Discrete Mathematics: Lecture 25

Part IV. Graph Theory

Graph Coloring

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Spring Semester, 2025

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Dual Graph

- Properties of dual graph
  - If G is a planar connected graph, then $v^* = f$, $e^* = e$, and $f^* = v$
  - If G is a planar connected graph then $G^{**} = G$ (not true if the graph is disconnected!)

Definition

A planar graph is said **self-dual** if it is isomorphic to its dual.

Example: The wheels W_n are self-dual graphs.

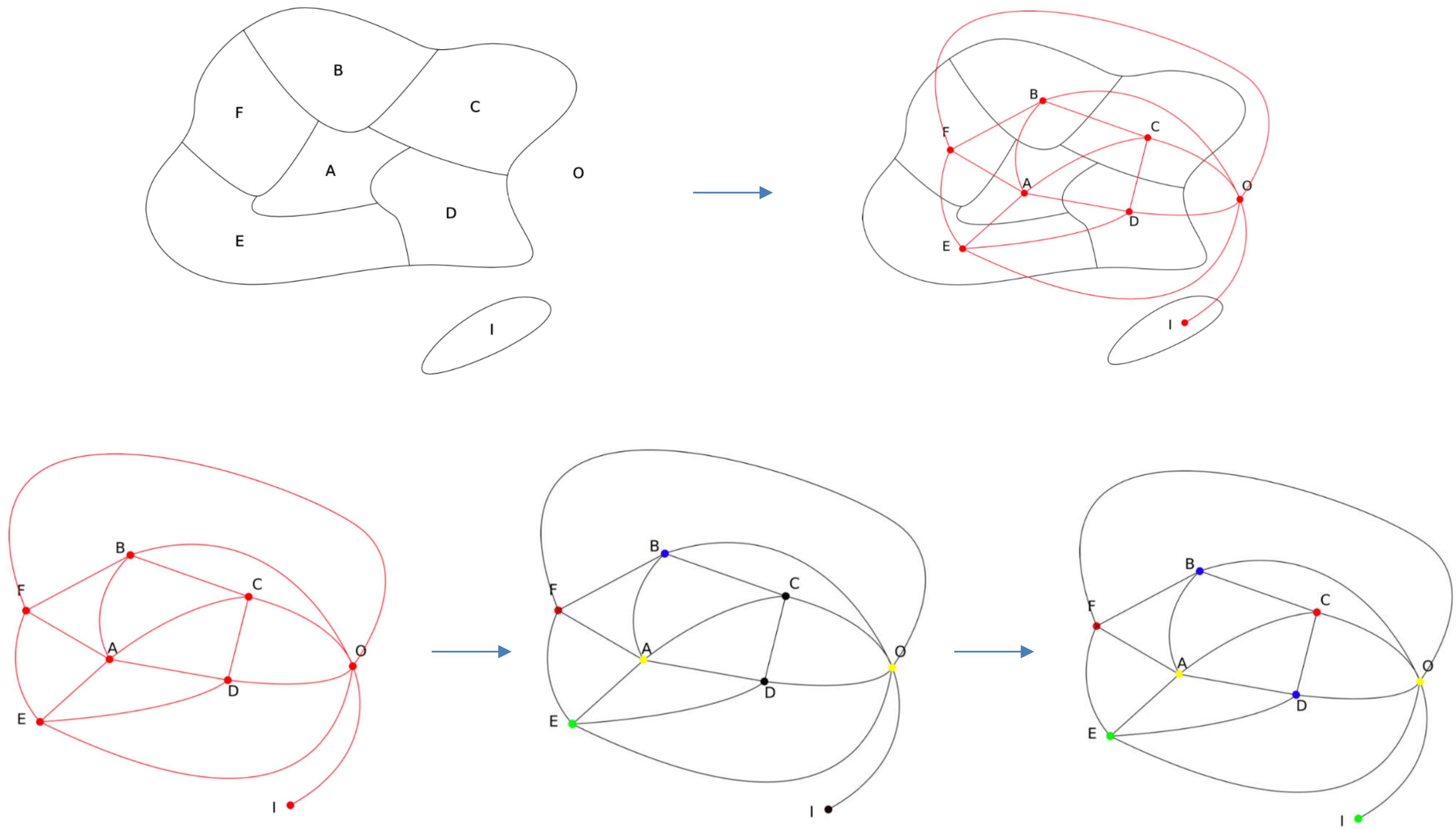
Proposition

A self-dual graph with v vertices has $2v - 2$ edges.

Proof: We have $v^* = v = f^* = f$ because the graph is self-dual. By Euler's formula

