Discrete Mathematics: Lecture 23

Part IV. Graph Theory

Hamilton Paths and Circuits, Shortest Paths and Djikstra’s Algorithm, Traveling Salesperson Problem

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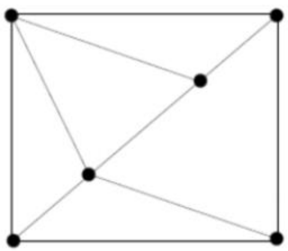
Notes by Prof. Liangfeng Zhang

Q1 Which of the following graphs is biparatite but not complete bipartite?

A

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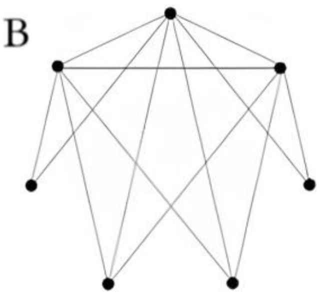
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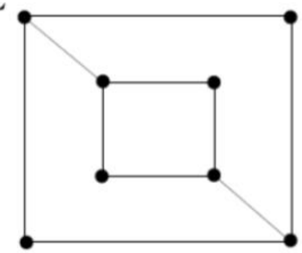
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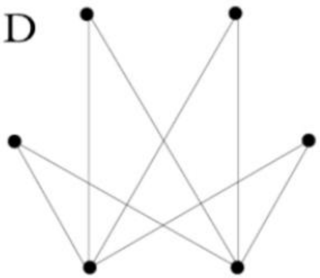
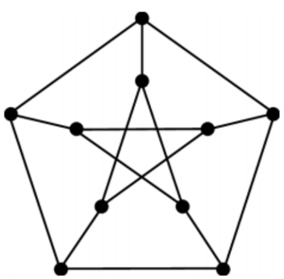


Q2

What is the vertex connectivity of the graph pictured below?



A. 1

B. 2

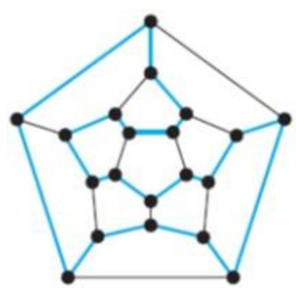
C. 3

D. 4

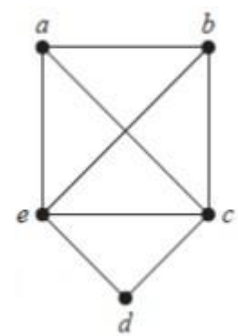
Hamilton Paths and Circuits

DEFINITION: Let $G = (V, E)$ be a graph.

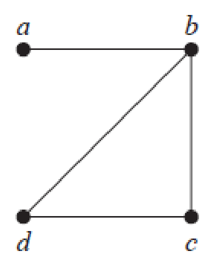
- **Hamilton Path**: A **simple path** that passes through every vertex exactly once.
- **Hamilton Circuit:** A **simple circuit** that passes through every vertex exactly once.



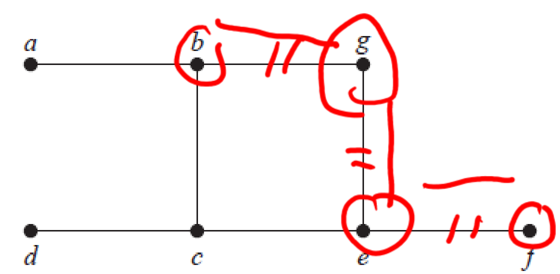
√ Hamilton path
√ Hamilton circuit



√ Hamilton path
√ Hamilton circuit



√ Hamilton path
×Hamilton circuit



× Hamilton path
× Hamilton circuit

Hamilton Circuits

Determine if there is a Hamilton circuit in a given graph G?

- This problem is NP-Complete. //that means very difficult

Necessary conditions on Hamilton circuit.

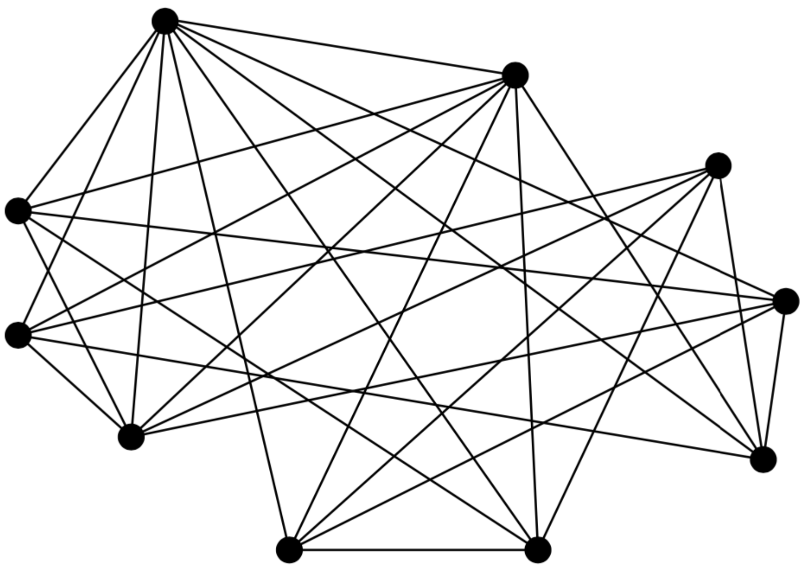
- If G has a vertex of degree 1, then G cannot have a Hamilton circuit.
- If G has a vertex of degree 2, then a Hamilton circuit of G traverses both edges.

