Discrete Mathematics: Lecture 24

Part IV. Graph Theory

Planar Graph, Euler’s Formula, Homeomorphic, Kuratowski's Theorem

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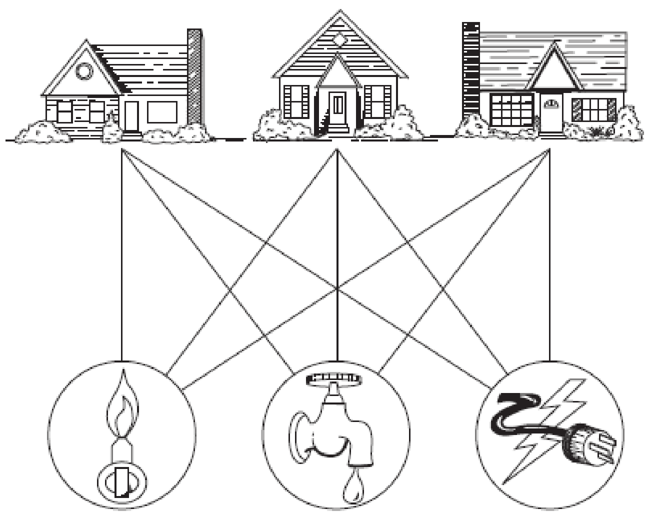
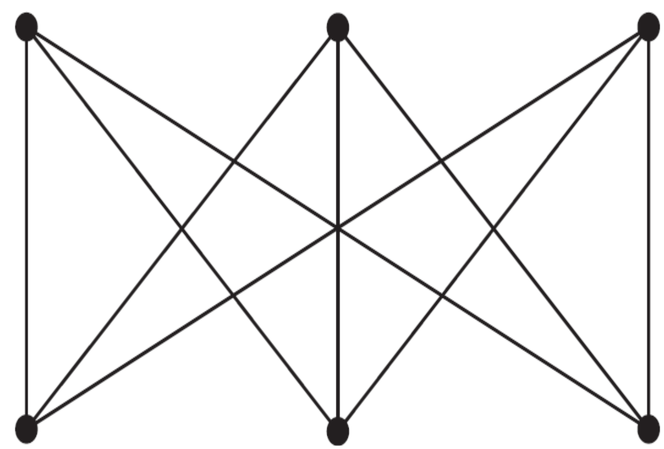
Planar Graph

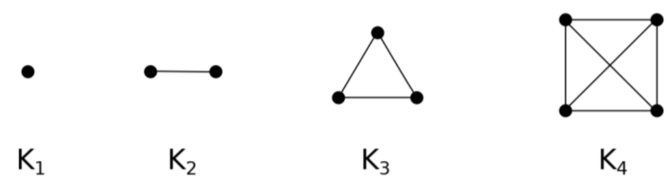
DEFINITION: Let $G = (V, E)$ be an undirected graph. G is called a **planar graph**平面图 if it can be drawn in the plane without any edges crossing.

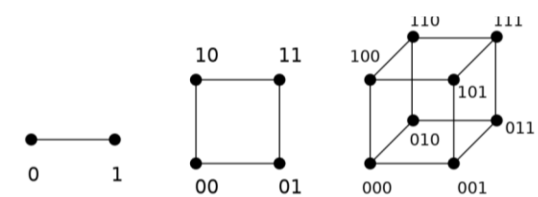
- Crossing of edges: an intersection other than endpoints (vertices)
- **planar representation**平面表示: a drawing w/o edge crossing; **nonplanar**非平面的





