Solutions to Problems from Wald's Book "General Relativity"

Lucas Braune[1]

September 29, 2010

[1]E-mail address: lucasbraune@hotmail.com

Contents

Chapter 1

Introduction

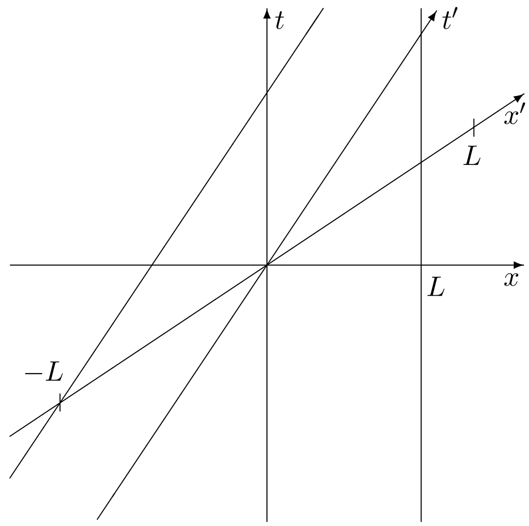
Problem 1

Car and garage paradox: The lack of a notion of absolute simultaneity in special relativity leads to many supposed paradoxes. One of the most famous of these involves a car and a garage of equal proper length. The driver speeds toward the garage, and a doorman at the garage is instructed to slam the door shut as soon as the back end of the car enters the garage. According to the doorman, "the car Lorentz contracted and easily fitted into the garage when I slammed the door." According to the driver, "the garage Lorentz contracted and was too small for the car when I entered the garage." Draw a spacetime diagram showing the above events and explain what really happens. Is the doorman's statement correct? Is the driver's statement correct? For definiteness, assume that the car crashes through the back wall of the garage without stopping or slowing down.

Solution

Let $c = 1$. The spacetime diagram can be found below. In it the primed coordinates are those assigned to events by the driver and the unprimed ones are those assigned by the doorman. At the origin is the event "the driver has just reached the doorman and is about to enter the garage". It follows that the world lines of the driver and doorman are the t'-axis and the t-axis, respectively.

Let L denote the common proper length of the car and the garage. The world line of the back wall of the garage is thus the one parallel to the t-axis and passing through the point $(x, t) = (L, 0)$. Similarly, the world line of the rear end of the car is the one parallel to the t'-axis and passing through $(x', t') = (-L, 0)$.



A justification for the placing of the point $(x', t') = (L, 0)$ where it is in the diagram given the position of $(x, t) = (L, 0)$ is that the interval between each of these points and the origin is $L^2 - 0^2 = L^2$. So they lie on the hyperbole $x^2 - t^2 = L^2$ (doorman coordinates). The intersection of this hyperbole's branches and the x'-axis are the points $(x', t') = (L, 0)$ and $(x', t') = (-L, 0)$ indicated on the diagram.

Analyzing the spacetime diagram, one concludes that both statements are correct.

The intersection of the $t' = 0$ line with the world line of the back wall of the garage is at a smaller value of x' than L. This agrees with the driver's account.

Let t_{slam} be time coordinate recorded by the doorman as he slams the door shut. This event is at the intersection of the world lines of the doorman and that of the car's rear end. The line $t = t_{\text{slam}}$ intersects the world line of the driver at a smaller value of x than that of the back wall of the garage. This agrees with the doorman's statement.

So what happens here? The driver is correct to say he had crashed through the back wall of the garage by the time the doorman shuts the door. The doorman is also correct when he says the the back wall was intact by the time he closed the door. This seems to be a contradiction since from both statements one would conclude that the car would and would have not crashed by the time the door was closed. It turns out this conclusion would be wrong. This is because the expression "by the time the door was closed" means different things to the driver and the doorman. To the driver, it refers to a line parallel to the x'-axis. To the doorman, the expression refers to those events in a line parallel to the x-axis.

Chapter 2

Manifolds and Tensor Fields

Problem 1

(a) Show that the overlap functions $f_i^{\pm} \circ (f_j^{\pm})^{-1}$ are C^∞, thus completing the demonstration given in section 2.1 that S^2 is a manifold.

(b) Show by explicit construction that two coordinate systems (as opposed to the six used in the text) suffice to cover S^2. (It is impossible to cover S^2 with a single chart, as follows from the fact S^2 is compact, but every open subset of $\mathbb{R}^2$ is noncompact; see appendix A.)

Solution of (a)

This is done by finding an expression for $f_i^{\pm} \circ (f_j^{\pm})^{-1}$ and identifying it as C^∞. Take for example the case $i = 2$, $j = 3$ with both signs being '+'. The functions $f_i^{\pm}$ act as projections of the $\pm x^i > 0$ portion of the sphere into the $x^i = 0$ plane, so that

$$f_2^+(x^1, x^2, x^3) = (x^1, x^3)$$
$$f_3^+(x^1, x^2, x^3) = (x^1, x^2) \ ,$$

It follows that

$$(f_3^+)^{-1}(x^1, x^2) = (x^1, x^2, \sqrt{1 - (x^1)^2 - (x^2)^2}) \ .$$

It is clear that the composition $f_2^+ \circ (f_3^+)^{-1}$ is C^∞.

Solution of (b)

A couple of stereographic projections which omit different points from the sphere will do. Books on Complex Analysis usually include very nice descriptions of the stereographic projection.

Problem 2

Prove that any smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be written in the form equation (2.2.2). (Hint: For $n = 1$, use the identity

$$F(x) - F(a) = (x-a)\int_0^1 F'[t(x-a)+a]dt \ ;$$

then prove it for general n by induction.)

Solution

I didn't follow the Hint exactly to do this. Let γ be the path $t \longmapsto (1-t)a + tx$, where $t \in [0,1]$. The desired result follows from considering the following integrals.

$$\begin{aligned} F(x) - F(a) &= \int_0^1 \frac{d}{dt}(F \circ \gamma(t))dt \\ &= \int_0^1 \nabla F((1-t)a + tx) \cdot (x-a)dt \\ &= \sum_{\mu=1}^{n} (x^\mu - a^\mu) \underbrace{\int_0^1 \frac{\partial F}{\partial x^\mu}((1-t)a + tx)dt}_{H_\mu(x)} \ . \end{aligned}$$

Problem 3

(a) Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field.

