Discrete Mathematics: Lecture 21

Part IV. Graph Theory

Handshaking Theorem, Graph Transform, Graph Isomorphism, Bipartite Graph, Matching

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1. Graph Type Identification

Consider a graph G with the following properties:

- Vertex set: $V = \{v_1, v_2, v_3\}$
- Edge set: $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_2), (v_2, v_2)\}$

Which of the following best describes the type of graph G?

A. Simple directed graph
B. Directed multigraph
C. Pseudograph
D. Mixed graph

2. Handshaking Theorem

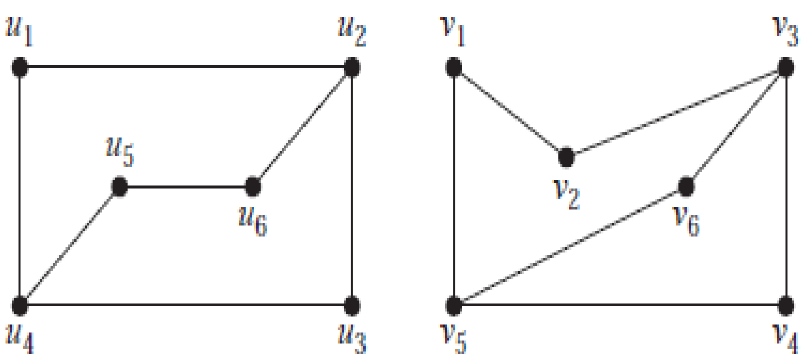
Given G being a undirected graph, what is the possible number of vertices with odd degree?

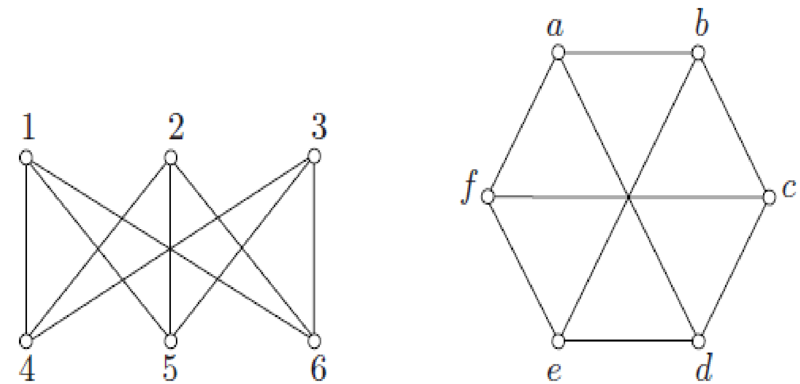
A. 0
B. 1
C. 3
D. 5

Review: Graph Isomorphism

DEFINITION: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**同构 if there is a bijection $\sigma: V_1 \to V_2$ such that $\{u, v\} \in E_1 \Leftrightarrow \{\sigma(u), \sigma(v)\} \in E_2$.

- σ is called an **isomorphism**同构映射
- **nonisomorphic:** not isomorphic





u_1	u_2	u_3	u_4	u_5	u_6
v_6	v_3	v_4	v_5	v_1	v_2

Isomorphism σ

1	2	3	4	5	6
a	c	e	b	d	f

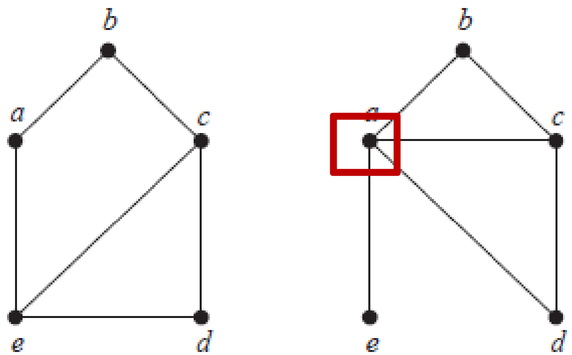
Isomorphism σ

Graph Invariants

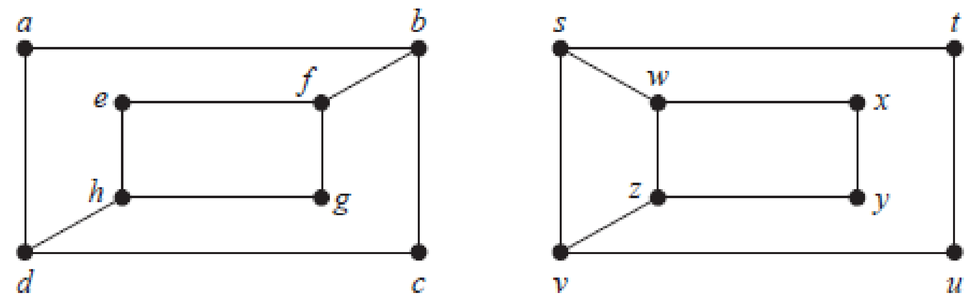
DEFINITION: **Graph invariants** are properties preserved by graph isomorphism. For example,

- The number of vertices
- The number of edges
- The number of vertices of each degree

REAMRKS: The graph invariants can be used to determine if two graphs are isomorphic or not.