$$f = e - v + 2 \Rightarrow v = e - v + 2 \Rightarrow 2v - 2 = e$$

Coloring a Map

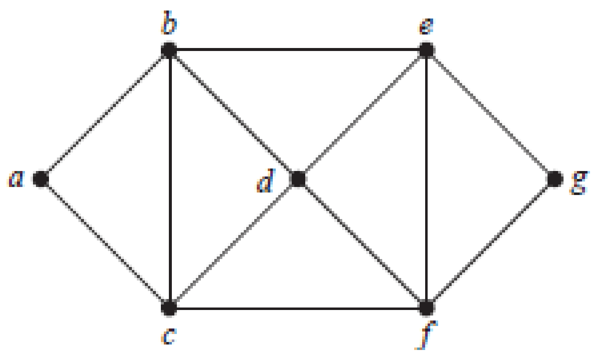


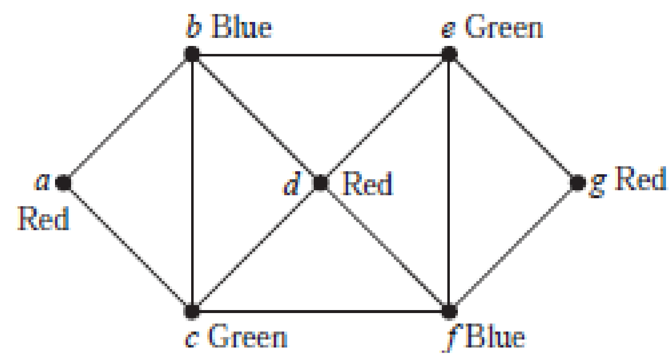
Coloring regions of the map $\Leftrightarrow$ Coloring vertices of the dual graph

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

- **chromatic number ($\chi(G)$)$_{色数}$:** the least k s.t. G has a k-coloring.





$$\chi(G) = 3$$

The chromatic number is at least 3 because *a; b; c* is a circuit of length 3

Graph Coloring

DEFINITION: Let $G = (V, E)$ be a simple graph. A ***k*-coloring**$_{k\text{-着色}}$ of G is a map $f: V \to [k]$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$.

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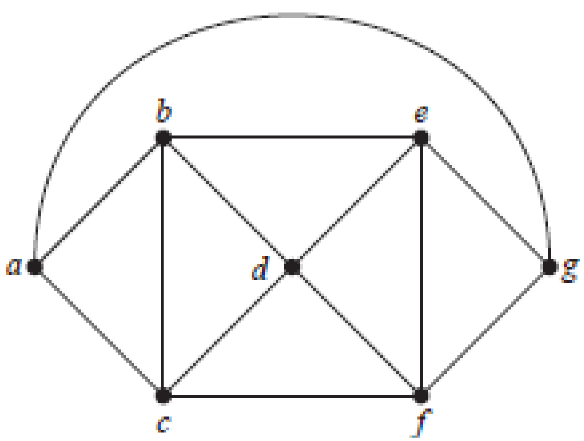
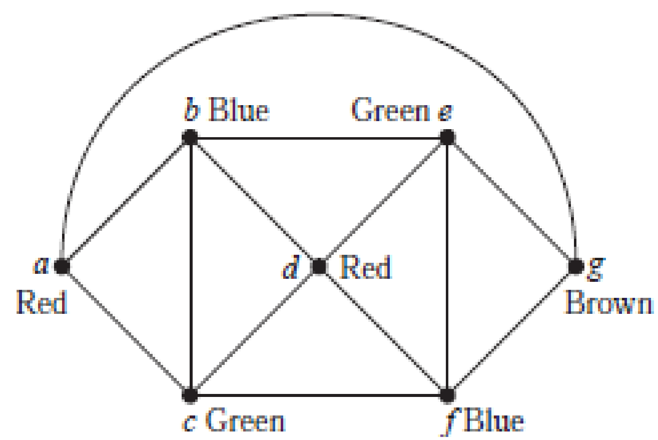


