S10

Population Growth
Linear Stability Analysis
Potentials

population of orgamisms at time t

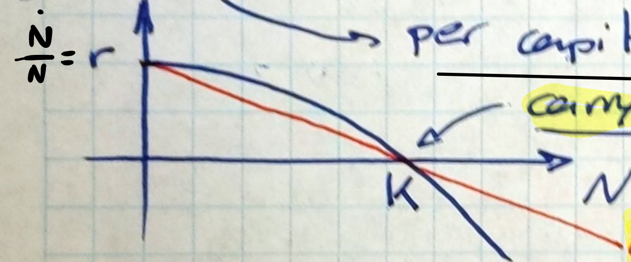
$\dot{N} = rN$ → $r > 0$ is the growth rate

$\frac{dN}{dt} = rN.$

$\rightsquigarrow \frac{dN}{dt} = rN \Rightarrow \boxed{N = N_0 e^{rt}}$ unreasonable!

* the growth rate cannot be constant

growth rate → per capita growth rate, $\frac{\dot{N}}{N}$



carrying capacity

linearly-decreasing.

a simple model (linear) : $\alpha = r(1 - \frac{N}{K})$

$\rightsquigarrow \frac{\dot{N}}{N} = r(1 - \frac{N}{K}) \Rightarrow \boxed{\dot{N} = rN(1 - \frac{N}{K})}$

when $N = K$, $1 - \frac{N}{K} = 0$.
$\Rightarrow r > 0$

Analytical solution: $\frac{dN}{dt} = rN(1 - \frac{N}{K}) \rightsquigarrow \frac{dN}{N(1-\frac{N}{K})} = r\,dt$

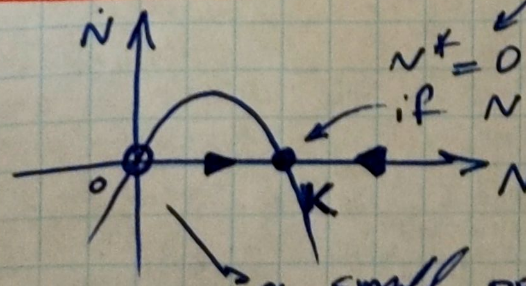
partial fractions:

$$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}} = \frac{A - \frac{A}{K}N + BN}{N(1-\frac{N}{K})} \Rightarrow \begin{array}{l} A = 1 \\ B = \frac{A}{K} = \frac{1}{K} \end{array}$$

$$\Rightarrow \int \frac{dN}{N(1-\frac{N}{K})} = \int \frac{1}{N}\,dN + \int \frac{\frac{1}{K}}{1-\frac{N}{K}}\,dN = \boxed{\ln\left|\frac{N}{1-\frac{N}{K}}\right| = rt + C}$$

Alternative → Let $x = \frac{1}{N} \Rightarrow \dots$

Graphical :



unstable stable

$N^* = 0$ and $N^* = K$ are the fixed points

if N is disturbed from $K \Rightarrow N \to K$ as $t \to \infty$

monotonically

a small population grows exponentially fast

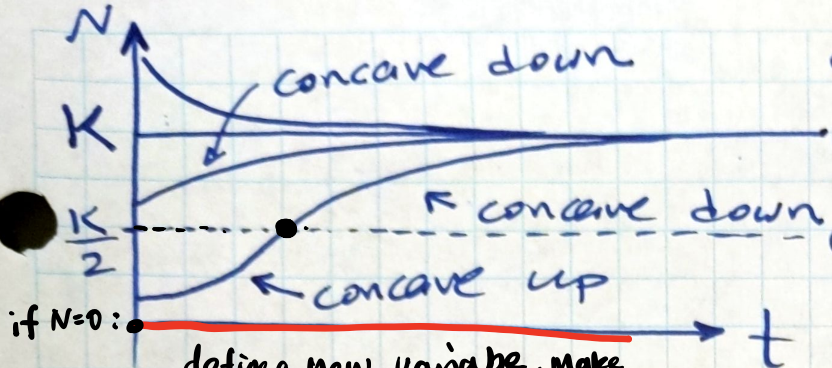
$N_0 = 0$? ⟹ there's nobody around to reproduce!

$$\frac{\dot{N}}{N} = r\left(1 - \frac{N}{K}\right) \Rightarrow \dot{N} = rN\left(1 - \frac{N}{K}\right)$$

$$\frac{dN}{N\left(1-\frac{N}{K}\right)} = r\,dt$$

partial fraction: $$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}.$$

$$\Rightarrow \ln \frac{N}{1-\frac{N}{K}} = rt + C.$$



$$\dot{N} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{f(N)}$$

$$\ln\left|\frac{N}{1-\frac{N}{K}}\right| = rt+c$$

$$f'(N) = r\left(1-\frac{N}{K}\right) - \frac{rN}{K}$$

$$f'(0) = r > 0, \quad f'(K) = -r < 0$$

define a new variable. Make it easier to solve.