There is no vertex of degree 4 in the 1st graph



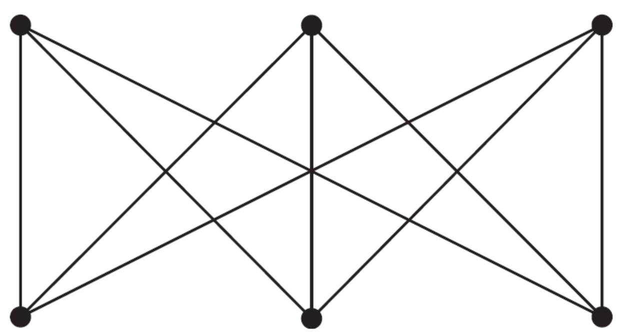
Induced subgraphs are not isomorphic. Original graphs are not isomorphic.

The subgraphs induced by the vertices of degree 3 must be isomorphic to each other.

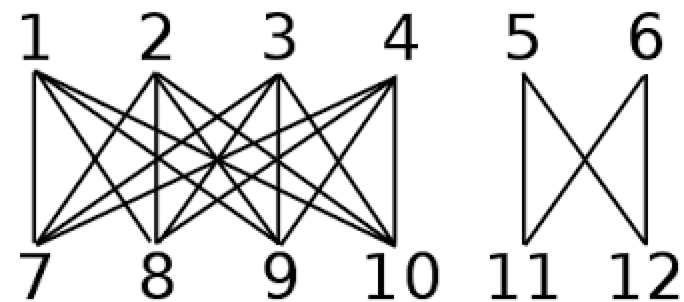
Bipartite Graph

DEFINITION: $G = (V, E)$ is a **bipartitie graph**二分图、二部图 if V has a partition $\{V_1, V_2\}$ such that $E \subseteq \{\{u_1, u_2\}: u_1 \in V_1, u_2 \in V_2\}$.

- (V_1, V_2) is a **bipartition**二划分 of the vertex set V.



A bipartite graph of order 6



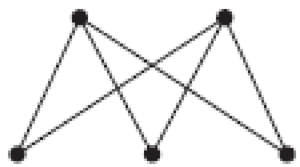
A bipartite graph of order 12

- $V_1 = \{1,2,3,4,5,6\}$
- $V_2 = \{7,8,9,10,11,12\}$

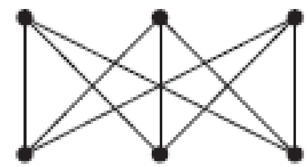
Complete Bipartite Graph

DEFINITION: A **complete bipartite graph**完全二部图 is a graph $K_{m,n} = (V, E)$ with $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$ and $E = \left\{\{x_i, y_j\}: i \in [m], j \in [n]\right\}$

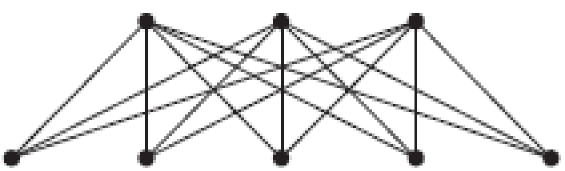
- Every vertex in V_1 is adjacent to every vertex in V_2



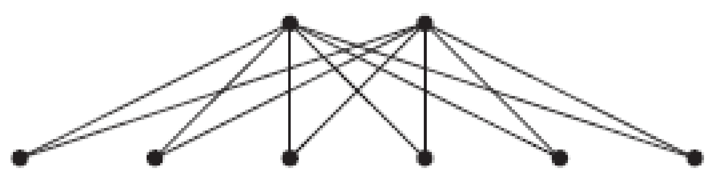
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Bipartite Graph

Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex such that no two adjacent vertices have the same color.

