0)$.

Exercise 5.2.10.

Exercise 5.2.11. (a) First let's prove that there exists $x \in (a, b)$ where $g(x) < g(a)$. Let (x_n) be a sequence in (a, b) satisfying $(x_n) \to a$. Then we have

$$g'(a) = \lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} < 0.$$

The denominator is always positive. If the numerator were always positive then the Order Limit Theorem would imply $g'(a) \geq 0$. Because we know this is not the case, we may conclude that the numerator is eventually negative and thus $g(x) < g(a)$ for some x near a.

The proof that there exists $y \in (a, b)$ where $g(y) < g(b)$ is similar.

(b) We must show that $g'(c) = 0$ for some $c \in (a, b)$. Because g is differentiable on the compact set $[a, b]$ it must also be continuous here, and so by the Extreme Value Theorem (Theorem 4.4.2), g attains a minimum at a point $c \in [a, b]$. From our work in (a) we know that the minimum of g is neither $g(a)$ nor $g(b)$, and therefore $c \in (a, b)$. Finally, the Interior Extremum Theorem (Theorem 5.2.6) allows us to conclude $g'(c) = 0$.

To prove the general result stated in the theorem we just observe that $g'(c) = 0$ is equivalent to the conclusion $f'(c) = \alpha$.

Exercise 5.2.12.

5.3 The Mean Value Theorems

Exercise 5.3.1. (a) Because f' is continuous on the compact set $[a, b]$, we know that it is bounded. Thus, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$.

Now, given $x < y$ in the interval $[a, b]$, the Mean Value Theorem says that there exists a point $c \in (a, b)$ for which

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

Because $|f'(c)| \leq M$ (regardless of the value of c), it follows that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M.$$

(b) As before, the Extreme Value Theorem can be used to conclude that f' attains a maximum and a minimum value on $[a, b]$. Thinking in terms of absolute value, this means that there exists a point x_0 where $|f'(x)| \leq |f'(x_0)|$ for all $x \in [a, b]$. Setting $s = |f'(x_0)|$, we see from our hypothesis that $0 \leq s < 1$.

Now, given $x < y$ in $[a, b]$, the Mean Value Theorem tells us that there exists a point $c \in (x, y)$ where

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq |f'(x_0)| = s.$$

It follows that

$$|f(x) - f(y)| \leq s|x - y|,$$

and f is contractive on $[a, b]$.

Exercise 5.3.2.

Exercise 5.3.3. (a) Set $g(x) = x - h(x)$. Because $g(1) = -1$ and $g(3) = 1$, by the Intermediate Value Theorem (Theorem 4.5.1), there must exist a $d \in [0, 3]$ where $g(d) = 0$. In terms of h, we note that this implies $h(d) = d$, as desired.

(b) Applying the Mean Value Theorem to h on the interval $[0, 3]$ implies that there exists a point $c \in (0, 3)$ where

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(c) Applying Rolle's Theorem to h on the interval $[1, 3]$, we see that there must exist a point $a' \in (1, 3)$ where $h'(a) = 0$. In (b), we found a point where $h'(c) = 1/3$. Because $1/4$ falls between 0 and $1/3$, we can appeal to Darboux's Theorem to assert that $h'(x) = 1/4$ at some point between c and a.

Exercise 5.3.4.

Exercise 5.3.5. (a) Let

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

From the many algebraic limit theorems we know that h is continuous on $[a, b]$ and differentiable on (a, b). We also have $h(a) = g(a)f(b) - f(a)g(b) = h(b)$. Thus by Rolle's Theorem, there exists a $c \in (a, b)$ where $h'(c) = 0$. Because

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x),$$

we see that

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0,$$

and the result follows.

(b) Set $x = g(t)$ and $y = f(t)$ and consider the parametric curve in the xy plane drawn as t ranges over the interval $[a, b]$. The quantity $(f(b)-f(a))/(g(b)-g(a))$ corresponds to the slope of the segment joining the endpoints of this curve, while $f'(c)/g'(c)$ gives the slope of the line tangent to the curve at the point $(g(c), f(c))$. In this context, the Generalized Mean Value Theorem says that if g' is never zero, then at some point along the parametric curve, the tangent line must be parallel to the segment joining the two endpoints.

Exercise 5.3.6.

Exercise 5.3.7. Assume, for contradiction, that f has two distinct fixed points x_1 and x_2. Noting that $f(x_1) = x_1$ and $f(x_2) = x_2$, the Mean Value Theorem implies that there exists c where

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1.$$

Because this is impossible, we conclude that f can have at most one fixed point.

Exercise 5.3.8.

Exercise 5.3.9. For all $x \neq a$ we can write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)}.$$

Because f and g are differentiable at a, we may use the Algebraic Limit Theorem to conclude

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Finally, the continuity of f' and g' at a implies

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)},$$

and the result follows.

Exercise 5.3.10.

Exercise 5.3.11. (a) Let $\epsilon > 0$. Because $L = \lim_{x \to a} f'(x)/g'(x)$, we know that there exists a $\delta > 0$ such that

$$\left| \frac{f'(t)}{g'(t)} - L \right| < \epsilon \quad \text{provided } 0 < |t - a| < \delta.$$

This δ is going to suffice to prove $L = \lim_{x \to a} f(x)/g(x)$ as well. To see why, pick $x \in V_\delta(a)$ with $a < x$ (the case $x < a$ is similar) and apply GMVT to f and g on the interval $[a, x]$. In this case we get a point $c \in (a, x)$ where

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Because c must satisfy $0 < |c - a| < \delta$, it follows that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon$$

whenever $0 < |x - a| < \delta$. This completes the proof.

(b) Yes, the proof goes through exactly the same way in this case.

Exercise 5.3.12.

5.4 A Continuous Nowhere-Differentiable Function

Exercise 5.4.1. The graph of $h_1(x)$ is similar to the sawtooth function $h(x)$ except that the maximum height is now $1/2$ and the length of the period is 1. For each n, the maximum height of $h_n(x)$ is $1/2^n$ and the period is $1/2^{n-1}$. Note that the slopes of the segments that make up $h_n(x)$ continue to be ± 1 for all values of n.

Exercise 5.4.2.

Exercise 5.4.3. For each n, the linear function $l(x) = 2^n x$ is certainly continuous. Then the Composition of Continuous Functions Theorem (Theorem 4.3.9) implies $h(2^n x)$ is continuous. The Algebraic Continuity Theorem (Theorem 4.3.4) part (i) implies $\frac{1}{2^n} h(2^n x)$ is continuous. Finally, part (ii) of the same theorem says

$$g_m(x) = h(x) + \frac{1}{2} h(2x) + \cdots + \frac{1}{2^m} h(2^m x)$$

is continuous as long as the sum is finite.

Exercise 5.4.4.

Exercise 5.4.5. For $g'(0)$ to exist, the sequential criterion for limits requires that

$$g'(0) = \lim_{m\to\infty} \frac{g(x_m) - g(0)}{x_m - 0}$$

exist for any sequence $(x_m) \to 0$. Fix $m \in \mathbf{N}$ and consider $x_m = 1/2^m$. Then

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}).$$

If $n > m$ then $h(2^{n-m}) = 0$ because the sawtooth function is zero at any multiple of 2. If $n \leq m$ then we are on the part of the graph where $h(x) = x$ and we get

$$\frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}.$$

It follows that $g(x_m)$ can be represented with the finite sum

$$g(x_m) = \sum_{n=0}^{m} \frac{1}{2^m}.$$

Turning our attention to the difference quotient, we get

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{\sum_{n=0}^{m} 1/2^m}{1/2^m} = \sum_{n=0}^{m} 1 = m + 1.$$

Because this quantity increases without bound, it is impossible for $\lim_{m\to\infty} g(x_m)/x_m$ to exist. It follows that g is not differentiable at zero.

Exercise 5.4.6.

Exercise 5.4.7. (a) To prove the lemma, first note that

$$\begin{aligned}\frac{f(b_n) - f(a_n)}{b_n - a_n} &= \frac{f(b_n) - f(a)}{b_n - a_n} + \frac{f(a) - f(a_n)}{b_n - a_n} \\ &= \frac{f(b_n) - f(a)}{b_n - a}\left(\frac{b_n - a}{b_n - a_n}\right) + \frac{f(a) - f(a_n)}{a - a_n}\left(\frac{a - a_n}{b_n - a_n}\right).\end{aligned}$$

Then substitute the identity $\frac{b_n - a}{b_n - a_n} = 1 - \left(\frac{a - a_n}{b_n - a_n}\right)$ into the previous equation, and do a little algebra to get

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{f(b_n) - f(a)}{b_n - a} + \left(\frac{f(a) - f(a_n)}{a - a_n} - \frac{f(b_n) - f(a)}{b_n - a}\right)\left(\frac{a - a_n}{b_n - a_n}\right).$$

What happens when we take the limit as $n \to \infty$? Well, the assumption that $f'(a)$ exists tells us that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a)}{b_n - a} = f'(a),$$

and
$$\lim_{n\to\infty}\left(\frac{f(a)-f(a_n)}{a-a_n}-\frac{f(b_n)-f(a)}{b_n-a}\right)=f'(a)-f'(a)=0.$$

The Algebraic Limit Theorem doesn't quite apply here, but the fact that $\left|\frac{a-a_n}{b_n-a_n}\right|\leq 1$ for all n is enough to let us conclude that

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(a),$$

as desired.

(b) Now we return to the business of showing that g is not differentiable at the non-dyadic point x. According to our lemma from (a), if g were differentiable at x then it would follow that

$$g'(x)=\lim_{m\to\infty}\frac{g(y_m)-g(x_m)}{y_m-x_m}.$$

So we just need to argue that this limit does NOT exist for the particular sequences (x_m) and (y_m) that we constructed earlier.

Looking back at the construction of g, we have seen that $h_n(x_m)=h_n(y_m)=0$ for all $n>m$. It follows that $g_m(x_m)=g(x_m)$ and $g_m(y_m)=g(y_m)$. Focusing on the graphs over the interval $[x_m,y_m]$, what we see is that g_m is the line segment connecting the points $(x_m,g(x_m))$ and $(y_m,g(y_m)$ and thus

$$g_m'(x)=\frac{g(y_m)-g(x_m)}{y_m-x_m}.$$

And what happens as m increases? Well,

$$|g_{m+1}'(x)-g_m'(x)|=|h_{m+1}'(x)|$$

and $h_{m+1}'(x)=\pm 1$ because it is a piecewise linear function consisting of segments of slope $\pm$ 1. Said another way, the slopes of the approximating line segments that make up the graphs of g_m are always integers that change by ± 1 at each iteration. Thus, the sequence $g_m'(x)$ has no chance of being Cauchy and thus cannot converge.

This means
$$\lim_{m\to\infty}\frac{g(y_m)-g(x_m)}{y_m-x_m},$$

does not exist, which implies that g is not differentiable at x.

Exercise 5.4.8.

Chapter 6

Sequences and Series of Functions

6.1 Discussion: The Power of Power Series

6.2 Uniform Convergence of a Sequence of Functions

Exercise 6.2.1. (a) By dividing the numerator and denominator by n, we can compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{1/n + x^2} = \frac{1}{x}.$$

Therefore, the pointwise limit of $f_n(x)$ is $f(x) = 1/x$.

(b) The convergence of $(f_n(x))$ is not uniform on $(0,\infty)$. To see this write

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \frac{1}{x+nx^3}.$$

In order to make $|f_n(x) - f(x)| < \epsilon$ we must choose

$$N \geq \frac{1-\epsilon x}{\epsilon x^3}.$$

For a fixed $\epsilon > 0$, the expression $(1-\epsilon x)/(\epsilon x^3)$ grows without bound as x tends to zero, and thus there is no way to pick a value of N that will work for every value of x in $(0,\infty)$.

(c) The convergence is not uniform on $(0,1)$ either. As seen in (b), the problem arises when x tends to zero and this is equally relevant over the domain $(0,1)$.

(d) The convergence is uniform on the interval $(1,\infty)$. If $x > 1$ then it follows that

$$|f_n(x) - f(x)| = \frac{1}{x+nx^3} \leq \frac{1}{1+n}.$$

Given $\epsilon > 0$, choose N large enough so that $1/(1+n) < \epsilon$ whenever $n \geq N$. It follows that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and thus $(f_n) \to f$ uniformly on $(1, \infty)$.

Exercise 6.2.2.

Exercise 6.2.3. For the sequence $g_n(x)$:

(a) The pointwise limit of (g_n) on $[0, \infty)$ is

$$g(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

(b) Theorem 6.2.6 tells us that if the convergence were uniform then $g(x)$ would be continuous. However, $g(x)$ is not continuous at $x = 1$ and so the convergence cannot be uniform on any domain containing this point. In fact, the convergence is not uniform over any domain that has $x = 1$ as a limit point.

(c) Consider the set $[2, \infty)$. If $x \geq 2$ then

$$|g_n(x) - g(x)| = \left| \frac{x}{1+x^n} - 0 \right| < \frac{x}{x^n} \leq \frac{1}{2^{n-1}}.$$

Given $\epsilon > 0$, pick N so that $n \geq N$ implies $1/2^{n-1} < \epsilon$. Then $|g_n(x) - g(x)| < \epsilon$ for all $n \geq N$, and we conclude that $g_n \to g$ uniformly on $[2, \infty)$.

For the sequence $h_n(x)$:

(a) Taking the limit for each fixed value of x we find that $h_n(x)$ converges pointwise to

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(b) Each of the functions h_n is continuous, but the limit function h is not. Therefore, Theorem 6.2.6 tells us that the convergence cannot be uniform.

(c) It is possible to prove uniform convergence on any set of the form $[a, \infty)$ where $a > 0$. On such a set, we can choose N large enough so that $1/N < a$. Then for all $n \geq N$, it follows that $|h_n(x) - h(x)| = |1 - 1| = 0$ whenever $x \geq a$. This shows that N is a suitable response for any value of $\epsilon > 0$ on the set $[a, \infty)$.

Exercise 6.2.4.

Exercise 6.2.5. ($\Rightarrow$) This is the easier of the two directions. Let $\epsilon > 0$ be arbitrary. Given that (f_n) converges uniformly on A, our job is to produce an N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and $x \in A$.

Because we are given that (f_n) converges uniformly, we may let $f(x) = \lim_{n\to\infty} f_n(x)$. By the definition of uniform convergence, there exists an N with the property that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } n \geq N \text{ and } x \in A.$$

Now given $m, n \geq N$, it follows that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $x \in A$. This completes the proof in the forward direction.

($\Leftarrow$) In this direction we assume that, given $\epsilon > 0$, there exists an N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and $x \in A$. Our goal is to prove that (f_n) converges uniformly.

To produce a candidate for the limit, notice that for each $x \in A$ our hypothesis tells us that the sequence $(f_n(x))$ is a Cauchy sequence. Because Cauchy sequences converge, it makes sense to define the limit function

$$f(x) = \lim_{n\to\infty} f_n(x).$$

It is important to realize that, because we are applying the Cauchy Criterion to sequences generated at each point $x \in A$, all we have proved thus far is that $f_n(x) \to f(x)$ pointwise on A.

Let $\epsilon > 0$. Using our hypothesis again (in its full strength this time), we know that there exists an N such that

$$-\epsilon < f_n(x) - f_m(x) < \epsilon \quad \text{for all } m, n \geq N \text{ and } x \in A.$$

The Algebraic Limit Theorem says that for each $x \in A$,

$$\lim_{m\to\infty} (f_n(x) - f_m(x)) = f_n(x) - f(x),$$

and the Order Limit Theorem then implies

$$-\epsilon \leq f_n(x) - f(x) \leq \epsilon \quad \text{for all } n \geq N \text{ and } x \in A.$$

This is sufficient to conclude that $f_n \to f$ uniformly on A.

Exercise 6.2.6.

Exercise 6.2.7. Let $\epsilon > 0$ be arbitrary. We need to show that there exists an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$.

Because f is uniformly continuous on all of $\mathbf{R}$, we can pick δ so that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } |x - y| < \delta.$$

Now choose $N > 1/\delta$. If $n \geq N$ then $|(x + 1/n) - x| < \delta$ and it follows that

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \epsilon,$$

as desired.

This proposition fails if f is not uniformly continuous. Consider $f(x) = x^2$ which is continuous but not uniformly continuous on all of $\mathbf{R}$. In this case we see

$$|f_n(x) - f(x)| = |(x + 1/n)^2 - x^2| = |2x/n + 1/n^2|.$$

Although for each $x \in R$, this expression tends to zero as $n \to \infty$, we see that larger values of x require larger values of n and the convergence is not uniform.

Exercise 6.2.8.

Exercise 6.2.9. Without the limit functions mentioned, it is a little smoother to argue in terms of the Cauchy Criterion. Let $\epsilon > 0$. Choose N_1 so that

$$|f_n - f_m| < \epsilon/2 \quad \text{for all } n, m \geq N_1,$$

and choose N_2 so that

$$|g_n - g_m| < \epsilon/2 \quad \text{for all } m, n \geq N_2.$$

Letting $N = \max\{N_1, N_2\}$ we see that

$$\begin{aligned} |(f_n + g_n) - (f_m + g_m)| &\leq |f_n - f_m| + |g_n - g_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $m, n \geq N$. It follows from the Cauchy Criterion for Uniform Convergence that $(f_n + g_n)$ converges uniformly.