(b) Let X, Y, Z be smooth vector fields on a manifold M. Verify that their commutator satisfies the Jacobi identity:

$$[[X,Y],Z] + [[Y,Z],X] + [Z,X],Y] = 0 \ .$$

(c) Let $Y_1, \ldots, Y_n$ be smooth vector fields on an n-dimensional manifold M such that at each $p \in M$ they form a basis of the tangent space V_p. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ by

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma{}_{\alpha\beta} Y_\gamma \ .$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$. (This equation is a useful algebraic relation if the $C^\gamma{}_{\alpha\beta}$ are constants, as will be the case if $Y_1, \ldots, Y_n$ are left [or right] invariant vector fields on a Lie group [see section 7.2].)

Solution of (a)

First we check linearity. Let u and v be vector fields, f and g be functions and a and b be real numbers.

$$\begin{aligned}[v,w](\alpha f+\beta g) &= v[w(\alpha f+\beta g)] - w[v(\alpha f+\beta g)] \\ &= \alpha\ v[w(f)] + \beta\ v[w(g)] - \alpha\ w[v(f)] - \beta\ w[v(g)] \\ &= \alpha[v,w](f) + \beta[v,w](g)\end{aligned}$$

Next we check the Leibniz property.

$$\begin{aligned}[v,w](fg) =& v[w(fg)] - w[v(fg)] \\ =& \cancel{v(f)w(g)} + f\ v[w(g)] + \cancel{v(g)w(f)} + g\ v[w(f)] - \\ &- \cancel{w(f)v(g)} - f\ w[v(g)] - \cancel{w(g)v(f)} - g\ w[v(f)] \\ =& f\ [v,w](g) + g\ [v,w](f)\end{aligned}$$

Solution of (b)

This is just a boring computation. Let us compute the action of $[[X,Y],Z]$ on a function f.

$$\begin{aligned}[[X,Y],Z](f) &= [X,Y]\{Z(f)\} - Z\{[X,Y](f)\} \\ &= X[Y\{Z(f)\}] - Y[X\{Z(f)\}] - Z[X\{Y(f)\}] + Z[Y\{X(f)\}]\end{aligned}$$

The formula for $[[Z,X],Y](f)$ is the same as the above, but with the substitutions

$$X \longrightarrow Z\ , \quad Y \longrightarrow X\ , \quad Z \longrightarrow Y\ .$$

The formula for $[[Y,Z],X]$ is obtained the same way. The proof of the Jacobi identity is completed by taking the formulas corresponding to each term in the cyclic sum and checking that they add to zero.

Solution of (c)

I'm not sure what equation Wald expected us to prove. The one I will derive follows from the Jacobi identity applied to the fields Y_α, Y_β and Y_δ.

First note that if v and w are vector fields and f is a function, then $[f\ v,w] = f[v,w] - w(f)v$. It is easy to check that both sides of this equation applied to any function yield the same result.

Next we obtain a formula for the terms involved in the Jacobi identity.

$$\begin{aligned}[[Y_\alpha,Y_\beta],Y_\delta] &= \sum_\gamma [C^\gamma{}_{\alpha\beta}Y_\gamma, Y_\delta] \\ &= \sum_\gamma C^\gamma{}_{\alpha\beta}[Y_\gamma,Y_\delta] - Y_\delta(C^\gamma{}_{\alpha\beta})[Y_\alpha,Y_\beta] \\ &= \sum_{\gamma,\omega} C^\gamma{}_{\alpha\beta}C^\omega{}_{\gamma\delta}Y_\omega - Y_\delta(C^\gamma{}_{\alpha\beta})C^\omega{}_{\alpha\beta}Y_\omega\end{aligned}$$

Taking a cyclic sum over α, β and δ and equating the result to zero, one finds that the coefficient of each Y_ω vanishes, that is

$$\begin{aligned}\sum_\gamma \{ & C^\gamma{}_{\alpha\beta} C^\omega{}_{\gamma\delta} + C^\gamma{}_{\delta\alpha} C^\omega{}_{\gamma\beta} + C^\gamma{}_{\beta\delta} C^\omega{}_{\gamma\alpha} \\ & - Y_\delta(C^\gamma{}_{\alpha\beta}) C^\omega{}_{\alpha\beta} - Y_\alpha(C^\gamma{}_{\beta\delta}) C^\omega{}_{\beta\delta} - Y_\beta(C^\gamma{}_{\delta\alpha}) C^\omega{}_{\gamma\beta} \} = 0 \ .\end{aligned}$$

If the $C^\gamma{}_{\alpha\beta}$ terms are constant, the last 3 terms vanish.

Problem 4

(a) Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by

$$[v, w]^\mu = \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \ .$$

(b) Let $Y_1, \ldots, Y_n$ be as in problem 3(c). Let $Y^{1^*}, \ldots, Y^{n^*}$ be the dual basis. Show that the components $(Y^{\gamma^*})_\mu$ of Y^{γ^*} in any coordinate basis satisfy

$$\frac{\partial (Y^{\gamma^*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma^*})_\nu}{\partial x^\mu} = \sum_{\alpha,\beta} C^\gamma{}_{\alpha\beta} (Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu \ .$$

(Hint: Contract both sides with $(Y_\sigma)^\mu (Y_\rho)^\nu$.)

Solution of (a)

Even though Wald does not use the Einstein summation convention in his book, I will use it in this problem. Moreover, I will use ∂_ν as shorthand for $\partial/\partial x^\nu$.

Let $\{X_\mu\} = \{\partial/\partial x^\mu\}$ be a coordinate basis.

$$\begin{aligned}[v, w](f) =& [v^\mu X_\mu, w^\nu X_\nu](f) \\ =& v^\mu X_\mu [w^\nu X_\nu(f)] - w^\nu X_\nu [v^\mu X_\mu(f)] \\ =& v^\mu \{X_\mu(w^\nu) X_\nu(f) + \cancel{w^\nu X_\mu X_\nu(f)}\} - \\ & - w^\nu \{X_\nu(v^\mu) X_\mu(f) + \cancel{v^\mu X_\nu X_\mu(f)}\} \\ =& \{v^\nu \partial_\nu - w^\nu \partial_\nu v^\mu\} X_\mu(f)\end{aligned}$$

The terms on the third line canceled because of the equality of mixed partial derivatives for smooth functions in $\mathbb{R}^n$. The last line is the desired result.