Proof:

- If $G = (V, E)$ is bipartite, $V = V_1 \cup V_2$. Assign color c_1 to vertices of V_1 and color c_2 to vertices of V_2.
- Reversely, suppose we can assign colors c_1 and c_2 to the vertices such that no two adjacent have the same. Let V_i be the set of vertices of color c_i, for $i = 1, 2$. Then $V = V_1 \cup V_2$. By assumption there are no edges connecting two vertices of V_1 or two vertices of V_2, so each edge connects one vertex of V_1 with one vertex of V_2. $\square$

Bipartite Graph*

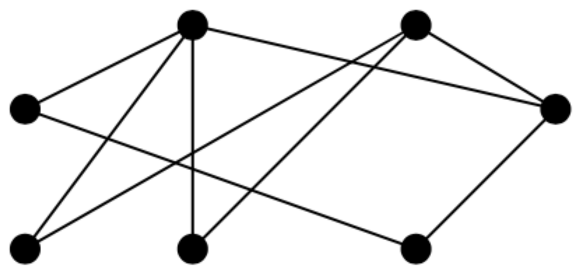
THEOREM: A simple graph $G = (V, E)$ is a bipartite graph iff there is a map $f: V \to \{1,2\}$ such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"

- Only if: $G = (V_1 \cup V_2,\ E)$, where $V_1 \cap V_2 = \emptyset$.
  - Define $f: V \to \{1,2\}$ such that $f(x) = \begin{cases} 1 & \text{if } x \in V_1 \\ 2 & \text{if } x \in V_2 \end{cases}$
  - $\{x, y\} \in E \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - $f(x) \neq f(y)$
- If: $f: V \to \{1,2\}$ is a map such that "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)$"
  - Let $V_1 = f^{-1}(1), V_2 = f^{-1}(2)$
    - $V = V_1 \cup V_{2,} V_1 \cap V_2 = \emptyset$
      - $\{V_1, V_2\}$ is a bipartition of V
  - $\{x, y\} \in E \Rightarrow f(x) \neq f(y) \Rightarrow x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$
    - G is a bipartite graph.

***: This indicates that this slide is an optional topic, which can be skipped.**

Bipartite Graph

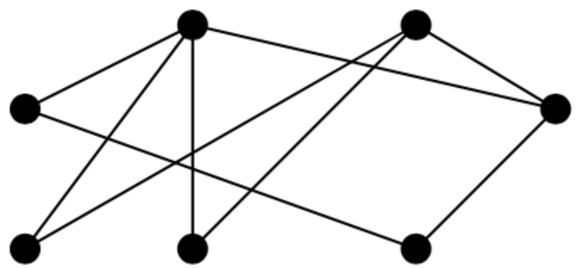
Example: Is the graph G bipartite?



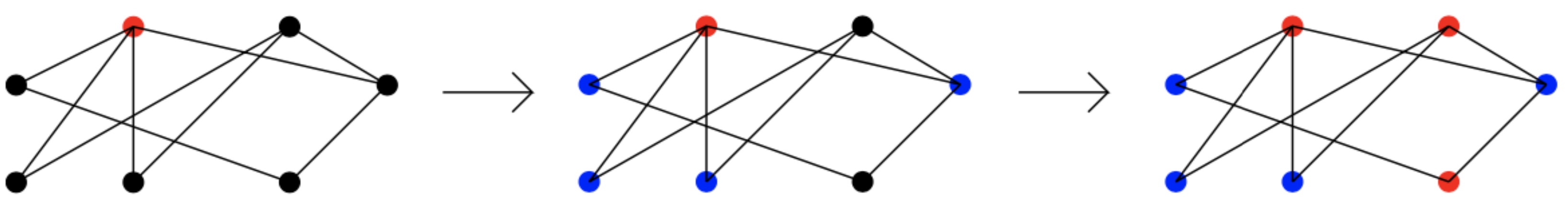
G

Bipartite Graph

Example: Is the graph G bipartite?



G



Motivation: Job Assignment

Suppose there are m employees and n different jobs to be done, with $m \geq n$.

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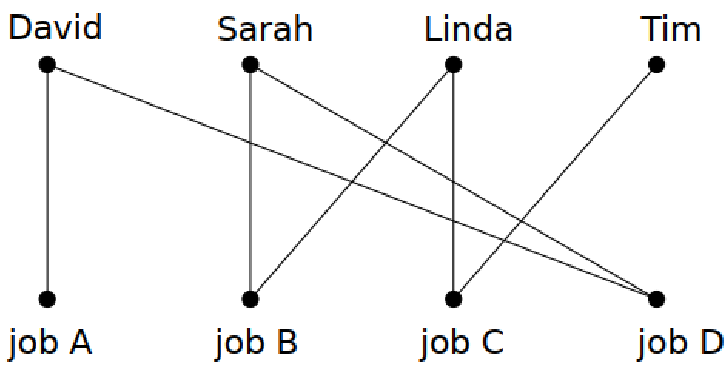

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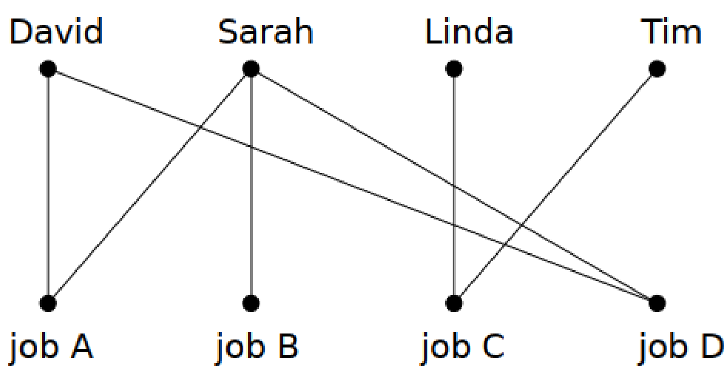