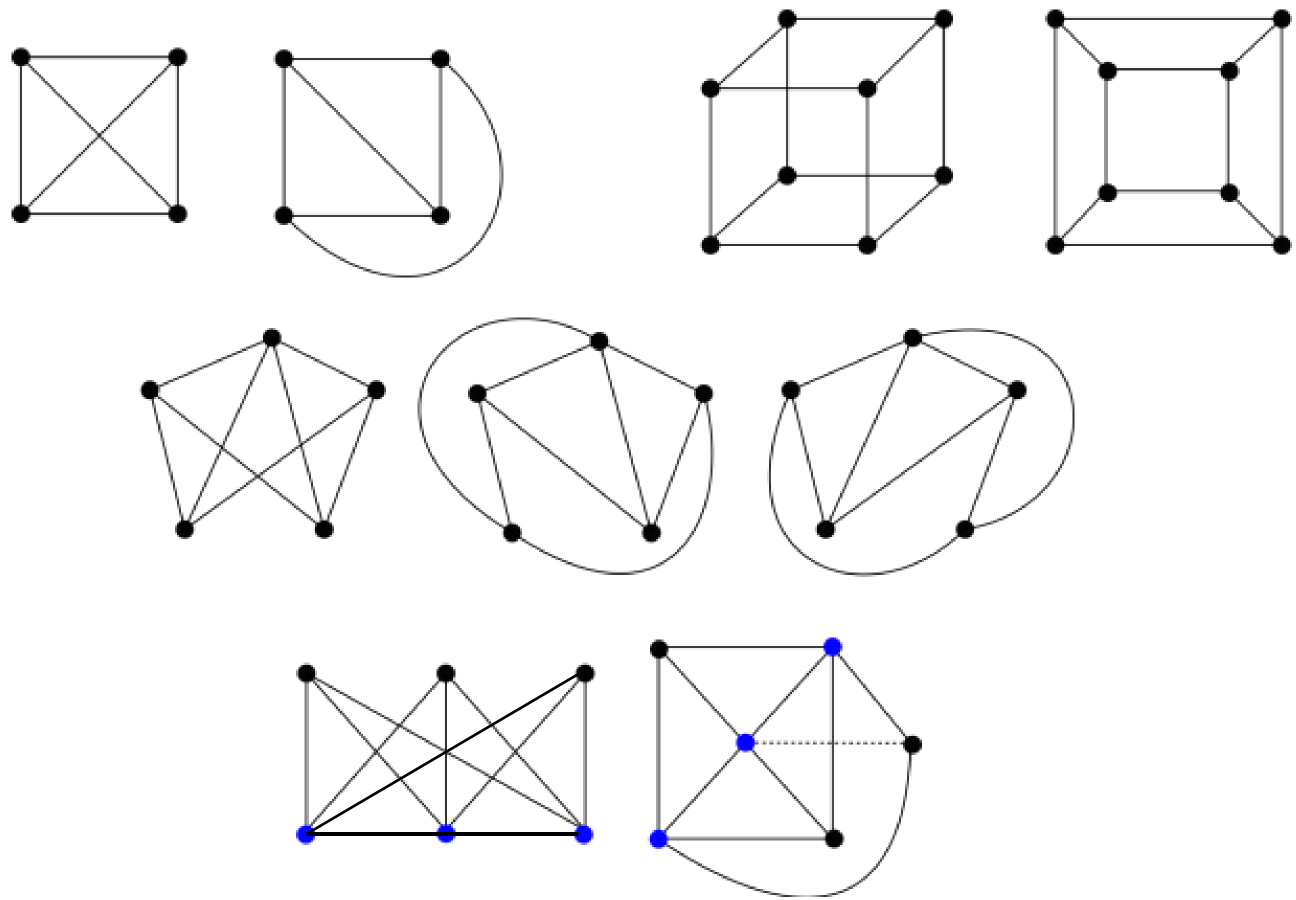
- K_1, K_2, K_3, K_4 are planar graphs
- $K_{1,n}, K_{2,n}$ are planar graphs
- C_n $(n \geq 3), W_n (n \geq 3)$ are planar graphs
- Q_1, Q_2, Q_3 are planar graphs





