Discrete Mathematics: Lecture 19

Part III. Mathematical Logic

logic equivalence , tautological implication, building arguments

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Spring Semester, 2025

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Interpretation

DEFINITION: an **interpretation**解释 requires one to (remove all uncertainty)

- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

EXAMPLE: $\exists x P(x) \to q$

- Domain of x={Alice, Bob, Eve}
- $P(x) =$"x gets A+"
- $q =$"I get A+"
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

Types of WFFs

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\,(P(x) \vee \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\,(P(x) \wedge \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\,(x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

<u>**Rule of Substitution**</u>: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \vee \neg p$ is a tautology; hence, $P(x) \vee \neg P(x)$ is logically valid

Logical Equivalence

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$;
- example: $\forall x\ P(x) \wedge \forall x\ Q(x) \equiv \forall x\ (P(x) \wedge Q(x))$

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

Rule of Substitution

METHOD: Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.

$P \vee Q \equiv Q \vee P$	$A(x) \vee B(y) \equiv B(y) \vee A(x)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	$\big(A(x) \wedge B(y)\big) \wedge c \equiv A(x) \wedge (B(y) \wedge c)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	$A(x) \wedge (B(y) \vee c) \equiv \big(A(x) \wedge B(y)\big) \vee (A(x) \wedge c)$
$P \wedge (P \vee Q) \equiv P$	$A(x) \wedge \big(A(x) \vee B(y)\big) \equiv A(x)$
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	$\neg\big(A(x) \wedge B(y)\big) \equiv \neg A(x) \vee \neg B(y)$
$P \rightarrow Q \equiv \neg P \vee Q$	$A(x) \rightarrow (\forall y\, B(y)) \equiv \neg A(x) \vee \big(\forall y\, B(y)\big)$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	$A(x) \leftrightarrow c \equiv (A(x) \rightarrow c) \wedge (c \rightarrow A(x))$

De Morgan's Laws for Quantifiers

THEOREM: $\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$

- Show that $\neg \forall x\ P(x) \rightarrow \exists x\ \neg P(x)$ is logically valid
  - Suppose that $\neg \forall x\ P(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\ P(x)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - $\exists x\ \neg P(x)$ is $\mathbf{T}$ in I
- Show that $\exists x\ \neg P(x) \rightarrow \neg \forall x\ P(x)$ is logically valid
  - Suppose that $\exists x \neg P(x)$ is $\mathbf{T}$ in an interpretation I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - $\forall x P(x)$ is $\mathbf{F}$ in I
  - $\neg \forall x P(x)$ is $\mathbf{T}$ in I

THEOREM: $\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$.

De Morgan's Laws for Quantifiers

EXAMPLE: $R(x)$:"x is a real number"; $Q(x)$:"x is a rational number"

- $\neg\forall x\ (R(x) \rightarrow Q(x))$
  - Not all real numbers are rational numbers
- Negation: $\exists x\ \neg\big(R(x) \rightarrow Q(x)\big) \equiv \exists x\ (R(x) \wedge \neg Q(x))$
  - There is a real number which is not rational

EXAMPLE: Let the domain be the set of all real numbers. Let $Q(x)$:"x is a rational number" and $I(x)$:"x is an irrational number"

- $\neg\exists x\ (Q(x) \wedge I\ (x))$
  - No real number is both rational and irrational.
- Negation: $\forall x\ \neg\big(Q(x) \wedge I(x)\big) \equiv \forall x\ (\neg Q(x) \vee \neg I(x))$
  - Any real number is either not rational or not irrational.

Distributive Laws for Quantifiers

THEOREM: $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- Show that $\forall x\,(P(x) \wedge Q(x)) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$ is logically valid
  - Suppose that $\forall x\,(P(x) \wedge Q(x))$ is $\mathbf{T}$ in an interpretation I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in I
- Show that $\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\,(P(x) \wedge Q(x))$ is logically valid.
  - Suppose that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\,(P(x) \wedge Q(x))$ is $\mathbf{T}$ in I

THEOREM: $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$.

Tautological Implication

DEFINITION: Let A and B be WFFs in predicate logic. A **tautologically implies** (重言蕴涵) B if every interpretation that causes A to be true causes B to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蕴涵)

THEOREM: $A \Rightarrow B$ iff $A \to B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes A to be true causes B to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \to B$ is true in every interpretation
- iff $A \to B$ is logically valid

THEOREM: $A \Rightarrow B$ iff $A \land \neg B$ is unsatisfiable.

- $A \to B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$

Rule of Substitution

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

EXAMPLE: $P \wedge (P \rightarrow Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \wedge \big(A(x) \rightarrow B(y)\big) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

Tautological Implications

a.	$\forall x P(x) \lor \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \lor Q(x))$
b.	$\exists x \big(P(x) \land Q(x)\big) \Rightarrow \exists x\, P(x) \land \exists x Q(x)$
c.	$\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$
d.	$\forall x\, \big(P(x) \to Q(x)\big) \Rightarrow \exists x\, P(x) \to \exists x\, Q(x)$
e.	$\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
f.	$\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
g.	$\forall x\, \big(P(x) \to Q(x)\big) \land \forall x\, \big(Q(x) \to R(x)\big) \Rightarrow \forall x\, (P(x) \to R(x))$
h.	$\forall x\, \big(P(x) \to Q(x)\big) \land P(a) \Rightarrow Q(a)$

Additional rule: Conclusion Premise (附加前提)

- The following two tautological implications are equivalent
  - $P \Rightarrow A \to B$
  - $P \land A \Rightarrow B$

Examples

EXAMPLE: $\underline{\forall x \left(P(x) \to Q(x)\right) \land P(a)} \Rightarrow \underline{Q(a)}$

