Bayesian Scientific Computing
Exercise 2 for Day 4

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Deconvolution with a hierarchical model

Consider the deconvolution problem of estimating a signal $f : [0, 1] \to \mathbb{R}$ from noisy observation of the signal

$$b(t) = \int_0^1 a(t, s) f(s) ds,$$

where $a(t, s)$ is the *Airy kernel* describing diffraction in a thin slit,

$$a(t, s) = \left(\frac{J_1(k|t - s|)}{k|t - s|} \right)^2 ,$$

where J_1 is the order one Bessel function of the first kind and $k = 1/w$ is the inverse of the width parameter. We define $a(t, t) = 1$. It is assumed that outside the interval $[0, 1]$, $f(s) = 0$. We set the width parameter to $w = 0.03$.

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The noisy data is defined as

$$b_j = \int_0^1 a(t_j, s) f(s) ds + \varepsilon_j, \quad 1 \leq j \leq m,$$

where the noise is zero mean Gaussian noise, $\varepsilon \sim \mathcal{N}(0, \sigma^2 \mathrm{I}_m)$. We set $m = 50$, define

$$t_j = \frac{j - 1/2}{m}, \quad 1 \leq j \leq m,$$

and set the noise level to

$$\sigma = 0.03;$$

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To generate the data, choose a discretization level N, subdivide the interval $[0,1]$ into N equal intervals, and write an approximation for the noiseless data,

$$b_j^* = \int_0^1 a(t_j,s)f(s)ds \approx \frac{1}{N}\sum_{k=1}^N a(t_j,s_k)f(s_k), \quad s_k = \frac{k-1/2}{N},$$

or, in matrix notation,

$$b^* = \mathsf{A}_N x_N,$$

where $\mathsf{A}_N(j,k) = a(t_j,s_k)$ and $x_N(k) = f(s_k)$. To generate the data, use the boxcar signal,

$$x_{N,j} = \begin{cases} 1, & N/2 < j < 2N/3, \\ 0, & \text{elsewhere.} \end{cases}$$

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To solve the inverse problem of recovering x_N from the data, we use a different discretization to avoid the so called *inverse crime*, comprising of using the same discrete model for generating the data and solving the inverse problem. To this end set $n = 200$, and define the forward matrix $\mathsf{A} \in \mathbb{R}^{m \times n}$ analogously to how you defined A_N. We seek to solve the problem

$$b = \mathsf{A}x + \varepsilon,$$

where b is one of the noisy signals generated in the previous problem.

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Defining

$$\mathsf{L} = n \begin{bmatrix} 1 & 0 & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix},$$

introduce the hierarchical prior model,

$$\pi_X(x \mid \theta) \propto \exp\left(-\frac{1}{2}\|\mathsf{D}_\theta^{-1/2}\mathsf{L}x\|^2 - \frac{1}{2}\sum_{k=1}^{n} \log \theta_k \right),$$

where

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$$
\mathsf{D}_\theta = \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix} \in \mathbb{R}^{n\times n}.
$$

We assume that the components of the vector θ in the diagonal of D_θ follow the gamma distribution

$$
\pi_\Theta(\theta) \propto \exp\left((\beta - 1)\sum_{k=1}^{n} \log \theta_k - \sum_{k=1}^{n} \frac{\theta_k}{\theta^*} \right).
$$

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Hence, the posterior density for the pair (X, Θ) is

$$\pi_{X,\Theta|B} \propto \exp\left(-\frac{1}{2}\|\mathsf{D}_\theta^{-1/2}\mathsf{L}x\|^2 - \frac{1}{2}\sum_{k=1}^{n}\log\theta_k + \eta\sum_{k=1}^{n}\frac{\theta_k}{\theta^*}\right),$$

where $\eta = \beta - 3/2 > 0$.

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Implement the IAS algorithm by performing the following operations:

1. Initialization: Set $\theta_j^{(0)} = \theta^*$. Set the counter $k = 1$.
2. Given $\theta = \theta^{(k-1)}$, update x by solving

$$w = \operatorname{argmin} \left(\frac{1}{\sigma^2} \|b - \mathsf{A}\mathsf{L}^{-1} w\|^2 + \|\mathsf{D}_\theta^{-1/2} w\|^2 \right), \quad \mathsf{L}x = w;$$

3. Given $w = w^k$, update θ by evaluating

$$\theta_j = \frac{\theta^*}{2} \left(\eta + \sqrt{\eta^2 + \frac{2w_j^2}{\theta^*}} \right), \quad \eta = \beta - 3/2.$$

Repeat Step 2 and Step 3 until the relative change of the parameter vector θ is below a threshold, that you can set to 0.001.
Here are some suggested values for the parameters of the hyperprior:

$$\eta = 0.01, \quad \theta^* = 10^{-4}.$$

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In Step 2, write the objective function as

$$\frac{1}{\sigma^2}\|b - \mathsf{A}\mathsf{L}^{-1}w\|^2 + \|\mathsf{D}_\theta^{-1/2}w\|^2 = \left\| \begin{bmatrix} \sigma^{-1}\mathsf{A}\mathsf{L} \\ \mathsf{D}_\theta^{-1/2} \end{bmatrix} x - \begin{bmatrix} \sigma^{-1}b \\ 0 \end{bmatrix} \right\|^2,$$

and compute the solution as a least squares solution to

$$\begin{bmatrix} \sigma^{-1}\mathsf{A}\mathsf{L} \\ \mathsf{D}_\theta^{-1/2} \end{bmatrix} x = \begin{bmatrix} \sigma^{-1}b \\ 0 \end{bmatrix}$$

using the “backslash” of Matlab.