Discrete Mathematics: Lecture 22

Part IV. Graph Theory

nonseparable, vertex connectivity, k-connected, cut edge, edge cut, edge connectivity,

Edge connectivity, Paths and Isomorphism, Counting Paths, Euler Paths and Circuits

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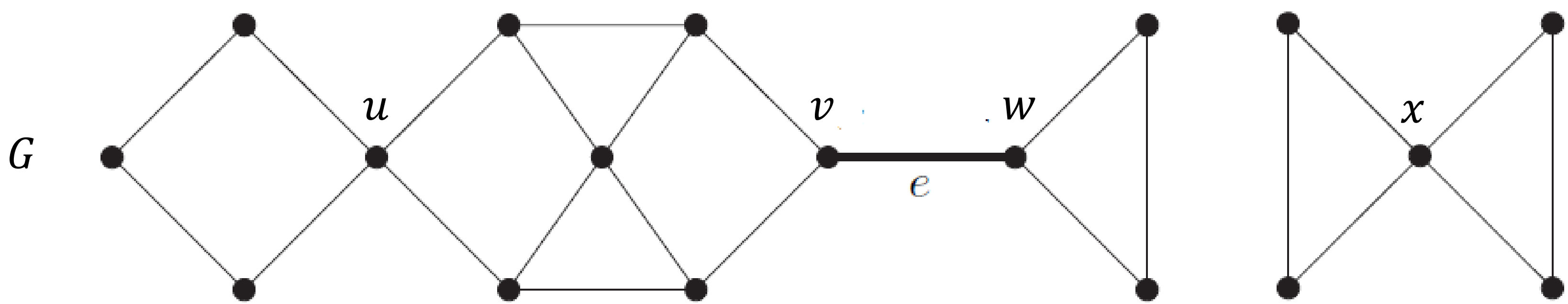
Notes by Prof. Liangfeng Zhang

Review: Connected Component

DEFINITION: A **connected component**连通分支 of a graph $G = (V, E)$ is a connected subgraph of G that is not a proper subgraph of a connected subgraph of G. //i.e., maximal极大 connected subgraph

- $v \in V$ is a **cut vertex**割点 if $G - v$ has more connected components than G
- $e \in E$ is a **cut edge**割边、**bridge**桥 if $G - e$ has more connected components than G



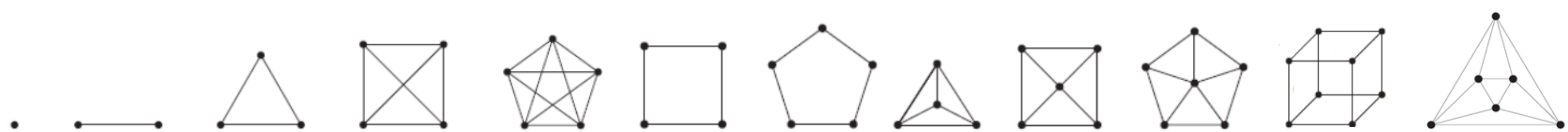
- There are 2 connected components in the graph G
- cut vertices: u, v, w, x
- cut edge: e

Vertex Connectivity

DEFINITION: A connected undirected graph $G = (V, E)$ is said to be **nonseparable**不可分的 if G has no **cut vertex.**

DEFINITION: Let $G = (V, E)$ be a connected simple graph.

- **vertex cut**点割集: A subset $V' \subseteq V$ such that $G - V'$ is disconnected
- **vertex connectivity**点连通度 $\kappa(G)$: the minimum number of vertices whose removal disconnect G or results in K_1; equivalently,
  - if G is disconnected, $\kappa(G) = 0$; //additional definition
  - if $G = K_n, \kappa(G) = n - 1$ $//K_n$ has no vertex cut
  - else, $\kappa(G)$ is the minimum size of a vertex cut of G



0 1 2 3 4 2 2 3 3 3 3 4

These graphs are all nonseparable

Vertex Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

- $0 \leq \kappa(G) \leq n - 1$
  - Removing $n - 1$ vertices gives K_1
    - $\kappa(G) \leq n - 1$
- $\kappa(G) = 0$ iff G is disconnected or $G = K_1$
  - trivial
- $\kappa(G) = n - 1$ iff $G = K_n (n \geq 2)$
  - If: obvious
  - Only if:
    - $n = 2$: $\kappa(G) = 1 \Rightarrow G = K_2$
    - $n \geq 3$: Prove by contradiction. Suppose that $G \neq K_n$.
      - There exist distinct $u, v \in V$ such that $u \neq v$ and $\{u, v\} \notin E$
        - Let $X = V - \{u, v\}$. Then $G - X$ is disconnected.
          - $\kappa(G) \leq |X| = n - 2 < n - 1.$
            - This contradicts the condition $\kappa(G) = n - 1.$

Vertex Connectivity

DEFINITION: A simple graph $G = (V, E)$ is called $\boldsymbol{k}$**-connected**k连通的 ($\boldsymbol{k}$**-vertex-connected**) k点连通的 if $\kappa(G) \geq k$.