Planar Graph

Examples

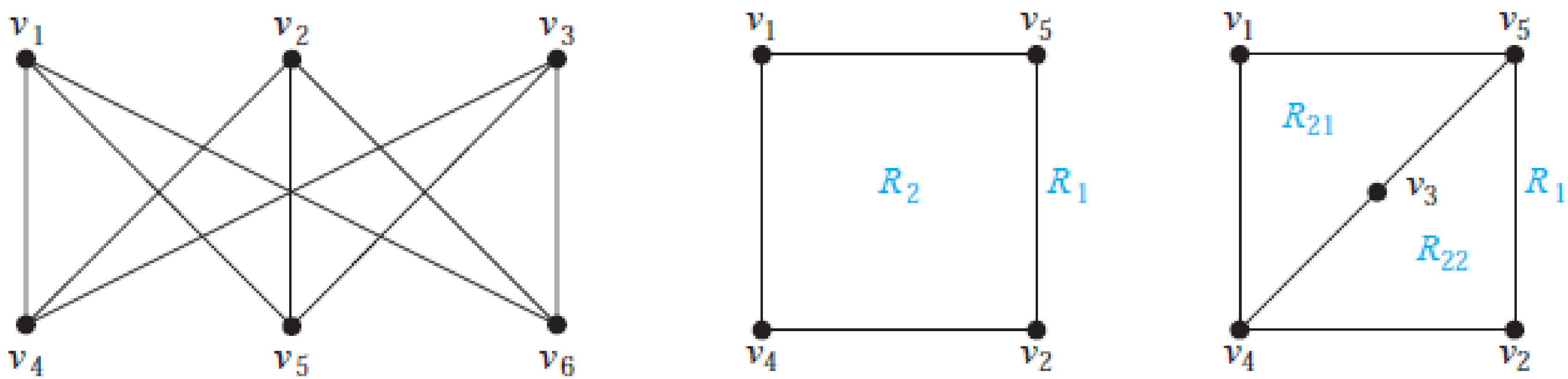


A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

Nonplanar Graph

Jordan Curve Theorem: Every **simple closed planar curve** Γ separates the plane into a bounded interior region and an unbounded exterior region. Any planar curve connecting the two regions must intersect Γ.

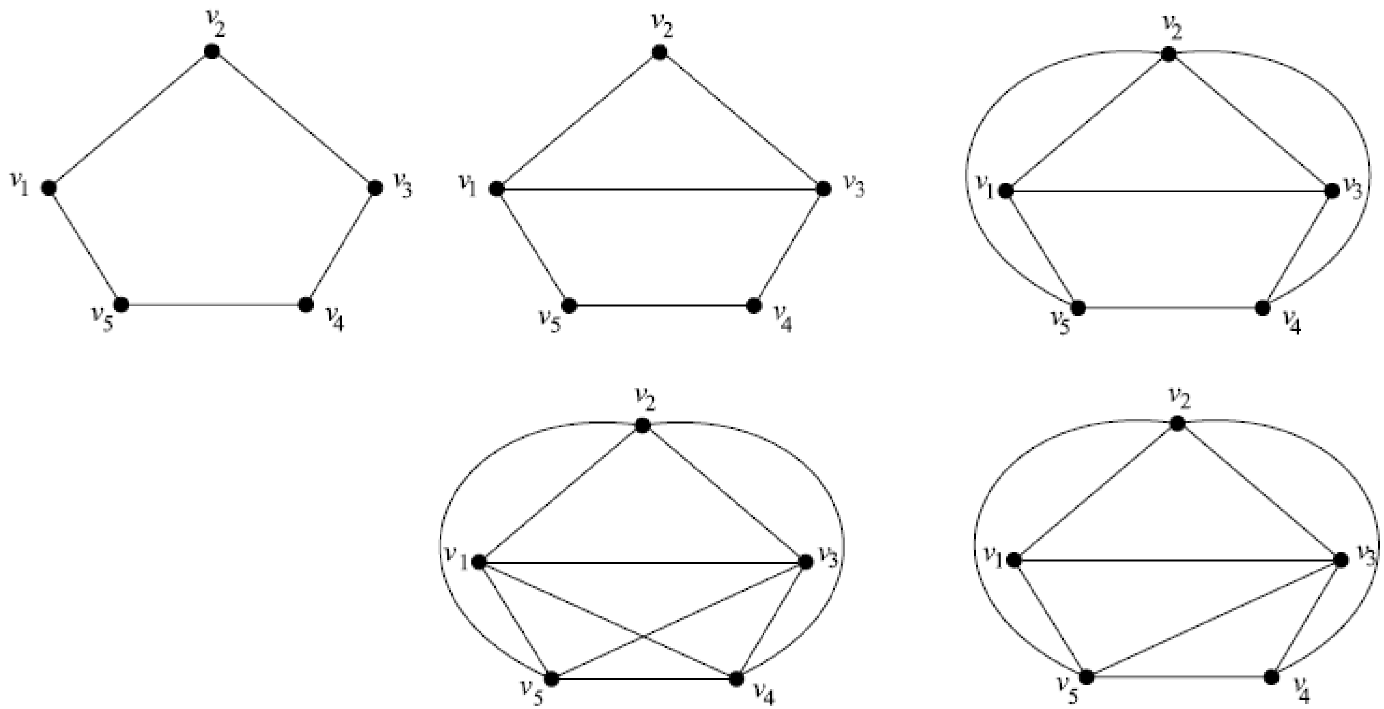
EXAMPLE: The bipartite graph $K_{3,3}$ is not planar.



- choose a simple circuit v_1, v_5, v_2, v_4, v_1 in $K_{3,3}$
- If $K_{3,3}$ is a planar, then the circuit forms a simple closed planar curve
- Add v_3, v_6 and the edges incident with them.
  - Intersection occurs (due to the Jordan curve Theorem).

