NYU PARIS

Linear Algebra

Math -UA 9140

Exercises Sets

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Contents

Part I

Assignements

Exercises Set # 1
Vectors & Scalar product

Exercise 1.

In the xy plane mark all nine of these linear combinations:

$$c \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + d \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } c \text{ in } \{0, 1, 2\} \text{ and } d \text{ in } \{0, 1, 2\}.$$

Exercise 2.

If three corners of a parallelogram are $(1, 1)$, $(4, 2)$ and $(1, 3)$, what are all three of the possible fourth corners?

Exercise 3.

In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} := (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} := (1, 1, 0)$?

Exercise 4.

What combination $c \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + d \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ produces $\begin{pmatrix} 14 \\ 8 \end{pmatrix}$?

Exercise 5.

Find the angle θ (from its cosine) between these pairs of vectors:

a) $\boldsymbol{v} := \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **b)** $\boldsymbol{v} := \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

c) $\boldsymbol{v} := \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$ **d)** $\boldsymbol{v} := \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Exercise 6.

1. How long is the vector $\boldsymbol{v} := (1, 1, \cdots, 1)$ in 9 dimensions?
2. Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a unit vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

Exercise 7.

Let n be a positive integer and let $\boldsymbol{u} := (u_1, u_2, \cdots, u_n)$, $\boldsymbol{v} := (v_1, v_2, \cdots, v_n)$ and $\boldsymbol{w} := (w_1, w_2, \cdots, w_n)$ be three vectors of $\mathbf{R}^n$. Having in mind that the dot product $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{k=1}^{n} u_i v_i$, prove the following equalities:

1. (a) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.
   (b) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.
   (c) $(c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{u} \cdot \boldsymbol{v})$.
2. Use **1.**. with $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{w}$ to prove that

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}.$$

EXERCISES SET # 2
MATRICES

Exercise 1.

For which of x, y, z and w are the matrices $A := \begin{pmatrix} x+y & x-z \\ y+w & x+2w \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ equal ?

Exercise 2.

Which of the following pairs of matrices commute under matrix multiplication?

$a)$ $A := \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ & $B := \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}$;

$b)$ $C := \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 2 & 4 \end{pmatrix}$ & $D := \begin{pmatrix} 4 & 2 & -2 \\ 5 & 2 & 4 \end{pmatrix}$;

$c)$ $F := \begin{pmatrix} 3 & 0 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & 0 \end{pmatrix}$ & $E := \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}$.

Exercise 3.

Find all matrices B that commute (under matrix multiplication) with $A := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Exercise 4.

Find a non zero matrix A in $\mathscr{M}_2(\mathbf{R})$ such that $A^2 = 0$.

Exercise 5.

Find all non zero matrix A in $\mathscr{M}_2(\mathbf{R})$ such that $A^2 = 0$.

Exercise 6.

The commutator of two matrices A, B is defined to be the matrix $C := [A, B] := AB - BA$.

1. Explain why $[A, B]$ is defined if and only if A and B are square matrices of the same size.
2. Show that A and B commute under matrix multiplication if and only if $[A, B] = 0$.
3. Compute the commutator of the following matrices (denoted A_1 & B_1, A_2 & B_2 and A_3 & B_3).

$$i)\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix} \& \begin{pmatrix}2 & 1\\ -2 & 0\end{pmatrix};\quad ii)\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix} \& \begin{pmatrix}1 & 7\\ 7 & 1\end{pmatrix};\quad iii)\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \& \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.$$

Exercise 7.

Let $P(x) = c_0 + c_1 x + \cdots c_{n-1}x^{n-1} + c_n x^n$ be a polynomial function. If A is a square matrix with n rows, we define the corresponding matrix polynomial $P(A) = c_0 I_n + c_1 A + \cdots c_{n-1}A^{n-1} + c_n A^n$; the constant term becomes a scalar multiple of the identity matrix I_n. For instance, if $P(x) = x^2 - 2x + 3$, then $P(A) = A^2 - 2A + 3I_n$.

1. Write out the matrix polynomials $P(A), Q(A)$ when $P(x) = x^3 - 3x + 2$, $Q(x) = 2x^2 + 1$.
2. Evaluate $P(A)$ and $Q(A)$ when $A := \begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$.
3. Show that the matrix product $P(A)\cdot Q(A)$ is the matrix polynomial corresponding to the product polynomial $R(x) = P(x)\cdot Q(x)$.
4. True or false: If $B = P(A)$ and $C = Q(A)$, then $BC = CB$. Check your answer in the particular case of Question **2.**.

Exercise 8.

1. Show that if $D := \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ is a 2×2 diagonal matrix with $a \neq b$, then the only matrices that commute (under matrix multiplication) with D are other 2×2 diagonal matrices.
2. What if $a = b$?
3. Find all matrices that commute with $D := \begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}$ where a, b and c are all different.
4. Answer the same question for the case when $a \neq b = c$.
5. Prove that a matrix A commutes with an $n\times n$ diagonal matrix D with all distinct diagonal entries if and only if A is a diagonal matrix.

Exercise 9.

True or false : If A and B are square matrices of the same size, then:

$$A^2 - B^2 = (A - B) \cdot (A + B) \tag{1}$$

Exercise 10.

1. Under what conditions is the square A^2 of a matrix defined?
2. Show that A and A^2 commute.
3. How many matrix multiplications are needed to compute A^n, where n belongs to $\mathbf{N}^*$?

EXERCISES SET # 3
GAUSSIAN ELIMINATION

Exercise 1.

Solve the following linear systems using Gaussian Elimination:

$a)\ \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix};$ $\quad b)\ \begin{pmatrix} 6 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix};$

$c)\ \begin{pmatrix} 2 & 1 & 2 \\ -1 & 3 & 3 \\ 4 & -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix};$ $\quad d)\ \begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -1 \end{pmatrix};$

$e)\ \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \end{pmatrix} \cdot A = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix};$ $\quad f)\ \begin{pmatrix} 2 & -3 & 1 & 1 \\ 1 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} -1 \\ 0 \\ 5 \\ 3 \end{pmatrix},$

where $A := {}^t(a, b, c, d)$ and $X := {}^t(x, y, z, t)$.

Exercise 2.

Solve the following linear systems using Gaussian Elimination.

$(\mathscr{S}_1) \begin{cases} a + 7b = 4 \\ -2a - 9b = 2 \end{cases}$ $\quad (\mathscr{S}_2) \begin{cases} 3z - 5w = -1 \\ 2z + w = 8 \end{cases}$ $\quad (\mathscr{S}_3) \begin{cases} x - 2y + z = 0 \\ 2y - 8z = 8 \\ -4x + 5y + 9z = -9 \end{cases}$

$(\mathscr{S}_4) \begin{cases} p + 4q - 2r = 1 \\ -2p - 3r = -7 \\ 3p - 2q + 2r = -1 \end{cases}$ $\quad (\mathscr{S}_5) \begin{cases} x_1 - 2x_3 = -1 \\ x_2 - x_4 = 2 \\ -3x_2 + 2x_3 = 0 \\ -4x_1 + 7x_4 = -5 \end{cases}$ $\quad (\mathscr{S}_6) \begin{cases} -x + 3y - z + w = -2 \\ x - y + 3z - w = 0 \\ y - z + 4w = 7 \\ 4x - y + z = 5 \end{cases}$

Exercise 3.

One says that a square Matrix A is regular if the Gaussian Elimination algorithm sucessfully reduces to upper triangular form U with. All non-zero pivots on the diagonal.

Which of the following matrices is regular?
Check your answers using Maple.

$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$; $b)\ \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix}$;

$c)\ \begin{pmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & -2 & 5 \end{pmatrix}$; $d)\ \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$; $e)\ \begin{pmatrix} 1 & 3 & -3 & 0 \\ -1 & 0 & -1 & 2 \\ 3 & 3 & -6 & 1 \\ 2 & 3 & -3 & 5 \end{pmatrix}$;

Exercise 4.

1. Write down an example of a system of 5 linear equations in 5 unknowns with regular diagonal coefficient matrix.
2. Solve your system.
3. Explain why solving a system whose coefficient matrix is diagonal is very easy.

Exercise 5.

A linear system is called *homogeneous* if all the right-hand sides are zero, and so takes the matrix form $A \cdot X = 0_{\mathbf{R}^n}$.
Explain why the solution to a homogeneous system with regular coefficient matrix is $X = 0_{\mathbf{R}^n}$.

Exercise 6.

A square matrix is called *strictly lower (upper) triangular* if all entries on or above (below) the main diagonal are 0.

1. Prove that every square matrix can be uniquely written as a sum $A = L + D + U$, with L strictly lower triangular, D diagonal, and U strictly upper triangular.
2. Decompose $A := \begin{pmatrix} 3 & 1 & -1 \\ 1 & -4 & 2 \\ -2 & 0 & 5 \end{pmatrix}$ in this manner.

Exercise 7.

A square matrix N is called nilpotent if $N^k = O_{\mathscr{M}_n(\mathbf{R})}$, for some $k \geqslant 1$.

(a) Show that $N := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.

(b) Show that every strictly upper triangular matrix, as defined in Exercise 144 is nilpotent.

(c) Find a nilpotent matrix which is neither lower nor upper triangular.

EXERCISES SET # 4
MATRIX INVERSES

1 Basic Exercises

Exercise 1.

Verifiy by direct multiplication that the following matrices inverses.

$$a)\ A := \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix},\ A^{-1} := \begin{pmatrix} -1 & -3 \\ 1 & 2 \end{pmatrix};$$

$$b)\ A := \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix},\ A^{-1} := \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix};$$

$$c)\ A := \begin{pmatrix} -1 & 3 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix},\ A^{-1} := \begin{pmatrix} -1 & 1 & 1 \\ 4/7 & -1/7 & -3/7 \\ -6/7 & 5/7 & 8/7 \end{pmatrix};$$

Exercise 2.

Write down the inverse of each of the following matrices

$$a)\quad A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b)\quad A := \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \qquad c)\quad A := \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

$$d)\quad A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad e)\quad A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f)\quad A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$g)\quad A := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{pmatrix}.$$

Exercise 3.

Explain why a matrix with a row of all zeros does not have an inverse.

Exercise 4.

Find all real 2×2 matrices that are their own inverses: *i.e.* $A^{-1} = A$.

Exercise 5.

Show that if a square matrix A of $\mathscr{M}_n(\mathbf{R})$ satisfies the equality: $A^2 - 3A + I_n = 0_{\mathscr{M}_n(\mathbf{R})}$, then $A^{-1} = 3I_n - A$.

Exercise 6.

Show that $A := \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$ is not invertible for any value of the entries.

2 Intermediate Level Exercises

Exercise 7.

Prove that a diagonal matrix $D := \text{diag}(d_1, d_2, \cdots, d_n)$ is invertible if and only if all its diagonal entries are nonzero, in which case $D^{-1} = \text{diag}(1/d_1, 1/d_2, \cdots, 1/d_n)$.

Exercise 8.

Two matrices A and B are said to be *similar*, written $A \sim B$ if there exists an invertible matrix S such that $B = S^{-1}AS$. Prove

(a) $A \sim A$.

(b) If $A \sim B$, then $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Exercise 9.

(a) A Block matrix $D := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is called *block diagonal* if A and B are square matrices, not necessarily of the same size, while the 0's are zero matrices of the appropriate size.

Prove that D has an inverse if and only if both A and B do, and $D^{-1} := \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.

(b) Find the inverse of $C := \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $D := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix}$ by using this method.

3 Other Exercises

Exercise 10.

Let A be the element of $\mathscr{M}_n(\mathbf{R})$ defined by setting: $A := \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$.

Compute A^2 and deduce that A is invertible and give its inverse.

Exercise 11.

Let $A := (a_{i,j})_{1\leqslant i,j\leqslant n}$ be the element of $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$\begin{cases} a_{i,j} := 1, & \text{if } i \neq j, \\ a_{i,i} := 0 \text{ for all } i \text{ in } [\![1, n]\!]. \end{cases}$$

Show that A is invertible and give its inverse.

Exercise 12.

Let n be in $\mathbf{N}^*$, a and b be two scalars and let $A := (a_{i,j})_{1\leqslant i,j\leqslant n}$ be the matrix in $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$\begin{cases} a_{i,j} := a, & \text{if } i \neq j, \\ a_{i,i} := b. \end{cases}$$

Define $J := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathscr{M}_n(\mathbf{R})$ and let I_n be the $n \times n$ matrice identity.

1. Show that $A = \alpha(\frac{J}{n}) + \beta(I_n - \frac{J}{n})$, where $\alpha := b + (n-1)a$ and $\beta := b - a$.

2. Express J^2 in function of J.

3. Show that

$$\left(\frac{J}{n}\right)^2 = \frac{J}{n}, \quad \left(I_n - \frac{J}{n}\right)^2 = I_n - \frac{J}{n}, \quad \frac{J}{n}\left(I_n - \frac{J}{n}\right) = \left(I_n - \frac{J}{n}\right)\frac{J}{n} = 0.$$

4. Show that, for any m in $\mathbf{N}^*$,

$$A^m = \alpha^m \left(\frac{J}{n}\right) + \beta^m \left(I_n - \frac{J}{n}\right). \tag{1}$$

5. Deduce from the previous question that:

$$A^2 = (b-a)^2 I_n + (2ab - 2a^2 + na^2)J. \tag{2}$$

6. Using the equality

$$A = aJ + (b-a)I_n, \tag{3}$$

show that:

$$A \cdot (A - (2b - 2a + na)I_n) = (b-a)(a-b-na)I_n. \tag{4}$$

7. Find a necessary and sufficient condition on a and b such that A is invertible and give its inverse.

Exercise 13.

Let A be a matrix of $\mathscr{M}_p(\mathbf{K})$ and let B be a matrix of $\mathscr{M}_p(\mathbf{K})$ such that:

$$B \cdot A = I_p.$$

Show that A is invertible and that $A^{-1} = B$.

EXERCISES SET # 5
LU FACTORIZATION, TRANSPOSES & SYMMETRIC MATRICES

Exercise 1.

For each of the listed matrices A and vectors $\boldsymbol{b}$ below, find a permuted LU factorization of the matrix, and use your factorization to solve the system $A\boldsymbol{x} = \boldsymbol{b}$.

a) $A := \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $\boldsymbol{b} := \begin{pmatrix} 3 \\ 2 \end{pmatrix}$;

b) $A := \begin{pmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix}$, $\boldsymbol{b} := \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$;

c) $A := \begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, $\boldsymbol{b} := \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$;

d) $A := \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 6 & 2 & -1 \\ 1 & 1 & -7 & 2 \\ 1 & -1 & 2 & 1 \end{pmatrix}$, $\boldsymbol{b} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$;

e) $A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 4 & -1 & 2 \\ 7 & -1 & 2 & 3 \end{pmatrix}$, $\boldsymbol{b} := \begin{pmatrix} -1 \\ -4 \\ 0 \\ 5 \end{pmatrix}$;

Exercise 2.

1. Explain why

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix},$$

are all legitimate permuted LU factorizations of the same matrix. List the elementary row operations that are being used in each case.

2. Use each of the factorizations to solve the linear system

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}.$$

Do you obtain the same result? Explain why or why not.

Exercise 3.

Let p, q and r be three integers and let A be in $\mathscr{M}_{p,q}(\mathbf{R})$ and B be in $\mathscr{M}_{q,r}(\mathbf{R})$. Show the following equality:

$$^t(A \cdot B) = {}^tB \cdot {}^tA.$$

Exercise 4.

A square matrix is called *normal* if it commutes with its transpose. In other words, A is *normal* if the following equality holds:

$$^tA \cdot A = A \cdot {}^tA.$$

Find all normal 2×2 matrices.

Exercise 5.

Let A and B be two elements of $\mathscr{M}_{p,q}(\mathbf{R})$.

(a) Suppose that

$$^t\boldsymbol{v} A \boldsymbol{w} = {}^t\boldsymbol{v} B \boldsymbol{w}$$

for all vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. Prove that $A = B$.

(b) Give an example of two matrices such that:

$$^t\boldsymbol{v} A \boldsymbol{v} = {}^t\boldsymbol{v} B \boldsymbol{v}$$

for all vectors $\boldsymbol{v}$ but $A \neq B$.

Exercise 6.

Find all values of a, b, and c for which the following matrices are symmetric:

$$a) \begin{pmatrix} 3 & a \\ 2a-1 & a-2 \end{pmatrix}, \quad b) \begin{pmatrix} 1 & a & 2 \\ -1 & b & c \\ b & 3 & 0 \end{pmatrix}, \quad c) \begin{pmatrix} 3 & a+2b-2c & -4 \\ 6 & 7 & b-c \\ -a+b+c & 4 & b+3c \end{pmatrix}.$$

Exercise 7.

Find all values of a, b, and c for which the following matrices are symmetric:

$$a)\ \begin{pmatrix} 3 & a \\ 2a-1 & a-2 \end{pmatrix},\quad b)\ \begin{pmatrix} 1 & a & 2 \\ -1 & b & c \\ b & 3 & 0 \end{pmatrix},\quad c)\ \begin{pmatrix} 3 & a+2b-2c & -4 \\ 6 & 7 & b-c \\ -a+b+c & 4 & b+3c \end{pmatrix}.$$

Exercise 8.

Find the LD^tL factorization of the following matrices:

$$a)\ A := \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},\qquad b)\ B := \begin{pmatrix} -2 & 3 \\ 3 & -1 \end{pmatrix},$$

$$c)\ C := \begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix},\qquad d)\ D := \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix},$$

Exercise 9.

Prove that the matrix $A := \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ can not be factorized as $A = LD^tL$.

EXERCISES SET # 6
VECTOR SUBSPACES & BASIS OF A VECTOR SPACES

1 Basic Exercises

Exercise 1.

Are the following family of vectors linearly independent?

- in $\mathbf{R}^2$, $((1,2),(3,5))$;
- in $\mathbf{R}^3$, $((1,2,3),(1,-2,-3),(1,4,3))$;
- in $\mathbf{R}^4$, $((1,2,3,4),(5,6,7,8),(11,14,16,20))$.
- in $\mathbf{R}^4$, $((1,2,3,4),(5,6,7,8),(-4,-8,-12,-16))$.

Exercise 2.

In $\mathbf{R}^4$, let us consider the set of all vectors, denoted (x,y,z,t) such that:

$$x+y+z+t=0.$$

1. Show that $F := \{(x,y,z,t) \in \mathbf{R}^4,\ x+y+z+t=0\}$ is a vector subspace of $\mathbf{R}^4$.
2. Give a set of vectors, linearly independent, which spans F.

Exercise 3.

In $\mathbf{R}^3$, let us consider the set of all vectors, denoted (x,y,z) such that:

$$(\mathscr{S}) : \begin{cases} x+y+z=0 \\ 2x-y+z=0. \end{cases}$$

Show that $F := \{(x,y,z) \in \mathbf{R}^3,\ (\mathscr{S}) \text{ is fulfilled }\}$ is a vector subspace of $\mathbf{R}^3$ and give a set of vectors, linearly independent, which spans F.

Exercise 4.

Let $u := (1, 2, 3)$ and $v := (3, 2, 1)$ be two vectors of $\mathbf{R}^3$.
Give a condition on x, y and z so the vector (x, y, z) belongs to the subspace spanned by u and v (denoted $\operatorname{Span}\{u, v\}$).

Exercise 5.

Define

$$F := \left\{ \begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix}, (\alpha, \beta) \in \mathbf{R}^2 \right\} \quad \text{and} \quad G := \left\{ \begin{pmatrix} 0 & \gamma-\delta \\ \gamma+\delta & 0 \end{pmatrix}, (\gamma, \delta) \in \mathbf{R}^2 \right\}.$$

Show that F and G below are two supplementary vector subspaces of of $\mathscr{M}_2(\mathbf{R})$.

2 Intermediate Level Exercises

Exercise 6.

Let F and G be two vector subspaces of a vector space E.
Show that $F \cup G$ is a vector subspace de E if and only if $F \subset G$ or $G \subset F$.

Exercise 7.

Let n be in $\mathbb{N}^*$ and $a_1, a_2, \cdots a_n$ n real numbers such that $a_1 < a_2 < \cdots < a_n$. Show that the family $(f_1, f_2, \cdots, f_n)$, where:

$$\begin{array}{rccl} f_k : & \mathbf{R} & \to & \mathbf{R} \\ & x & \mapsto & e^{a_k x}, \ \forall k \in [\![1, n]\!], \end{array}$$

is linearly independent in $\mathscr{F}(\mathbf{R}, \mathbf{R})$.