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

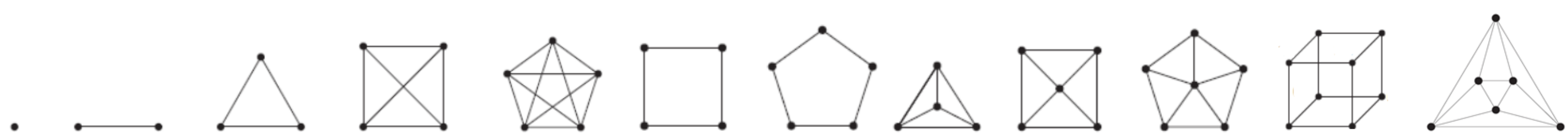
- G is 1-connected iff G is connected and $G \neq K_1$.
  - **Only if**: G disconnected or $G = K_1 \Rightarrow \kappa(G) = 0$
  - **If** : $G \neq K_1 \Rightarrow n \geq 2$; G is connected $\Rightarrow$ removing 0 vertex cannot disconnect G or give $K_1 \Rightarrow \kappa(G) \geq 1$
- G is 2-connected iff G is nonseparable and $n \geq 3$.
  - **Only if**: $n \leq 2 \Rightarrow \kappa(G) \leq 1$; G not nonseparable $\Rightarrow$ G has cut vertex $\Rightarrow \kappa(G) \leq 1$.
  - **If**: $n \geq 3 \Rightarrow$ removing ≤ 1 vertex cannot result in K_1; G nonseparable$\Rightarrow$ removing ≤ 1 vertex cannot disconnect G; Hence. $\kappa(G) \geq 2$.
- G is k-connected iff G is j-connected for all $j \in \{0,1, \dots, k\}$
  - **Only if**: $\kappa(G) \geq k \Rightarrow \kappa(G) \geq j$ for all $j \in \{0,1, \dots, k\} \Rightarrow G$ is j connected
  - **If**: G is obviously k-connected

Edge Connectivity

DEFINITION: Let $G=(V,E)$ be a connected simple graph. $E' \subseteq E$ is an **edge cut**边割集 of G if $G-E'$ is disconnected.

DEFINITION: Let $G=(V,E)$ be a simple graph.
The **edge connectivity**边连通度 $(\lambda(G))$ of G is defined as below:

- G disconnected: $\lambda(G)=0$
- G connected:
  - $|V|=1$: $\lambda(G)=0$
  - $|V|>1$: $\lambda(G)$ is the minimum size of edge cuts of G.



0 1 2 3 4 2 2 3 3 3 3 4

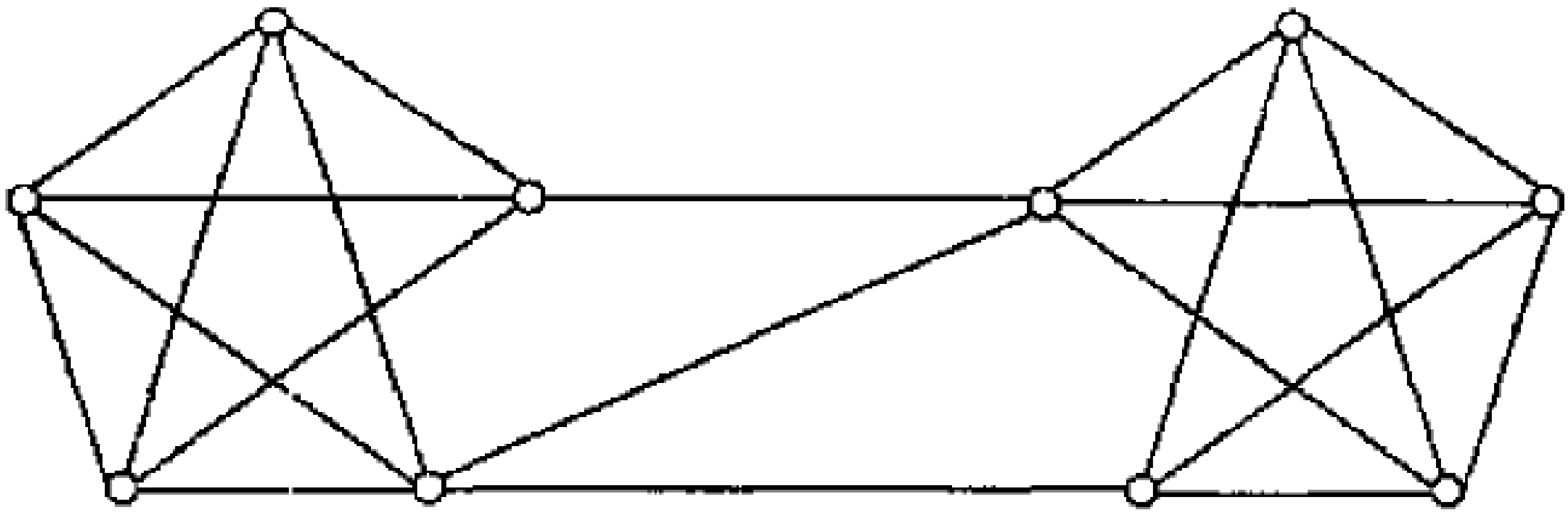
Edge Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph of order n. Then

- $0 \leq \lambda(G) \leq n - 1$
  - $n = 1$: $G = K_1$ and $\lambda(G) = 0$
  - $n > 1$: $\deg(u) \leq n - 1$ for every $u \in V$
    - By removing $\{\{u, x\}: \{u, x\} \in E\}$, we can disconnect G.
      - Hence, $\lambda(G) \leq n - 1$.
- $\lambda(G) = 0$ iff G is disconnected or $G = K_1$
  - Only if: $n > 1$ and G connected $\Rightarrow \lambda(G) \geq 1$;
  - If: definition
- $\lambda(G) = n - 1$ iff $G = K_n$ $(n \geq 2)$
  - Only if: if $G \neq K_n$, then $\deg(u) < n - 1$ for some $u \in V$.
    - Remove $\big\{\{u, x\}: \{u, x\} \in E\big\}$. Then G is disconnected. $\lambda(G) < n - 1$
  - If: $\lambda(K_n) \geq \kappa(K_n) = n - 1$. (see the next theorem)

Connectivity

THEOREM: Let $G = (V, E)$ be a simple graph. Then $\kappa(G) \le \lambda(G) \le \delta(G)$, where $\delta(G) = \min_{v \in V} \deg(v)$ is the least degree of $G's$ vertices.