Nonplanar Graph

EXAMPLE: The complete graph K_5 is not planar.

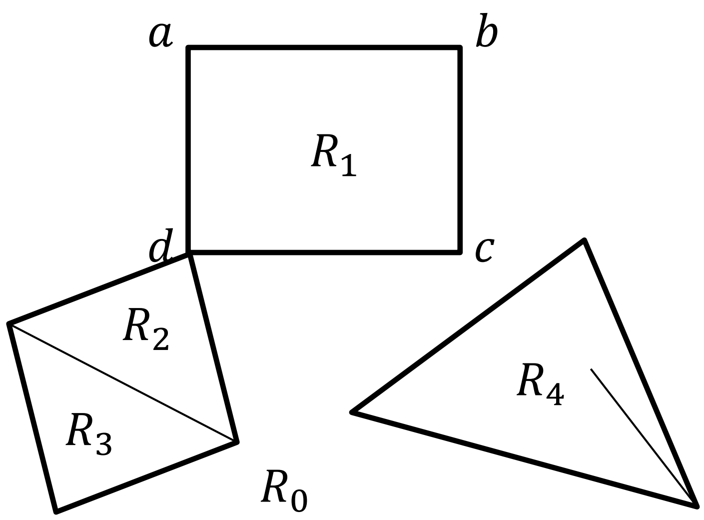


- $v_1, v_2, v_3, v_4, v_5, v_1$ is a simple closed curve in the planar representation of K_5
- Every remaining edge is in the interior region or in the exterior region
  - at least one is in the interior region
- No matter how you draw the remaining edges, crossing occurs.

Regions

DEFINITION: Let $G = (V, E)$ be a planar graph. Then the plane is divided into several **regions**面 by the edges of G.

- The infinite region is **exterior region**外部面. The others are **interior regions**内部面.
- The **boundary**边界 of a region is a subset of E.
- The **degree**度数 of a region is the number of edges on its boundary.
  - If an edge is shared by R_i, R_j, then it contributes 1 to $\deg(R_i)$, $\deg(R_j)$
  - If an edge is on the boundary of a single region R_i, then it contributes 2 to $\deg(R_i)$



- The plane is divided into 5 regions R_0, R_1, R_2, R_3, R_4
  - R_0 is the exterior region
  - R_1, R_2, R_3, R_4 are interior regions
- The boundary of R_1; $\deg(R_1) = 4$
- There are 4 edges on the boundary of R_4
  - $\deg(R_4) = 1 + 1 + 1 + 2 = 5$ because one of the edges contribute 2 to $\deg(R_4)$
- $\deg(R_0) = 11, \deg(R_1) = 4, \deg(R_2) = 3, \deg(R_3) = 3, \deg(R_4) = 5$

Euler's Formula

THEOREM: Let $G = (V, E)$ be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

THEOREM: Let G be a planar simple graph with p connected components. Then $|V(G)| - |E(G)| + |R(G)| = p + 1$.

- Let $G_1, G_2, \dots, G_p$ be the connected components of G.
  - By Euler's formula, $|R(G_i)| = |E(G)_i| - |V(G_i)| + 2$ for all $i \in [p]$
- $|V(G)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_p)|$
- $|E(G)| = |E(G_1)| + |E(G_2)| + \cdots + |E(G_p)|$
- $|R(G)| = |R(G_1)| + |R(G_2)| + \cdots + \left|R\left(G_p\right)\right| - p + 1$
- $|V(G)| - |E(G)| + |R(G)| = \sum_{i=1}^{p}(|V(G_i)| - |E(G_i)| + |R(G_i)|) - p + 1$
  $= 2p - p + 1 = p + 1$

Euler's Formula: Proof

Proof of Euler's formula by induction on the number e of edges

- A simple connected planar graph with 0 edges has only one vertex and one face (unbounded). The relation $f = e - v + 2$ is satisfied.

- Suppose the relation is satisfied for all simple connected planar graphs with k edges.
Consider a simple connected planar graph G with $k + 1$ edges, $k \geq 0$. This graph can be seen as a simple connected planar graph G' with k edges (satisfying the relation by induction hypothesis) to which we add one edge. There are two ways to add an edge to G' to get G:

  - either the two endpoints of the edge are already in G': in this case, adding the edge adds also one face,
  - either only one of the endpoint is already in G': in this case, adding the edge adds also one vertex but no other face.

In both cases, the relation $f = e - v + 2$ is satisfied by G.