(b) Looking ahead to (c), we see that problems can arise when at least one of the limit functions is unbounded. For example, let $f_n(x) = x + 1/n$ and $g_n(x) = 1/n$. Then $f_n(x) \to x$ uniformly on $\mathbf{R}$ and $g_n(x) \to 0$ uniformly on $\mathbf{R}$. However $f_n(x)g_n(x) = x/n + 1/n^2$. Although $f_n g_n \to 0$ pointwise on $\mathbf{R}$, the convergence is not uniform.

(c) The first step is to write

$$\begin{aligned} |f_n g_n - f_m g_m| &= |f_n g_n - f_n g_m + f_n g_m - f_m g_m| \\ &\leq |f_n||g_n - g_m| + |g_m||f_n - f_m|. \end{aligned}$$

Given $\epsilon > 0$, choose N_1 so that

$$|f_n - f_m| < \frac{\epsilon}{2M} \quad \text{for all } n, m \geq N_1.$$

Also, choose N_2 so that

$$|g_n - g_m| < \frac{\epsilon}{2M} \quad \text{for all } m, n \geq N_2.$$

Letting $N = \max\{N_1, N_2\}$ we see that

$$\begin{aligned} |f_n g_n - f_m g_m| &\leq |f_n||g_n - g_m| + |g_m||f_n - f_m| \\ &< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon \end{aligned}$$

whenever $n \geq N$, as desired.

Exercise 6.2.10.

Exercise 6.2.11. (a) Setting $g_n = f - f_n$ we see that

(i) g_n is continuous for each $n \in N$,

(ii) $g_n(x)$ is decreasing for all $x \in K$, and

(iii) $g_n(x) \to 0$ for all $x \in K$.

(b) We first prove that each K_n is closed. To see this, assume (x_m) is a convergent sequence in K_n. If $x = \lim x_m$, then $x \in K$ (because K is closed) and the fact that g_n is continuous on all of K allows us to write $g_n(x) = \lim_{m\to\infty} g_n(x_m)$. Because $g_n(x_m) \geq \epsilon$ for all m, it follows that the limit $g_n(x)$ also satisfies $g_n(x) \geq \epsilon$. But this implies $x \in K_n$ and we see that K_n contains its limit points and thus is closed.

Each K_n is also bounded because it is a subset of the bounded set K, and it follows that K_n is compact.

The nested property $K_n \supseteq K_{n+1}$ is a direct consequence of our assumption that $g_n(x) \geq g_{n+1}(x)$ for all $x \in K$, and so we are prepared to use Theorem 3.3.5. Assume, for contradiction, that K_n is nonempty for every $n \in \mathbf{N}$. Then Theorem 3.3.5 implies there exists a point x satisfying $x \in K_n$ for every n. But this means $g_n(x) \geq \epsilon$ for every n, contradicting our assumption that $g_n(x) \to 0$. We conclude that there must exist an N for which $K_n = \emptyset$ for all $n \geq N$, and this is equivalent to asserting that

$$|g_n(x)| < \epsilon \quad \text{for all } n \geq N \text{ and } x \in K.$$

We conclude that $g_n \to 0$ uniformly, and thus $f_n \to f$ uniformly as well.

Exercise 6.2.12.

Exercise 6.2.13. (a) The sequence of real numbers $f_n(x_1)$ is bounded by M. The Bolzano–Weierstrass Theorem implies that there is a convergent subsequence.

(b) Focusing on the sequence $f_{1,k}(x_2)$, we again use the Bolzano–Weierstrass Theorem to conclude that there is a convergent subsequence which we write as $f_{2,k}(x_2)$.

(c) Keep in mind that if $m' > m$ then $(f_{m',k})$ is a subsequence of $(f_{m,k})$. The key idea is to let

$$f_{n_k} = f_{k,k} = (f_{1,1}, f_{2,2}, f_{3,3}, \ldots).$$

The nested quality shows that $(f_{k,k})$ is a subsequence of $f_{1,k}$ and thus $f_{k,k}(x_1)$ converges. But what about $f_{k,k}(x_m)$ for an arbitrary $x_m \in A$? Well, after the first m terms, we see that $f_{k,k}$ becomes a proper subsequence of $f_{m,k}$ (i.e., $f_{k,k}$ is eventually in $f_{m,k}$), and it follows that $f_{k,k}(x_m)$ converges. This shows $f_{k,k}$ converges pointwise on A.

Exercise 6.2.14.

Exercise 6.2.15. (a) Because the set of rational numbers in $[0,1]$ is countable, Exercise 6.2.13 gives us exactly what we need to produce the sequence (g_k).

(b) Consider a fixed r_i from our finite set $\{r_1, r_2, \ldots, r_m\}$. Because (g_k) converges pointwise at every rational, the sequence $(g_k(r_i))$ is a Cauchy sequence. Thus we can choose N_i such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3} \quad \text{for all } s, t \geq N_i.$$

Letting $N = \max\{N_1, N_2, \ldots, N_m\}$ produces the desired N.

Note that if the set $\{r_1, r_2, \ldots, r_m\}$ were infinite then N would be the maximum of an infinite set which is problematic to say the least.

(c) Given $x \in [0,1]$, we know there exists a rational r_i from our designated set satisfying $|r_i - x| < \delta$. It follows that

$$|g_s(x) - g_s(r_i)| < \frac{\epsilon}{3} \quad \text{for all } s \in \mathbf{N}.$$

Using this fact (twice) and the result in (b) we see that $s, t \geq N$ implies

$$\begin{aligned} |g_s(x) - g_t(x)| &= |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)| \\ &\leq |g_s(x) - g_s(r_i)| + |g_s(r_i) - g_t(r_i)| + |g_t(r_i) - g_t(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

It follows that (g_k) converges uniformly using the Cauchy criterion in Theorem 6.2.5.

6.3 Uniform Convergence and Differentiation

Exercise 6.3.1. (a) First we deduce that $g = \lim g_n = 0$, and the convergence is uniform on $[0,1]$. To prove this, we must find an N such that $n \geq N$ implies $|x^n/n - 0| < \epsilon$. But notice that

$$|\frac{x^n}{n} - 0| \leq \frac{1}{n} \quad \text{for all } x \in [0,1].$$

Given $\epsilon > 0$, pick $N > 1/\epsilon$. Then $n \geq N$ implies $|g_n| < \epsilon$ for all $x \in [0,1]$, as desired.

Because $g(x) = 0$ for all $x \in [0,1]$ it is differentiable and, furthermore, $g'(0) = 0$.

(b) Writing

$$g_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1},$$

we see that the sequence (g_n') converges pointwise on $[0,1]$, to

$$h(x) = \lim_{n \to \infty} g_n'(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

The convergence is not uniform over $[0,1]$, and in fact it is not uniform over any set that contains 1 as a limit point. Comparing $h = \lim g_n'$ to g' is illuminating. Note in particular that $h(1) \neq g(1)$, so that it is possible for the sequence of derivatives to converge to the "wrong" value when the convergence of g_n' is not uniform. On the other hand, the convergence of g_n' is uniform on sets of the form $[0,c]$ where $c < 1$, and this is reflected by the fact that $h(x) = g(x)$ on $[0,1)$.

Exercise 6.3.2.

Exercise 6.3.3. (a) Taking the derivative we find

$$f_n'(x) = \frac{1 - x^2 n}{(x^2 n + 1)^2}$$

which yields critical points $\pm 1/\sqrt{n}$. Using the standard techniques from calculus we can determine that the maximum of f occurs at $1/\sqrt{n}$ and the minimum at $-1/\sqrt{n}$. Because $f_n(1/\sqrt{n}) = |f_n(-1/\sqrt{n})| = 1/(2\sqrt{n})$, we see that

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \qquad \text{for all } x \in \mathbf{R}.$$

To show $(f_n) \to 0$ uniformly on $\mathbf{R}$, we let $\epsilon > 0$ and choose N large enough so that $n \geq N$ implies $1/(2\sqrt{n}) < \epsilon$. It follows that $|f_n(x) - 0| < \epsilon$ whenever $n \geq N$, as desired.

(b) Because $f_n \to 0$ uniformly, the limit $f = \lim f_n$ satisfies $f'(x) = 0$ for all values of x.

Taking the derivative of each f_n we get

$$f_n'(x) = \frac{1 - nx^2}{1 + 2nx^2 + n^2 x^4}.$$

If $x \neq 0$ then we can show $\lim f_n'(x) = 0 = f'(x)$. However, for $x = 0$ we get $f_n'(0) = 1$ for all n and thus $f'(0) \neq \lim f_n'(0)$.

Exercise 6.3.4.

Exercise 6.3.5. (a) We have

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{2} + \frac{x^2}{2n} = \frac{x}{2},$$

so $g'(x) = 1/2$.

(b) This time we compute $g_n'(x)$ first to get

$$g_n'(x) = \frac{1}{2} + \frac{x}{n},$$

and note that the pointwise limit of this sequence is $1/2$. For $x \in [-M, M]$ we can write

$$|g_n'(x) - 1/2| = \left|\frac{x}{n}\right| \leq \frac{M}{n}.$$

Given $\epsilon > 0$, choose $N > M/\epsilon$, independent of x. Then $n \geq N$ implies $|g_n'(x) - 1/2| < \epsilon$, and we conclude that $g_n' \to 1/2$ uniformly on $[-M, M]$. It follows from Theorem 6.3.3 that $g'(x) = 1/2$.

(c) Taking the pointwise limit of $f_n(x)$ gives

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + 1/n}{2 + x/n} = \frac{x^2}{2}.$$

Thus, $f'(x) = x$.

Computing the derivative sequence first we get

$$f_n'(x) = \frac{4n^2x + 3nx^2 + 1}{4n^2 + 4nx + x^2},$$

so that

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{4x + 3x^2/n + 1/n^2}{4 + 4x/n + x^2/n^2} = x.$$

Arguing for uniform convergence on intervals of the form $[-M, M]$ is less elegant for this example but no harder really. For values of x satisfying $|x| < M$ we have

$$|f_n'(x) - x| = \left| \frac{-nx^2 - x^3 + 1}{4n^2 + 4nx + x^2} \right| \leq \frac{nM^2 + M^3 + 1}{4n^2 - 4nM},$$

as long as $n > M$. Because this estimate does not depend on x and tends to zero as $n \to \infty$, it follows that $f_n'(x) \to x$ uniformly on $[-M, M]$.

Exercise 6.3.6.

Exercise 6.3.7. Let $x \in [a, b]$ and assume, without loss of generality, that $x > x_0$. Applying the Mean Value Theorem to the function $f_n - f_m$ on the interval $[x_0, x]$, we get that there exists a point α such that

$$(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) = (f_n'(\alpha) - f_m'(\alpha))(b - a).$$

Let $\epsilon > 0$. Because (f_n') converges uniformly, the Cauchy Criterion asserts that there exists an N_1 such that

$$|f_n'(c) - f_m'(c)| < \frac{\epsilon}{2(b-a)} \quad \text{for all } n, m \geq N \text{ and } c \in [a, b].$$

Our hypothesis states that $(f_n(x_0))$ converges so there exists an N_2 such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \quad \text{for all } n, m \geq N_2.$$

Finally, let $N = \max\{N_1, N_2\}$. Then if $n, m \geq N$ it follows that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\
&= |(f_n'(\alpha) - f_m'(\alpha))(b - a)| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\epsilon}{2(b-a)}(b - a) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Because our choice of N is independent of the point x, the Cauchy Criterion implies that the sequence (f_n) converges uniformly on $[a, b]$.

6.4 Series of Functions

Exercise 6.4.1. The key idea is to use the Cauchy criterion for convergence of a series of real numbers given in Theorem 2.7.2. Let $\epsilon > 0$ be arbitrary. Because $\sum_{n=1}^{\infty} M_n$ converges, there exists an N such that $n > m \geq N$ implies

$$M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon.$$

Because

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + M_{n+2} + \cdots + M_n,$$

we can appeal to the Cauchy criterion for uniform convergence of series (Theorem 6.4.4) to conclude that $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Exercise 6.4.2.

Exercise 6.4.3. (a) Because

$$\left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n},$$

and we know $\sum_{n=1}^{\infty} 1/2^n$ converges, it follows by the Weierstrass M–Test that $\sum_{n=1}^{\infty} \cos(2^n x)/2^n$ converges uniformly. Because each of the summands is continuous, the limit is as well according to Theorem 6.4.2.

(b) Taking the term-by-term derivative of the series for g yields the series

$$\sum_{n=0}^{\infty} -\sin(2^n x).$$

Without the compression factor of 2^n there is no way to use the M-Test on the differentiated series. In fact, it is clear that, pointwise, the terms don't go to zero except in a few rare exceptions (e.g., $x = 0$, multiples of π, $\pi/2$, $\pi/4$, etc..) Without uniform convergence of the differentiated series we certainly can't apply the Term-by-term Differentiability Theorem. To be clear, this is not a proof that g is nowhere differentiable—that turns out to be a more difficult and delicate argument.

Exercise 6.4.4.

Exercise 6.4.5. (a) This is straightforward application of the M-Test. For $x \in [-1, 1]$, we have $|x^n/n^2| \leq 1/n^2$. Since $\sum 1/n^2$ converges, the series converges uniformly. Because each term in the series is a continuous function, the Term-by-term Continuity Theorem implies that h is continuous as well.

(b) First fix $x_0 \in (-1, 1)$. Now choose c to satisfy $|x_0| < c < 1$ and apply the M–Test on $[-c, c]$. Over this interval we get the estimate $|x^n/n| \leq c^n/n$. Because $\sum_{n=1}^{\infty} c^n/n$ converges, the M-Test implies the convergence is uniform and thus f is continuous at $x_0 \in [-c, c]$.

Exercise 6.4.6.

Exercise 6.4.7. (a) The series for f certainly converges uniformly, but Theorem 6.4.3 requires us to look at the differentiated series

$$\sum_{n=1}^{\infty} \frac{\cos(kx)}{k^2}. \tag{1}$$

For this series we can make the estimate

$$\left| \frac{\cos(kx)}{k^2} \right| \leq \frac{1}{k^2}.$$

Because $\sum_{n=1}^{\infty} 1/k^2$ converges, the M–Test asserts that the series above in (1) converges uniformly. Now Theorem 6.4.3 asserts that f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

Finally, we note that the uniform convergence also implies (via Theorem 6.4.2) that $f'(x)$ is continuous because each of the summands is.

(b) To use Theorem 6.4.3 to determine whether $f'(x)$ is differentiable requires that we differentiate the series for f' term-by-term and consider

$$\sum_{n=1}^{\infty} \frac{-\sin(kx)}{k}.$$

Unfortunately, the Weierstrass M–Test cannot be used here because $\sum_{n=1}^{\infty} 1/k$ diverges.

The differentiability of f' turns out to be a very deep question that has been studied in depth by Riemann and Hardy, among others.

Exercise 6.4.8.

Exercise 6.4.9. (a) First observe that the summands are continuous functions and satisfy

$$\left| \frac{1}{x^2+n^2} \right| \leq \frac{1}{n^2} \qquad \text{for all } x \in \mathbf{R}.$$

Because $\sum_{n=1}^{\infty} 1/n^2$ converges, the M–Test implies the convergence is uniform and hence $h(x)$ is continuous on $\mathbf{R}$.

(b) To determine if h is differentiable we consider the differentiated series

$$\sum_{n=1}^{\infty} \frac{-2x}{(x^2+n^2)^2}.$$

Restricting our attention to an interval $[-M, M]$, we get the estimate

$$\left| \frac{-2x}{x^2+n^2} \right| \leq \frac{2M}{n^2},$$

and, as before, we note $\sum_{n=1}^{\infty} 2M/n^2$ converges. This proves that the differentiated series converges uniformly to $h'(x)$ and that h' is continuous on $[-M, M]$. Because the interval $[-M, M]$ is arbitrary in this argument, we conclude that h' exists and is continuous on all of $\mathbf{R}$.

Exercise 6.4.10.

6.5 Power Series

Exercise 6.5.1. (a) The series for g converges for all $x \in (-1, 1]$ and is continuous over this interval. The series does not converge when $x = -1$ and Theorem 6.5.1 implies that it then does not converge for any values of x satisfying $|x| > 1$.

(b) Theorem 6.5.6 implies g is differentiable on $(-1, 1)$ with

$$g'(x) = 1 - x + x^2 - x^3 + \cdots .$$

Notice that our formula for $g'(x)$ no longer converges when $x = 1$, but with some cleverness we can actually argue that $g(x)$ is in fact differentiable at $x = 1$. Using results about geometric series, we see that $g'(x) = 1/(1 + x)$, and this version of the formula extends to the value $x = 1$. We can then use an argument like the one in Exercise 5.3.8 with 1 in place of 0 to show that g is differentiable at $x = 1$.

Exercise 6.5.2.

Exercise 6.5.3. Set $M_n = |a_n x_0^n|$ and note that absolute convergence at x_0 implies

$$\sum_{n=0}^{\infty} |a_n x_0^n| = \sum_{n=0}^{\infty} M_n$$

converges. If $x \in [-c, c]$ then we get the estimate

$$|a_n x^n| \le |a_n x_0^n| = M_n,$$

and the Weierstrass M–Test implies that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$.

Exercise 6.5.4.

Exercise 6.5.5. (a) Let's set $y_n = ns^{n-1}$. We want to argue that (y_n) is eventually decreasing, and in fact converges to 0. A good way to see this is to look at the ratio $y_{n+1}/y_n = s(n+1)/n$, and note that it is eventually less than 1. In particular, if $n > s/(1-s)$ then $y_{n+1}/y_n < 1$ at which point the sequence starts decreasing. Since all we really need to know is that (y_n) bounded, we can stop here.