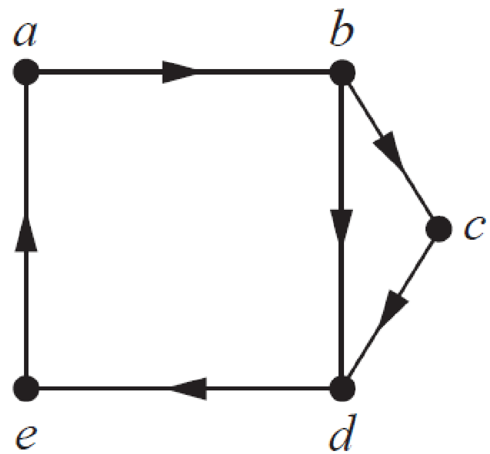
- $\kappa(G) = 2$
- $\lambda(G) = 3$
- $\delta(G) = 4$

https://cp-algorithms.com/graph/edge_vertex_connectivity.html
http://www.math.caltech.edu/~2014-15/2term/ma006b/05%20connectivity%201.pdf

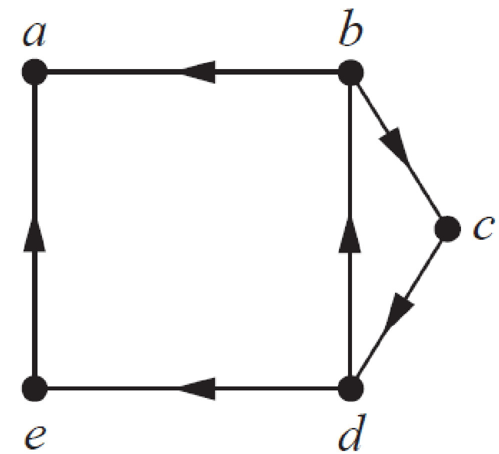
Connected Directed Graphs

DEFINITION: Let $G = (V, E)$ be a directed graph. G is said to be **strongly connected** if there is a path from u to v and a path from v to u for all $u, v \in V$ $(u \neq v)$.

- **weakly connected***:* the graph is connected if we remove the directions of all direct edges.



Strongly connected

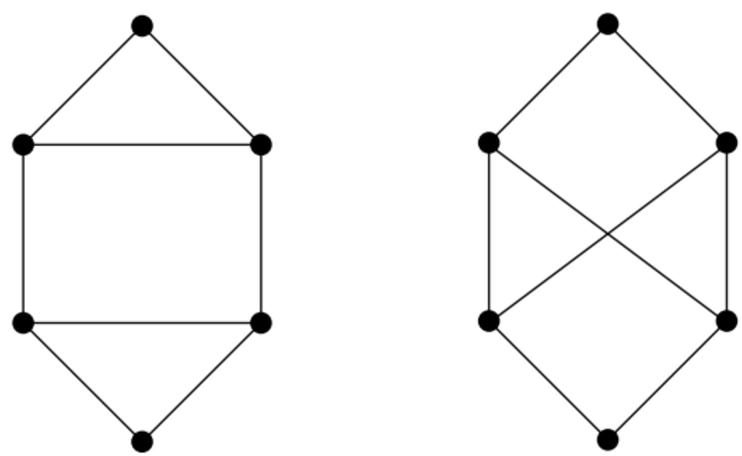


Weakly connected

Paths and Isomorphism

Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.



G_1 G_2

6 vertices, 8 edges

Degree sequence: 3, 3, 3, 3, 2, 2

Paths and Isomorphism

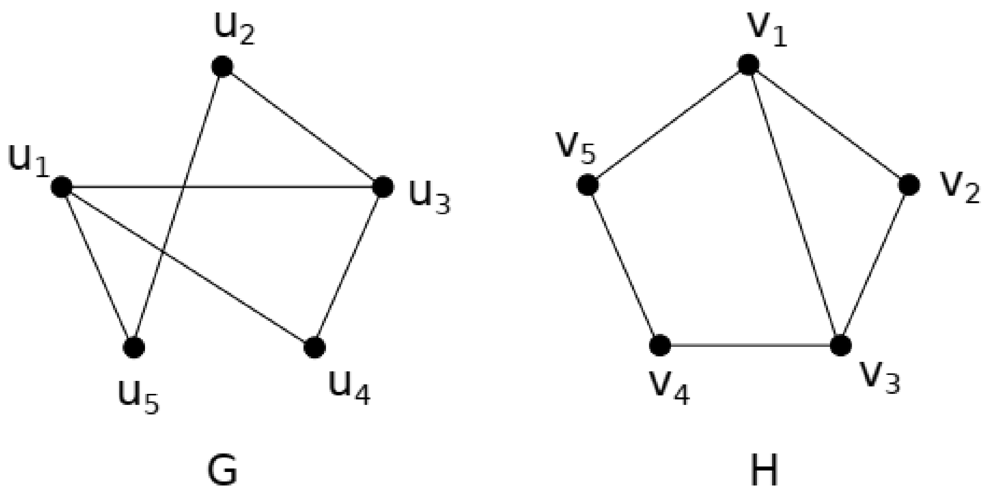
Theorem

The existence of a simple circuit of length k, $k \geq 3$ is an isomorphism invariant for simple graphs.