Let $n = \frac{N}{K} \Rightarrow \frac{dN}{dt} = K\frac{dn}{dt} = rKn(1-n)$

$\Rightarrow \frac{dn}{dt} = rn(1-n)$ see populationGrowth.py

$\frac{1}{r}$ is the time scale : $[r] = \frac{1}{time}$

Linear stability Analysis

$x^* =$ constant

Let x^* be a fixed point of the system $\dot{x} = f(x)$.
Let $\eta = x - x^*$ be a small perturbation away from x^*.

$\hookrightarrow \dot{\eta} = \dot{x} - 0 = \dot{x} = f(x) = f(x^* + \eta)$

x^* is constant

$f(x+h) = f(x) + f'(x)h$

for fix point: 0; negligible

From Taylor series expansion: $f(x^*+\eta) = \boxed{f(x^*)} + f'(x^*)\eta + \ldots$

Linearization

$\Rightarrow \dot{\eta} \approx f'(x^*)\eta \longrightarrow \eta(t) = \eta_0 \exp(f'(x^*)t)$ same as growth model.

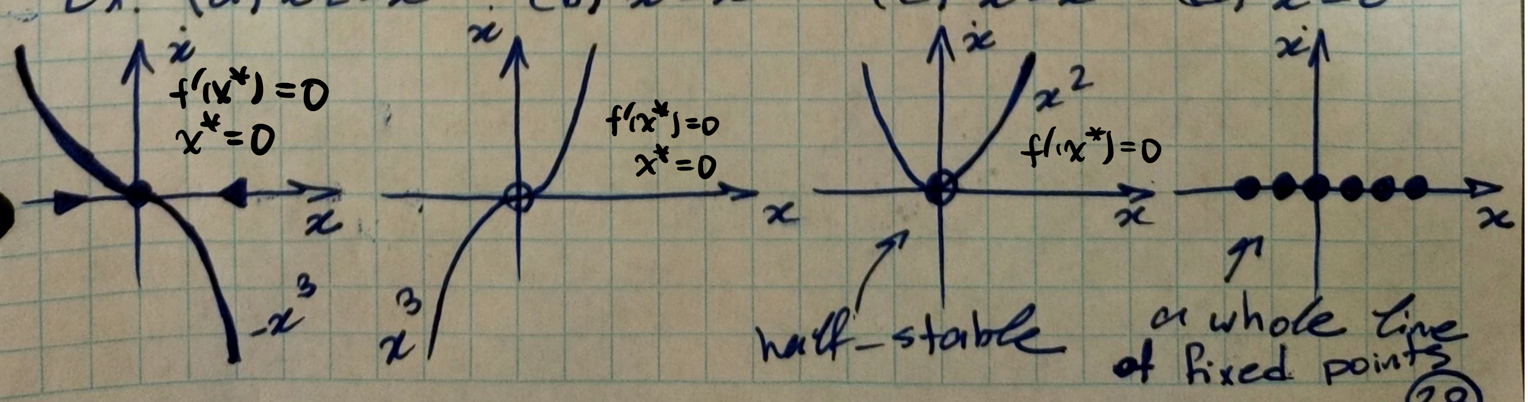
$f'(x^*) = 0 \Rightarrow \eta = \eta_0$

$f'(x^*) < 0 \longrightarrow$ decays (stable); $f'(x^*) > 0 \longrightarrow$ grows (unstable)

If $f'(x^*) = 0$, then $O(\eta^2)$ is not negligible and a nonlinear analysis is needed. (or via a case-by-case analysis)

Q: what does the magnitude of $f'(x^*)$ tell us?

