# Discrete Mathematics: Lecture 20

## Part IV. Graph Theory

graph, vertex, edge, endpoints, directed, undirected, multiple edge, loop,

complete graph, cycle, wheel, cube, graph representation

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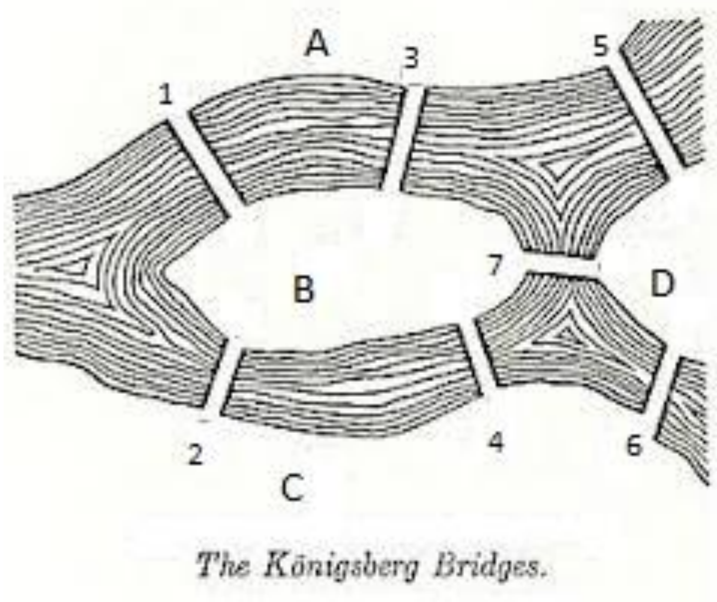
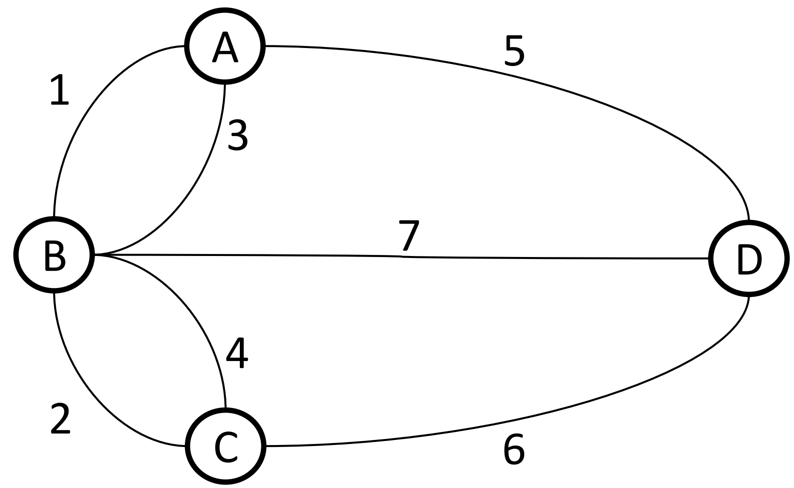
Notes by Prof. Liangfeng Zhang and Xuming He

# Seven Bridges of Königsberg



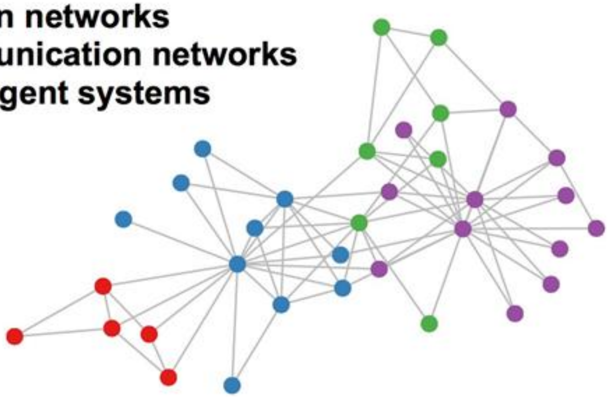

**QUESTION:** Is it possible to travel all seven bridges without repetition?

- Start at one of the four locations $A, B, C, D$
- Travel across every bridge exactly once
- Return to the starting point

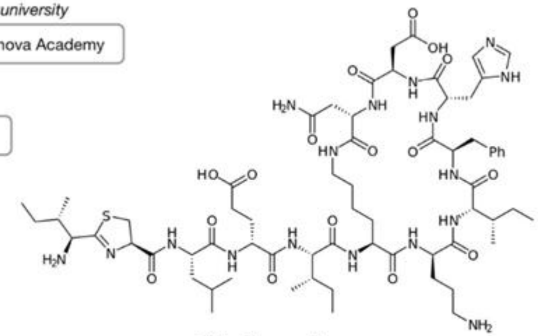
**Graph Notion**: Euler Circuit (1736)

# Real-world Graphs

**Social networks**
**Citation networks**
**Communication networks**
**Multi-agent systems**

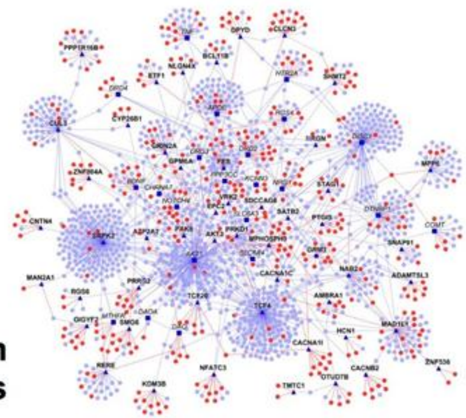



**Knowledge graphs**



**Molecules**

**Protein interaction networks**

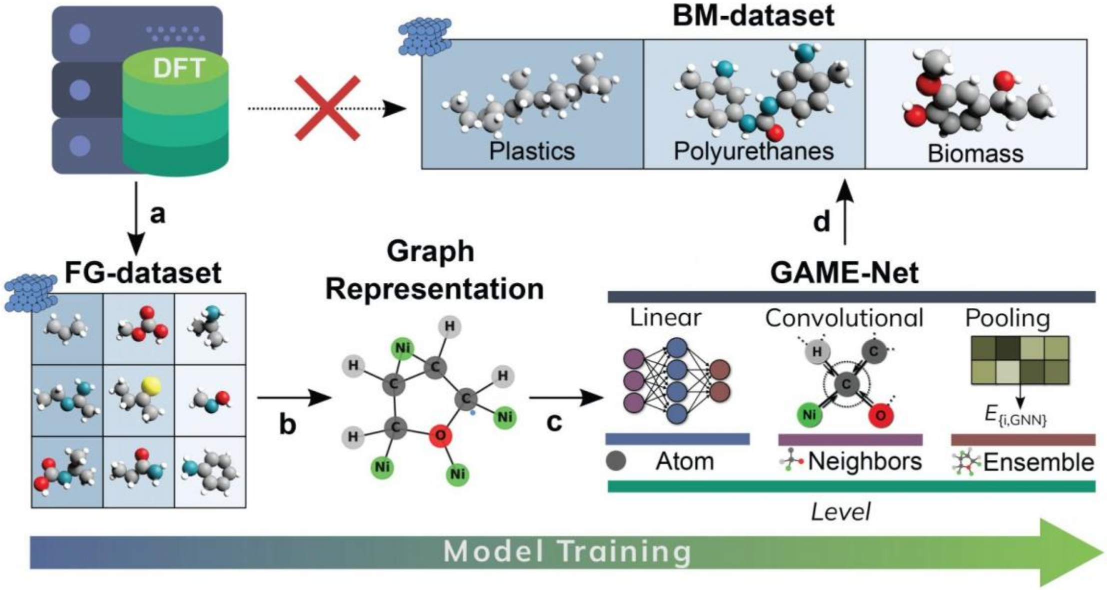


**Road maps**

# AI and Graphs

**"Fast Evaluation of the Adsorption Energy of Organic Molecules on Metals via Graph Neural Networks"**

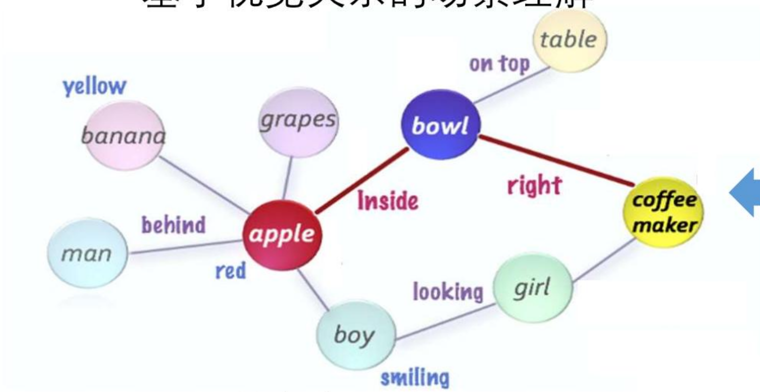
*Nature Comp. Sci. 2023*

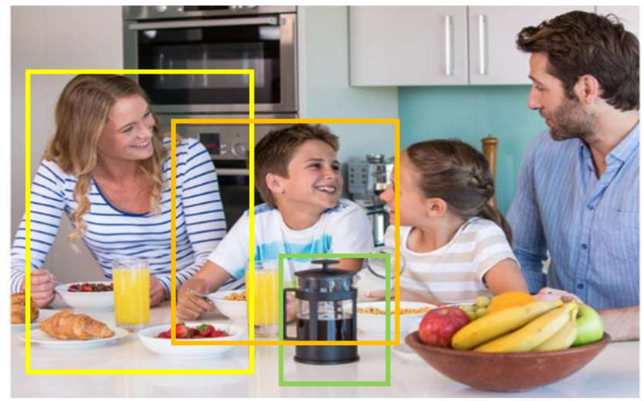



ICIQ Institute of Chemical Research of Catalonia

# AI and Graphs

场景图:
基于视觉关系的场景理解



视觉定位

The **female** is **at the left**.
The **cup is next to the coffeemaker**.
The **male is smiling.**

文本视觉生成

A **family is sitting at the table**, and there are **apples**, **bananas**, and **grapes in the bowl** on the table.