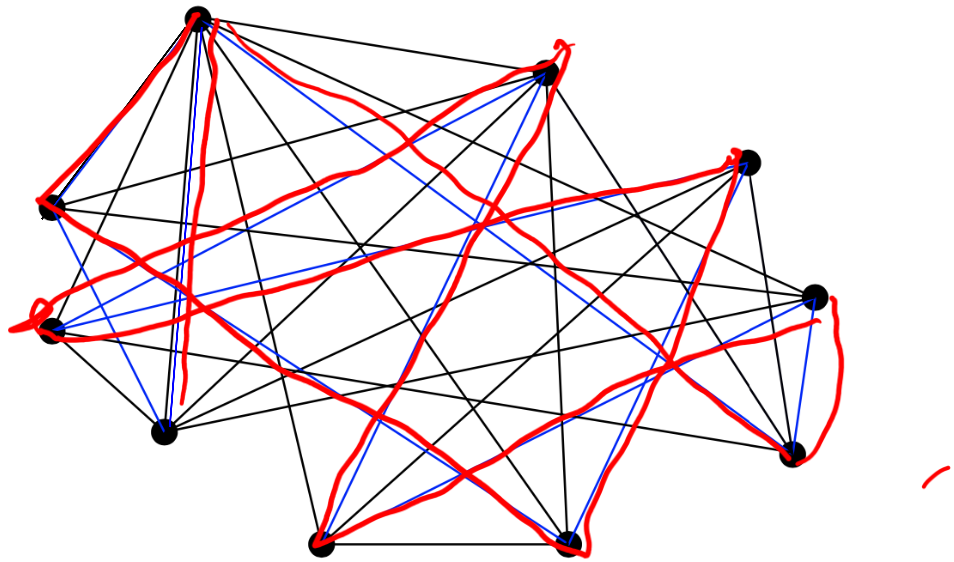
Sufficient conditions on Hamilton circuit.

- **Ore's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for all $\{u, v\} \notin E$, then G has a Hamilton circuit.
- **Dirac's Theorem:** Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\deg(u) \geq n/2$ for every $u \in V$, then G has a Hamilton circuit.
  - This is a corollary of Ore's Theorem
    - $\forall u \in V,\ \deg(u) \geq n/2 \Rightarrow \forall u, v \in V, \deg(u) + \deg(v) \geq n$

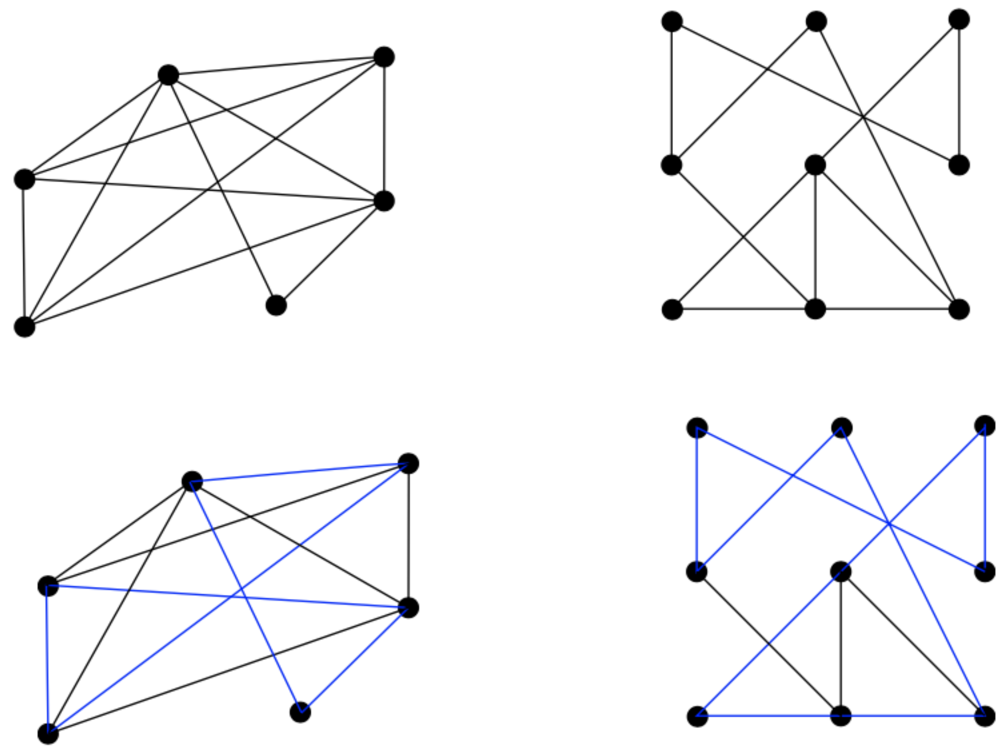
Hamilton Circuits

- Examples (sufficient condition)