Solution of (b)

In order to use the Hint, we define two tensor fields T_{ab} and S_{ab} by

$$T_{ab} = \{\partial_\nu (Y^{\gamma^*})_\mu - \partial_\mu (Y^{\gamma^*})_\nu\}(dx^\mu)_a (dx^\nu)_b \ ,$$
$$S_{ab} = \{C^\gamma{}_{\alpha\beta} (Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu\}(dx^\mu)_a (dx^\nu)_b \ .$$

Now we show that $T_{ab}(Y_\sigma)^a(Y_\rho)^b = S_{ab}(Y_\sigma)^a(Y_\rho)^b$ for all σ and ρ. Because the vector fields $Y_1, \ldots, Y_n$ form basis for each tangent space, it follows that T_{ab} and S_{ab} coincide. This implies the desired result since the expression of a tensor of type $(0,2)$ as a sum of terms $(dx^\mu)_a(dx^\nu)_b$ is unique.

First we compute $S_{ab}(Y_\sigma)^a(Y_\rho)^b$.

$$\begin{aligned} S_{ab}(Y_\sigma)^a(Y_\rho)^b &= C^\gamma{}_{\alpha\beta}(Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu (Y_\sigma)^\mu (Y_\rho)^\nu \\ &= C^\gamma{}_{\alpha\beta} Y^{\alpha^*}(Y_\sigma)\ Y^{\beta^*}(Y_\rho) \\ &= C^\gamma{}_{\sigma\rho} \ . \end{aligned}$$

The last equation follows from $Y^{\beta^*}(Y_\rho) = \delta^\beta{}_\rho$. This of course implies

$$0 = \partial_\nu\{(Y^{\beta^*})_\mu (Y_\rho)^\mu\} = (Y^{\beta^*})_\mu \partial_\nu (Y_\rho)^\mu + (Y_\rho)^\mu \partial_\nu (Y^{\beta^*})_\mu \ .$$

We use this relation (in the second equation below) to show that $T_{ab}(Y_\sigma)^a(Y_\rho)^b$ also equals $C^\gamma{}_{\sigma\rho}$.

$$\begin{aligned} T_{ab}(Y_\sigma)^a(Y_\rho)^b &= (Y_\rho)^\nu (Y_\sigma)^\mu \partial_\nu (Y^{\gamma^*})_\mu - (Y_\sigma)^\mu (Y_\rho)^\nu \partial_\mu (Y^{\gamma^*})_\nu \\ &= -(Y_\rho)^\nu (Y^{\gamma^*})_\mu \partial_\nu (Y_\sigma)^\mu + (Y_\sigma)^\mu (Y^{\gamma^*})_\nu \partial_\mu (Y_\rho)^\nu \\ &= (Y^{\gamma^*})_\nu \{(Y_\sigma)^\mu \partial_\mu (Y_\rho)^\nu - (Y_\rho)^\mu \partial_\mu (Y_\sigma)^\nu\} \\ &= (Y^{\gamma^*})_\nu [Y_\sigma, Y_\rho]^\nu \\ &= C^\gamma{}_{\sigma\rho} \end{aligned}$$

Problem 5

Let $Y_1, \ldots, Y_n$ be smooth vector fields on an n-dimensional manifold M which form a basis of V_p at each $P \in M$. Suppose $[Y_\alpha, Y_\beta] = 0$ for all α,β. Prove that in a neighborhood of each $p \in M$ there exist coordinates $y_1, \ldots, y_n$ such that $Y_1, \ldots, Y_n$ are the coordinate vector fields $Y_\mu = \partial/\partial y^\mu$. (Hint: In an open ball of $\mathbb{R}^n$, the equations $\partial f/\partial x^\mu = F_\mu$ with $\mu = 1, \ldots, n$ for the unknown functions f have a solution if and only if $\partial F_\mu/\partial x^\nu = \partial F_\nu/\partial x^\mu$. [See the end of section B.1 of appendix B for a statement of generalizations of this result.] Use this fact together with the results of problem 4(b) to obtain the new coordinates.)

Solution

Let $p \in M$ be an arbitrary point in spacetime. We wish to construct a coordinate system about p which satisfies a certain condition. To do so we must begin with an arbitrary one $\psi : O \longrightarrow U \subset \mathbb{R}^n$ and then change variables in U so as to construct a new coordinate system which satisfies the desired condition. This was (somewhat) clear to me when I first did this problem. But how I was supposed to come up with the change of variables was not. Because this problem is of course solvable, I just kept putting the pieces from the previous problems together kind of randomly until the following solution came to be.

Problem 3(c) shows that, because $[Y_\alpha, Y_\beta] = 0$, the coefficients $C^\gamma{}_{\alpha\beta}$ must vanish. By problem 4(b), this implies that

$$\frac{\partial (Y^{\gamma^*})_\mu}{\partial x^\nu} = \frac{\partial (Y^{\gamma^*})_\nu}{\partial x^\mu} \ .$$

Here $(x^1, \ldots, x^n)$ are the coordinates associated with ψ. We now use the mathematical fact described in the Hint (which goes by the name of Poincaré Lemma). It guarantees the existence of a function $y^\gamma : U \longrightarrow \mathbb{R}$ such that

$$\frac{\partial y^\gamma}{\partial x^\sigma} = (Y^{\gamma^*})_\sigma$$

provided that U is an open ball. This is an assumption which we can and will make. Define $F : U \longrightarrow \mathbb{R}^n$ by $F(x^1, \ldots, x^n) = (y^1, \ldots, y^n)$. The claim is that F is a diffeomorphism (change of variables) and that $F \circ \psi$ is the desired coordinate system.

The tool to be used to show that F is diffeomorphism is the Inverse Function Theorem. It states that F will be a diffeomorphism provided that $\det DF(\psi(p)) \neq 0$ and that we sufficiently restrict F's domain to a smaller neighborhood of $\psi(p)$ if necessary. Now

$$\det DF = \det \begin{bmatrix} (Y^{1^*})_1 & \ldots & (Y^{1^*})_n \\ \vdots & & \vdots \\ (Y^{n^*})_1 & \ldots & (Y^{n^*})_n \end{bmatrix} \neq 0$$

because the vectors $Y^{1^*}, \ldots, Y^{n^*}$ are linearly independent. Restricting the domain if necessary, we get that F is a diffeomorphism.