多模态模型

图文匹配

The **boy** is **eating banana**. X
The **boy** is **looking** at **apple**. X
The **boy** is **looking** at **girl**. √

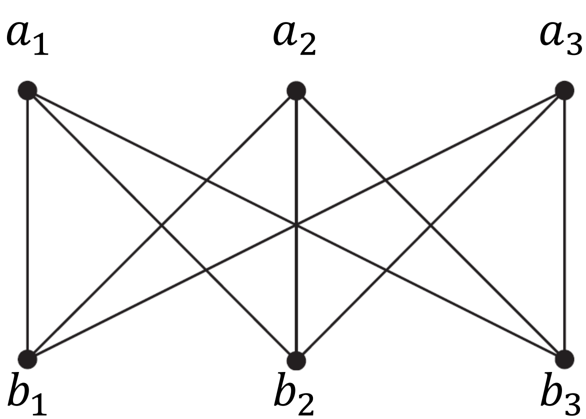
视觉问答

Q: Does **the boy eat bananas**?
A: No.
Q: What **fruit** is **in the bowl**?
A: **Apple/Banana**

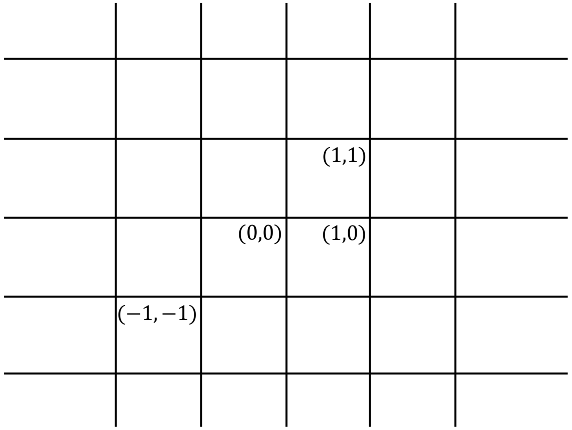
# Graph

**DEFINITION:** A **graph** $G = (V, E)$ is defined by a nonempty set $V$ of **vertices**顶点 and a set $E$ of **edges**边, where each edge is associated with one or two vertices (called **endpoints**端点 of the edge).

- **Infinite Graph**无限图: $|V| = \infty$ or $|E| = \infty$
- **Finite Graph**有限图: $|V| < \infty$ and $|E| < \infty$;  //$|V|$ is called the **order**阶数 of $G$




$V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$
$E = \left\{\{a_i, b_j\}: i, j = 1,2,3\right\}$




$V = \{(i, j): i, j \in \mathbb{Z}\}$
$E = \{\{(a, b), (c, d)\}: |a - c| = 1 \text{ or } |b - d| = 1\}$

# Graphs

- Loop & multiple edge

An edge with one endpoint is called a **loop**.
If there is more than one edge between two distinct vertices, it is called a **multiple edge**.

- Simple graph

A **simple graph** is a finite graph with no loops nor multiple edges.

- Weighted graph

A **weighted graph** is a graph $G = (V, E)$ such that each edge is assignated with a strictly positive number.

# Graphs

- Directed graph

A **directed graph** $G = (V, E)$ consists of:

- $V$ a non empty set of **vertices**,
- $E$ a set of **directed edges**

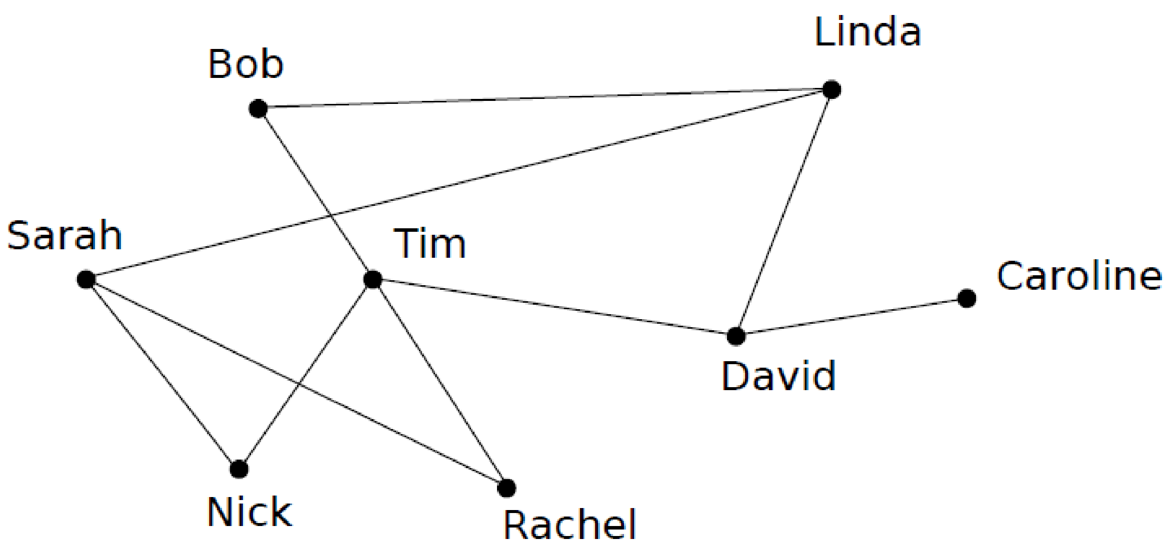
Each edge $e$ is associated with an **ordered pair of vertices** $(u, v)$, we say that $e$ **starts at** $u$ and **ends at** $v$.

- Subgraph

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subset V$, $F \subset E$. A subgraph $H$ of $G$ is a **proper subgraph** if $H \neq G$.

# Graph Examples

**Acquaintanceship Graph:**



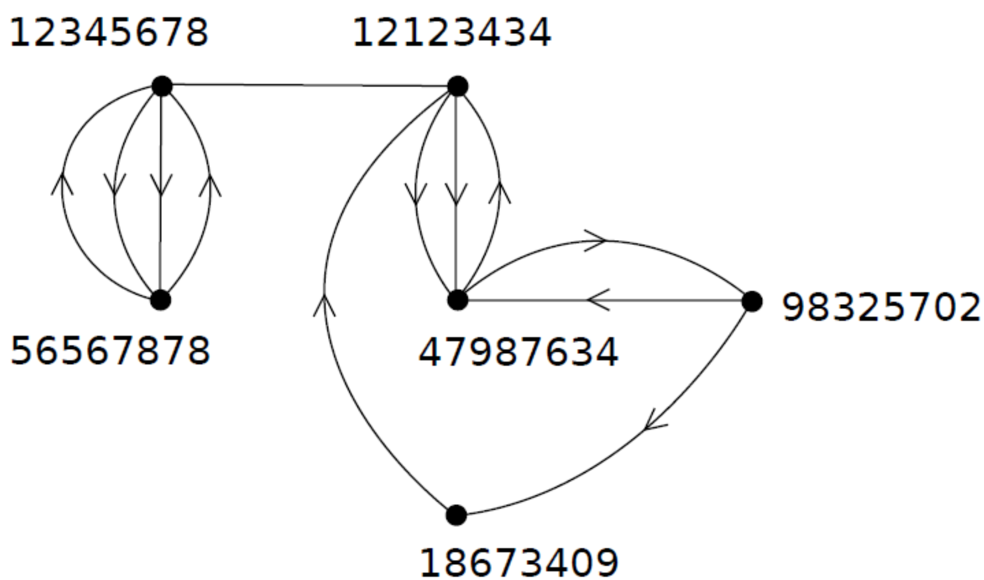

Tim knows Bob, David, Rachel and Nick. But Tim doesn't know Linda neither Caroline.

**Simple graph, undirected**

# Graph Examples

**Call Graphs:** directed edges; the same edge may appear multiple times

- Vertices: telephone numbers
- Edges: there is an arc $(u, v)$ if $u$ called $v$
- AT&T experiment: calls during 20 days (290 million vertices and 4 billion edges)




