Homework 1

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The 1-D expression of the wave equation.

$$\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)], \quad (x \in [0,L], t \in [0,+\infty)) \tag{1}$$

First, let's look at the second order system. In the homogeneous system, this equation can be simplified as equation (2).

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) \tag{2}$$

Using discretization,

$$\begin{aligned}
\partial_t^2 u(x_i,t_n) &\approx \frac{u(x_i,t_{n+1}) - 2u(x_i,t_n) + u(x_i,t_{n-1})}{\Delta t^2} = \frac{1}{\Delta t^2}[u_i^{n+1} - 2u_i^n + u_i^{n-1}] \\
\partial_x^2 u(x_i,t_n) &\approx \frac{u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)}{\Delta x^2} = \frac{1}{\Delta x^2}[u_{i+1}^n - 2u_i^n + u_{i-1}^n]
\end{aligned} \tag{3}$$

Therefore,

$$u_i^{n+1} = \frac{c^2\Delta t^2}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + 2u_i^n - u_i^{n-1} \tag{4}$$

Next, the wave equation can be solved in the first order system with two equations.

$$\begin{aligned}
\rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\
\partial_t T(x,t) &= \kappa(x)\partial_x v(x,t)
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
v(x,t) &= \partial_t u(x,t) \\
T(x,t) &= \kappa(x)\partial_x u(x,t)
\end{aligned} \tag{6}$$

Take the centered difference scheme and use discretization,

$$\begin{aligned}
\rho(x_i)\frac{v(x_i,t_{n+1}) - v(x_i,t_{n-1})}{2\Delta t} &= \frac{T(x_{i+1},t_n) - T(x_{i-1},t_n)}{2\Delta x} \\
\frac{T(x_i,t_{n+1}) - T(x_i,t_{n-1})}{2\Delta t} &= \kappa(x_i)\frac{v(x_{i+1},t_n) - v(x_{i-1},t_n)}{2\Delta x}
\end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned}
v_i^{n+1} &= \frac{\Delta t}{\Delta x \rho(x_i)}(T_{i+1}^n - T_{i-1}^n) + v_i^{n-1} \\
T_i^{n+1} &= \frac{\kappa(x_i)\Delta t}{\Delta x}(v_{i+1}^n - v_{i-1}^n) + T_i^{n-1}
\end{aligned} \tag{8}$$

Problem 1. Homogeneous Medium

The setting of this model is $c = 1$, $\rho = 1$, $\kappa = 1$. The evolution setting is $L = 100$, $\Delta x = 0.1$,$\Delta t = \frac{\Delta x}{c} = 0.1$. The initial condition is

$$\begin{aligned} u(x,0) &= exp(-0.1(x-50)^2) \\ \partial_t u(x,0) &= 0 \end{aligned} \tag{9}$$

Equation (9) is the initial condition for the second order system. As in the first order system. $T(x,0)$ is calculated by $u(x,0)$ through definition. In addition, $v(x,0) = 0$ exists.

* **Case 1** Dirichlet boundary conditions on both ends of the string

$$u(0,t) = u(L,t) = 0 \tag{10}$$

This equation provide boundary conditions for the second order system. As to the first order system, $v(0,t) = v(L,0) = 0$ always exists. $T(0,t)$ and $T(L,t)$ should be calculated using $\partial_x v(0,t)$ and $\partial_x v(L,t)$ based on the second equation in (8).

* **Case 2** Neumann boundary conditions on both ends of the string

$$T(0,t) = T(L,t) = 0 \tag{11}$$

$v(0,t)$ and $v(L,t)$ should be calculated accordingly based on the first equation in (8). This is the boundary condition in the first order system. In terms of the boundary conditions in the second order system, it should be,

$$u(0,t) = u(dx,t) \quad and \quad u(L,t) = u(L-dx,t) \tag{12}$$

Problem 2. Heterogeneous Medium

The initial condition remains unchanged. With the Dirichlet boundary conditions in $x = 0$ and the Neumann boundary conditions in $x = 100$, the boundary condition for the second order system is,

$$u(0,t) = 0 \quad and \quad u(L,t) = u(L-dx,t) \tag{13}$$

The boundary condition for the first order system is,

$$v(0,t) = 0 \quad and \quad T(L,t) = 0 \tag{14}$$

$T(0,t)$ and $v(L,t)$ can be calculated accordingly.