- Suppose that the left hand side is true in an interpretation I (domain=D)
  - $\forall x \left(P(x) \to Q(x)\right)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
    - $P(a) \to Q(a)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
      - $Q(a)$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All rational numbers are real numbers $\boxed{\forall x \left(P(x) \to Q(x)\right)}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x) =$"x is a rational number"
  - $Q(x) =$"x is a real number"
  - rule of inference: $\forall x \left(P(x) \to Q(x)\right) \land P(1/3) \Rightarrow Q(1/3)$

Examples

EXAMPLE: $\underline{\forall x\left(P(x) \rightarrow Q(x)\right) \wedge \forall x\left(Q(x) \rightarrow R(x)\right)} \Rightarrow \underline{\forall x\,(P(x) \rightarrow R(x))}$

- Suppose that the left hand side is $\mathbf{T}$ in an interpretation I (domain=D)
  - $\forall x\left(P(x) \rightarrow Q(x)\right)$ is $\mathbf{T}$ and $\forall x\left(Q(x) \rightarrow R(x)\right)$ is $\mathbf{T}$
  - $P(x) \rightarrow Q(x)$ is $\mathbf{T}$ for all $x \in D$ and $Q(x) \rightarrow R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $P(x) \rightarrow R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $\forall x\,(P(x) \rightarrow R(x))$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All integers are rational numbers. $\boxed{\forall x\left(P(x) \rightarrow Q(x)\right)}$
- All rational numbers are real numbers. $\boxed{\forall x\left(Q(x) \rightarrow R(x)\right)}$
- All integers are real numbers. $\boxed{\forall x\left(P(x) \rightarrow R(x)\right)}$
  - $P(x) =$"x is an integer"
  - $Q(x) =$"x is a rational number"
  - $R(x) =$"x is a real number"
  - rule of inference: $\forall x\left(P(x) \rightarrow Q(x)\right) \wedge \forall x\left(Q(x) \rightarrow R(x)\right) \Rightarrow \forall x\,(P(x) \rightarrow R(x))$

Building Arguments

QUESTION: Given the premises $P_1, \dots, P_n$, show a conclusion Q, that is, show that $P_1 \wedge \cdots \wedge P_n \Rightarrow Q$.

Name	Operations
Premise	Introduce the given formulas $P_1, \dots, P_n$ in the process of constructing proofs.
Conclusion	Quote the intermediate formula that have been deducted.
Rule of replacement	Replace a formula with a logically equivalent formula.
Rules of Inference	Deduct a new formula with a tautological implication.
Rule of substitution	Deduct a formula from a tautology.

Rules of Inference for ∀, ∃

Name	Rules of Inference	NO.
Universal Instantiation 全称量词消去	$\forall\, x\, P(x) \Rightarrow P(a)$ a is any individual in the domain of x	1
Universal Generalization 全称量词引入	$P(a) \Rightarrow \forall x\, P(x)$ a takes any individual in the domain of x	2
Existential Instantiation 存在量词消去	$\exists\, x\, P(x) \Rightarrow P(a)$ a is a specific individual in the domain of x	3
Existential Generalization 存在量词引入	$P(a) \Rightarrow \exists x\, P(x)$ a is a specific individual in the domain of x	4

Building Arguments

EXAMPLE: Show that the following premises 1, 2 lead to conclusion 3.

1. "A student in this class has not read the book," $\exists x\big(C(x) \wedge \neg B(x)\big)$
2. "Everyone in this class passed the exam," $\forall x(C(x) \rightarrow P(x))$
3. "Someone who passed the exam has not read the book." $\exists x(P(x) \wedge \neg B(x))$

- **Translate the premises and the conclusion into formulas.**
  - $C(x)$: "x is in the class"; $B(x)$: "x has read the book"; $P(x)$: "x passed the exam"

- **?**$\exists x\big(C(x) \wedge \neg B(x)\big) \wedge \forall x(C(x) \rightarrow P(x)) \Rightarrow \exists x(P(x) \wedge \neg B(x))$

(1)	$\exists x\big(C(x) \wedge \neg B(x)\big)$	Premise
(2)	$C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3)	$C(a)$	Simplification from (2)
(4)	$\forall x(C(x) \rightarrow P(x))$	Premise
(5)	$C(a) \rightarrow P(a)$	Universal instantiation from (4)
(6)	$P(a)$	Modus ponens from (3) and (5)
(7)	$\neg B(a)$	Simplification from (2)
(8)	$P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
(9)	$\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Building Arguments

EXAMPLE: Show that the following premises lead to conclusion.

- $\exists x F(x) \to \forall y \big(G(y) \to H(y)\big),\ \exists x M(x) \to \exists y G(y)$
- $\Rightarrow \exists x (F(x) \wedge M(x)) \to \exists y H(y)$

(1)	$\exists x F(x) \to \forall y \big(G(y) \to H(y)\big)$	Premise
(2)	$\exists x M(x) \to \exists y G(y)$	Premise
(3)	$\exists x (F(x) \wedge M(x))$	Conclusion Premise
(4)	$\exists x F(x) \wedge \exists x M(x)$	Rule b from (3)
(5)	$\exists x F(x)$	Simplification from (4)
(6)	$\forall y \big(G(y) \to H(y)\big)$	Modus ponens from (1) and (5)
(7)	$\exists x M(x)$	Simplification from (4)
(8)	$\exists y G(y)$	Modus ponens from (7) and (2)
(9)	$G(\mathrm{c})$	Existential instantiation from (8)
(10)	$G(c) \to H(c)$	Universal instantiation from (6)
(11)	$H(c)$	Modus ponens from (9) and (10)
(12)	$\exists y H(y)$	Existential generalization from (11)
(13)	$\exists x (F(x) \wedge M(x)) \to \exists y H(y)$	Conclusion Premise from (3) (12)