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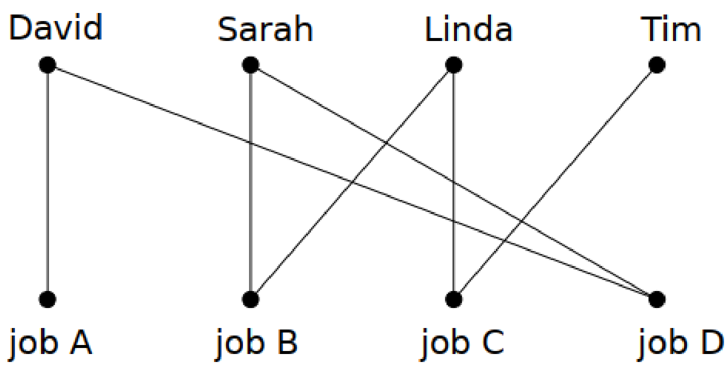

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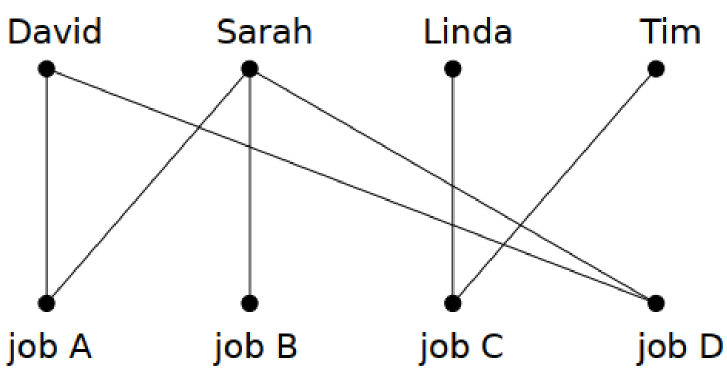
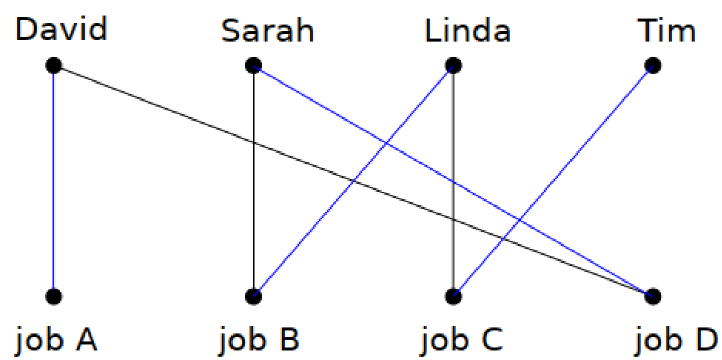


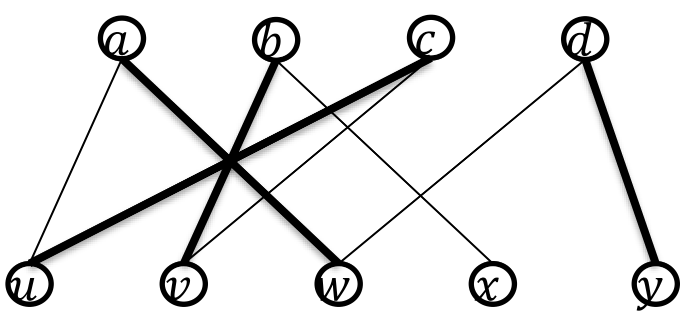


Possible solution for situation 1

Matching

DEFINITION: Let $G = (V, E)$ be a simple graph. $M \subseteq E$ is a **matching**匹配 if $e \cap e' = \emptyset$ for every $e, e' \in M$. A vertex $v \in V$ is **matched** in M if $\exists e \in M$ such that $v \in e$, otherwise, v is **not matched.**

- **maximum matching**最大匹配**:** a matching with largest number of edges.
- In a bipartite graph $G = (A \cup B, E)$, $M \subseteq E$ is a **complete matching**完全匹配 from A to B if every $u \in A$ is matched.