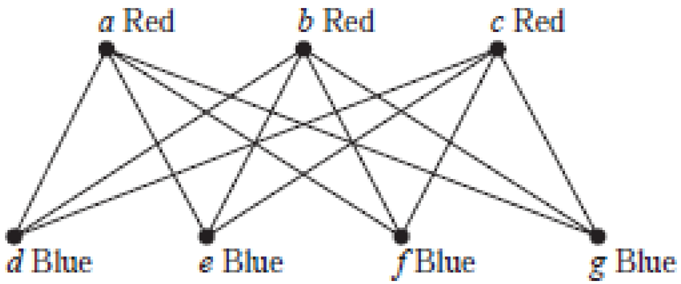


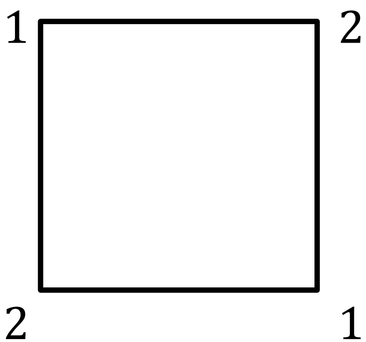
$\chi(G) = 4$

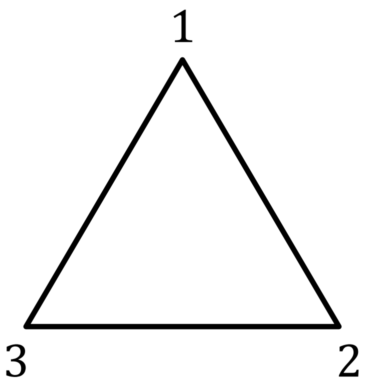
Graph Coloring

THEOREM: Let $G = (V, E)$ be a simple graph.

- $1 \leq \chi(G) \leq |V|$
- $\chi(G) = 1$ iff $E = \emptyset$
- $\chi(G) = 2$ iff G is bipartite and $|E| \geq 1$.
- $\chi(K_n) = n$ for every integer $n \geq 1$.
    - $\chi(G) \geq n$ if G has a subgraph isomorphic to K_n
- $\chi(C_n) = 2$ if $2|n$; $\chi(C_n) = 3$ if $2|(n-1)$; $(n \geq 3)$
- $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg(v) : v \in V\}$.