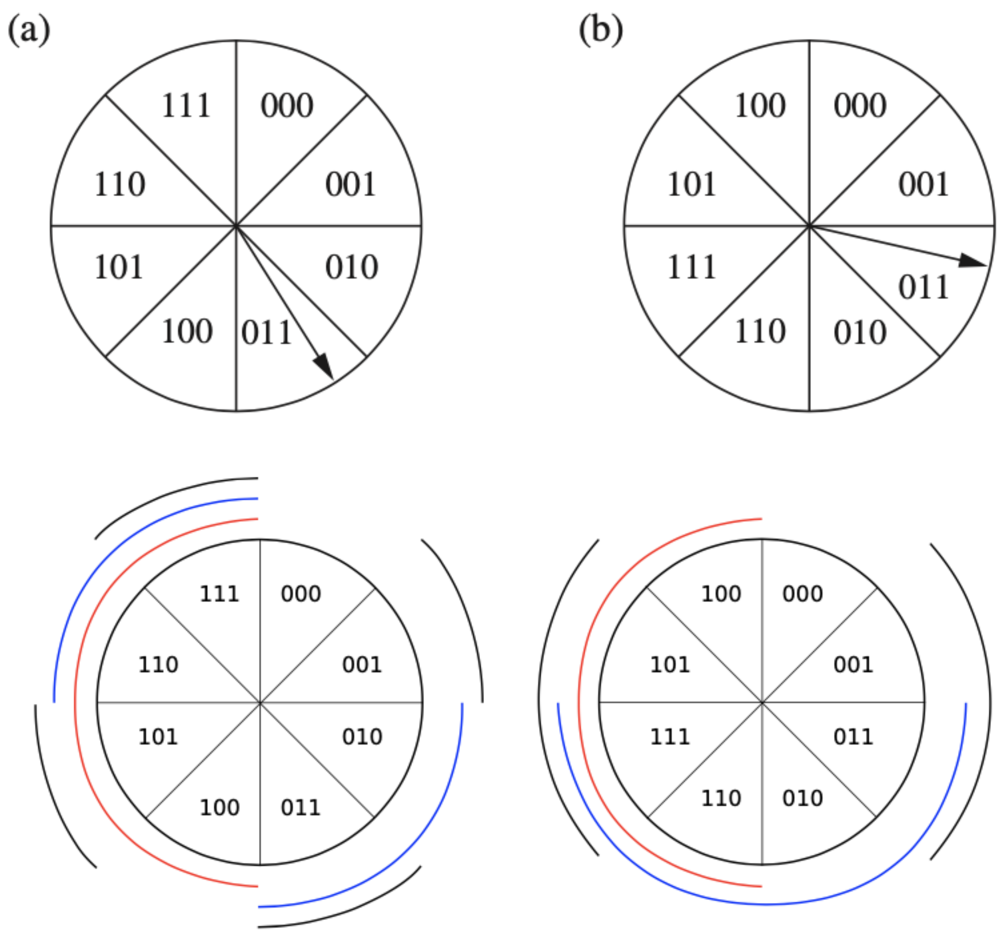




Hamilton Circuits



Remark: Dirac's and Ore's Theorems do not give a necessary condition for the existence of a Hamilton circuit!



Position of a rotating pointer encoded by a bit string of length n

Gray code: Labeling of the arcs of the circle such that adjacent arcs are labeled with bit string that differ exactly in one bit.

$\Rightarrow$ Hamilton cycle in $\mathcal{Q}_n$.

