SSY130 Applied Signal Processing
Lecture Notes

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Lecture 1

Introduction

Signals are everywhere. For example voltage, current, electromagnetic fields, acoustic signals, sound waves, vibrations, angular motions, velocities, they are all continuous in time. Some signals are continuous in space, like sound pressure.

Sampling leads to the ability to process the signals algorithmically perhaps in a computer. The availability of low cost low power digital devices has lead to an enormous development of DSP application: speech, audio, music, noise reduction, coding compression(MPEG, MP3, CD, DVD); image, movie; radio, mobile phones, modulations, transmission; radar, sonar, target detection, tracking; control, servo, mechanics, plant control; biomedical, analysis diagnosis, monitoring, telemedicine;

But what on earth is a DSP system? Basically it covers the following functions: processing of signals : filtering, modulation, etc, and these are characterized as change of the signals; analysis of signals : transform, model based analysis, etc, and these are characterized as extracting features; the last category is detection and classification, and we will not discuss this topic in this course.

In DSP implementation, the basic building block is the gate logic, this is almost the lowest level of the system. There are other different layers of abstraction. Desktop computer can be put at the higest level, where the development relies purely on software. Dedicated signal processor, which is processor+software style, is in the middle. ASIC/FPGA is at the bottom.

A DSP system has certain properties. In a DSP system, signal is always quantized(in amplitude) and sampled(in temporal), which leads to not exact response of its continuous counterpart. AD/DA converters impose extra costs to the system. Besides the system has limited bandwidth, because of limited clock frequencies of digital circuits. Although at all those costs, a DSP system has many attracting features. It has a good control of accuracy, no drift-away temperature affections that usually encountered in analog circuits. Many complex algorithms can be realized. Flexibility and adaptivity can also be achieved. Another interesting pro is that low frequency is easy to achieve, which is hard for analog circuits.

Fourier Transfrom

signal processing classically deals with spectral contents. Why? **The complex exponential $e^{j\omega t}$ forms a basis for all solutions to linear dynamic systems**. The

Fourier or Laplace transform forms the analytical means of such solutions.

$$\begin{aligned} X(\omega) = \mathcal{F}[x(t)] &\triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ x(t) = \mathcal{F}^{-1}[X(\omega)] &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{j\omega t}d\omega \end{aligned} \tag{1}$$

The signal energy can be written as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{2}$$

where $|X(\omega)|$ as the amplitude of Fourier transform of $x(t)$ shows how the signal can be decomposed into different frequency components. Similarly, $|X(\omega)|^2$ shows how the power is distributed along the frequency axis. Here we introduce Dirac delta function for analog signals.

$$\begin{aligned} &\delta(t) = 0, t \neq 0 \\ &\delta(t) = \infty, t = 0 \\ &\int_{-\infty}^{\infty} \delta(t)dt = 1 \\ &\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a) \end{aligned} \tag{3}$$

These are some properties for Fourier transform.

Table 1: Properties of Fourier Transform

$x(t)$		$X(\omega)$
linearity	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
swapping	$X(t)$	$2\pi x(-\omega)$
scale in time domain	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$
delay-modulation dual	$x(t-t_0)$	$X(\omega)e^{-jt_0\omega}$
delay-modulation dual	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
differentiate dual	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(\omega)$
differentiate dual	$t^n x(t)$	$j^n \frac{d^n X(\omega)}{d\omega^n}$
multiplication-convolution dual	$\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$	$H(\omega)X(\omega)$
multiplication-convolution dual	$h(t)x(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\tau)X(\omega-\tau)d\tau$
constant(infinite energy)	1	$2\pi\delta(\omega)$
delta function	$\delta(t)$	1

Lecture 2

Linear Systems

Linear systems, also called filtering, has its output calculated as convolution of the input signal and the impulse response of the system(filter) in time domain, as shown in equation 4.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = x(t) * h(t) \tag{4}$$

where $x(t)$ is the input signal and $h(t)$ is the impulse response of the linear system. Equation 4 implies that:

(1) if $x(t) = \delta(t)$, then $y(t) = h(t)$. Because

$$\int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\lambda)\delta(\lambda)d\lambda = h(t) \tag{5}$$

(2) Linearity. $kx(t) \implies ky(t)$, $x_1(t) + x_2(t) \implies y_1(t) + y_2(t)$.