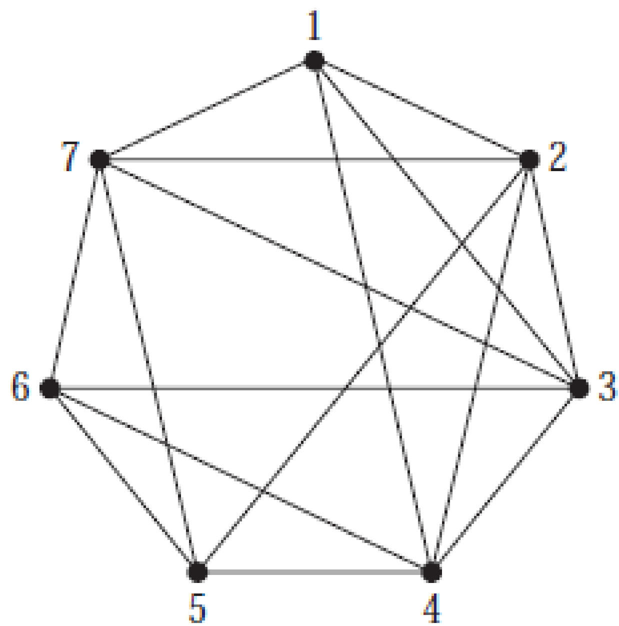


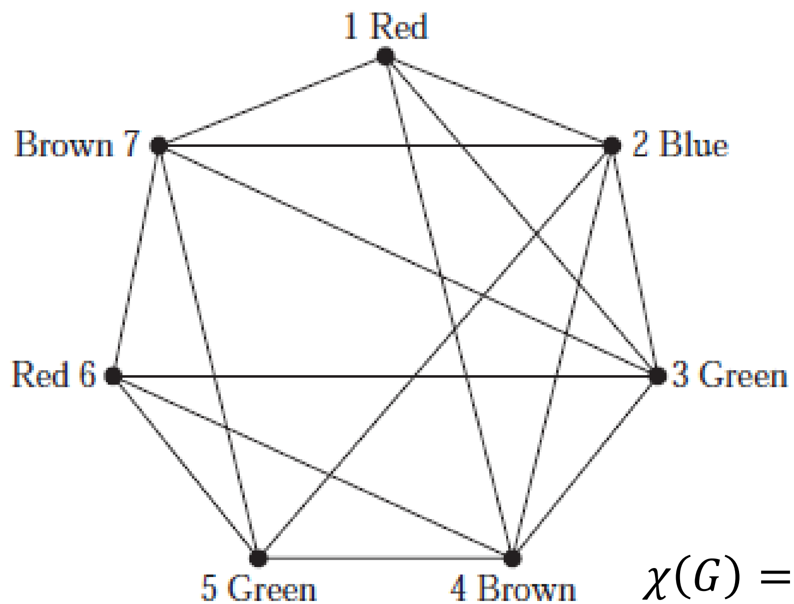




Application

PROBLEM: How can the final exams at a university be scheduled so that no student has two exams at the same time?





$\chi(G) = 4$

- There are 7 different courses, they are vertices of a graph.
- Two courses are adjacent if there is a student registered both courses.
- Choose time slots for the courses such that no two adjacent courses take place at the same time.
  - $\chi(G)$ time slots is needed.

$1 \leq \chi(G) \leq 7$

$1 \leq \chi(G) \leq \Delta(G) + 1 = 6$

$\chi(G) \geq 4$: G has a subgraph isomorphic to K_4

4-coloring Theorem

Theorem (Four coloring Theorem)

The chromatic number of a simple planar graph is no greater than 4.

Remarks: The proof of the 4-coloring Theorem depends on a computer. The two previous theorems are true for planar graphs only. A non planar graph can have an arbitrarily large chromatic number.

5-coloring Theorem*

Theorem (5-coloring Theorem)

The chromatic number of a planar simple graph is no greater than 5.

Proof: Induction on the number of vertices of the graph

- If $v \leq 5$ the theorem is true.
- Assume that all planar simple graph with k vertices can be 5-colored.

Consider a simple planar graph with $k+1$ vertices. G has a vertex u of degree at most 5 (by Corollary 2 of Euler's formula).
Let G' be the subgraph of G obtained by removing u (and the edges incident to it).
G' is a planar simple graph with k vertices $\Rightarrow$ it can be 5-colored by induction hypothesis.
Case 1: if the neighbours of u in G do not use all the 5 colors, we use one remaining color for u and the theorem is true.
Case 2: u has exactly 5 neighbors using the 5 colors we have.