- $V = \{a, b, c, d, u, v, w, x, y\}$
- $V_1 = \{a, b, c, d\}$;
- $V_2 = \{u, v, w, x, y\}$
- $E = \{au, aw, bv, bx, cu, cv, dw, dy\}$

- $M = \{au, bv\}$ is a matching
  - a, b, u, v are matched in M
  - c, d, x, y are not matched in M
  - M is not a maximum matching
- $M' = \{aw, bv, cu, dy\}$ is a maximum matching
- M' is a complete matching from V_1 to V_2

Matching

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Example: Marriages. Suppose there are m men and n women on an island. Each person has a list of people of the opposite gender acceptable as a spouse $\Rightarrow$ bipartite graph.

- matching $\Leftrightarrow$ marriages
- maximum matching $\Leftrightarrow$ largest possible number of marriages
- complete matching from women to men $\Leftrightarrow$ marriages such that every women is married but possibly not all men.

Hall's Theorem

EXAMPLE: Job assignment in a company

- There are m members $X = \{x_1, \ldots, x_m\}$ and n jobs $Y = \{y_1, \ldots, y_n\}$
- $G = \left(X \cup Y, E = \left\{\{x_i, y_j\}: x_i \text{ and } y_j \text{ are compatible}\right\}\right)$
- What is the largest number of jobs that can be completed?

THEOREM (Hall 1935): A bipartitie graph $G = (X \cup Y, E)$ has a complete matching from X to Y iff $|N(A)| \geq |A|$ for any $A \subseteq X$.

- $\Rightarrow$: Let $\{\{x_1, y_1\}, \ldots, \{x_m, y_m\}\}$ be a complete matching from X to Y
  - For any $A = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq X$, $N(A) \supseteq \{y_{i_1}, \ldots, y_{i_s}\}$
    - $|N(A)| \geq s = |A|$
- $\Leftarrow$: suppose that $|N(A)| \geq |A|$ for any $A \subseteq X$. Find a complete matching M.
  - By induction on $|X|$
  - $|X| = 1$: Let $X = \{x\}$.
    - $|N(X)| \geq 1$
      - $\exists y \in Y$ such that $e = \{x, y\} \in E$.
        - $M = \{e\}$ is a complete matching from X to Y

Hall's Theorem

- **Induction hypothesis**: "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" is true when $|X| \leq k$
- Prove that "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" when $|X| = k + 1$
  - Let $X = \{x_1, \dots, x_k, x_{k+1}\}$.
  - **Case 1:** $\forall\ A \subseteq X$ with $1 \leq |A| \leq k$, $|N_G(A)| \geq |A| + 1$
    - $N_G(A)$: A's neighborhood in G
    - Say $y_{k+1} \in N_G(\{x_{k+1}\})$.
    - Let $V' = (X \setminus \{x_{k+1}\}) \cup (Y \setminus \{y_{k+1}\})$; $E' = \{e \in E: e \subseteq V' \times V'\}$
    - Let $G' = (V', E') = G - \{x_{k+1}\} - \{y_{k+1}\}$.
      - $\forall\ A \subseteq \{x_1, \dots, x_k\}, \left|N_{G'}(A)\right| \geq |N_G(A)| - |\{y_{k+1}\}| \geq |A| + 1 - 1 = |A|$
        - $\exists$ a complete matching M' from $X - \{x_{k+1}\}$ to $Y - \{y_{k+1}\}$ in G' **(IH)**
    - $M = M' \cup \left\{\{x_{k+1}, y_{k+1}\}\right\}$ is a complete matching from X to Y in G

Hall's Theorem

- **Induction hypothesis**: "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" is true when $|X| \leq k$
- Prove that "$\forall A \subseteq X, |N(A)| \geq |A| \Rightarrow \exists$ complete matching" when $|X| = k + 1$

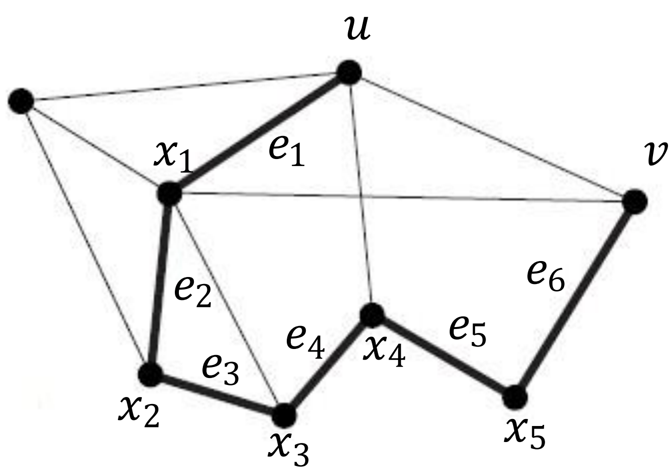
  - **Case 2:** $\exists A \subseteq X, 1 \leq |A| \leq k$ such that $|N_G(A)| = |A|$
    - Say $A = \{x_1, \dots, x_j\}$ and $N_G(A) = \{y_1, \dots, y_j\}$, where $1 \leq j \leq k$
    - Let $V' = A \cup N_G(A), E' = \{e \in E: e \subseteq V' \times V'\}$ and $G' = (V', E')$
      - $\forall A' \subseteq A,\ |N_{G'}(A')| = |N_G(A')| \geq |A'|$
      - There is a complete matching M' from A to $N_G(A)$ in G' **(IH)**
    - Let $V'' = (X \setminus A) \cup (Y \setminus N_G(A)), E'' = \{e \in E: e \subseteq V'' \times V''\}$,
    - Let $G'' = (V'', E'') = G - A - N_G(A)$
      - Then $\forall\ A'' \subseteq X \setminus A, |N_{G''}(A'')| \geq |A''|$.
        - Otherwise, $|N_G(A'' \cup A)| = |N_{G''}(A'')| + |N_G(A)| < |A''| + |A|$
          - $\exists$ a complete matching M'' from $X \setminus A$ to $Y \setminus N_G(A)$ (IH)
    - $M = M' \cup M''$ is a complete matching from X to Y