(3) Time invariant. $x(t-\Delta) \implies y(t-\Delta)$, i.e. $h(\tau)$ does not depend on t in equation 4.

(4) Causality. $h(\tau) = 0$ for $\tau < 0$. To make the filter physically realizable, this constraint must be fulfilled.

The Fourier transform of the convolution equation, 4 will be

$$Y(\omega) = H(\omega)X(\omega) \tag{6}$$

where $H(\omega) = \mathbb{FT}[h(t)]$ is called the frequency function of the filter(linear system).

For example,

$$\begin{aligned} &x(t) = e^{j\omega_0 t} \implies X(\omega) = 2\pi\delta(\omega-\omega_0) \\ &Y(\omega) = H(\omega)2\pi\delta(\omega-\omega_0) \\ &y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = H(\omega_0)e^{j\omega_0 t} \end{aligned} \tag{7}$$

For every frequency function of linear system, $|H(\omega)| = A(\omega)$ is called the amplitude function, and $\angle H(\omega) = \Phi(\omega)$ is called the phase function of the linear system. Thus $H(\omega_0)$ can be written as $A(\omega_0)e^{j\Phi(\omega_0)}$, and the output will be $A(\omega_0)e^{j\omega_0 t + j\Phi(\omega_0)}$.

Sampled Data System and Discrete-Time Systems

Discrete time signal $\{x(n)\}$, $n = 0 \pm 1 \pm 2 \ldots$ is often associated with a continuous time signal $x(t)$ by sampling. i.e. $x(n) \triangleq x(nT_s)$, where T_s is the sampling period. $f_s = 1/T_s$ is the sampling frequency in Hz. $\omega_s = 2\pi f_s = 2\pi/T_s$ is the sampling angular frequency in radians per second. $f_s/2$ is called Nyquist frequency. Please note the difference between Nyquist frequency and Nyquist rate.

$$X(\omega) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega T_s n} \tag{8}$$

The Discrete-Time Fourier Transform of a sampled sequence is defined in equation 8, where ωT_s has the unit of radians per sample. It can also be regarded as the normalized frequency because of the other form ω/f_s. We can further expand the ω into $2\pi f$, then we have $\omega T_s = 2\pi f/f_s$. Note that $X\omega = X(\omega + k\omega_s)$, $k = 0 \pm 1 \pm 2 \ldots$, which can be proved by the periodicity of the complex exponential function. The periodicity of the frequency function and the sampling frequency also explain the aliasing effects during sampling of real-numbered signals.

The inverse DTFT is defined as

$$x(n) = \frac{1}{2\pi f_s} \int_0^{2\pi f_s} X(\omega)e^{j\omega T_s n} d\omega \tag{9}$$

Take unit delay for instance, we have $y(n) \triangleq x(n-1)$, then

$$Y(\omega) = \sum_{-\infty}^{\infty} x(n-1)e^{-j\omega T_s n} = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega T_s m}e^{-j\omega T_s} = e^{-j\omega T_s}X(\omega) \tag{10}$$

The general difference equation of a linear system can be written as

$$y(n) = -\sum_{k=1}^{N_a} a_k y(n-k) + \sum_{k=0}^{N_b} b_k x(n-k) \tag{11}$$

By the result of example 10, we take the Fourier transform of equation 11 to obtain the transfer function of the linear system, described in equation 12.

$$Y(\omega) = \frac{\sum_{k=0}^{N_b} b_k e^{-j\omega T_s k}}{\sum_{k=1}^{N_a} a_k e^{-j\omega T_s k} + 1} X(\omega) \tag{12}$$

If $z = e^{j\omega T_s}$, then we can change the transfer function 12 of ω into Z transformation.

$$H(z) = \frac{\sum_{k=0}^{N_b} b_k z^{-k}}{1 + \sum_{k=1}^{N_a} a_k z^{-k}} = \frac{b(z)}{a(z)} \tag{13}$$