Finally, we show that $F \circ \psi$ is the desired coordinate system.

$$\begin{aligned} Y_\alpha &= \sum_\mu (Y_\alpha)^\mu \frac{\partial}{\partial x^\mu} \\ &= \sum_{\mu,\nu} (Y_\alpha)^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \end{aligned}$$

$$= \sum_{\mu,\nu} (Y_\alpha)^\mu (Y^{\nu^*})_\mu \frac{\partial}{\partial y^\nu}$$
$$= \sum_\nu \delta^\nu{}_\alpha \frac{\partial}{\partial y^\nu} = \frac{\partial}{\partial y^\alpha}$$

Problem 6

(a) Verify that the dual vectors $\{v^{\mu^*}\}$ defined by equation (2.3.1) constitute a basis of V^*.

(b) Let $v_1, \ldots, v_n$ be a basis of the vector space V and let $v^{1^*}, \ldots, v^{n^*}$ be the dual basis. Let $w \in V$ and let $\omega \in V^*$. Show that

$$w = \sum_\alpha v^{\alpha^*}(w) v_\alpha \ ,$$
$$\omega = \sum_\alpha \omega(v_\alpha) v^{\alpha^*} \ .$$

(c) Prove that the operation of contraction, equation (2.3.2), is independent of the choice of basis.

Solution of (a)

The set $\{v^{\mu^*}\}$ is linearly independent: Suppose $T = \alpha_1 v^{1^*} + \cdots + \alpha_n v^{n^*} = 0$. It follows that $T(v_i) = \alpha_i = 0$ for all i.

Also, the vectors $\{v^{\mu^*}\}$ span V^*: Let S be any linear functional in V^*. Then S and $\sum_\alpha S(v_\alpha) v^{\alpha^*}$ agree on the basis $\{v_1, \ldots, v_n\}$ of V. It follows that they are the same linear functional.

Please note that "linear functional", "dual vector" and "element of V^*" are different names for the same object.

Solution of (b)

Write $w = c_1 v_1 + \cdots + c_n v_n$. Applying v^{α^*} to both sides of this equation yields $v^{\alpha^*}(w) = c_\alpha$. This proves the first equation. The second follows from the fact that both sides agree on a basis of V.

Solution of (c)

Let $\{v_\mu\}$ and $\{w_\nu\}$ be bases for V; let T be any tensor defined of V. We wish to show that contracting T using one basis or the other yields the same result.

$$\begin{aligned}
\sum_{\mu=1}^{n} T(\ldots, w^{\mu^*}, \ldots; \ldots, w_\mu, \ldots) &= \sum_{\mu,\sigma,\gamma} T(\ldots, w^{\mu^*}(v_\sigma)v^{\sigma^*}, \ldots; \ldots, v^{\gamma^*}(w_\mu)v_\gamma, \ldots) \\
&= \sum_{\mu,\sigma,\gamma} w^{\mu^*}(v_\sigma)v^{\gamma^*}(w_\mu)T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\gamma, \ldots) \\
&= \sum_{\mu,\sigma,\gamma} v^{\gamma^*}(w^{\mu^*}(v_\sigma)w_\mu)T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\gamma, \ldots) \\
&= \sum_{\sigma,\gamma} v^{\gamma^*}(v_\sigma)T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\gamma, \ldots) \\
&= \sum_{\sigma,\gamma} \delta^\gamma{}_\sigma T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\gamma, \ldots) \\
&= \sum_{\sigma} T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\sigma, \ldots)
\end{aligned}$$

Problem 7

Let V be an n-dimensional vector space and let g be a metric on V.

(a) Show that one always can find an orthonormal basis $v_1, \ldots, v_n$ of V,i.e., a basis such that $g(v_\alpha, v_\beta) = \pm\delta_{\alpha\beta}$. (Hint: Use induction.)

(b) Show that the signature of g is independent of the choice of orthonormal basis.

Solution

I don't know how to do this without having to reproduce here a few definitions and propositions of Linear Algebra. Instead of doing so, I refer the reader to section 2 in chapter 7 of Michael Artin's book "Algebra". The solutions to items (a) and (b) are given in Proposition (2.9) and Theorem (2.11), respectively, though in a slightly more general form. These two theorems build on Propositions (2.2) and (2.4), and also on the concept of the direct sum of vector subspaces. A discussion of direct sums which is sufficient for our purposes can be found on the first page of section 6 in chapter 3.

Artin considers symmetric bilinear forms in general in (2.9) and (2.11), dropping the requirement of nondegeneracy. To apply these results to the case of a metric, we show that there can be no self-orthogonal (null) vector in a orthonormal basis of a space V with metric g. Let $v_1, \ldots, v_n$ be a orthonormal basis. If v_i were self-orthogonal, then the linear functional $g(v_i, \ \cdot\ )$ would vanish on a basis of V, contradicting the nondegeneracy of g.

Problem 8

(a) The metric of flat, three-dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2 \ .$$

Show that the metric components $g_{\mu\nu}$ in spherical polar coordinates r, θ, ϕ defined by

$$\begin{aligned} r &= (x^2 + y^2 + z^2)^{1/2} \ , \\ \cos\theta &= z/r \ , \\ \tan\phi &= y/x \end{aligned}$$

is given by

$$ds^2 = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ .$$

(b) The spacetime metric of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ .$$

Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates," defined by

$$\begin{aligned} t' &= t \ , \\ x' &= (x^2 + y^2)^{1/2} \cos(\phi - \omega t) \ , \\ y' &= (x^2 + y^2)^{1/2} \sin(\phi - \omega t) \ , \\ z' &= z \ , \end{aligned}$$

where $\tan\phi = y/x$.

Solution of (a)

Let $(x^1, \ldots, x^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$ be coordinates on some manifold M. According to the tensor transformation law (2.3.7), one has

$$dx^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^1} \, d\tilde{x}^1 + \cdots + \frac{\partial x^\mu}{\partial \tilde{x}^n} \, d\tilde{x}^n \ , \tag{2.1}$$

as should be.