Following are the simulation results. The first column plots the displacement profiles in the second order system. The second column plots the velocity profiles, with blue line calculated in the second order system and red line calculated in the first order system. The third column plots the stress profiles, which is calculated in the first order system.

The first figure is the case of homogeneous medium with Dirichlet boundary conditions. The second figure is the case of homogeneous medium with Neumann boundary conditions.

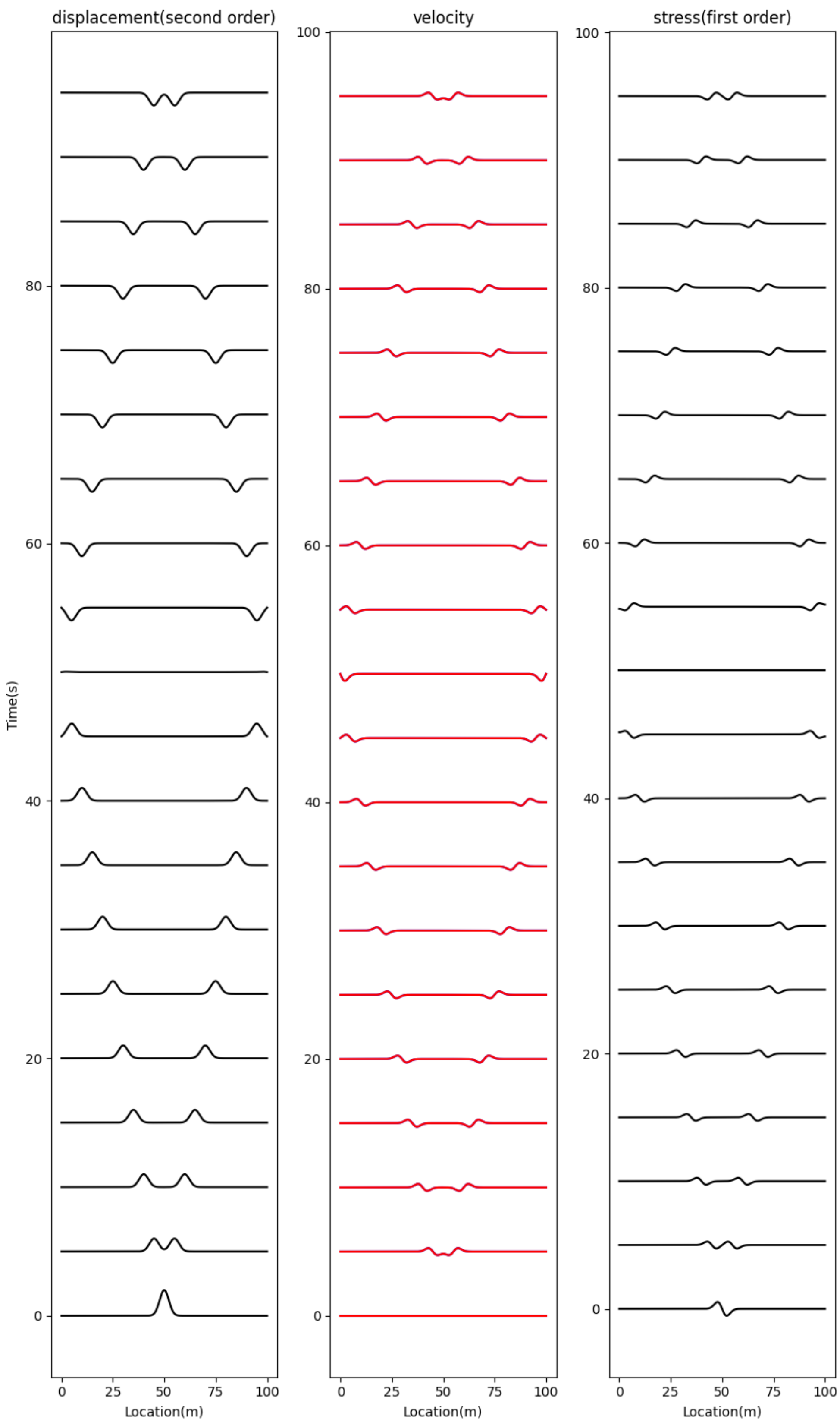

Figure 1: Homogeneous Medium with Dirichlet boundary conditions

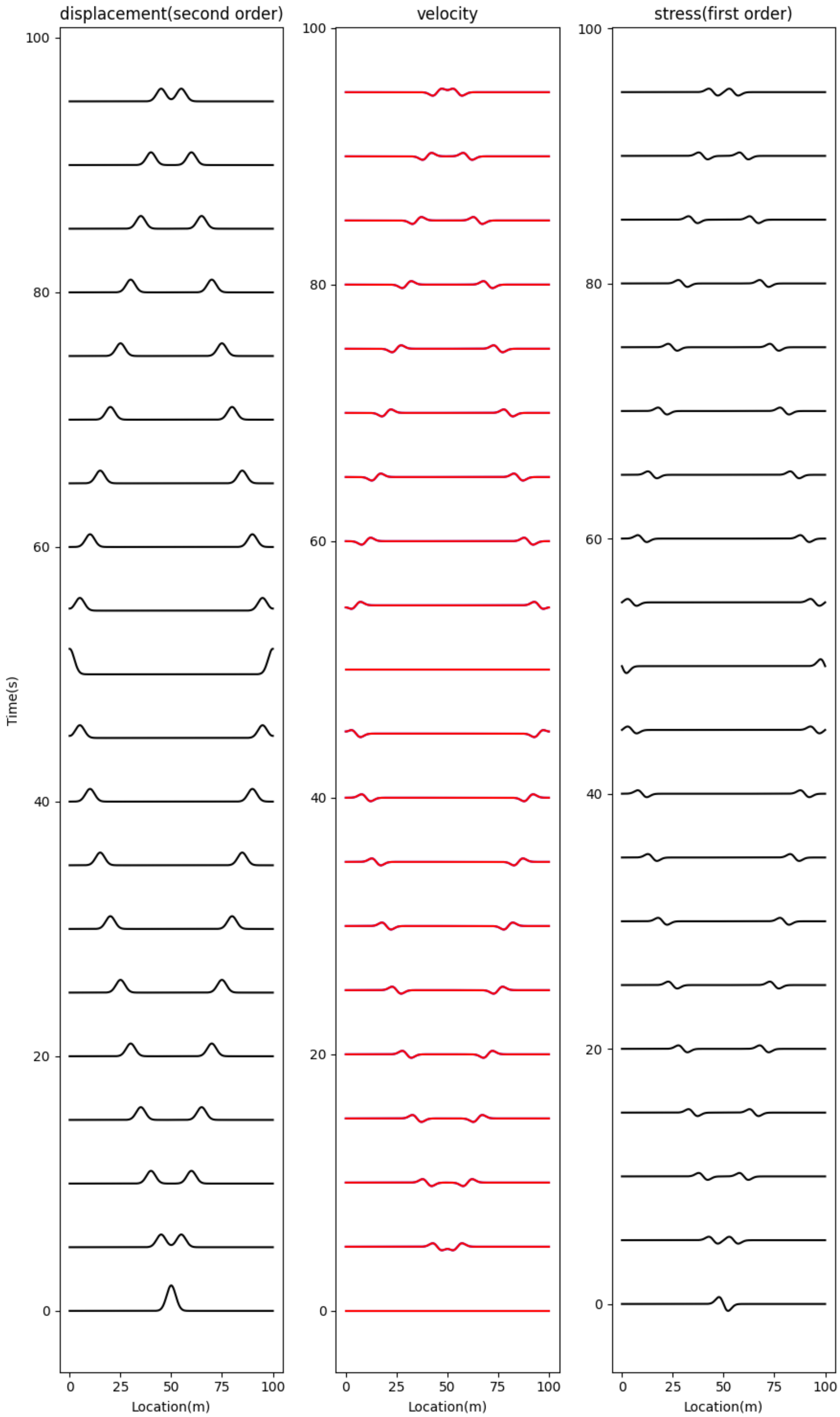

Figure 2: Homogeneous Medium with Neumann boundary conditions

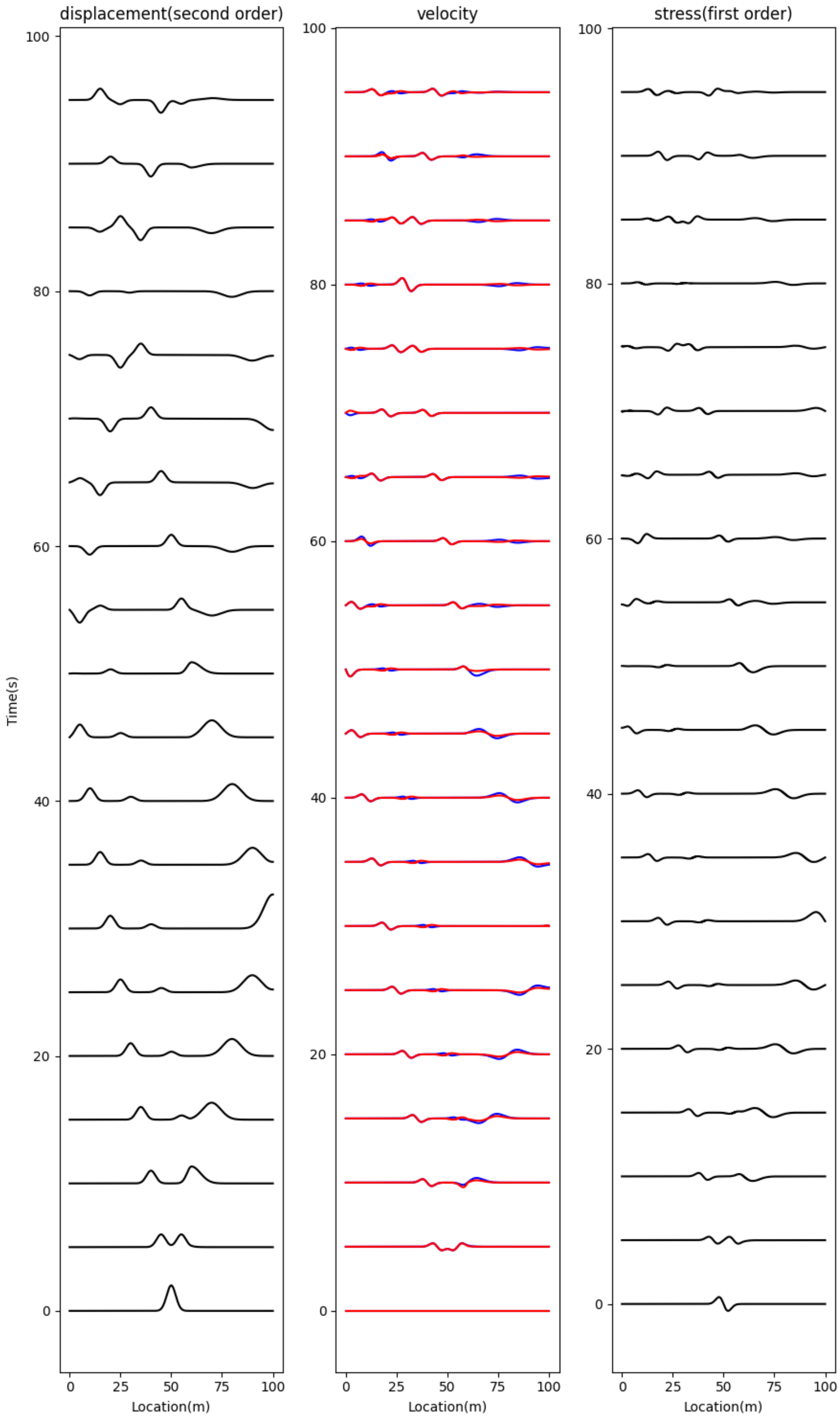

Figure 3: Heterogeneous Medium

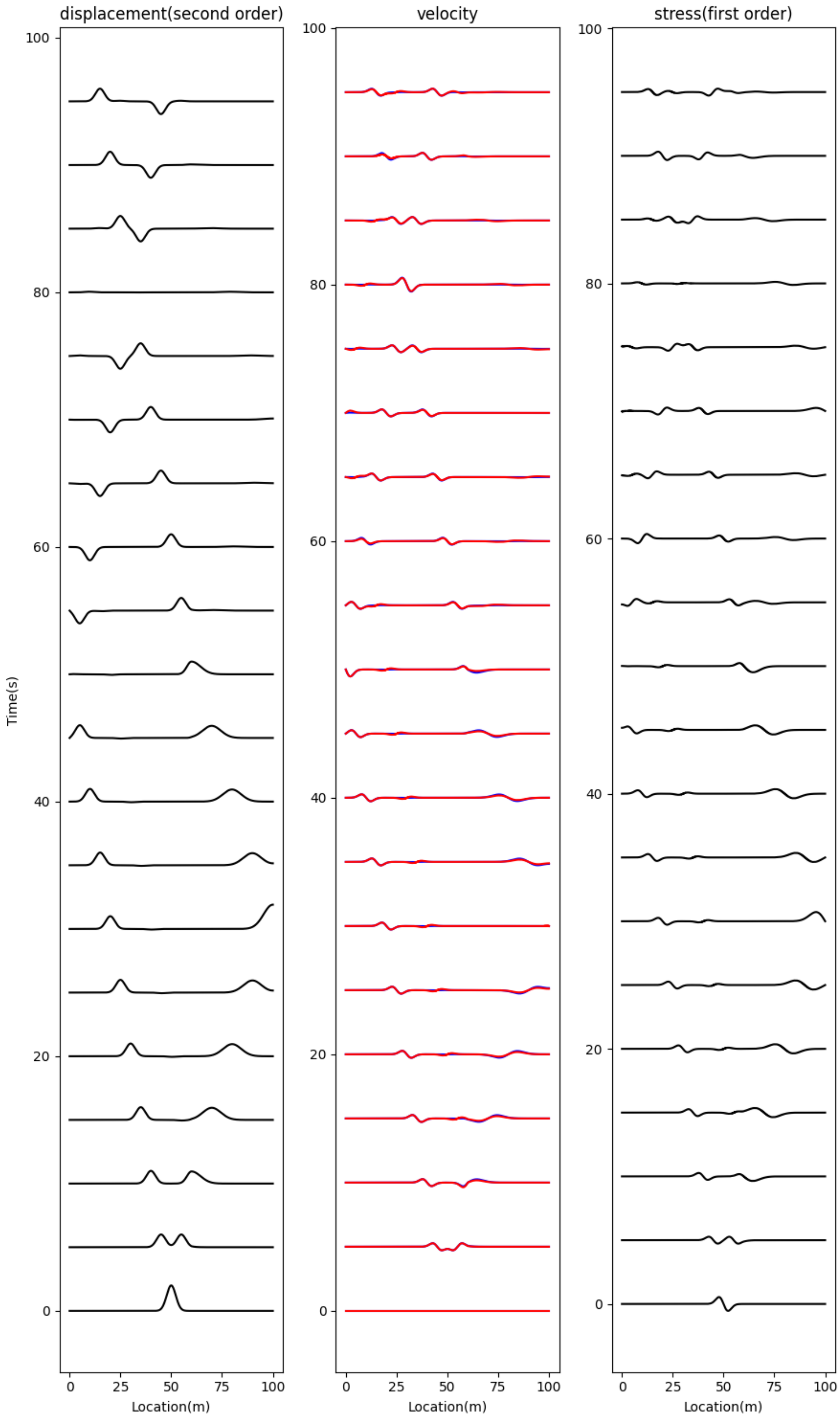

Figure 4: Heterogeneous Medium after the improvement

The third figure is the case of heterogeneous medium, and the fourth figure is the case in the heterogeneous medium with improved second order system.

In the heterogeneous medium, the velocity generated using the second order system is different from the counterpart using the first order system. The first order system is still correct while the second order system isn't. This is because equation (2) should be modified as,

$$\rho(x)\partial_t^2 u(x,t) = \kappa(x)\partial_x^2 u(x,t) + \partial_x \kappa(x)\partial_x u(x,t) \tag{15}$$

In most locations, euqtion (16) exists. However, in the heterogeneity boundary $x = 60m$, it can be improved as,

$$u_i^{n+1} = \frac{c^2\Delta t^2}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \frac{\Delta t^2 \Delta\kappa}{2\rho\Delta x^2}(u_{i+1}^n - u_{i-1}^n) + 2u_i^n - u_i^{n-1} \tag{16}$$

Comparing Figure 3 and Figure 4, there is an apparent improvementent.