Roots of $a(z)$ are called poles and roots of $b(k)$ are called zeros. The linear system is stable if all poles are inside the unit circle. We can expend the close form of the impulse response in equation 12 and 13 into series, as shown in equation 14.

$$
\begin{aligned}
H(\omega) &= \frac{\sum_{k=0}^{N_b} b_k e^{-j\omega T_s k}}{\sum_{k=1}^{N_a} a_k e^{-j\omega T_s k} + 1} = \sum_{k=0}^{\infty} h(k) e^{-j\omega T_s k} \\
H(z) &= \frac{\sum_{k=0}^{N_b} b_k z^{-k}}{1 + \sum_{k=1}^{N_a} a_k z^{-k}} = \sum_{k=0}^{\infty} h(k) z^{-k}
\end{aligned} \tag{14}
$$

where $h(k)$ is the impulse response of the system in time doamin. The system output will be

$$
Y(\omega) = H(\omega)X(\omega) = \sum_{k=0}^{\infty} h(k) e^{-j\omega T_s k} X(\omega) \tag{15}
$$

After inverse DTFT, we have

$$
\begin{aligned}
y(n) &= \frac{T_s}{2\pi} \int_0^{2\pi/T_s} \sum_{k=0}^{\infty} h(k) e^{-j\omega T_s k} X(\omega) e^{j\omega T_s n} d\omega \\
&= \sum_{k=0}^{\infty} h(k) \frac{T_s}{2\pi} \int_0^{2\pi/T_s} X(\omega) e^{j\omega T_s (n-k)} d\omega \\
&= \sum_{k=0}^{\infty} h(k) x(n-k)
\end{aligned} \tag{16}
$$

This is the convolution(filtering) of the input signal and the linear system.

Sampling Process

The mathematical model for sampling can be derived by multiplication of the continuous input signal $x(t)$ by Dirac Delta pulse train $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$. Thus there will be equation 17

$$
\begin{aligned}
x_c(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) x(t) \\
&= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) x(nT_s)
\end{aligned} \tag{17}
$$

Special care must be taken here: $x_c(t)$ is a continuous signal having the same value with the sampled sequence $x_d(n) = x(nT_s)$ at sample instants. So only Fourier transform can be exercised here for $x_c(t)$, as shown in equation 18.

$$
\begin{aligned}
FT\{x_c(t)\} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - nT_s)x(nT_s)e^{-j\omega t}dt \\
&= \sum_{n=-\infty}^{\infty} \int_{n=-\infty}^{\infty} \delta(t - nT_s)x(nT_s)e^{-j\omega t}dt \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega T_s n} \\
&= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega_d n}
\end{aligned} \tag{18}
$$

where $\omega_d = \omega T_s$ is the digital angular frequency in radians per sample, and $x(n)$ is the discrete sequence of sampled $x(t)$, as introduced before.

Therefore we define that the Fourier transform of the sampled signal to be discrete time Fourier transform, A.K.A. DTFT. This can be described by equation below.

$$FT\{x_c(t)\} = X_c(\omega) = X_d(\omega) = DTFT\{x_d(n)\} \tag{19}$$

But, what is the relationship between the spectral of the sample sequence and that of the continuous signal? Again we use Fourier transform to make the analysis.

$$
\begin{aligned}
x(nT_s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega nT_s}d\omega \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{\frac{k2\pi}{T_s}}^{\frac{(k+1)2\pi}{T_s}} X(\omega)e^{j\omega nT_s}d\omega \\
&\text{assume} \quad \omega = \omega' + \frac{2\pi k}{T_s} \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{0}^{\frac{2\pi}{T_s}} X(\omega' + \frac{2\pi k}{T_s})e^{j(\omega' + \frac{2\pi k}{T_s})nT_s}d\omega' \\
&= \frac{T_s}{2\pi} \int_{0}^{\frac{2\pi}{T_s}} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega' + \frac{2\pi k}{T_s})e^{j\omega' nT_s}d\omega' \\
&= DTFT^{-1}\{\frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega' + \frac{2\pi k}{T_s})\}
\end{aligned} \tag{20}
$$