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs: there is a bijective function $f : V_1 \to V_2$ respecting adjacency conditions.
Assume G_1 has a simple circuit of length k: $u_0, u_1, \ldots, u_k = u_0$, with $u_i \in V_1$ for $0 \leq i \leq k$. Let's denote $v_i = f(u_i)$, for $0 \leq i \leq k$.

$$(u_i, u_{i+1}) \in E_1 \Rightarrow (f(u_i), f(u_{i+1})) = (v_i, v_{i+1}) \in E_2, \text{ for } 0 \leq i \leq k-1.$$

So $v_0, \ldots, v_k$ is a path of length k in G_2.
It is a circuit because $v_k = f(u_k) = f(u_0) = v_0$.
It is simple: if not, at least one edge is traversed more than once, so it would mean that there exist $0 \leq i \neq j \leq k-1$ such that $(v_i, v_{i+1}) = (v_j, v_{j+1})$. But this implies $(u_i, u_{i+1}) = (u_j, u_{j+1})$ by bijectivity of f. This is impossible because $u_0, u_1, \ldots, u_k$ is simple.



5 vertices, 6 edges
Degree sequence: 3, 3, 2, 2, 2
1 simple circuit of length 3,
1 simple circuit of length 4,
1 simple circuit of length 5.

Isomorphic graphs ?

If there is an iso $f : V_G \to V_H$, the simple circuit of length 5 u_1, u_4, u_3, u_2, u_5 must be sent to the simple circuit of length 5 in H, respecting the degrees of vertices.
Check that $f(u_1) = v_1, f(u_4) = v_2, f(u_3) = v_3, f(u_2) = v_4, f(u_5) = v_5$ is an isomorphism by writing adjacency matrices.

Counting Paths Between Vertices

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering of vertices $v_1, \ldots, v_n$. The number of different paths of length $r \geq 1$ from v_i to v_j equals the (i,j) entry of the matrix A^r.

Proof: By induction

- $r = 1$: the number of paths of length 1 from v_i to v_j is equal to the (i,j) entry of A by definition of A, as it corresponds to the number of edges from v_i to v_j.

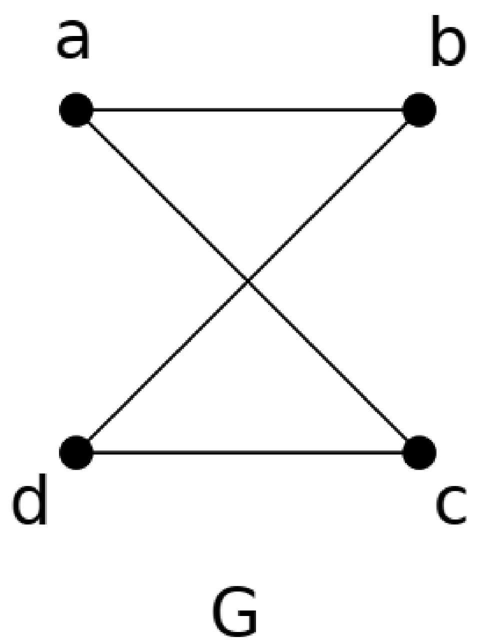
• Assume the (i, j) entry of the matrix A^r is the number of different paths of length r from v_i to v_j.
We can write $A^{r+1} = A^r A$
Let's denote $A^r = (b_{ij})_{1 \le i,j \le n}$, and $A = (a_{ij})_{1 \le i,j \le n}$. The (i, j) entry of A^{r+1} is given by:

$$\sum_{k=1}^{n} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + \cdots + b_{in} a_{nj} \tag{1}$$

By hypothesis: b_{ik} equals the number of paths of length r from v_i to v_k.
"Path of length $r+1$ from v_i to v_j = path of length r from v_i to any vertex v_k + an edge from v_k to v_j."
This is equal to the sum (1).

Example

How many paths of length four are there from a to d in the simple graph G



with ordering of vertices (a, b, c, d, e):

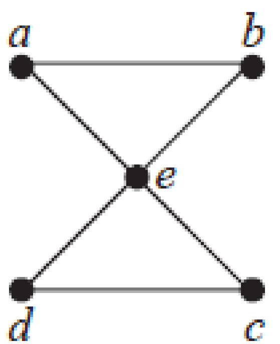
$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

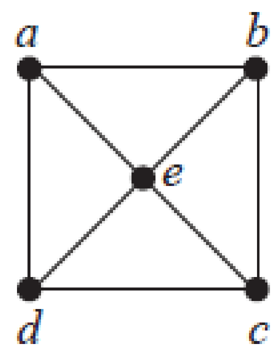
$$A_G^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \quad A_G^3 = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{pmatrix} \quad A_G^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

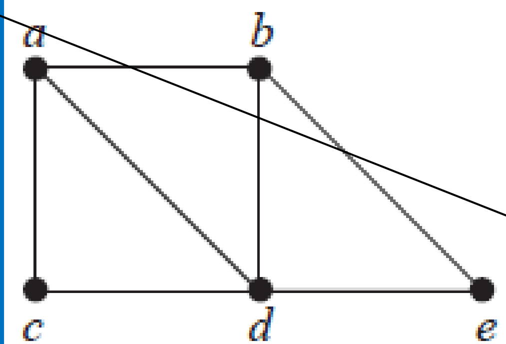
Euler Paths and Circuits

DEFINITION: Let $G = (V, E)$ be a graph.