5-coloring Theorem*

Denote b, r, g, y, p the neighbors of u in clockwise order. Assume that b is colored in blue and g is colored in green.
Take the subgraph of G formed by vertices in blue and green (and edges between them).

1. if the vertices b and g are not in the same connected component: in the component containing b, interchange the colors blue and green on the vertices. The graph G' is still 5-colored, and the color blue is now available for u.

5-coloring Theorem*

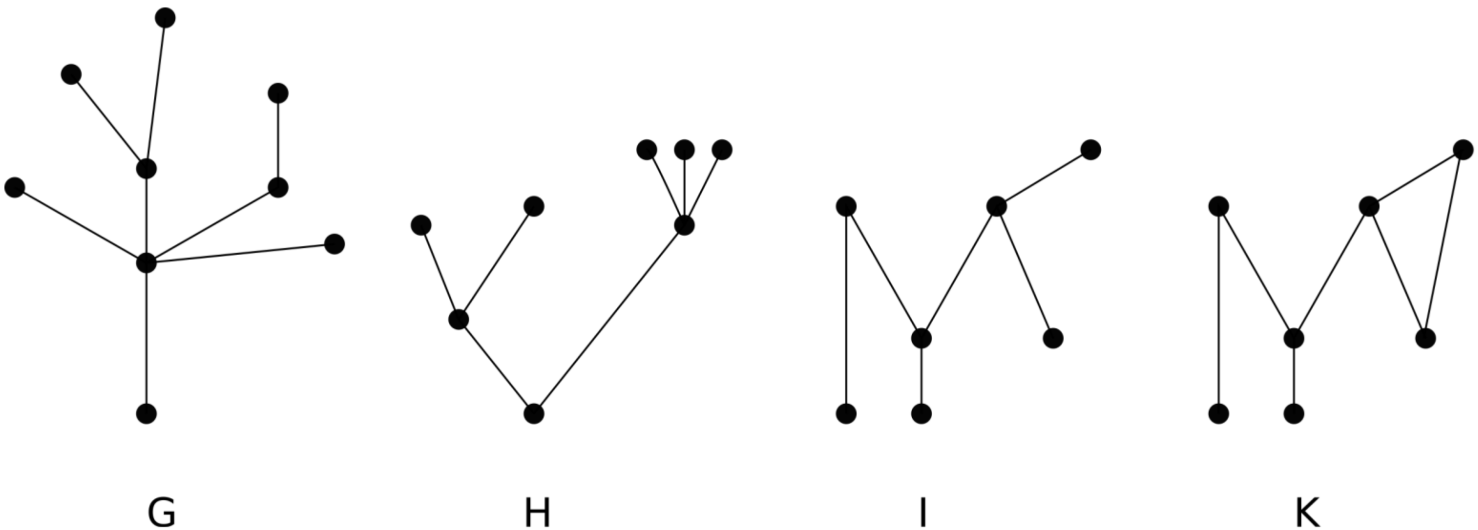
Denote b, r, g, y, p the neighbors of u in clockwise order. Assume that b is colored in blue and g is colored in green.
Take the subgraph of G formed by vertices in blue and green (and edges between them).

2. if the vertices b and g are in the same connected component: consider a path from b to g in the "blue-green subgraph". This path with edges (g, u) and (u, b) forms a circuit in G.
   This circuit divides the plane into two regions. The vertex y (colored in yellow) is in one of them, r (colored in red) in the other. In the region containing r, exchange the colors yellow and red. Now red is available for u.

Tree

Definition

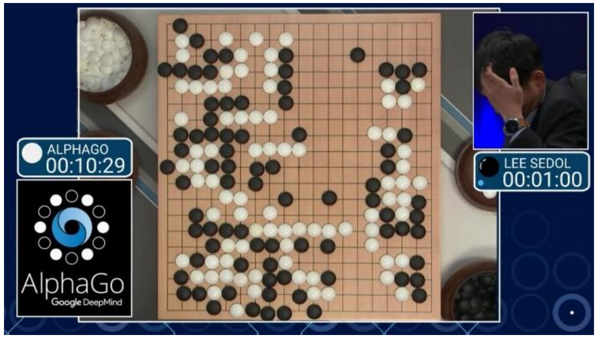
- A **tree** is a connected undirected graph with no simple circuits.
- A **forest** is an graph such that each of its connected components is a tree.