EX. (a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

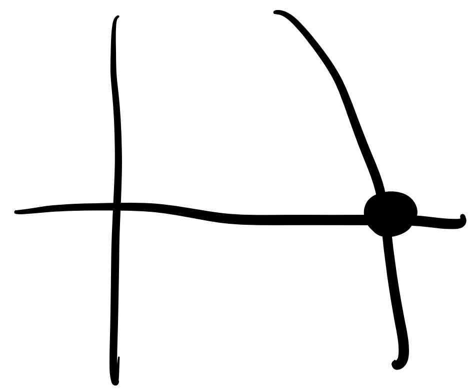


$f(x)t$ should be dimensionless

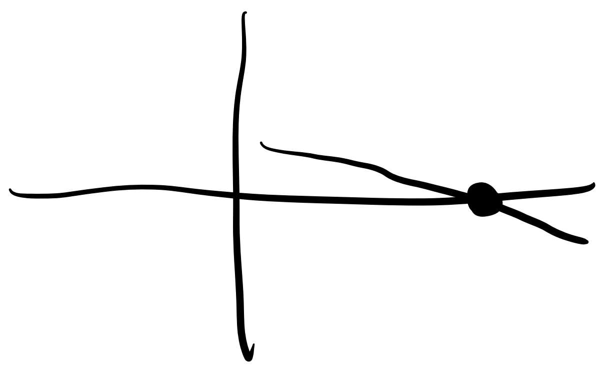
$$[f'(x^*)] = \frac{1}{\text{time}}$$

$$\tau = \frac{1}{f'(x^*)}.$$

if $f'(x^*)$ is large



if $f'(x^*)$ is small



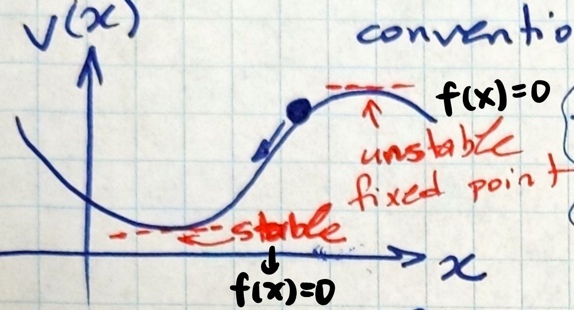
Potentials (oscillation)

$$\dot{x} = f(x) = -\frac{dV}{dx}$$

potential energy

V = potential energy

convention: particle always move downhill



* no inertia → impossibility of oscillation (the only force on the particle is from the potential energy)

topic of a future session

see x as a function of t

$$\rightarrow \frac{dV(x(t))}{dt} = \frac{dV}{dx}\underbrace{\boxed{\frac{dx}{dt}}}_{= -\frac{dV}{dx} = \dot{x}} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

$$\frac{dV}{dt} \leq 0.$$

⇒ V(t) decreases along trajectories!

$\boxed{\text{EX.}}$ Graph the potential for the system $\dot{x} = -x$

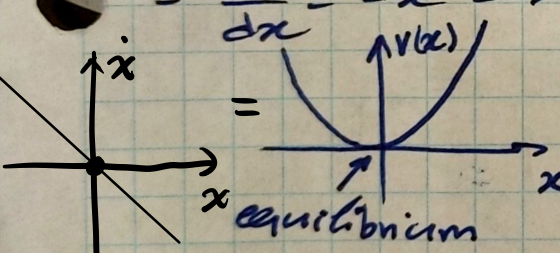
$$-\frac{dV}{dx} = -x \Rightarrow V = \frac{1}{2}x^2 + \cancel{C}$$

set 0 for convenience

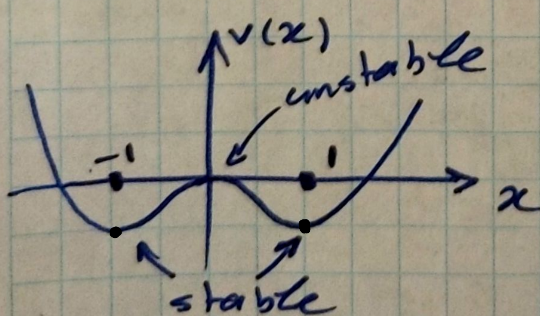
↓

move C will not change the pattern of graph.

(potential is only defined up to an additive constant)



EX. $\dot{x} = x - x^3 \Rightarrow -\frac{dV}{dx} = x - x^3 \Rightarrow V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \cancel{C}$ (C → 0)



. double-well potential

. the system is bistable