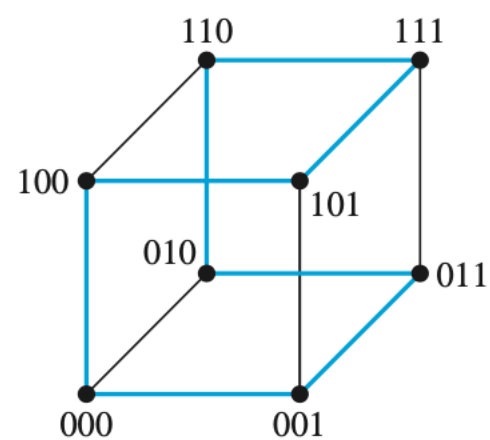
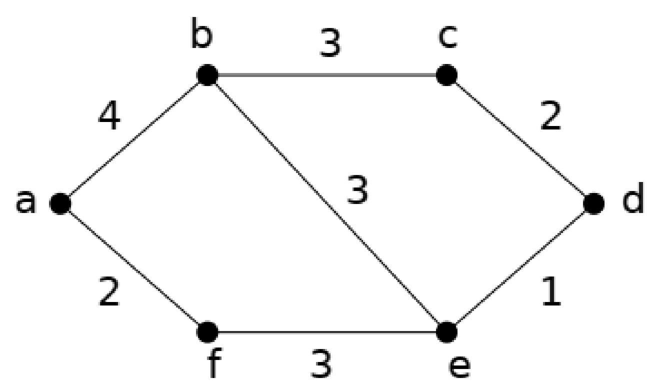


FIGURE 14 A Hamilton Circuit for Q_3.

Shortest Path Problem

Definition

A **weighted graph** is a graph $G = (V, E)$ such that each edge is assigned with a strictly positive number.
The **length** of a path in weighted graph is the sum of the weights of the edges of this path.

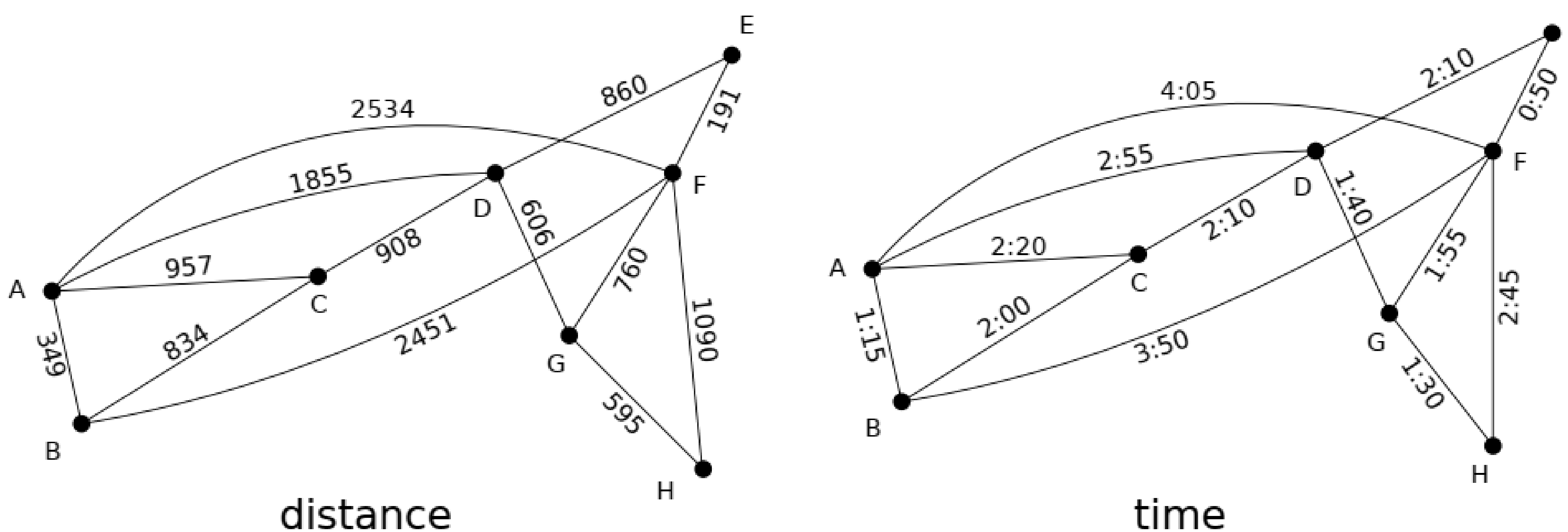


a, b, c is a path of length 7 and b, e, d, c is a path of length 6

Remark: Observe that in a non-weighted graph the length of a path is the number of edges in the path!

Shortest Path Problem

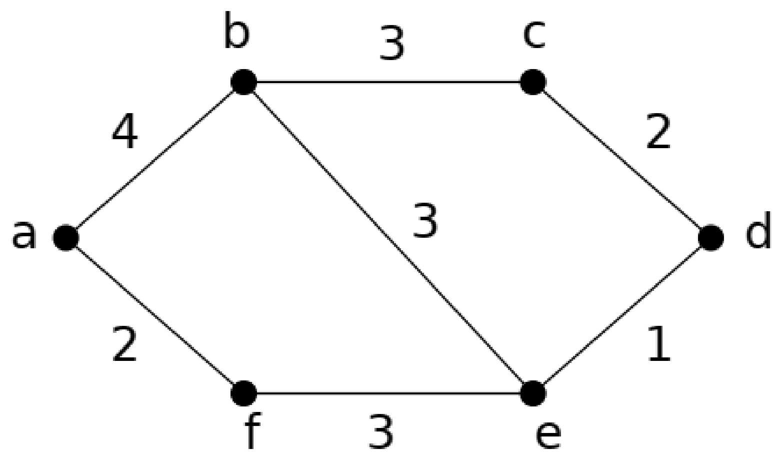
Examples



What is the shortest path in air distance between cities A and E?
What combination of flights has the smallest total flight time?