Application

THEOREM: Let G be a **connected planar simple graph**. If every region has degree $\geq l$ in a planar representation of G, then

then $|E(G)| \leq \frac{l}{l-2}(|V(G)| - 2)$.

- Let $R_1, \ldots R_t$ be the regions given by a planar representation of G $//t = |R(G)|$
  - $\deg(R_i) \geq l$ for every $i = 1,2, \ldots, t$
- Let $r = \deg(R_1) + \deg(R_2) + \cdots + \deg(R_t)$. Then $r = 2|E(G)|$.
  - Every edge contributes 2 to r
    - If $e \in E$ is on the boundary of a single region R_i, then e contributes 2 to $\deg(R_i)$;
    - If $e \in E$ is shared by R_i and R_j, then e contributes 1 to $\deg(R_i)$ and 1 to $\deg(R_j)$;
- $2|E(G)| = r = \deg(R_1) + \deg(R_2) + \cdots + \deg(R_t) \geq lt = l|R(G)|$
- $|R(G)| = |E(G)| - |V(G)| + 2$
- Hence, $|E(G)| \leq \frac{l}{l-2}(|V(G)| - 2)$

Application

COROLLARY: Let G be **a connected planar simple graph**. If $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$.

- Every region has degree ≥ 3 in a planar representation of G
- Let $l = 3$ in the previous theorem
  - $|E(G)| \leq \frac{3}{3-2}(|V(G)| - 2) = 3|V(G)| - 6.$

EXAMPLE: The complete graph K_5 is not planar.

- $|E(K_5)| = \binom{5}{2} = 10, |V(K_5)| = 5, K_5$ is connected simple and of order ≥ 3
- $|E(K_5)| > 3|V(K_5)| - 6$
  - Hence, K_5 cannot be planar

COROLLARY: Let G be a connected planar simple graph. Then G has a vertex of degree ≤ 5.

- $|V(G)| < 3$: the statement is true.
- $|V(G)| \geq 3$: $\forall u \in V(G), \deg(u) \geq 6 \Rightarrow 2|E(G)| = \sum_u \deg(u) \geq 6|V(G)|$
  - G cannot be planar

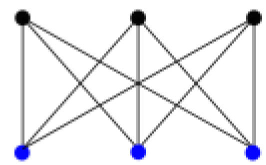
Application

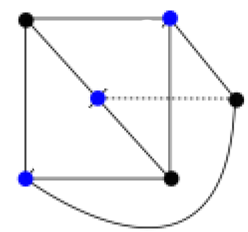
COROLLARY: Let G be **a connected planar simple graph**. If $|V(G)| \geq 3$ and there is no circuits of length 3 in G, then $|E(G)| \leq 2|V(G)| - 4$.

- Let $R_1, \ldots R_t$ be the regions given by a planar representation of G $\quad //t = |R(G)|$
    - $\deg(R_i) \geq 4$ for every $i = 1,2, \ldots, t$
- Hence, $|E(G)| \leq \frac{4}{4-2}(|V(G)| - 2) = 2|V(G)| - 4$

EXAMPLE: The complete bipartite graph $K_{3,3}$ is not planar.

- $\left|E\left(K_{3,3}\right)\right| = 3 \times 3 = 9, \left|V\left(K_{3,3}\right)\right| = 3 + 3 = 6 \geq 3$
- $K_{3,3}$ is connected, simple and of order ≥ 3.
- There is no circuits of length 3 in $K_{3,3}$
- $\left|E\left(K_{3,3}\right)\right| = 9 > 8 = 2\left|V\left(K_{3,3}\right)\right| - 4$
- Hence, $K_{3,3}$ cannot be planar



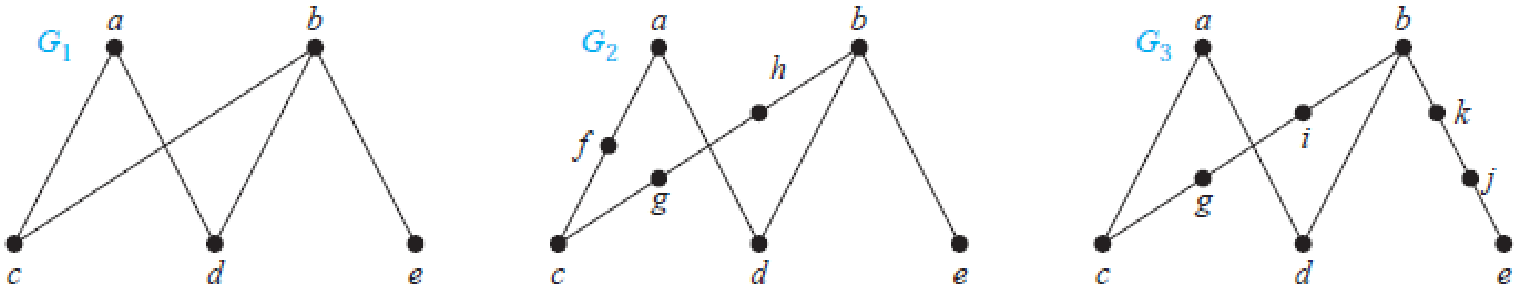


REMARKS: K_5 and $K_{3,3}$ are fundamental nonplanar graphs.