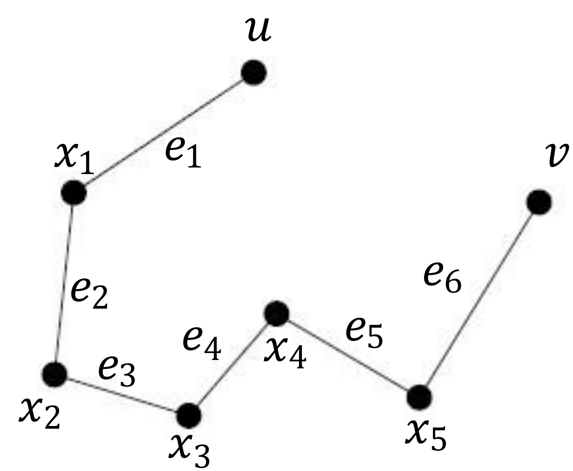
Path (Undirected)

DEFINITION: Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{N}$. A **path**路径 **of length $\boldsymbol{k}$** from u to v in G is a sequence of k edges $e_1, \dots, e_k$ of G for which there exist vertices $x_0 = u, x_1, \dots, x_{k-1}, x_k = v$ such that $e_i = \{x_{i-1}, x_i\}$ for every $i \in [k]$.

- The path is **circuit**回路 if $u = v$ and $k > 0$
- The path **passes through**经过 $x_1, \dots, x_{k-1}$
- The path **traverses**遍历 $e_1, e_2, \dots, e_k$
- The path is **simple**简单 if it doesn't contain an edge more than once.
- If G is simple, the path can be denoted as $x_0, x_1, \dots, x_k$

Example





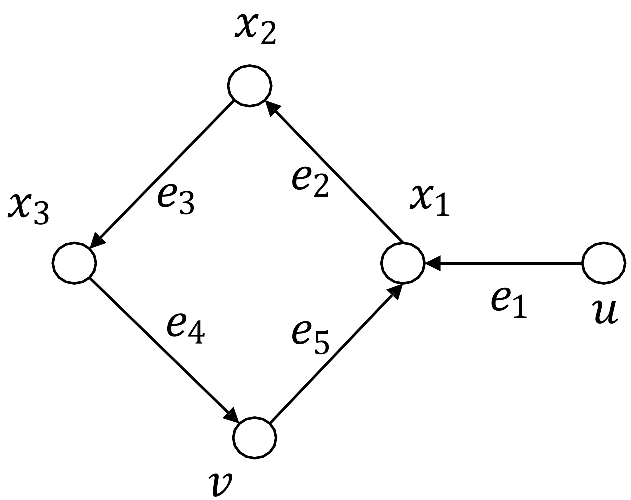
- The right-hand side graph is a path from u to v
- The path is $e_1, e_2, e_3, e_4, e_5, e_6$
- The path is simple
- The path can be denoted by $u, x_1, x_2, x_3, x_4, x_5, v$
- The path passes through x_1, x_2, x_3, x_4, x_5
- The path traverses $e_1, e_2, e_3, e_4, e_5, e_6$
- $e_1, e_2, e_3, e_4, e_5, e_6, e_7 = \{v, u\}$ is a (simple) circuit