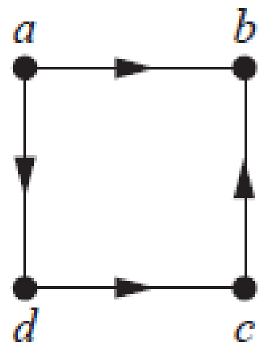
- **Euler Path**欧拉路径: a **simple path** that traverses every edge of G.
- **Euler Circuit**欧拉回路: a **simple circuit** that traverses every edge of G.

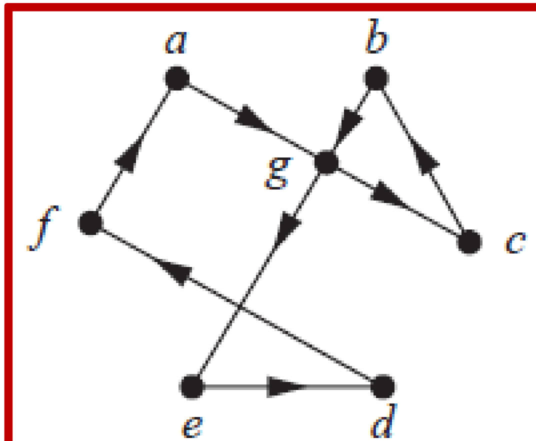


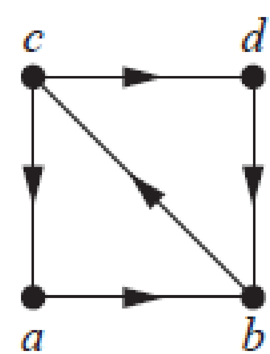




Remark: When G has multiple edges, these edges will be given different names and considered as different. This is implicit in the textbook.







Euler Circuits

THEOREM: Let $G = (V, E)$ be a **connected multigraph** of order ≥ 2.

Then G has an Euler circuit iff $2|\deg(x)$ for every $x \in V$.

- $\Rightarrow$: Let P: $\{x_0, x_1\}, \ldots, \{x_{i-1}, x_i\}, \ldots, \{x_{n-1}, x_n\}$ be an Euler circuit, $x_0 = x_n$
  - Every occurrence of x_i in P contributes 2 to $\deg(x_i)$
    - Every vertex x_i has an even degree
- $\Leftarrow$: Let P: $\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}$ be a longest simple path in G.
  - Let $H = G[P]$, the subgraph of G induced by all edges in P
    - If $x_n \neq x_0$, then $\deg_H(x_n)$ is odd and so P cannot be longest.
      - $x_n = x_0$, P is a simple circuit, and $2|\deg_H(x_i)$ for all i.
    - If $\exists i \in \{0,1, \ldots, n-1\}$ such that $\deg_H(x_i) < \deg_G(x_i)$,
      - then $\exists y \in V$ such that $\{x_i, y\} \notin P$
        - $y, x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1}, x_i$ is longer than P
    - Hence, $\deg_H(x_i) = \deg_G(x_i)$ for all $i \in \{0,1, \ldots, n-1\}$.
      - $V = \{x_0, x_1, \ldots, x_{n-1}\}$ and $H = G$.
        - P is an Euler circuit.

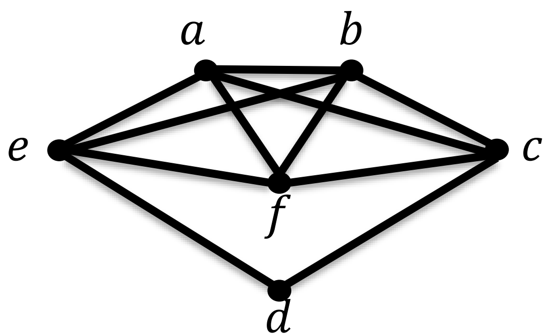
Remark: H contains all vertices of G. Otherwise, P can be extended.

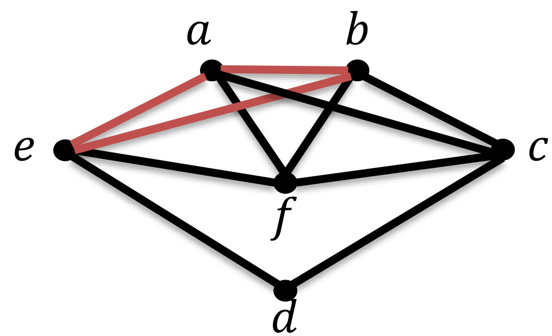
Construction

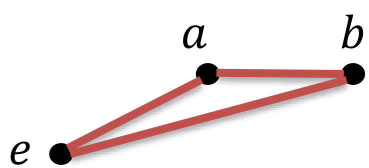
ALGORITHM (Hierholzer):

- **Input**: $G = (V, E)$, a connected multigraph, $2 | \deg(x)$, $\forall x \in V$
- **Output**: an Euler circuit
  - $\mathbf{circuit} := $ a circuit in G
  - $H := G - \mathbf{circuit} - $ isolated vertices
  - while H has edges do
    - $\mathbf{subcircuit} := $ a circuit in H that intersects $\mathbf{circuit}$
    - $H := H - \mathbf{subcircuit} - $ isolated vertices
    - $\mathbf{circuit} := \mathbf{circuit} \cup \mathbf{subcircuit}$
  - return $\mathbf{circuit}$