Then we can conclude that the relationship between the continuous signal and its sample sequence is much like the process of down sampling, as described in equation 21.

$$X_c(\omega) = X_d(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi k}{T_s}) \tag{21}$$

Here comes the Nyquist sampling theorem,

theorem 1. *if $|X(\omega)| = 0$ for all $|\omega| \geq \pi/T_s = \omega_s/2$ (Nyquist frequency), then $X_d(\omega) = 1/T_s X(\omega)$ for $|\omega| < \omega_s/2$, and all information in $x(t)$ is preserved in $x_d(n)$.*

Reconstruction

A practical implementation of D/A conversion is the zero-order hold reconstruction, which is piecewise constant within each sample, i.e. $y_r(t) = y_d(t)$, where $nT_s \leq t < (n+1)T_s$. Zero-order hold reconstruction can be modeled as filtering, which can be described as:

$$\begin{aligned} y_c(t) &= \sum_{n=-\infty}^{\infty} y_d(n)\delta(t - nT_s) \\ h_{zoh}(t) &= 1 \quad \text{when} 0 \leq t < T_s \\ h_{zoh}(t) &= 0 \quad \text{otherwise} \end{aligned} \tag{22}$$

Then the Fourier transform of the convolution can be calculated in frequency domain as multiplication.

$$\begin{aligned} &Y_r(\omega) = H_{zoh}(\omega)Y_c(\omega) \\ &H_{zoh}(\omega) = FT\{h_{zoh}(t)\} = T_s e^{\frac{-j\omega T_s}{2}} \frac{sin(\frac{\omega T_s}{2})}{\frac{\omega T_s}{2}} \\ &H_{zoh}(0) = T_s \\ &H_{zoh}(\frac{2\pi}{T_s}) = 0 \end{aligned} \tag{23}$$

To sum up, for the entire DSP system, the reconstructed output can be calculated in equation 24. Note that $P(\omega)$ and $X_d(\omega)$ are $2\pi/T_s$ periodic.

$$\begin{aligned} &Y_r(\omega) = H_{zoh}(\omega)P(\omega)[\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi k}{T_s})] \\ &\text{where} \quad P(\omega) \quad \text{is the DTFT of the processing units} \\ &\text{and} \quad X_d(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi k}{T_s}) \end{aligned} \tag{24}$$

Sidelobes are caused by ZOH. Since $H_{zoh}(\omega)$ is non-zero outside $[-\pi/T_s \quad \pi/T_s]$, low-pass filter is need to eliminate this effect. In the sampling, aliasing is avoided or reduced by a low pass filter before the sampling A/D converter.

Frequency analysis of signals

We have already familiar with the convolution-multiplication transformation pair, as shown below.

$$\begin{aligned} &y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \iff Y(\omega) = H(\omega)X(\omega) \\ &y(n) = x(n)w(n) \iff Y(\omega) = \frac{T_s}{2\pi}\int_0^{2\pi f_s} X(\lambda)W(\omega - \lambda)d\lambda \end{aligned} \tag{25}$$

For practical cases, we always focus on certain interval of a infinite long sample sequence, and this is known as windowing. Hence the signal we actually interest in is the multiplication of the ideal infinite long sample sequence and the window functions. For example, the rectangular window function can be written as:

$$\begin{aligned} r_w(n) &= 1 \quad \text{when} \quad n = 0 \ldots N-1 \\ r_w(n) &= 0 \quad \text{otherwise} \end{aligned} \tag{26}$$

Let $\hat{x}(n) = r_w(n)x(n)$, and $R_w(\omega) = \sum_{n=-\infty}^{\infty} r_w(n)e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{j\omega N}}{1-e^{j\omega}} = e^{-j(N-2)/2\omega} sin(N\omega/2)/sin(\omega/2)$. We can see that $R_w(0) = N$, and $R_w(\omega) = 0$ for $\omega = 2\pi/N \quad 4\pi/N \ldots.$ $R_w(\omega)$ is periodic with 2π. The mainlobe width is proportional to $1/N$, when N increases to infinity, the Fourier transform of the window function converges to Dirac delta function. The peak level ratio between mainlobe and sidelobe are constant 13dB.