We apply this formula in the case where $(x^1, x^2, x^3) = (x, y, z)$ and $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (r, \theta, \phi)$. To compute the derivatives involved, we use the formulas

$$\begin{aligned} x &= r \cos\phi \sin\theta \ , \\ y &= r \sin\phi \sin\theta \ , \\ z &= r \cos\theta \ . \end{aligned}$$

The resulting equations are

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial r}\ dr + \frac{\partial x}{\partial \theta}\ d\theta + \frac{\partial x}{\partial \phi}\ d\phi \\
&= \cos\phi \sin\theta\ dr + r\cos\phi\cos\theta\ d\theta - r\sin\phi\sin\theta\ d\phi\ , \\
dy &= \sin\phi\sin\theta\ dr + r\sin\phi\cos\theta\ d\theta + r\cos\phi\sin\theta\ d\phi\ , \\
dz &= \cos\theta\ dr - r\sin\theta\ d\theta\ .
\end{aligned}
$$

We now expand $dx^2 + dy^2 + dz^2$ using these expressions and find

$$dr^2 + r^2\ d\theta^2 + r^2 \sin^2\theta\ d\phi^2\ .$$

Solution of (b)

This item can be solved by proceeding as in the solution of item (a). To compute the relevant derivatives, we use the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega t' & -\sin\omega t' \\ \sin\omega t' & \cos\omega t' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}\ .$$

One finds

$$
\begin{aligned}
dt &= dt'\ , \\
dx &= [-\omega x'\sin\omega t' - \omega y'\cos\omega t']\ dt' + \cos\omega t'\ dx' - \sin\omega t'\ dy'\ , \\
dy &= [\omega x'\cos\omega t' - \omega y'\sin\omega t']\ dt' + \sin\omega t'\ dx' + \cos\omega t'\ dy'\ , \\
dz &= dz'\ .
\end{aligned}
$$

From this, we compute

$$
\begin{aligned}
-dt^2 + dx^2 + dy^2 + dz^2 &= (-1 + \omega^2 x'^2 + \omega^2 y'^2)\ dt'^2 + dx'^2 + dy'^2 + dz'^2 \\
&\quad -\omega y'(dt'\ dx' + dx'\ dt') + \omega x'(dt'\ dy' + dy'\ dt')\ .
\end{aligned}
$$

The components $g_{\mu\nu}$ of the metric in rotating coordinates are given in the last equation above. The components of the inverse metric can now be found using the relation $\left[g^{\mu\nu}\right] = \left[g_{\mu\nu}\right]^{-1}$. (This is supposed to be read as a matrix equation).

$$
\begin{aligned}
\left[g^{\mu\nu}\right] &= \begin{bmatrix} \omega^2(x'^2 + y'^2) - 1 & -\omega y' & \omega x' & 0 \\ -\omega y' & 1 & 0 & 0 \\ \omega x' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & -\omega y' & \omega x' & 0 \\ -\omega y' & 1 - \omega^2 y'^2 & \omega^2 x' y' & 0 \\ \omega x' & \omega^2 x' y' & 1 - \omega^2 x'^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

The matrix inversion can be easily done using Crammer's rule.

Chapter 3

Curvature

Problem 1

Let property (5) (the "torsion free" condition) be dropped from the definition of derivative operator ∇_a in section 3.1.

(a) Show that there exists a tensor $T^c{}_{ab}$ (called the *torsion tensor*) such that for all smooth functions, f, we have $\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T^c{}_{ab} \nabla_c f$. (Hint: Repeat the derivation of eq. [3.1.8], letting $\tilde{\nabla}_a$ be a torsion-free derivative operator.)

(b) Show that for any smooth vector fields X^a, Y^a we have

$$T^c{}_{ab} X^a Y^b = X^a \nabla_a Y^c - Y^a \nabla_a X^c - [X, Y]^c \ .$$

(c) Given a metric, g_{ab}, show that there exists a unique derivative operator ∇_a with torsion $T^c{}_{ab}$ such that $\nabla_c g_{ab} = 0$. Derive the analog of equation (3.1.29), expressing this derivative operator in terms of an ordinary derivative ∂_a and $T^c{}_{ab}$.

Solution of (a)

Let $\tilde{\nabla}_a$ be some torsion free derivative operator. There is no need to repeat here the derivation of eq. (3.1.8), since the one in text uses nowhere that the operators ∇_a and $\tilde{\nabla}_a$ satisfy property (5). So we have

$$\nabla_a \nabla_b f = \tilde{\nabla}_a \tilde{\nabla}_b f - C^c{}_{ab} \nabla_c f$$

for some tensor field $C^c{}_{ab}$. It follows that

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = \cancel{\tilde{\nabla}_a \tilde{\nabla}_b f - \tilde{\nabla}_b \tilde{\nabla}_a f} - (C^c{}_{ab} - C^c{}_{ba}) \nabla_c f \ .$$

This is the desired result with $T^c{}_{ab} = C^c{}_{ab} - C^c{}_{ba}$.

Solution of (b)

To do this we use property (4): For all vector fields X^a and functions f, one has $X(f) = X^a \nabla_a f$.

$$\begin{aligned}
[X,Y](f) &= X^a\nabla_a(Y^b\nabla_b f) - Y^a\nabla_a(X^b\nabla_b f) \\
&= X^a\nabla_a Y^b\nabla_b f + X^a Y^b\nabla_a\nabla_b f - Y^a\nabla_a X^b\nabla_b f - Y^a X^b\nabla_a\nabla_b f \\
&= (X^a\nabla_a Y^b - Y^a\nabla_a X^b)\nabla_b f + X^a Y^b(\nabla_a\nabla_b f - \nabla_b\nabla_a f) \\
&= [X^a\nabla_a Y^c - Y^a\nabla_a X^c](f) - T^c{}_{ab}X^a Y^b(f)
\end{aligned}$$

This shows that both sides of the equation we wish to prove applied on an arbitrary function yield the same result. Because tangent vectors are maps of functions into numbers, the above derivation is in fact a proof that the equation holds.

Solution of (c)

Suppose one has a derivative operator ∇_a with torsion $T^c{}_{ab}$ such that $\nabla_c g_{ab} = 0$. We show that ∇_a is unique.