Exercise 8.

Let $\mathscr{P} := (P_1, P_2, \cdots, P_n)$ be a family of n polynomials on $\mathbf{R}[X]$ such that:

$$\deg(P_1) < \deg(P_2) < \cdots < \deg(P_n).$$

Show that the vectors of the family of $\mathscr{P}$ are linearly independent. We recall that, by convention, the degree of the null polynomial equals 0. In other words $\deg(0_{\mathbf{R}[X]}) = -\infty$.

Exercise 9.

Let u, v and w be three vectors of a vector space E.

1. Show that

$$\operatorname{Span}\{u, v\} = \operatorname{Span}\{u, w\}$$

if and only if
$$\exists(\alpha,\beta,\gamma)\in\mathbf{R}^3,\ \beta\gamma\neq 0,\ \alpha u+\beta v+\gamma w=0.$$

2. Let F be a subspace of a vector space E. For any vector v of E, define
$$F+\mathbf{R}v := \{f+\gamma v,\ (f,\gamma)\in F\times\mathbf{R}\}.$$

3. Show that $F+\mathbf{R}v$ is a subspace of E.

4. Show that:
$$F+\mathbf{R}v = F+\mathbf{R}w$$
if and only if
$$\exists u\in F,\ \exists(\alpha,\beta)\in\mathbf{R}^2,\ \alpha\beta\neq 0,\ u+\alpha v+\beta w=0.$$

Exercise 10.

Let F, G and H be three subspaces of a vector space E.
Make a comparison between $F\cup(G+H)$ and $(F\cup G)+(F\cup H)$.

Exercise 11.

Denote $E := \mathscr{F}(\mathbf{R},\mathbf{R})$ the set of functions from $\mathbf{R}$ to $\mathbf{R}$. Among the follwoing subsets of $(\mathscr{F}(\mathbf{R},\mathbf{R}),+,\cdot)$, which ones are vector subspaces of E.

1. The set of functions f, for which $f(0)=0$.
2. The set of functions f, for which $f(0)=1$.
3. The set of monotonic functions.
4. The set of differentiable functions.

3 Other Exercises

Exercise 12.

Let n be in $\mathbb{N}$. Show that the family $(f_1, f_2, \cdots, f_n)$, where:
$$\begin{array}{cccl} f_k : & \mathbf{R} & \to & \mathbf{R} \\ & x & \mapsto & \sin(kx),\ \forall k\in[\![1,n]\!], \end{array}$$
is linearly independent in the set, denoted $\mathscr{F}(\mathbf{R},\mathbf{R})$, of functions from $\mathbf{R}$ to $\mathbf{R}$.

Exercise 13.

In $\mathbf{R}^3$, let us consider the set:
$$F := \{(x,y,z)\in\mathbf{R}^3,\ x^2+y^2+2z^2-2xy-2yz+2xz=0\}.$$

Is F a vector subspace of $\mathbf{R}^3$?

EXERCISES SET # 7
FINITE-DIMENSIONAL VECTOR SPACES & LINEAR MAPS

1 Rank Nullity Theorem

Exercise 1.

Let E be a $\mathbf{K}$ vector space with finite dimension, and let f be in $\mathscr{L}(E)$. Show the following equivalence

$$E = \mathrm{Im}(f) \oplus \mathrm{Ker}(f) \iff \mathrm{Im}(f) = \mathrm{Im}(f^2).$$

Is this equivalence true with spaces of infinite dimension?

2 Linear maps in finite-dimensional Spaces

Exercise 2.

Show that the linear maps from $\mathbf{R}$ to $\mathbf{R}$ are the maps $x \mapsto kx$, with k in $\mathbf{R}$.

Exercise 3.

Let $u : \mathbf{R}^3 \to \mathbf{R}^2$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ by setting:

$$u(\boldsymbol{x}) := (x_1 + x_2 + x_3, 2x_1 + x_2 - x_3).$$

1. Show that u is a linear map.
2. Determine $\mathrm{Ker}(u)$.
3. Determine $\mathrm{Im}(u)$.
4. Give the dimension of both $\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$.Is their some conform to what one can expect from the Rank Nullity Theorem?
5. (bonus question) Give a matrix A in $\mathscr{M}_{2,3}(\mathbf{R})$ such that $u(\boldsymbol{x}) = A\boldsymbol{x}$.

Exercise 4.

Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be defined, for all (x, y, z) in $\mathbf{R}^3$, by setting:

$$f(x, y, z) := (x + y + z, -x + 2y + 2z).$$

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and $\mathscr{B}' := (\boldsymbol{e}'_1, , \boldsymbol{e}'_2)$ the standard basis of $\mathbf{R}^2$.

1. Show that f is a linear map.
2. Give a basis of $\mathrm{Ker}(f)$ and deduce [a] $\dim(\mathrm{Im}(f))$.
3. Give a basis of $\mathrm{Im}(f)$.
4. Give a matrix A such that $f(x, y, z) = A \cdot (x, y, z)$.
5. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A. Make a comparison with $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

[a] One may admit the fundamental result which states that $\dim(E) = \dim(\mathrm{Im}(u)) + \dim(\mathrm{Ker}(u))$, for any linear map $u : E \to F$, where the vector spaces E and F both fulfill $\dim(E) < \infty$ and $\dim(F) < \infty$.

Exercise 5.

Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be defined, for all (x, y) in $\mathbf{R}^2$, by setting:

$$h(x, y) := (x - y, -3x + 3y).$$

1. Show that h is a linear map.
2. Show that h is not injective nor surjective.
3. Give a basis of $\mathrm{Ker}(h)$ and of $\mathrm{Im}(h)$.
4. Give a matrix A such that $h(x, y) = A \cdot (x, y)$.
5. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.

Exercise 6. (Practice Final Exam)

Define $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. Denote $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{x}) := A \cdot \boldsymbol{x}.$$

1. Compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.
2. Determine the coordinates of $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ in the standard basis.
3. By only studying Matrix A, determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

4. Give a basis of both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

Exercise 7.

Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(x_1, x_2, x_3) := (x_1 - x_3, 2x_1 + x_2 - 3x_3, -x_2 + 2x_3).$$

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$.

1. Give a basis of $\mathrm{Ker}(f)$ and give $\dim(\mathrm{Im}(f))$ without making $\mathrm{Im}(f)$ explicit.
2. Compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.
3. Determine the coordinates of $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ in the standard basis.
4. Give a basis of $\mathrm{Im}(f)$.
5. Give a matrix A such that $f(\boldsymbol{x}) = A \cdot \boldsymbol{x}$.
6. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.

Exercise 8.

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map defined, for all $\boldsymbol{u} := (x, y, z)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{u}) := (6x - 4y - 4z, 5x - 3y - 4z, x - y).$$

1. Show that there is non null vector $\boldsymbol{a}$ in $\mathbf{R}^3$, such that $\mathrm{Ker}(f) = \mathrm{Span}\{\boldsymbol{a}\}$. Give explicitely such a vector $\boldsymbol{a}$.
2. Define $\boldsymbol{b} := \boldsymbol{e}_1 + \boldsymbol{e}_2$ and $\boldsymbol{c} := \boldsymbol{e}_2 - \boldsymbol{e}_3$.
   (a) Compute $f(\boldsymbol{b})$ and $f(\boldsymbol{c})$.
   (b) Deduce that $(\boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathrm{Im}(f)$. One can you an another method to do so.
3. Give one or several equations that characterize $\mathrm{Im}(f)$.
4. Do we have the equality
$$\mathrm{Im}(f) \bigoplus \mathrm{Ker}(f) = \mathbf{R}^3?$$
5. Give a matrix A such that $f(\boldsymbol{u}) = A \cdot \boldsymbol{u}$.
6. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.
7. Do we have the equality
$$N(A) \bigoplus C(A) = \mathbf{R}^3?$$

Exercise 9. (Practice Final Exam)

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map such that:

$$\begin{cases} f(\boldsymbol{e}_1) := \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_2) := \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_3) := \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3. \end{cases}$$

Define $E_{-1} := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = -\boldsymbol{u}\}$ and $E_1 := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = \boldsymbol{u}\}$.

1. Show that E_{-1} and E_1 are vector subspaces of $\mathbf{R}^3$.
2. Prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1} and that $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.
3. What can we deduce, from the previous question, about the dimension of E_{-1} and E_1?
4. Determine $E_{-1} \cap E_1$.
5. Does the equality $E_{-1} \bigoplus E_1 = \mathbf{R}^3$ hold?
6. Compute $f^2 = f \circ f$. Deduce that f is bijective and determine f^{-1}.

Exercise 10.

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let u be an endomorphism of $\mathbf{R}^3$ defined by setting:

$$\begin{cases} u(\boldsymbol{e}_1) := 2\boldsymbol{e}_1 + \boldsymbol{e}_2 + 3\boldsymbol{e}_3 \\ u(\boldsymbol{e}_2) := \boldsymbol{e}_2 - 3\boldsymbol{e}_3 \\ u(\boldsymbol{e}_3) := -2\boldsymbol{e}_2 + 2\boldsymbol{e}_3. \end{cases}$$

1. Let $\boldsymbol{x} := (x_1, x_2, x_3)$ be a vector in $\mathbf{R}^3$. Compute $u(\boldsymbol{x})$.
2. Give a matrix A such that $u(\boldsymbol{x}) = A \cdot \boldsymbol{x}$.
3. Define $E := \{\boldsymbol{x} \in \mathbf{R}^3,\ u(\boldsymbol{x}) = 2\boldsymbol{x}\}$ and $F := \{\boldsymbol{x} \in \mathbf{R}^3,\ u(\boldsymbol{x}) = -\boldsymbol{x}\}$.
   Show that E and F are vector subspaces of $\mathbf{R}^3$.
4. Give a basis of E and a basis of F.
5. Does the equality $E \bigoplus F = \mathbf{R}^3$ holds?

Intermediate-level Exercises

Exercise 11.

Let n be in $\mathbf{N}^*$ and let M be the matrix defined by setting:

$$M := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & C_1^1 & C_2^1 & \cdots & C_n^1 \\ \vdots & \ddots & C_2^2 & \cdots & C_n^2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & C_n^n \end{pmatrix} \in \mathscr{M}_{n+1}(\mathbf{R}),$$

where C_n^k is the binomial coefficient (*i.e.* $C_n^k := \frac{n!}{k!(n-k)!}$).
Show that M is invertible and compute its inverse, denoted M^{-1}.

Exercise 12.

Let M be the matrix of $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$M := \begin{pmatrix} 0 & m_{1,2} & \cdots & m_{1,n} \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & m_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

1. Without using the Theorem of Cayley-Hamilton, show that M is nilpotent.
2. Let $M := \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ be in $\mathscr{M}_3(\mathbf{R})$. Compute, for all p in $\mathbf{N}^*$, M^p.

Exercise 13.

Let E be a $\mathbf{K}$ vector space with finite dimension. Let f be in $\mathscr{L}(E)$. Assume that there exists x_0 in E such that $\mathscr{B} = (f(x_0), f^2(x_0), \cdots, f^n(x_0))$ is a basis of E.

1. Show that f is bijective.
2. Without using Cayley-Hamilton's Theorem, show that there exists $(a_0, a_1, \cdots, a_{n-1}) \in \mathbf{K}^n$ such that
$$f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 \operatorname{Id}_E = 0.$$

Exercise 14.

The goal of this exercise is to determine all the matrices in $\mathscr{M}_n(\mathbf{R})$ such that

$$A^2 = 0_{\mathscr{M}_n(\mathbf{R})}. \tag{1}$$

1. Denote $f : \mathbf{R}^n \to \mathbf{R}^n$ the linear map, the matrix representation of which is A, in the standard basis.

2. Show that $\mathrm{Im}(f) \subset \ker(f)$.

3. Denote r the dimension of $\mathrm{Im}(f)$ and let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r)$ be a basis of $\mathrm{Im}(f)$. Using the Rank Nullity theorem, show that $n - r \geqslant r$ and that there exists a family of vectors, denoted $(\boldsymbol{e}_{r+1}, \boldsymbol{e}_{r+2}, \cdots, \boldsymbol{e}_{n-r})$, such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{n-r})$ is a basis of $\ker(f)$.

4. Denote $(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r)$ the vectors of $\mathbf{R}^n$ such that:

$$\boldsymbol{e}_i = f(\boldsymbol{u}_i), \ \forall i \in [\![1, r]\!].$$

5. Show that $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n)$ is a basis of $\mathbf{R}^n$.

6. Give the matrix which represents f from $\mathscr{B}$ to $\mathscr{B}$.

7. Finally determine all the matrices in $\mathscr{M}_n(\mathbf{R})$ such that (1) holds.

Exercise 15.

Define $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. Denote $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{x}) := A \cdot \boldsymbol{x}.$$

1. Compute $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.

2. Determine the coordinates of $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ in the standard basis.

3. By only studying Matrix A, determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

4. Give a basis of both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

Exercise 16.

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map such that:

$$\begin{cases} f(\boldsymbol{e}_1) := \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_2) := \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_3) := \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3. \end{cases}$$

Define $E_{-1} := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = -\boldsymbol{u}\}$ and $E_1 := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = \boldsymbol{u}\}$.

1. Show that E_{-1} and E_1 are vector subspaces of $\mathbf{R}^3$.

2. Prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1} and that $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.

3. What can we deduce, from the previous question, about the dimension of E_{-1} and E_1?

4. Determine $E_{-1} \cap E_1$.

5. Does the equality $E_{-1} \bigoplus E_1 = \mathbf{R}^3$ hold?

6. Compute $f^2 = f \circ f$. Deduce that f is bijective and determine f^{-1}.

Exercises Set # 8
Determinants

1 Basic Exercises

Exercise 1.

Compute all the determinants below:

1.
   - By cofactor expansion along the first column
   - By cofactor expansion along the second row.
   - By Gaussian Elimination
2. Which method is the fastest?

$$\delta_1 := \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix}, \quad \delta_2 := \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{vmatrix}, \quad \delta_3 := \begin{vmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & -2 & -2 \end{vmatrix}, \quad \delta_4 := \begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix},$$

$$\delta_5 := \begin{vmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix}, \quad \delta_6 := \begin{vmatrix} -1 & -2 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix}, \quad \delta_7 := \begin{vmatrix} 0 & -2 & 2 \\ -1 & -1 & 0 \\ 2 & 2 & 2 \end{vmatrix}, \quad \delta_8 := \begin{vmatrix} 1 & -2 & -1 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{vmatrix}.$$

Exercise 2.

Compute all the determinants below:

1.
   - By cofactor expansion along the first column
   - By cofactor expansion along the second row.
   - By Gaussian Elimination
2. Which method is the fastest?

$$\delta_1' := \begin{vmatrix} -1 & -1 & -2 & -1 \\ -2 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 2 & 0 \end{vmatrix}, \quad \delta_2' := \begin{vmatrix} 2 & 2 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ -2 & 2 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{vmatrix}, \quad \delta_3' := \begin{vmatrix} 2 & 0 & -1 & 2 \\ 2 & -1 & 1 & -1 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & -2 & 2 \end{vmatrix},$$

$$\delta_4' := \begin{vmatrix} -2 & -1 & 0 & 2 \\ 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & -1 \end{vmatrix}, \quad \delta_5' := \begin{vmatrix} -2 & -1 & -2 & 1 \\ 2 & 2 & -1 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{vmatrix}, \quad \delta_6' := \begin{vmatrix} -2 & -1 & 0 & 2 \\ -1 & 1 & -1 & 0 \\ 2 & 2 & 2 & -1 \\ -1 & 2 & 2 & -2 \end{vmatrix}.$$

Exercise 3.

Define, for any matrice M, the Characteristic polynomial of M, denoted χ_M, by setting

$$\chi_M(X) := \mathbf{det}(M - XI_3).$$

1. Determine the polynomial characteristic of all matrices $(M_i)_{i\in[\![1,3]\!]}$ given below.

$$M_1 := \begin{pmatrix} 2 & 2 & -3 \\ 5 & 1 & -5 \\ -3 & 4 & 0 \end{pmatrix}; \quad M_2 := \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}; \quad M_3 := \begin{pmatrix} 1 & 4 & -2 \\ 0 & 6 & -3 \\ -1 & 4 & 0 \end{pmatrix}$$

2. Find the roots of these characteristic polynomials.
3. For any root λ of χ_M, denote $E_\lambda := \mathrm{Ker}(M - \lambda I_3)$. Determine E_λ for any root λ of χ_{M_1}.
4. Determine E_λ for any root λ_i of the family $(\chi_{M_i})_{i\in[\![1,3]\!]}$.

Exercise 4.

1. Give an example of a non-diagonal 2×2 matrix for which $A^2 = I_2$.
2. In general, if $A^2 = I_2$, show that $\mathbf{det}(A) \in \{-1, 1\}$
3. If $A^2 = A$, what can you say about $\mathbf{det}(A)$?

Exercise 5.

Two matrices A and B in $\mathscr{M}_n(\mathbf{R})$ are said to be similar, in $\mathbf{R}$, if there exists S in $\mathbf{GL}_n(\mathbf{R})$ such that $B = S^{-1}AS$.
Show that all similar matrices have the same determinant.

Exercise 6.

1. Show that if $D := \begin{pmatrix} A & 0_{\mathscr{M}_n(\mathbf{R})} \\ 0_{\mathscr{M}_n(\mathbf{R})} & B \end{pmatrix}$ is block diagonal matrix, where A and B are square matrices (which both belong to $\mathscr{M}_n(\mathbf{R})$), then

$$\mathbf{det}(A\cdot B) = \mathbf{det}(A)\cdot\mathbf{det}(B). \tag{1}$$

2. Prove that the same holds for a block upper triangular matrix *i.e.* that Equality

$$\mathbf{det}\begin{pmatrix} A & C \\ 0_{\mathscr{M}_n(\mathbf{R})} & B \end{pmatrix} = \mathbf{det}(A)\cdot\mathbf{det}(B). \tag{2}$$

holds.

Exercise 7.

Let A, B, C and D be elements of $\mathscr{M}_n(\mathbf{R})$ such that $DC = CD$.
If D is invertible, show that

$$\mathbf{det}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbf{det}(AD - BC) \tag{1}$$

In order to so, we might try to find matrices, denoted A', B', C', D', E', in $\mathscr{M}_n(\mathbf{R})$ such that:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AD - BC & B'' \\ O_{\mathscr{M}_n(\mathbf{R})} & I_n \end{pmatrix}$$

and such that the determinant of $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ is easily computable.

Exercise 8.

Use the method seen in Exercise 203 to compute the determinant of the following matrices:

$$M_1 := \begin{pmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \\ 0 & 3 & 7 \end{pmatrix}; \qquad M_2 := \begin{pmatrix} 1 & 2 & -2 & 5 \\ -3 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix};$$

$$M_3 := \begin{pmatrix} 1 & 2 & 0 & 4 \\ -3 & 1 & 4 & -1 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}; \qquad M_4 := \begin{pmatrix} 5 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 2 & 4 & 4 & -2 \\ 3 & -2 & 9 & -5 \end{pmatrix}.$$

Exercise 9.

Exercise 10.

Exercise 11.

Exercise 12.

Exercise 13.

EXERCISES SET # 9
REDUCTION OF ENDOMORPHISMS

1 Basic Exercises

Change of Basis

Exercise 1.

Let $u : \mathbf{R}^p \to \mathbf{R}^q$, be a linear map. Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_p)$ be the standard basis of $\mathbf{R}^p$ and $\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_q)$ be the standard basis of $\mathbf{R}^q$.

1. **Case where $p = 3$ and $q = 2$:**
   Define u by setting:
$$\begin{cases} u(\boldsymbol{e}_1) := \boldsymbol{f}_1 + 2\boldsymbol{f}_2 \\ u(\boldsymbol{e}_2) := \boldsymbol{f}_1 - \boldsymbol{f}_2 \\ u(\boldsymbol{e}_3) := -\boldsymbol{f}_1 + \boldsymbol{f}_2. \end{cases}$$
   (i) Determine the image of any vector $\boldsymbol{x} = (x_1, x_2, x_3)$ by u.
   (ii) Determine the matrix of the linear map u, from basis $\boldsymbol{e}$ to basis $\boldsymbol{f}$.
   (iii) Determine $\mathrm{Ker}(u)$.