Since $\hat{X}(\omega) = 1/2\pi \int X(\omega - \lambda)R_w(\lambda)d\lambda$, when $\omega \to \infty$, $R_w(\lambda) = 2\pi\delta(\lambda)$, then $\Longrightarrow \hat{X}(\omega) = X(\omega)$.

If we sample the windowed DTFT at frequencies $\omega_k = 2\pi k/N$, then we have defined the Discrete Fourier Transform(DFT), shown in equation 27.

$$\begin{aligned} &\hat{X}(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \\ &X(k) = \frac{2\hat{\pi} k}{N} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n} \\ &k = 0 \ldots N-1 \end{aligned} \tag{27}$$

The inverse DFT has been defined in equation 28.

$$\begin{aligned} &x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi k}{N}n} \\ &n = 0 \ldots N-1 \end{aligned} \tag{28}$$

DFT is linear invertible transformation. $X(N) = X(0)$ because $X(k)$ is N periodic function. $x(n)$ is also N periodic if calculated by IDFT. The IDFT relation is a Fourier series representation of the N-periodic signal.

Lecture 3

More on DFT

When we have collected $\{x(n)\} \quad n = 0 \dots N-1$, the DFT of $x(n)$ can be obtained as $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0 \dots N-1$. What does this periodic $X(k)$ tell us about $X(\omega)$, $x(n)$?

(1) $x(n)$ is N-periodic. Then the corresponding DFT yields the Fourier series coefficients. DTFT can be calculated as:

$$\begin{aligned} X(\omega) &= \sum_{k=0}^{N-1} X(k) 2\pi\delta(\omega - \frac{2\pi k}{N}) \\ \omega &= [0 \quad 2\pi] \quad \text{and periodic} \quad k = 0 \dots N-1 \end{aligned} \tag{29}$$

(2) $x(n) = 0$ when $n < 0$ and $n > N-1$. We have DFT as samples of DTFT.

$$X(k) = X(\omega = \frac{2\pi k}{N}) \tag{30}$$

(3) $x(n)$ is unknown outside $[0 \dots N-1]$. Then the sample sequence we study is the windowed sequence, shown below.

$$\begin{aligned} &\hat{x}(n) = r_N(n)x(n) \\ &\text{where} \quad r_N(n) \quad \text{is the rectangular function} \\ &\text{and} \quad \hat{x}(n) \quad \text{is the windowed signal} \\ &X(k) = \hat{X}(\frac{2\pi k}{N}) = \frac{1}{2\pi}\int_0^{2\pi} X(\frac{2\pi k}{N} - \lambda)R_N(\lambda)d\lambda \end{aligned} \tag{31}$$

Hence DFT here is the sample of the windowed DTFT.

Assume $x(n)$ and $n = 0 \dots N-1$. Let $N_z > N$,

$$\begin{aligned} \hat{x}(n) &= x(n) \quad \text{when} n = 0 \dots N-1 \\ \hat{x}(n) &= 0 \quad \text{when} n = N \dots N_z - 1 \end{aligned} \tag{32}$$

calculate the N_z long DFT of $\hat{x}(n)$.

$$\begin{aligned} &\hat{X}(k) = \sum_{n=0}^{N_z-1} \hat{x}(n)e^{\frac{-j2\pi kn}{N_z}} = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N_z}} \\ &k = 0 \dots N_z - 1 \end{aligned} \tag{33}$$

By zero padding, without change of the information, the resolution of the frequency function has increased.

FFT

$$
\begin{aligned}
&X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \\
&k = 0 \dots N-1 \\
&W_N = e^{\frac{-j2\pi}{N}}
\end{aligned} \tag{34}
$$

If $N = 2^P$ where P is an integer, then

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N/2-1} x(2n) W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x(2n+1) W_{N/2}^{kn} \\
&= X_e(k) + W_N^k X_o(k)
\end{aligned} \tag{35}
$$

The key observation for such radix-2 FFT is that $X_e(k)$ and $X_o(k)$ are periodic with period $N/2$. This means that it only needs to evaluate from $0 \dots N/2-1$, and uses periodicity for $N/2 \dots N-1$.

