§9 Flows on the line

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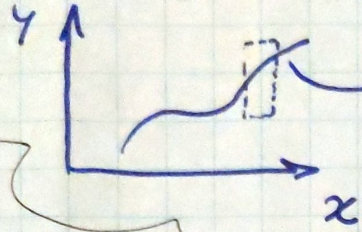
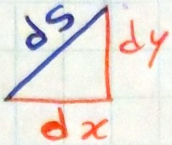
A digression on calculus (re HW #2)





$$\rightsquigarrow ds = \sqrt{dx^2 + dy^2}$$
$$= |dx| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \to \dots$$

$\dot{x} = f(x)$

$\cancel{\dot{x} = f(x,t)}$?

not for now

$x(t) \to$ a real valued function of t

$f(x) \to$ a $\underbrace{\quad}_{\text{smooth}}$ " " " " x

usually nonlinear

$t \to$ independent variable.

Interpreting a differential eqn as a vector field

$$\dot{x} = \sin x \rightsquigarrow dt = \frac{dx}{\sin x} \rightsquigarrow t = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx \to \dots$$

$$= \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}dx + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}dx$$
$$= \ln\left|\sin\tfrac{x}{2}\right| - \ln\left|\cos\tfrac{x}{2}\right|$$
$$= \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right|$$

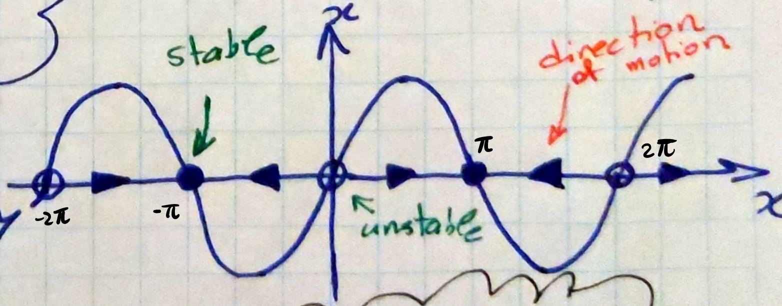
$$t = \ln\left|\tan\frac{x}{2}\right| + C$$

Let $x = x_0$ at $t = 0 \Rightarrow C = -\ln\left|\tan\frac{x_0}{2}\right|$

$$\Rightarrow t = \ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x_0}{2}}\right|$$

$\leftarrow$ exact but very difficult to interpret

Alternative:



$t \rightsquigarrow$ time

$x \rightsquigarrow$ position of an imaginary particle moving along the real line

$\dot{x} \rightsquigarrow$ velocity of that particle

$\dot{x} > 0 \to$ flow to the right

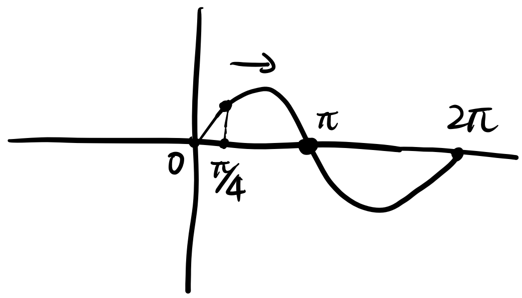
$\dot{x} < 0 \to$ " " " left

$\dot{x} = 0 \to$ no flow (fixed points)

a vector field on the line $\to$ dictates the velocity vector at each x

stable FP: attractors, sinks   unstable FP: repellers, sources

Eg: If $x = \pi/4$:



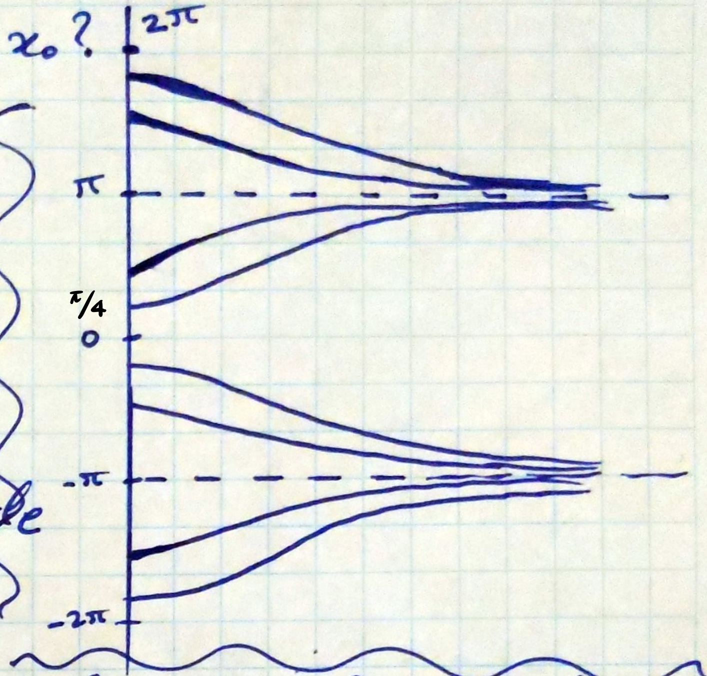
it will stop at π.