(b) Let $x \in (-R, R)$ be arbitrary and pick t to satisfy $|x| < t < R$. We will show that $\sum |na_n x^{n-1}|$ converges, implying $\sum na_n x^{n-1}$ converges. First write

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n|.$$

Because $|x/t| < 1$, by part (a) we can pick a bound L satisfying

$$n\left|\frac{x}{t}\right|^{n-1} \leq L \quad \text{for all } n \in \mathbf{N}.$$

Now we have

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t}\left(n\left|\frac{x}{t}\right|^{n-1}\right)|a_n t^n| \leq \frac{L}{t}\sum_{n=1}^{\infty}|a_n t^n|$$

where the last sum converges because $t \in (-R, R)$. Therefore, $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges absolutely and thus converges.

Exercise 6.5.6.

Exercise 6.5.7. (a) For a fixed x, apply the Ratio Test to the series $\sum a_n x^n$ to get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x| = L|x|.$$

If $|x| < 1/L$ then $L|x| < 1$ and the series converges.

(b) If $L = 0$ then $L|x| = 0 < 1$ for every value of x and the Ratio Test implies that the series converges on all of $\mathbf{R}$.

(c) This will follow using the same proofs if we can prove the following modified version of the Ratio Test:

Given a sequence (b_n), let

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{\left|\frac{b_{k+1}}{b_k}\right| : k \geq n\right\}.$$

If $L' < 1$ then $\sum b_n$ converges.

The proof is very similar to the proof of the Ratio Test in Exercise 2.7.9. First choose R to satisfy $L' < R < 1$. The sequence (s_n) is decreasing, and because it converges to L' we know there exists an N such that

$$\left|\frac{b_{k+1}}{b_k}\right| \leq R \quad \text{for all } k \geq N.$$

An induction proof like the one before shows

$$|b_k| \leq |b_N| R^{k-N} \quad \text{for all } k \geq N,$$

and then we may compare the series $\sum b_k$ to the convergent geometric series $|a_N| \sum R^k$ to conclude that $\sum b_k$ converges.

Exercise 6.5.8.

Exercise 6.5.9. We are assuming $\sum a_n$, $\sum b_n$ and $\sum d_n$ each converge which, according to Abel's Theorem, tells us that the respective series for f, g, and h converge uniformly on $[0, 1]$. Among other things, this implies that f, g and h are all continuous functions over the closed interval $[0, 1]$.

Fix $x \in [0, 1)$. Because we know we have convergence at 1, Theorem 6.5.1 implies that $\sum a_n x^n$, $\sum b_n x^n$ and $\sum d_n x^n$ each converge absolutely. This fact means that we can invoke the result in Exercise 2.8.7 to assert that

$$h(x) = \sum d_n x^n = f(x)g(x).$$

Because this is true for all $x \in [0, 1)$, and because f, g and h are continuous on the closed interval $[0, 1]$, it follows that $h(1) = f(1)g(1)$ or

$$\sum d_n = \left(\sum a_n\right)\left(\sum b_n\right),$$

as desired.

Exercise 6.5.10.

Exercise 6.5.11. (a) Assume $\sum a_n$ converges to L. If we set $f(x) = \sum a_n x^n$, then Abel's Theorem implies that the series for f converges uniformly on the interval $[0, 1]$. Because the summands are continuous polynomials, this proves that f is continuous on $[0, 1]$. In particular, this implies $\lim_{x\to 1^-} f(x) = f(1)$. But notice that $f(1) = L$ and thus we have shown that $\sum a_n$ is Abel-summable to L.

(b) Using some familiar facts about geometric series, observe that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n x^n &= 1 - x + x^2 - x^3 + x^4 - \\ &= \frac{1}{1-(-x)} \\ &= \frac{1}{1+x},\end{aligned}$$

provided $|x| < 1$. Then $\lim_{x\to 1^-} 1/(1 + x) = 1/2$ shows that our series is Abel-summable to $1/2$.

6.6 Taylor Series

Exercise 6.6.1. Because the series converges when $x = 1$, Abel's Theorem implies that we get uniform convergence over the interval $[0, 1]$ and thus the series represents a continuous function over the interval $[0, 1]$. Assuming that $\arctan(x)$ is continuous over $[0, 1]$, it follows that if these two continuous functions agree for all values of $x \in [0, 1)$, then they must also agree when $x = 1$. A similar argument can be made at $x = -1$.

Setting $x = 1$ into this formula gives "Leibniz's formula,"

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Exercise 6.6.2.

Exercise 6.6.3. The key idea is to take the derivative of each side of equation (2) using a term-by-term approach for the series on the right (this is justified by Theorem 6.5.7). Setting $x = 0$ after n derivatives gives the formula for a_n.

Exercise 6.6.4.

Exercise 6.6.5. (a) Because $f^{(n)}(x) = e^x$ for every n, we get $a_n = e^0/n! = 1/n!$ which yields

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

To show that the series converges uniformly to e^x on any interval of the form $[-R, R]$ use Lagrange's Remainder formula to write

$$|E_N(x)| = \left| \frac{e^c}{(N+1)!} x^{N+1} \right| \leq \frac{e^R}{(N+1)!} R^{N+1}$$

for all $x \in [-R, R]$. Now, just as in the previous exercise, this error bound tends to zero as $N \to \infty$. Because this bound is independent of x, it follows that $E_N(x) \to 0$ uniformly on $[-R, R]$ and we get that $S_N(x) \to e^x$ uniformly on $[-R, R]$ as well.

(b) To verify the formula $f'(x) = e^x$ we differentiate the series representation

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

term-by-term to get

$$\begin{aligned} (e^x)' &= 0 + 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + 4\frac{x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= e^x. \end{aligned}$$

(c) Starting from the formula $e^x = \sum_{n=0}^{\infty} x^n/n!$ we get the formula

$$e^{-x} = \sum_{n=0}^{\infty} (-x)^n/n! = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots .$$

Reviewing the material on Cauchy products from the end of Section 2.8 we now write

$$\begin{aligned} (e^x)(e^{-x}) &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots) \\ &= 1 + (-1+1)x + (\frac{1}{2!} - 1 + \frac{1}{2!})x^2 + (\frac{-1}{3!} + \frac{1}{2!} - \frac{1}{2!} + \frac{1}{3!})x^3 + \cdots \\ &= 1 + 0x + 0x^2 + 0x^3 + \cdots \\ &= 1 \end{aligned}$$

The key to the above calculation is to use the binomial formula to show that the coefficient for x^n is

$$\sum_{k=0}^{n} \frac{1}{k!}\frac{(-1)^{n-k}}{(n-k)!} = \frac{1}{n!}(-1+1)^n = 0 \qquad \text{for all } n \geq 1.$$

The point of this exercise is to illustrate that if we take the power series representation for e^x to be the *definition* of the exponential function, then familiar statements such as $(e^x)' = e^x$ and $e^{-x} = 1/e^x$ follow naturally from the definition.

Exercise 6.6.6.

Exercise 6.6.7. (a) One example would be $g(x) = \frac{1}{1+x^2}$. Notice that g is defined and differentiable on all of $\mathbf{R}$, but the Taylor series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ "stops working" outside of the interval $(-1, 1)$. (The reason for this is revealed when we look at g as a function on the complex plane. In this setting, the function is undefined at $\pm i$, which is a distance of 1 to the origin.)

(b) Let $h(x) = \sin(x) + g(x)$, where g is the function in the "Counterexample" discussion at the end of this section.

(c) This time let $f(x) = 0$ for all $x \leq 0$, and let $f(x) = e^{-1/x^2}$ for all $x > 0$. This infinitely differentiable function has Taylor coefficients all equal to zero, and so the series only converges to f if $x \leq 0$.

Exercise 6.6.8.

Exercise 6.6.9. (a) In this case,

$$E_N(x, x) = f(x) - S_N(x, x) = f(x) - f(x) = 0.$$

The error function is always zero at the point where the series is centered.

(b) Write

$$S_N(x, a) = f(a) + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and note that $S_N(x, a)$ is an algebraic combination of differentiable functions, and thus differentiable. Then $E_N(x, a)$ is differentiable as well, and

$$\begin{aligned}
E_N'(x, a) &= (f(x) - S_N(x, a))' \\
&= -S_N'(x, a) \\
&= -f'(a) - \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!} n(x-a)^{n-1}(-1) + \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= -f'(a) + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} - \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= -f'(a) + \left(f'(a) - \frac{f^{(N+1)}(a)}{N!}(x-a)^N \right) \\
&= -\frac{f^{(N+1)}(a)}{N!}(x-a)^N.
\end{aligned}$$

(c) Applying MVT to $E_N(x,a)$ on the interval $[0,x]$ (or $[x,0]$), we find

$$\frac{E_N(x,x) - E_N(x,0)}{x-0} = E_N'(x,c),$$

for some $c \in (0,x)$, and the result follows immediately from the formula in (b).

Exercise 6.6.10.

6.7 The Weierstrass Approximation Theorem

Exercise 6.7.1. Apply WAT repeatedly with $\epsilon_n = 1/n$ to get a sequence of polynomials (p_n) satisfying

$$|p_n(x) - f(x)| < \frac{1}{n}.$$

Then $p_n \to f$ uniformly.

Exercise 6.7.2.

Exercise 6.7.3. (a) Setting $p(0) = 0, p(-1) = 1, p(1) = 1$ and solving for the coefficients yields the polynomial $p(x) = x^2$.

(b) This time we have five equations and five unknowns. But $g(0) = 0$ implies the constant term is zero, and we know ahead of time that the polynomial is even. So we really just need to find $h(x) = ax^2 + bx^4$ such that $h(1/2) = 1/2$ and $h(1) = 1$. The result is $h(x) = (7/3)x^2 - (4/3)x^4$.

Exercise 6.7.4.

Exercise 6.7.5. (a) See the solution for Exercise 6.6.9.

(b) Fix $x \in (-1,1)$. By the Cauchy formula in (a) we have

$$\begin{aligned}
E_N(x) &= \frac{f^{(N+1)}(c)}{N!}(x-c)^N x \\
&= \frac{1}{N!}\left(\frac{-1 \cdot 3 \cdot 5 \cdots (2N-1)}{2^{N+1}(1-c)^{N+1/2}}\right)(x-c)^N x \\
&= \left(\left(\frac{1}{2}\right)\frac{-1 \cdot 3 \cdot 5 \cdots (2N-1)}{2 \cdot 4 \cdot 6 \cdots (2N)}\right)\left(\frac{x-c}{1-c}\right)^N \frac{x}{\sqrt{1-c}}.
\end{aligned}$$

The key observation is that $\left|\frac{x-c}{1-c}\right|$ is largest when $c = 0$, so we know that

$$\left|\frac{x-c}{1-c}\right| \leq |x|$$

which leads to the estimate

$$|E_N(x)| \leq \frac{1}{2}|x|^N \frac{1}{\sqrt{1-|x|}}.$$

Letting $N \to \infty$ it follows that $E_N(x) \to 0$.

Exercise 6.7.6.

Exercise 6.7.7. (a) The idea is to exploit the fact that

$$|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)}.$$

Let $\epsilon > 0$. By our previous work we know there is a partial sum of the Taylor series $S_N(x) = \sum_{n=0}^{N} a_n x^n$ satisfying

$$|\sqrt{1-y} - S_N(y)| < \epsilon$$

for all $y \in [-1, 1]$. Now let $y = 1 - x^2$ and observe that $x \in [-1, 1]$ implies $y \in [0, 1]$. Substiting then gives

$$||x| - S_N(1 - x^2)| < \epsilon$$

for all $x \in [-1, 1]$. Because $S_N(1-x^2)$ is a polynomial, this completes the proof.

(b) Let $l(x) = mx + B$ be the linear function that maps the interval $[-1, 1]$ to $[a, b]$. (So $m = (b - a)/2$ and $B = (b + a)/2$.) Given a continuous function f on $[a, b]$ and an $\epsilon > 0$, the composition $f(l(x))$ is continuous on $[-1, 1]$, so there exists a polynomial $p(x)$ satisfying

$$|f(l(x)) - p(x)| < \epsilon$$

for all $x \in [-1, 1]$. Setting $y = l(x)$, this same inequality can be restated as

$$|f(y) - p(l^{-1}(y)| < \epsilon$$

for all $y \in [a, b]$. The function $l^{-1}(y)$ is just the linear function that maps $[a, b]$ onto $[-1, 1]$, and so the composition $p(l^{-1}(y))$ is still a polynomial.

Exercise 6.7.8.

Exercise 6.7.9. (a) Consider $f(x) = 1/x$ on the open interval $(0, 1)$. Because f is unbounded, there is no way for a (most definitely bounded) polynomial to uniformly approximate f over this domain.

(b) Again, WAT fails. The growth rate of $g(x) = e^x$ outpaces any polynomial that might try to uniformly approximate it. A function like $\sin(x)$ poses the opposite problem in that it stays bounded as $x \to \infty$, which no polynomial can do.

Exercise 6.7.10.

Exercise 6.7.11. Focus first on the continuous derivative function f', and let q be a polynomial that uniformly approximates f' better than $\min\{\epsilon, \epsilon'\}$, where $\epsilon' = \epsilon/(b - a)$.

Now let $p(x)$ be the polynomial satisfying $p'(x) = q(x)$ and $p(a) = f(a)$. Clearly $|f'(x) - p'(x)| < \epsilon$, so we just need to prove that $|f(x) - p(x)| < \epsilon$.

To this end, let $g(x) = f(x) - p(x)$, and note that $g(a) = 0$ and $|g'(x)| < \epsilon/(b - a)$. For any $x \in (a, b]$ we can apply MVT to g over the interval $[a, x]$ to conclude

$$|g(x) - g(a)| < \frac{\epsilon}{b - a}(x - a).$$

It follows that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$, as desired.

Chapter 7

The Riemann Integral

7.1 Discussion: How Should Integration be Defined?

7.2 The Definition of the Riemann Integral

Exercise 7.2.1. Momentarily fix the partition P. Then Lemma 7.2.4 implies

$$L(f, P) \leq U(f, P') \quad \text{for all partitions } P'.$$

Because $L(f, P)$ is a lower bound for the set of upper sums, it must be less than the greatest lower bound for this set; i.e., $L(f, P) \leq U(f)$. But P is arbitrary in this discussion meaning that $U(f)$ is an upper bound on the set of lower sums. From the definition of the supremum we get $L(f) \leq U(f)$ as desired.

Exercise 7.2.2.

Exercise 7.2.3. (a) ($\Rightarrow$) Assume there exists a sequence of partitions (P_n) satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Given $\epsilon > 0$, choose P_N from this sequence so that $U(f, P_N) - L(f, P_N) < \epsilon$. Then Theorem 7.2.8 implies f is integrable.

($\Leftarrow$) Conversely, if f is integrable then given $\epsilon_n = 1/n$, Theorem 7.2.8 implies that there exists a partition P_n satisfying $U(f, P_n) - L(f, P_n) < 1/n$. It follows that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0,$$

as desired.

Finally, observe that if we have such a sequence of partitions for an integrable function f, then for every $n \in \mathbf{N}$,

$$L(f, P_n) \leq \int_a^b f \leq U(f, P_n).$$

Equivalently,

$$0 \leq \int_a^b f - L(f, P_n) \leq U(f, P_n) - L(f, P_n),$$

and a Squeeze Theorem argument shows

$$\int_a^b f = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

(b) For the partition P_n we have $x_k = k/n$, $m_k = (k-1)/n$ and $M_k = k/n$. Then

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k}{n}(1/n) = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right),$$

and

$$L(f, P_n) = \sum_{k=1}^{n} \frac{(k-1)}{n}(1/n) = \frac{1}{n^2}\sum_{k=1}^{n}(k-1) = \frac{1}{n^2}\left(\frac{(n-1)n}{2}\right).$$

(c) Now we may compute

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty}\left[\frac{n(n+1)}{2n^2} - \frac{(n-1)n}{2n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\right] = 0.$$

The result in (a) now implies that $f(x) = x$ is integrable, and

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{(n-1)n}{2}\right) = 1/2.$$

Exercise 7.2.4.

Exercise 7.2.5. We shall use the criterion in Theorem 7.2.8. The shape of the proof is determined by the triangle inequality estimate

$$\begin{aligned} U(f,P) - L(f,P) &= U(f,P) - U(f_N,P) + U(f_N,P) - L(f_N,P) \\ &\qquad + L(f_N,P) - L(f,P) \\ &\leq |U(f,P) - U(f_N,P)| + (U(f_N,P) - L(f_N,P)) \\ &\qquad + |L(f_N,P) - L(f,P)|. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Because $f_n \to f$ uniformly, we can choose N so that

$$|f_N(x) - f(x)| \leq \frac{\epsilon}{3(b-a)} \quad \text{for all } x \in [a,b].$$

Now the function f_N is integrable and so there exists a partition P for which

$$U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}.$$

Let's consider a particular subinterval $[x_{k-1}, x_k]$ from this partition. If

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad N_k = \sup\{f_N(x) : x \in [x_{k-1}, x_k]\},$$

then our choice of f_N guarantees that

$$|M_k - N_k| \leq \frac{\epsilon}{3(b-a)}.$$

From this estimate we can argue that

$$\begin{aligned} |U(f,P) - U(f_N,P)| &= \left|\sum_{k=1}^{n}(M_k - N_k)\Delta x_k\right| \\ &\leq \sum_{k=1}^{n}\frac{\epsilon}{3(b-a)}\Delta x_k = \frac{\epsilon}{3}. \end{aligned}$$

Similarly we can show

$$|L(f_N,P) - L(f,P)| \leq \frac{\epsilon}{3}.$$

Putting this altogether, we see that using our choices of f_N and P in the preliminary estimate gives

$$U(f,P) - L(f,P) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

By the criterion in Theorem 7.2.8 we conclude that the uniform limit of integrable functions is integrable.