**Directed graph, multiple edges**

# Graph Examples

**Precedence Graph**

$S_1$ $a := 0$

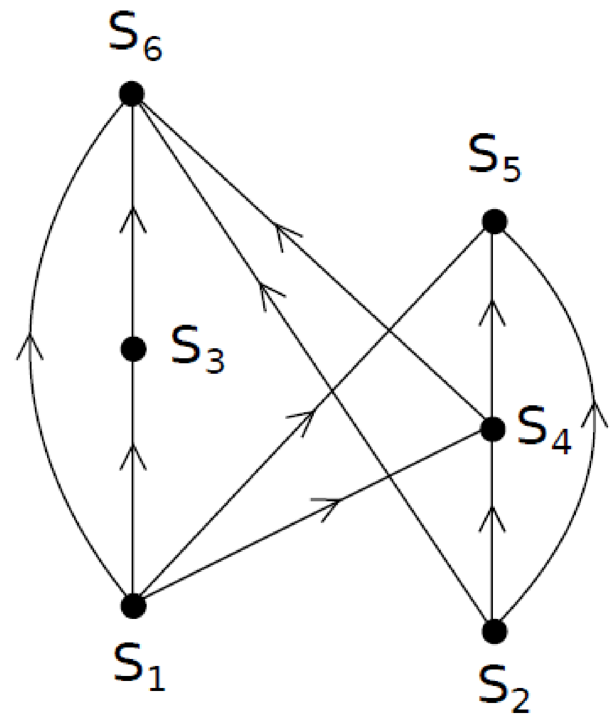
$S_2$ $b := 1$

$S_3$ $c := a + 1$

$S_4$ $d := b + a$

$S_5$ $e := d + 1$

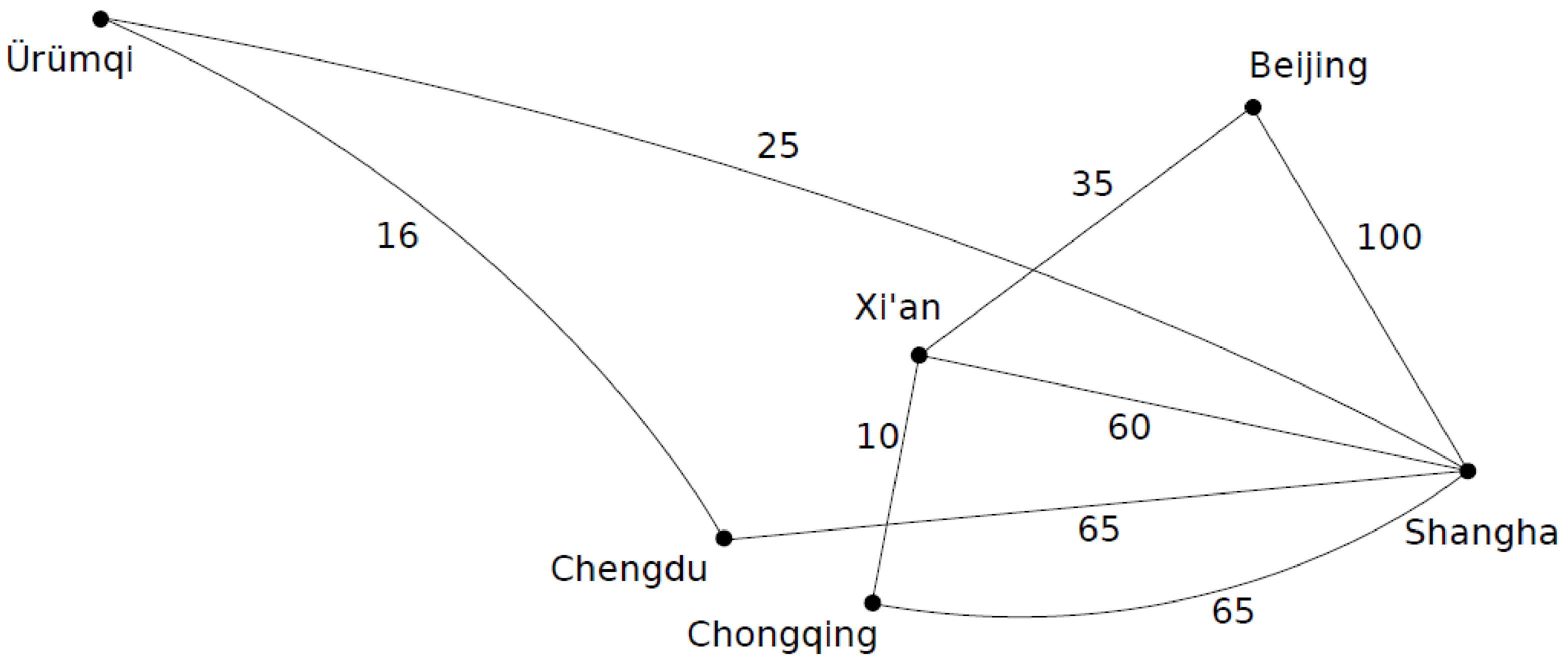
$S_6$ $f := c + d$




**Directed simple graph**

# Graph Examples

**Flights**




**Weighted graph**

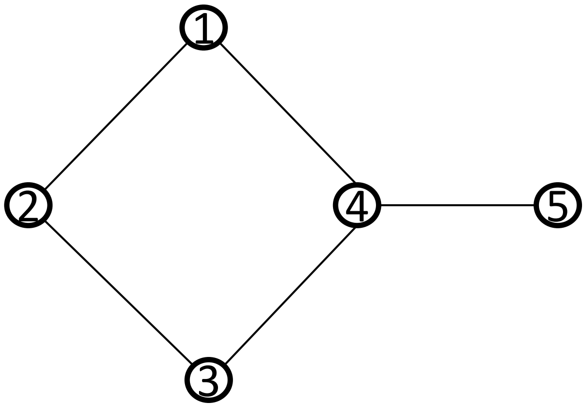
# Types of Graphs

**DEFINITION:** Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \dots, v_n\}$.

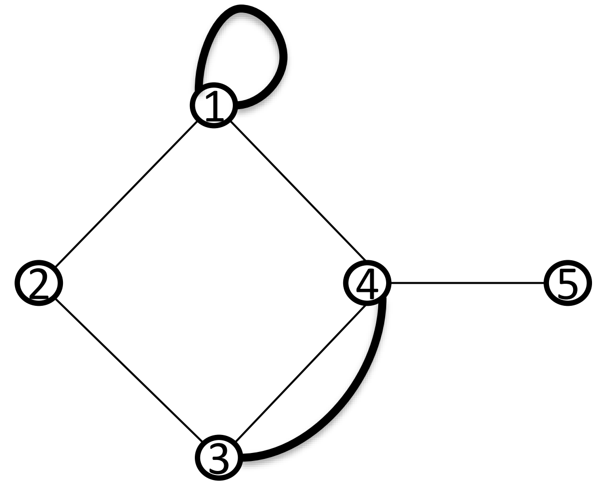
- **Question 1**: are the edges of $G$ **directed**有向的?
  - No: $G$ is an **undirected graph**无向图; the edge connecting $v_i, v_j$: $\{v_i, v_j\}$
  - Yes: $G$ is a **directed graph**有向图; the edge starting at $v_i$ and ending at $v_j$: $(v_i, v_j)$
- **Question 2**: are there **multiple edges**多重边connecting two different vertices $v_i, v_j$?
  - No: $G$ is a **simple graph**简单图**;** Yes: $G$ is a **multigraph**多重图
- **Question 3:** are there **loops**自环 connecting a vertex $v_i$ to itself?
  - Yes: $G$ is a **pseudograph**伪图

| Type | Edges | Multiple Edges Allowed? | Loops Allowed? |
|---|---|---|---|
| Simple graph | undirected | No | No |
| Multigraph | undirected | Yes | No |
| Pseudograph | undirected | Yes | Yes |
| Simple directed graph | directed | No | No |
| Directed multigraph | directed | Yes | Yes |
| Mixed graph | undirected + directed | Yes | Yes |

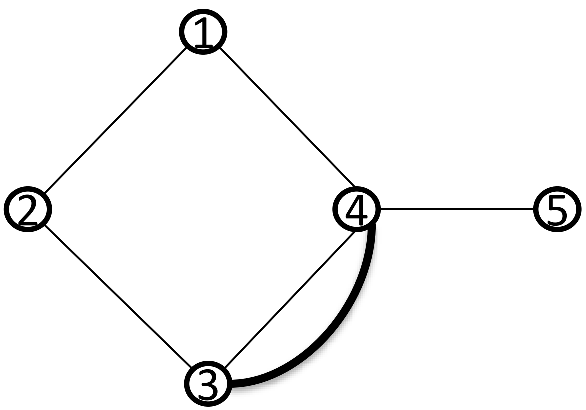
# Types of Graphs




A Simple Graph ($G_1$)



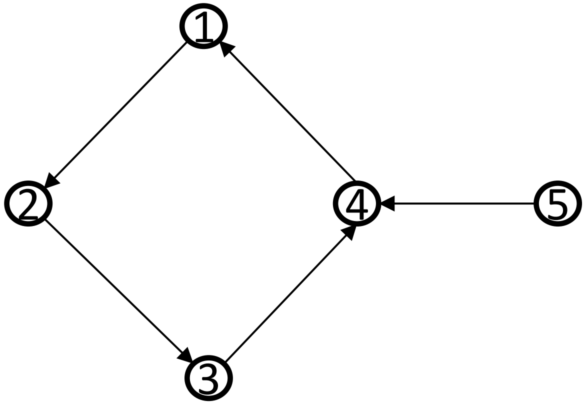

A Pseudograph ($G_3$)



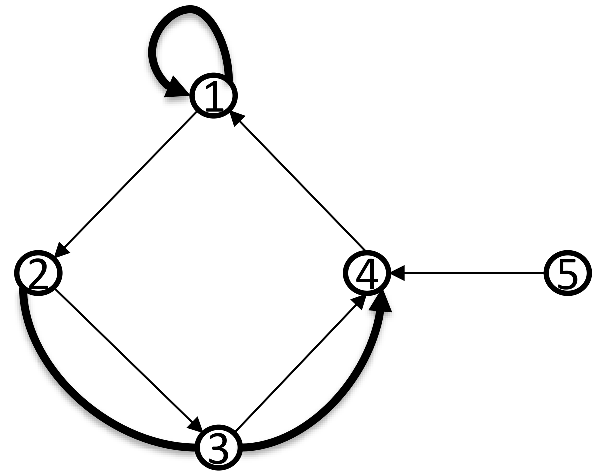

A Multigraph ($G_2$)

- Vertex set: $V = \{1,2,3,4,5\}$
- Edge set of $G_1$: $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{4,5\}\}$
- $\{4,5\}$ is an edge of the simple graph $G_1$
  - 4,5 are endpoints of the edge $\{4,5\}$
  - $\{4,5\}$ connects 4 and 5.
- $\{3,4\}$ is a multiple edge of the multigraph $G_2$
- There is a loop connecting 1 to itself in $G_3$