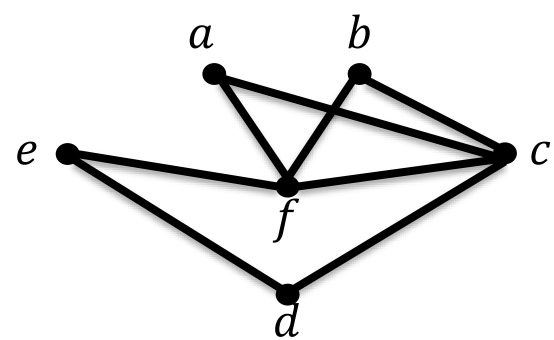
Example





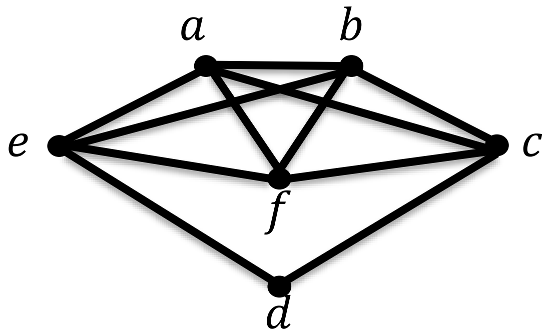


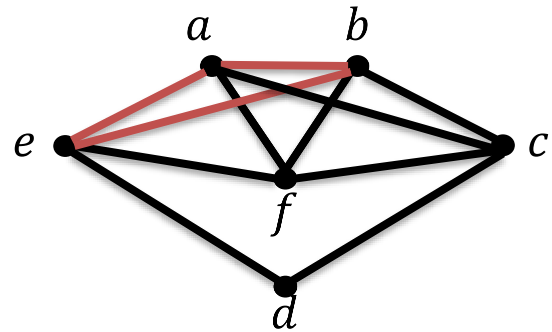
circuit $= a, b, e, a$

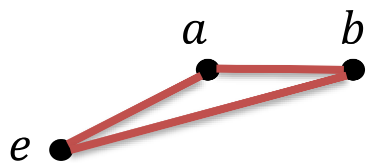


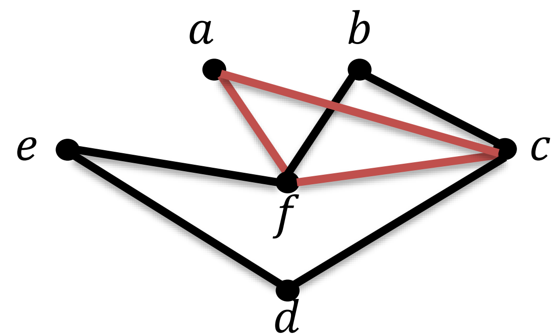
H

Example







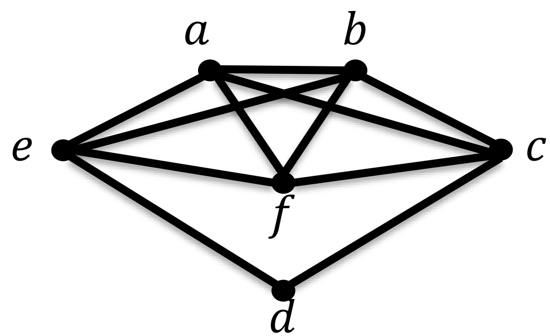


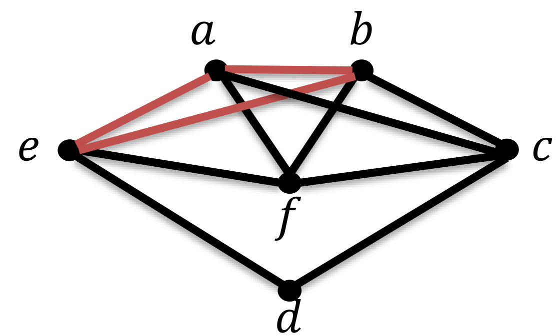
circuit $= a, b, e, a$

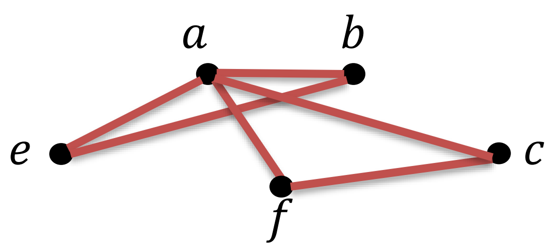
H

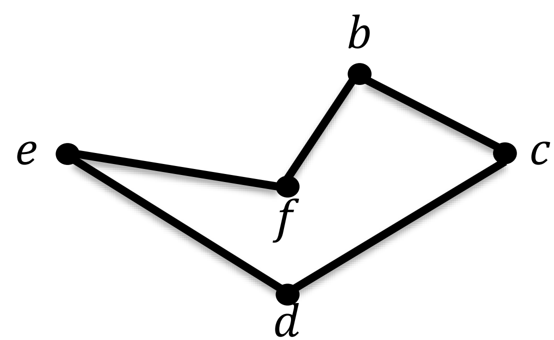
subcircuit $= a, c, f, a$

Example







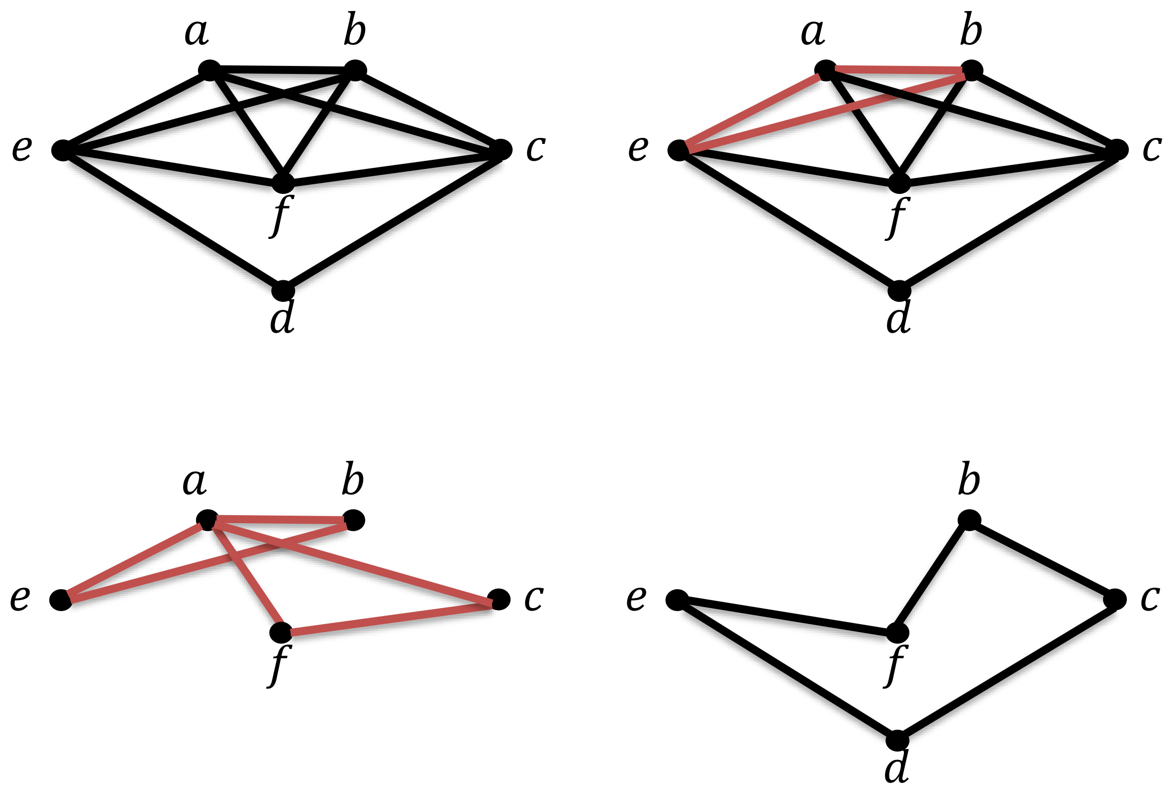


circuit $= a, b, e, a$

H

subcircuit $= a, c, f, a$

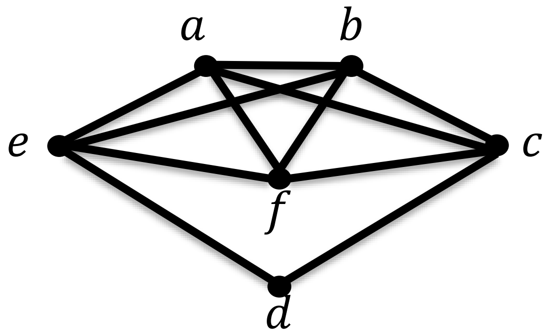
Example