Exercise 7.2.6.

Exercise 7.2.7. As in the previous exercise, we shall use the criterion in Theorem 7.2.8. Let P be a partition where all the subintervals have equal length $\Delta x = x_k - x_{k-1}$. Because the function is increasing, on each subinterval $[x_{k-1}, x_k]$ we have

$$M_k = f(x_k) \quad \text{and} \quad m_k = f(x_{k-1}).$$

Thus,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n}(M_k - m_k)\Delta x \\ &= \Delta x\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) \\ &= \Delta x(b-a). \end{aligned}$$

Given $\epsilon > 0$, choose a partition P_ϵ to have equal subintervals with common length satisfying $\Delta x < \epsilon/(b-a)$. The previous calculation then shows

$$U(f,P_\epsilon) - L(f,P_\epsilon) = \Delta x(b-a) < \frac{\epsilon}{b-a}(b-a) = \epsilon.$$

7.3 Integrating Functions with Discontinuities

Exercise 7.3.1. (a) Let P be an arbitrary partition of $[0,1]$. On any subinterval $[x_{k-1}, x_k]$, it must be that $m_k = \inf\{h(x) : x \in [x_{k-1}, x_k]\} = 1$, and it follows that

$$L(h,P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1.$$

(b) Consider the partition $P = \{0, .95, 1\}$. Then

$$U(h,P) = (1)(.95) + (2)(.05) = 1.05.$$

(c) Consider the partition $P_\epsilon = \{0, 1-\epsilon/2, 1\}$. Then

$$U(h,P_\epsilon) = (1)\left(1 - \frac{\epsilon}{2}\right) + (2)\left(\frac{\epsilon}{2}\right) = 1 + \frac{\epsilon}{2}.$$

The implication is that for this partition we have $U(h,P_\epsilon) - L(h,P_\epsilon) < \epsilon$, proving that h is integrable.

Exercise 7.3.2.

Exercise 7.3.3. Every interval contains points where $f(x) = 0$, and thus it follows that $L(f,P) = 0$ for every partition P. This implies that $L(f) = 0$. It remains to show that $U(f) = 0$.

Let $\epsilon > 0$ be arbitrary and consider the *finite* set $\{1, 1/2, 1/3, \ldots, 1/N\}$ consisting of points of the form $1/n$ that satisfy $1/n \geq \epsilon/2$. Because this set is finite, we may construct a set of disjoint intervals around each of these points with the property that the sum of the lengths of these intervals comes to less than $\epsilon/2$. Letting P be the partition that results from taking the union of these intervals together with the interval $[0, \epsilon/2]$, it follows that

$$U(f,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and f integrates to zero.

Exercise 7.3.4.

Exercise 7.3.5. (a) First write

$$\mathbf{Q} \cap [0,1] = \{r_1, r_2, r_3, \ldots\},$$

which is allowed because $\mathbf{Q} \cap [0,1]$ is a countable set. For each $n \in \mathbf{N}$ define

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Because g_n has only a finite number of discontinuities we know it is integrable, and $g_n \to g$ pointwise is easy to verify. This example is discussed explicitly in the Epilogue to Chapter 7.

(b) This is impossible. Because each g_n has only a finite number of discontinuities, it must be integrable. And the uniform limit of integrable functions is integrable, as was proved in Exercise 7.2.5.

(c) Let g be the Dirichlet function, and set $h_n(x) = g(x)/n$. Then h_n converges uniformly to zero, which is very integrable.

Exercise 7.3.6.

Exercise 7.3.7. (a) Assume f is integrable so that $U(f) = L(f) = \int_a^b f$. Now let g differ from f at the single point x_0. Set $D = |f(x_0) - g(x_0)|$. We want to prove that $U(g) = U(f)$ and $L(g) = L(f)$.

Let $\epsilon > 0$ be arbitrary. To argue that $U(g) = U(f)$, it is sufficient to find a partition for which $U(g,P) < U(f) + \epsilon$. Because $U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$, we know there exists a partition P where

$$U(f,P) < U(f) + \epsilon/2.$$

The first step is let P' be a refinement of P with the property that the interval(s) containing x_0 have width less than $\epsilon/(4D)$. Because $P \subseteq P'$ we know $U(f,P') \leq U(f,P)$. Now observe that because f and g agree everywhere except at x_0 it follows that

$$|U(f,P') - U(g,P')| < D(2\Delta x) < \frac{\epsilon}{2}.$$

(The extra 2 is needed in case the point x_0 is an endpoint of an interval in P' and is thus contained in two subintervals.) Finally, we see that

$$U(g,P') < U(f,P') + \frac{\epsilon}{2} \leq \left(U(f) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = U(f) + \epsilon,$$

and we conclude that $U(g) = U(f)$.. The proof that $L(g) = L(f)$ is similar.

The general case for a finite number of points where g and f differ follows using an induction argument.

(b) Dirichlet's function differs from the zero function in only a countable number of points but is not integrable.

Exercise 7.3.8.

Exercise 7.3.9. (a) This proof follows the typical model where we build a partition P in two steps: first handling the "bad" or discontinuous points, and then handling the "good" or continuous parts of the interval $[a,b]$.

Assume f is bounded by M and let $\epsilon > 0$. Because our set of discontinuities has content zero, we may let $\{O_1, \ldots, O_N\}$ be a collection of open intervals that covers the set of discontinuous points and satisfies

$$\sum_{n=1}^{N} |O_n| \leq \frac{\epsilon}{4M}.$$

Focusing on just these subintervals we see that $|O_n| = \Delta x_n$ and

$$\sum_{\text{bad pts}} (M_n - m_n)\Delta x_n \leq \sum_{\text{bad pts}} 2M\Delta x_n = 2M\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

If $O = \bigcup_{n=1}^{N} O_n$ then $[a,b]\backslash O$ is a compact set. Because f is continuous on this set, it is uniformly continuous and so there exists a $\delta > 0$ with the property that

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \qquad \text{whenever } |x-y| < \delta.$$

Focusing on the intervals that make up $[a,b]\backslash O$ (the "good points"), we partition these so that all the resulting subintervals have length less than δ. This puts us into a situation like the one in Theorem 7.2.9. In particular we get that

$$\sum_{\text{good pts}} (M_k - m_k)\Delta x_k < \frac{\epsilon}{2(b-a)} \sum_{\text{good pts}} \Delta x_k < \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2}.$$

Putting these two parts together we see

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{\text{bad pts}} (M_k - m_k)\Delta x_k + \sum_{\text{good pts}} (M_k - m_k)\Delta x_k \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and by the criterion in Theorem 7.2.8, f is integrable.

(b) Given a finite set $\{z_1, z_2, \ldots, z_N\}$ and $\epsilon > 0$, let $O_n = V_{\epsilon'}(z_n)$ where $\epsilon' = \epsilon/(2N)$. Then $|O_n| = \epsilon/N$ and the sum of these lengths is equal to ϵ, as desired.

(c) Recall that we defined the Cantor set C as the intersection

$$C = \bigcap_{n=0}^{\infty} C_n,$$

where C_n consists of 2^n closed intervals of length $1/3^n$. Given $\epsilon > 0$, choose m so that $2^m(1/3^m) < \epsilon/2$. Now it would be nice if we could just use the intervals that make up C_m as our covering set. However, the definition of content zero requires that we use *open* intervals. To fix this, we can imbed each of the 2^m closed intervals that make up C_m in a slightly larger open interval whose length is equal to $1/3^m + (\epsilon/2)2^{-m}$. This collection of open intervals will then contain C (because C_m does) and the lengths will sum to

$$2^m[1/3^m + \frac{\epsilon}{2}2^{-m}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

(d) The fact that h is integrable will follow immediately from (a) and (c), provided we can show that the set of discontinuities of h is precisely the Cantor set. To this end, first consider a point $c \in C$. Because the Cantor set contains no intervals, we know there exists a sequence (y_n) converging to c consisting of points from the complement of C. Thus, $\lim_{n\to\infty} h(y_n) = 0 \neq h(c)$, and it follows that h is not continuous at c.

Now fix a point $z \notin C$. The complement of the Cantor set is open, so we know there exists a neighborhood $V_\delta(z)$ contained in the complement of C. Because $h(x) = 0$ for all $x \in V_\delta(z)$, we conclude that h is continuous at z.

Having shown that h is integrable, we can now appeal (once again) to the fact that C contains no intervals to see that $L(h, P) = 0$ for every partition P. Thus it must be that $L(f) = \int_0^1 h = 0$.

7.4 Properties of the Integral

Exercise 7.4.1. (a) Let $\epsilon > 0$ be arbitrary and choose x_1 and x_2 so that $M' - \epsilon/2 < |f(x_1)|$ and $m' + \epsilon/2 > |f(x_2)|$. Then using Exercise 1.2.6 (d) we can write

$$\begin{aligned}(M' - m') - \epsilon &\leq |f(x_1)| - |f(x_2)| \\ &\leq |f(x_1) - f(x_2)| \leq M - m.\end{aligned}$$

(b) Let $\epsilon > 0$. Because f is integrable, there exists a partition P satisfying $U(f, P) - L(f, P) < \epsilon$. But now from part (a) it follows that

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \epsilon,$$

and the result follows.

(c) Because $-|f| \leq f \leq |f|$ and all of these functions are integrable, we know from Theorem 7.4.2 (iv) and (ii) that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|.$$

Exercise 7.4.2.

Exercise 7.4.3. (a) False. A Dirichlet-type function that is 1 on the rationals and -1 on the irrationals is a counterexample.

(b) False. The functions in Exercise 7.3.3 and Exercise 7.3.9 are counterexamples.

(c) True. Because g is continuous at x_0 with $g(x_0) > 0$, there exists a δ-neighborhood $V_\delta(x_0)$ with the property that $g(x) \geq g(x_0)/2$ for all $x \in V_\delta(x_0)$. Now let P be a partition that contains the interval $V_\delta(x_0)$. When we compute the lower sum $L(f, P)$ with respect to this partition, the contribution from the subinterval $V_\delta(x_0)$ is at least $[g(x_0)/2]2\delta > 0$. The assumption that $g(x) \geq 0$ on the rest of $[a, b]$ guarantees that there are no negative terms in the sum $L(f, P)$, and it follows that

$$\int_a^b f = L(f) \geq L(f, P) > 0.$$

Exercise 7.4.4.

Exercise 7.4.5. (a) Consider a particular subinterval $[x_{k-1}, x_k]$ of P and let

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k' = \sup\{g(x) : x \in [x_{k-1}, x_k]\}, \text{ and}$$
$$M_k'' = \sup\{f(x) + g(x) : x \in [x_{k-1}, x_k]\}.$$

Because $M_k + M_k'$ is an upper bound for the set $\{f(x) + g(x) : x \in [x_{k-1}, x_k]\}$ it follows that $M_k'' \leq M_k + M_k'$. This inequality leads directly to the conclusion that $U(f+g, P) \leq U(f, P) + U(g, P)$.

The two sides are usually not equal because the functions f and g could easily take on their larger values in different places of each subinterval. For example, consider $f(x) = x$ and $g(x) = 1 - x$ on the interval $[0, 1]$. Then

$$M_k = 1, \quad M_k' = 1 \quad \text{and} \quad M_k'' = 1,$$

so we have $M_k'' < M_k + M_k'$.

The inequality for lower sums takes the form $L(f+g, P) \geq L(f, P) + L(g, P)$.

(b) Because f and g are integrable, there exist sequences of partitions (P_n) and (Q_n) such that

$$\text{(1)} \qquad \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0 \quad \text{and} \quad \lim_{n\to\infty} [U(g, Q_n) - L(g, Q_n)] = 0.$$

For each n, let R_n be the common refinement $R_n = P_n \cup Q_n$. Then part (a) of this exercise and Lemma 7.2.3 imply

$$\begin{aligned} U(f+g, R_n) - L(f+g, R_n) &\leq [U(f, R_n) + U(g, R_n)] - [L(f, R_n) + L(g, R_n)] \\ &\leq [U(f, P_n) + U(g, Q_n)] - [L(f, P_n) + L(g, Q_n)] \\ &\leq [U(f, P_n) - L(f, P_n)] + [U(g, Q_n) - L(g, Q_n)]. \end{aligned}$$

From (1) it now follows that

$$\lim_{n\to\infty} [U(f+g, R_n) - L(f+g, R_n)] = 0,$$

and the result follows.

Exercise 7.4.6.

Exercise 7.4.7. (a) Set

$$f_n(x) = \begin{cases} (-1)^n n & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } x \geq 1/n. \end{cases}$$

Then $\int_0^1 f_n = (-1)^n$, and the limit of these integrals does not exist.

(b) The notation is a bit cumbersome by the idea for this example is not too hard. Set $g_1(x) = 1$ on $[0, 1]$. Let $g_2(x) = 1$ for $0 \leq x < 1/2$ and $g(x) = 0$ otherwise. Let $g_3(x) = 1$ for $1/2 \leq x \leq 1$ and equal 0 otherwise. For g_4, g_5, g_6, g_7 we divide the interval $[0, 1]$ into equal pieces of length $1/4$ and take each of the four functions (g_4 to g_7) to equal 1 on precisely one of these four pieces. For g_8 through g_{15} we repeat this procedure, setting each of these eight functions equal to 1 on one of eight distinct intervals of length $1/8$ that make up $[0, 1]$. Continuing on, it should be clear that $\int_0^1 g_n \to 0$, but for each $x \in [0, 1]$, the sequence $g_n(x)$ is equal to one infinitely often and thus cannot converge to zero.

Exercise 7.4.8.

Exercise 7.4.9. As a first step we use the properties of the integral proved in this section to write

$$\left|\int_0^1 g_n - \int_0^1 g\right| \leq \int_0^1 |g_n - g| = \int_0^\alpha |g_n - g| + \int_\alpha^1 |g_n - g|.$$

Let $\epsilon > 0$. Let's first pick α so that $1-\alpha < \epsilon/(4M)$. Having chosen α, we know $g_n \to g$ uniformly on $[0,\alpha]$, so there exists an N such that $|g_n - g| < \epsilon/2$ for all $n \geq N$. It follows that if $n \geq N$ then

$$\begin{aligned}\left|\int_0^1 g_n - \int_0^1 g\right| &\leq \int_0^\alpha |g_n - g| + \int_\alpha^1 |g_n - g| \\ &\leq \int_0^\alpha \epsilon/2 + \int_\alpha^1 2M \\ &\leq \epsilon/2 + 2M(1-\alpha) < \epsilon,\end{aligned}$$

and the result follows.

Exercise 7.4.10.

Exercise 7.4.11. The problem is with linearity. Properties (i) and(ii) both fail with this definition of the integral.

To see that (i) fails, let g be the Dirichlet function and let $f = 1 - g$. Then both $\int_0^1 g = 1$ and $\int_0^1 f = 1$, but notice that $\int_0^1 (f+g) = \int_0^1 1 = 1$. To see that (ii) also fails, continue to let g be the Dirichlet function and set $k = -1$. Then $\int_0^1 (-1)g = 0$ which is clearly not equal to $(-1)\int_0^1 g = -1$.

For what it's worth, properties (iii), (iv), and (v) continue to hold if we define $\int_a^b f = U(f)$.

7.5 The Fundamental Theorem of Calculus

Exercise 7.5.1. (a) For $f(x) = |x|$ we get

$$F(x) = \begin{cases} -x^2/2 + 1/2 & \text{if } x < 0 \\ x^2/2 + 1/2 & \text{if } x \geq 0 \end{cases}$$

In this case, F is continuous and differentiable with $F'(x) = f(x)$ for all $x \in \mathbf{R}$. This follows from FTC but it is interesting to check this directly from the formula for F, especially at $x = 0$ where we get $F'(0) = 0$ from both sides.

(b) This time we get

$$F(x) = \begin{cases} x+1 & \text{if } x < 0 \\ 2x+1 & \text{if } x \geq 0 \end{cases}$$

A sketch of F is valuable and illustrates in particular that F is continuous on all of $\mathbf{R}$ but fails to be differentiable at $x = 0$ due to the "corner" on the graph. If $x \neq 0$, then we certainly get $F'(x) = f(x)$ as predicted by FTC.

Exercise 7.5.2.

Exercise 7.5.3. The Mean Value Theorem does not require $F(x)$ to be differentiable at the endpoints so we could get by with assuming that F is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$. By appealing to Theorem 7.4.1 we could in fact weaken the hypothesis even more to allow $F'(x) = f(x)$ to fail at an arbitrary finite number of points.

Exercise 7.5.4.

Exercise 7.5.5. Because $f_n' \to g$ uniformly on any interval of the form $[a, x]$, it follows from Theorem 7.4.4 that

$$\lim_{n\to\infty} \int_a^x f' = \int_a^x g.$$

Taking the limit as $n \to \infty$ on each side of the equation $\int_a^x f_n' = f_n(x) - f_n(a)$ leads to the equation

$$f(x) = f(a) + \int_a^x g.$$

But g is the uniform limit of continuous functions and so g must also be continuous. Part (ii) of the Fundamental Theorem of Calculus then implies that $f'(x) = g(x)$, as desired.

Exercise 7.5.6.