$$
\begin{aligned}
&X_o(k) = X_o(k + \frac{N}{2}) \\
&X(k) = X_e(k) + W_N^k X_o(k) \quad \text{for} \quad k = 0 \dots N/2 - 1 \\
&X(k + N/2) = X_e(k) - W_N^k X_o(k) \quad \text{since} \quad W_N^{k+N/2} = -W_n^k
\end{aligned} \tag{36}
$$

Only $N/2$ multiplications are needed plus $2 \times N/2$ DFT operations. Recursively we can do the same over and again, until $N = 2$ and $W_2 = -1$.

$$
\begin{aligned}
&X(0) = X(0) + X(1) \\
&X(1) = X(0) - X(1)
\end{aligned} \tag{37}
$$

Finally we have $T(n) = 2T(n/2) + n/2$, which belongs to $O(nlogn)$, compared to the $O(N^2)$ complexity of default calculation.

Filtering With FFT

Now we simplify the filtering(convolution) model as finite impulse response(FIR), which can be described as:

$$
\begin{aligned}
&y(n) = \sum_{k=0}^{M-1} h(k) x(n-k) \\
&\{x(n)\} \quad n = 0 \dots N-1 \quad \text{and} \quad \{h(n)\} \quad n = 0 \dots M-1 \\
&M < N
\end{aligned} \tag{38}
$$

There are total two cases to solve the equation in 38:

(1) $x(n) = 0 \quad n < 0$

(2) $x(n)$ is partially periodic, i.e. $x(-n) = x(-n+N) \quad n = 0 \dots M-1$

For the first case, by DTFT we have $Y(\omega) = H(\omega)X(\omega)$. To calculate $y(n)$, possibly non-zero after $n = 0 \ldots N + M - 2$, it can be derived as:

$$y(n) = \frac{1}{N + M - 1} \sum_{k=0}^{N+M-2} Y(k) W_{N+M-1}^{-nk} \tag{39}$$

So we need to know $Y(k) = Y(\frac{2\pi k}{N+M-1}) \quad k = 0 \ldots N + M - 2$. We also have to calculate $H(\frac{2\pi k}{N+M-1})$ and $X(\frac{2\pi k}{N+M-1})$. As a result, the signals $x(n)$ and $h(n)$ both can be zero padded to length $N + M - 1$, and then the DFT can be calculated, as demonstrated in equation 40.

$$\begin{aligned} &y(n) = DFT^{-1}\{DFT_{N+M-1}\{x(n)\}DFT_{N+M-1}\{h(n)\}\} \\ &Y(K) = H(k)X(k) \end{aligned} \tag{40}$$

For the second case, if $x(n)$ is periodic, or at least partially periodic, then it can be inferred that $y(n)$ is also periodic. Consequently, just the FIR filter is required to be zero padded to length N.

$$\begin{aligned} &y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi kn}{N}} \\ &Y(k) = H(k)X(k) \end{aligned} \tag{41}$$

Lecture 4

Filter Design

In order to design digital filters, we start by revisiting the linear convolution:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{42}$$

Infinite length of the filter and non-causality are not suitable for real implementation. Thus we have classified the filters by two categories both with causal structure (impulse response be zero in time domain when $t < 0$) and finite filter length: finite impulse response and infinite impulse response filter.

FIR filter has the form of equation 43.

$$y(n) = \sum_{k=0}^{M} h(k)x(n-k) \tag{43}$$

While IIR filter has the form of equation 44,

$$y(n) = -\sum_{k=1}^{M_a} a(k)y(n-k) + \sum_{k=0}^{M_b} b(k)x(n-k) \tag{44}$$

which can also be regarded as a recursive filter since it uses the history values of output for present output calculation, also known as feedback (loop) structure.

As can be seen from equations above, IIR filter is a generalization of FIR filter. Most filters are designed to be frequency selective, so we design and analyze the filter properties in frequency domain. Sometimes however, an alternative class of filters are designed based on time domain properties, for example signal matched filters.