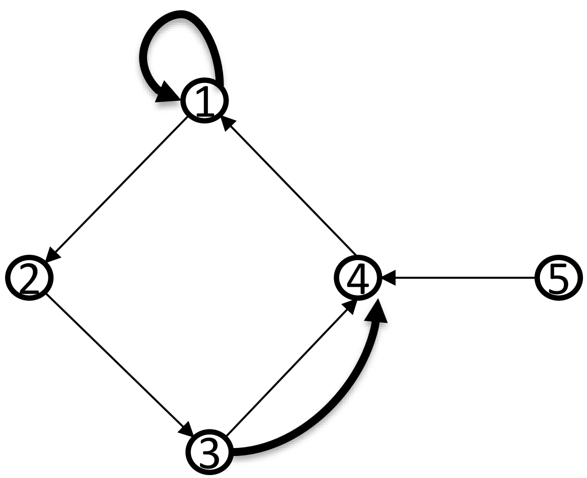
# Types of Graphs




A Simple Directed Graph ($G_4$)



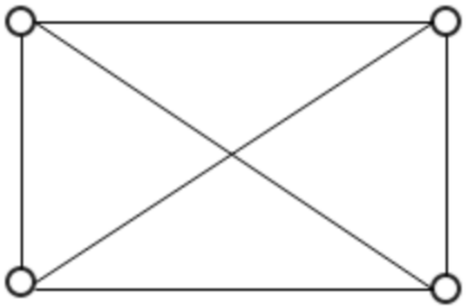

A Mixed Graph ($G_6$)



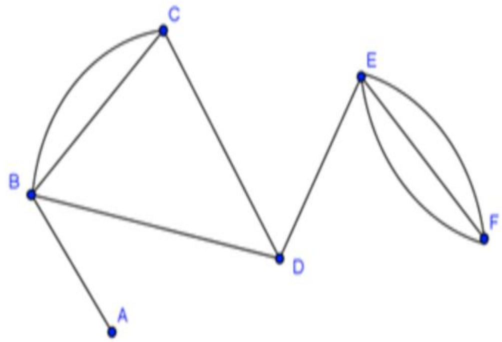

A Directed Multigraph ($G_5$)

- Vertex set: $V = \{1,2,3,4,5\}$
- Edge set of $G_4$: $E = \{(1,2), (2,3), (3,4), (4,1), (5,4)\}$
  - $(5,4)$ is a directed edge
  - $(5,4)$ starts at 5 and ends at 4
- $(3,4)$ is a directed multiple edge in $G_5$
- There is a loop connecting 1 to itself in $G_5$