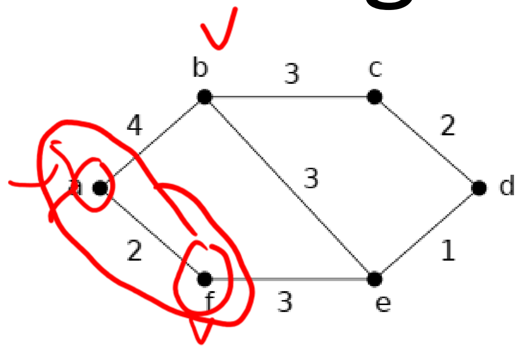
Shortest Path Problem

Question: Find the shortest path from a to d.



Method: Find the closest vertex to a, then the second closest, the third closest... until we reach d.
$\Rightarrow$ Dijkstra's algorithm

Djikstra’s Algorithm



1. Find the closest vertex to $a \rightsquigarrow$ analyse all the edges starting from a:
   a, b of length 4
   a, f of length 2
   $\Rightarrow$ f is the closest vertex to a. The shortest path from a to f has length 2.

2. Find the second closest vertex to $a \rightsquigarrow$ shortest paths from a to a vertex in $\{a, f\}$ followed by an edge from a vertex in $\{a, f\}$ to a vertex not in this set:
   a, b of length 4
   a, f, e of length 5
   $\Rightarrow$ b is the second closest vertex to a. The shortest path from a to b has length 4.

Djikstra's Algorithm

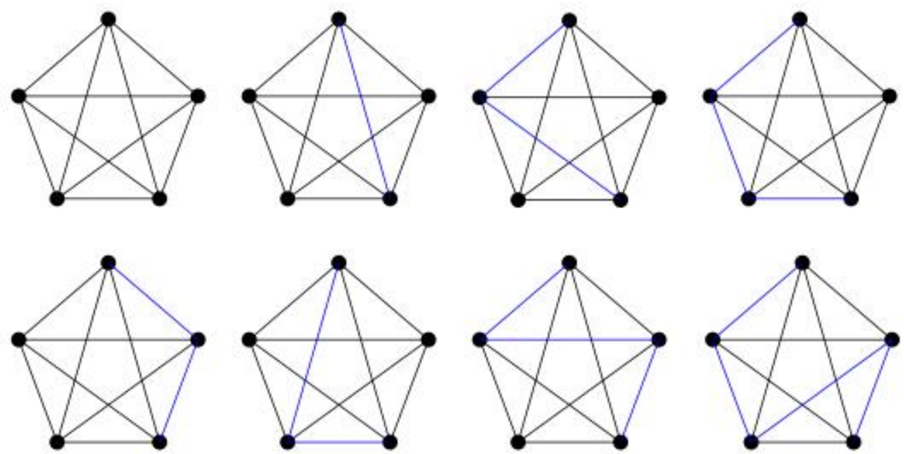
3. Find the third closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b\}$ followed by an edge from a vertex in $\{a, f, b\}$ to a vertex not in this set:
   a, b, c of length 7
   a, b, e of length 7
   a, f, e of length 5
   $\Rightarrow$ e is the third closest vertex to a. The shortest path from a to e has length 5.

4. Find the fourth closest vertex to $a \rightsquigarrow$ shortest path from a to a vertex in $\{a, f, b, e\}$ followed by an edge from a vertex in $\{a, f, b, e\}$ to a vertex not in this set:
   a, b, c of length 7
   a, f, e, d of length 6
   $\Rightarrow$ d is the fourth closest vertex to a. The shortest path from a to d has length 6.

Shortest Path Problem

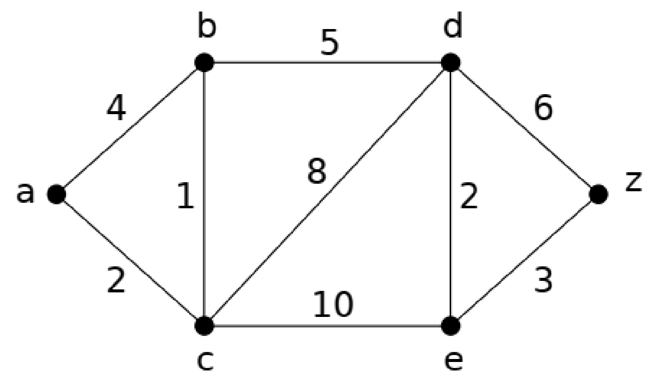
Remarks:

- Of course in the example above, we could have looked at all the paths between a and d and compute their length, but too complicated if the graph has a lot of edges.