Homeomorphic

DEFINITION: Let $G = (V, E)$ be a graph and $\{u, v\} \in E$.

- **elementary subdivision**初等细分: $G' = (V \cup \{w\}, E - \{u, v\} + \{u, w\} + \{v, w\})$
- Two graphs are **homeomorphic**同胚的 if they can be obtained from the same graph via elementary subdivisions

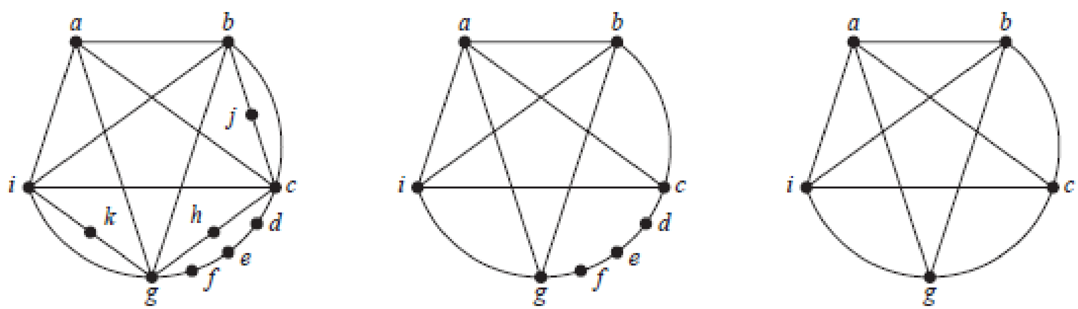


G_2 and G_3 are homeomorphic

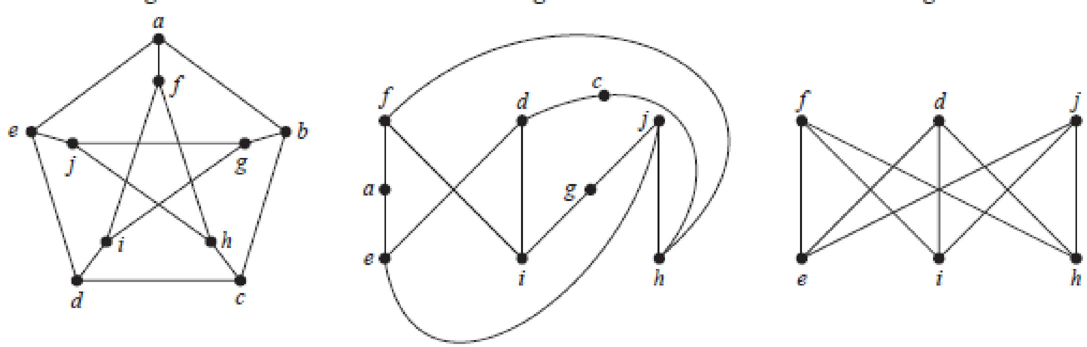
Kuratowski's Theorem

THEOREM: A graph G is nonplanar if and only if it has a subgraph homeomorphic to $K_{3,3}$ or K_5.

EXAMPLE: The following graph is nonplanar.