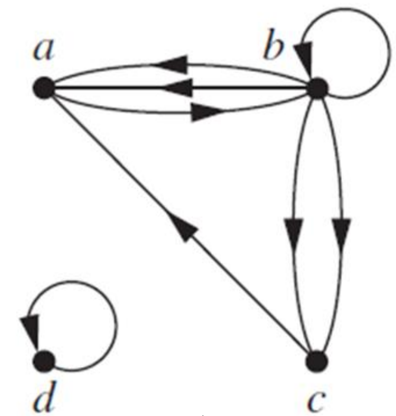
# Bonus exercise



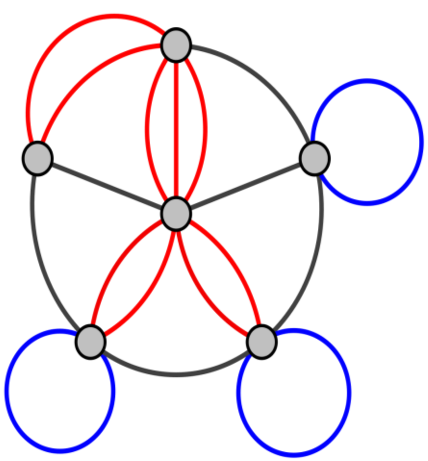
（1）



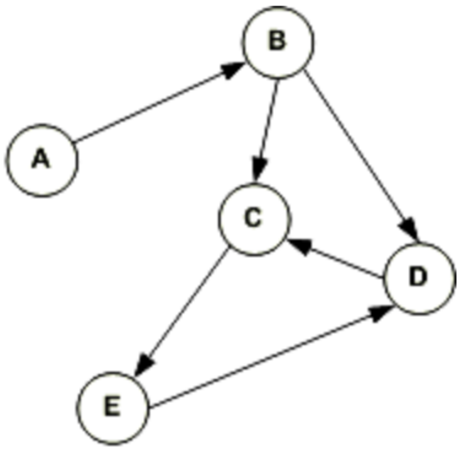

（3）




（5）

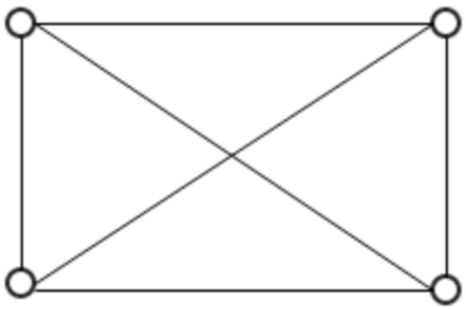


（2）

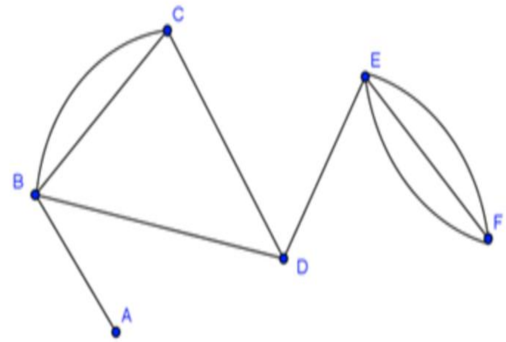



（4）

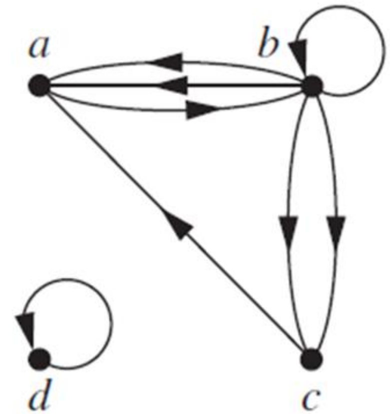
# Bonus exercise



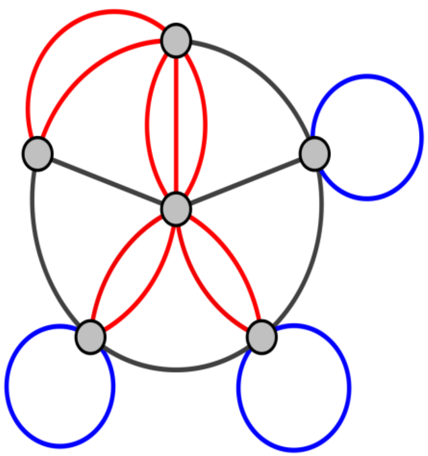
（1） simple graph



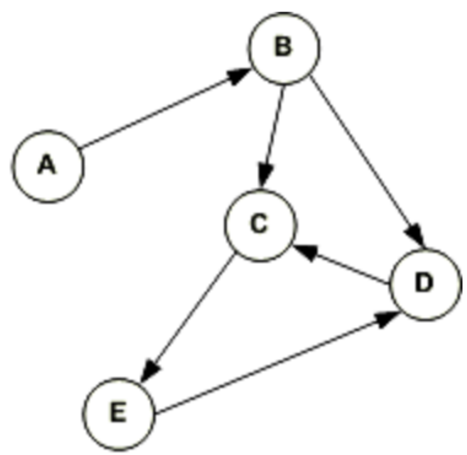

（3） multigraph




（5） directed multigraph



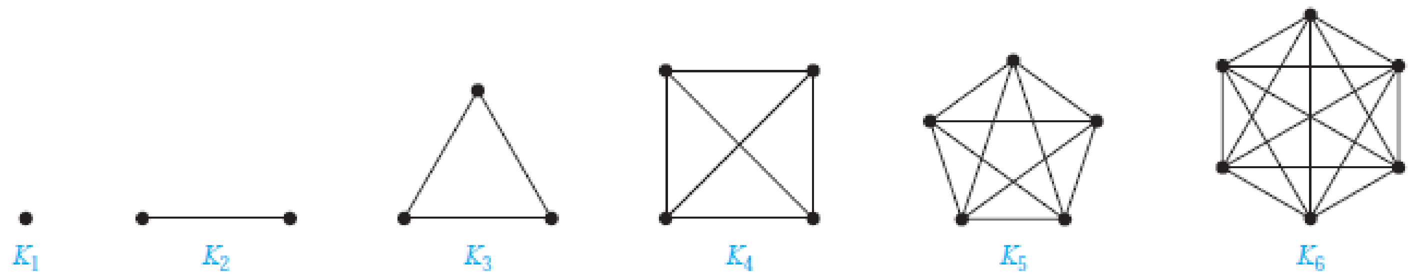
（2） pseudograph



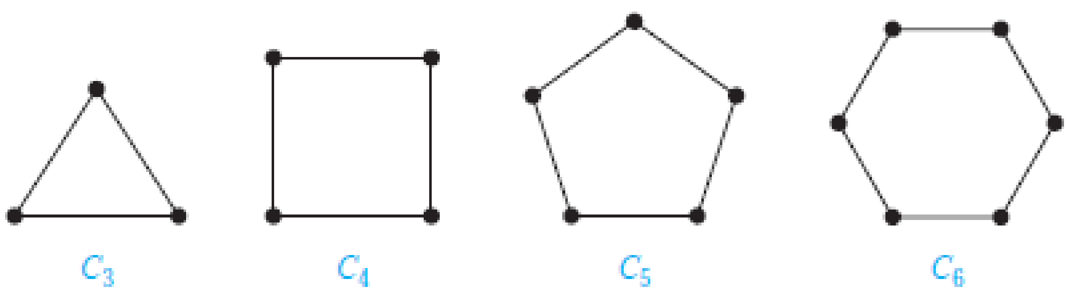

（4） simple directed graph

# Special Simple Graphs

**Complete Graph**完全图 $K_n$: $V = \{v_1, \dots, v_n\}$; $E = \left\{\{v_i, v_j\}: 1 \leq i \neq j \leq n\right\}$

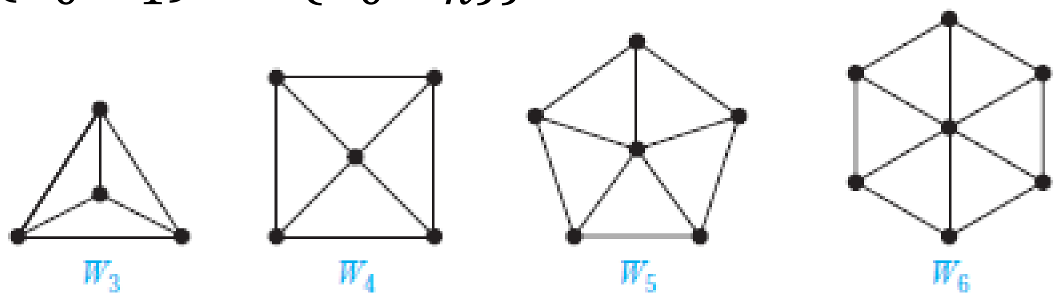



**Cycle**环,圈 $C_n$: $V = \{v_1, v_2, \dots, v_n\}$; $E = \left\{\{v_1 v_2\}, \{v_2, v_3\}, \dots, \{v_n, v_1\}\right\}$



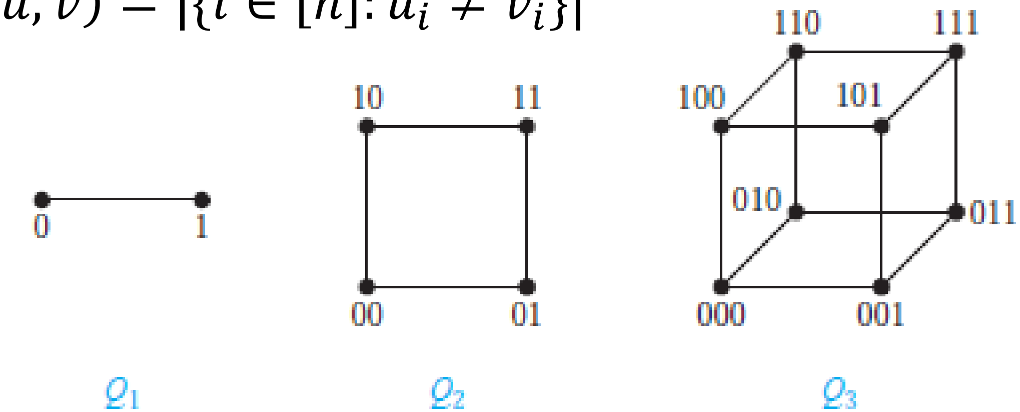
# Special Simple Graphs

**Wheel**轮 $W_n$: $V = \{v_0, v_1, v_2, \dots, v_n\}$; $E = \{\{v_1, v_2\}, \dots, \{v_n, v_1\}\} \cup \{\{v_0, v_1\}, \dots, \{v_0, v_n\}\}$




$n$-**Cubes**方体 $Q_n$: $V = \{0,1\}^n$; $E = \{\{u, v\}: d(u, v) = 1\}$

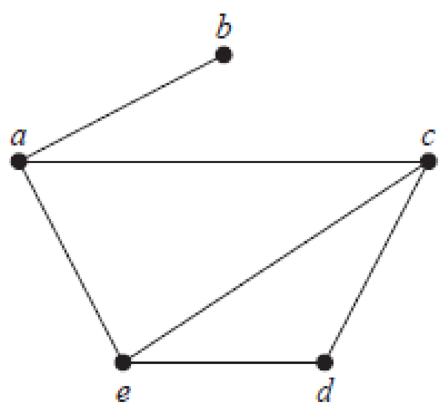
- $d(u, v) = |\{i \in [n]: u_i \neq v_i\}|$



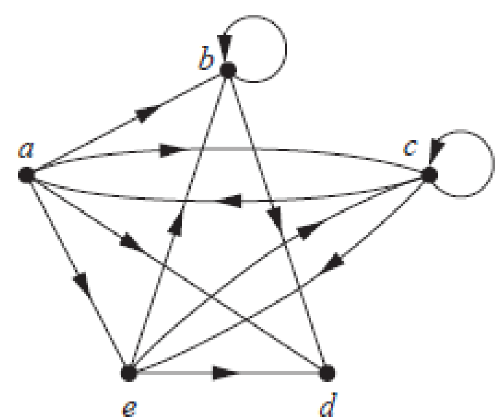
# Adjacency List

**DEFINITION:** Let $G = (V, E)$ be a graph with no multiple edges. The **adjacency list**邻接表 of $G$ is a list the vertices of the graph and all adjacent vertices

- $v_i, v_j \in V$ are **adjacent**相邻的 if $\{v_i, v_j\}$ or $(v_i, v_j)$ is an edge




| | |
|---|---|
| $a$ | $b, c, e$ |
| $b$ | $a$ |
| $c$ | $a, d, e$ |
| $d$ | $c, e$ |
| $e$ | $a, c, d$ |

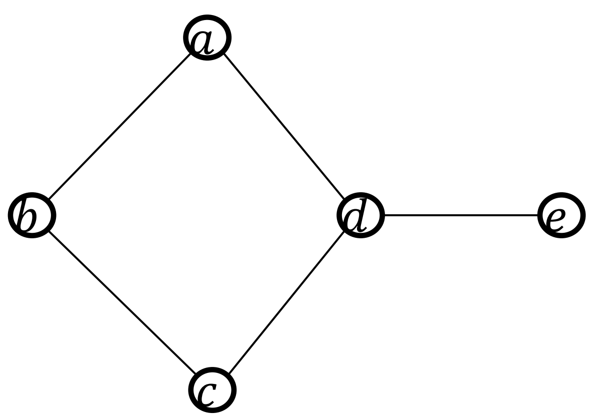



| | |
|---|---|
| $a$ | $b, c, d, e$ |
| $b$ | $b, d$ |
| $c$ | $a, c, e$ |
| $d$ | |
| $e$ | $b, c, d$ |

# Adjacency Matrix

**DEFINITION:** Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a simple graph. The **adjacency matrix**邻接矩阵 of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & \{v_i, v_j\} \notin E \end{cases}$$

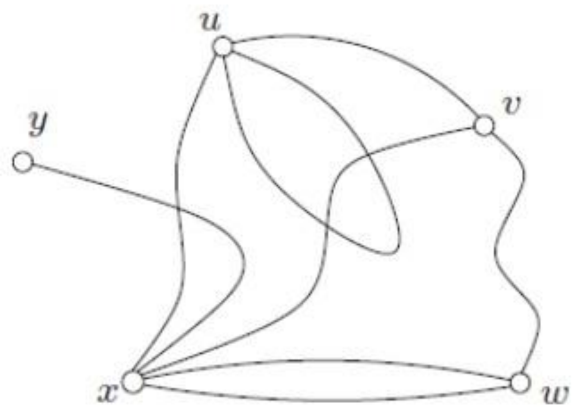



| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | 0 | 1 | 0 |
| $b$ | 1 | 0 | 1 | 0 | 0 |
| $c$ | 0 | 1 | 0 | 1 | 0 |
| $d$ | 1 | 0 | 1 | 0 | 1 |
| $e$ | 0 | 0 | 0 | 1 | 0 |

# Adjacency Matrix

**DEFINITION:** Let $G = (V = \{v_1, \dots, v_n\}, E)$ be an undirected graph. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where

- $a_{ij} =$ **multiplicity**重数 of $\{v_i, v_j\}$ when $i \neq j$
- $a_{ii} = 1$ if ∃ a loop from $v_i$ to itself; $a_{ii} = 0$, otherwise.