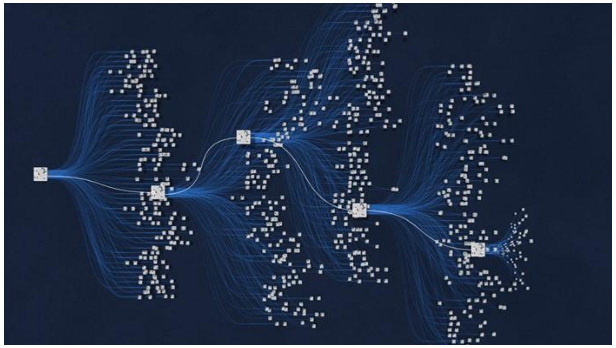
G, H, I are trees, but K is not a tree.

Applications

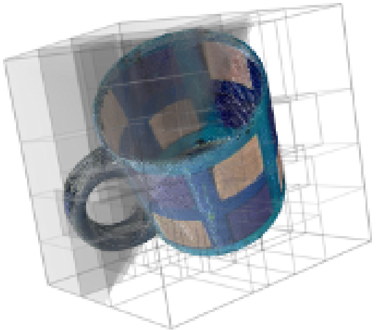
- Trees are widely used in AI & other CS sub-areas



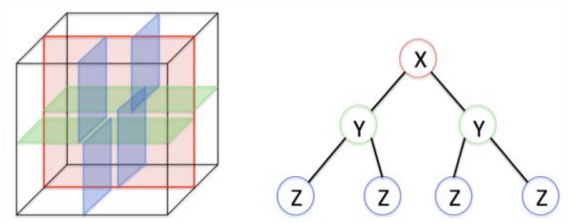
AlphaGo



Monte Carlo Tree Search



Point Cloud/3D Shape



KD Tree

Characterization of Tree

Theorem

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Proof: ($\Rightarrow$) Assume T is a tree and let u and v be two vertices. T is connected so there is a *simple path* P_1 from u to v. Assume there is a second simple path P_2 from u to v.

Claim: There is a simple circuit in T.
Let $u = x_0, x_1, \ldots, x_n = v$ denote the vertices of P_1 and $u = y_0, y_1, \ldots,$ $y_m = v$ the vertices of P_2.
P_1 and P_2 start at u but are not equal so must diverge at some point.

- If they diverge after one of them has ended, then the remaining part of the other path is a circuit from v to v.

Characterization of Tree

- Otherwise, we can assume

$$x_0 = y_0, x_1 = y_1, \ldots, x_i = y_i$$

and $x_{i+1} \neq y_{i+1}$.

We follow then $y_{i+1}, y_{i+2}, \ldots$ until we reach a vertex of P_1.
Then go back to x_i following P_1 forwards or backwards.

This gives a circuit which is simple because P_1 and P_2 are, and we stop using edges of P_2 as soon as we hit P_1.

($\Leftarrow$) Assume there is a unique simple path between any two vertices of the graph T. Then:

- T is connected (by definition)
- if T has a simple circuit containing the vertices x and y $\rightsquigarrow$ two simple paths between x and y. $\square$

Properties of Tree

Theorem

A tree with n vertices has $n-1$ edges.

Properties of Tree

Theorem

A tree with n vertices has $n-1$ edges.

Proof: By induction on the number of vertices.

- $n = 1$: A tree with one vertex has no edge.

- $k \rightsquigarrow k+1$: Assume every tree with k vertices has $k-1$ edges.

