Derivation of Jacobian Matrices

1 Propagation

The derivation of the Jacobian matrices for IEKF propagation is introduced here. We begin with the derivation of the Jacobian matrix $\mathbf{F}_k$. Assuming the state before augmentation is $[\mathbf{R}_k, \cdots, \mathbf{R}_{k-m}, \mathbf{x}_{int}^T, t_r, t_d, \boldsymbol{b}_w^T, \mathbf{R}_{\mathrm{I}}^{\mathrm{C}}]$, after augmentation, we will have $\mathbf{x}_k = [\mathbf{R}_k, \mathbf{R}_k, \cdots, \mathbf{R}_{k-m}, \mathbf{x}_{int}^T, t_r, t_d, \boldsymbol{b}_w^T, \mathbf{R}_{\mathrm{I}}^{\mathrm{C}}]$. Note that only the augmented state (the first one) involves in the propagation, i.e.

$$\mathbf{R}_{k+1} = \mathbf{R}_k \exp\left(\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\left(\boldsymbol{w} - \boldsymbol{b}_w\right) d_t\right)^{\wedge}\right) \tag{1}$$

By using the right invariant error and the corresponding retraction [2, 4], we compute $\mathbf{F}_k$ as follows:

$$\mathbf{F}_k = \lim_{\boldsymbol{e}\to 0} \frac{f(\mathbf{x}_k \oplus \boldsymbol{e}, \mathbf{w}) \ominus f(\mathbf{x}_k, \mathbf{w})}{\boldsymbol{e}} \tag{2}$$

for $\mathbf{R}_{k+1}$ part, we will have:

$$\begin{aligned} f(\mathbf{x}_k \oplus \boldsymbol{e}, \mathbf{w}) &= \exp(\boldsymbol{e}_1^{\wedge})\mathbf{R}_k \exp\left(\left(\exp(\boldsymbol{e}_2^{\wedge})\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\left(\boldsymbol{w} - \boldsymbol{b}_w - \boldsymbol{e}_3\right) d_t\right)^{\wedge}\right) \\ &\approx \exp(\boldsymbol{e}_1^{\wedge})\mathbf{R}_k \exp\left(\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w) - \mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\boldsymbol{e}_3 + \boldsymbol{e}_2^{\wedge}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)\right)^{\wedge} d_t\right) \\ &\approx \exp(\boldsymbol{e}_1^{\wedge})\mathbf{R}_{k+1} \exp\left(\left(-\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\boldsymbol{e}_3 + \boldsymbol{e}_2^{\wedge}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)\right)^{\wedge} d_t\right) \end{aligned} \tag{3}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the small perturbation inserted to $\mathbf{R}_k, \mathbf{R}_{\mathrm{I}}^{\mathrm{C}}, \boldsymbol{b}_w$. Then,

$$\begin{aligned} f(\mathbf{x}_k \oplus \boldsymbol{e}, \mathbf{w}) \ominus f(\mathbf{x}_k, \mathbf{w}) &\approx \exp(\boldsymbol{e}_1^{\wedge})\mathbf{R}_{k+1} \exp\left(\left(-\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\boldsymbol{e}_3 + \boldsymbol{e}_2^{\wedge}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)\right)^{\wedge} d_t\right)\mathbf{R}_{k+1}^T \\ &= \exp(\boldsymbol{e}_1^{\wedge}) \exp\left(\mathbf{R}_{k+1}\left(-\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\boldsymbol{e}_3 + \boldsymbol{e}_2^{\wedge}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)\right)^{\wedge} d_t\right) \\ &\approx \exp\left(\boldsymbol{e}_1^{\wedge} + \mathbf{R}_{k+1}\left(-\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\boldsymbol{e}_3 + \boldsymbol{e}_2^{\wedge}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)\right)^{\wedge} d_t\right) \end{aligned} \tag{4}$$

Therefore, we will have

$$\begin{aligned} \mathbf{F}_1 &= \lim_{\boldsymbol{e}_1\to 0} \frac{f(\mathbf{x}_k \oplus \boldsymbol{e}_1, \mathbf{w}) \ominus f(\mathbf{x}_k, \mathbf{w})}{\boldsymbol{e}_1} = \mathbf{I} \\ \mathbf{F}_2 &= \lim_{\boldsymbol{e}_2\to 0} \frac{f(\mathbf{x}_k \oplus \boldsymbol{e}_2, \mathbf{w}) \ominus f(\mathbf{x}_k, \mathbf{w})}{\boldsymbol{e}_2} = -\mathbf{R}_{k+1}\boldsymbol{m}^{\wedge} \\ \mathbf{F}_3 &= \lim_{\boldsymbol{e}_3\to 0} \frac{f(\mathbf{x}_k \oplus \boldsymbol{e}_3, \mathbf{w}) \ominus f(\mathbf{x}_k, \mathbf{w})}{\boldsymbol{e}_3} = -\mathbf{R}_{k+1}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}} d_t \end{aligned} \tag{5}$$

where $\boldsymbol{m} = \mathbf{R}_{\mathrm{I}}^{\mathrm{C}}(\boldsymbol{w} - \boldsymbol{b}_w)d_t$, $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are the corresponding terms of the Jacobian matrix with respect to the augmented $\mathbf{R}_k$.