2. **Case where $p = 3$ and $q = 3$:**
   In this question $\boldsymbol{e} = \boldsymbol{f}$. Define u by setting:
$$\begin{cases} u(\boldsymbol{e}_1) := 3\boldsymbol{f}_1 + 2\boldsymbol{f}_2 + 2\boldsymbol{f}_3 \\ u(\boldsymbol{e}_2) := 2\boldsymbol{f}_1 + 3\boldsymbol{f}_2 + 2\boldsymbol{f}_3 \\ u(\boldsymbol{e}_3) := 2\boldsymbol{f}_1 + 2\boldsymbol{f}_2 + 3\boldsymbol{f}_3. \end{cases}$$
   (i) Determine the image of any vector $\boldsymbol{x} = (x_1, x_2, x_3)$ by u.
   (ii) Determine the matrix of the linear map u, from basis $\boldsymbol{e}$ to basis $\boldsymbol{e}$.
   (iii) Determine $\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$.

Exercise 2.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$, the representative matrix of which, in the standard basis, is

$$A := \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$$

1. Show that $E_1 := \{x \in \mathbf{R}^3,\ u(\boldsymbol{x}) = \boldsymbol{x}\}$ is a vector subspace of $\mathbf{R}^3$ and give a basis of it, denoted $\boldsymbol{a}$.
2. Define $\boldsymbol{b} := (0, 1, 1)$ and $\boldsymbol{c} := (1, 1, 2)$ two vectors of $\mathbf{R}^3$. Compute $u(\boldsymbol{b})$ and $u(\boldsymbol{c})$.
3. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.
4. Determine the matrix, denoted P, representing a change of basis from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$.
5. Compute P^{-1}.
6. Determine the matrix D of u, in the basis $\boldsymbol{\beta}'$.
7. Give the relation that exists between A, P and D.
8. Compute A^n, for every integer n in $\mathbf{N}$.

Exercise 3.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$ defined by:

$$u(x_1, x_2, x_3) := (x_2 - 2x_3, 2x_1 - x_2 + 4x_3, x_1 - x_2 + 3x_3).$$

1. Determine the matrix A of u in the standard basis.
2. Determine a basis $(\boldsymbol{a}, \boldsymbol{b})$ of $\mathrm{Ker}(u - Id_{\mathbf{R}^3})$.
3. Give a vector $\boldsymbol{c}$ such that $\mathrm{Ker}(u) = \mathrm{Span}\{\boldsymbol{c}\}$.
4. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.
5. Determine the matrix D of u in the basis $\boldsymbol{\beta}'$. In other words, $\mathrm{Mat}_{\boldsymbol{\beta}', \boldsymbol{\beta}'}(u) = D$.
6. Show that $\mathrm{Im}(u) = \mathrm{Ker}(u - Id_{\mathbf{R}^3})$.
7. Show that $\mathrm{Ker}(u) \bigoplus \mathrm{Im}(u) = \mathbf{R}^3$ holds.
8. Compute A^n, for every integer n in $\mathbf{N}$.

Exercise 4.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let f be the endomorphism of $\mathbf{R}^3$ defined by:

$$\begin{cases} f(\boldsymbol{e}_1) := 2\boldsymbol{\beta}_2 + 3\boldsymbol{\beta}_3 \\ f(\boldsymbol{e}_2) := 2\boldsymbol{\beta}_1 - 5\boldsymbol{\beta}_2 - 8\boldsymbol{\beta}_3 \\ f(\boldsymbol{e}_3) := -\boldsymbol{\beta}_1 + 4\boldsymbol{\beta}_2 + 6\boldsymbol{\beta}_3. \end{cases}$$

Denote $f^2 := f \circ f$.

1. Determine the matrix of f in basis $\boldsymbol{\beta}$ (*i.e.* $\mathrm{Mat}_{\boldsymbol{\beta},\boldsymbol{\beta}}(f)$).
2. Show that $E_1 := \mathrm{Ker}(f - Id_{\mathbf{R}^3})$ and $N_{-1} := \mathrm{Ker}(f^2 + Id_{\mathbf{R}^3})$ are vector subspaces of $\mathbf{R}^3$.
3. Determine two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $E_1 = \mathrm{Span}\{\boldsymbol{a}\}$ and $N_{-1} = \mathrm{Span}\{\boldsymbol{b}, \boldsymbol{f}(b)\}$. Does the equality $E_1 \bigoplus N_{-1} = \mathbf{R}^3$ hold?
4. Define $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, f(\boldsymbol{b}))$. Show that $\boldsymbol{\beta}'$ is a basis of $\mathbf{R}^3$.
5. Give $\mathrm{Mat}_{\boldsymbol{\beta}',\boldsymbol{\beta}'}(f)$.
6. Give $\mathrm{Mat}_{\boldsymbol{\beta}',\boldsymbol{\beta}'}(f^2)$.

Exercise 5.

For each of the following matrices determine the eigenvalues and precise if they are diagonalizable?

$$A_1 := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad A_2 := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}; \quad A_3 := \begin{pmatrix} -1 & -4 & -4 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix};$$

$$A_4 := \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \quad A_5 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}; \quad A_6 := \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & -4 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix};$$

Exercise 6.

Let A be in $\mathscr{M}_n(\mathbf{R})$. Prove that if A is a diagonalizable matrix, then so is $cA + dI_n$, for any real numbers c and d.

Exercise 7.

1. Prove that if A is diagonalizable then so is A^2.
2. Give an example of a non diagonalizable matrix A such that A^2 is diagonalizable.

Exercise 8.

Let A be in $\mathscr{M}_n(\mathbf{R})$. Prove that if A is a diagonalizable then so is every similar matrix $B := S^{-1}AS$, where S belongs to $\mathbf{GL}_n(\mathbf{R})$.

Exercise 9.

Let u be the endomorphism of $\mathbf{R}^4$, which representative matrix in the standard basis $\boldsymbol{\$} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ of $\mathbf{R}^4$ is:

$$A := \begin{pmatrix} -7 & 6 & 6 & 6 \\ 0 & 2 & 0 & 0 \\ -3 & 3 & 2 & 3 \\ -6 & 3 & 6 & 5 \end{pmatrix}$$

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ be the four vectors defined by setting:

$$\boldsymbol{a} := -2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_4; \quad \boldsymbol{b} := \boldsymbol{e}_2 - \boldsymbol{e}_4; \quad \boldsymbol{c} := 2\boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_4; \quad \boldsymbol{d} := 3\boldsymbol{e}_1 + \boldsymbol{e}_3 + 2\boldsymbol{e}_4.$$

1. Show that $\boldsymbol{e}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is a basis of $\mathbf{R}^4$.
2. Compute $u(\boldsymbol{a}), u(\boldsymbol{b}), u(\boldsymbol{c})$ and $u(\boldsymbol{d})$ and write them in the basis $\boldsymbol{e}'$.
3. Deduce the matrix D of u in the basis $\boldsymbol{e}'$.
4. Determine the matrix, denoted P, representing the change of basis from $\boldsymbol{e}$ to $\boldsymbol{e}'$.
5. Compute P^{-1}.
6. Compute $P^{-1}AP$.

Exercise 10.

Let u be the endomorphism of $\mathbf{R}^4$, which representative matrix in the standard basis $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4)$ is:

$$A := \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Define $\boldsymbol{a} := \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$, $\boldsymbol{b} := \boldsymbol{e}_1$, $\boldsymbol{c} := u(\boldsymbol{b})$ and $\boldsymbol{d} := u^2(\boldsymbol{b})$.

1. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is a basis of $\mathbf{R}^4$.
2. Determine the matrix, denoted P, representing a change of basis from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$. Compute P^{-1}
3. Compute $u(\boldsymbol{a}), u(\boldsymbol{b}), u(\boldsymbol{c})$ and $u(\boldsymbol{d})$ and write them in the basis $\boldsymbol{\beta}'$.
4. Compute N^4 and deduce A^4.
5. Determine a basis of $\mathrm{Ker}(u)$.
6. Determine a basis of $\mathrm{Im}(u)$.

Exercise 11.

Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ be the standard basis of $\mathbf{R}^4$. Let u be the endomorphism of $\mathbf{R}^4$, which representative matrix in the standard basis $\boldsymbol{e}$ is given by:

$$A := \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 0 & -2 \end{pmatrix}$$

1. Determine a vector $\boldsymbol{a}$ which spans the kernel of u.
2. Let λ be a real number. Show that $E_\lambda := \{\boldsymbol{x} \in \mathbf{R}^4,\ u(\boldsymbol{x}) = \lambda \boldsymbol{x}\}$ is a vector subspace of $\mathbf{R}^4$.
3. Find a vector $\boldsymbol{b}$ such that $E_{-1} = \mathrm{Span}\{\boldsymbol{b}\}$
4. Determine a basis $(\boldsymbol{c}, \boldsymbol{d})$ of E_1.
5. Show that $\boldsymbol{e}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is a basis of $\mathbf{R}^4$.
6. Determine the matrix D of u in the basis $\boldsymbol{e}'$. In other words, give $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u)$.

Exercise 12.

Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$, which representative matrix in the standard basis is given by:

$$A := \begin{pmatrix} -10 & -3 & -12 \\ 5 & 0 & 7 \\ 6 & 2 & 7 \end{pmatrix}$$

1. Determine all the λ in $\mathbf{R}$ such that $A - \lambda I_3$ is not invertible. Then determine $\mathrm{Ker}(A - \lambda I_3)$.
2. Let $\boldsymbol{a} := (-3, 1, 2)$, compute $u(\boldsymbol{a})$.
3. Determine $\boldsymbol{b}$ in $\mathbf{R}^3$ such that $u(\boldsymbol{b}) = \boldsymbol{a} - \boldsymbol{b}$, and then $\boldsymbol{c}$ in $\mathbf{R}^3$ such that $u(\boldsymbol{c}) = b - c$.
4. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.
5. Determine $T := \mathrm{Mat}_{\boldsymbol{\beta}',\boldsymbol{\beta}'}(u)$.
6. Show that $(T + I_3)^3 = \mathbf{0}_{\mathscr{M}_3(\mathbf{R})}$. Then deduce $(A + I_3)^3$.
7. Determine A^{-1} in function of A^2, A and I_3.

Problem 1.

Let m be a real number and let A_m be the element of $\mathscr{M}_3(\mathbf{R})$ defined by:

$$A_m := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1+m & m \\ 1 & -1 & 1 \end{pmatrix}$$

Part I: Solving the general regular case: $(\mathscr{S}_m)$, for m in $\mathbf{R}$.

1. Compute the determinant of A_2.
2. Is A_2 invertible?
3. Compute the determinant of A_m.
4. For which values of m is A_m invertible? (**One will not try to compute the inverse of** A_m)?
5. Prove that
$$A_m^{-1} := \begin{pmatrix} \frac{1+2m}{2m} & -\frac{1}{m} & \frac{-1}{2m} \\ 1/2 & 0 & -1/2 \\ \frac{-(1+m)}{2m} & \frac{1}{m} & \frac{1+m}{2m} \end{pmatrix}$$
whenever A_m is invertible.
6. Solve the linear system
$$(\mathscr{S}_m) \begin{cases} x+y+z=1 \\ (1+m)y+mz=2 \\ x-y+z=3. \end{cases}$$
for all real m. We might want to treat separately the cases $m=0$ and $m \neq 0$.

Part II: Diagonalizing (A_m), for m in $\mathbf{R}$.

7. Denote χ_{A_m} the Characteristic polynomial of A_m. Show that
$$\chi_{A_m}(X) = -X^3 + (3+m)X^2 + (-2-3m)X + 2m.$$
8. Prove that $1, 2$ and m are roots of χ_{A_m}.
9. Deduce that
$$\chi_{A_m}(X) = -(X-1)\cdot(X-2)\cdot(X-m). \tag{1}$$

Diagonalizing A_m, for m in $\mathbf{R}\backslash\{1,2\}$.

Denote $\boldsymbol{e} := (e_1, e_2, e_3)$ the standard basis of $\mathbf{R}^3$ and define $u_m : \mathbf{R}^3 \to \mathbf{R}^3$ the endomorphism, the matrix representation of which is A_m (in the standard basis). In other words,

$$\mathrm{Mat}_{\boldsymbol{e},\boldsymbol{e}}(u_m) = A_m. \tag{2}$$

First case: Diagonalizing A_0.

We here assume that $m=0$.

10. Define $\boldsymbol{f}_1 := (1, 1, -1)$, $\boldsymbol{f}_2 := (1, 0, 1)$ and $\boldsymbol{f}_3 := (-1, 0, 1)$ and define $\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Using determinant, show that the vectors of $\boldsymbol{f}$ are linearly independent.

11. Show that $\boldsymbol{f}$ is a basis of $\mathbf{R}^3$.

12. Denote
$$Q := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$
Compute the inverse, denoted Q^{-1}, of Q.

13. Give the matrix representation of u_0 in the basis $\boldsymbol{f}$. Denote it $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

14. Define $D' := Q^{-1} A_0 Q$. Compute D'.

15. Make a comparison between D' and $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

16. Compute D'^4 and deduce the fourth power of A_0, denoted A_0^4.

17. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0))^4$.

Second Case: Diagonalizing A_m, with m in $\mathbf{R} \backslash \{0, 1, 2\}$.

We here assume that m belongs to $\mathbf{R} \backslash \{0, 1, 2\}$.

18. Define $\boldsymbol{e}'_1 := (1, 1, -1)$, $\boldsymbol{e}'_2 := (\frac{1}{m}, 1, \frac{1-m}{m})$ and $\boldsymbol{e}'_3 := (-1, -m, 1)$ and define $\boldsymbol{e}' := (\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$. Show that the vectors of $\boldsymbol{e}'$ are linearly independent.

19. Show that $\boldsymbol{e}'$ is a basis of $\mathbf{R}^3$.

20. Denote
$$P_m := \begin{pmatrix} 1 & m^{-1} & -1 \\ 1 & 1 & -m \\ -1 & \frac{1-m}{m} & 1 \end{pmatrix}$$
Show that P_m is invertible and that its inverse, denoted P_m^{-1}, verifies the following equality:
$$P_m^{-1} = \begin{pmatrix} \frac{m}{-1+m} & -(-1+m)^{-1} & 0 \\ -\frac{m}{m-2} & 0 & -\frac{m}{m-2} \\ -\left(m^2 - 3m + 2\right)^{-1} & -(-1+m)^{-1} & -(m-2)^{-1} \end{pmatrix}. \tag{3}$$

21. Give the matrix representation of u_m in the basis $\boldsymbol{e}'$. Denote it $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

22. Define $D_m := P_m^{-1} A_m P_m$. Compute D_m.

23. Make a comparison between D_m and $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

24. Compute $(D_m)^4$ and deduce the fourth power of A_m, denoted A_m^4.

25. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m))^4$.

EXERCISES SET # 10
EUCLIDEAN SPACES

To be filled

Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4.

Exercise 5.

Exercise 6.

Exercise 7.

Exercise 8.

Exercise 9.

Exercise 10.

Exercises Set # 11
Isometries

To be filled

Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4.

Exercise 5.

Exercise 6.

Exercise 7.

Exercise 8.

Exercise 9.

Exercise 10.

Exercise 11.

Part II

Correction

CORRECTION OF EXERCISES SET # 1
VECTORS & SCALAR PRODUCT

Exercise 1.

In the xy plane mark all nine of these linear combinations:

$$c \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + d \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } c \text{ in } \{0,1,2\} \text{ and } d \text{ in } \{0,1,2\}.$$

Solution of Exercise 1.

All we have to do is drawing the following points:

$A_1(0,0)$; $A_2(0,1)$; $A_3(0,2)$; $A_4(2,1)$; $A_5(2,2)$; $A_6(2,3)$;
$A_7(4,2)$; $A_8(4,3)$; $A_9(4,4)$.

Exercise 2.

If three corners of a parallelogram are $(1,1)$, $(4,2)$ and $(1,3)$, what are all three of the possible fourth corners?

Solution of Exercise 2.

Let us call $A(1,3)$, $B(1,1)$ and $C(4,2)$. Let $D(x,y)$ be a point of the plane. $ABCD$ is a parallelogram if and only if one of the following cases is fulfilled:

(a) $\overrightarrow{AB} = \overrightarrow{CD}$

(b) $\overrightarrow{AB} = \overrightarrow{DC}$

(c) $\overrightarrow{AD} = \overrightarrow{CB}$

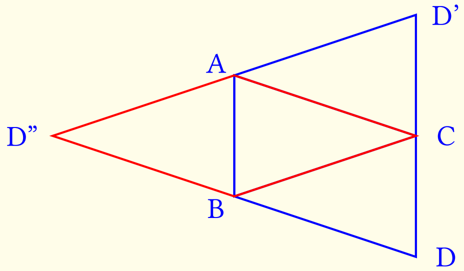
This leads to:

(a) $\overrightarrow{AB} = \overrightarrow{CD} \iff \begin{cases} x-4=0 \\ y-2=-2 \end{cases} \iff \begin{cases} x=4 \\ y=0 \end{cases}$ and thus $D(4,0)$.

(b) $\overrightarrow{AB} = \overrightarrow{D'C} \iff \begin{cases} x-4=0 \\ 2-y=-2 \end{cases} \iff \begin{cases} x=4 \\ y=4 \end{cases}$ and thus $D'(4,4)$.

(c) $\overrightarrow{AD''} = \overrightarrow{BC} \iff \begin{cases} x - 1 = -3 \\ y - 3 = -1 \end{cases} \iff \begin{cases} x = -2 \\ y = 2 \end{cases}$ and thus $D''(-2, 2)$.

Thus the three possible points, denoted D, D' and D". Their coordinates are: D$(4,0)$, D'$(4,4)$ and D"$(-2,2)$. These three points are represented on the graph below.



Exercise 3.

In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} := (1,0,0)$ and $\boldsymbol{i}+\boldsymbol{j} := (1,1,0)$?

Solution of Exercise 3.

We want to characterize the set $\Gamma := \{\alpha\boldsymbol{i} + \beta(\boldsymbol{i}+\boldsymbol{j});\ (\alpha,\beta) \in \mathbf{R}^2\}$. We have the following equalities.

$$\begin{aligned}\Gamma &= \{\alpha\boldsymbol{i} + \beta(\boldsymbol{i}+\boldsymbol{j});\ (\alpha,\beta) \in \mathbf{R}^2\} = \{\alpha\cdot(1,0,0) + \beta\cdot(1,1,0);\ (\alpha,\beta) \in \mathbf{R}^2\} \\ &= \{(\alpha,0,0) + (\beta,\beta,0);\ (\alpha,\beta) \in \mathbf{R}^2\} = \{(\alpha+\beta,\beta,0);\ (\alpha,\beta) \in \mathbf{R}^2\} \\ &= \{(\gamma,\delta,0);\ (\gamma,\delta) \in \mathbf{R}^2\}.\end{aligned}$$

Thus $\{\alpha\boldsymbol{i} + \beta(\boldsymbol{i}+\boldsymbol{j});\ (\alpha,\beta) \in \mathbf{R}^2\}$ is nothing but the plane containing all points (a,b,c) with the third coordinate equals to 0, *i.e.* the plane which corresponds to $z = 0$ in the xyz space.

Exercise 4.

What combination $c \cdot \begin{pmatrix}1\\2\end{pmatrix} + d \cdot \begin{pmatrix}3\\1\end{pmatrix}$ produces $\begin{pmatrix}14\\8\end{pmatrix}$?

Solution of Exercise 4.

One just needs to solve the system

$$(\mathcal{S}) : \begin{cases} c + 3d = 14 \\ 2c + d = 8 \end{cases}$$

the solution of which is $(2,4)$.

$$\boxed{\text{Thus the combination } (c,d) \text{ which produces } \begin{pmatrix}14\\8\end{pmatrix} \text{ is } (2,4).}$$

Exercise 5.

Find the angle θ (from its cosine) between these pairs of vectors:

a) $\boldsymbol{v} := \begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix}1\\0\end{pmatrix}$ **b)** $\boldsymbol{v} := \begin{pmatrix}2\\2\\-1\end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix}2\\-1\\2\end{pmatrix}$

c) $\boldsymbol{v} := \begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix}$ **d)** $\boldsymbol{v} := \begin{pmatrix}3\\1\end{pmatrix}$ and $\boldsymbol{w} := \begin{pmatrix}-1\\-2\end{pmatrix}$.

Solution of Exercise 5.