The Window Method

We try to explain this method by example demonstration, where a low pass filter is designed. The ideal mathematical model of low pass filter is good starting point. An ideal low pass filter can be modeled as a "brick wall" in its frequency function, shown in equation 45,

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_k \\ 0 & \omega_k < |\omega| \leq \pi \end{cases} \tag{45}$$

given unity sampling frequency. Hence the inverse DTFT of this frequency function would be

$$h(n) = \begin{cases} \frac{\omega_k}{\pi} & n = 0 \\ \frac{sin(\omega_k n)}{\pi n} & n \neq 0 \end{cases} \tag{46}$$

Note that resulting $h(n)$ is so far non-causal and infinite long, which should be modified for physical implementation.

Consequently, the task of filter design now has been reduced to how to select $h(k) \quad k = 0 \ldots M$ such that the filter approximates the ideal behavior. This definitely involves trade-off. Even more, the inverse DFT we developed in previous lectures now turns out to be an improper technique for filter design because it only defines the frequency behavior at limited points.

The window method for FIR design now becomes finding the finite and causal impulse response $H_d(\omega)$ as the desired response, in equation 47.

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \quad n = 0, \pm 1, \pm 2 \ldots \tag{47}$$

Say

$$H_d(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < 2\pi \end{cases} \tag{48}$$

we have $h_d(n)$ calculated as

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1(cos(\omega n) + jsin(\omega n))d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} cos(\omega n)d\omega \\ &= \frac{sin(\omega_c n)}{\pi n} \end{aligned} \tag{49}$$

Then we make a windowed and delayed version of $h_d(n)$, i.e. we (1) truncate the impulse response of the desired filter with certain symmetrical interval and (2) then make a time shift of length $M/2$ to the right, where M is the total length of the filter response.

(1) Truncation.

$$h_T(n) = w(n)h_d(n) = \begin{cases} w(n)h_d(n) & n = 0, \pm 1 \cdots \pm M/2 \\ 0 & otherwise \end{cases} \tag{50}$$

This indicates that the truncated filter has the frequency behavior characterized as the convolution of the desired filter and the window function in frequency domain.

$$H_T(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\lambda)W(\omega - \lambda)d\lambda \tag{51}$$

If $w(n)$ is rectangular window, then we have

$$W(\omega) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} 1e^{-j\omega n} = \frac{sin(\omega(N-1)/2)}{sin(\omega/2)} \tag{52}$$

(2) Time shift, delay $M/2$.

$$h(n) = h_T(n - \frac{M}{2}) \implies H(\omega) = e^{-j\omega M/2} H_T(\omega) \tag{53}$$

which introduces a linear phase shift.

The window function $w(n)$ is a design parameter, which introduces trade-off between main-lobe width (the width of transition region); the side-lobe height is related to stop-band attenuation.

Table 2: Properties of several window functions

window function	main-lobe width	side-lobe	transition band	stop band rejection
rectangular	$4\pi/M$	-13dB	$1/N$	21dB
Hamming	$8\pi/M$	-31dB	$33/N$	53dB
Blackman	$12\pi/M$	-57dB	$5.5/N$	74dB

Symmetric FIR filters

$$h(n) = h(M - n) \implies H(\omega) = H_r(\omega) e^{-j\omega M/2} \tag{54}$$

where $H_r(\omega)$ is a real function. We can see that $H(\omega)$ always has a frequency response with linear phase shift.

$$\angle H(\omega) = \begin{cases} -\omega M/2 & if \quad H_r(\omega) > 0 \\ -\omega M/2 + \pi & if \quad H_r(\omega) < 0 \end{cases} \tag{55}$$

Note that there might be break points in phase plots when using matlab function "angle".

Lecture 5

In this chapter, we introduce some alternative FIR filter design methods, as well as IIR design method.

In the beginning, let's recall the window method discussed in previous lecture.

(1) Select cut-off frequency ω_c

(1) Select model order N

(1) Select window function

In Matlab, function "fir1()" may help finish the design procedure with given parameters. For more information please refer to related help files.

Frequency sampling method Decide $H(k)$, samples of desired DTFT $H(\omega)$, then use IDFT, we have $h(n)$, which is multiplied by the window function $w(n)$, resulting in $h'(n) = w(n)h(n)$. In matlab the corresponding function is "fir2()".