For other terms including the bias, the derivation is fairly easy and the corresponding terms are the identity matrix. Thus, the complete Jacobian matrix of $\mathbf{F}_k$ can be given as

$$\mathbf{F}_k = \begin{bmatrix} \mathbf{F}_1 \cdots \mathbf{0}\ 0\ 0\ \mathbf{F}_2\ \mathbf{F}_3 \\ \mathbf{I} \end{bmatrix} \tag{6}$$

Now we derive the Jacobian matrix $\mathbf{G}_k$ for the input $\boldsymbol{w}$ and the process noise $\boldsymbol{n}_w$ as

$$\mathbf{G}_k = \lim_{\boldsymbol{e}\to 0} \frac{f(\mathbf{x}_k, \mathbf{w} + \boldsymbol{e}) \ominus f(\mathbf{x}_k, \mathbf{w})}{\boldsymbol{e}} \tag{7}$$

The corresponding term for $\boldsymbol{b}_w$ is easy to derive (which is $\mathbf{I}$ actually). Therefore, we only show the derivation of the term with respect to the augmented $\mathbf{R}_k$.

$$\begin{aligned} f(\mathbf{x}_k, \mathbf{w} + \boldsymbol{e}_w) \ominus f(\mathbf{x}_k, \mathbf{w}) =& \mathbf{R}_k \exp\left(\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{C}} \left(\boldsymbol{w} - \boldsymbol{b}_w + \boldsymbol{e}_w\right) d_t\right)^{\wedge}\right) \\ & \exp\left(-\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{C}} \left(\boldsymbol{w} - \boldsymbol{b}_w\right) d_t\right)^{\wedge}\right) \mathbf{R}_k^T \\ \approx & \mathbf{R}_k \exp\left(\left(\mathbf{J}_l \mathbf{R}_{\mathrm{I}}^{\mathrm{C}} \boldsymbol{e}_w d_t\right)^{\wedge}\right) \mathbf{R}_k^T \\ =& \exp(\left(\mathbf{R}_k \mathbf{J}_l \mathbf{R}_{\mathrm{I}}^{\mathrm{C}} \boldsymbol{e}_w d_t\right)^{\wedge}) \end{aligned} \tag{8}$$

Thus,

$$\mathbf{G}_1 = \mathbf{R}_k \mathbf{J}_l \mathbf{R}_{\mathrm{I}}^{\mathrm{C}} \tag{9}$$

Put them together, we have $\mathbf{G}_k$ as

$$\mathbf{G}_k = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{I} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{10}$$

The process covariance $\mathbf{Q}$ is given as

$$\mathbf{Q} = \begin{bmatrix} d_t \boldsymbol{\sigma}_{\boldsymbol{w}}^2 & \mathbf{0} \\ \mathbf{0} & d_t \boldsymbol{\sigma}_{\boldsymbol{n}_{\boldsymbol{b}_{\boldsymbol{w}}}}^2 \end{bmatrix} \tag{11}$$

2 Update

The measurement implicit function can be written in various explicit form, e.g.

1. $\boldsymbol{u}' - \mathbf{K}\pi(\mathbf{R}_{\boldsymbol{u}'}^T \mathbf{R}_{\boldsymbol{u}} \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u})) = \mathbf{0}$ or backward.
2. $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, where $\boldsymbol{v}_1 = \frac{\mathbf{R}_{\boldsymbol{u}} \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u})}{|\mathbf{R}_{\boldsymbol{u}} \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u})|}$ and $\boldsymbol{v}_2 = \frac{\mathbf{R}_{\boldsymbol{u}'} \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u}')}{|\mathbf{R}_{\boldsymbol{u}} \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u}')|}$.