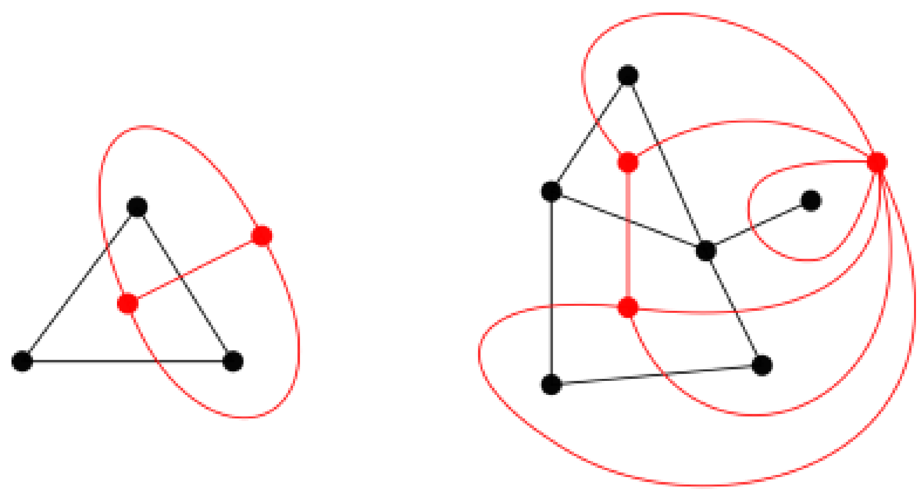
There is a subgraph homoemorphic to K_5



There is a subgraph homoemorphic to $K_{3,3}$

Dual Graph

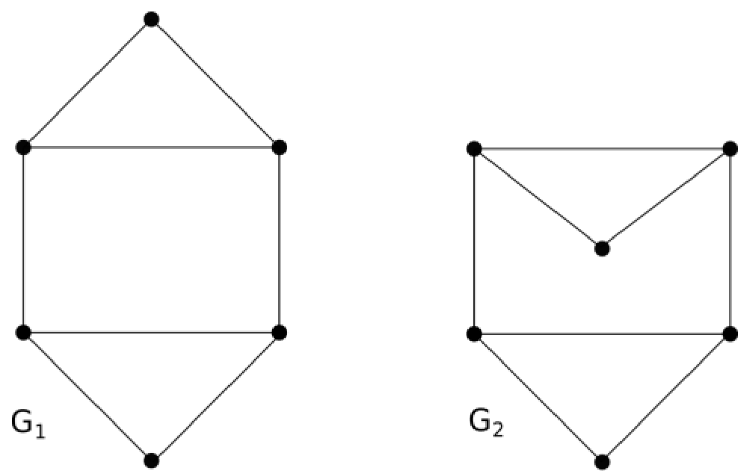
Let G be a planar graph and assume we take a planar representation of G that we denote also G. The **dual of** G is the graph G^* that has a vertex for each face of G and an edge connecting two vertices if the corresponding faces in G have a common edge in their boundary.



Remark: The dual of a planar simple graph is not necessarily simple.

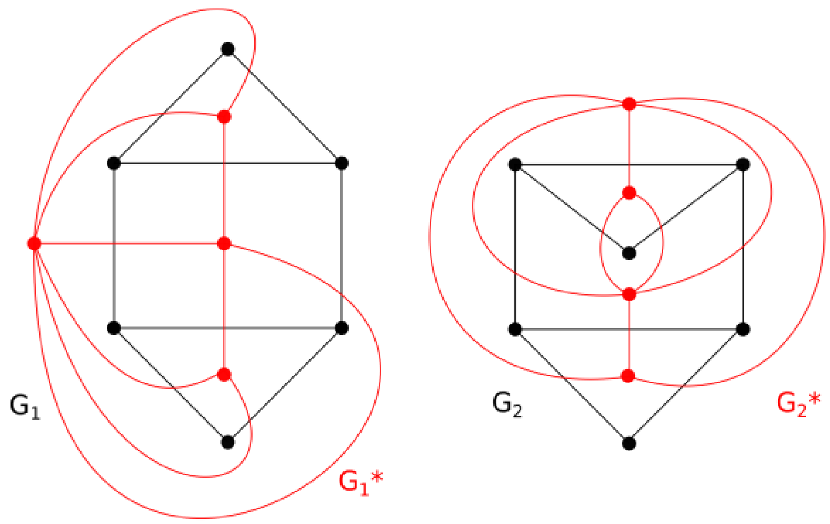
Dual Graph

- Properties of dual graph
  - The dual G^* of a planar graph G is a planar graph.
  - The dual of a planar graph is always connected.
  - The graphs G_1 and G_2 below are isomorphic. What about their duals?



Dual Graph

- Properties of dual graph
  - The graphs G_1 and G_2 below are isomorphic. What about their duals?



$\Rightarrow$ The dual of a planar graph depends on the planar representation of the graph.

Dual Graph

- Properties of dual graph
  - If G is a planar connected graph, then $v^* = f$, $e^* = e$, and $f^* = v$
  - If G is a planar connected graph then $G^{**} = G$ (not true if the graph is disconnected!)

Dual Graph

- Properties of dual graph
  - If G is a planar connected graph, then $v^* = f$, $e^* = e$, and $f^* = v$
  - If G is a planar connected graph then $G^{**} = G$ (not true if the graph is disconnected!)