We just have to use he equality

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|\cdot\cos(\boldsymbol{u},\boldsymbol{v}) \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are two 2-dimensional vectors.

a) Applying (1) to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ defined in **a)**, we get:

$$1 = \boldsymbol{v}\cdot\boldsymbol{w} = 2\cdot 1\cos(\boldsymbol{v},\boldsymbol{w}).$$

We then deduce that $\cos(\boldsymbol{v},\boldsymbol{w}) = 1/2$.

$$\boxed{\text{Thus } \theta := (\boldsymbol{v},\boldsymbol{w}) \in \{-\pi/3, \pi/3\}.}$$

b) Applying (1) to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ defined in **b)**, we get:

$$0 = \boldsymbol{v}\cdot\boldsymbol{w} = \sqrt{5}\cdot\sqrt{5}\cos(\boldsymbol{v},\boldsymbol{w}).$$

We then deduce that $\cos(\boldsymbol{v},\boldsymbol{w}) = 0$.

$$\boxed{\text{Thus } \theta := (\boldsymbol{v},\boldsymbol{w}) \in \{-\pi/2, \pi/2\}.}$$

c) Applying (1) to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ defined in **c)**, we get:

$$2 = \boldsymbol{v}\cdot\boldsymbol{w} = 2\cdot 2\cos(\boldsymbol{v},\boldsymbol{w}).$$

We then deduce that $\cos(\boldsymbol{v},\boldsymbol{w}) = 1/2$.

$$\boxed{\text{Thus } \theta := (\boldsymbol{v},\boldsymbol{w}) \in \{-\pi/3, \pi/3\}.}$$

d) Applying (1) to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ defined in **d)**, we get:

$$-5 = \boldsymbol{v} \cdot \boldsymbol{w} = \sqrt{10} \cdot \sqrt{5} \cos(\boldsymbol{v}, \boldsymbol{w}).$$

We then deduce that $\cos(\boldsymbol{v}, \boldsymbol{w}) = -\sqrt{2}/2$. $\boxed{\text{Thus } \theta := (\boldsymbol{v}, \boldsymbol{w}) \in \{-3\pi/4, 3\pi/4\}.}$

Exercise 6.

1. How long is the vector $\boldsymbol{v} := (1, 1, \cdots, 1)$ in 9 dimensions?
2. Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a unit vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

Solution of Exercise 6.

$\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^{9} 1^2} = \sqrt{9} = 3$. Let u be non zero real number. It is obvious that $\boldsymbol{u} := (\frac{u}{\|u\|}, \frac{u}{\|u\|}, \cdots, \frac{u}{\|u\|})$ fulfills the requirement. Finally, define $\boldsymbol{w} := (w_1, w_2, \cdots, w_8)$ by setting:

$$w_{2i} = \frac{1}{2\sqrt{2}}, \qquad w_{2i+1} = \frac{1}{2\sqrt{2}}, \qquad \forall i \in [\![1, 4]\!].$$

and $w_9 = 0$. Then Vector $\boldsymbol{w}$ is perpendicular to $\boldsymbol{v}$ since $\boldsymbol{v}.\boldsymbol{w} = 0$. Moreover it is a unit vector.

Exercise 7.

Let n be a positive integer and let $\boldsymbol{u} := (u_1, u_2, \cdots, u_n)$, $\boldsymbol{v} := (v_1, v_2, \cdots, v_n)$ and $\boldsymbol{w} := (w_1, w_2, \cdots, w_n)$ be three vectors of $\mathbf{R}^n$. Having in mind that the dot product $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{k=1}^{n} u_i v_i$, prove the following equalities:

1. (a) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

   (b) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

   (c) $(c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{u} \cdot \boldsymbol{v})$.

2. Use **1.**. with $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{w}$ to prove that

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}.$$

Solution of Exercise 7.

The proofs of all the equalities given here lie on the commutativity of addition of real numbers. let $\boldsymbol{u} := (u_1, u_2, \cdots, u_n)$, $\boldsymbol{v} := (v_1, v_2, \cdots, v_n)$ and $\boldsymbol{w} := (w_1, w_2, \cdots, w_n)$ be three vectors of $\mathbf{R}^n$ and let c be a real number. One has:

1a) $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i\, v_i = \sum_{i=1}^{n} v_i\, u_i = \boldsymbol{u} \cdot \boldsymbol{v}$.

1b) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \displaystyle\sum_{i=1}^{n} u_i\, (\boldsymbol{v} + \boldsymbol{w})_i = \sum_{i=1}^{n} u_i\, (v_i + w_i) = \sum_{i=1}^{n} u_i\, v_i + \sum_{i=1}^{n} u_i\, w_i = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

1c) $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \sum_{i=1}^{n} c u_i v_i = c\, \sum_{i=1}^{n} u_i v_i = c(\boldsymbol{u} \cdot \boldsymbol{v})$.

2. Using **1.**, one can write:

$$\|\boldsymbol{v}+\boldsymbol{w}\|^2 = (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}.$$

CORRECTION OF EXERCISES SET # 2
MATRICES

Exercise 1.

For which of x, y, z and w are the matrices $A := \begin{pmatrix} x+y & x-z \\ y+w & x+2w \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ equal ?

Solution of Exercise 1.

To be written $\cdots$

Exercise 2.

Which of the following pairs of matrices commute under matrix multiplication?

$$a) \quad A := \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \& \quad B := \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix};$$

$$b) \quad C := \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 2 & 4 \end{pmatrix} \quad \& \quad D := \begin{pmatrix} 4 & 2 & -2 \\ 5 & 2 & 4 \end{pmatrix};$$

$$c) \quad F := \begin{pmatrix} 3 & 0 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \quad \& \quad E := \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}.$$

Solution of Exercise 2.

To be written

Exercise 3.

Find all matrices B that commute (under matrix multiplication) with $A := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Solution of Exercise 3.

To be written

Exercise 4.

Find a non zero matrix A in $\mathscr{M}_2(\mathbf{R})$ such that $A^2 = 0$.

Solution of Exercise 4.

Obviously the following matrice satisfies the requirement.

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Exercise 5.

Find all non zero matrix A in $\mathscr{M}_2(\mathbf{R})$ such that $A^2 = 0$.

Solution of Exercise 5.

Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathscr{M}_2(\mathbf{R})$. We know that:

$$A^2 = 0 \iff \begin{pmatrix} a^2 + bc & (a+d) \cdot b \\ c \cdot (a+d) & bc + d^2 \end{pmatrix} = 0 \iff \begin{cases} a^2 + bc = 0 \\ (a+d) \cdot b = 0 \\ c \cdot (a+d) = 0 \\ bc + d^2 = 0 \end{cases} \tag{1}$$

We can therefore consider two cases.

-1$^{\text{st}}$ case: $a + d \neq 0$: This entails that $b = c = 0$ and then that $a^2 = d^2 = 0$. Thus $a = b = c = d = 0$ *i.e.* $A = 0$. But this is absurd since A should be a non zero matrix. **Thus 1$^{\text{st}}$ case is impossible.**

-2$^{\text{nd}}$ case: $a + d = 0$: In this case (1) reduces to

$$\begin{cases} bc = -a^2 \\ a = -d. \end{cases} \tag{2}$$

- If $b = 0$, then $a = d = 0$ and thus:

$$A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \tag{3}$$

- If $c = 0$ then $a = d = 0$ and thus A=

$$A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \tag{4}$$

.

Finally, if none of b or c equals 0 then we get:

$$A = \begin{pmatrix} a & \frac{-a^2}{c} \\ c & -a \end{pmatrix} \tag{5}$$

Conversely, the null matrix and every matrix of the form (3), (4) or (5) has a square equal to 0.

$$\boxed{\text{Conclusion: } \begin{array}{c} \{A \in \mathscr{M}_2(\mathbf{R}),\ \text{s.t. } A^2 = 0\} = \\ \left\{ 0_{\mathscr{M}_2(\mathbf{R})}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & \frac{-a^2}{c} \\ c & -a \end{pmatrix},\ (a,c) \in \mathbf{R} \times \mathbf{R}^* \right\}. \end{array}}$$

.

Exercise 6.

The commutator of two matrices A, B is defined to be the matrix $C := [A, B] := AB - BA$.

1. Explain why $[A, B]$ is defined if and only if A and B are square matrices of the same size.
2. Show that A and B commute under matrix multiplication if and only if $[A, B] = 0$.
3. Compute the commutator of the following matrices (denoted A_1 & B_1, A_2 & B_2 and A_3 & B_3).

$$i) \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \& \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}; \quad ii) \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \& \begin{pmatrix} 1 & 7 \\ 7 & 1 \end{pmatrix}; \quad iii) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution of Exercise 6.

1. Let (A, B) be in $\mathscr{M}_{p,q}(\mathbf{R}) \times \mathscr{M}_{r,s}(\mathbf{R})$. Since one has to compute $A \cdot B$ one must have $q = r$. Moreover, since one has to compute $B \cdot A$ one must have $s = p$. Thus AB belongs to $\mathscr{M}_{p,p}(\mathbf{R})$ and BA belongs to $\mathscr{M}_{q,q}(\mathbf{R})$. In order to compute $AB - BA$, one needs that AB and BA have the same dimension *i.e.* $p = q$.
2. We have the following equivalences.

$$\begin{aligned} A \text{ and } B \text{ commute under matrix multiplication } &\iff AB = BA \\ &\iff AB - BA = 0 \iff [A, B] = 0. \end{aligned}$$

3. Define $C_1 := [A_1, B_1], C_2 := [A_2, B_2]$ and $C_3 := [A_3, B_3]$

$$C_1 = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}; \quad C_2 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_3 := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Exercise 7.

Let $P(x) = c_0 + c_1 x + \cdots c_{n-1}x^{n-1} + c_n x^n$ be a polynomial function. If A is a square matrix with n rows, we define the corresponding matrix polynomial $P(A) = c_0 I_n + c_1 A + \cdots c_{n-1}A^{n-1} + c_n A^n$; the constant term becomes a scalar multiple of the identity matrix I_n. For instance, if $P(x) = x^2 - 2x + 3$, then $P(A) = A^2 - 2A + 3I_n$.

1. Write out the matrix polynomials $P(A), Q(A)$ when $P(x) = x^3 - 3x + 2$, $Q(x) = 2x^2 + 1$.
2. Evaluate $P(A)$ and $Q(A)$ when $A := \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$.
3. Show that the matrix product $P(A) \cdot Q(A)$ is the matrix polynomial corresponding to the product polynomial $R(x) = P(x) \cdot Q(x)$.
4. True or false: If $B = P(A)$ and $C = Q(A)$, then $BC = CB$. Check your answer in the particular case of Question **2.**.

Solution of Exercise 7.

To be written $\cdots$

Exercise 8.

1. Show that if $D := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a 2×2 diagonal matrix with $a \neq b$, then the only matrices that commute (under matrix multiplication) with D are other 2×2 diagonal matrices.
2. What if $a = b$?
3. Find all matrices that commute with $D := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where a, b and c are all different.
4. Answer the same question for the case when $a \neq b = c$.
5. Prove that a matrix A commutes with an $n \times n$ diagonal matrix D with all distinct diagonal entries if and only if A is a diagonal matrix.

Solution of Exercise 8.

To be written $\cdots$

Exercise 9.

True or false : If A and B are square matrices of the same size, then:

$$A^2 - B^2 = (A - B) \cdot (A + B) \tag{1}$$

Solution of Exercise 9.

To be written $\cdots$

Exercise 10.

1. Under what conditions is the square A^2 of a matrix defined?
2. Show that A and A^2 commute.
3. How many matrix multiplications are needed to compute A^n, where n belongs to $\mathbf{N}^*$?

Solution of Exercise 10.

Of course $n-1$ multiplications is a possible answer! However, one can also notice that:

$$A^2 = A \cdot A, \qquad A^4 = A^2 \cdot A^2 = \left(A^2\right)^2, \qquad A^8 = A^{2^3} = \left(\left(A^2\right)^2\right)^2,$$

One can therefore assume that A^{2^r} can be computed with only r multiplications. The proof can be done by induction on r in $\mathbf{N}^*$.

$$A$$

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Math -UA 9140
Linear Algebra
Cor.of Ex.Set#3
J. Lebovits

Correction of Exercises Set # 3
Gaussian Elimination

Exercise 1.

Solve the following linear systems using Gaussian Elimination:

$a)$ $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$; $b)$ $\begin{pmatrix} 6 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$;

$c)$ $\begin{pmatrix} 2 & 1 & 2 \\ -1 & 3 & 3 \\ 4 & -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$; $d)$ $\begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -1 \end{pmatrix}$;

$e)$ $\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \end{pmatrix} \cdot A = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $f)$ $\begin{pmatrix} 2 & -3 & 1 & 1 \\ 1 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} -1 \\ 0 \\ 5 \\ 3 \end{pmatrix}$,

where $A := {}^t(a, b, c, d)$ and $X := {}^t(x, y, z, t)$.

Solution of Exercise 1.

$\boxed{\mathscr{S}_{a)} := \{(11, 4)\}.}$ $\boxed{\mathscr{S}_{b)} := \{(\frac{17}{3}, -\frac{4}{3})\}.}$ $\boxed{\mathscr{S}_{c)} := \{(\frac{3}{2}, -\frac{1}{3}, \frac{1}{6})\}.}$

$\boxed{\mathscr{S}_{d)} := \{(\frac{11}{3}, -\frac{10}{3}, -\frac{2}{3})\}.}$ $\boxed{\mathscr{S}_{e)} := \{(\frac{1}{3}, 0, 4/3, -2/3)\}.}$ $\boxed{\mathscr{S}_{f)} := \{(-8, -3, -11, 17)\}.}$

Exercise 2.

Solve the following linear systems using Gaussian Elimination.

$$(\mathscr{S}_1)\begin{cases} a+7b=4 \\ -2a-9b=2 \end{cases} \qquad (\mathscr{S}_2)\begin{cases} 3z-5w=-1 \\ 2z+w=8 \end{cases} \qquad (\mathscr{S}_3)\begin{cases} x-2y+z=0 \\ 2y-8z=8 \\ -4x+5y+9z=-9 \end{cases}$$

$$(\mathscr{S}_4)\begin{cases} p+4q-2r=1 \\ -2p-3r=-7 \\ 3p-2q+2r=-1 \end{cases} \qquad (\mathscr{S}_5)\begin{cases} x_1-2x_3=-1 \\ x_2-x_4=2 \\ -3x_2+2x_3=0 \\ -4x_1+7x_4=-5 \end{cases} \qquad (\mathscr{S}_6)\begin{cases} -x+3y-z+w=-2 \\ x-y+3z-w=0 \\ y-z+4w=7 \\ 4x-y+z=5 \end{cases}$$

Solution of Exercise 2.

$\boxed{\mathscr{S}ol_{(\mathscr{S}_1)} := \{(-10,2)\}.}$ $\boxed{\mathscr{S}ol_{(\mathscr{S}_2)} := \{(3,2)\}.}$ $\boxed{\mathscr{S}ol_{(\mathscr{S}_3)} := \{(43/8,5/2,-3/8)\}.}$

$\boxed{\mathscr{S}ol_{(\mathscr{S}_4)} := \{(-1,2,3)\}.}$ $\boxed{\mathscr{S}ol_{(\mathscr{S}_5)} := \{(-4,-1,-3/2,-3)\}.}$ $\boxed{\mathscr{S}ol_{(\mathscr{S}_6)} := \{([1,-1,0,2])\}.}$

Exercise 3.

One says that a square Matrix A is regular if the Gaussian Elimination algorithm sucessfully reduces to upper triangular form U with. All non-zero pivots on the diagonal.
Which of the following matrices is regular?
Check your answers using Maple.

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}; \qquad b)\ \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix};$$

$$c)\ \begin{pmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & -2 & 5 \end{pmatrix}; \quad d)\ \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}; \quad e)\ \begin{pmatrix} 1 & 3 & -3 & 0 \\ -1 & 0 & -1 & 2 \\ 3 & 3 & -6 & 1 \\ 2 & 3 & -3 & 5 \end{pmatrix};$$

Solution of Exercise 3.

You may check your computations using Maple.

Exercise 4.

1. Write down an example of a system of 5 linear equations in 5 unknowns with regular diagonal coefficient matrix.

2. Solve your system.

3. Explain why solving a system whose coefficient matrix is diagonal is very easy.

Solution of Exercise 4.

To be written

Exercise 5.

A linear system is called *homogeneous* if all the right-hand sides are zero, and so takes the matrix form $A \cdot X = 0_{\mathbf{R}^n}$.
Explain why the solution to a homogeneous system with regular coefficient matrix is $X = 0_{\mathbf{R}^n}$.

Solution of Exercise 5.

To be written

Exercise 6.

A square matrix is called *strictly lower (upper) triangular* if all entries on or above (below) the main diagonal are 0.

1. Prove that every square matrix can be uniquely written as a sum $A = L + D + U$, with L strictly lower triangular, D diagonal, and U strictly upper triangular.

2. Decompose $A := \begin{pmatrix} 3 & 1 & -1 \\ 1 & -4 & 2 \\ -2 & 0 & 5 \end{pmatrix}$ in this manner.

Solution of Exercise 6.

To be written

Exercise 7.

A square matrix N is called nilpotent if $N^k = O_{\mathscr{M}_n(\mathbf{R})}$, for some $k \geqslant 1$.

(a) Show that $N := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.

(b) Show that every strictly upper triangular matrix, as defined in Exercise 144 is nilpotent.

(c) Find a nilpotent matrix which is neither lower nor upper triangular.

Solution of Exercise 7.

To be written

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Linear Algebra
Cor.of Ex.Set#4
J. Lebovits

Correction of Exercises Set # 4
Matrix Inverses

1 Basic Exercises

Exercise 1.

Verifiy by direct multiplication that the following matrices inverses.

$$a)\ A := \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix},\ A^{-1} := \begin{pmatrix} -1 & -3 \\ 1 & 2 \end{pmatrix};$$

$$b)\ A := \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix},\ A^{-1} := \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix};$$

$$c)\ A := \begin{pmatrix} -1 & 3 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix},\ A^{-1} := \begin{pmatrix} -1 & 1 & 1 \\ 4/7 & -1/7 & -3/7 \\ -6/7 & 5/7 & 8/7 \end{pmatrix};$$

Solution of Exercise 1.

No need!

Exercise 2.

Write down the inverse of each of the following matrices

$a)\quad A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b)\quad A := \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \qquad c)\quad A := \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$

$d)\quad A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad e)\quad A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f)\quad A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$

$g)\quad A := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{pmatrix} \quad .$

Solution of Exercise 2.

$\boxed{a)\ A^{-1} = A.}$

$\boxed{b)\ A^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}.}$

$\boxed{c)\ A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.}$

$\boxed{d)\ A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.}$

$\boxed{e)\ A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.}$

$\boxed{f)\ A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.}$

Exercise 3.

Explain why a matrix with a row of all zeros does not have an inverse.

Solution of Exercise 3.

To be written

Exercise 4.

Find all real 2×2 matrices that are their own inverses: *i.e.* $A^{-1} = A$.

Solution of Exercise 4.

To be written

Exercise 5.

Show that if a square matrix A of $\mathscr{M}_n(\mathbf{R})$ satisfies the equality: $A^2 - 3A + I_n = 0_{\mathscr{M}_n(\mathbf{R})}$, then $A^{-1} = 3I_n - A$.

Solution of Exercise 5.

To be written

Exercise 6.

Show that $A := \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$ is not invertible for any value of the entries.

Solution of Exercise 6.

To be written.

2 Intermediate Level Exercises

Exercise 7.

Prove that a diagonal matrix $D := \mathrm{diag}(d_1, d_2, \cdots, d_n)$ is invertible if and only if all its diagonal entries are nonzero, in which case $D^{-1} = \mathrm{diag}(1/d_1, 1/d_2, \cdots, 1/d_n)$.

Solution of Exercise 7.

Assume that D is invertible. Then, in the Gaussian elimination of D, every single column of D must have a non zero element. Since D has only one single element on each column, it must be non zero. Thus $d_i \neq 0, \forall i \in [\![1, n]\!]$.
Conversely, assume that the statement

$$d_i \neq 0, \forall i \in [\![1, n]\!]$$

is true. Then define $B := \mathrm{diag}(1/d_1, 1/d_2, \cdots, 1/d_n)$. It is clear that $DB = BD = I_n$. Thus D is invertible and $D^{-1} = B$.

Exercise 8.

Two matrices A and B are said to be *similar*, written $A \sim B$ if there exists an invertible matrix S such that $B = S^{-1}AS$. Prove

(a) $A \sim A$.

(b) If $A \sim B$, then $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Solution of Exercise 8.

To be written.

Exercise 9.

(a) A Block matrix $D := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is called *block diagonal* if A and B are square matrices, not necessarily of the same size, while the 0's are zero matrices of the appropriate size. Prove that D has an inverse if and only if both A and B do, and $D^{-1} := \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.

(b) Find the inverse of $C := \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $D := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix}$ by using this method.

Solution of Exercise 9.

To be written.

3 Other Exercises

Exercise 10.