The first step is to note that equation (3.1.14) continues to hold even if the torsion free condition is dropped from the definition of a derivative operator. Then we proceed as in the proof of Theorem 3.1.1. Let ∂_a be an ordinary derivative operator. We have

$$0 = \nabla_a g_{bc} = \partial_c g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{db}\ .$$

This is the same as

$$\partial_a g_{bc} = C_{cab} + C_{bac}\ . \tag{3.1}$$

By index substitution,

$$\partial_b g_{ac} = C_{cba} + C_{abc}\ , \tag{3.2}$$
$$\partial_c g_{ab} = C_{bca} + C_{acb}\ . \tag{3.3}$$

By adding equations (3.1) and (3.2) and then subtracting equation (3.3), we find

$$\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} = C_{cab} + C_{bac} + C_{cba} + C_{abc} - C_{bca} - C_{acb}\ .$$

We now use the relation $T^c{}_{ab} = C^c{}_{ab} - C^c{}_{ba}$ proved at the end of the solution of item (a). This relation implies

$$\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} = T_{bac} + T_{abc} + T_{cab} + 2C_{cba}\ ,$$

or

$$C_{cba} = \frac{1}{2}\{\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} - T_{bac} - T_{abc} - T_{cab}\}\ . \tag{3.4}$$

Because the tensor $C^c{}_{ab}$ determines ∇_a, this completes the proof of the uniqueness. On the other hand, we can use equations (3.4) and (3.1.14) to define a derivative operator with the desired property, so that the above derivation also settles the question of existence.

Problem 2

Let M be a manifold with metric g_{ab} and associated derivative operator ∇_a. A solution of the equation $\nabla_a \nabla^a \alpha = 0$ is called a *harmonic function.* In the case where M is a two-dimensional manifold, let α be harmonic and let ϵ_{ab} be an antisymmetric tensor field satisfying $\epsilon_{ab}\epsilon^{ab} = 2(-1)^s$, where s is the number of minuses occurring in the signature of the metric. Consider the equation $\nabla_a \beta = \epsilon_{ab} \nabla^b \alpha$.

(a) Show that the integrability conditions (see problem 5 of chapter 2 or appendix B) for this equations are satisfied, and thus, locally, there exists a solution, β. Show that β is also harmonic, $\nabla_a \nabla^a \beta = 0$. ($\beta$ is called the harmonic function *conjugate* to α.)

(b) By choosing α and β as coordinates, show that the metric takes the form
$$ds^2 = \pm\Omega^2(\alpha, \beta)[d\alpha^2 + (-1)^s \, d\beta^2] \ .$$

Solution of (a)

In order to make progress in this problem, it is necessary to know $\nabla_c \epsilon_{ab} = 0$. This is the two-dimensional case of equation (B.2.11). A derivation of this case follows. From $\epsilon_{ab}\epsilon^{ab} = 2(-1)^s$, one finds $\epsilon^{ab}\nabla_c \epsilon_{ab} = 0$. This equation reads in any coordinate system as
$$\epsilon^{12}\nabla_\sigma \epsilon_{12} + \epsilon^{21}\nabla_\sigma \epsilon_{21} = 2\epsilon^{12}\nabla_\sigma \epsilon_{12} = 0 \ .$$
Because $\epsilon^{12} \neq 0$, the desired result, $\nabla_\sigma \epsilon_{12} = 0$, follows.

The Poincaré Lemma states that equation $(d\beta)_a = \nabla_a \beta = \epsilon_{ab}\nabla^b \alpha$ locally has a solution β if and only if
$$d(\epsilon_{ce}\nabla^e \alpha)_{ab} = \nabla_a(\epsilon_{bc}\nabla^c \alpha) - \nabla_b(\epsilon_{ac}\nabla^c \alpha) = 0 \ . \tag{3.5}$$
The exterior derivative operator d and the Poincaré Lemma are introduced and very briefly discussed (two pages) in section 1 of appendix B. Please note the definitions of $(d\beta)_a$ as the dual of $(\partial/\partial\beta)^a$ and as the exterior derivative of the function β agree. Equation (2.1) from problem 8 of chapter 2 shows that
$$(d\beta)_a^{\text{dual}} = \partial_a \beta = \nabla_a \beta = (d\beta)_a^{\text{exterior}} \ ,$$
where ∂_a is the derivative operator associated with some coordinate system.

We use the fact that α is harmonic to show that the integrability condition (3.5) holds. The harmonicity of α is expressed in coordinates as

$$\nabla_1 \nabla^1 \alpha + \nabla_2 \nabla^2 \alpha = 0 \ .$$

We now compute

$$\begin{aligned}
\nabla_1(\epsilon_{2\sigma}\nabla^\sigma \alpha) &= \epsilon_{2\sigma}\nabla_1\nabla^\sigma\alpha \\
&= \epsilon_{21}\nabla_1\nabla^1\alpha \ , \\
\nabla_2(\epsilon_{1\sigma}\nabla^\sigma \alpha) &= \epsilon_{12}\nabla_2\nabla^2\alpha \\
&= (-\epsilon_{21})(-\nabla_1\nabla^1\alpha) \ .
\end{aligned}$$

This shows $d(\epsilon_{ce}\nabla^e\alpha)_{12} = 0$, which implies the desired result $d(\epsilon_{ce}\nabla^e\alpha) = 0$.

Finally, β is also harmonic.

$$\nabla^a \nabla_a \beta = \nabla^a(\epsilon_{ab}\nabla^b\alpha) = \epsilon^{ab}\nabla_a\nabla_b\alpha = 0$$

The equation $\epsilon^{ab}\nabla_a\nabla_b\alpha = 0$ holds because the tensor ϵ^{ab} is antisymmetric while $\nabla_a\nabla_b\alpha$ is symmetric.

Solution of (b)

Instead of computing the coefficients of the metric in the coordinates (α, β) directly, we first find those of the inverse metric g^{ab}. These coefficients can then be used to show that the metric takes the desired form in (α, β) coordinates.

It is not hard to see that the off-diagonal coefficients $g^{\alpha\beta} = g^{\beta\alpha}$ vanish:

$$\begin{aligned}
g^{\alpha\beta} &= g^{ab}(d\alpha)_a(d\beta)_b \\
&= \nabla^b\alpha\nabla_b\beta \\
&= \nabla^b\alpha \ \epsilon_{bc}\nabla^c\alpha \\
&= 0 \ .
\end{aligned}$$

The expression in the third line vanishes because $\nabla^b\nabla^c\alpha$ is symmetric while ϵ_{bc} is antisymmetric.