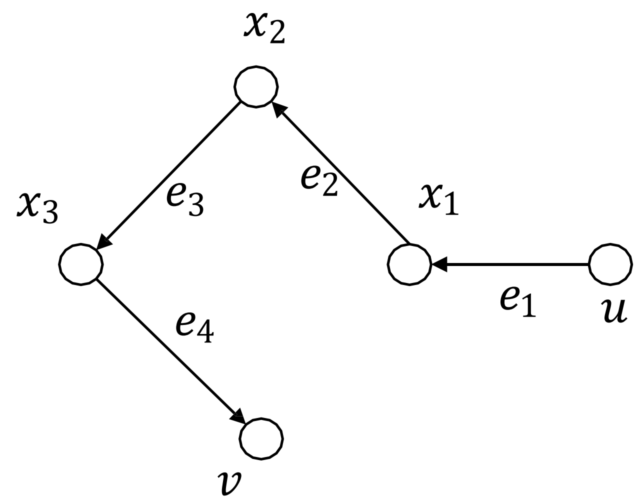
Path (Directed)

DEFINITION: Let $G = (V, E)$ be a directed graph and let $k \in \mathbb{N}$. A **path of length k** from u to v in G is a sequence of k edges $e_1, \dots, e_k$ of G for which there exist vertices $x_0 = u, x_1, \dots, x_{k-1}, x_k = v$ such that $e_i = (x_{i-1}, x_i)$ for every $i \in [k]$.

- The path is a **circuit** if $u = v$ and $k > 0$
- The path **passes through** $x_1, \dots, x_{k-1}$
- The path **traverses** $e_1, e_2, \dots, e_k$
- The path is **simple** if it doesn't contain an edge more than once.
- If G has no multiple edges, the path can be denoted as $x_0, \dots, x_k$

Example



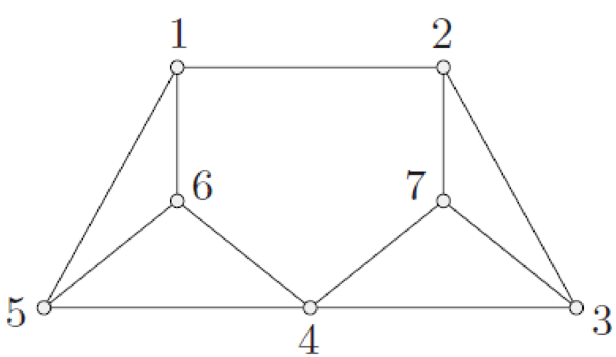


- e_1, e_2, e_3, e_4 is a path
- The path is simple
- The path can be denoted by u, x_1, x_2, x_3, v
- The path passes through x_1, x_2, x_3
- The path traverses e_1, e_2, e_3, e_4
- e_2, e_3, e_4, e_5 is a (simple) circuit

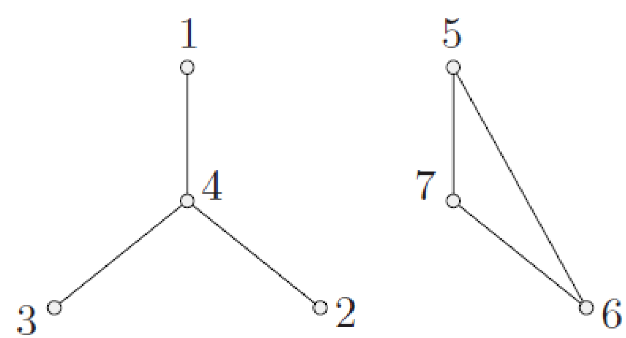
Connectivity

DEFINITION: An undirected graph G is said to be **connected**连通的 if there is a path between any pair of distinct vertices.

- Graph of order 1 is connected; the complete graph K_n is connected
- **disconnected**非连通的: not connected
- **disconnect** G**:** remove vertices or edges to produce a disconnected subgraph



A Connected Graph



A Disconnected Graph

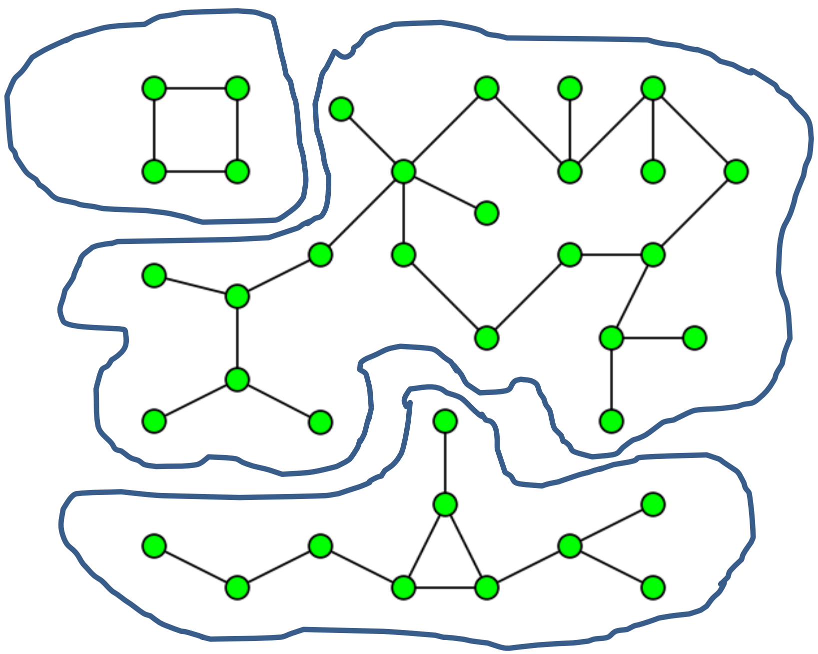
Connectivity

THEOREM: Let $G = (V, E)$ be a connected undirected graph. Then there is a simple path between any pair of distinct vertices.