Let A be the element of $\mathscr{M}_n(\mathbf{R})$ defined by setting: $A := \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$.

Compute A^2 and deduce that A is invertible and give its inverse.

Solution of Exercise 10.

Exercise 11.

Let $A := (a_{i,j})_{1 \leqslant i,j \leqslant n}$ be the element of $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$\begin{cases} a_{i,j} := 1, & \text{if } i \neq j, \\ a_{i,i} := 0 \text{ for all } i \text{ in } [\![1, n]\!]. \end{cases}$$

Show that A is invertible and give its inverse.

Solution of Exercise 11.

To be written

Exercise 12.

Let n be in $\mathbf{N}^*$, a and b be two scalars and let $A := (a_{i,j})_{1 \leqslant i,j \leqslant n}$ be the matrix in $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$\begin{cases} a_{i,j} := a, & \text{if } i \neq j, \\ a_{i,i} := b. \end{cases}$$

Define $J := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathscr{M}_n(\mathbf{R})$ and let I_n be the $n \times n$ matrice identity.

1. Show that $A = \alpha(\frac{J}{n}) + \beta(I_n - \frac{J}{n})$, where $\alpha := b + (n-1)a$ and $\beta := b - a$.
2. Express J^2 in function of J.
3. Show that
$$\left(\frac{J}{n}\right)^2 = \frac{J}{n}, \quad \left(I_n - \frac{J}{n}\right)^2 = I_n - \frac{J}{n}, \quad \frac{J}{n}\left(I_n - \frac{J}{n}\right) = \left(I_n - \frac{J}{n}\right)\frac{J}{n} = 0.$$
4. Show that, for any m in $\mathbf{N}^*$,
$$A^m = \alpha^m \left(\frac{J}{n}\right) + \beta^m \left(I_n - \frac{J}{n}\right). \tag{1}$$
5. Deduce from the previous question that:
$$A^2 = (b-a)^2 I_n + (2ab - 2a^2 + na^2)J. \tag{2}$$
6. Using the equality
$$A = aJ + (b-a)I_n, \tag{3}$$
show that:
$$A \cdot (A - (2b - 2a + na)I_n) = (b-a)(a - b - na)I_n. \tag{4}$$
7. Find a necessary and sufficient condition on a and b such that A is invertible and give its inverse.

Solution of Exercise 12.

1. Note that $A = aJ + (b-a)I_n$. Moreover, one has the equality
$$\alpha\left(\frac{J}{n}\right) + \beta\left(I_n - \frac{J}{n}\right) = \frac{b+(n-1)a-(b-a)}{n}J + (b-a)I_n = aJ + (b-a)I_n = A.$$

2. It is clear that $J^2 = nJ$.

3. From the previous question we immediately get that $\left(\frac{J}{n}\right)^2 = \frac{J}{n}$. Moreover, using this last equality, we easily get:
$$\left(I_n - \frac{J}{n}\right)^2 = I_n - \frac{2J}{n} + \left(\frac{J}{n}\right)^2 = I_n - \frac{J}{n}$$
$$\frac{J}{n}\left(I_n - \frac{J}{n}\right) = \frac{J}{n} - \left(\frac{J}{n}\right)^2 = \frac{J}{n} - \frac{J}{n} = 0.$$

4. Since I_n commutes with any other matrix and since J always commutes with itself, one can use the Binomial formula to get, for any $n \geqslant 2$:
$$\begin{aligned}A^m &= \left(\alpha(\frac{J}{n}) + \beta(I_n - \frac{J}{n})\right)^m = \sum_{k=0}^{m}\binom{m}{k}\left(\alpha\frac{J}{n}\right)^k \cdot \left(\beta(I_n - \frac{J}{n})\right)^{m-k}\\ &= \beta^m\left(I_n - \frac{J}{n}\right)^m + \alpha^m\left(\frac{J}{n}\right)^m + \sum_{k=1}^{m-1}\binom{m}{k}\left(\alpha\frac{J}{n}\right)^k \cdot \left(\beta(I_n - \frac{J}{n})\right)^{m-k}\\ &= \beta^m(I_n - \frac{J}{n}) + \alpha^m\left(\frac{J}{n}\right) + 0 = \alpha^m\left(\frac{J}{n}\right) + \beta^m\left(I_n - \frac{J}{n}\right).\end{aligned}$$
Since the equality is obviously true for m in $\{0,1\}$, the proof is achieved.

5. Applying (1) with $m = 2$, we easily deduce (2).

6. Assuming $a \neq 0$, we deduce, from (3), that
$$J = \frac{A-(b-a)I_n}{a}. \tag{5}$$
Thus, replacing J, by the expression given in (5), in (2), we get:
$$A^2 = (b-a)^2 I_n + (2ab - 2a^2 + na^2)\left(\frac{A-(b-a)I_n}{a}\right)$$
$$A^2 = (an + 2(b-a)) \cdot A + (b-a)\cdot(b - a - an - 2b + 2a)\, I_n$$
This last equality can be rewritten as:
$$A \cdot (A - (an + 2(b-a)) \cdot I_n)) = (b-a)\cdot(a-b-an)I_n$$

7. From (4) we deduce that: A is invertible if and only if both $b-a$ and $a-b-an$ are non zero. In this latter case we have:
$$A^{-1} = \frac{1}{(b-a)\cdot(a-b-an)}(A - (an+2(b-a))\cdot I_n)).$$

A is invertible if and only if both $b-a$ and $a-b-an$ are non zero. In this latter case we have:
$$A^{-1} = \frac{1}{(b-a)\cdot(a-b-an)}(A - (an+2(b-a))\cdot I_n)).$$

Exercise 13.

Let A be a matrix of $\mathscr{M}_p(\mathbf{K})$ and let B be a matrix of $\mathscr{M}_p(\mathbf{K})$ such that:

$$B \cdot A = I_p.$$

Show that A is invertible and that $A^{-1} = B$.

Solution of Exercise 13.

1ère méthode:En fait il faut utiliser soit les déterminants et c'est réglé!
2ème méthode: Démontrons qu'il existe une matrice carrée C de taille p, telle que $BC = I_p$.
Pour cela, on considère l'équation $BC = aI_p$, d'inconnues C et $a \in \mathbf{K}$. Elle se traduit par un système de p^2 équations à $p^2 + 1$ inconnues. Ce système admet donc une solution non triviale (C_0, a_0). On a $a_0 \neq 0$. En effet sinon on aurait $C_0 = ABC_0 = a_0A = 0$ ce qui voudrait dire que (C_0, a_0) est triviale.
Maintenant $C := a_0^{-1}C_0$ vérifie $BC = I_p$.

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Math -UA 9140

Linear Algebra

Cor. of Ex. Set #5

J. Lebovits

Correction of Exercises Set # 5

LU Factorization, Transposes & Symmetric Matrices

Exercise 1.

For each of the listed matrices A and vectors $\boldsymbol{b}$ below, find a permuted LU factorization of the matrix, and use your factorization to solve the system $A\boldsymbol{x} = \boldsymbol{b}$.

$a)\ A := \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \ \boldsymbol{b} := \begin{pmatrix} 3 \\ 2 \end{pmatrix};$ $\qquad b)\ A := \begin{pmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix}, \ \boldsymbol{b} := \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix};$

$c)\ A := \begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \ \boldsymbol{b} := \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix};$ $\qquad d)\ A := \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 6 & 2 & -1 \\ 1 & 1 & -7 & 2 \\ 1 & -1 & 2 & 1 \end{pmatrix}, \ \boldsymbol{b} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix};$

$e)\ A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 4 & -1 & 2 \\ 7 & -1 & 2 & 3 \end{pmatrix}, \ \boldsymbol{b} := \begin{pmatrix} -1 \\ -4 \\ 0 \\ 5 \end{pmatrix};$

Solution of Exercise 1.

a) $(P, L, U, \boldsymbol{x}) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5/2 \\ 3 \end{pmatrix} \right).$

b) $(P, L, U, \boldsymbol{x}) := \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 5/4 \\ 3/4 \\ -1/4 \end{pmatrix} \right).$

c) $(P, L, U, \boldsymbol{x}) := \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -9/2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -0 \end{pmatrix} \right).$

d) $(P, L, U, \boldsymbol{x}) := \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 3 & \frac{21}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -6 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -4/5 \end{pmatrix} \begin{pmatrix} 22 \\ -13 \\ -5 \\ -22 \end{pmatrix} \right).$

e) $(P, L, U, \boldsymbol{x}) := \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 5/2 & 1 & 0 \\ 7/2 & -23/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3/2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} \right).$

Exercise 2.

1. Explain why

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix},$$

are all legitimate permuted LU factorizations of the same matrix. List the elementary row operations that are being used in each case.

2. Use each of the factorizations to solve the linear system

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}.$$

Do you obtain the same result? Explain why or why not.

Solution of Exercise 2.

To be written!

Exercise 3.

Let p, q and r be three integers and let A be in $\mathscr{M}_{p,q}(\mathbf{R})$ and B be in $\mathscr{M}_{q,r}(\mathbf{R})$. Show the following equality:

$$^t(A \cdot B) = {}^tB \cdot {}^tA.$$

Solution of Exercise 3.

To be written!

Exercise 4.

A square matrix is called *normal* if it commutes with its transpose. In other words, A is *normal* if the following equality holds:

$$^tA \cdot A = A \cdot {}^tA.$$

Find all normal 2×2 matrices.

Solution of Exercise 4.

To be written!

Exercise 5.

Let A and B be two elements of $\mathscr{M}_{p,q}(\mathbf{R})$.

(a) Suppose that

$$^t\boldsymbol{v} A \boldsymbol{w} = {}^t\boldsymbol{v} B \boldsymbol{w}$$

for all vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. Prove that $A = B$.

(b) Give an example of two matrices such that:

$$^t\boldsymbol{v} A \boldsymbol{v} = {}^t\boldsymbol{v} B \boldsymbol{v}$$

for all vectors $\boldsymbol{v}$ but $A \neq B$.

Solution of Exercise 5.

To be written!

Exercise 6.

Find all values of a, b, and c for which the following matrices are symmetric:

$$a)\ \begin{pmatrix} 3 & a \\ 2a-1 & a-2 \end{pmatrix},\quad b)\ \begin{pmatrix} 1 & a & 2 \\ -1 & b & c \\ b & 3 & 0 \end{pmatrix},\quad c)\ \begin{pmatrix} 3 & a+2b-2c & -4 \\ 6 & 7 & b-c \\ -a+b+c & 4 & b+3c \end{pmatrix}.$$

Solution of Exercise 6.

To be written!

Exercise 7.

Find all values of a, b, and c for which the following matrices are symmetric:

$$a)\ \begin{pmatrix} 3 & a \\ 2a-1 & a-2 \end{pmatrix},\quad b)\ \begin{pmatrix} 1 & a & 2 \\ -1 & b & c \\ b & 3 & 0 \end{pmatrix},\quad c)\ \begin{pmatrix} 3 & a+2b-2c & -4 \\ 6 & 7 & b-c \\ -a+b+c & 4 & b+3c \end{pmatrix}.$$

Solution of Exercise 7.

Exercise 8.

Find the LD^tL factorization of the following matrices:

$$a)\ A := \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},\qquad b)\ B := \begin{pmatrix} -2 & 3 \\ 3 & -1 \end{pmatrix},$$

$$c)\ C := \begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix},\qquad d)\ D := \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix},$$

Solution of Exercise 8.

Exercise 9.

Prove that the matrix $A := \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ can not be factorized as $A = LD^tL$.

Solution of Exercise 9.

CORRECTION OF EXERCISES SET # 6
VECTOR SUBSPACES & BASIS OF A VECTOR SPACES

1 Basic Exercises

Exercise 1.

Are the following family of vectors linearly independent?

- in $\mathbf{R}^2$, $((1,2),(3,5))$;
- in $\mathbf{R}^3$, $((1,2,3),(1,-2,-3),(1,4,3))$;
- in $\mathbf{R}^4$, $((1,2,3,4),(5,6,7,8),(11,14,16,20))$.
- in $\mathbf{R}^4$, $((1,2,3,4),(5,6,7,8),(-4,-8,-12,-16))$.

Solution of Exercise 1.

- For any (α,β) in $\mathbf{R}^2$,

$$\alpha(1,2)+\beta(3,5)=(0,0) \Longrightarrow (\alpha+3\beta,2\alpha+5\beta)=(0,0) \Longrightarrow \alpha=-3\beta \text{ and } -6\beta+5\beta=0$$
$$\Longrightarrow \alpha=-3\beta \text{ and } \beta=0 \Longrightarrow \alpha=\beta=0.$$

  This implication precisely means that $(1,2)$ and $(3,5)$ are linearly independent vectors of $\mathbf{R}^2$.

- For any (α,β,γ) in $\mathbf{R}^3$,

$$\alpha\begin{pmatrix}1\\2\\3\end{pmatrix}+\beta\begin{pmatrix}1\\-2\\-3\end{pmatrix}+\gamma\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow \begin{cases}\alpha+\beta+\gamma=0\\2\alpha-2\beta+4\gamma=0\\3\alpha-3\beta+3\gamma=0\end{cases}$$

  Solving this system provides us with a unique solution which is $\alpha=\beta=\gamma=0$. This proves that these three vectors are linearly independent in $\mathbf{R}^3$.

- For any (α, β, γ) in $\mathbf{R}^3$,

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} + \gamma \begin{pmatrix} 11 \\ 14 \\ 16 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} \alpha + 5\beta + 11\gamma = 0 \\ 2\alpha + 6\beta + 14\gamma = 0 \\ 3\alpha + 7\beta + 16\gamma = 0 \\ 4\alpha + 8\beta + 20\gamma = 0 \end{cases}$$

$$\implies \begin{cases} \alpha = -5\beta - 11\gamma \\ -4\beta - 8\gamma = 0 \\ -8\beta - 17\gamma = 0 \\ -12\beta - 24\gamma = 0 \end{cases} \implies \begin{cases} \alpha = -5\beta - 11\gamma \\ \beta = -2\gamma \\ \beta = -\frac{17}{8}\gamma \end{cases} \implies \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

This proves that these three vectors are linearly independent in $\mathbf{R}^4$.

- Denote $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ these three vectors. We immediately see that $\boldsymbol{w} = -2\boldsymbol{u}$. Thus $2\boldsymbol{u} + 0\boldsymbol{v} + 1\boldsymbol{w} = 0_{\mathbf{R}^4}$. Since $(2, 0, 1) \neq (0, 0, 0)$, this proves that $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ are not linearly independent vectors.

Exercise 2.

In $\mathbf{R}^4$, let us consider the set of all vectors, denoted (x, y, z, t) such that:

$$x + y + z + t = 0.$$

1. Show that $F := \{(x, y, z, t) \in \mathbf{R}^4,\ x + y + z + t = 0\}$ is a vector subspace of $\mathbf{R}^4$.
2. Give a set of vectors, linearly independent, which spans F.

Solution of Exercise 2.

Exercise 3.

In $\mathbf{R}^3$, let us consider the set of all vectors, denoted (x, y, z) such that:

$$(\mathcal{S}) : \begin{cases} x + y + z = 0 \\ 2x - y + z = 0. \end{cases}$$

Show that $F := \{(x, y, z) \in \mathbf{R}^3,\ (\mathcal{S}) \text{ is fulfilled } \}$ is a vector subspace of $\mathbf{R}^3$ and give a set of vectors, linearly independent, which spans F.

Solution of Exercise 3.

- For any (α, β) in $\mathbf{R}^2$,

$$\alpha(1, 2) + \beta(3, 5) = (0, 0) \implies (\alpha + 3\beta, 2\alpha + 5\beta) = (0, 0) \implies \alpha = -3\beta \text{ and } -6\beta + 5\beta = 0$$
$$\implies \alpha = -3\beta \text{ and } \beta = 0 \implies \alpha = \beta = 0.$$

This implication precisely means that $(1,2)$ and $(3,5)$ are linearly independent vectors of $\mathbf{R}^2$.

- For any (α,β,γ) in $\mathbf{R}^3$,

$$\alpha\begin{pmatrix}1\\2\\3\end{pmatrix}+\beta\begin{pmatrix}1\\-2\\-3\end{pmatrix}+\gamma\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\implies\begin{cases}\alpha+\beta+\gamma=0\\2\alpha-2\beta+4\gamma=0\\3\alpha-3\beta+3\gamma=0\end{cases}$$

Solving this system provides us with a unique solution which is $\alpha=\beta=\gamma=0$. This proves that these three vectors are linearly independent in $\mathbf{R}^3$.

- For any (α,β,γ) in $\mathbf{R}^3$,

$$\alpha\begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\beta\begin{pmatrix}5\\6\\7\\8\end{pmatrix}+\gamma\begin{pmatrix}11\\14\\16\\20\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\implies\begin{cases}\alpha+5\beta+11\gamma=0\\2\alpha+6\beta+14\gamma=0\\3\alpha+7\beta+16\gamma=0\\4\alpha+8\beta+20\gamma=0\end{cases}$$

$$\implies\begin{cases}\alpha=-5\beta-11\gamma\\-4\beta-8\gamma=0\\-8\beta-17\gamma=0\\-12\beta-24\gamma=0\end{cases}\implies\begin{cases}\alpha=-5\beta-11\gamma\\\beta=-2\gamma\\\beta=-\frac{17}{8}\gamma\end{cases}\implies\begin{cases}\alpha=0\\\beta=0\\\gamma=0\end{cases}$$

This proves that these three vectors are linearly independent in $\mathbf{R}^4$.

- Denote $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$ these three vectors. We immediately see that $\boldsymbol{w}=-2\boldsymbol{u}$. Thus $2\boldsymbol{u}+0\boldsymbol{v}+1\boldsymbol{w}=0_{\mathbf{R}^4}$. Since $(2,0,1)\neq(0,0,0)$, this proves that $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$ are not linearly independent vectors.

Exercise 4.

In $\mathbf{R}^3$, let us consider the set of all vectors, denoted (x,y,z) such that:

$$(\mathscr{S}):\begin{cases}x+y+z=0\\2x-y+z=0.\end{cases}$$

Show that $F:=\{(x,y,z)\in\mathbf{R}^3,\ (\mathscr{S})\text{ is fulfilled }\}$ is a vector subspace of $\mathbf{R}^3$ and give a set of vectors, linearly independent, which spans F.

Exercise 5.

Define

$$F:=\left\{\begin{pmatrix}\alpha+\beta & 0\\0 & \alpha-\beta\end{pmatrix},\ (\alpha,\beta)\in\mathbf{R}^2\right\}\quad\text{and}\quad G:=\left\{\begin{pmatrix}0 & \gamma-\delta\\\gamma+\delta & 0\end{pmatrix},\ (\gamma,\delta)\in\mathbf{R}^2\right\}.$$

Show that F and G below are two supplementary vector subspaces of of $\mathscr{M}_2(\mathbf{R})$.

Solution of Exercise 4.

2 Intermediate Level Exercises

Exercise 6.

Let F and G be two vector subspaces of a vector space E.
Show that $F \cup G$ is a vector subspace de E if and only if $F \subset G$ or $G \subset F$.

Solution of Exercise 5.

Exercise 7.

Let n be in $\mathbb{N}^*$ and $a_1, a_2, \cdots a_n$ n real numbers such that $a_1 < a_2 < \cdots < a_n$. Show that the family $(f_1, f_2, \cdots, f_n)$, where:

$$\begin{array}{rccl} f_k : & \mathbf{R} & \to & \mathbf{R} \\ & x & \mapsto & e^{a_k x}, \ \forall k \in [\![1, n]\!], \end{array}$$

is linearly independent in $\mathscr{F}(\mathbf{R}, \mathbf{R})$.

Solution of Exercise 6.