$0, \pi, 2\pi, \dots$ are called fixed points. $\dot{x} = 0$.

but for, $0, 2\pi, 4\pi$, if gives the particle an small initial speed, it will drive the point to $\pi, -\pi, 3\pi \dots$, so the points are unstable.

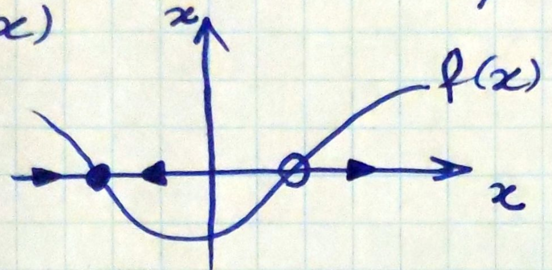
Q: Suppose $x_0 = \frac{\pi}{4}$ → describe the qualitative features of the solution $x(t)$ for all $t > 0$. What happens as $t \to \infty$?

Q: What about an arbitrary x_0?



Fixed Points & Stability

$\dot{x} = f(x)$

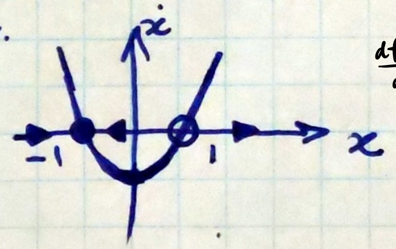


phase point: our imaginary particle

trajectory: $x(t)$ (based on x_0)

phase portrait (this kind of problems)

EX. Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

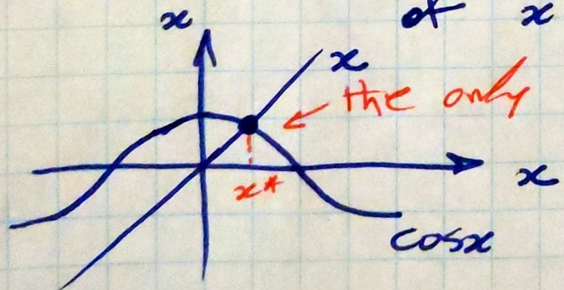


$$\frac{df(x^*)}{dx^*} = 2x^* = \begin{cases} 2 & x^* = 1 \text{ (unstable)} \\ -2 & x^* = -1 \text{ (stable)} \end{cases}$$

$f(x^*) = 0 \rightsquigarrow x^* = \pm 1$

fixed point

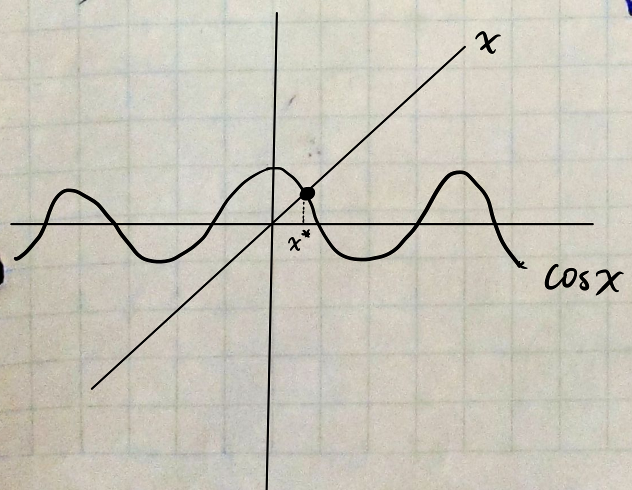
EX. Determine the stability of the fixed points of $\dot{x} = x - \cos x$