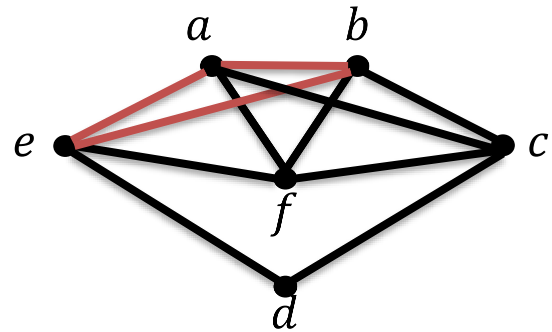


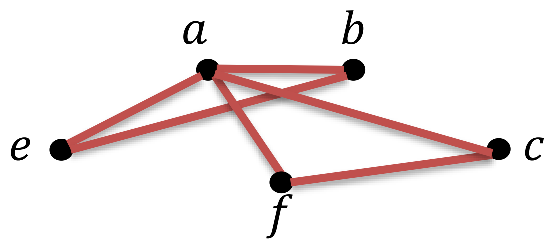
circuit $= a, b, e, a, c, f, a$

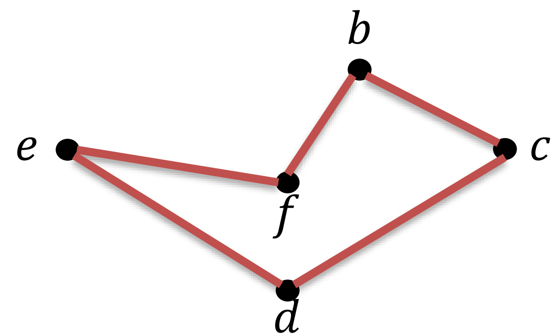
H

Example







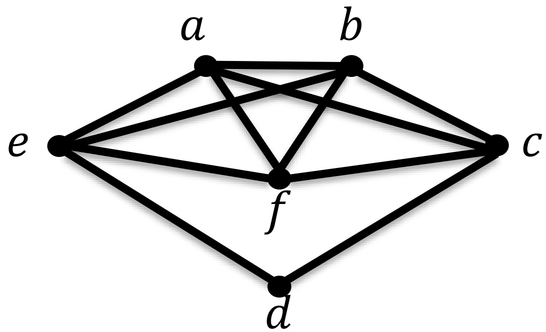


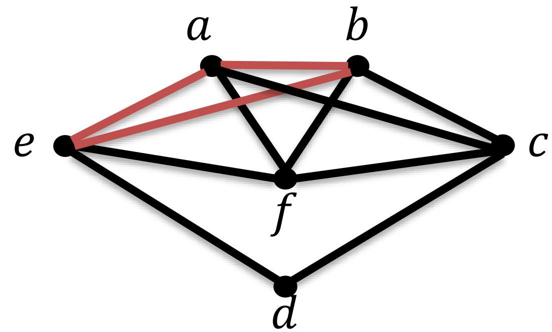
circuit $= a, b, e, a, c, f, a$

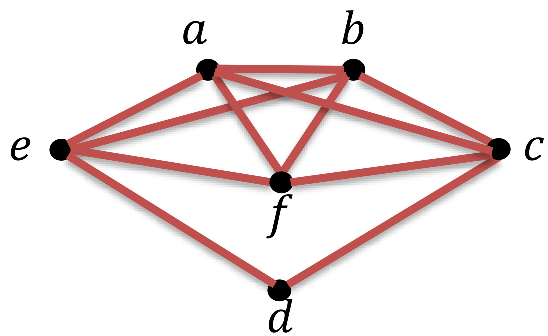
H

subcircuit $= c, d, e, f, b, c$

Example





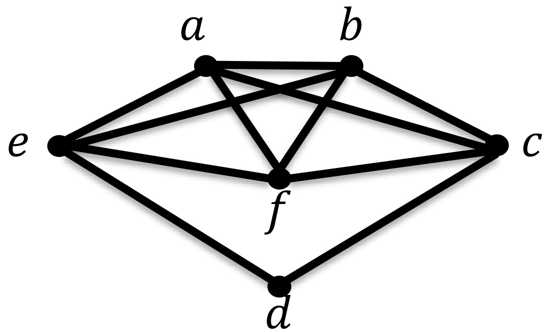


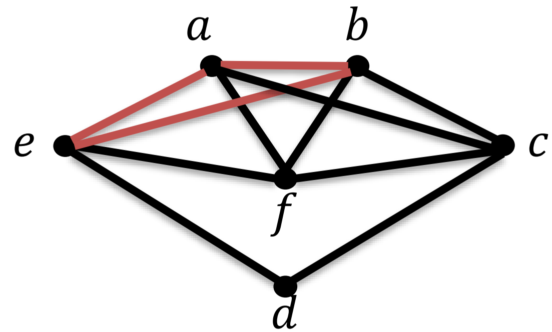
circuit $= a, b, e, a, c, f, a$

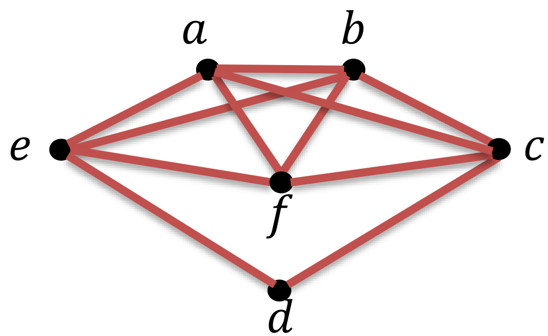
H

subcircuit $= c, d, e, f, b, c$

Example







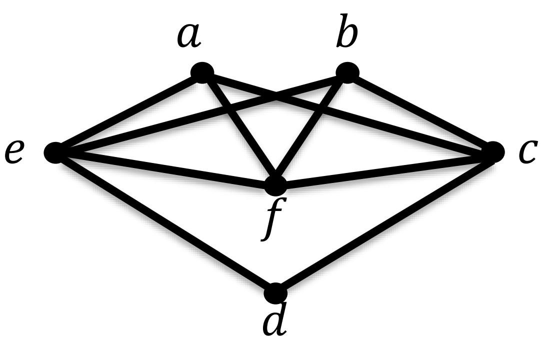