Let T be a tree with $k+1$ vertices, and v a leaf (which exists because the tree has a finite number of vertices).

Let T' be the tree obtained from T by removing v (and the edge incident to it). T' is a connected tree with k vertices $\Rightarrow$ it has $k-1$ edges by induction hypothesis.

$\Rightarrow$ T has $k+1$ vertices and k edges.

Properties of Tree

Tree $=$ connected with no simple circuit (definition)

(1) connected
(2) no simple circuit
(3) $(n-1)$ edges (n=nb of vertices)

Previous theorem: $(1) + (2) \Rightarrow (3)$

We also have: $(1) + (3) \Rightarrow (2)$
$(2) + (3) \Rightarrow (1)$

Example: For what value of m, n the complete bipartite graph $K_{m,n}$ is a tree?

$K_{m,n}$ is connected, has $m + n$ vertices and $m \times n$ edges.
It is a tree if:

$$m \times n = m + n - 1 \Longleftrightarrow (n-1)m = n - 1$$

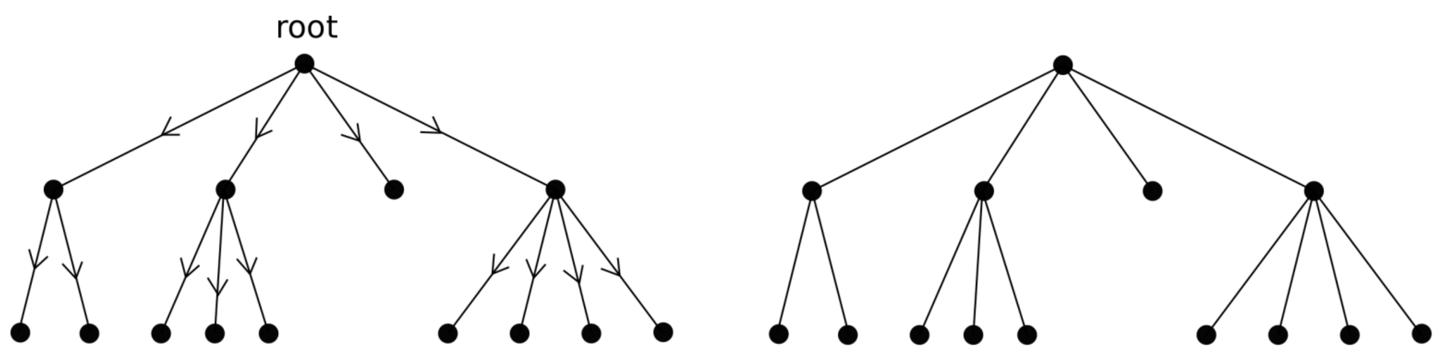
If $n \neq 1$: $m = 1$
If $n = 1$: $m \in \mathbb{N}^*$

Rooted Tree

Definition

A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Remarks: • A rooted tree is a directed graph.
• We usually draw a rooted tree with its root at the top of the graph.
• We usually omit the arrows on the edges to indicate the direction because it is uniquely determined by the choice of the root.
• Any non rooted tree can be changed to a rooted tree by choosing a vertex for the root.

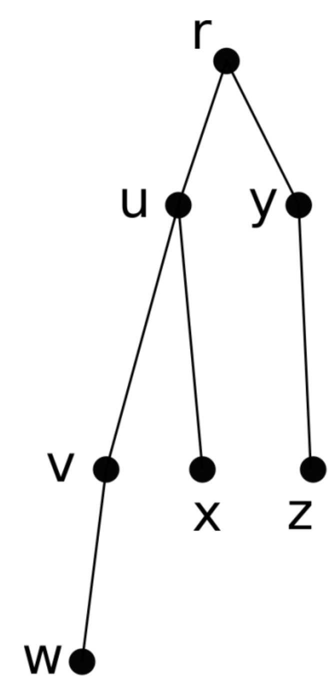


Rooted Tree

Definition

Let T be a rooted tree and v a vertex which is not the root. We call

- **parent** of v the *unique* vertex u such that there is an edge from u to v,
- **child** of v a vertex w such that there is an edge from v to w,
- **siblings** vertices with the same parent,
- **ancestors** of v all vertices in the path from the root to v,
- **descendants** of v all vertices that have v as an ancestor,
- **leaf** a vertex which has no children,
- **internal vertex** a vertex that has children,
- **subtree with v at its root** the subgraph of T consisting of v and its descendants and the edges incident to them.