- Advantage of Dijkstra's algorithm: we can compute the length of a shortest path from one vertex to all other vertices of the graph.

Djikstra's Algorithm



Goal: find the length of a shortest path from *a* to *z* with a series of iterations.

- A distinguished set of vertices is constructed by adding one vertex at each iteration.
- A labeling procedure is carried out at each iteration: a vertex *w* is labeled with the length of a shortest path from *a* to *w* that contains only vertices in the distinguished set.
- The vertex added to the distinguished set is one with minimal label among those vertices not already in the set.

Djikstra's Algorithm

Notations: $S_k :=$ distinguished set after k iterations, $L_k(v) :=$ length of a shortest path from a to v containing only vertices in S_k ("label" of v).

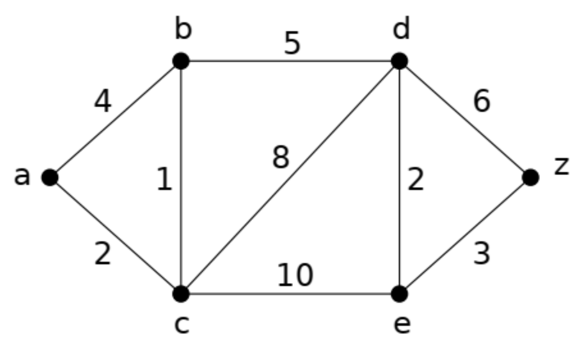
Initialization: $L_0(a) = 0$,
$L_0(v) = \infty$ for every vertex $v \neq a$,
$S_0 = \emptyset$.

***k*th iteration:**

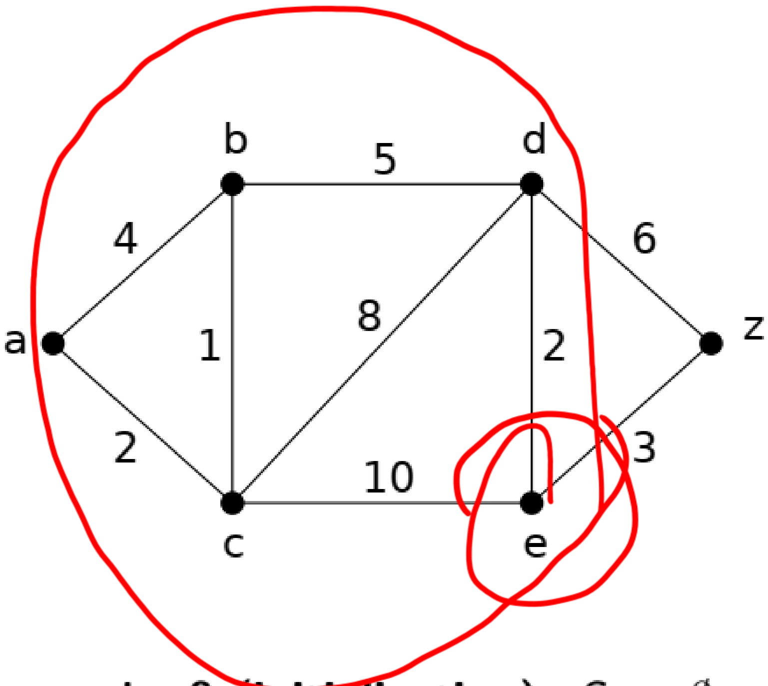
• S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with smallest label,

• Update the labels of all vertices not in S_k so that $L_k(v)$ is the length of a shortest path from a to v containing only vertices in S_k, i.e.

$L_k(v) = min\{L_{k-1}(v), L_{k-1}(u) + w(u,v)\}$ (with $w(u,v)$ length of the edge (u,v))



Djikstra's Algorithm



- **k=0 (initialization):** $S_0 = \emptyset$, $L_0(a) = 0$, $L_0(b) = L_0(c) = L_0(d) = L_0(e) = L_0(z) = \infty$