Both $\boldsymbol{u}$ and $\boldsymbol{u'}$ are in their homogeneous coordinates. $\pi(\cdot)$ applies the possible distortion. Here, we select the second form to derive the Jacobian matrix since the first form can be derived similarly. Using the chain-rule, we will have:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \boldsymbol{v}_1}\frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}} + \frac{\partial f}{\partial \boldsymbol{v}_2}\frac{\partial \boldsymbol{v}_2}{\partial \boldsymbol{x}} \tag{12}$$

Since $\frac{\partial f}{\partial \boldsymbol{v}_1}$ is easy to compute, we focus on the second part $\frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}}$. By denoting $\mathbf{R}_{\boldsymbol{u}}\pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u})$ as $\boldsymbol{d}_1$, we will have

$$\begin{aligned}\frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}} &= \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{d}_1}\frac{\partial \boldsymbol{d}_1}{\partial \boldsymbol{x}} \\ \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{d}_1} &= \frac{1}{|\boldsymbol{d}_1|^3}(\boldsymbol{d}_1^T\boldsymbol{d}_1 \cdot \mathbf{I} - \boldsymbol{d}_1\boldsymbol{d}_1^T)\end{aligned} \tag{13}$$

Then, what we need to focus is the term $\frac{\partial \boldsymbol{d}_1}{\partial \boldsymbol{x}}$. Let's fist list the involved states in $\boldsymbol{d}_1$:

a. f, c_u, c_v and possible distortion parameters.
b. $\boldsymbol{u}$.
c. two neighbouring pose for interpolation, denoted as $\mathbf{R}_0$ and $\mathbf{R}_1$.
d. t_r and t_d.

Among them, the Jacobian term with respect to (a) is fairly easy to compute (since they have no relation ship with the group states). The second easiest one is the (d) part. By denoting $\boldsymbol{e} = \pi^{-1}(\mathbf{K}^{-1}\boldsymbol{u'})$ and $\boldsymbol{\xi} = \log(\mathbf{R}_0^T\mathbf{R}_1)^\vee/(t_1 - t_0)$, we will have

$$\begin{aligned}\lim_{e_t\to 0}\frac{\boldsymbol{d}_1(e_t) - \boldsymbol{d}_1}{e_t} &= \lim_{e_t\to 0}\frac{\mathbf{R}_0\exp(\boldsymbol{\xi}(t+e_t)^\wedge)\boldsymbol{e} - \mathbf{R}_0\exp(\boldsymbol{\xi}(t)^\wedge)\boldsymbol{e}}{e_t} \\ &\approx \lim_{e_t\to 0}\frac{\mathbf{R}_0\exp(\boldsymbol{\xi}(t)^\wedge)(\mathbf{I}+(\boldsymbol{\xi}e_t)^\wedge)\boldsymbol{e} - \mathbf{R}_0\exp(\boldsymbol{\xi}(t)^\wedge)\boldsymbol{e}}{e_t} \\ &= -\mathbf{R}_0\exp(\boldsymbol{\xi}(t)^\wedge)\boldsymbol{e}^\wedge\boldsymbol{\xi} = -\mathbf{R}\boldsymbol{e}^\wedge\boldsymbol{\xi}\end{aligned} \tag{14}$$

where $\mathbf{R} = \mathbf{R}_0\exp(\boldsymbol{\xi}(t)^\wedge)$. With (14), it is fairly to compute the Jacobian term with respect to t_r and t_d since $t = t_f + t_d + v/h \cdot t_r$ where t_f is the frame time, h is the image height, v is the row coordinate of $\boldsymbol{u}$.

With the intermediate results of terms (a) and (d), (b) can be computed by some combinations. Note that $\boldsymbol{u}$ appears implicitly in t, which should be taken into consideration. Otherwise, we may underestimate the uncertainty of the measurements.

The most complex part is (c), which is detailed as follows. Let $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ are two small perturbations to $\mathbf{R}_0$ and $\mathbf{R}_1$, respectively. We have the following approximate (we omit unimportant $\wedge$ and $\vee$ for making it more clear):

