Group Theory in Physics

A *not so* Short Introduction

Hsia Mingyu

Hangzhou Dianzi University

Hangzhou in Late Spring 2024

Group Theory in Physics

Contents

This page was intentionally left blank.

1

CHAPTER

The Basis Concepts of Group

1.1 The Introduction to group

Definition 1.1. Group is a mathematical structure, it's a combine of set and binary operation.

For example, we define the *Multiplication Group* $(G, *)$, and G is commonly expressed as Eq. (1.1.1)

$$G = \{e, a_1, a_2, \ldots, a_n\} \tag{1.1.1}$$

A group should meet 4 requirements.

Proposition 1.1 (4 - requirements for a group).

1. A group should have the unit element e. For the *Multiplication Group*, we have $e * a = a * e = a$.

2. A group should satisfy that all the elements in it have the corresponding inverse elements, that is $a^{-1}a = aa^{-1} = e$, while the order of the multiplication of an element and its inverse is exchangeable (we call this commutative). However, most of the time elements are not commutable.

3. Associative law: for three or more elements, we have $abc = (ab)c = a(bc)$.

4. Closure. If $a, b \in G$, we have $ab \in G$.

Following are some examples of group.

Example 1. The addition integer group $(\mathbb{Z}, +)$.

- Unit element: $e = 0$, $e + a = a + e = a$.

- Inverse element: $a^{-1} = -a$, $a^{-1} + a = a + a^{-1} = e = 0$.

- Combine law: $a + b + c = (a + b) + c = a + (b + c)$.

- Closure: If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, the group satisfies $a + b \in \mathbb{Z}$.

Example 2. The multiplication integer group $(\mathbb{Z}, *)$.

- For any element $a \in \mathbb{Z}$, its inverse $a^{-1} = 1/a \notin \mathbb{Z}$. So this is not a group.

Example 3. The multiplication real number group $(\mathbb{R}, *)$.

- Unit element: $e = 1$. Inverse element: $a^{-1} = 1/a$, however 0 has not inverse element.

- So this is not a group. Certainly, $(\mathbb{R}/\{0\}, *)$ is a group.

Example 4. The Hilbert space addition group $(\mathcal{H}, +)$.
For the Hilbert space, there are a set of basis in it. For finite dimension H-space

$$\mathcal{H} = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$$

Any element in it is a linear combination

$$|\psi_i\rangle = \sum_i c_i |\phi_i\rangle$$

So all the elements in $\mathcal{H}$ form a group under addition.

Example 5. Consider a 2nd order ordinary group $G = \{e, a\}$. We use the multiplication **Table 1.1** to acknowledge its structure.

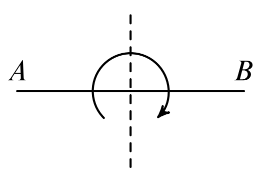
Table 1.1. The multiplication table.

	e	a
e	e	a
a	a	a or e?

- $a * a$ has 2 possibilities: $a * a = a$ or $a * a = e$.
- If $a * a = a$, then we have $a^{-1}a * a = a^{-1}a, a = e$.
- So $a * a = e$, means the 2nd order ordinary group has only one structure, it corresponds to the group $G = \{1, -1\}$.

Now we discuss the relation between the 2nd order group and the geometry.

1. The symmetry of line segments.



- Identity operation: do nothing to the line segment (e).
- Flip operation: rotate the line segment 180° around the center, or mirror the line segment with the central mirror (a).

After we label the line segment, we can write e and a into matrices

$$e = \begin{bmatrix} A & B \\ A & B \end{bmatrix}, \; a = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

While $a * a = e$ means that flip twice is equivalent to do nothing.

2. For the CO_2 molecule, its Hamiltonian has the symmetry of flip, which stands for some degeneracy.

We consider the parity. A 1-dimension Hamiltonian is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

If $V(x) = V(-x)$, then we can define the parity operator P, exists that

$$Pf(x) = f(-x)$$

And we have $P\mathcal{H} = \mathcal{H}P$.

Table 1.2. The multiplication table.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

- We know $a * a$ cannot equal to a. If $a * a = e$, the following two situations are rejected
  - If $a * b = b$, then $b = e$.
  - If $a * b = e = a * a$, then $a = b$
- So we choose $a * a = b$. Symmetrically, $b * b = a$.
- We know ab cannot equal to a or b, so $a * b = b * a = e$.

Example 6. Consider a 3rd order group $G = \{e, a, b\}$. We also use the multiplication **Table 1.2** to acknowledge its structure.

Due to the relation in the 3rd order group $b = a * a$, we can form the 3rd order group by complex numbers, $G = \{1, \omega, \omega^2\}$ while $\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$.