Definition

A planar graph is said **self-dual** if it is isomorphic to its dual.

Example: The wheels W_n are self-dual graphs.

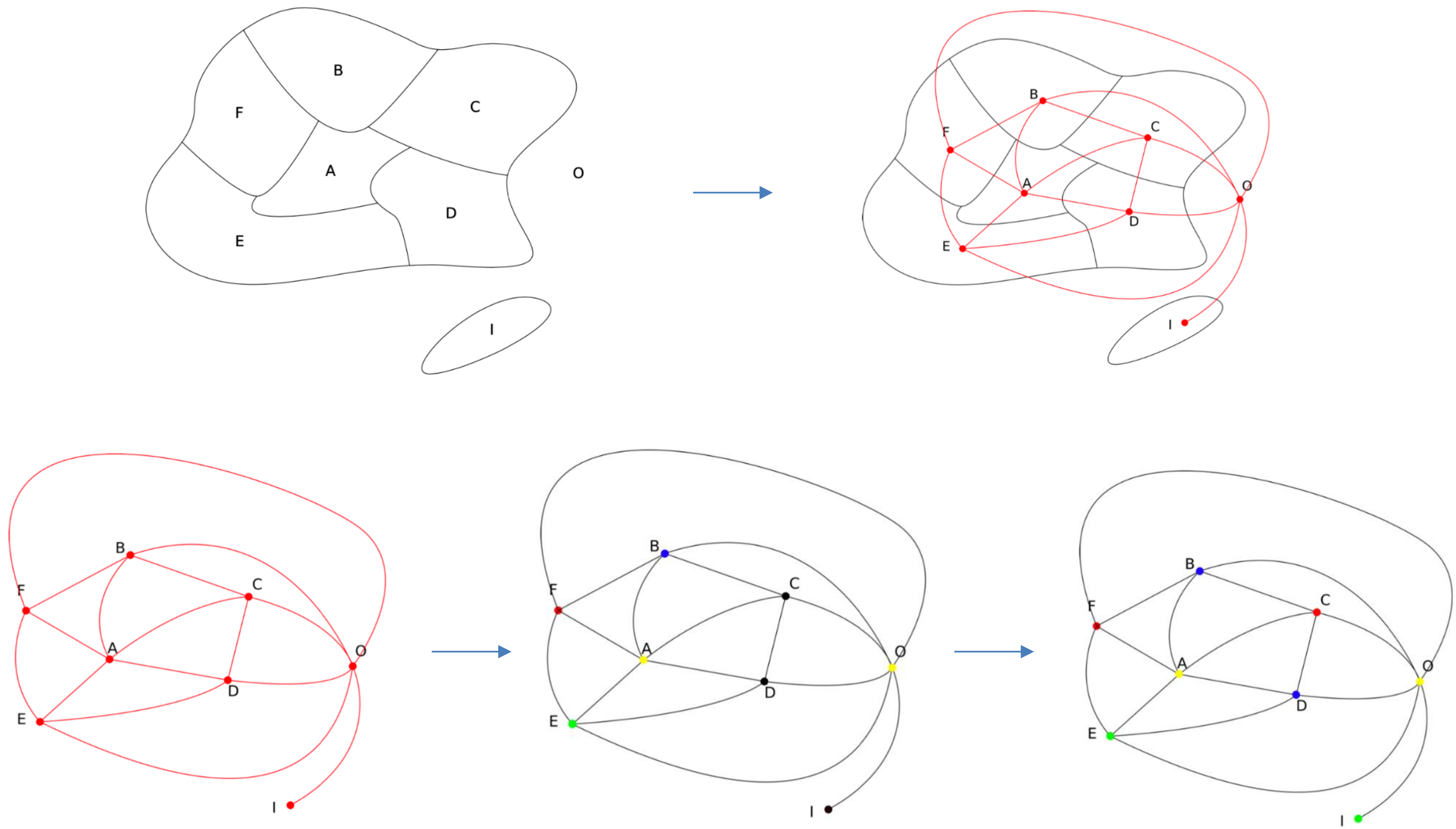
Proposition

A self-dual graph with v vertices has $2v - 2$ edges.

Proof: We have $v^* = v = f^* = f$ because the graph is self-dual. By Euler's formula

$$f = e - v + 2 \Rightarrow v = e - v + 2 \Rightarrow 2v - 2 = e$$

Coloring a Map



Coloring regions of the map ⇔ Coloring vertices of the dual graph