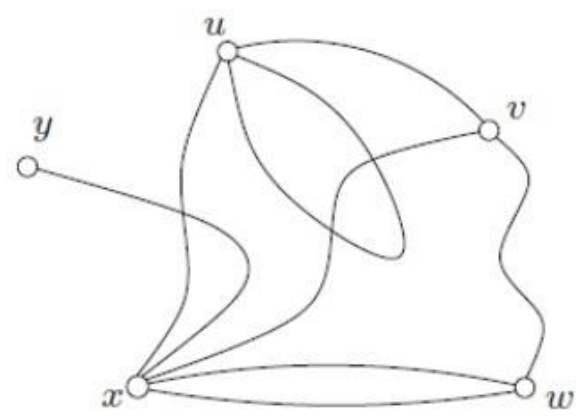



| | $u$ | $v$ | $w$ | $x$ | $y$ |
|---|---|---|---|---|---|
| $u$ | $\boxed{1}$ | 1 | 0 | 1 | 0 |
| $v$ | 1 | 0 | 1 | 1 | 0 |
| $w$ | 0 | 1 | 0 | 2 | 0 |
| $x$ | 1 | 1 | 2 | 0 | 1 |
| $y$ | 0 | 0 | 0 | 1 | 0 |

# Adjacency Matrix

**DEFINITION:** Let $G = (V = \{v_1, \ldots, v_n\}, E)$ be an undirected graph. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where

- $a_{ij} = \textbf{multiplicity}$ 重数 of $\{v_i, v_j\}$ when $i \neq j$
- $a_{ii} = 1$ if $\exists$ a loop from $v_i$ to itself; $a_{ii} = 0$, otherwise.




| | $x$ | $y$ | $u$ | $v$ | $w$ |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 1 | 1 | 2 |
| $y$ | 1 | 0 | 0 | 0 | 0 |
| $u$ | 1 | 0 | 1 | 1 | 0 |
| $v$ | 1 | 0 | 1 | 0 | 1 |
| $w$ | 2 | 0 | 0 | 1 | 0 |

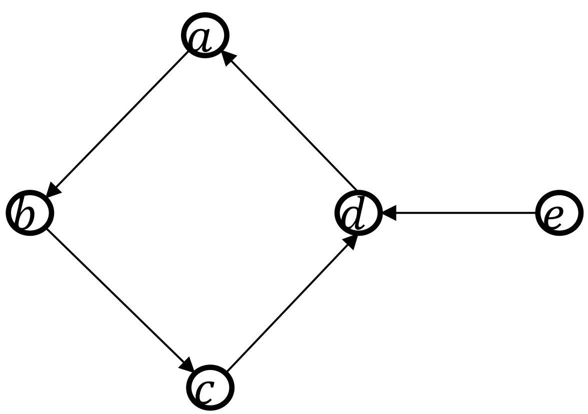
**REMARKs**: features of the adjacency matrices of undirected graphs

- The adjacency matrix depends on the ordering of the vertices
- The adjacency matrix of a simple graph is always symmetric
- The $(i, j)$ entry counts the multiplicity of $\{v_i, v_j\}$, $i \neq j$

# Adjacency Matrix

**DEFINITION:** Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a simple directed graph. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & (v_i, v_j) \in E \\ 0 & (v_i, v_j) \notin E \end{cases}$$



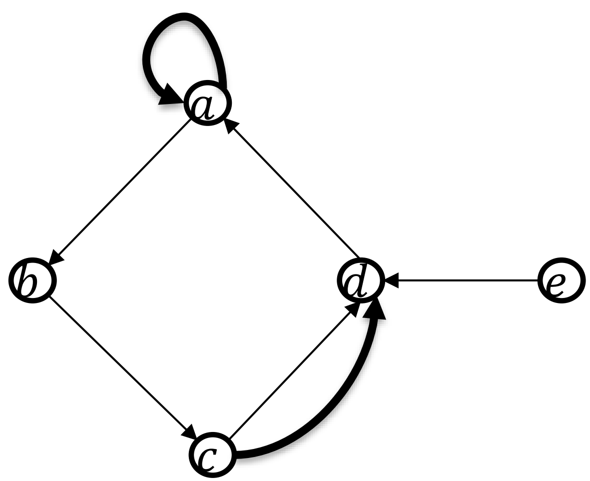

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 1 | 0 | 0 |
| $c$ | 0 | 0 | 0 | 1 | 0 |
| $d$ | 1 | 0 | 0 | 0 | 0 |
| $e$ | 0 | 0 | 0 | 1 | 0 |

**REMARKS**: The adjacency matrix is no longer symmetric

# Adjacency Matrix

**DEFINITION:** Let $G = (V = \{v_1, \ldots, v_n\}, E)$ be a directed multigraph. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} \text{multiplicity of } (v_i, v_j) & (v_i, v_j) \in E \\ 0 & (v_i, v_j) \notin E \end{cases}$$



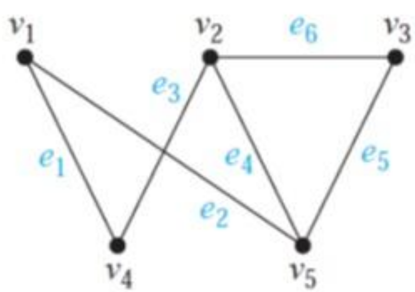

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 1 | 1 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 1 | 0 | 0 |
| $c$ | 0 | 0 | 0 | 2 | 0 |
| $d$ | 1 | 0 | 0 | 0 | 0 |
| $e$ | 0 | 0 | 0 | 1 | 0 |

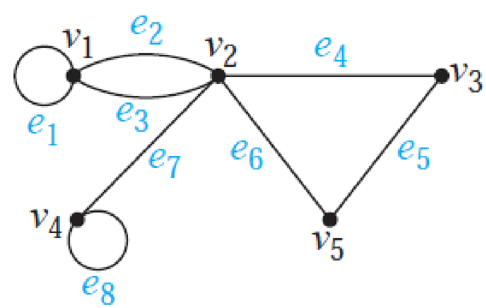
# Incidence Matrix

**DEFINITION:** Let $G = (V = \{v_1, \ldots, v_n\}, E = \{e_1, \ldots, e_m\})$ be <u>undirected</u>. The **incidence matrix**关联矩阵 of $G$ is an $n \times m$ matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

- $e_j$ **incident with** $v_i$: $v_i$ is an endpoint of $e_j$



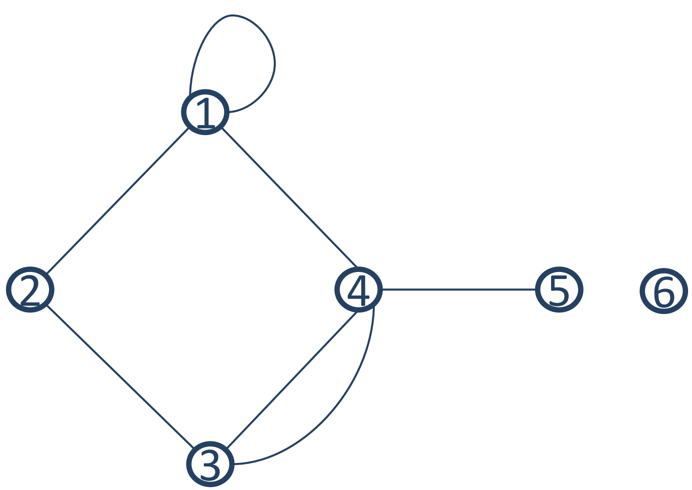

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

# Degree

**DEFINITION:** Let $G = (V, E)$ be an <u>undirected</u> graph. We say that two vertices $u, v \in V$ are **adjacent**相邻的 (or **neighbors**邻居) if $\{u, v\} \in E$.

- **neighborhood**邻域 of $v$ in $G$: $N(v) = \{u \in V : \{u, v\} \in E\}$
  - $N(A) = \cup_{v \in A} N(v)$ for $A \subseteq V$
- the **degree**度 $\deg(v)$ of $v \in V$ in $G$, is the number of edges incident with $v$
  - every loop from $v$ to $v$ contributes 2 to $\deg(v)$
- $v$ is **isolated**孤立的 if $\deg(v) = 0$; $v$ is **pendant**悬挂的 if $\deg(v) = 1$

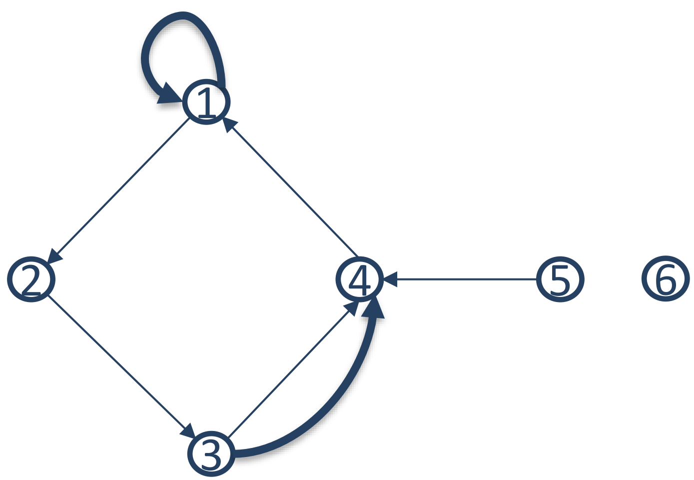



- 4 and 5 are adjacent
- $\{4,5\}$ is incident with 4 and 5
- $N(4) = \{1,3,5\}$; $N(\{1,4\}) = \{1,2,3,4,5\}$
- $\deg(1) = 4, \deg(2) = 2, \deg(3) = 3, \deg(4) = 4, \deg(5) = 1$
- 6 is isolated; 5 is pendant

# Degree

**DEFINITION: Let** $G = (V, E)$ be a <u>directed</u> graph. If $(u, v) \in E$, we say that $u$ is **adjacent to** $v$ and $v$ is **adjacent from** $u$.

- $u$ is the **initial vertex**起始点 of $(u, v)$; $v$ is the **terminal vertex**终点 of $(u, v)$
  - $u = v$: $u$ is the initial vertex and the terminal vertex
- **in-degree**入度 $\deg^{-}(v)$ : the number of edges where $v$ is the terminal vertex
- **out-degree**出度 $\deg^{+}(v)$ : the number of edges where $v$ is the initial vertex
  - $u = v$: the loop contributes 1 to $\deg^{-}(v)$ and 1 to $\deg^{+}(v)$




- 5 is adjacent to 4; 4 is adjacent from 5
- 5 is the initial vertex of (5,4)
- 4 is the terminal vertex of (5,4)
- 1 is the initial and terminal vertex of a loop
- $\deg^{-}(1) = 2\,; \deg^{+}(1) = 2$
- $\deg^{-}(4) = 3\,; \deg^{+}(4) = 1$

# Handshaking Theorem

**THEOREM:** Let $G = (V, E)$ be an undirected graph. Then

$2|E| = \sum_{v \in V} \deg(v)$ and $|\{v \in V : \deg(v) \text{ is odd}\}|$ is even.