2

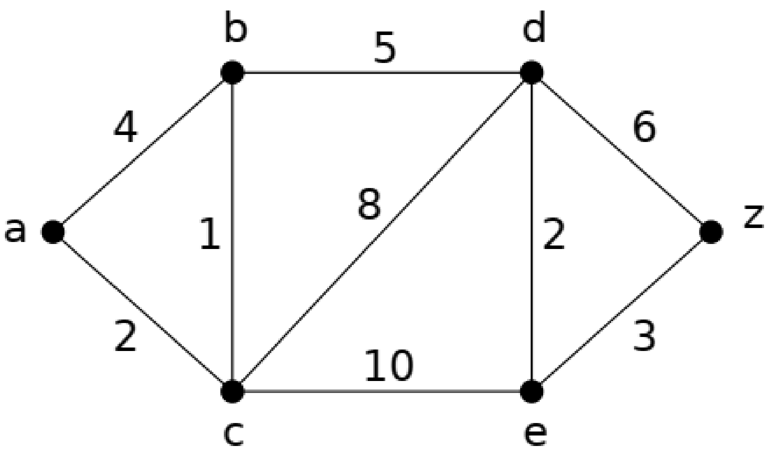
3

4

5

6 $\{a, c, b, d, e, z\}$

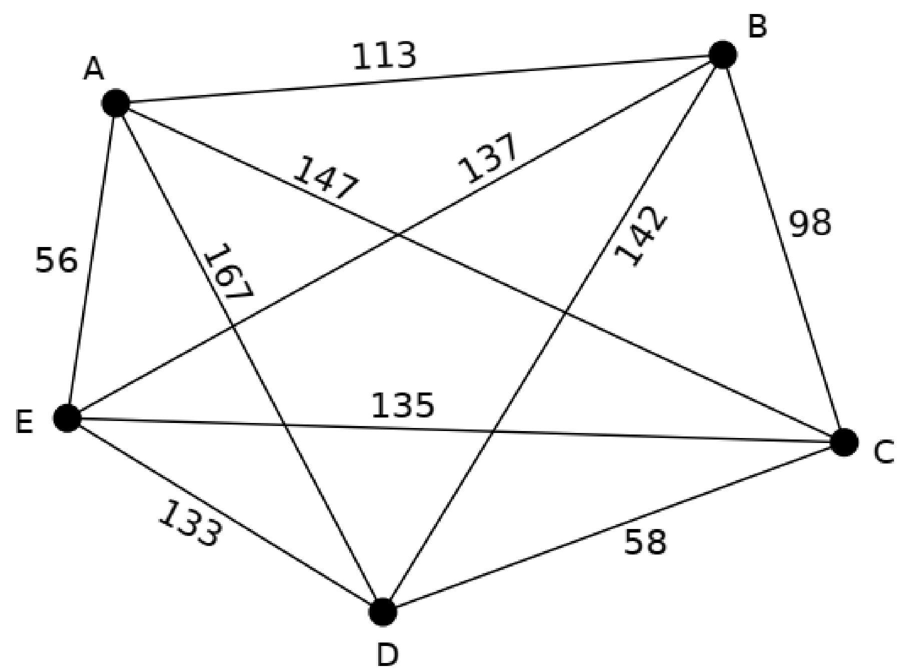
Djikstra's Algorithm



- **k=0 (initialization):** $S_0 = \emptyset$, $L_0(a) = 0,\ L_0(b) = L_0(c) = L_0(d) = L_0(e) = L_0(z) = \infty$

- **k=1:** $u := a \rightsquigarrow S_1 = \{a\}$,
  $L_0(a) + w(a,b) = 4 < L_0(b) \rightsquigarrow L_1(b) = 4$
  $L_0(a) + w(a,c) = 2 < L_0(c) \rightsquigarrow L_1(c) = 2$
- **k=2:** $u := c \rightsquigarrow S_2 = \{a, c\}$,
  $L_1(c) + w(c,b) = 3 < L_1(b) \rightsquigarrow L_2(b) = 3$
  $L_1(c) + w(c,d) = 10 < L_1(d) \rightsquigarrow L_2(d) = 10$
  $L_1(c) + w(c,e) = 12 < L_1(e) \rightsquigarrow L_2(e) = 12$
- **k=3:** $u := b \rightsquigarrow S_3 = \{a, c, b\}$,
  $L_2(b) + w(b,d) = 8 < L_2(d) \rightsquigarrow L_3(d) = 8$
- **k=4:** $u := d \rightsquigarrow S_4 = \{a, c, b, d\}$,
  $L_3(d) + w(d,e) = 10 < L_3(e) \rightsquigarrow L_4(e) = 10$
  $L_3(d) + w(d,z) = 14 < L_3(z) \rightsquigarrow L_4(z) = 14$
- **k=5:** $u := e \rightsquigarrow S_5 = \{a, c, b, d, e\}$,
  $L_4(e) + w(e,z) = 13 < L_4(z) \rightsquigarrow L_5(z) = 13$
- **k=6:** $u := z \rightsquigarrow S_6 = \{a, c, b, d, e, z\}$
- **return:** $L(z) = 13$

