Bayesian Scientific Computing
Exercise for Day 3

Daniela Calvetti, Erkki Somersalo

Case Western Reserve University
Department of Mathematics, Applied Mathematics and Statistics

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Bayesian interpolation

Consider the problem of interpolation a function between noisy observations, with the prior belief that the function is smooth. The Bayesian interpolation method discussed below is known as **Kriging**, or **Wiener-Kolmogorov prediction.**

Assume, for simplicity, that the function to be interpolated is defined over the unit interval, and the end point values vanish,

$$u(0) = u(1) = 0.$$

We discretize the function, writing

$$u_k = u(t_k), \quad t_k = hk, \quad 0 \leq h \leq n,$$

where $h = 1/n$.

Bayesian interpolation

For simplicity, assume that the function is observed at few of the discretization points, $t_{k_1} < t_{k_2} < \ldots < t_{k_m} < 1$, where m is small.

we write the observation model

$$b_j = u_{k_j} + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \sigma^2).$$

In matrix form, we define the matrix $\mathsf{A} \in \mathbb{R}^{m\times(n-1)}$ with zero entries except for

$$\mathsf{A}_{j,k_j} = 1, \quad j = 1, 2, \ldots, m.$$

Thus, the matrix A picks the observed values of the vector u.

Bayesian interpolation

We write

1. the likelihood,

$$\pi_{B|U} \propto \exp\left(-\frac{1}{2\sigma^2}\|b - Au\|^2\right),$$

2. the prior using the one-dimensional Whittle-Mátern prior,

$$\pi_U \propto \exp\left(-\frac{1}{2}\|\mathsf{M}_\lambda u\|^2\right),$$

where

$$\mathsf{M}_\lambda = \left(\mathsf{L}_1 - \frac{1}{\lambda^2}\mathsf{I}\right),$$

see the Exercises for Day 2.

Bayesian interpolation

The posterior density is, by Bayes' formula,

$$\pi_{X|B}(x \mid b) \propto \exp\left(-\frac{1}{2}\|\mathsf{M}_\lambda u\|^2 - \frac{1}{2\sigma^2}\|\mathsf{A}u\|^2\right).$$

The **Maximum A Posteriori** (MAP) estimate is defined as the maximizer of the above expression, or, equivalently , the minimizer of

$$F(u) = \|\mathsf{M}_\lambda u\|^2 + \frac{1}{\sigma^2}\|\mathsf{A}u - b\|^2 = \left\|\left[\begin{array}{c} \mathsf{M}_\lambda \\ (1/\sigma)\mathsf{A} \end{array}\right] x - \left[\begin{array}{c} 0 \\ (1/\sigma)b \end{array}\right]\right\|^2.$$

Bayesian interpolation

We conclude that the MAP estimate is the least squares solution of the problem

$$\left[\begin{array}{c} \mathsf{M}_\lambda \\ (1/\sigma)\mathsf{A} \end{array}\right] x = \left[\begin{array}{c} 0 \\ (1/\sigma)b \end{array}\right]. \tag{1}$$

In the following, test the performance of Bayesian interpolation.

Bayesian interpolation

1. Choose n, the number of discretization intervals, e.g., $n = 100$, and write the matrix M_λ.
2. Choose a couple of points at which the underlying function is evaluated. For instance, set $n = 100$, and $k_1 = 25$, $k_2 = 35$, $k_3 = 50$ and $k_4 = 80$ (four observations).
3. Give the four values that hypothetically have been observed, e.g., $b_1 = 0.5$, $b_2 = 1$, $b_3 = 0.2$, $b_4 = 2$.
4. Write the matrix $\mathsf{A} \in \mathbb{R}^{4\times 99}$.
5. Choose the presumed error level σ in the measurement.
6. Solve the problem (1) in the least squares sense.

Test different values of λ (correlation length) and γ (scaling of the prior).