It remains to show that $g^{\beta\beta} = (-1)^s g^{\alpha\alpha}$. From this it follows that

$$ds^2 = \frac{1}{g^{\alpha\alpha}}[d\alpha^2 + (-1)^s d\beta^2] \ ,$$

since the matrix with components of the metric is the inverse of the matrix with components of the inverse metric. Writing $1/g^{\alpha\alpha}$ as $\pm\Omega^2$ is the same as saying that $g^{\alpha\alpha}$ does not change sign in M. Because $g^{\alpha\alpha}$ is continuous and never zero, this is the case if one makes the hypothesis that M is connected. I don't think this hypothesis can be dropped.

We show $g^{\beta\beta} = (-1)^s g^{\alpha\alpha}$ by working with (α, β) components. In these coordinates the relation $(d\beta)_a = \epsilon_{ab}(d\alpha)^b$ is expressed as

$$(d\beta)_\alpha = \sum_{\mu,\nu} \epsilon_{\alpha\mu} g^{\mu\nu} (d\alpha)_\nu = 0 \ ,$$
$$(d\beta)_\beta = \sum_{\mu,\nu} \epsilon_{\beta\mu} g^{\mu\nu} (d\alpha)_\nu = \epsilon_{\beta\alpha} g^{\alpha\alpha} \ .$$

Now, of course, even if α and β were not conjugate harmonic functions, one would have $(d\beta)_\alpha = 0$ and $(d\beta)_\beta = 1$, so the condition that they are is equivalent to $\epsilon_{\beta\alpha} g^{\alpha\alpha} = 1$. The relation $\epsilon^{ab}\epsilon_{ab} = 2(-1)^s$ is expressed as

$$2(-1)^s = \sum_{\mu,\nu,\sigma,\rho} g^{\mu\sigma} g^{\nu\rho} \epsilon_{\sigma\rho}\epsilon_{\mu\nu} = g^{\alpha\alpha} g^{\beta\beta} \epsilon_{\alpha\beta}\epsilon_{\alpha\beta} + g^{\beta\beta} g^{\alpha\alpha} \epsilon_{\beta\alpha}\epsilon_{\beta\alpha} = 2 g^{\alpha\alpha} g^{\beta\beta} \epsilon_{\beta\alpha}\epsilon_{\beta\alpha} \ .$$

Multiplying this equation by $g^{\alpha\alpha}/2$, one finds

$$g^{\alpha\alpha}(-1)^s = (\epsilon_{\beta\alpha} g^{\alpha\alpha})(\epsilon_{\beta\alpha} g^{\alpha\alpha}) g^{\beta\beta} = g^{\beta\beta} \ .$$

Problem 3

(a) Show that $R_{abcd} = R_{cdab}$.

(b) In n dimensions, the Riemann tensor has n^4 components. However, on account of the symmetries (3.2.13), (3.2.14) and (3.2.15), not all of these components are independent. Show that the number of independent components is $n^2(n^2-1)/12$.

Solution of (a)

Equations (3.2.13), (3.2.14) and (3.2.15) are equivalent to

$$\begin{aligned} R_{abcd} &= -R_{bacd} \ , \\ R_{abcd} &= -R_{acdc} \ , \\ R_{abcd} &= R_{acbd} + R_{cbad} \ . \end{aligned} \tag{3.6}$$

These equations imply

$$R_{cbad} = R_{bcda} = R_{bdca} + R_{dcba} = R_{bdca} + R_{cdab} \ .$$

Plugging this in (3.6) yields

$$R_{abcd} = R_{acbd} + R_{bdca} + R_{cdab} \ ,$$

or

$$R_{abcd} - R_{cdab} = R_{acbd} - R_{bdac} \ . \tag{3.7}$$

Writing $R_{abcd} - R_{cdab}$ as $R_{badc} - R_{dcba}$ and applying equation (3.7) results in

$$\begin{aligned} R_{abcd} - R_{cdab} &= R_{badc} - R_{dcba} \\ &= R_{bdac} - R_{acbd} \ . \end{aligned}$$

This together with equation (3.7) shows that $R_{abcd} = R_{cdab}$.

Solution of (b)

The word "independence" here has the meaning given to it in the context of Linear Algebra. The components $R_{\mu\nu\sigma\rho}$ satisfy

$$\begin{cases} R_{\mu\nu\sigma\rho} + R_{\nu\mu\sigma\rho} = 0 \ , \\ R_{\mu\nu\sigma\rho} + R_{\mu\nu\rho\sigma} = 0 \ , \\ R_{\mu\nu\sigma\rho} - R_{\mu\sigma\nu\rho} + R_{\sigma\mu\nu\rho} - R_{\sigma\nu\mu\rho} + R_{\nu\sigma\mu\rho} - R_{\nu\mu\sigma\rho} = 0 \ . \end{cases} \tag{3.8}$$

The first two of these equations are (3.2.13) and (3.2.15); the third is (3.2.14). So there are 3 equations for each multi-index $\mu\nu\sigma\rho$, giving in total $3n^4$ equations. The equations aren't independent of course; the number of independent components is the number of components minus the number of independent equations.

There is a number of solutions to this problem on the internet. Most of them involve picking some subset of the $3n^4$ equations (3.6) and claiming that it contains all the information of (3.6), but no redundancies. Then the number of equations in this subset is subtracted from n^4 yielding the correct number $n^2(n^2 - 1)/12$. Perhaps because of notational difficulties, I've had a hard time coming up with or finding a clear argument for the claim that some specific subset of equations has the nice properties above.

Instead of solving this problem completely, I convinced myself that the number of independent components is $n^2(n^2-1)/12$ for some cases of interest by asking Mathematica to compute the dimension of the null space of the matrix associated with the linear system (3.6). This dimension equals the number of independent components.

The code used follows.

```
n = 4;
index[a_, b_, c_, d_] := d + (c - 1)*n + (b - 1)*n^2 + (a - 1)*n^3;

M = ConstantArray[0, {3*n^4, n^4}];

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <= n, k++,
   For[l = 1, l <= n, l++,
    {
```

```
      M[[ 0*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 0*n^4 + index[i, j, k, l], index[j, i, k, l] ]] = 1;

      M[[ 1*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 1*n^4 + index[i, j, k, l], index[i, j, l, k] ]] = 1;

      M[[ 2*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[j, i, k, l] ]] = -1;
      M[[ 2*n^4 + index[i, j, k, l], index[k, i, j, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[i, k, j, l] ]] = -1;
      M[[ 2*n^4 + index[i, j, k, l], index[j, k, i, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[k, j, i, l] ]] = -1;
      }
     ]
    ]
   ]
  ]

Length[NullSpace[M]]
```

The first line sets the dimension to 4. Any positive integer (of reasonable size) could go here. The second line associates each multi-index $\mu\nu\rho\sigma$ with a number between 1 and n^4. The third line creates a matrix with a column for each of the n^4 variables and an equation for each of the $3n^4$ equations. The nested 'for' loops set the coefficients for the three equations each index satisfy. The equations for multi-index $\mu\nu\rho\sigma$ are represented in lines J, $J+n^4$ and $J+2n^4$, where J is the number associated with $\mu\nu\rho\sigma$.