circuit $= a, b, e, a, c, d, e, f, b, c, f, a$

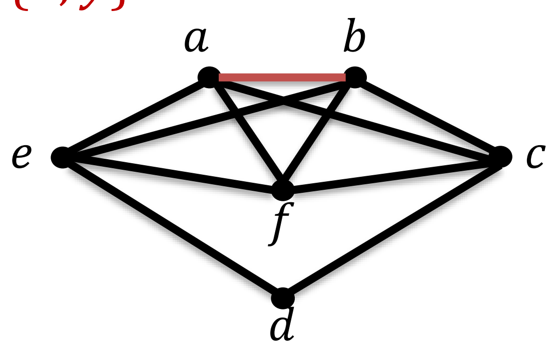
Euler Paths

THEOREM: Let $G = (V, E)$ be a connected multigraph of order ≥ 2. Then G has an Euler path (not Euler circuit) iff G has exactly 2 vertices of odd degree.

ALGORITHM:

- **Input**: $G = (V, E)$, a connected multigraph, $x, y \in V$ have odd degrees
- **Output**: an Euler path
    - $H \coloneqq G + \{x, y\}$
    - $\text{find an Euler circuit using Hierholzer's algorithm}$
    - $\text{remove the edge } \{x, y\} \text{ from the circuit}$





$a, c, d, e, f, b, a, e, b, c, f, a$

$a, c, d, e, f, b, a, e, b, c, f, a$

$a, e, b, c, f, a, c, d, e, f, b$