Let us prove this result by induction on the integer n, which represents the number of elements of the family $(a_i)_{i \in [\![1,n]\!]}$. Let $(\lambda_i)_{i \in [\![1,n]\!]}$ be a family of real numbers and denote $(\mathscr{P}_n)$ the following statement.

$$\left[\sum_{k=1}^{n} \lambda_k e^{a_k x} = 0, \ \forall x \in \mathbf{R}\right] \Longrightarrow [\lambda_1 = \cdots = \lambda_n = 0]\,. \tag{$\mathscr{P}_n$}$$

Our goal is to show, by induction on n, that $(\mathscr{P}_n)$ holds.
If $n = 1$, $(\mathscr{P}_n)$ reduces to

$$\left[\lambda_1 e^{a_1 x} = 0, \ \forall x \in \mathbf{R}\right] \Longrightarrow [\lambda_1 = 0]\,.$$

This last implication being obvious, $(\mathscr{P}_1)$ is true.
Assume that $(\mathscr{P}_n)$ is true for some positive integer n. Let us show that $(\mathscr{P}_{n+1})$ holds. Let $(\lambda_i)_{i \in [\![1,n+1]\!]}$ be a family of real numbers such that the following implication holds.

$$\sum_{k=1}^{n+1} \lambda_k e^{a_k x} = 0, \ \forall x \in \mathbf{R}. \tag{1}$$

By multiplying both sides of (1) by $e^{-a_n x}$, we get:

$$\sum_{k=1}^{n} \lambda_k e^{(a_k - a_n)x} = 0, \ \forall x \in \mathbf{R}. \tag{2}$$

Since the sequence $(a_i)_{i\in[\![1,n]\!]}$ is increasing, $\lim\limits_{x\to\infty} e^{(a_k-a_n)x} = 0$ for every k in $[\![1, n-1]\!]$. Thus, taking the limit on both sides of (2) provides us with:

$$\lambda_n = 0.$$

Exercise 8.

Let $\mathscr{P} := (P_1, P_2, \cdots, P_n)$ be a family of n polynomials on $\mathbf{R}[X]$ such that:

$$\deg(P_1) < \deg(P_2) < \cdots < \deg(P_n).$$

Show that the vectors of the family of $\mathscr{P}$ are linearly independent. We recall that, by convention, the degree of the null polynomial equals 0. In other words $\deg(0_{\mathbf{R}[X]}) = -\infty$.

Solution of Exercise 7.

Let $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be in $\mathbf{K}^n$ such that:

$$\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = 0.$$

Denote Q the polynomial defined by setting $Q := \sum_{k=0}^{n} P_k$. If $\lambda_n \neq 0$ then $\deg(Q) = n$ and $\deg(0_{\mathbf{R}[X]}) = -\infty$. Since $n \neq -\infty$ we have a contradiction. Thus $\lambda_n = 0$. Iterate this very reasoning will draw us to the conclusion that $\lambda_k = 0$, for all k in $[\![1, n]\!]$. This is exactly the definition of the linearly independent family.

Exercise 9.

Let u, v and w be three vectors of a vector space E.

1. Show that
$$\mathrm{Span}\{u, v\} = \mathrm{Span}\{u, w\}$$
if and only if
$$\exists(\alpha, \beta, \gamma) \in \mathbf{R}^3,\ \beta\gamma \neq 0,\ \alpha u + \beta v + \gamma w = 0.$$
2. Let F be a subspace of a vector space E. For any vector v of E, define
$$F + \mathbf{R}v := \{f + \gamma v,\ (f, \gamma) \in F \times \mathbf{R}\}.$$
3. Show that $F + \mathbf{R}v$ is a subspace of E.
4. Show that:
$$F + \mathbf{R}v = F + \mathbf{R}w$$
if and only if
$$\exists u \in F,\ \exists(\alpha, \beta) \in \mathbf{R}^2,\ \alpha\beta \neq 0,\ u + \alpha v + \beta w = 0.$$

Solution of Exercise 8.

Exercise 10.

Let F, G and H be three subspaces of a vector space E.
Make a comparison between $F \cup (G + H)$ and $(F \cup G) + (F \cup H)$.

Solution of Exercise 9.

Exercise 11.

Denote $E := \mathscr{F}(\mathbf{R}, \mathbf{R})$ the set of functions from $\mathbf{R}$ to $\mathbf{R}$. Among the follwoing subsets of $(\mathscr{F}(\mathbf{R}, \mathbf{R}), +, \cdot)$, which ones are vector subspaces of E.

1. The set of functions f, for which $f(0) = 0$.
2. The set of functions f, for which $f(0) = 1$.
3. The set of monotonic functions.
4. The set of differentiable functions.

Solution of Exercise 10.

1. Denote $F := \{f : \mathbf{R} \to \mathbf{R},\ f(0) = 0\}$. Let us show that F is a vector subspace of E. F is clearly a subset of E. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by setting $g(x) := 0$, for all real x. Function g clearly belongs to F. Moreover, for any (f, h) in F^2 and α in $\mathbf{R}$, $\alpha f + h$ clearly belongs to E. Besides, since both f and h belong to F, we have:
$$(\alpha f + h)(0) = \alpha f(0) + h(0) = \alpha 0 + 0 = 0.$$
Thus $\alpha f + h$ belongs to F. The associativity and commutativity of both laws $+$ and $\cdot$ come from the associativity and commutativity of E.
2.
3.
4.

3 Other Exercises

Exercise 12.

Let n be in $\mathbb{N}$. Show that the family $(f_1, f_2, \cdots, f_n)$, where:

$$\begin{array}{rccl} f_k : & \mathbf{R} & \to & \mathbf{R} \\ & x & \mapsto & \sin(kx),\ \forall k \in [\![1, n]\!], \end{array}$$

is linearly independent in the set, denoted $\mathscr{F}(\mathbf{R}, \mathbf{R})$, of functions from $\mathbf{R}$ to $\mathbf{R}$.

Solution of Exercise 11.

Exercise 13.

In $\mathbf{R}^3$, let us consider the set:

$$F := \{(x, y, z) \in \mathbf{R}^3,\ x^2 + y^2 + 2z^2 - 2xy - 2yz + 2xz = 0\}.$$

Is F a vector subspace of $\mathbf{R}^3$?

Solution of Exercise 12.

NYU PARIS
SPRING 2021

Math -UA 9140
LINEAR ALGEBRA
COR.OF EX.SET#7
J. LEBOVITS

CORRECTION OF EXERCISES SET # 7
FINITE-DIMENSIONAL VECTOR SPACES & LINEAR MAPS

1 Rank Nullity Theorem

Exercise 1.

Let E be a $\mathbf{K}$ vector space with finite dimension, and let f be in $\mathscr{L}(E)$. Show the following equivalence

$$E = \mathrm{Im}(f) \oplus \mathrm{Ker}(f) \iff \mathrm{Im}(f) = \mathrm{Im}(f^2).$$

Is this equivalence true with spaces of infinite dimension?

Solution of Exercise 1.

2 Linear maps in finite-dimensional Spaces

Exercise 2.

Show that the linear maps from $\mathbf{R}$ to $\mathbf{R}$ are the maps $x \mapsto kx$, with k in $\mathbf{R}$.

Solution of Exercise 2.

These maps are obviously linear. Conversely if f is an endomorphism of $\mathbf{R}$, then

$$\forall x \in \mathbf{R},\ f(x) = f(x \cdot 1) = k \cdot x \quad \text{with } k = f(1).$$

Exercise 3.

Let $u : \mathbf{R}^3 \to \mathbf{R}^2$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ by setting:

$$u(\boldsymbol{x}) := (x_1 + x_2 + x_3, 2x_1 + x_2 - x_3).$$

1. Show that u is a linear map.
2. Determine $\mathrm{Ker}(u)$.

3. Determine $\mathrm{Im}(u)$.
4. Give the dimension of both $\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$.Is their some conform to what one can expect from the Rank Nullity Theorem?
5. (bonus question) Give a matrix A in $\mathscr{M}_{2,3}(\mathbf{R})$ such that $u(\boldsymbol{x}) = A\boldsymbol{x}$.

Solution of Exercise 3.

cf. [Lainé, 2010, Cor. Ex. 1]

Exercise 4.

Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be defined, for all (x, y, z) in $\mathbf{R}^3$, by setting:

$$f(x, y, z) := (x + y + z, -x + 2y + 2z).$$

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and $\mathscr{B}' := (\boldsymbol{e}'_1, , \boldsymbol{e}'_2)$ the standard basis of $\mathbf{R}^2$.

1. Show that f is a linear map.
2. Give a basis of $\mathrm{Ker}(f)$ and deduce [a] $\dim(\mathrm{Im}(f))$.
3. Give a basis of $\mathrm{Im}(f)$.
4. Give a matrix A such that $f(x, y, z) = A \cdot (x, y, z)$.
5. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A. Make a comparison with $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

[a] One may admit the fundamental result which states that $\dim(E) = \dim(\mathrm{Im}(u)) + \dim(\mathrm{Ker}(u))$, for any linear map $u : E \to F$, where the vector spaces E and F both fulfill $\dim(E) < \infty$ and $\dim(F) < \infty$.

Solution of Exercise 4.

cf. [Lainé, 2010, Cor. Ex. 2]

Exercise 5.

Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be defined, for all (x, y) in $\mathbf{R}^2$, by setting:

$$h(x, y) := (x - y, -3x + 3y).$$

1. Show that h is a linear map.
2. Show that h is not injective nor surjective.
3. Give a basis of $\mathrm{Ker}(h)$ and of $\mathrm{Im}(h)$.
4. Give a matrix A such that $h(x, y) = A \cdot (x, y)$.
5. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.

Solution of Exercise 5.

[Lainé, 2010, Cor. Ex. 4]

Exercise 6. (Practice Final Exam)

Define $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. Denote $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{x}) := A \cdot \boldsymbol{x}.$$

1. Compute $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.
2. Determine the coordinates of $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ in the standard basis.
3. By only studying Matrix A, determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.
4. Give a basis of both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

Solution of Exercise 6.

1. One just has to read on the matrix since the columns of A are the vectors $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$. Thus we have:
$$\boxed{f(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} ; f(\boldsymbol{e}_2) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} ; f(\boldsymbol{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.}$$
2. From the previous question we easily deduce that:
$$\boxed{f(\boldsymbol{e}_1) = \boldsymbol{e}_1 - \boldsymbol{e}_3; \qquad f(\boldsymbol{e}_2) = 2\boldsymbol{e}_1 + \boldsymbol{e}_2 - 3\boldsymbol{e}_3; \qquad f(\boldsymbol{e}_3) = -\boldsymbol{e}_2 + 2\boldsymbol{e}_3.}$$
3. It is easy to see that $\mathrm{Ker}(A) = \{0_{\mathbf{R}^3}\}$. Moreover, According to the Rank-nullity theorem [a], we see that $\mathrm{rk}(A) = 3 - \dim(\mathrm{Ker}(A)) = 3$. Thus $\mathrm{rk}(A) = 3$ and thus $C(A) = \mathbf{R}^3$. we can thus conclude that $\boxed{\mathrm{Ker}(f) = \{0_{\mathbf{R}^3}\} \text{ and } \mathrm{Im}(f) = \mathbf{R}^3.}$
4. $\boxed{\begin{array}{c}\text{Since } \mathrm{Ker}(f) = \{0_{\mathbf{R}^3}\}\text{, its basis is } 0_{\mathbf{R}^3}\text{. On the other hand,} \\ \mathscr{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \text{ constitutes a basis of } \mathrm{Im}(f).\end{array}}$

Exercise 7.

Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(x_1, x_2, x_3) := (x_1 - x_3, 2x_1 + x_2 - 3x_3, -x_2 + 2x_3).$$

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$.

1. Give a basis of $\mathrm{Ker}(f)$ and give $\dim(\mathrm{Im}(f))$ without making $\mathrm{Im}(f)$ explicit.
2. Compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.
3. Determine the coordinates of $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ in the standard basis.
4. Give a basis of $\mathrm{Im}(f)$.
5. Give a matrix A such that $f(\boldsymbol{x}) = A \cdot \boldsymbol{x}$.
6. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.

Solution of Exercise 7.

[Lainé, 2010, Cor. Ex. 5]

Exercise 8.

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map defined, for all $\boldsymbol{u} := (x, y, z)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{u}) := (6x - 4y - 4z, 5x - 3y - 4z, x - y).$$

1. Show that there is non null vector $\boldsymbol{a}$ in $\mathbf{R}^3$, such that $\mathrm{Ker}(f) = \mathrm{Span}\{\boldsymbol{a}\}$. Give explicitely such a vector $\boldsymbol{a}$.
2. Define $\boldsymbol{b} := \boldsymbol{e}_1 + \boldsymbol{e}_2$ and $\boldsymbol{c} := \boldsymbol{e}_2 - \boldsymbol{e}_3$.
   (a) Compute $f(\boldsymbol{b})$ and $f(\boldsymbol{c})$.
   (b) Deduce that $(\boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathrm{Im}(f)$. One can you an another method to do so.
3. Give one or several equations that characterize $\mathrm{Im}(f)$.
4. Do we have the equality
$$\mathrm{Im}(f) \bigoplus \mathrm{Ker}(f) = \mathbf{R}^3?$$
5. Give a matrix A such that $f(\boldsymbol{u}) = A \cdot \boldsymbol{u}$.
6. Determine $N(A)$ and $C(A)$ the null space of A and the column space of A.
7. Do we have the equality
$$N(A) \bigoplus C(A) = \mathbf{R}^3?$$

Solution of Exercise 8.

Cf. [Lainé, 2010, Cor. Ex. 5]

Exercise 9. (Practice Final Exam)

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map such that:

$$\begin{cases} f(\boldsymbol{e}_1) := \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_2) := \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_3) := \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3. \end{cases}$$

Define $E_{-1} := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = -\boldsymbol{u}\}$ and $E_1 := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = \boldsymbol{u}\}$.

1. Show that E_{-1} and E_1 are vector subspaces of $\mathbf{R}^3$.
2. Prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1} and that $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.
3. What can we deduce, from the previous question, about the dimension of E_{-1} and E_1?
4. Determine $E_{-1} \cap E_1$.
5. Does the equality $E_{-1} \bigoplus E_1 = \mathbf{R}^3$ hold?
6. Compute $f^2 = f \circ f$. Deduce that f is bijective and determine f^{-1}.

Solution of Exercise 9.

1. It is clear that

$$E_{-1} = \{\boldsymbol{u} \in \mathbf{R}^3,\ (f + Id_{\mathbf{R}^3})(\boldsymbol{u}) = 0_{\mathbf{R}^3}\} = \mathrm{Ker}(f + Id_{\mathbf{R}^3});$$
$$E_1 = \{\boldsymbol{u} \in \mathbf{R}^3,\ (f - Id_{\mathbf{R}^3})(\boldsymbol{u}) = 0_{\mathbf{R}^3}\} = \mathrm{Ker}(f - Id_{\mathbf{R}^3}).$$

Since both $f - Id_{\mathbf{R}^3}$ and $f + Id_{\mathbf{R}^3}$ are linear maps, their Kernel are both vector spaces.

2. One can obviously write:

$$\begin{aligned} f(\boldsymbol{e}_1 - \boldsymbol{e}_2) = f(\boldsymbol{e}_1) - f(\boldsymbol{e}_2) &= \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 - (\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) \\ &= -\boldsymbol{e}_1 + \boldsymbol{e}_2 = -(\boldsymbol{e}_1 - \boldsymbol{e}_2). \end{aligned}$$

and

$$\begin{aligned} f(\boldsymbol{e}_1 - \boldsymbol{e}_3) = f(\boldsymbol{e}_1) - f(\boldsymbol{e}_2) &= \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 - (\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3) \\ &= -\boldsymbol{e}_1 + \boldsymbol{e}_3 = -(\boldsymbol{e}_1 - \boldsymbol{e}_3). \end{aligned}$$

These two equalities prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1}.

Moreover, we have:

$$\begin{aligned} f(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) &= f(\boldsymbol{e}_1) + f(\boldsymbol{e}_2) + f(\boldsymbol{e}_3) \\ &= \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 + \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 + \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3. \end{aligned}$$

Thus $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.

3. From the previous question one can deduce that:

$$\boxed{\dim(E_{-1}) \geqslant 2 \text{ and that } \dim(E_1) \geqslant 1.}$$

4. Let $\boldsymbol{u}$ be an element of $E_{-1} \cap E_1$. We have $f(\boldsymbol{u}) = \boldsymbol{u}$ and $f(\boldsymbol{u}) = -u$. Thus $\boldsymbol{u} = -u$ *i.e.* $\boldsymbol{u} = 0_{\mathbf{R}^3}$. $\boxed{\text{Therefore } E_{-1} \cap E_1 = \{\mathbf{0}_{\mathbf{R}^3}\}.}$

5. In view of the previous question, we know that the equality $E_{-1}+E_1 = E_{-1} \bigoplus E_1$ holds. Moreover, thanks to Question 2, we know that $\dim(E_{-1}+E_1) \geqslant 3$. Since $E_{-1}+E_1 \subset \mathbf{R}^3$ it is clear that $\dim(E_{-1}+E_1) \leqslant 3$ and thus that $\dim(E_{-1}+E_1) = 3$.

$$\boxed{\text{Thus } E_{-1}+E_1 = E_{-1} \bigoplus E_1 = \mathbf{R}^3.}$$

6. We easily get:

$$\begin{aligned}
f(f(\boldsymbol{e}_1)) &= f(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) = \frac{-1}{3}f(\boldsymbol{e}_1) + \frac{2}{3}f(\boldsymbol{e}_2) + \frac{2}{3}f(\boldsymbol{e}_3)\\
&= \frac{-1}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)\\
&= \boldsymbol{e}_1;\\
f(f(\boldsymbol{e}_2)) &= f(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) = \frac{2}{3}f(\boldsymbol{e}_1) + \frac{-1}{3}f(\boldsymbol{e}_2) + \frac{2}{3}f(\boldsymbol{e}_3)\\
&= \frac{2}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{-1}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)\\
&= \boldsymbol{e}_2;\\
f(f(\boldsymbol{e}_3)) &= f(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3) = \frac{2}{3}f(\boldsymbol{e}_1) + \frac{2}{3}f(\boldsymbol{e}_2) + \frac{-1}{3}f(\boldsymbol{e}_3)\\
&= \frac{2}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{-1}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)\\
&= \boldsymbol{e}_3.
\end{aligned}$$

$$\boxed{\begin{array}{c}\text{In other words, } f^2 = Id_{\mathbf{R}^3}\text{. Form the previous equality, we easily deduce}\\ \text{that } f \text{ is invertible and that } f^{-1} = f.\end{array}}$$

Exercise 10.

Define $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let u be an endomorphism of $\mathbf{R}^3$ defined by setting:

$$\begin{cases} u(\boldsymbol{e}_1) := 2\boldsymbol{e}_1 + \boldsymbol{e}_2 + 3\boldsymbol{e}_3\\ u(\boldsymbol{e}_2) := \boldsymbol{e}_2 - 3\boldsymbol{e}_3\\ u(\boldsymbol{e}_3) := -2\boldsymbol{e}_2 + 2\boldsymbol{e}_3.\end{cases}$$

1. Let $\boldsymbol{x} := (x_1, x_2, x_3)$ be a vector in $\mathbf{R}^3$. Compute $u(\boldsymbol{x})$.

2. Give a matrix A such that $u(\boldsymbol{x}) = A \cdot \boldsymbol{x}$.

3. Define $E := \{\boldsymbol{x} \in \mathbf{R}^3,\ u(\boldsymbol{x}) = 2\boldsymbol{x}\}$ and $F := \{\boldsymbol{x} \in \mathbf{R}^3,\ u(\boldsymbol{x}) = -\boldsymbol{x}\}$.

   Show that E and F are vector subspaces of $\mathbf{R}^3$.

4. Give a basis of E and a basis of F.

5. Does the equality $E \bigoplus F = \mathbf{R}^3$ holds?

Solution of Exercise 10.

1.
$$\begin{aligned} u(\boldsymbol{x}) &= u(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = x_1u(\boldsymbol{e}_1) + x_2u(\boldsymbol{e}_2) + x_3u(\boldsymbol{e}_3) \\ &= x_1(2\boldsymbol{e}_1 + \boldsymbol{e}_2 + 3\boldsymbol{e}_3) + x_2(\boldsymbol{e}_2 - 3\boldsymbol{e}_3) + x_3(-2\boldsymbol{e}_2 + 2\boldsymbol{e}_3) \\ &= 2x_1\boldsymbol{e}_1 + (x_1 + x_2 - 2x_3)\boldsymbol{e}_2 + (3x_1 - 3x_2 + 2x_3)\boldsymbol{e}_3 \\ &= (2x_1, x_1 + x_2 - 2x_3, 3x_1 - 3x_2 + 2x_3). \end{aligned}$$

2. Define $A := \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -2 \\ 3 & -3 & 2 \end{pmatrix}$. It is obvious that
$$u(\boldsymbol{x}) = A \cdot \boldsymbol{x},$$
for any $\boldsymbol{x}$ in $\mathbf{R}^3$.