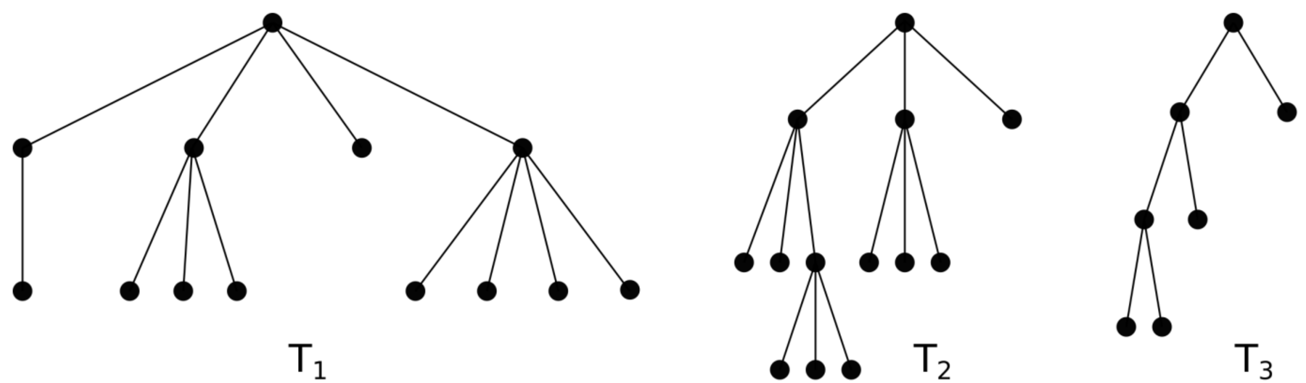


- r is the root
- v is child of u and parent of w
- v and x are siblings

Rooted Tree

Definition

- A rooted tree is called an *m*-**ary tree** if every internal vertex has no more than m children.
- A rooted tree is called a **full m-ary tree** if every internal vertex has exactly m children.
- An m-ary tree with $m = 2$ is called a **binary tree**. In this case if an internal vertex has two children, they are called **left child** and **right child**. The subtree rooted at the left (resp. right) child of a vertex is called the **left (resp. right) subtree** of this vertex.



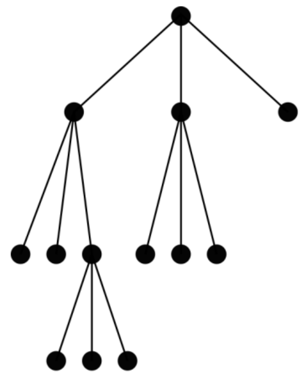
T_1 is a 4-ary tree, T_2 a full 3-ary tree, T_3 a full binary tree.

Properties of Tree

Theorem

A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Proof: Each vertex (except the root) is the child of an internal vertex.
There are i internal vertices, each with m children
$\Rightarrow mi$ vertices + root $= mi + 1$ vertices



A full m-ary tree with

1. *n vertices has $i = (n-1)/m$ internal vertices and $\ell = \big((m-1)n + 1\big)/m$ leaves,*
2. *i internal vertices has $n = mi + 1$ vertices and $\ell = (m-1)i + 1$ leaves,*
3. *ℓ leaves has $n = (m\ell - 1)/(m-1)$ vertices and $i = (\ell - 1)/(m-1)$ internal vertices.*