fixed point: $x^* = \cos x^* \Rightarrow \dot{x} = 0$ ✓

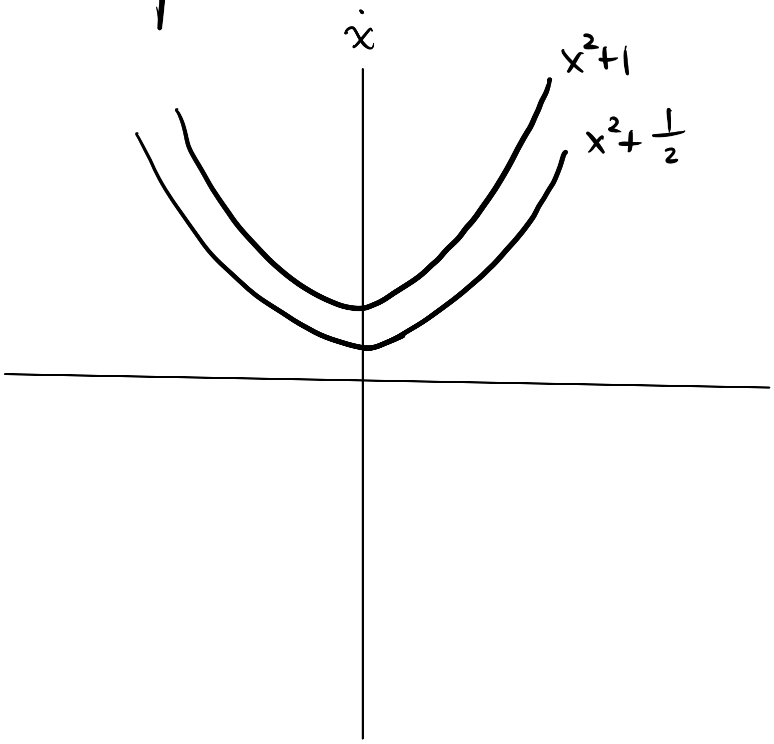
$$\left.\begin{array}{l} x > x^* \Rightarrow \dot{x} > 0 \\ x < x^* \Rightarrow \dot{x} < 0 \end{array}\right\} \Rightarrow \text{unstable}$$

but what is the value of x^*?

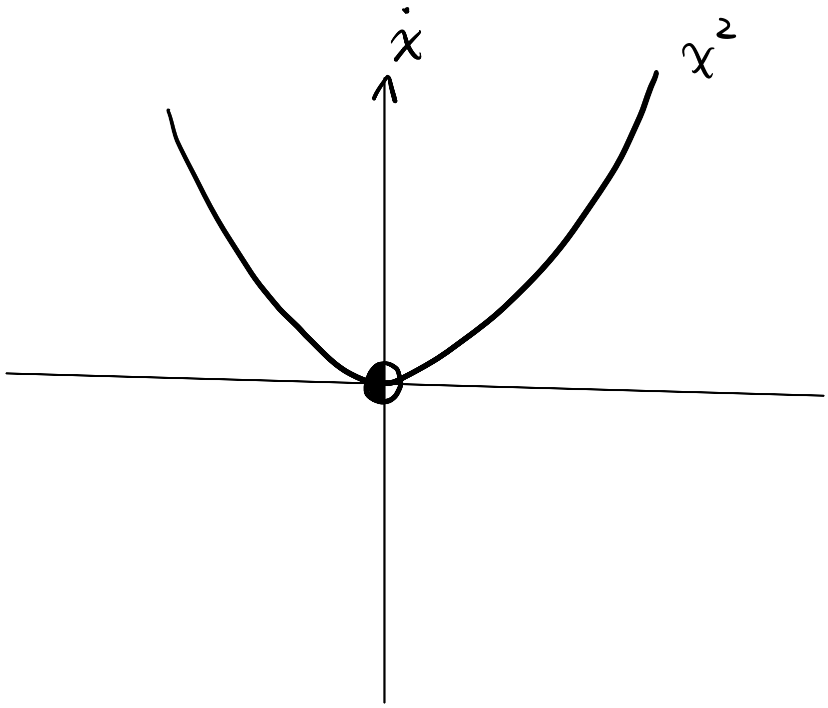


x^* is the only fixed point

Example :



no fixed points



one half-stable point.