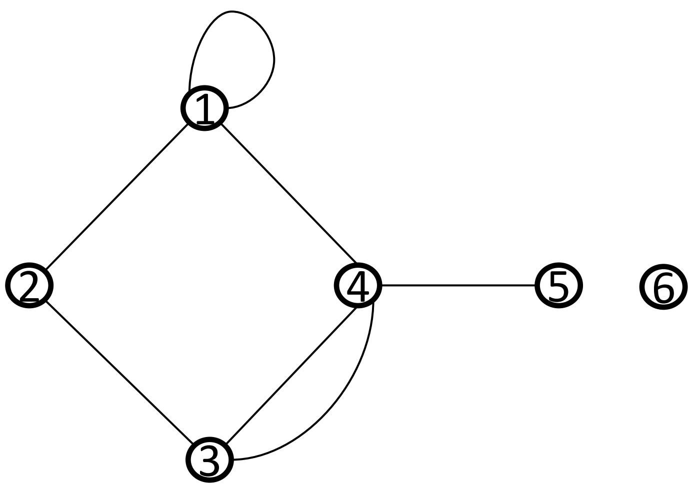
- Any edge $e \in E$ contribute 2 to the sum $\sum_{v \in V} \deg(v)$
  - $e = \{v_i, v_j\}$: $e$ contributes 1 to $\deg(v_i)$ and 1 to $\deg(v_j)$
  - $e = \{v_i\}$: $e$ contributes 2 to $\deg(v_i)$
- The $m$ edges contribute $2|E|$ to $\sum_{v \in V} \deg(v)$.
  - Hence, $\sum_{v \in V} \deg(v) = 2|E|$
- $\sum_{v \in V} \deg(v) = \sum_{v \in V : 2 | \deg(v)} \deg(v) + \sum_{v \in V : 2 \nmid \deg(v)} \deg(v)$
  - $2 | \sum_{v \in V} \deg(v)$; $2 | \sum_{v \in V : 2 | \deg(v))} \deg(v)$
    - $2 | \sum_{v \in V : 2 \nmid \deg(v)} \deg(v)$
      - $|\{v \in V : \deg(v) \text{ is odd}\}|$ must be even

# Handshaking Theorem

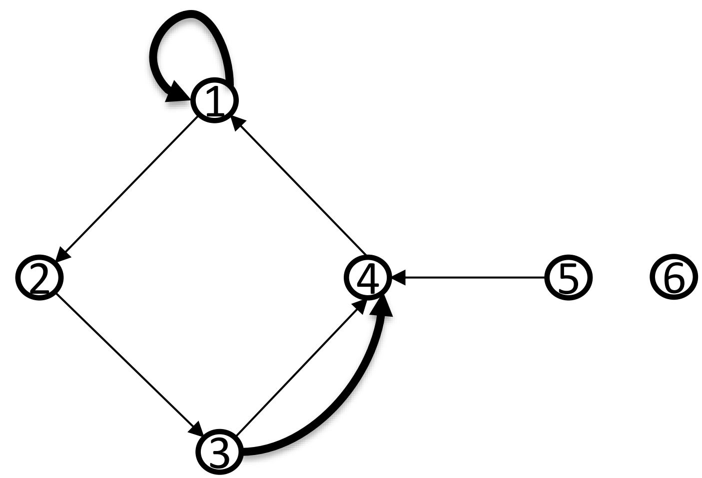
**THEOREM:** Let $G = (V, E)$ be a <u>directed</u> graph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

- Every edge $e \in E$ contributes 1 to $\sum_{v \in V} \deg^-(v)$
  - $e = (v_i, v_j)$ contributes 1 to $\deg^-(v_i)$
- Hence, $\sum_{v \in V} \deg^-(v) = |E|$




| $v$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\deg(v)$ | 4 | 2 | 3 | 4 | 1 | 0 |



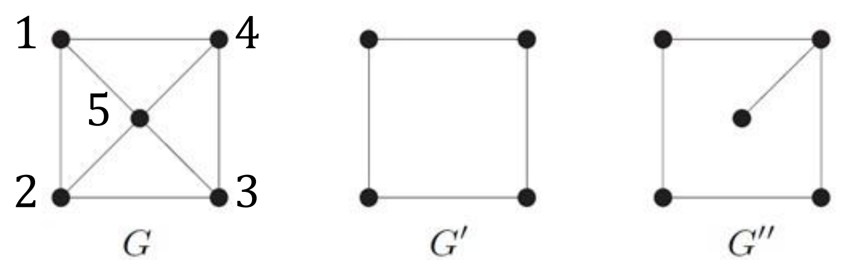

| $v$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\deg^-(v)$ | 2 | 1 | 1 | 3 | 0 | 0 |
| $\deg^+(v)$ | 2 | 1 | 2 | 1 | 1 | 0 |

# Subgraph

**DEFINITION:** Let $G = (V, E)$ be a simple graph. $H = (W, F)$ is a **subgraph**子图 of $G$ if $W \subseteq V$ and $F \subseteq E$.

- **proper subgraph**真子图: $H$ is a subgraph of $G$ and $H \neq G$.
- The **subgraph induced**导出子图 by $W \subseteq V$ is $(W, F)$, where $F = \{e: e \in E, e = \{u, v\} \subseteq W\}$. //Notation: $G[W]$
- The **subgraph induced**导出子图 by $F \subseteq E$ is $(W, F)$, where $W = \{v: v \in V, v \in e \text{ for some } e \in F\}$. //Notation: $G[F]$

**EXAMPLE:** Let $G, G', G''$ be three graphs as below.

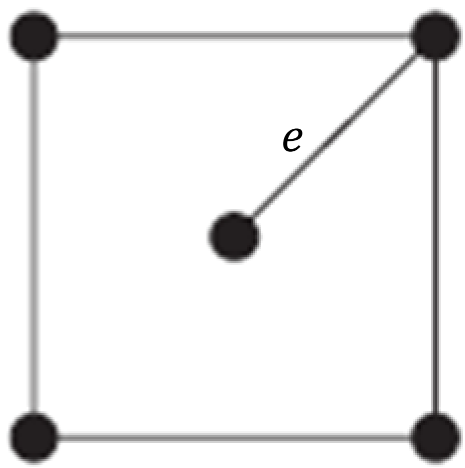



- $G', G''$ are subgraphs of $G$; $G', G''$ are proper subgraphs of $G$
- $G'$ is a subgraph induced by $W = \{1,2,3,4\}$, i.e., $G' = G[W]$
- $G''$ is a subgraph induced by $F = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{4,5\}\}$, i.e., $G'' = G[F]$