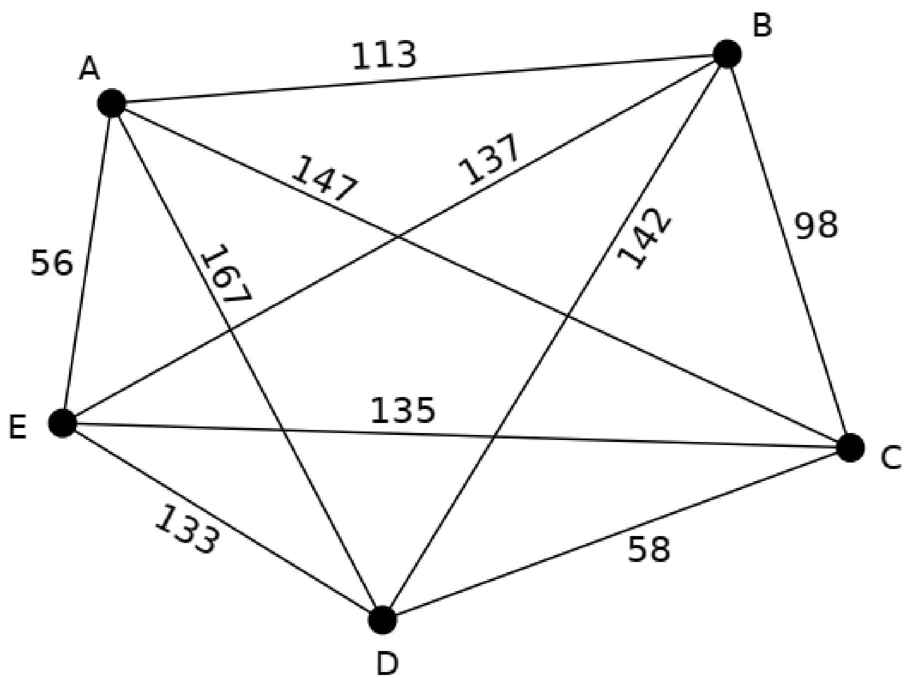
Traveling Salesperson Problem



Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**

Traveling Salesperson Problem



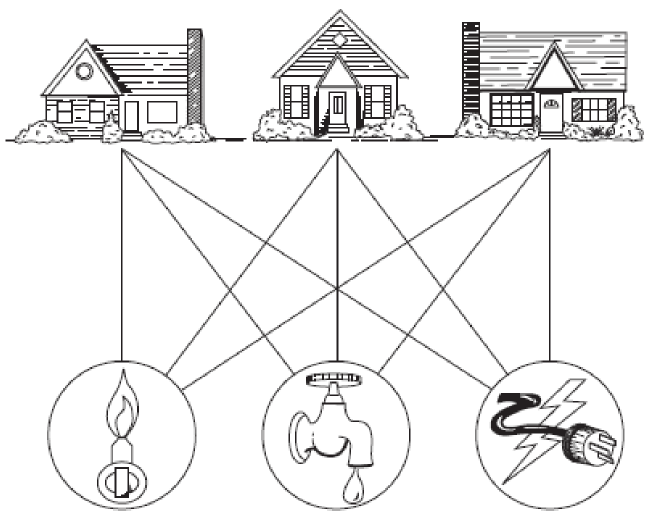
Route	Tot. dist.
A, B, C, D, E, A	610
A, B, C, E, D, A	516
A, B, E, D, C, A	588
A, B, E, C, D, A	458
A, B, D, E, C, A	540
A, B, D, C, E, A	504
A, D, B, C, E, A	598
A, D, B, E, C, A	576
A, D, E, B, C, A	682
A, D, C, B, E, A	646
A, C, D, B, E, A	670
A, C, B, D, E, A	728

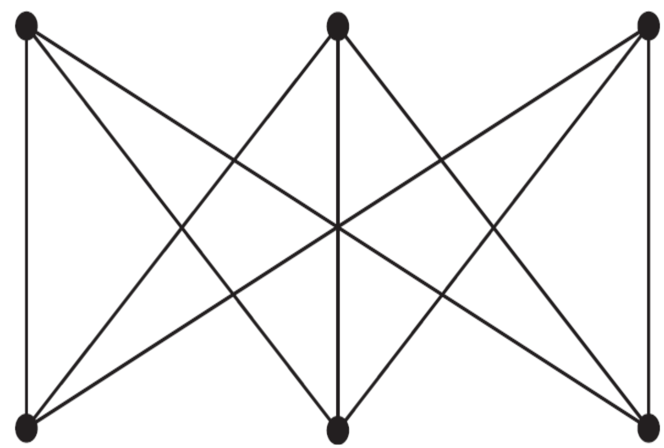
Traveling salesperson problem: a traveling salesperson wants to visit each of the cities once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?
$\Rightarrow$ **Hamiltonian circuit with minimum total weight in the complete graph.**

Planar Graph

DEFINITION: Let $G = (V, E)$ be an undirected graph. G is called a **planar graph**平面图 if it can be drawn in the plane without any edges crossing.

- Crossing of edges: an intersection other than endpoints (vertices)
- **planar representation**平面表示: a drawing w/o edge crossing; **nonplanar**非平面的





- K_1, K_2, K_3, K_4 are planar graphs
- $K_{1,n}, K_{2,n}$ are planar graphs
- $C_n\ (n \geq 3), W_n (n \geq 3)$ are planar graphs
- Q_1, Q_2, Q_3 are planar graphs