$$\begin{aligned}\mathbf{R}_{\boldsymbol{u}}(\boldsymbol{e}_0,\boldsymbol{e}_1) &= \exp(\boldsymbol{e}_0)\mathbf{R}_0\exp(\log(\mathbf{R}_0^T\exp(-\boldsymbol{e}_0)\exp(\boldsymbol{e}_1)\mathbf{R}_1)\frac{t-t_0}{t_1-t_0}) \\ &= \exp(\boldsymbol{e}_0)\mathbf{R}_0\exp(\log(\exp(-\mathbf{R}_0^T\boldsymbol{e}_0)\mathbf{R}_0^T\mathbf{R}_1\exp(\mathbf{R}_1^T\boldsymbol{e}_1))\frac{t-t_0}{t_1-t_0}) \\ &\approx \exp(\boldsymbol{e}_0)\mathbf{R}_0\exp((\log(\mathbf{R}_0^T\mathbf{R}_1)-\mathbf{J}_{l1}^{-1}\mathbf{R}_0^T\boldsymbol{e}_0^\wedge+\mathbf{J}_{r1}^{-1}\mathbf{R}_1^T\boldsymbol{e}_1^\wedge)\frac{t-t_0}{t_1-t_0}) \\ &\approx \exp(\boldsymbol{e}_0)\mathbf{R}(\mathbf{I}+\mathbf{J}_{r2}(-\mathbf{J}_{l1}^{-1}\mathbf{R}_0^T\boldsymbol{e}_0^\wedge+\mathbf{J}_{r1}^{-1}\mathbf{R}_1^T\boldsymbol{e}_1^\wedge)\frac{t-t_0}{t_1-t_0}) \\ &\approx \mathbf{R}+\mathbf{R}(\mathbf{J}_{r2}(-\mathbf{J}_{l1}^{-1}\mathbf{R}_0^T\boldsymbol{e}_0^\wedge+\mathbf{J}_{r1}^{-1}\mathbf{R}_1^T\boldsymbol{e}_1^\wedge)\frac{t-t_0}{t_1-t_0})+\boldsymbol{e}_0^\wedge\mathbf{R}+\mathcal{O}(\boldsymbol{e}_0,\boldsymbol{e}_1)\end{aligned} \tag{15}$$

where $\mathcal{O}(\boldsymbol{e}_0,\boldsymbol{e}_1)$ represents the second order or higher order terms. $\mathbf{J}_{l1}$ and $\mathbf{J}_{r1}$ are the left and right Jacobian matrices of the term $\log(\mathbf{R}_0^T\mathbf{R}_1)$. $\mathbf{J}_{r2}$ is the right Jacobian matrix of the term $\log(\mathbf{R}_0^T\mathbf{R}_1)\frac{t-t_0}{t_1-t_0}$. If $\mathbf{R}_0$ and $\mathbf{R}_1$ are close (the IMU's sampling frequency is high enough), they can be replaced with the identity matrix. Combining with $\boldsymbol{e}$, we will have

$$\begin{aligned}\boldsymbol{d}_1(\boldsymbol{e}_0,\boldsymbol{e}_1) &= \mathbf{R}(\boldsymbol{e}_0,\boldsymbol{e}_1)\boldsymbol{e} \\ &\approx \mathbf{R}\boldsymbol{e}+\mathbf{R}(\mathbf{J}_{r2}(-\mathbf{J}_{l1}^{-1}\mathbf{R}_0^T\boldsymbol{e}_0^\wedge+\mathbf{J}_{r1}^{-1}\mathbf{R}_1^T\boldsymbol{e}_1^\wedge)\frac{t-t_0}{t_1-t_0})\boldsymbol{e}+\boldsymbol{e}_0^\wedge\mathbf{R}\boldsymbol{e} \\ &= \mathbf{R}\boldsymbol{e}+\underbrace{\left(\mathbf{R}\boldsymbol{e}^\wedge\mathbf{J}_{r2}\mathbf{J}_{l1}^{-1}\mathbf{R}_0^T\frac{t-t_0}{t_1-t_0}-(\mathbf{R}\boldsymbol{e})^\wedge\right)}_{\mathbf{H}_{\mathbf{R}_0}}\boldsymbol{e}_0+\underbrace{\left(-\mathbf{R}\boldsymbol{e}^\wedge\mathbf{J}_{r2}\mathbf{J}_{r1}^{-1}\mathbf{R}_1^T\frac{t-t_0}{t_1-t_0}\right)}_{\mathbf{H}_{\mathbf{R}_1}}\boldsymbol{e}_1\end{aligned} \tag{16}$$

Therefore, we have $\mathbf{H}_{\mathbf{R}_0}$ and $\mathbf{H}_{\mathbf{R}_1}$ being the corresponding terms with respect to $\mathbf{R}_0$ and $\mathbf{R}_1$ in the Jacobian matrix $\mathbf{H}_k$.

Put all term together, we will have a Jacobian matrix like

$$\mathbf{H}_k=\begin{bmatrix}\mathbf{0} & \cdots & \mathbf{H}_{\mathbf{R}_1} & \mathbf{H}_{\mathbf{R}_0} & \cdots & \mathbf{0} & \mathbf{H}_{\mathbf{x}_{int}} & \mathbf{H}_{t_r} & \mathbf{H}_{t_d} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{H}_{\mathbf{R}_1} & \mathbf{H}_{\mathbf{R}_0} & \cdots & \cdots & \mathbf{0} & \mathbf{H}_{\mathbf{x}_{int}} & \mathbf{H}_{t_r} & \mathbf{H}_{t_d} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix} \tag{17}$$

As can be seen, $\mathbf{H}_k$ is very sparse (note that only a few poses will be involved).