3.
$$\boldsymbol{x} \in E \iff u(\boldsymbol{x}) = 2\boldsymbol{x} \iff (2x_1, x_1 + x_2 - 2x_3, 3x_1 - 3x_2 + 2x_3) = 2(x_1, x_2, x_3)$$
$$\iff \begin{cases} 2x_1 = 2x_1 \\ x_1 + x_2 - 2x_3 = 2x_2 \\ 3x_1 - 3x_2 + 2x_3 = 2x_3 \end{cases} \iff \begin{cases} x_1 - x_2 - 2x_3 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases} \iff \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases}$$

Besides, one has:
$$\boldsymbol{x} \in F \iff u(\boldsymbol{x}) = -\boldsymbol{x} \iff (2x_1, x_1 + x_2 - 2x_3, 3x_1 - 3x_2 + 2x_3) = -(x_1, x_2, x_3)$$
$$\iff \begin{cases} 2x_1 = -x_1 \\ x_1 + x_2 - 2x_3 = -x_2 \\ 3x_1 - 3x_2 + 2x_3 = -x_3 \end{cases} \iff \begin{cases} 3x_1 = 0 \\ x_1 + 2x_2 - 2x_3 = 0 \\ 3x_1 - 3x_2 + 3x_3 = 0 \end{cases} \iff \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$$

$$\boxed{\text{Thus } E = \mathrm{Span}_{\mathbf{R}}\{(1,1,0)\} \text{ and } F = \mathrm{Span}_{\mathbf{R}}\{(0,1,1)\}.}$$

4. We immediately deduce that $(1,1,0)$ is a basis of E while $(0,1,1)$ is a basis of F.

5. Since $E \cap F = \{0_{\mathbf{R}^3}\}$ it is clear that $E + F = E \bigoplus F$. Moreover, since
$$\dim(E + F) = \dim(E) + \dim(F) - \dim(E \cap F) \leqslant \dim(E) + \dim(F) = 2 < 3 = \dim(\mathbf{R}^3),$$
it is therefore clear that $E + F \subsetneq \mathbf{R}^3$ and thus that
$$\boxed{E \bigoplus F \neq \mathbf{R}^3.}$$

Intermediate-level Exercises

Exercise 11.

Let n be in $\mathbf{N}^*$ and let M be the matrix defined by setting:

$$M := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & C_1^1 & C_2^1 & \cdots & C_n^1 \\ \vdots & \ddots & C_2^2 & \cdots & C_n^2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & C_n^n \end{pmatrix} \in \mathscr{M}_{n+1}(\mathbf{R}),$$

where C_n^k is the binomial coefficient (*i.e.* $C_n^k := \frac{n!}{k!(n-k)!}$).
Show that M is invertible and compute its inverse, denoted M^{-1}.

Solution of Exercise 11.

Exercise 12.

Let M be the matrix of $\mathscr{M}_n(\mathbf{R})$ defined by setting:

$$M := \begin{pmatrix} 0 & m_{1,2} & \cdots & m_{1,n} \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & m_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

1. Without using the Theorem of Cayley-Hamilton, show that M is nilpotent.

2. Let $M := \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ be in $\mathscr{M}_3(\mathbf{R})$. Compute, for all p in $\mathbf{N}^*$, M^p.

Solution of Exercise 12.

To be written $\cdots$

Exercise 13.

Let E be a $\mathbf{K}$ vector space with finite dimension. Let f be in $\mathscr{L}(E)$. Assume that there exists x_0 in E such that $\mathscr{B} = (f(x_0), f^2(x_0), \cdots, f^n(x_0))$ is a basis of E.

1. Show that f is bijective.

2. Without using Cayley-Hamilton's Theorem, show that there exists $(a_0, a_1, \cdots, a_{n-1}) \in$

$\mathbf{K}^n$ such that

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \operatorname{Id}_E = 0.$$

Solution of Exercise 13.

Exercise 14.

The goal of this exercise is to determine all the matrices in $\mathscr{M}_n(\mathbf{R})$ such that

$$A^2 = 0_{\mathscr{M}_n(\mathbf{R})}. \tag{1}$$

1. Denote $f : \mathbf{R}^n \to \mathbf{R}^n$ the linear map, the matrix representation of which is A, in the standard basis.
2. Show that $\operatorname{Im}(f) \subset \ker(f)$.
3. Denote r the dimension of $\operatorname{Im}(f)$ and let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r)$ be a basis of $\operatorname{Im}(f)$. Using the Rank Nullity theorem, show that $n - r \geqslant r$ and that there exists a family of vectors, denoted $(\boldsymbol{e}_{r+1}, \boldsymbol{e}_{r+2}, \cdots, \boldsymbol{e}_{n-r})$, such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{n-r})$ is a basis of $\ker(f)$.
4. Denote $(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r)$ the vectors of $\mathbf{R}^n$ such that:
$$\boldsymbol{e}_i = f(\boldsymbol{u}_i),\ \forall i \in [\![1, r]\!].$$
5. Show that $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, e_n)$ is a basis of $\mathbf{R}^n$.
6. Give the matrix which represents f from $\mathscr{B}$ to $\mathscr{B}$.
7. Finally determine all the matrices in $\mathscr{M}_n(\mathbf{R})$ such that (1) holds.

Solution of Exercise 14.

To be written!

Exercise 15.

Define $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. Denote $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for all $\boldsymbol{x} := (x_1, x_2, x_3)$ in $\mathbf{R}^3$, by setting:

$$f(\boldsymbol{x}) := A \cdot \boldsymbol{x}.$$

1. Compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$.
2. Determine the coordinates of $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ in the standard basis.
3. By only studying Matrix A, determine $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$.
4. Give a basis of both $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$.

Solution of Exercise 15.

1. One just has to read on the matrix since the columns of A are the vectors $f(\boldsymbol{e}_1), f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$. Thus we have:
$$\boxed{f(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix};\ f(\boldsymbol{e}_2) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix};\ f(\boldsymbol{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.}$$

2. From the previous question we easily deduce that:
$$\boxed{f(\boldsymbol{e}_1) = \boldsymbol{e}_1 - \boldsymbol{e}_3; \qquad f(\boldsymbol{e}_2) = 2\boldsymbol{e}_1 + \boldsymbol{e}_2 - 3\boldsymbol{e}_3; \qquad f(\boldsymbol{e}_3) = -\boldsymbol{e}_2 + 2\boldsymbol{e}_3.}$$

3. It is easy to see that $\mathrm{Ker}(A) = \{0_{\mathbf{R}^3}\}$. Moreover, According to the Rank-nullity theorem [a], we see that $\mathrm{rk}(A) = 3 - \dim(\mathrm{Ker}(A)) = 3$. Thus $\mathrm{rk}(A) = 3$ and thus $C(A) = \mathbf{R}^3$. we can thus conclude that $\boxed{\mathrm{Ker}(f) = \{0_{\mathbf{R}^3}\} \text{ and } \mathrm{Im}(f) = \mathbf{R}^3.}$

4. $\boxed{\begin{array}{c}\text{Since } \mathrm{Ker}(f) = \{0_{\mathbf{R}^3}\}, \text{ its basis is } 0_{\mathbf{R}^3}. \text{ On the other hand,} \\ \mathscr{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \text{ constitutes a basis of } \mathrm{Im}(f).\end{array}}$

Exercise 16.

Let $\mathscr{B} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$ and let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map such that:
$$\begin{cases} f(\boldsymbol{e}_1) := \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_2) := \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 \\ f(\boldsymbol{e}_3) := \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3. \end{cases}$$
Define $E_{-1} := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = -\boldsymbol{u}\}$ and $E_1 := \{\boldsymbol{u} \in \mathbf{R}^3,\ f(\boldsymbol{u}) = \boldsymbol{u}\}$.

1. Show that E_{-1} and E_1 are vector subspaces of $\mathbf{R}^3$.
2. Prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1} and that $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.
3. What can we deduce, from the previous question, about the dimension of E_{-1} and E_1?
4. Determine $E_{-1} \cap E_1$.
5. Does the equality $E_{-1} \bigoplus E_1 = \mathbf{R}^3$ hold?
6. Compute $f^2 = f \circ f$. Deduce that f is bijective and determine f^{-1}.

Solution of Exercise 16.

1. It is clear that
$$E_{-1} = \{\boldsymbol{u} \in \mathbf{R}^3,\ (f + Id_{\mathbf{R}^3})(\boldsymbol{u}) = 0_{\mathbf{R}^3}\} = \mathrm{Ker}(f + Id_{\mathbf{R}^3});$$
$$E_1 = \{\boldsymbol{u} \in \mathbf{R}^3,\ (f - Id_{\mathbf{R}^3})(\boldsymbol{u}) = 0_{\mathbf{R}^3}\} = \mathrm{Ker}(f - Id_{\mathbf{R}^3}).$$

Since both $f - Id_{\mathbf{R}^3}$ and $f + Id_{\mathbf{R}^3}$ are linear maps, their Kernel are both vector spaces.

2. One can obviously write:

$$f(\boldsymbol{e}_1 - \boldsymbol{e}_2) = f(\boldsymbol{e}_1) - f(\boldsymbol{e}_2) = \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 - (\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3)$$
$$= -\boldsymbol{e}_1 + \boldsymbol{e}_2 = -(\boldsymbol{e}_1 - \boldsymbol{e}_2).$$

and

$$f(\boldsymbol{e}_1 - \boldsymbol{e}_3) = f(\boldsymbol{e}_1) - f(\boldsymbol{e}_2) = \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 - (\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)$$
$$= -\boldsymbol{e}_1 + \boldsymbol{e}_3 = -(\boldsymbol{e}_1 - \boldsymbol{e}_3).$$

These two equalities prove that $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_1 - \boldsymbol{e}_3$ both belong to E_{-1}.

Moreover, we have:

$$f(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = f(\boldsymbol{e}_1) + f(\boldsymbol{e}_2) + f(\boldsymbol{e}_3)$$
$$= \frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 + \frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3 + \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3.$$

Thus $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ belongs to E_1.

3. From the previous question one can deduce that:

$\dim(E_{-1}) \geqslant 2$ and that $\dim(E_1) \geqslant 1$.

4. Let $\boldsymbol{u}$ be an element of $E_{-1} \cap E_1$. We have $f(\boldsymbol{u}) = \boldsymbol{u}$ and $f(\boldsymbol{u}) = -u$. Thus $\boldsymbol{u} = -u$ *i.e.* $\boldsymbol{u} = 0_{\mathbf{R}^3}$. Therefore $E_{-1} \cap E_1 = \{\mathbf{0}_{\mathbf{R}^3}\}$.

5. In view of the previous question, we know that the equality $E_{-1}+E_1 = E_{-1} \bigoplus E_1$ holds. Moreover, thanks to Question 2, we know that $\dim(E_{-1}+E_1) \geqslant 3$. Since $E_{-1}+E_1 \subset \mathbf{R}^3$ it is clear that $\dim(E_{-1} + E_1) \leqslant 3$ and thus that $\dim(E_{-1} + E_1) = 3$.

Thus $E_{-1} + E_1 = E_{-1} \bigoplus E_1 = \mathbf{R}^3$.

6. We easily get:

$$f(f(\boldsymbol{e}_1)) = f(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) = \frac{-1}{3}f(\boldsymbol{e}_1) + \frac{2}{3}f(\boldsymbol{e}_2) + \frac{2}{3}f(\boldsymbol{e}_3)$$
$$= \frac{-1}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)$$
$$= \boldsymbol{e}_1;$$
$$f(f(\boldsymbol{e}_2)) = f(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) = \frac{2}{3}f(\boldsymbol{e}_1) + \frac{-1}{3}f(\boldsymbol{e}_2) + \frac{2}{3}f(\boldsymbol{e}_3)$$
$$= \frac{2}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{-1}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)$$
$$= \boldsymbol{e}_2;$$
$$f(f(\boldsymbol{e}_3)) = f(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3) = \frac{2}{3}f(\boldsymbol{e}_1) + \frac{2}{3}f(\boldsymbol{e}_2) + \frac{-1}{3}f(\boldsymbol{e}_3)$$
$$= \frac{2}{3}(\frac{-1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{2}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{-1}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3) + \frac{-1}{3}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{-1}{3}\boldsymbol{e}_3)$$
$$= \boldsymbol{e}_3.$$

In other words, $f^2 = Id_{\mathbf{R}^3}$. Form the previous equality, we easily deduce that f is invertible and that $f^{-1} = f$.

NYU PARIS

SPRING 2021

Math -UA 9140
LINEAR ALGEBRA
COR. OF EX. SET#8
J. LEBOVITS

CORRECTION OF EXERCISES SET # 8
DETERMINANTS

1 Basic Exercises

Exercise 1.

Compute all the determinants below:

1.
   - By cofactor expansion along the first column
   - By cofactor expansion along the second row.
   - By Gaussian Elimination
2. Which method is the fastest?

$$\delta_1 := \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix}, \quad \delta_2 := \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{vmatrix}, \quad \delta_3 := \begin{vmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & -2 & -2 \end{vmatrix}, \quad \delta_4 := \begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix},$$

$$\delta_5 := \begin{vmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix}, \quad \delta_6 := \begin{vmatrix} -1 & -2 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix}, \quad \delta_7 := \begin{vmatrix} 0 & -2 & 2 \\ -1 & -1 & 0 \\ 2 & 2 & 2 \end{vmatrix}, \quad \delta_8 := \begin{vmatrix} 1 & -2 & -1 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{vmatrix}.$$

Solution of Exercise 1.

Exercise 2.

Compute all the determinants below:

1.
   - By cofactor expansion along the first column
   - By cofactor expansion along the second row.
   - By Gaussian Elimination
2. Which method is the fastest?

$$\delta_1' := \begin{vmatrix} -1 & -1 & -2 & -1 \\ -2 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 2 & 0 \end{vmatrix}, \quad \delta_2' := \begin{vmatrix} 2 & 2 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ -2 & 2 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{vmatrix}, \quad \delta_3' := \begin{vmatrix} 2 & 0 & -1 & 2 \\ 2 & -1 & 1 & -1 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & -2 & 2 \end{vmatrix},$$

$$\delta_4' := \begin{vmatrix} -2 & -1 & 0 & 2 \\ 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & -1 \end{vmatrix}, \quad \delta_5' := \begin{vmatrix} -2 & -1 & -2 & 1 \\ 2 & 2 & -1 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{vmatrix}, \quad \delta_6' := \begin{vmatrix} -2 & -1 & 0 & 2 \\ -1 & 1 & -1 & 0 \\ 2 & 2 & 2 & -1 \\ -1 & 2 & 2 & -2 \end{vmatrix}.$$

Solution of Exercise 2.

Exercise 3.

Define, for any matrice M, the Characteristic polynomial of M, denoted χ_M, by setting

$$\chi_M(X) := \mathbf{d}\text{et}(M - XI_3).$$

1. Determine the polynomial characteristic of all matrices $(M_i)_{i\in[\![1,3]\!]}$ given below.

$$M_1 := \begin{pmatrix} 2 & 2 & -3 \\ 5 & 1 & -5 \\ -3 & 4 & 0 \end{pmatrix}; \quad M_2 := \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}; \quad M_3 := \begin{pmatrix} 1 & 4 & -2 \\ 0 & 6 & -3 \\ -1 & 4 & 0 \end{pmatrix}$$

2. Find the roots of these characteristic polynomials.
3. For any root λ of χ_M, denote $E_\lambda := \mathrm{Ker}(M - \lambda I_3)$. Determine E_λ for any root λ of χ_{M_1}.
4. Determine E_λ for any root λ_i of the family $(\chi_{M_i})_{i\in[\![1,3]\!]}$.

Solution of Exercise 3.

Exercise 4.

1. Give an example of a non-diagonal 2×2 matrix for which $A^2 = I_2$.
2. In general, if $A^2 = I_2$, show that $\mathbf{d}\text{et}(A) \in \{-1, 1\}$
3. If $A^2 = A$, what can you say about $\mathbf{d}\text{et}(A)$?

Solution of Exercise 4.

1. Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathscr{M}_2(\mathbf{R})$. We know that:

$$A^2 = I_2 \Longleftrightarrow \begin{pmatrix} a^2+bc & (a+d)\cdot b \\ c\cdot(a+d) & bc+d^2 \end{pmatrix} = I_2 \Longleftrightarrow \begin{cases} a^2+bc=1 \\ (a+d)\cdot b = 0 \\ c\cdot(a+d)=0 \\ bc+d^2=1 \end{cases} \tag{1}$$

Since A must be non-diagonal, it requires that c or b equals 0. Let's take both c and b different from 0. The system of equations (1) requires that $a+d=0$ *i.e.* that $a=-d$. It hen reduces to

$$a^2+bc=1.$$

Let's take for example $a = 1/2$. We then get $bc = 3/4$. One can choose $b = 1/2$ and $c = 3/2$. Thus one has:

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}$$

2.

$$\begin{aligned} A^2 = I_2 &\Longrightarrow \mathbf{det}(A^2) = \mathbf{det}(I_2) \Longrightarrow (\mathbf{det}(A))^2 = 1 \\ &\Longrightarrow (\mathbf{det}(A))^2 - 1 = 0 \Longrightarrow (\mathbf{det}(A)-1)\cdot(\mathbf{det}(A)+1) = 0 \Longrightarrow \mathbf{det}(A) \in \{-1,1\}. \end{aligned}$$

3.

$$\begin{aligned} A^2 = A &\Longrightarrow \mathbf{det}(A^2) = \mathbf{det}(A) \Longrightarrow (\mathbf{det}(A))^2 = \mathbf{det}(A) \\ &\Longrightarrow \mathbf{det}A\cdot(\mathbf{det}(A)-1) = 0 \Longrightarrow \mathbf{det}(A) \in \{0,1\}. \end{aligned}$$

Exercise 5.

Two matrices A and B in $\mathscr{M}_n(\mathbf{R})$ are said to be similar, in $\mathbf{R}$, if there exists S in $\mathbf{GL}_n(\mathbf{R})$ such that $B = S^{-1}AS$.
Show that all similar matrices have the same determinant.

Solution of Exercise 5.

Exercise 6.

1. Show that if $D := \begin{pmatrix} A & 0_{\mathscr{M}_n(\mathbf{R})} \\ 0_{\mathscr{M}_n(\mathbf{R})} & B \end{pmatrix}$ is block diagonal matrix, where A and B are square matrices (which both belong to $\mathscr{M}_n(\mathbf{R})$), then

$$\mathbf{det}(A\cdot B) = \mathbf{det}(A)\cdot\mathbf{det}(B). \tag{1}$$

2. Prove that the same holds for a block upper triangular matrix *i.e.* that Equality

$$\mathbf{det}\begin{pmatrix} A & C \\ 0_{\mathscr{M}_n(\mathbf{R})} & B \end{pmatrix} = \mathbf{det}(A) \cdot \mathbf{det}(B). \tag{2}$$

holds.

Solution of Exercise 6.

Exercise 7.

Let A, B, C and D be elements of $\mathscr{M}_n(\mathbf{R})$ such that $DC = CD$.
If D is invertible, show that

$$\mathbf{det}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbf{det}(AD - BC) \tag{1}$$

In order to so, we might try to find matrices, denoted A', B', C', D', E', in $\mathscr{M}_n(\mathbf{R})$ such that:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AD - BC & B'' \\ O_{\mathscr{M}_n(\mathbf{R})} & I_n \end{pmatrix}$$

and such that the determinant of $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ is easily computable.

Solution of Exercise 7.

Exercise 8.

Use the method seen in Exercise 203 to compute the determinant of the following matrices:

$$M_1 := \begin{pmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \\ 0 & 3 & 7 \end{pmatrix}; \qquad M_2 := \begin{pmatrix} 1 & 2 & -2 & 5 \\ -3 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix};$$

$$M_3 := \begin{pmatrix} 1 & 2 & 0 & 4 \\ -3 & 1 & 4 & -1 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}; \qquad M_4 := \begin{pmatrix} 5 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 2 & 4 & 4 & -2 \\ 3 & -2 & 9 & -5 \end{pmatrix}.$$

Solution of Exercise 8.

Exercise 9.

Solution of Exercise 9.

Exercise 10.

Solution of Exercise 10.

Exercise 11.

Solution of Exercise 11.

Exercise 12.

Solution of Exercise 12.

Exercise 13.

Solution of Exercise 13.

CORRECTION OF EXERCISES SET # 9
REDUCTION OF ENDOMORPHISMS

1 Basic Exercises

Change of Basis

Exercise 1.

Let $u : \mathbf{R}^p \to \mathbf{R}^q$, be a linear map. Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_p)$ be the standard basis of $\mathbf{R}^p$ and $\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_q)$ be the standard basis of $\mathbf{R}^q$.