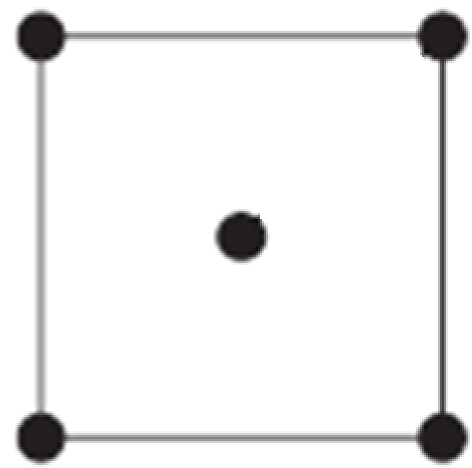
# Removing An Edge

**DEFINITION:** Let $G = (V, E)$ be a simple graph and $e \in E$. Define

$$G - e = (V, E - \{e\})$$




$G = (V, E)$

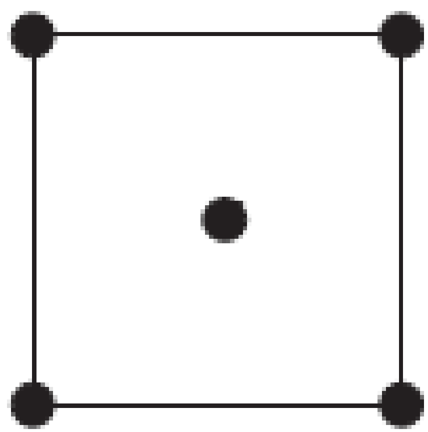


$G - e = (V, E - \{e\})$

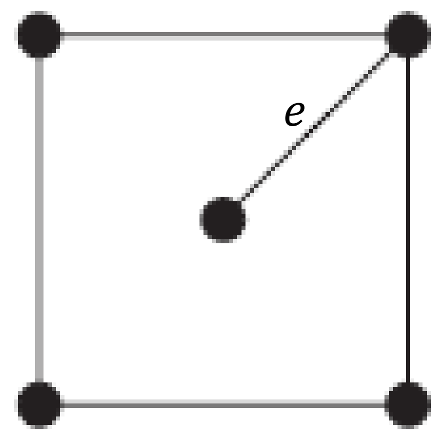
# Adding An Edge

**DEFINITION:** Let $G = (V, E)$ be a simple graph and $e \notin E$. Define

$$G + e = (V, E \cup \{e\})$$



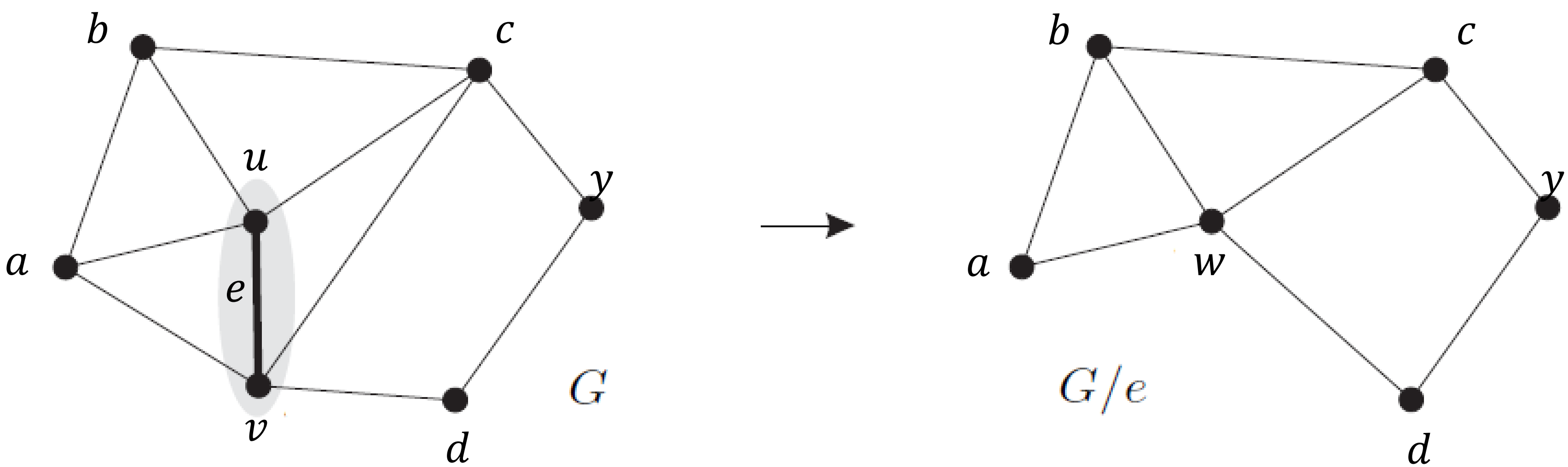
$G = (V, E)$




$G + e = (V, E \cup \{e\})$

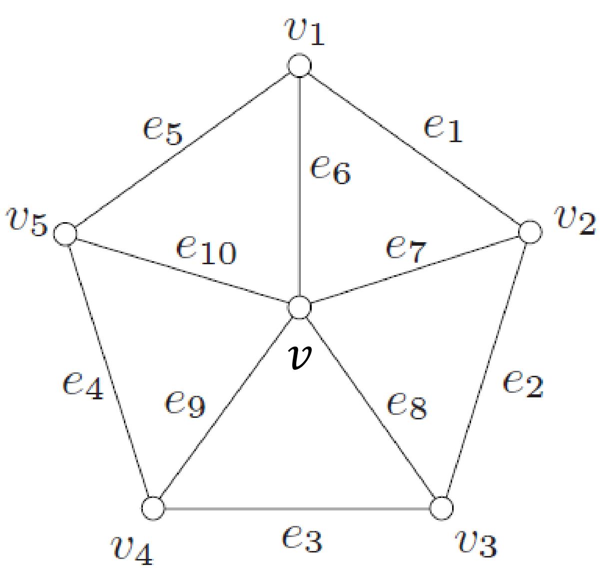
# Edge Contraction

**DEFINITION:** Let $G = (V, E)$ be a simple graph and $e = \{u, v\} \in E$. Define $G/e = (V', E')$, where $V' = (V - \{u, v\}) \cup \{w\}$ and $E' = \{e' \in E : e' \cap e = \emptyset\} \cup \left\{\{w, x\} : \{u, x\} \in E \text{ or } \{v, x\} \in E\right\}$

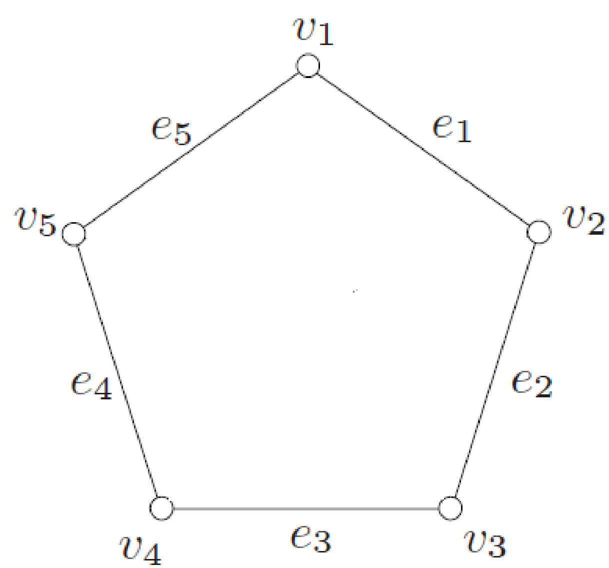


# Removing A Vertex

**DEFINITION:** Let $G = (V, E)$ be a simple graph and let $v \in V$. Define $G - v = (V - \{v\}, E')$, where $E' = \{e \in E : v \notin e\}$



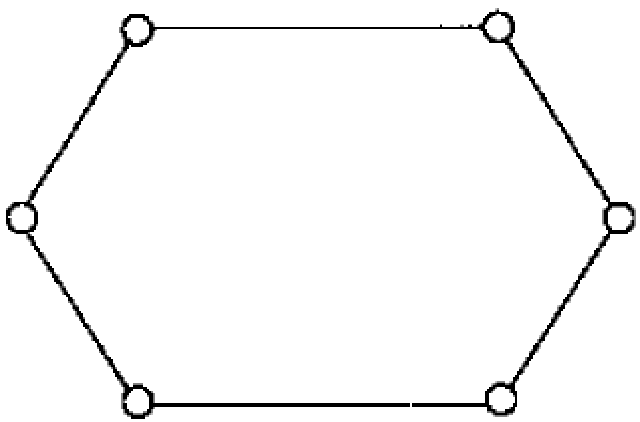

$G = (V, E)$



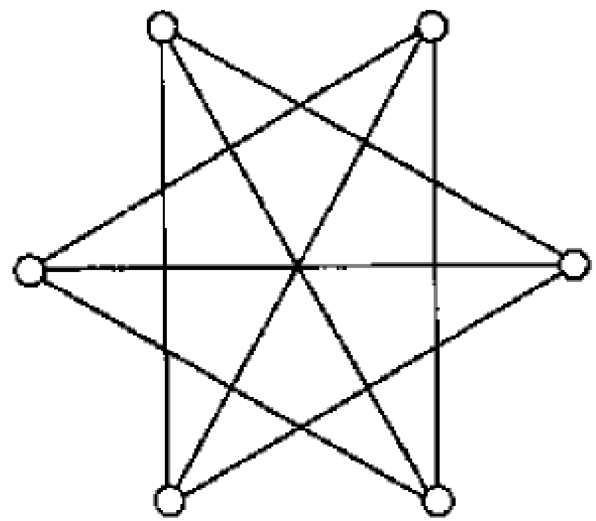

$G - v$

# Complement

**DEFINITION:** Let $G = (V, E)$ be a simple graph of order $n$. Define the **complement graph**补图 of $G$ as $\bar{G} = (V, E')$, where
$E' = \big\{\{u, v\}: u, v \in V,\ u \neq v, \{u, v\} \notin E\big\}$



$G$

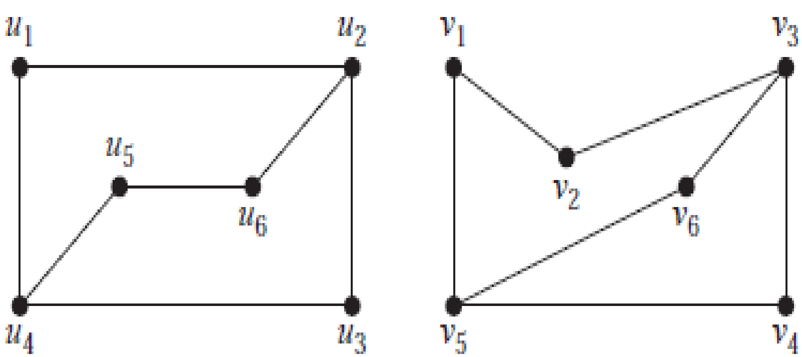


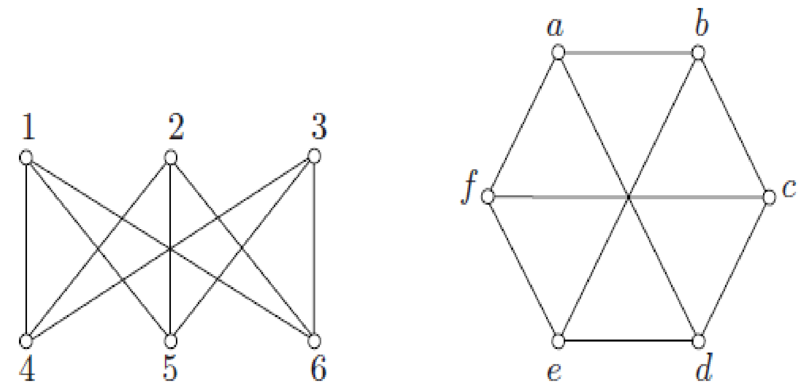
$\bar{G}$

# Graph Isomorphism

**DEFINITION:** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**同构 if there is a bijection $\sigma: V_1 \to V_2$ such that $\{u, v\} \in E_1 \Leftrightarrow \{\sigma(u), \sigma(v)\} \in E_2$.

- $\sigma$ is called an **isomorphism**同构映射
- **nonisomorphic:** not isomorphic




| | | | | | |
|---|---|---|---|---|---|
| $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
| $v_6$ | $v_3$ | $v_4$ | $v_5$ | $v_1$ | $v_2$ |

Isomorphism $\sigma$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| $a$ | $c$ | $e$ | $b$ | $d$ | $f$ |

Isomorphism $\sigma$