Now, we act the elements of the 3rd order group to a complex number $z = r\mathrm{e}^{\mathrm{i}\theta}$, then we have

$$ez = z = r\mathrm{e}^{\mathrm{i}\theta},\ \omega z = \omega r\mathrm{e}^{\mathrm{i}\theta} = r\mathrm{e}^{\mathrm{i}(\theta+\frac{2\pi}{3})},\ \omega^2 z = \omega^2 r\mathrm{e}^{\mathrm{i}\theta} = r\mathrm{e}^{\mathrm{i}(\theta+\frac{4\pi}{3})}$$

In geometry, the *Graphene's Hexagonal Lattice* has $2\pi/3$ rotational symmetry around its center and the 3rd order group can be expressed as $G = \{1, C_3^1, C_3^2\}$. Here, the point group notation C_n^i means rotate $2\pi\mathrm{i}/n$.

Definition 1.2 (The Isomorphism groups). If elements in two groups are one to one corresponded, then we call the two groups are the Isomorphism groups.

Table 1.3. G_1 and G_2 are isomorphism groups can be considered the same.

$G_1 = \{1, -1\} \backsimeq G_2 = \{e, P\}$	$1 \to e, -1 \to P$
$G_1 = \{1, \omega, \omega^2\} \backsimeq G_2 = \{e, C_3^1, C_3^2\}$	$1 \to e, \omega \to C_3^1, \omega^2 \to C_3^2$

Definition 1.3 (The homomorphism groups). Different from the Isomorphism groups, the homomorphism groups allow 'n to 1', that is several elements in a group can correspond to the same element in another group.

The homomorphism groups also satisfy the closure. Such as if $G \xrightarrow{\varphi} H$ and $\varphi(g_1) = h_1, \varphi(g_2) = h_2$, then there should be $\varphi(g_1 g_2) = h_1 h_2$. The simplest homomorphism group for any group G is $\{e\}$.

Theorem 1.1 (The rearrange theorem). If T is any element from the group $G = \{E, R, S, \ldots\}$, then we can use the closure

$$TG = \{T, TR, TS, \ldots\} = G,\ GT = \{T, RT, ST, \ldots\} = G,\ G^{-1} = \{E, R^{-1}, S^{-1}, \ldots\} = G$$

That is $TG = GT = G^{-1} = G$, which is equivalent to disrupt the elements in the group ant then rearrange them in another order.

Definition 1.4 (The cyclic group). All the elements in the group satisfy

$$G_n = \{e, R, R^2, \ldots, R^{n-1}\}$$

We call R the 2nd generator and $R^n = e$, elements in the cyclic group are commutative.

The 4th group has 2 structures, one is the cyclic group while the other is not. The cyclic 4th group is $G = \{1, \omega, \omega^2, \omega^3\}$ while $\omega = \sqrt[4]{1} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

1.2 The subsets of group

Definition 1.5 (The subgroup). If $H \subset G$, and $(H, *)$ is also a group, then we call H the subgroup of G. Also, the elements in H satisfy

- If $a \in$ H, then $a^{-1} \in H$.
- If $a, b \in H$, then $ab \in H$.

For example, the 4th cyclic group $G = \{1, \omega, \omega^2, \omega^3\}$, its subgroup H can be $\{1, \omega^2\} = \{1, -1\}$. Here, we express the order number of G as $|G|$. Then $|G| = 4$, $|H| = 2$. We can have the following theorem.

Theorem 1.2. The order of the subgroup must be the factor of the order of the original group.

Certainly, for the 3rd order group $G = \{1, \omega, \omega^2\}$, its subgroup can be $\{e\}$ and G. For $H = \{1, \omega\}$, $H = \{1, \omega^2\}$ or $H = \{\omega, \omega^2\}$ are not its subgroup.

Before we proof this theorem, we define the coset first.

Definition 1.6 (The coset). For the group G and its subgroup H, we define any element g from G left multiples H, that is gH, is the left coset of the group H.

$$gH = \{gh_1, gh_2, \dots, gh_n\}$$

Due to the rearrange theorem, if $h_i \in H$, then $h_i H = H$ and $HH = H$.

For the two elements g_i, g_j, which are from G, if $g_i H = g_j H$, then $g_j^{-1} g_i H = H$ and $g_j^{-1} g_i h_i = h_i'$.

Due to $h_i, h_i' \in H$, using the closure we have $g_j^{-1} g_i \in H$, $g_i \in g_j H$. By exchanging the label i and j, we can also obtain that if $g_i H = g_j H$, then $g_j \in g_i H$.

1.3 The homomorphic relationship of group

Problems - 12/03/2024

Problem 1. The group of all symmetry operations on a square is called the D_4 group, write down the multiplication table of the D_4 group.

Solution.

Problem 2. Prove that all the non-zero integers form a group under the multiplication.

Solution. *Proof.* □