Exercise 7.5.7. If we set $G(x) = \int_a^x f$, then it follows from part (ii) of FTC that $G'(x) = f(x)$. Because $F'(x) = f(x)$ as well, F and G have the same derivative and a corollary to the Mean Value Theorem implies

$$G(x) = F(x) + k, \tag{1}$$

for some constant k. To compute k, set $x = a$ in equation (1) to get $0 = F(a)+k$ or $k = -F(a)$. Substituting this back into (1) and setting $x = b$ we find

$$\int_a^b f = G(b) = F(b) - F(a).$$

Exercise 7.5.8.

Exercise 7.5.9. (a) Let P be a partition of $[a, b]$ and consider a particular subinterval $[x_{k-1}, x_k]$ of P. Because f' is continuous, we may use FTC to write

$$f(x_k) - f(x_{k-1}) = \int_{x_{k-1}}^{x_k} f'.$$

Computing the variation with respect to this particular partition, we get

$$\begin{aligned} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \\ &\le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|. \end{aligned}$$

What we discover is that $\int_a^b |f'|$ is an upper bound on the set of variations, and it follows that $Vf \leq \int_a^b |f'|$ because Vf is the least upper bound of this set.

(b) Given a partition P, this time we apply MVT to an arbitrary subinterval $[x_{k-1}, x_k]$ to get

$$f(x_k) - f(x_{k-1}) = f'(c_k)\Delta x_k \quad \text{for some } c_k \in (x_{k-1}, x_k).$$

Because lower sums are computed by taking the infimum over each subinterval, this allows us to write

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} |f'(c_k)|\Delta x_k \geq L(|f'|, P).$$

It follows that Vf is an upper bound for the set of lower sums for $|f'|$ and we immediately get $Vf \geq \int_a^b |f'|$.

Parts (a) and (b) then imply $Vf = \int_a^b |f'|$.

Exercise 7.5.10.

Exercise 7.5.11. (a) Let $L_1 = \lim_{x\to c^-} f(x)$. If we insist that $f(c) = L_1$, then the argument in the text for FTC part (ii) can be used to show

$$\lim_{x\to c^-} \frac{F(x) - F(c)}{x - c} = f(c) = L_1.$$

On the other hand, if we let $L_2 = \lim_{x\to c+} f(x)$, and set $f(c) = L_2$, then the same argument also shows that

$$\lim_{x\to c^+} \frac{F(x) - F(c)}{x - c} = f(c) = L_2.$$

Because $L_1 \neq L_2$ the result is that the graph of F has a "corner" at $x = c$ and is not differentiable.

(b) Let $h(x) = \sum_{n=1}^{\infty} u_n(x)$ be the function defined in Exercise 6.4.10. Note that $0 < h(x) < 1$ and h is increasing. By Exercise 7.2.7, h is integrable over any interval and thus we can set

$$H(x) = \int_0^x h(t)\, dt.$$

Part (ii) of FTC implies that H is continuous (and differentiable at every irrational point.) Also, if $x < y$ then

$$H(y) - H(x) = \int_x^y h(t) \geq 0,$$

and it follows that H is increasing. Now fix a rational number r_N from the enumeration in Exercise 6.4.10. The fact that h is increasing implies that both $\lim_{x\to r_N^-} h(x)$ and $\lim_{x\to r_N^+} h(x)$ exist, and we can show that they must differ by r_N. Then Exercise 7.5.11 implies that H is not differentiable at r_N, and hence at any rational point in $\mathbf{R}$.

7.6 Lebesgue's Criterion for Riemann Integrability

Exercise 7.6.1. (a) Because $t(x) = 0$ for every irrational and the irrationals are dense in $\mathbf{R}$, it follows that $L(t, P) = 0$ for every partition P.

(b) If $x \in D_{\epsilon/2}$ then x must be a rational number of the form $x = m/n$ with $n \leq 2/\epsilon$. The number of such points in the interval $[0, 1]$ is finite.

(c) Let $\{x_1, x_2, \ldots, x_N\}$ be the finite set of points in $D_{\epsilon/2} \cap [0, 1]$. Now build a partition P by constructing small, disjoint intervals around each x_k with length less than $\epsilon/(2N)$. Because $|t(x)| \leq 1$, the contribution of all of the intervals containing "bad points" to the upper sum will be at most $N \cdot (\epsilon/(2N)) \cdot 1 = \epsilon/2$. On all of the other intervals we have $|t(x)| \leq \epsilon/2$ and so, taken altogether, these contribute at most $\epsilon/2$ to the value of the upper sum. It follows that $U(t, P) \leq \epsilon/2 + \epsilon/2 = \epsilon$. Thus, t is integrable and $\int_0^1 t = 0$.

Exercise 7.6.2.

Exercise 7.6.3. Let $A = \{a_1, a_2, a_3, \ldots\}$ be a countable set. Given $\epsilon > 0$, let $O_n = V_{\epsilon_n}(a_n)$ where $\epsilon_n = \epsilon/2^{n+1}$. Clearly the collection $\{O_n : n \in \mathbf{N}\}$ covers A and we have

$$\sum_{n=1}^{\infty} |O_n| = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Exercise 7.6.4.

Exercise 7.6.5. Given $\epsilon > 0$, let $\{O_n : n \in \mathbf{N}\}$ be a collection of open intervals that cover A with the property that $\sum_{n=1}^{\infty} |O_n| \leq \epsilon/2$. Likewise, let $\{P_n : n \in \mathbf{N}\}$ be a collection of open intervals that cover B satisfying $\sum_{n=1}^{\infty} |P_n| \leq \epsilon/2$. Then the collection $\{O_n, P_n : n \in \mathbf{N}\}$ is still countable (the union of countable sets is countable), it forms a cover for the union $A \cup B$, and

$$\sum_{n=1}^{\infty} |O_n| + |P_n| = \sum_{n=1}^{\infty} |O_n| + \sum_{n=1}^{\infty} |P_n| \leq \epsilon$$

as desired.

Now assume we are given a countable collection $\{A_1, A_2, A_3, \ldots\}$ of sets of measure zero. Let $\epsilon > 0$. For each A_k, let $\{O_{k,n} : n \in \mathbf{N}\}$ be a countable collection of open intervals that cover A_k and satisfies $\sum_{n=1}^{\infty} |O_{k,n}| \leq \epsilon/2^k$. It follows that $\{O_{k,n} : n, k \in \mathbf{N}\}$ is a countable collection of open intervals (Theorem 1.5.8 (ii)) whose union certainly covers $\bigcup_{k=1}^{\infty} A_k$. Finally, taking the sum of the lengths of all of the intervals in $\{O_{k,n} : k, n \in \mathbf{N}\}$ involves reordering this set, but the content of Theorem 2.8.1 is that we are justified in simply computing the iterated sum

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |O_{k,n}| = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

This shows $\bigcup_{k=1}^{\infty} A_k$ has measure zero.

Exercise 7.6.6.

Exercise 7.6.7. (a) Assume f is continuous at x. Then given our fixed $\alpha > 0$, we know there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\alpha}{2} \quad \text{provided } y \in V_\delta(x).$$

Thus, if $y, z \in V_\delta(x)$ we then get

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha, \end{aligned}$$

and we conclude that f is α-continuous at x. The contrapositive of this conclusion is that if f is not α-continuous at x, then it certainly cannot be continuous at x. This is precisely what it means to say $D^\alpha \subseteq D$.

(b) Assume f is not continuous at x. Negating the ϵ–δ definition of continuity we get that there exists an $\epsilon_0 > 0$ with the property that for all $\delta > 0$ there exists a point $y \in V_\delta(x)$ where $|f(y) - f(x)| \geq \epsilon_0$. Noting simply that both $x, y \in V_\delta(x)$, we conclude that f is not α-continuous for $\alpha = \epsilon_0$ (or anything smaller.)

To prove $D = \bigcup_{n=1}^\infty D^{1/n}$ we argue for inclusion each way. If $x \in D$, then we have just shown that $x \in D^{\epsilon_0}$ for some $\epsilon_0 > 0$. Choosing $n_0 \in \mathbf{N}$ small enough so that $1/n_0 \leq \epsilon_0$, it follows that $x \in D^{1/n_0}$. This proves $D \subseteq \bigcup_{n=1}^\infty D^{1/n}$.

For the reverse inclusion we observe that part (a) implies $D^{1/n} \subseteq D$ for all $n \in \mathbf{N}$, and the result follows.

Exercise 7.6.8.

Exercise 7.6.9. Because D has measure zero, we know there exists a *countable* collection of open intervals $\{G_1, G_2, \ldots\}$ whose union contains D and that satisfies

$$\sum_{n=1}^\infty |G_n| < \frac{\epsilon}{4M}. \tag{1}$$

But $D^\alpha \subseteq D$ is closed and hence compact. This means we can find a finite collection $\{G_1, \ldots, G_N\}$ that covers D^α and the inequality above in (1) is certainly true for this smaller set.

Exercise 7.6.10.

Exercise 7.6.11. As a first step in constructing P_ϵ we include the intervals from the open cover $\{G_1, G_2, \ldots, G_N\}$. Because $\sum_{n=1}^N |G_n| < \epsilon/(4M)$ the contribution of these subintervals to $U(f, P_\epsilon) - L(f, P_\epsilon)$ can be estimated by

$$\sum (M_k - m_k)\Delta x_k < (2M) \sum \Delta x_k \leq (2M)\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

Now consider the set $K = [a, b] \backslash \bigcup_{n=1}^N G_n$. The function f is uniformly α-continuous on K and so there exists $\delta > 0$ such that $|f(x) - f(y)| < \alpha$ whenever

$|x - y| < \delta$. To finish constructing the partition P_ϵ we take each interval in K and subdivide it until all of the subintervals have length less than δ. The implication here is that on each of these subintervals we get $M_k - m_k \leq \alpha$. Thus, the contribution of all of the subintervals that make up K is less than

$$\sum (M_k - m_k)\Delta x_k \leq \alpha \sum \Delta x_k < \left(\frac{\epsilon}{2(b-a)}\right)(b-a) = \frac{\epsilon}{2}.$$

Altogether then we get

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and it follows that f is Riemann-integrable.

Exercise 7.6.12.

Exercise 7.6.13. (a) If both f and g are integrable, then the set of discontinuities for each has measure zero. The Algebraic Continuity Theorem tells us that the set of discontinuities for the product fg must be contained in the the union of the sets of discontinuity for f and g. Because the union of two sets of measure zero has measure zero, it follows that fg is integrable by Lebesgue's Theorem.

(b) Because f is continuous, the set of discontinuities of the composition $f(g(x))$ must be a subset of the set of discontinuities of g. Since the set of discontinuities of g has measure zero by hypothesis, so does the set of discontinuities of $f(g(x))$, and thus this composition is integrable.

Exercise 7.6.14.

Exercise 7.6.15. (a) If $c \in C$ then $f_n(c) = 0$ for all $n \in \mathbf{N}$. It follows that $\lim_{n\to\infty} f_n(c) = 0$.

(b) If $x \notin C$ then choose N to be the smallest natural number for which $x \notin C_N$. Then, by its construction, $f_n(x) = f_N(x)$ for all $n \geq N$ and $\lim_{n\to\infty} f_n(x) = f_N(x)$.

Exercise 7.6.16.

Exercise 7.6.17. The set of discontinuities of f' is precisely the Cantor set C. Because C has measure zero (see Exercise 7.6.4), Lebesgue's Theorem (Theorem 7.6.5) implies f' is Riemann-integrable.

Exercise 7.6.18.

Exercise 7.6.19. Start with the Cantor-type set $F \subseteq [0, 1]$ constructed in the previous exercise. What is important is that F have strictly positive measure while also being closed, perfect (i.e., contains no isolated points), and nowhere dense (i.e., contains no intervals). By Exercise 4.3.12, we know there exists a continuous function g satisfying $g(x) = 0$ for all $x \in F$ and $g(x) \neq 0$ for all $x \notin F$.

Define f by setting $f(0) = 0$ and $f(x) = 1$ for all $x \neq 0$. With just one discontinuity, f is clearly integrable. Now consider the composition $f \circ g$ on $[0, 1]$. If $x \in F$ then $f(g(x)) = 0$ and if $x \notin F$ then $f(g(x)) = 1$. Because every point in F is a limit point of both F and F^c, it is straightforward to show that $f \circ g$ is not continuous on F, and it follows from Lebesgue's Theorem that $f \circ g$ is not integrable.

Chapter 8

Additional Topics

8.1 The Generalized Riemann Integral

Exercise 8.1.1. (a) For any tagged partition $(P, \{c_k\})$, it is certainly true that $m_k \leq f(c_k) \leq M_k$, and this is enough to conclude

$$L(f, P) \leq R(f, P) \leq U(f, P).$$

The fact that $L(f, P) \leq \int_a^b f \leq U(f, P)$ follows from Definition 7.2.7.

(b) Because P' is a refinement of P_ϵ, we can use Lemma 7.2.3 to argue

$$U(f, P') - L(f, P') \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{3}.$$

Exercise 8.1.2.

Exercise 8.1.3. (a) To form P' from P we added the points of P_ϵ. This means adding $(n-1)$ potentially new points to the interior of $[a, b]$. Now each new point adds two terms to $U(f, P')$ that do not appear in $U(f, P)$ and also creates one term in $U(f, P)$ that is no longer in $U(f, P')$. Thus, there can be at most $3(n-1)$ terms of the form $M_k \Delta x_k$ that appear in one of $U(f, P')$ or $U(f, P)$ but not the other.

(b) Because P is assumed to be δ-fine and $P \subseteq P'$, any term from either $U(f, P')$ or $U(f, P)$ can be estimated by

$$|M_k(x_x - x_{k-1})| \leq M\delta = \frac{\epsilon}{9n}.$$

Using our conclusion from (a), we then get

$$U(f, P) - U(f, P') \leq 3(n-1)\frac{\epsilon}{9n} < \frac{\epsilon}{3}.$$

Exercise 8.1.4.

Exercise 8.1.5. We shall prove f is integrable using the criterion in Theorem 7.2.8. Let $\epsilon > 0$. From our hypothesis we know that there exists a $\delta > 0$ such that

$$|R(f,P) - A| < \frac{\epsilon}{4} \tag{1}$$

for all δ-fine partitions P regardless of the choice of tags. So let P_ϵ be δ-fine and use the previous exercise to pick tags $\{c_k\}$ so that

$$U(f,P_\epsilon) - R(f,P_\epsilon,\{c_k\}) < \frac{\epsilon}{4}.$$

Now we can also pick tags $\{d_k\}$ so that

$$R(f,P_\epsilon,\{d_k\}) - L(f,P_\epsilon) < \frac{\epsilon}{4},$$

and from (1) above it must be that

$$|R(f,P_\epsilon,\{c_k\}) - R(f,P_\epsilon,\{d_k\})| < \frac{\epsilon}{2}.$$

A triangle inequality argument then implies

$$U(f,P_\epsilon) - L(f,P_\epsilon) < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon,$$

and we conclude that f is integrable. A second implication from this string of inequalities is that

$$L(f,P_\epsilon) \leq A \leq U(f,P_\epsilon),$$

from which may conclude that $\int_a^b f = A$.

Exercise 8.1.6.

Exercise 8.1.7. Assume, for contradiction, that this process of bisection does not terminate after a finite number of steps. Then we obtain a sequence of nested intervals (I_n) satisfying $|I_n| \to 0$ and

$$\delta(x) \leq |I_n| \quad \text{for all } x \in I_n.$$

From the Nested Interval Property we know that there exists a point $x_0 \in \bigcap_{n=1}^{\infty} I_n$. But then $\delta(x_0) \leq |I_n|$ for all $n \in \mathbf{N}$, and it follows that $\delta(x_0) = 0$. Because this is not allowed in the definition of a gauge, we conclude that the algorithm does terminate after a finite number of steps and we obtain a $\delta(x)$-fine tagged partition.

Exercise 8.1.8.

Exercise 8.1.9. Looking at Theorem 8.1.2, we just observe that the constant δ can also serve as the gauge function $\delta = \delta(x)$ required in Definition 8.1.6.

Exercise 8.1.10.

Exercise 8.1.11. This is due to the fact that we have a "telescoping" sum. Writing out the terms in the finite sum $\sum_{k=1}^{n} F(x_k) - F(x_{k-1})$, we can check that all of the summands cancel out except $F(x_n) = F(b)$ and $-F(x_0) = -F(a)$.

Exercise 8.1.12.

Exercise 8.1.13. (a) Let's first apply the result in Exercise 8.1.12 with $x = x_k$ and $c = c_k$. Because our tagged partition is assumed to be $\delta(c)$-fine we know that $(x_k - c_k) \leq (x_k - x_{k-1}) < \delta(c_k)$ and so

$$\left| \frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c_k) \right| < \epsilon.$$

Multiplying by the positive number $(x_k - c_k)$ gives the first requested inequality. To obtain the second one we again apply Exercise 8.1.12, this time with $x = x_{k-1}$ and $c = c_k$.

(b) An equivalent way to write these two inequalities is

$$-\epsilon(x_k - c_k) < F(x_k) - F(c_k) - f(c_k)(x_k - c_k) < \epsilon(x_k - c_k)$$

$$-\epsilon(c_k - x_{k-1}) < F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1}) < \epsilon(c_k - x_{k-1}),$$

and adding along the respective columns yields

$$-\epsilon \Delta x_k < F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k < \epsilon \Delta x_k.$$

Now this is equivalent to $|F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k| < \epsilon \Delta x_k$ and taking a sum over k gives us

$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k| < \epsilon(b - a).$$

Looking back at the beginning of the proof in the text, we see that we have now derived the inequality requested in (2) albeit with $\epsilon(b-a)$ in place of ϵ. This completes the proof.