Appendix

Matrix Lie Group

The Special Orthogonal group $\mathbb{SO}(3)$ is defined as the closed group established by all 3×3 rotation matrices as:

$$\mathbb{SO}(3):\{\mathbf{R}\in\mathbb{R}^{3\times3}|\mathbf{R}\mathbf{R}^T=\mathbf{I}_{3\times3},\ \det(\mathbf{R})=1\} \tag{18}$$

On any Lie group, the tangent space at the group identity has an associated Lie algebra $\mathfrak{g}$. The mapping relationships are shown in Fig. ??. The $\exp(\cdot)$ and $\log(\cdot)$ operations establish a local diffeomorphism between a neighborhood of $\mathbf{0}_{n\times n}$ in the tangent space to a local neighborhood of the identity on the manifold. $\mathfrak{g}$ associates to its vector space $\mathbb{R}^n$ by $(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^n$ and $(\cdot)^\wedge : \mathbb{R}^n \to \mathfrak{g}$. In the case of 3D rotations, the Lie algebra is denoted as $\mathfrak{so}(3)$ and given as follows:

$$\boldsymbol{\phi}^\wedge = \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix} \in \mathfrak{so}(3) \tag{19}$$

where $\boldsymbol{\phi} \in \mathbb{R}^3$ is the coefficient of the basis in the corresponding vector space. On $\mathbb{SO}(3)$, the exponential mapping exists a closed-form formula:

$$\exp(\boldsymbol{\phi}^\wedge) = \mathbf{I} + \frac{\sin(||\boldsymbol{\phi}||)}{||\boldsymbol{\phi}||}\boldsymbol{\phi}^\wedge + \frac{1-\cos(||\boldsymbol{\phi}||)}{||\boldsymbol{\phi}||^2}\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge \tag{20}$$

where $||\boldsymbol{\phi}||$ is the standard Euclidean norm. Conversely, the logarithm mapping is given by

$$\log(\mathbf{R}) = \frac{\theta \cdot (\mathbf{R} - \mathbf{R}^T)}{2\sin(\theta)} \ , \ \theta = \arccos(\frac{\mathrm{tr}(\mathbf{R}) - 1}{2}) \tag{21}$$

where $\mathrm{tr}(\cdot)$ takes the trace of a matrix. The Lie bracket of two elements corresponds to their vector product is given by:

$$[\boldsymbol{\phi}_1^\wedge, \boldsymbol{\phi}_2^\wedge] = \boldsymbol{\phi}_1^\wedge\boldsymbol{\phi}_2^\wedge - \boldsymbol{\phi}_2^\wedge\boldsymbol{\phi}_1^\wedge = (\boldsymbol{\phi}_1^\wedge\boldsymbol{\phi}_2)^\wedge \tag{22}$$

The *Baker-Campbell-Hausdorff (BCH)* formula [3] gives the following result:

$$\log(\exp(\boldsymbol{\phi}_1^\wedge)\exp(\boldsymbol{\phi}_2^\wedge))^\vee \approx \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \frac{1}{2}[\boldsymbol{\phi}_1^\wedge, \boldsymbol{\phi}_2^\wedge] + \mathrm{h.o.t} \tag{23}$$

where h.o.t stands for the 1^{st} and higher order terms. If $\boldsymbol{\phi}_1$ is assumed to be small, a tighter approximation by *(BCH)* is given by

$$\log(\exp(\boldsymbol{\phi}_1^\wedge)\exp(\boldsymbol{\phi}_2^\wedge))^\vee \approx \boldsymbol{\phi}_2 + \mathbf{J}_l^{-1}(\boldsymbol{\phi}_2)\boldsymbol{\phi}_1 \tag{24}$$

where $\mathbf{J}_l$ is the left Jacobian of $\mathbb{SO}(3)$ defined as (25):

$$\mathbf{J}_l = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n \tag{25}$$

The analytical solution for $\mathbf{J}_l$ is detailed in [1]. The meaning of (24) can be interpreted as follows: a multiplication increment on $\mathbb{SO}(3)$ will associate to a corresponding additive increment in the vector space. A similar result can be obtained when $\boldsymbol{\phi}_2$ is small using the right Jacobian, which is omitted for brevity.

References

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