- Let $u, v \in V$ and $u \neq v$. Find a simple path from u to v.
- G is connected $\Rightarrow$ there are paths from u to v.
  - Let $x_0 = u, x_1, \ldots, x_{k-1}, x_k = v$ be one that has least length k.
    - This path must be simple.
      - otherwise, the path contains some edge more than once
        - $\exists i, j \in \{0,1, \ldots, k\}$, say $i < j$, such that $x_i = x_j$
          - $x_0, x_1, \ldots, x_{i-1}, x_j, \ldots, x_k$ is a shorter path from u to v
      - The contradiction shows that the path must be simple

Connected Component

DEFINITION: A **connected component**连通分支 of a graph $G = (V, E)$ is a connected subgraph of G that is not a proper subgraph of a connected subgraph of G. //i.e., maximal极大 connected subgraph



Connected Component

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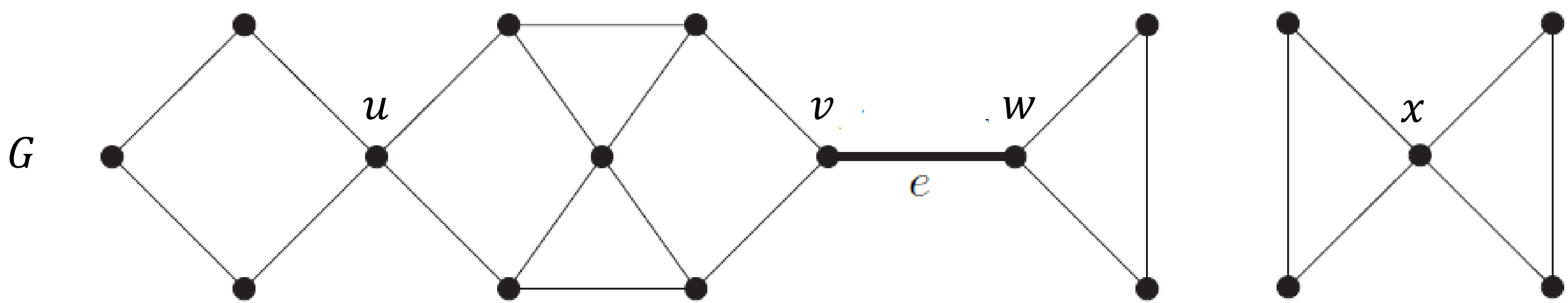
- $v \in V$ is a **cut vertex**割点 if $G - v$ has more connected components than G
- $e \in E$ is a **cut edge**割边、**bridge**桥 if $G - e$ has more connected components than G



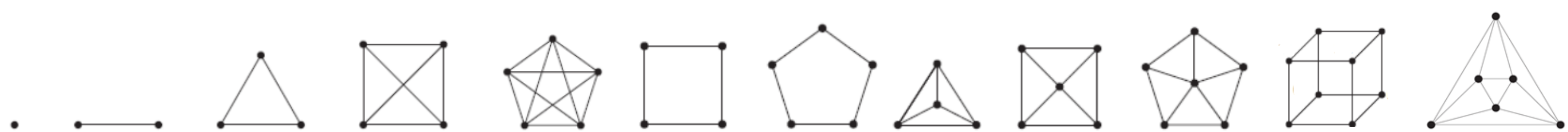
- There are 2 connected components in the graph G
- cut vertices: u, v, w, x
- cut edge: e

Vertex Connectivity

DEFINITION: A connected undirected graph $G = (V, E)$ is said to be **nonseparable**不可分的 if G has no cut vertex.

DEFINITION: Let $G = (V, E)$ be a connected simple graph.

- **vertex cut**点割集: A subset $V' \subseteq V$ such that $G - V'$ is disconnected
- **vertex connectivity**点连通度 $\kappa(G)$: the minimum number of vertices whose removal disconnect G or results in K_1; equivalently,
  - if G is disconnected, $\kappa(G) = 0$; //additional definition
  - if $G = K_n, \kappa(G) = n - 1$ //K_n has no vertex cut
  - else, $\kappa(G)$ is the minimum size of a vertex cut of G



0 1 2 3 4 2 2 3 3 3 3 4

These graphs are all nonseparable