Exercise 8.1.14.

8.2 Metric Spaces and the Baire Category Theorem

Exercise 8.2.1. (a) This is a metric. In fact this is the standard Euclidean distance function on $\mathbf{R}^2$. Conditions (i) and (ii) are straightforward. The most common way to prove (iii) is to introduce the scalar product from vector calculus. Squaring both sides of (iii) gives in equivalent inequality that can be derived using the so-called Schwartz inequality. An alternative proof can be derived by first considering the special case where the point z falls on the line

$$l(t) = (x_1, x_2) + t(y_1 - x_1, y_2 - x_2), \qquad t \in \mathbf{R}$$

through the points x and y. In this case $z = l(t_0)$ for some $t_0 \in \mathbf{R}$ and it follows that $d(x,z) = |t_0|d(x,y)$ and $d(z,y) = |1-t_0|d(x,y)$. Then the triangle inequality in $\mathbf{R}$ implies

$$\begin{aligned} d(x,y) &= (t_0 + 1 - t_0)\, d(x,y) \\ &\leq (|t_0| + |1-t_0|)\, d(x,y) \\ &= d(x,z) + d(z,y). \end{aligned}$$

To prove the general case, we let $z \in \mathbf{R}^2$ be arbitrary, and pick z_t to be the point on the line $l(t)$ such that the line through z and z_t is perpendicular to $l(t)$. Because x and y are both on the line $l(t)$, we can use the Pythagorean Theorem to show that $d(x,z_t) \leq d(x,z)$ and $d(y,z_t) \leq d(y,z)$. Applying the previous result about collinear points we get

$$d(x,y) \leq d(x,z_t) + d(z_t,y) \leq d(x,z) + d(z,y).$$

(b) This is a metric. It is clear that $d(x,y) \geq 0$. Also, if $\max\{|x_1-y_1|, |x_2-y_2|\} = 0$, then $|x_1-y_1| = 0$ and $|x_2-y_2| = 0$. But this is true if and only if $x_1 = y_1$ and $x_2 = y_2$. This proves (i). Because $|x_i - y_i| = |y_i - x_i|$, condition (ii) holds. For (iii), consider the case where $\max\{|x_1-y_1|, |x_2-y_2|\} = |x_1-y_1|$. The triangle inequality from $\mathbf{R}^1$ implies

$$|x_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1|.$$

Because $|x_1 - z_1| \leq d(x,z)$ and $|z_1 - y_1| \leq d(z,y)$, it follows that

$$|x_1 - y_1| \leq d(x,z) + d(z,y),$$

and similar argument works in the other case.

(c) This is not a metric, for it fails conditions (i) and (iii). This example fails (i) because we can have have $d(x,y) = 0$ where $x \neq y$. For instance, let $x = (1,-1)$ and let $y = (1,1)$. Then $d(x,y) = |1(-1)+1(1)| = 0$, but $x_2 \neq y_2$, so $x \neq y$. Part (iii) also does not hold in general. Consider $x = (1,-1), y = (4,-1)$, and $z = (1,1)$. Then

$$d(x,y) = 6 > 5 = d(x,z) + d(z,y),$$

which violates the triangle inequality.

Exercise 8.2.2.

Exercise 8.2.3. Conditions (i) and (ii) can be easily verified. For (iii), we must consider 5 distinct cases. First, suppose that x, y and z are all distinct. Then

$$d(x,y) = 1 < 2 = d(x,z) + d(z,y).$$

If $x = y \neq z$, then

$$d(x,y) = 0 < 2 = d(x,z) + d(z,y).$$

If $x \neq y = z$, then

$$d(x, y) = 1 \leq 1 = d(x, z) + d(z, y),$$

which is identical to the case where $y \neq x = z$. Finally, if $x = y = z$, then

$$d(x, y) = 0 \leq 0 = d(x, z) + d(z, y).$$

Thus the triangle inequality holds for all possible scenarios.

Exercise 8.2.4.

Exercise 8.2.5. (a) By considering values of ϵ less than one, we can show that Cauchy sequences in this metric space are eventually constant sequences. Because such a sequence converges (to this constant value), $\mathbf{R}^2$ is complete with respect to this metric.

(b) Assume that (f_n) is a Cauchy sequence in the metric of Exercise 8.2.2 (a). Then given $\epsilon > 0$, there exists an N such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$. This implies that

$$|f_m(x) - f_n(x)| < \epsilon \quad \text{for all } m, n \geq N \text{ and } x \in [0, 1].$$

Thus (f_n) converges uniformly according to the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5). What is really happening here is that convergence with respect to this metric is equivalent to uniform convergence on $[0, 1]$. If $f = \lim_{n\to\infty} f_n$ uniformly, then f is continuous by Theorem 6.2.6. Hence f is an element of $C[0, 1]$ and the metric is complete.

(c) Let's start with a Cauchy sequence (f_n) in $C[0, 1]$. In (b) we saw that there exists $f \in C[0, 1]$ such that $f_n \to f$ uniformly, but it does not have to be the case that $f \in C^1[0, 1]$. A counterexample appears in Example 6.2.2 (iii).

Exercise 8.2.6.

Exercise 8.2.7. The ϵ-neighborhoods of the metric in (a) are discs with center x and radius ϵ. For the metric in (b), ϵ-neighborhoods form a square with sides of length 2ϵ and x in the center. Item (c) is not a metric.

For the discrete metric, $V_\epsilon(x) = \mathbf{R}^2$ if $\epsilon \geq 1$. If $\epsilon < 1$, then $V_\epsilon(x)$ is a singleton point.

Exercise 8.2.8.

Exercise 8.2.9. (a) Let h be a limit point of Y, and let $\epsilon > 0$ be arbitrary. Then $V_\epsilon(h)$ intersects Y at a point $g \neq h$. So $d(g, h) < \epsilon$, and $g \in Y$ meaning $|g(x)| \leq 1$ for all $x \in [0, 1]$. It follows that

$$\begin{aligned} |h(x)| &= |h(x) - g(x) + g(x)| \\ &\leq |h(x) - g(x)| + |g(x)| \\ &< \epsilon + 1. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, $|h(x)| \leq 1$ for all $x \in [0,1]$ and thus $h \in Y$. This proves Y is closed.

(b) This set is closed. Suppose that g is a limit point of T. Then, given $\epsilon > 0$, we know $V_\epsilon(g)$ intersects T at a point $h \neq g$. So

$$|g(0) - h(0)| \leq d(g,h)) < \epsilon.$$

But $h \in T$ so $h(0) = 0$ and this implies $|g(0)| < \epsilon$. Because ϵ is arbitrary, we conclude that $g(0) = 0$, and hence $g \in T$. Thus T is closed.

To see that T is not open, notice that any ϵ-neighborhood around a function $f \in T$ is going to include functions that are not zero at the origin. Thus, there is no $V_\epsilon(f) \subseteq T$ and T is not open.

Exercise 8.2.10.

Exercise 8.2.11. (a) ($\Rightarrow$) Assume that E is closed. Then E contains its limit points, so $E \cup L = E$, where L is the set of limit points of E. Therefore $E = \overline{E}$. ($\Leftarrow$) Now assume that $\overline{E} = E$. Then $E = E \cup L$, so E contains its limit points and hence it is closed.

($\Rightarrow$) Assume that E is open. Then for each $x \in E$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq E$. Hence $E = E^\circ$. ($\Leftarrow$) For the other direction, if $E^\circ = E$, then for each $x \in E$, $V_\epsilon(x) \subseteq E$. Hence E is open.

(b) Let $x \in \overline{E}^c$. Then $x \notin \overline{E}$. Hence there exists $\epsilon > 0$ such that $V_\epsilon(x)$ does not intersect E, so $V_\epsilon(x) \subseteq E^c$. By Definition 8.2.8, $x \in (E^c)^\circ$. This shows $\overline{E}^c \subseteq (E^c)^\circ$. To prove the other inclusion let $x \in (E^c)^\circ$. Then there exists $V_\epsilon(x) \subseteq E^c$. So $x \notin \overline{E}$, and hence $x \in \overline{E}^c$. Thus $(E^c)^\circ \subseteq \overline{E}^c$ and $\overline{E}^c = (E^c)^\circ$.

To prove the second statement let $x \in (E^\circ)^c$. Then $x \notin E^\circ$, so every $V_\epsilon(x)$ fails to be contained in E. Thus every $V_\epsilon(x)$ intersects E^c, and therefore $x \in \overline{E^c}$. This shows $(E^\circ)^c \subseteq \overline{E^c}$. Now assume $x \in \overline{E^c}$. Then every $V_\epsilon(x)$ intersects E^c, and so $V_\epsilon(x)$ is not contained in E. Thus $x \notin E^\circ$ implying $x \in (E^\circ)^c$. This proves $\overline{E^c} \subseteq (E^\circ)^c$ and hence we have $(E^\circ)^c = \overline{E^c}$.

Exercise 8.2.12.

Exercise 8.2.13. ($\Rightarrow$) Let E be nowhere-dense in X. Then $\overline{E}^\circ$ is empty. This means that given $x \in \overline{E}$, every $V_\epsilon(x)$ intersects $\overline{E}^c$. So x is a limit point of $\overline{E}^c$. It follows that $\overline{\overline{E}^c} = X$, and hence $\overline{E}^c$ is dense.

($\Leftarrow$) Now assume that $\overline{E}^c$ is dense. Then $\overline{\overline{E}^c} = X$. So every point $x \in X$ is either an element of $\overline{E}^c$ or a limit point of $\overline{E}^c$. This implies that for all $\epsilon > 0$, $V_\epsilon(x)$ is not contained in $\overline{E}$, which means that $\overline{E}^\circ$ is empty. Hence E is nowhere dense.

Exercise 8.2.14.

Exercise 8.2.15. If E is nowhere-dense in X, then $(\overline{E})^c$ is dense. We've also seen that the complement of a closed set (such as $\overline{E}$) is open.

Now suppose that E_n is a collection of nowhere dense sets and assume, for contradiction, that $X = \bigcup_{n=1}^\infty E_n$. Then certainly it is true that $X = \bigcup_{n=1}^\infty \overline{E_n}$.

By De Morgan's Law, this implies that $\bigcap_{n=1}^{\infty}(\overline{E_n})^c$ is empty. But since $(\overline{E_n})^c$ is dense and open, this intersection is not empty by Theorem 8.2.10, so we have reached a contradiction.

Exercise 8.2.16.

Exercise 8.2.17. (a) The sequence (x_k) is contained in $[0,1]$ and so the Bolzano–Weierstrass Theorem can be applied to deduce that there is a convergent subsequence.

(b) Let $\epsilon > 0$. Because $f_{k_l} \to f$ uniformly, we can pick L_1 so that $l \geq L_1$ implies $|f_{k_l}(y) - f(y)| < \epsilon/2$ for all $y \in [0,1]$. Now the limit function f is continuous at x and so there exists a $\delta > 0$ such that

$$|f(x_{k_l}) - f(x)| < \frac{\epsilon}{2} \quad \text{whenever } |x_{k_l} - x| < \delta.$$

Because $x_{k_l} \to x$, we can pick L_2 so that $|x_{k_l} - x| < \delta$ for all $l \geq L_2$. Finally, set $L = \max\{L_1, L_2\}$. Then $l \geq L$ implies

$$\begin{aligned} |f_{k_l}(x_{k_l}) - f(x)| &\leq |f_{k_l}(x_{k_l}) - f(x_{k_l})| + |f(x_{k_l}) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(c) If t satisfies $|x-t| < 1/m$, then there exists an L such that $|x_{k_l} - x| < 1/m$ for all $l \geq L$. In this case we have

$$\left| \frac{f_{k_l}(x_{k_l}) - f_{k_l}(t)}{x_{k_l} - t} \right| \leq n.$$

Now taking the limit as $l \to \infty$ and using (b) together with the Algebraic Limit Theorem and the Order Limit Theorem gives

$$\left| \frac{f(x) - f(t)}{x - t} \right| \leq n.$$

The conclusion is that $f \in A_{m,n}$ meaning that $A_{m,n}$ contains its limit points and thus is closed.

Exercise 8.2.18.

8.3 Euler's Sum

Exercise 8.3.1. Setting $x = \pi/2$ in equation (2) yields

$$\begin{aligned} 1 &= \frac{\pi}{2}\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4}\right)\left(1 - \frac{1}{6}\right)\left(1 + \frac{1}{6}\right)\cdots \\ &= \frac{\pi}{2}\left(\frac{1}{2} \cdot \frac{3}{2}\right)\left(\frac{3}{4} \cdot \frac{5}{4}\right)\left(\frac{5}{6} \cdot \frac{7}{6}\right)\cdots \\ &= \frac{\pi}{2} \lim_{n\to\infty} \left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{(2n-1)(2n+1)}{2n\cdot 2n}\right). \end{aligned}$$

Thus,

$$\frac{2}{\pi} = \lim_{n\to\infty} \left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{(2n-1)(2n+1)}{2n\cdot 2n}\right),$$

and taking reciprocals (with a nod to the Algebraic Limit Theorem) gives the result.

Exercise 8.3.2.

Exercise 8.3.3. (a) Applying the integration-by-parts formula in the suggested way, we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n(x)dx &= \int_0^{\frac{\pi}{2}} \sin^{n-1}(x)\sin(x)dx \\
&= -\sin^{n-1}(\pi/2)\cos(\pi/2) + \sin^{n-1}(0)\cos(0) + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\cos^2(x)dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)(1-\sin^2(x))dx \\
&= (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)dx - \int_0^{\frac{\pi}{2}} \sin^n(x)dx\right).
\end{aligned}$$

Combining like terms from each side yields

$$n\int_0^{\frac{\pi}{2}} \sin^n(x)dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)dx,$$

and the recurrence relation

$$b_n = \frac{n-1}{n} b_{n-2}$$

is immediate.

(b)

$$b_0 = \frac{\pi}{2}, \quad b_2 = \frac{1}{2}\cdot\frac{\pi}{2}, \quad b_4 = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$b_1 = 1, \quad b_3 = \frac{2}{3}\cdot 1, \quad b_5 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

(c)

$$b_{2n} = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} \qquad b_{2n+1} = 1\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1}.$$

Exercise 8.3.4.

Exercise 8.3.5. Using the previously derived formulas for the various factorial expressions we can write

$$\begin{aligned}
\frac{\pi}{2} &= \lim_{n\to\infty}\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right) \\
&= \lim_{n\to\infty}\frac{2^{3n}2^{n-1}(n!)^3(n-1)!}{(2n-1)!(2n+1)!}.
\end{aligned}$$

So,

$$\pi = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2 n} \cdot \frac{2n}{(2n+1)},$$

and by the continuity of $f(x) = \sqrt{x}$ we get

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} \cdot \sqrt{\frac{2n}{2n+1}}.$$

Now use the Algebraic Limit Theorem and the fact that $\lim(2n/(2n+1)) = 1$ to get the result.

Exercise 8.3.6.

Exercise 8.3.7. The basic idea here is that because

$$0 \neq \sqrt{\pi} = \lim_{n\to\infty} \frac{1}{c_n\sqrt{n}},$$

then when n gets large enough the sequence (c_n) "looks like" a constant multiple of $(1/\sqrt{n})$. As we have seen, this latter sequence goes to zero, but not fast enough to be summable.

To make this informal argument precise, take $\epsilon = 1$ (or anything less than $\sqrt{\pi}$). Then there exists an N such that $n \geq N$ implies

$$\sqrt{\pi} - 1 < \frac{1}{c_n\sqrt{n}} < \sqrt{\pi} + 1,$$

which is equivalent to

$$\left(\frac{1}{\sqrt{\pi}-1}\right)\frac{1}{\sqrt{n}} > c_n > \left(\frac{1}{\sqrt{\pi}+1}\right)\frac{1}{\sqrt{n}}.$$

The first inequality leads directly to a proof that $(c_n) \to 0$ while the second can be used in conjunction with the Comparison Test to conclude that $\sum c_n$ diverges.

Exercise 8.3.8.

Exercise 8.3.9. (a) This is the Fundamental Theorem of Calculus, part (i), applied on the interval $[0, x]$.

(b) The "previous result" is the integration-by-parts formula, and it should be applied to the integral

$$\int_0^x f'(t)dt$$

with $h(t) = f'(t)$ and $k(t) = -(x-t)$. By Exercise 8.3.2, we get

$$\int_0^x f'(t)dt = -f'(x)(x-x) + f'(0)x + \int_0^x f''(t)(x-t)dt,$$

and substituting into the formula from (a) gives the result.

(c) This time set $h(t) = f''(t)$ and $k(t) = -(x-t)^2/2$ (so $k'(t) = (x-t)$.) Then,

$$\int_0^x f''(t)(x-t)dt = 0 + \frac{f''(0)}{2}x^2 + \int_0^x f'''(t)\frac{(x-t)^2}{2}$$

and substituting yields

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{1}{2}\int_0^x f'''(t)(x-t)^2 dt \\ &= S_2(x) + E_2(x). \end{aligned}$$

Continuing on yields the desired formula for an arbitrary value of N, as an induction argument would formerly prove.

Exercise 8.3.10.

Exercise 8.3.11. This is an application of the term-by-term antidifferentiation result in Exercise 6.5.4. On the interior of its interval of convergence, taking the anti-derivative term-by-term indeed gives an antiderivative for the original power series. The Mean Value Theorem shows that antiderivatives are unique up to a constant, so we just need to check that equation (5) is valid at $x = 0$ to get equality on the interval $(-1, 1)$.