1. **Case where $p = 3$ and $q = 2$:**
   Define u by setting:
$$\begin{cases} u(\boldsymbol{e}_1) := \boldsymbol{f}_1 + 2\boldsymbol{f}_2 \\ u(\boldsymbol{e}_2) := \boldsymbol{f}_1 - \boldsymbol{f}_2 \\ u(\boldsymbol{e}_3) := -\boldsymbol{f}_1 + \boldsymbol{f}_2. \end{cases}$$
   (i) Determine the image of any vector $\boldsymbol{x} = (x_1, x_2, x_3)$ by u.
   (ii) Determine the matrix of the linear map u, from basis $\boldsymbol{e}$ to basis $\boldsymbol{f}$.
   (iii) Determine $\mathrm{Ker}(u)$.

2. **Case where $p = 3$ and $q = 3$:**
   In this question $\boldsymbol{e} = \boldsymbol{f}$. Define u by setting:
$$\begin{cases} u(\boldsymbol{e}_1) := 3\boldsymbol{f}_1 + 2\boldsymbol{f}_2 + 2\boldsymbol{f}_3 \\ u(\boldsymbol{e}_2) := 2\boldsymbol{f}_1 + 3\boldsymbol{f}_2 + 2\boldsymbol{f}_3 \\ u(\boldsymbol{e}_3) := 2\boldsymbol{f}_1 + 2\boldsymbol{f}_2 + 3\boldsymbol{f}_3. \end{cases}$$
   (i) Determine the image of any vector $\boldsymbol{x} = (x_1, x_2, x_3)$ by u.
   (ii) Determine the matrix of the linear map u, from basis $\boldsymbol{e}$ to basis $\boldsymbol{e}$.
   (iii) Determine $\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$.

Solution of Exercise 1.

Exercise 2.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$, the representative matrix of which, in the standard basis, is

$$A := \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$$

1. Show that $E_1 := \{x \in \mathbf{R}^3,\ u(\boldsymbol{x}) = \boldsymbol{x}\}$ is a vector subspace of $\mathbf{R}^3$ and give a basis of it, denoted $\boldsymbol{a}$.
2. Define $\boldsymbol{b} := (0, 1, 1)$ and $\boldsymbol{c} := (1, 1, 2)$ two vectors of $\mathbf{R}^3$. Compute $u(\boldsymbol{b})$ and $u(\boldsymbol{c})$.
3. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.
4. Determine the matrix, denoted P, representing a change of basis from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$.
5. Compute P^{-1}.
6. Determine the matrix D of u, in the basis $\boldsymbol{\beta}'$.
7. Give the relation that exists between A, P and D.
8. Compute A^n, for every integer n in $\mathbf{N}$.

Solution of Exercise 2.

Exercise 3.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$ defined by:

$$u(x_1, x_2, x_3) := (x_2 - 2x_3, 2x_1 - x_2 + 4x_3, x_1 - x_2 + 3x_3).$$

1. Determine the matrix A of u in the standard basis.
2. Determine a basis $(\boldsymbol{a}, \boldsymbol{b})$ of $\mathrm{Ker}(u - Id_{\mathbf{R}^3})$.
3. Give a vector $\boldsymbol{c}$ such that $\mathrm{Ker}(u) = \mathrm{Span}\{\boldsymbol{c}\}$.
4. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.
5. Determine the matrix D of u in the basis $\boldsymbol{\beta}'$. In other words, $\mathrm{Mat}_{\boldsymbol{\beta}', \boldsymbol{\beta}'}(u) = D$.
6. Show that $\mathrm{Im}(u) = \mathrm{Ker}(u - Id_{\mathbf{R}^3})$.
7. Show that $\mathrm{Ker}(u) \bigoplus \mathrm{Im}(u) = \mathbf{R}^3$ holds.
8. Compute A^n, for every integer n in $\mathbf{N}$.

Solution of Exercise 3.

Exercise 4.

Let $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ be the standard basis of $\mathbf{R}^3$. Let f be the endomorphism of $\mathbf{R}^3$ defined by:

$$\begin{cases} f(\boldsymbol{e}_1) := 2\boldsymbol{\beta}_2 + 3\boldsymbol{\beta}_3 \\ f(\boldsymbol{e}_2) := 2\boldsymbol{\beta}_1 - 5\boldsymbol{\beta}_2 - 8\boldsymbol{\beta}_3 \\ f(\boldsymbol{e}_3) := -\boldsymbol{\beta}_1 + 4\boldsymbol{\beta}_2 + 6\boldsymbol{\beta}_3. \end{cases}$$

Denote $f^2 := f \circ f$.

1. Determine the matrix of f in basis $\boldsymbol{\beta}$ (*i.e.* $\mathrm{Mat}_{\boldsymbol{\beta},\boldsymbol{\beta}}(f)$).
2. Show that $E_1 := \mathrm{Ker}(f - Id_{\mathbf{R}^3})$ and $N_{-1} := \mathrm{Ker}(f^2 + Id_{\mathbf{R}^3})$ are vector subspaces of $\mathbf{R}^3$.
3. Determine two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $E_1 = \mathrm{Span}\{\boldsymbol{a}\}$ and $N_{-1} = \mathrm{Span}\{\boldsymbol{b}, \boldsymbol{f}(b)\}$. Does the equality $E_1 \bigoplus N_{-1} = \mathbf{R}^3$ hold?
4. Define $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, f(\boldsymbol{b}))$. Show that $\boldsymbol{\beta}'$ is a basis of $\mathbf{R}^3$.
5. Give $\mathrm{Mat}_{\boldsymbol{\beta}',\boldsymbol{\beta}'}(f)$.
6. Give $\mathrm{Mat}_{\boldsymbol{\beta}',\boldsymbol{\beta}'}(f^2)$.

Solution of Exercise 4.

A taper je l'ai déjà corrigé en version manuscrite.

Exercise 5.

For each of the following matrices determine the eigenvalues and precise if they are diagonalizable?

$$A_1 := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad A_2 := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}; \quad A_3 := \begin{pmatrix} -1 & -4 & -4 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix};$$

$$A_4 := \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \quad A_5 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}; \quad A_6 := \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & -4 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix};$$

Solution of Exercise 5.

Exercise 6.

Let A be in $\mathscr{M}_n(\mathbf{R})$. Prove that if A is a diagonalizable matrix, then so is $cA + dI_n$, for any real numbers c and d.

Solution of Exercise 6.

Exercise 7.

1. Prove that if A is diagonalizable then so is A^2.
2. Give an example of a non diagonalizable matrix A such that A^2 is diagonalizable.

Solution of Exercise 7.

Exercise 8.

Let A be in $\mathscr{M}_n(\mathbf{R})$. Prove that if A is a diagonalizable then so is every similar matrix $B := S^{-1}AS$, where S belongs to $\mathbf{GL}_n(\mathbf{R})$.

Solution of Exercise 8.

Exercise 9.

Let A be in $\mathscr{M}_n(\mathbf{R})$. Prove that if A is a diagonalizable then so is every similar matrix $B := S^{-1}AS$, where S belongs to $\mathbf{GL}_n(\mathbf{R})$.

Exercise 10.

Let u be the endomorphism of $\mathbf{R}^4$, which representative matrix in the standard basis $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4)$ is:

$$A := \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Define $\boldsymbol{a} := \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$, $\boldsymbol{b} := \boldsymbol{e}_1$, $\boldsymbol{c} := u(\boldsymbol{b})$ and $\boldsymbol{d} := u^2(\boldsymbol{b})$.

1. Show that $\boldsymbol{\beta}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is a basis of $\mathbf{R}^4$.
2. Determine the matrix, denoted P, representing a change of basis from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$. Compute P^{-1}
3. Compute $u(\boldsymbol{a}), u(\boldsymbol{b}), u(\boldsymbol{c})$ and $u(\boldsymbol{d})$ and write them in the basis $\boldsymbol{\beta}'$.

4. Compute N^4 and deduce A^4.
5. Determine a basis of $\mathrm{Ker}(u)$.
6. Determine a basis of $\mathrm{Im}(u)$.

Solution of Exercise 9.

Exercise 11.

Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ be the standard basis of $\mathbf{R}^4$. Let u be the endomorphism of $\mathbf{R}^4$, which representative matrix in the standard basis $\boldsymbol{e}$ is given by:

$$A := \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 0 & -2 \end{pmatrix}$$

1. Determine a vector $\boldsymbol{a}$ which spans the kernel of u.
2. Let λ be a real number. Show that $E_\lambda := \{\boldsymbol{x} \in \mathbf{R}^4,\ u(\boldsymbol{x}) = \lambda \boldsymbol{x}\}$ is a vector subspace of $\mathbf{R}^4$.
3. Find a vector $\boldsymbol{b}$ such that $E_{-1} = \mathrm{Span}\{\boldsymbol{b}\}$
4. Determine a basis $(\boldsymbol{c}, \boldsymbol{d})$ of E_1.
5. Show that $\boldsymbol{e}' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is a basis of $\mathbf{R}^4$.
6. Determine the matrix D of u in the basis $\boldsymbol{e}'$. In other words, give $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u)$.

Solution of Exercise 10.

Exercise 12.

Let $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the standard basis of $\mathbf{R}^3$. Let u be the endomorphism of $\mathbf{R}^3$, which representative matrix in the standard basis is given by:

$$A := \begin{pmatrix} -10 & -3 & -12 \\ 5 & 0 & 7 \\ 6 & 2 & 7 \end{pmatrix}$$

1. Determine all the λ in $\mathbf{R}$ such that $A - \lambda I_3$ is not invertible. Then determine $\mathrm{Ker}(A - \lambda I_3)$.
2. Let $\boldsymbol{a} := (-3, 1, 2)$, compute $u(\boldsymbol{a})$.
3. Determine $\boldsymbol{b}$ in $\mathbf{R}^3$ such that $u(\boldsymbol{b}) = \boldsymbol{a} - \boldsymbol{b}$, and then $\boldsymbol{c}$ in $\mathbf{R}^3$ such that $u(\boldsymbol{c}) = b - c$.

4. Show that $\beta' := (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a basis of $\mathbf{R}^3$.

5. Determine $T := \mathrm{Mat}_{\beta',\beta'}(u)$.

6. Show that $(T + I_3)^3 = \mathbf{0}_{\mathscr{M}_3(\mathbf{R})}$. Then deduce $(A + I_3)^3$.

7. Determine A^{-1} in function of A^2, A and I_3.

Solution of Exercise 11.

Problem 1.

Let m be a real number and let A_m be the element of $\mathscr{M}_3(\mathbf{R})$ defined by:

$$A_m := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1+m & m \\ 1 & -1 & 1 \end{pmatrix}$$

Part I: Solving the general regular case: $(\mathscr{S}_m)$, for m in $\mathbf{R}$.

1. Compute the determinant of A_2.

2. Is A_2 invertible?

3. Compute the determinant of A_m.

4. For which values of m is A_m invertible? (**One will not try to compute the inverse of** A_m)?

5. Prove that
$$A_m^{-1} := \begin{pmatrix} \frac{1+2m}{2m} & -\frac{1}{m} & \frac{-1}{2m} \\ 1/2 & 0 & -1/2 \\ \frac{-(1+m)}{2m} & \frac{1}{m} & \frac{1+m}{2m} \end{pmatrix}$$
whenever A_m is invertible.

6. Solve the linear system
$$(\mathscr{S}_m) \begin{cases} x + y + z = 1 \\ (1+m)y + mz = 2 \\ x - y + z = 3. \end{cases}$$
for all real m. We might want to treat separately the cases $m = 0$ and $m \neq 0$.

Part II: Diagonalizing (A_m), for m in $\mathbf{R}$.

7. Denote χ_{A_m} the Characteristic polynomial of A_m. Show that
$$\chi_{A_m}(X) = -X^3 + (3+m)X^2 + (-2-3m)X + 2m.$$

8. Prove that $1, 2$ and m are roots of χ_{A_m}.

9. Deduce that

$$\chi_{A_m}(X) = -(X-1)\cdot(X-2)\cdot(X-m). \tag{1}$$

Diagonalizing A_m, for m in $\mathbf{R}\backslash\{1,2\}$.

Denote $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and define $u_m : \mathbf{R}^3 \to \mathbf{R}^3$ the endomorphism, the matrix representation of which is A_m (in the standard basis). In other words,

$$\mathrm{Mat}_{\boldsymbol{e},\boldsymbol{e}}(u_m) = A_m. \tag{2}$$

First case: Diagonalizing A_0.

We here assume that $m = 0$.

10. Define $\boldsymbol{f}_1 := (1, 1, -1)$, $\boldsymbol{f}_2 := (1, 0, 1)$ and $\boldsymbol{f}_3 := (-1, 0, 1)$ and define $\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Using determinant, show that the vectors of $\boldsymbol{f}$ are linearly independent.

11. Show that $\boldsymbol{f}$ is a basis of $\mathbf{R}^3$.

12. Denote

$$Q := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Compute the inverse, denoted Q^{-1}, of Q.

13. Give the matrix representation of u_0 in the basis $\boldsymbol{f}$. Denote it $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

14. Define $D' := Q^{-1}A_0Q$. Compute D'.

15. Make a comparison between D' and $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

16. Compute D'^4 and deduce the fourth power of A_0, denoted A_0^4.

17. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0))^4$.

Second Case: Diagonalizing A_m, with m in $\mathbf{R}\backslash\{0,1,2\}$.

We here assume that m belongs to $\mathbf{R}\backslash\{0,1,2\}$.

18. Define $\boldsymbol{e}'_1 := (1, 1, -1)$, $\boldsymbol{e}'_2 := (\frac{1}{m}, 1, \frac{1-m}{m})$ and $\boldsymbol{e}'_3 := (-1, -m, 1)$ and define $\boldsymbol{e}' := (\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$. Show that the vectors of $\boldsymbol{e}'$ are linearly independent.

19. Show that $\boldsymbol{e}'$ is a basis of $\mathbf{R}^3$.

20. Denote

$$P_m := \begin{pmatrix} 1 & m^{-1} & -1 \\ 1 & 1 & -m \\ -1 & \frac{1-m}{m} & 1 \end{pmatrix}$$

Show that P_m is invertible and that its inverse, denoted P_m^{-1}, verifies the following equality:

$$P_m^{-1} = \begin{pmatrix} \frac{m}{-1+m} & -(-1+m)^{-1} & 0 \\ -\frac{m}{m-2} & 0 & -\frac{m}{m-2} \\ -\left(m^2-3\,m+2\right)^{-1} & -(-1+m)^{-1} & -(m-2)^{-1} \end{pmatrix}. \tag{3}$$

21. Give the matrix representation of u_m in the basis $\boldsymbol{e}'$. Denote it $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

22. Define $D_m := P_m^{-1} A_m P_m$. Compute D_m.

23. Make a comparison between D_m and $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

24. Compute $(D_m)^4$ and deduce the fourth power of A_m, denoted A_m^4.

25. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m))^4$.

Solution of Problem 1.

Let m be a real number and let A_m be the element of $\mathscr{M}_3(\mathbf{R})$ defined by:

$$A_m := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1+m & m \\ 1 & -1 & 1 \end{pmatrix}$$

Part I: Solving the general regular case: $(\mathscr{S}_m)$, for m in $\mathbf{R}$.

1. Compute the determinant of A_2.

2. Is A_2 invertible?

3. Compute the determinant of A_m.

4. For which values of m is A_m invertible? **(One will not try to compute the inverse of** A_m)?

5. Prove that

$$A_m^{-1} := \begin{pmatrix} \frac{1+2m}{2m} & -\frac{1}{m} & \frac{-1}{2m} \\ 1/2 & 0 & -1/2 \\ \frac{-(1+m)}{2m} & \frac{1}{m} & \frac{1+m}{2m} \end{pmatrix}$$

whenever A_m is invertible.

6. Solve the linear system

$$(\mathscr{S}_m) \begin{cases} x+y+z=1 \\ (1+m)y+mz=2 \\ x-y+z=3. \end{cases}$$

for all real m. We might want to treat separately the cases $m = 0$ and $m \neq 0$.

Part II: Diagonalizing (A_m), for m in $\mathbf{R}$.

7. Denote χ_{A_m} the Characteristic polynomial of A_m. Show that

$$\chi_{A_m}(X) = -X^3 + (3+m)X^2 + (-2-3m)X + 2m.$$

8. Prove that $1, 2$ and m are roots of χ_{A_m}.

9. Deduce that

$$\chi_{A_m}(X) = -(X-1)\cdot(X-2)\cdot(X-m). \tag{1}$$

Diagonalizing A_m, for m in $\mathbf{R}\backslash\{1,2\}$.

Denote $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbf{R}^3$ and define $u_m : \mathbf{R}^3 \to \mathbf{R}^3$ the endomorphism, the matrix representation of which is A_m (in the standard basis). In other words,

$$\mathrm{Mat}_{\boldsymbol{e},\boldsymbol{e}}(u_m) = A_m. \tag{2}$$

First case: Diagonalizing A_0.

We here assume that $m = 0$.

10. Define $\boldsymbol{f}_1 := (1,1,-1)$, $\boldsymbol{f}_2 := (1,0,1)$ and $\boldsymbol{f}_3 := (-1,0,1)$ and define $\boldsymbol{f} := (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Using determinant, show that the vectors of $\boldsymbol{f}$ are linearly independent.

11. Show that $\boldsymbol{f}$ is a basis of $\mathbf{R}^3$.

12. Denote

$$Q := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Compute the inverse, denoted Q^{-1}, of Q.

13. Give the matrix representation of u_0 in the basis $\boldsymbol{f}$. Denote it $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

14. Define $D' := Q^{-1}A_0Q$. Compute D'.

15. Make a comparison between D' and $\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0)$.

16. Compute D'^4 and deduce the fourth power of A_0, denoted A_0^4.

17. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{f},\boldsymbol{f}}(u_0))^4$.

Second Case: Diagonalizing A_m, with m in $\mathbf{R}\backslash\{0,1,2\}$.

We here assume that m belongs to $\mathbf{R}\backslash\{0,1,2\}$.

18. Define $\boldsymbol{e}'_1 := (1,1,-1)$, $\boldsymbol{e}'_2 := (\frac{1}{m}, 1, \frac{1-m}{m})$ and $\boldsymbol{e}'_3 := (-1,-m,1)$ and define $\boldsymbol{e}' := (\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$. Show that the vectors of $\boldsymbol{e}'$ are linearly independent.

19. Show that $\boldsymbol{e}'$ is a basis of $\mathbf{R}^3$.

20. Denote

$$P_m := \begin{pmatrix} 1 & m^{-1} & -1 \\ 1 & 1 & -m \\ -1 & \frac{1-m}{m} & 1 \end{pmatrix}$$

Show that P_m is invertible and that its inverse, denoted P_m^{-1}, verifies the following equality:

$$P_m^{-1} = \begin{pmatrix} \frac{m}{-1+m} & -(-1+m)^{-1} & 0 \\ -\frac{m}{m-2} & 0 & -\frac{m}{m-2} \\ -\left(m^2-3\,m+2\right)^{-1} & -(-1+m)^{-1} & -(m-2)^{-1} \end{pmatrix}. \tag{3}$$

21. Give the matrix representation of u_m in the basis $\boldsymbol{e}'$. Denote it $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

22. Define $D_m := P_m^{-1} A_m P_m$. Compute D_m.

23. Make a comparison between D_m and $\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m)$.

24. Compute $(D_m)^4$ and deduce the fourth power of A_m, denoted A_m^4.

25. Deduce the value of $(\mathrm{Mat}_{\boldsymbol{e}',\boldsymbol{e}'}(u_m))^4$.

CORRECTION OF EXERCISES SET # 10
EUCLIDEAN SPACES

To be filled

Exercise 1.

Solution of Exercise 1.

Exercise 2.

Solution of Exercise 2.

Exercise 3.

Solution of Exercise 3.

Exercise 4.

Solution of Exercise 4.

Exercise 5.

Solution of Exercise 5.

Exercise 6.

Solution of Exercise 6.

Exercise 7.

Solution of Exercise 7.

Exercise 8.

Exercise 9.

Solution of Exercise 8.

Exercise 10.

Solution of Exercise 9.

CORRECTION OF EXERCISES SET # 11
ISOMETRIES

To be filled

Exercise 1.

Solution of Exercise 1.

Exercise 2.

Solution of Exercise 2.

Exercise 3.

Solution of Exercise 3.

Exercise 4.

Solution of Exercise 4.

Exercise 5.

Solution of Exercise 5.

Exercise 6.

Solution of Exercise 6.

Exercise 7.

Solution of Exercise 7.

Exercise 8.

Solution of Exercise 8.

Exercise 9.

Solution of Exercise 9.

Exercise 10.

Solution of Exercise 10.

Exercise 11.

Solution of Exercise 11.

References

[Lainé, 2010] Lainé, P. (2010). Applications linéaires, matrices, déterminants.