Exercise 8.3.12.

Exercise 8.3.13. (a) Because the series representation

$$\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta)$$

converges uniformly on $[\pi/2, \pi/2]$, the Integrable Limit Theorem (Theorem 7.4.4) can be applied (to the sequence of partial sums of the series) to get

$$\begin{aligned} \int_0^{\pi/2} \theta d\theta &= \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \int_0^{\pi/2} \sin^{2n+1}(\theta) \\ &= \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1}. \end{aligned}$$

(b) Computing the integral is simple enough:

$$\int_0^{\pi/2} \theta d\theta = \left.\frac{\theta^2}{2}\right|_0^{\pi/2} = \frac{\pi^2}{8}.$$

Looking back at our previous calculations, we see

$$\begin{aligned} c_n \cdot b_{2n+1} &= \frac{1 \cdot 3 \cdot 4 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \\ &= \frac{1}{2n+1} \end{aligned}$$

Putting these two calculations together gives the formula

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

For the final step in the argument, note that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \end{aligned}$$

which leads to the formula

$$\frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

By our earlier result then,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \left(\frac{\pi^2}{8} \right) = \frac{\pi^2}{6}.$$

8.4 Inventing the Factorial Function

Exercise 8.4.1. (a) There is nothing definitive to do here but one obvious idea is to look at the pattern of numbers. These are the so-called triangular numbers: $1\# = 1$, $2\# = 3$, $3\# = 6$, $4\# = 10, \ldots$.

The values are increasing, so a good guess for $\frac{7}{2}\#$ would be a number between 6 and 10, and by the looks of it something closer to 6. Let's guess $\frac{7}{2}\# = 7.5$.

How about 0 and -2? One idea we'll encounter in this section is the notion of a functional equation. In this case we have

$$(n+1)\# = (n+1) + n\#$$

for all $n \in \mathbf{N}$. One way to proceed is to assume that this relationship continues to hold on all of $\mathbf{Z}$. Setting $n = 0$ in this equation gives $1\# = 1 + 0\#$, and so $0\# = 0$.

Continuing in this fashion, $0\# = 0 + (-1)\#$ which implies $(-1)\# = 0$. Then $(-1)\# = (-1) + (-2)\#$ and so $(-2)\# = 1$.

(b) This familiar formula is often proved by induction. Another interesting way to prove it is write

$$2(n\#) = (1 + 2 + 3 + \cdots (n-1) + n) + (n + (n-1) + \cdots + 3 + 2 + 1).$$

Now taking one term from each group (in order) we get $1+n = n+1$, $2+(n-1) = n+1$, $3+(n-2) = n+1$, and so on. Taken together we get n sums of $(n+1)$, so $2(n\#) = n(n+1)$. The result follows.

Using this formula for non-natural numbers gives, $0\# = 0$, $\frac{7}{2}\# = 7.85$, and $(-2)\# = 1$.

Exercise 8.4.2.

Exercise 8.4.3. (a) Because both series converge absolutely for all x and y, Exercise 2.8.7 asserts that we can compute the product of $E(x)E(y)$ using any of the methods for computing a double summation discussed in that section. The method of choice here is the Cauchy product; that is, we want to write

$$E(x)E(y) = \sum_{k=0}^{\infty} d_k, \quad \text{where} \quad d_k = \sum_{j=0}^{k} \frac{x^j}{j!} \frac{y^{k-j}}{(k-j)!}.$$

Computing a few values of d_k by hand is illuminating: $d_0 = 1$, $d_1 = x + y$, and $d_2 = y^2/2 + yx + x^2/2 = (x + y)^2$.

In general, we can say

$$d_k = \sum_{j=0}^{k} \frac{x^j}{j!} \frac{y^{k-j}}{(k-j)!} = \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j} = \frac{1}{k!}(x+y)^k.$$

The last equality is a well-known identity referred to as the *binomial formula.* From this calculation it follows that

$$E(x)E(y) = \sum_{k=0}^{\infty} d_k = \sum_{k=0}^{\infty} \frac{1}{k!}(x+y)^k = E(x+y).$$

(b) Formula (1) leads immediately to the conclusion that $E(0) = 1$ and $E(x) > 0$ for all $x \geq 0$. Using the additive relationship proved in (a) we get that for all $x \in \mathbf{R}$,

$$1 = E(0) = E(x - x) = E(x)E(-x).$$

Thus $E(-x) = 1/E(x)$, as desired.

Exercise 8.4.4.

Exercise 8.4.5. For a fixed $n \in \mathbf{N}$, we can see that $E(x) > x^{n+1}/(n+1)!$ for $x \geq 0$ because all the terms in the power series for $E(x)$ are positive. Thus, $E(-x) = 1/E(x) < (n+1)!/x^{n+1}$.

It now follows that

$$x^n e^{-x} < x^n \frac{(n+1)!}{x^{n+1}} = \frac{(n+1)!}{x}$$

from which $\lim_{n\to\infty} x^n e^{-x} = 0$ can be proved in a straightforward way. Given $\epsilon > 0$, choose $M > (n+1)!/\epsilon$. Then $x \geq M$ implies $x^n e^{-x} < (n+1)!/x < \epsilon$, as desired.

Exercise 8.4.6.

Exercise 8.4.7. (a) To find an alternate expression for $t^{\frac{1}{n}}$ write

$$\begin{aligned} t = e^{\log t} &= e^{\frac{1}{n}\log t + \frac{1}{n}\log t + \cdots + \frac{1}{n}\log t} \qquad (n \text{ times}) \\ &= e^{\frac{1}{n}\log t} \cdot e^{\frac{1}{n}\log t} \cdots e^{\frac{1}{n}\log t} \\ &= \left(e^{\frac{1}{n}\log t}\right)^n, \end{aligned}$$

from which it follows that $\sqrt[n]{t} = e^{\frac{1}{n}\log t}$. Then

$$\left(\sqrt[n]{t}\right)^m = e^{\frac{1}{n}\log t} \cdot e^{\frac{1}{n}\log t} \cdots e^{\frac{1}{n}\log t} \qquad (m \text{ times})$$

and the additive property of the exponential function once again comes into play to give us

$$\left(\sqrt[n]{t}\right)^m = e^{\frac{m}{n}\log t} = t^{\frac{m}{n}}.$$

(b) To see this relationship write

$$\log\left(t^x\right) = \log\left(e^{x\log t}\right) = x\log t.$$

The first equality is the definition of t^x and the second is property (i) from Exercise 8.4.6 part (a).

(c) The function t^x is a composition of differentiable functions, so the Chain Rule implies t^x is differentiable with

$$(t^x)' = e^{x\log t} \cdot \log t = t^x \log t.$$

Exercise 8.4.8.

Exercise 8.4.9. (a) ($\Rightarrow$) Assume $\int_a^\infty f$ converges to L. Given $\epsilon > 0$ there exists $M > a$ such that $b \geq M$ implies

$$\left|\int_a^b f - L\right| < \frac{\epsilon}{2}.$$

If $d > c \geq M$, then

$$\begin{aligned} \left|\int_c^d f\right| &= \left|\int_a^d f - \int_a^c f\right| \\ &\leq \left|\int_a^d f - L\right| + \left|L - \int_a^c f\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

($\Leftarrow$) Assuming the Cauchy condition holds for the improper integral $\int_a^\infty f$, it is straightforward to show that the sequence $a_n = \int_a^{a+n} f$ is a Cauchy sequence. Thus $(a_n) \to L$, for some L. We'll show that the improper integral converges to L as well.

Let $\epsilon > 0$. By the Cauchy condition, there exists an $M > a$ such that $\left|\int_c^d f\right| < \epsilon/2$ for all $d > c \geq M$. Now go out the sequence (a_n) far enough and find a term a_N with the property that $a + N \geq M$ and $|a_N - L| < \epsilon/2$. Then for all $b \geq M$ we have that

$$\begin{aligned} \left|\int_a^b f - L\right| &\leq \left|\int_a^b f - \int_a^{a+N} f\right| + \left|\int_a^{a+N} f - L\right| \\ &< \left|\int_{a+N}^b f\right| + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and we conclude that $\int_a^\infty f$ converges to L.

(b) Let $\epsilon > 0$. The Cauchy criterion from (a) applied to the integral involving g says that there exists $M > a$ such that $d > c \geq M$ implies $\int_c^d g < \epsilon$. Because $\int_c^d f \leq \int_c^d g$, the Cauchy criterion (in the other direction) implies the integral involving f converges as well.

(c) The absolute convergence test says that if $\int_a^\infty |f|$ converges, then $\int_a^\infty f$ converges as well. (We need to assume that f is integrable.) To prove this, let $\epsilon > 0$ and use the Cauchy criterion from (a) to assert that there exists $M > a$ such that $\int_c^d |f| < \epsilon$ for all $d > c \geq M$. But in this case,

$$\left|\int_c^d f\right| \leq \int_c^d |f| < \epsilon,$$

and the Cauchy criterion implies that $\int_a^\infty f$ converges as well.

Exercise 8.4.10.

Exercise 8.4.11. (a) In the notation of Exercise 7.5.6, we set $h(t) = t$ and $k'(t) = e^{-\alpha t}$. This gives

$$\begin{aligned} \int_0^b te^{-\alpha t} dt &= \left(\frac{-b}{\alpha} e^{-\alpha b} - 0\right) + \frac{1}{\alpha}\int_0^b e^{-\alpha t} dt \\ &= \frac{-b}{\alpha} e^{-\alpha b} - \frac{1}{\alpha^2}\left(e^{-\alpha b} - 1\right). \end{aligned}$$

(b) Letting $b \to \infty$ in the above expression gives $\frac{1}{\alpha^2}$ as predicted by equation (4). Note that the result in Exercise 8.4.5 is being invoked to compute this limit.

Exercise 8.4.12.

Exercise 8.4.13. The key observation is that for all $x \neq y$ in $[a, b]$,

$$\begin{aligned} |F(x) - F(y)| &= \left|\int_c^d f(x,t) - f(y,t) dt\right| \\ &\leq \int_c^d |f(x,t) - f(y,t)| dt. \end{aligned}$$

Let $\epsilon > 0$. Because $f(x,t)$ is uniformly continuous on D, there exists a $\delta > 0$ such that

$$|f(x,t) - f(y,s)| < \frac{\epsilon}{d-c} \qquad \text{whenever} \qquad \|(x,t) - (y,s)\| < \delta.$$

Observe that $|x-y| < \delta$ implies $\|(x,t) - (y,t)\| < \delta$, and in this case we have

$$|F(x) - F(y)| < \int_c^d \frac{\epsilon}{d-c} dt = \epsilon.$$

This shows F is uniformly continuous on $[a,b]$.

Exercise 8.4.14.

Exercise 8.4.15. (a) For an arbitrary $x > 0$ we can write

$$\begin{aligned} \frac{1}{x} - \int_0^d e^{-xt} dt &= \frac{1}{x} - \left(\frac{-1}{x} e^{-xt} \Big|_0^d \right) \\ &= \frac{1}{x} - \left(\frac{-1}{x} e^{-xd} + \frac{1}{x} \right) = \frac{1}{x} e^{-xd}. \end{aligned}$$

If we restrict our attention to $x \geq 1/2$ then we have the estimate

$$0 < \frac{1}{x} e^{-xd} \leq 2e^{-d/2},$$

which goes to zero as $d \to \infty$. Because this does not depend on x, the convergence is uniform on $[1/2, \infty)$.

(b) On the larger interval $(0, \infty)$ the convergence is not uniform. Although $\frac{1}{x} e^{-xd} \to 0$ as $d \to \infty$ for every $x > 0$, values of x close to zero require larger values of d. Specifically,

$$\frac{1}{x} e^{-xd} < \epsilon \qquad \text{if and only if} \qquad d > \frac{-\log(\epsilon x)}{x},$$

and the latter expression is unbounded in any neighborhood of zero.

Exercise 8.4.16.

Exercise 8.4.17. By our earlier work in this section, $F_n(x) = \int_c^{c+n} f(x,t) dt$ is continuous for all $x \in [a,b]$ and for all $n \in \mathbf{N}$. We also have $F_n \to F$ uniformly on $[a,b]$. By the Continuous Limit Theorem, F is continuous on $[a,b]$, and thus uniformly continuous on this compact set.

Exercise 8.4.18.

Exercise 8.4.19. (a) The idea is to use Theorem 8.4.9 to justify differentiating under the integral sign. In order to apply the theorem, we first fix $\alpha > 0$ and consider $\int_0^\infty e^{-xt}dt$ over an interval $[a,b]$ with $0 < a < \alpha < b$. Taking the derivative with respect to x leads to the integral

$$\int_0^\infty te^{-xt}dt.$$

To invoke Theorem 8.4.9 we need to argue that this improper integral converges uniformly on $[a,b]$. Appealing to the result in Exercise 8.4.16, we just need to argue that $\int_0^\infty te^{-at}dt$ converges. This is a straightforward computation.

(b) The stated formula emerges naturally by repeated applications of Theorem 8.4.9. Similar to part (a), justifying the use of this theorem boils down to arguing that integrals of the form

$$\int_0^\infty t^n e^{-at}dt$$

converge, regardless of the value of n. One way to prove this without resorting to integration formulas involving repeated applications of the integration-by-part formula is to use the estimate

$$t^n e^{-at} = \left(t^n e^{-at/2}\right) e^{-at/2} \leq M_n e^{-at/2}.$$

The constant M_n exists because of the result in Exercise 8.4.5, and the same exercise leads to a demonstration that $\int_0^\infty e^{-at/2}dt$ converges.

Exercise 8.4.20.

Exercise 8.4.21. (a) Applying the convexity condition to the chords over each of the three intervals gives

$$\frac{\log(f(n)) - \log(f(n-1))}{1} \leq \frac{\log(f(n+x)) - \log(f(n))}{x} \leq \frac{\log(f(n+1)) - \log(f(n))}{1}.$$

Now the fact that $f(n) = n!$ can be combined with properties of $\log(x)$ to conclude $\log(f(n)) - \log(f(n-1)) = \log(n)$. This, and similar calculations lead to the desired inequalities.

(b) Applying the functional equation relationship in (ii) to the expression $f(x+n)$ exactly n times yields

$$f(x+n) = f(x)(x+1)(x+2)(x+3)\cdots(x+n),$$

and the mutiplicative property of the logarithm yields the given equation.

(c) The relationship $x\log(n) = \log(n^x)$ is proved in the earlier exercises in this chapter. Combining this with our other properties of $\log(x)$ allows us to rewrite the inequalities in (a) as

$$0 \leq \log(f(n+x)) - \log(n!n^x) \leq x\log(1+\frac{1}{n}).$$

Substituting our alternate expression for $f(n+x)$ in (b) and one more application of the properties of the logarithm produce the given inequalities.

(d) A straightforward application of the Squeeze Theorem gives us

$$\log(f(x)) = \lim_{n\to\infty} \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right), \quad \text{for all } x \in (0,1].$$

To recover $f(x)$ we make use of the fact that $f(x) = e^{\log(f(x))}$. Because the exponential function is continuous, $\lim a_n = a$ implies $\lim e^{a_n} = e^a$, which effectively tells us we can exponentiate the right side as well to get the given limit formula for $f(x)$.

(e) Let $x \in (1,2]$. The functional equation says $f(x) = xf(x-1)$. Because $(x-1) \in (0,1]$, the formula for part (d) can be used to replace $f(x-1)$ and we get

$$\begin{aligned} f(x) &= x \lim_{n\to\infty} \frac{n^{x-1}n!}{x(x+1)(x+2)\cdots(x-1+n)} \\ &= \lim_{n\to\infty} \frac{n^{x-1}n!}{(x+1)(x+2)\cdots(x-1+n)} \\ &= \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)} \cdot \frac{(x+n)}{n}. \end{aligned}$$

Because $\left(\frac{x+n}{n}\right) \to 1$, the Algebraic Limit Theorem can be used to conclude

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}, \quad \text{for all } x \in (0,2].$$

An induction argument extends this result to all $x \geq 0$.

Exercise 8.4.22.

Exercise 8.4.23. The information in the previous exercise leads to the conclusion that $(1/2)! = \sqrt{\pi}/2$. Setting $x = 1/2$ in the Gauss product formula gives

$$\begin{aligned} \frac{\pi}{2} &= \lim_{n\to\infty} \frac{\sqrt{n}\, n!}{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\cdots\left(\frac{(2n+1)}{2}\right)} \\ &= \lim_{n\to\infty} \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)} \cdot \sqrt{n} \end{aligned}$$

Squaring both sides of this limit equation and multiplying by 2 yields

$$\begin{aligned} \frac{\pi}{2} &= \lim_{n\to\infty} \frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{3^2\cdot 5^2\cdot 7^2\cdots(2n+1)^2} \cdot (2n) \\ &= \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right)\cdot\left(\frac{2n}{(2n+1)}\right). \end{aligned}$$

This is almost the Wallis formula, except it has one extra term at the end. However, notice that this term satisfies

$$\lim_{n\to\infty} \frac{2n}{2n+1} = 1,$$

and the Algebraic Limit Theorem justifies removing it from the above formula.

8.5 Fourier Series

Exercise 8.5.1. (a) Taking partial derivatives yields

$$\frac{\partial^2 u}{\partial x^2} = -b_n \sin(nx)\cos(nt) \cdot n^2 = \frac{\partial^2 u}{\partial t^2}.$$

Also

$$u(0,t) = b_n \sin(0)\cos(nt) = 0 \quad \text{and}$$
$$u(\pi,t) = b_n \sin(\pi n)\cos(nt) = 0.$$

Note that this second statement requires n be an integer. Finally,

$$\frac{\partial u}{\partial t} = -b_n \sin(nx)\sin(nt) \cdot n,$$

and setting $t = 0$ gives $\frac{\partial u}{\partial t}(x,0) = 0$.

(b) The derivative is a linear transformation meaning that the derivative of the sum of functions is the sum of the derivatives of each one. This property makes (1) and (3) true for a sum of solutions, and (2) is easy to check as well.

Exercise 8.5.2.

Exercise 8.5.3. Start with equation (6) in the text and multiply each side of this equation by $\cos(mx)$ to get

$$f(x)\cos(mx) = a_0\cos(mx) = \sum_{n=1}^{\infty} a_n \cos(nx)\cos(mx) + b_n \sin(nx)\cos(mx).$$

Now take the integral of each side of this equation from $-\pi$ to π and, as before, distribute the integral through the infinite sum. Using Exercise 8.5.2, we see that for a_0 and for every value of $n \in \mathbf{N}$ we get an integral that equals zero *except* the one where $n = m$. When $n = m$ we get

$$\int_{-\pi}^{\pi} a_m \cos^2(mx)\,dx = a_m\pi$$

and it follows that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\pi.$$

The formula for a_m is immediate. To get the formula for b_m we multiply across equation (6) by $\sin(mx)$ and follow the same procedure.

Exercise 8.5.4.

Exercise 8.5.5. Recall that any function continuous on a compact set is uniformly continuous. Thus h is uniformly continuous over, say, $[\pi, 3\pi]$. This means that given $\epsilon > 0$, there exists a $\delta > 0$ that "works" for all pairs x, y in this set. Now the fact that h is periodic implies that this δ suffices on all of $\mathbf{R}$.

Exercise 8.5.6.

Exercise 8.5.7. (a) Because f is continuous, the function $q_x(u) = f(u+x) - f(x)$ is continuous. It follows from the Riemann–Lebesgue Lemma (Theorem 8.5.2) that

$$\int_{-\pi}^{\pi} q_x(u)\cos(Nx)\,dx \to 0 \qquad \text{as } N \to \infty.$$

(b) The idea here is to show that the discontinuity of $p_x(u)$ at zero is removable; that is, that $p_x(u)$ can be defined at $u = 0$ in such a way that makes p_x continuous. To see how to do this write

$$\begin{aligned} p_x(u) &= \frac{(f(u+x) - f(x))\cos(u/2)}{\sin(u/2)} \\ &= 2\,\frac{f(u+x) - f(x)}{u} \cdot \frac{(u/2)}{\sin(u/2)} \cdot \cos(u/2). \end{aligned}$$

The fact that f is differentiable at x and the well-known limit $\lim_{t\to 0}\sin(t)/t = 1$ imply

$$\lim_{u\to 0} p_x(u) = 2f'(x).$$

Thus, defining $p_x(0) = 2f'(x)$ makes p_x continuous on $(-\pi, \pi]$ and it now follows from the Riemann–Lebesgue Lemma that

$$\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\,du \to 0 \qquad \text{as } N \to \infty.$$

Exercise 8.5.8.

Exercise 8.5.9. For $k = 1, 2, \ldots, N$ write

$$D_k(\theta) = \frac{1}{2}\left[\cos(k\theta) + \frac{\sin(k\theta)\cos(\theta/2)}{\sin(\theta/2)}\right]$$

as in the proof of Theorem 8.5.3. Then,

$$\begin{aligned} \frac{1}{N+1}\left[\frac{1}{2} + \sum_{k=1}^{N} D_k(\theta)\right] &= \frac{1}{2(N+1)}\left[1 + \sum_{k=1}^{N}\cos(k\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{k=1}^{N}\sin(k\theta)\right] \\ &= \frac{1}{2(N+1)}\left[\frac{1}{2} + D_N(\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\frac{\sin(N\theta/2)\sin((N+1)\theta/2)}{\sin(\theta/2)}\right] \\ &= \frac{1}{2(N+1)\sin^2(\theta/2)}\left[\mathbf{B}\right], \end{aligned}$$

where

$$\mathbf{B} = \frac{\sin^2(\theta/2)}{2} + \frac{\sin(\theta/2)\sin(N\theta + \frac{\theta}{2})}{2} + \cos(\theta/2)\sin(N\theta/2)\sin((N+1)\theta/2).$$

To finish the proof we must show that $\mathbf{B} = \sin^2((N+1)\theta/2)$. Using the identity $\sin(t)\cos(t) = (1/2)\sin(2t)$, we can write

$$\begin{aligned}\sin^2((N+1)\theta/2) &= [\sin(N\theta/2)\cos(\theta/2) + \cos(N\theta/2)\sin(\theta/2)]^2 \\ &= \sin^2(N\theta/2)\cos^2(\theta/2) + \frac{\sin(N\theta)\sin(\theta)}{2} + \cos^2(N\theta/2)\sin^2(\theta/2).\end{aligned}$$

Now we use Fact 1(b) from the text together with the identities $\sin(t)\cos(t) = (1/2)\sin(2t)$ and $1+\cos(t) = 2\cos^2(t/2)$ to write

$$\begin{aligned}\mathbf{B} &= \frac{\sin^2(\theta/2)}{2} + \frac{\sin(\theta/2)}{2}[\cos(N\theta)\sin(\theta/2) + \sin(N\theta)\cos(\theta/2)] \\ &\quad + \cos(\theta/2)\sin(N\theta/2)\,[\cos(N\theta/2)\sin(\theta/2) + \sin(N\theta/2)\cos(\theta/2)] \\ &= \frac{\sin^2(\theta/2)}{2}[1+\cos(N\theta)] + \frac{\sin(N\theta)\sin(\theta)}{4} \\ &\quad + \frac{\sin(N\theta)\sin(\theta)}{4} + \sin^2(N\theta/2)\cos^2(\theta/2) \\ &= \sin^2(\theta/2)\cos^2(N\theta/2) + \frac{\sin(N\theta)\sin(\theta)}{2} + \sin^2(N\theta/2)\cos^2(\theta/2).\end{aligned}$$

This completes the derivation.

Exercise 8.5.10.

Exercise 8.5.11. Let $\epsilon > 0$. From Fejér's theorem, we know there exists N such that

$$|\sigma_N(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in [0,\pi].$$

But $\sigma_N(x)$ is a linear combination of the partial sums $S_n(x)$ and each $S_n(x)$ is a linear combination of functions of the form $\cos(kx)$ and $\sin(kx)$. From the previous discussion about Taylor series, we know it is possible to find polynomials that are arbitrarily and uniformly close to the trigonometric functions that constitute each S_n. Because the sums in question are all finite, a repeated application of the triangle inequality implies that we can find a polynomial $p(x)$ satisfying

$$|p(x) - \sigma_N(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in [0,\pi].$$

Finally, one last triangle inequality argument shows

$$\begin{aligned}|p(x) - f(x)| &\le |p(x) - \sigma_N(x)| + |\sigma_N(x) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This proves the result on the interval $[0,\pi]$.

(b) To prove the general case we just use the change of variables $t = \pi(x - a)/(b-a)$ and observe that polynomials are preserved under this transformation.

8.6 A Construction of R From Q

Exercise 8.6.1. (a) We have to show C_r possesses the three properties of a cut. Property (c1) can be verified by noticing that C_r contains all rational $t < r$ and hence, it is not the empty set. Also, the set $C_r \neq \mathbf{Q}$ since all rational numbers greater than r are not contained in C_r.

To prove property (c2), fix $t \in C_r$ and assume $q < t$. Because $t \in C_r$ we have $q < t < r$ and thus q is an element of C_r, as desired.

Finally, let's show property(c3) holds for C_r. Note that for any $t \in C_r$ we can produce $q \in C_r$ with $t < q < r$ by letting $q = (t + r)/2$ .This shows C_r does not have a maximum.

(b) The set S is not a cut because it has a maximum.

(c) The set T is a cut.

(d) The set U is also a cut. It may seem as though $\sqrt{2}$ is a maximum, but our definition of a cut deals exclusively with rational numbers. At the moment there is no such thing as $\sqrt{2}$. In fact, this cut (which is equal to the cut in (c)) is to become $\sqrt{2}$ when we are finished.

Exercise 8.6.2.

Exercise 8.6.3. The operations of addition and multiplication are commutative and associative on all of these sets, and the distributive property holds. The set of natural numbers is not a field because there is no additive identity and no additive inverses. Although **N** has a multiplicative identity, it also fails to have multiplicative inverses. The set of integers is an improvement in that **Z** has an additive identity and additive inverses. However, multiplicative inverses do not exist for elements of **Z** (except for the numbers -1 and 1). The set of rational numbers **Q** possesses all the properties of a field.

Exercise 8.6.4.

Exercise 8.6.5. (a) The set $A+B$ is not the empty set because A is not empty and B is not empty. To argue $A + B \neq \mathbf{Q}$ pick $r_1 \notin A$ and $l_2 \notin B$. Given arbitrary elements $a \in A$ and $b \in B$, we again use Exercise 8.6.2 to say that $a < l_1$ and $b < l_2$. This implies $l_1 + l_2$ is an upper bound on $A + B$ meaning $A + B$ cannot be all of **Q**.

To show that $A + B$ does not have a maximum, fix $c \in A + B$ and write $c = a + b$ where $a \in A$ and $b \in B$. By property (c3) we know that there exists $s \in A$ with $a < s$. Also, there exists $r \in B$ with $b < r$. We can now conclude $s + r \in A + B$ with $c < s + r$.

(b) To show that addition is commutative we can write

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ &= \{b + a : a \in A, b \in B\} = B + A. \end{aligned}$$

The proof that addition is associative is similar in that it follows directly from the fact that addition of rational numbers is associative. In particular, we can show that $x \in (A+B)+C$ if and only if $x = a+b+c$ where $a \in A$, $b \in B$ and $c \in C$. Then $(a+b)+c = a+(b+c)$ and the rest is clear sailing.

(c) Let $A \subseteq B$ and let C be an arbitrary cut. To show $A+C \subseteq B+C$, let $a+c$ be an arbitrary element of $A+C$, meaning $a \in A$ and $c \in C$. By hypothesis, $a \in B$ as well, and so $a+c \in B+C$.

(d) Let's follow the advice to prove inclusions in each direction, starting with $A+O \subseteq A$. Given $a+b \in A+O$ where $a \in A$ and $b \in O$, we know $b < 0$. Thus, $a+b < a$, and by property (c2), $a+b \in A$.

To prove the reverse inclusion, fix $a \in A$. By property (c3) there must exist $s \in A$ satisfying $a < s$, from which it follows that $a - s \in O$. Then

$$a = s + (a-s) \in A + O,$$

which proves $A \subseteq A+O$. These two inclusions together show $A = A+O$.

Exercise 8.6.6.

Exercise 8.6.7. (a) We must show AB has the properties of a cut. Let's first verify property (c1). The set $AB \neq \emptyset$ because all rational $q < 0$ are in AB. Furthermore, because A and B are bounded above then so are products of the form ab where both $a, b \geq 0$ with $a \in A$ and $b \in B$. This implies $AB \neq \mathbf{Q}$.

To prove property (c2), we let $t \in AB$ be arbitrary and let $s \in \mathbf{Q}$ satisfy $s < t$. If $s < 0$ then $s \in AB$ by the way we have defined the product. For the case $0 \leq s < t$ it must be that $t = ab$ where $a \in A$ and $b \in B$ satisfy $a > 0$ and $b > 0$. Because $s < ab$ we have $s/b < a$ which implies $s/b \in A$. Then

$$s = \left(\frac{s}{b}\right)(b) \in AB,$$

and (c2) is proved.

To verify property (c3), consider $t \in AB$. If $t < 0$ then $t < t/2$ and $t/2 \in AB$ because $t/2 < 0$ as well. If $t \geq 0$ then $t = ab$ for some $a \in A$ and $b \in B$. Applying property (c3) to A and B we get $s \in A$ and $r \in B$ with $a < s$ and $b < r$. We conclude $sr \in A+B$ with $ab < sr$.

Property (o5) follows directly from the definition of the product of two positive cuts.

(b) The cut $I = \{p \in \mathbf{Q} : p < 1\}$ is the multiplicative identity. Exercise 8.6.1 contains the argument that $I = C_1$ is actually a cut. We now show $AI = A$ for all $A \geq O$ by demonstrating inclusion both ways.

Fix $q \in AI$. Because $I \geq 0$, then either $q < 0$ or $q = ab$ where $a, b \geq 0$ with $a \in A$ and $b < 1$. If $q < 0$ then $q \in A$ because $A \geq 0$. In the other case we have $q = ab < a$ and property (c2) implies $ab \in A$. Thus, $AI \subseteq A$.

In the other direction we consider $a \in A$. If $a < 0$ then $a \in AI$ by our definition of the product of positive cuts. If $a \geq 0$, then property (c3) says that we can pick a rational $p \in A$ with $a < p$. This implies $a/p < 1$ and hence $a/p \in I$. But then,

$$a = \left(\frac{a}{p}\right)(p) \in AI,$$

which shows $AI \subseteq A$, and we conclude that $A = AI$.

(c) As usual, we'll show $A(B+C) = AB + AC$ by demonstrating inclusion in each direction.

If $q \in A(B+C)$ then either $q < 0$ or $q = a(b+c)$ for some a, b, c contained in A, B, C respectively, with $a \geq 0$ and $(b+c) \geq 0$. In the case $q < 0$, it is certainly true that $q/2 < 0$ as well and

$$q = \frac{q}{2} + \frac{q}{2} \in AB + AC.$$

If $q = a(b+c)$, then $q = ab + ac$. Because $ab \in AB$ and $ac \in AC$ regardless of whether b or c is positive or negative, we conclude that $A(B+C) \subseteq AB + AC$.

For the other inclusion, we start with an arbitrary $e \in AB$ and $f \in AC$ and consider the various cases for $e + f$. If both $e < 0$ and $f < 0$ then $e + f < 0$ and thus is in $A(B+C)$. Next consider the case where $e = a_1 b$ and $f = a_2 c$ for nonnegative a_1, b, a_2, c in the obvious sets. If $a_1 = a_2$ then factor out this common factor to get $e + f = a_1(b+c) \in A(B+C)$. If $a_1 \neq a_2$—let's say, $a_1 > a_2$—then we factor out the larger factor to get

$$e + f = a_1 \left(b + \left(\frac{a_2}{a_1} \right) c \right).$$

Because $(a_2/a_1)c < c$ it follows that $(a_2/a_1)c \in C$, and once again we have $e + f \in A(B+C)$.

The last possibility is that $e < 0$ while $f = ac$ for nonnegative $a \in A$ and $c \in C$. If $a = 0$ then $e + ac < 0$ and is therefore in $A(B+C)$. If $a > 0$ then

$$e + ac = a\left(\frac{e}{a} + c\right) \in A(B+C).$$

This proves $AB + AC \subseteq A(B+C)$ and we conclude $A(B+C) = AB + AC$.

Exercise 8.6.8.

Exercise 8.6.9. (a) We first show $C_r + C_s = C_{r+s}$ by showing inclusion both ways. For the forward inclusion, let $t + p \in C_r + C_s$ where $t \in C_r$ and $p \in C_s$. Then $t < r$ and $p < s$, and we see $t + p < r + s$. This implies $t + p \in C_{r+s}$ and thus $C_r + C_s \subseteq C_{r+s}$.

For the reverse inclusion we start with $p \in C_{r+s}$. Then $p < r + s$ implies $r + s - p > 0$. Letting $\epsilon = r + s - p$, a little algebra yields $p = (r - \epsilon/2) + (s - \epsilon/2)$. Observe that $r - \epsilon/2 \in C_r$ and $s - \epsilon/2 \in C_s$, and this implies $p \in C_r + C_s$, as desired. We conclude $C_{r+s} \subseteq C_r + C_s$ and therefore the sets are equal.

To verify $C_r C_s = C_{rs}$ for positive r and s we fix $q \in C_r C_s$. If $q < 0$ then $q < rs$ which implies $q \in C_{rs}$. If $q \geq 0$ then $q = ap$ for some $a \in C_r$ and $p \in C_s$ where both $a, p \geq 0$. Because everything is positive, we get $ap < rs$ which implies $q = ap \in C_{rs}$. This shows $C_r C_s \subseteq C_{rs}$.

For the other inclusion we consider $p \in C_{rs}$. If $p < 0$ then the way we have defined the product ensures $p \in C_r C_s$. If $p \geq 0$ then observe that $p < rs$ implies $p/s < r$ from which we conclude that $p/s \in C_r$. Then

$$p = \left(\frac{p}{s}\right)(s) \in C_r C_s,$$

and it follows that $C_{rs} \subseteq C_r C_s$. Thus $C_r C_s = C_{rs}$.

(b) ($\Rightarrow$) For each $n \in \mathbf{N}$ the rational number $r - (1/n) \in C_r$. Because $C_r \subseteq C_s$, we see $r - (1/n) \in C_s$. This means

$$r - \frac{1}{n} < s \quad \text{for all } n \in \mathbf{N},$$

and a short contradiction argument shows $r \leq s$.

($\Leftarrow$) Conversely, assume $r \leq s$. If $a \in C_r$ then $a < r \leq s$ which implies $a \in C_s$. Therefore, $C_r \subseteq C_s$ or, equivalently, $C_r \leq C_s$.

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