Problem 4

(a) Show that in two dimensions, the Riemann tensor takes the form $R_{abcd} = Rg_{a[c}g_{d]b}$. (Hint: Use the result of problem 3(b) to show that $g_{a[c}g_{d]b}$ spans the vector space of tensors having the symmetries of the Riemann tensor.)

(b) By similar arguments, show that in three dimensions the Weyl tensor vanishes identically; i.e., for $n = 3$, equation (3.2.28) holds with $C_{abcd} = 0$.

Solution of (a)

It is easy to check that

$$g_{a[c}g_{d]b} = \frac{1}{2}(g_{ac}g_{db} - g_{ad}g_{cb})$$

has the symmetries of the Riemann tensor, i.e., that it satisfies (3.6) and the two equations preceding it. Because in two dimensions the vector space of tensor having the symmetries of the Riemann tensor has $2^2(2^2-1)/12 = 1$ dimension, it follows that

$$R_{abcd} = \lambda g_{a[c} g_{d]b} \ ,$$

where λ is some (scalar) function. To show that $\lambda = R$, we contract both sides. First b with d.

$$\begin{aligned} R_{ac} &= (\lambda/2)(g_{ac}\delta_d{}^d - g_{ad}\delta_c{}^d) \\ &= (\lambda/2) g_{ac} \end{aligned}$$

Then we contract a with c.

$$R = (\lambda/2)\delta_a{}^a = \lambda$$

Solution of (b)

The tensor

$$S_{abcd} = \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)} R \ g_{a[c}g_{d]b}$$

has the symmetries of the Riemann tensor, because the Weyl tensor does. In dimension n=3, tensors with the symmetries of the Riemann tensor have at most $n^2(n^2-1)/12 = 6$ independent components. Now, because the Weyl tensor is trace-free, the components $S_{\mu\nu\rho\sigma}$ satisfy

$$\sum_\sigma S_{\alpha\sigma\beta}{}^\sigma = R_{\alpha\beta} \ . \tag{3.9}$$

Because of symmetry in α and β, this adds at most $n(n+1)/2 = 6$ equations. If these equations are independent of one another and of the ones used to reduce the number of independent components to 6, then the number of free components drops to zero. It follows that a unique tensor has the symmetries of the R_{abcd} and satisfies (3.9). Because S_{abcd} and R_{abcd} have these properties,

$$S_{abcd} = R_{abcd} \ .$$

It turns out that the independence hypotheses of the previous paragraph are satisfied, as shown in the piece of code below for the case where the index in (3.9) was raised with the flat metric.

```
n = 3;
index[a_, b_, c_, d_] := d + (c - 1)*n + (b - 1)*n^2 + (a - 1)*n^3;

M = ConstantArray[0, {3*n^4 + n^2, n^4}];

For[i = 1, i <= n, i++,
```

```
For[j = 1, j <= n, j++,
 {
  M[[ 3*n^4 + (i - 1) n + j, index[i, 1, j, 1] ]] = -1;
  For[a = 2, a <= n, a++,
    {
     M[[ 3*n^4 + (i - 1) n + j, index[i, a, j, a] ]] = 1;
     }
    ]
   For[k = 1, k <= n, k++,
    For[l = 1, l <= n, l++,
     {
      M[[ 0*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 0*n^4 + index[i, j, k, l], index[j, i, k, l] ]] = 1;

      M[[ 1*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 1*n^4 + index[i, j, k, l], index[i, j, l, k] ]] = 1;

      M[[ 2*n^4 + index[i, j, k, l], index[i, j, k, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[j, i, k, l] ]] = -1;
      M[[ 2*n^4 + index[i, j, k, l], index[k, i, j, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[i, k, j, l] ]] = -1;
      M[[ 2*n^4 + index[i, j, k, l], index[j, k, i, l] ]] = 1;
      M[[ 2*n^4 + index[i, j, k, l], index[k, j, i, l] ]] = -1;
      }
     ]
    ]
  }
 ]
]

Length[NullSpace[M]]
```

(See explanation of the [very similar] code used in problem 3[b].)

Mathematica returns that the length of the null space is zero, confirming the uniqueness of the tensor having the symmetries of S_{abcd}.

Problem 5

(a) Show that any curve whose tangent satisfies equation (3.3.2) can be reparameterized so that equation (3.3.1) is satisfied.

(b) Let t be an affine parameter of a geodesic γ. Show that all other affine parameters of γ take the form $at + b$, where a and b are constants.

Solution of (a)

To do this we work with coordinates. Let $x^1(t), \dots, x^n(t)$ be the components of a curve parameterized by t. Let s be some other parameter for this curve, and let $T^\mu = dx^\mu/dt$, $S^\mu = dx^\mu/ds$ be the components of the tangents with respect to each parameter. Suppose T^a satisfies equation (3.3.2), that is,

$$\sum_\sigma T^\sigma \nabla_\sigma T^\mu = \alpha T^\mu$$

for some function α. We show that one can choose s so that S^a satisfies (3.3.1).

$$\begin{aligned}
\sum_\sigma S^\sigma \nabla_\sigma S^\mu &= \frac{dS^\mu}{ds} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} S^\sigma S^\nu \\
&= \frac{d}{ds}\left(T^\mu \frac{dt}{ds}\right) + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} T^\sigma T^\nu \left(\frac{dt}{ds}\right)^2 \\
&= T^\mu \frac{d^2t}{ds^2} + \frac{dT^\mu}{dt}\left(\frac{dt}{ds}\right)^2 + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} T^\sigma T^\nu \left(\frac{dt}{ds}\right)^2 \\
&= T^\mu \frac{d^2t}{ds^2} + \alpha T^\mu \left(\frac{dt}{ds}\right)^2
\end{aligned}$$

This shows that it is sufficient to choose s such that

$$\frac{d^2t}{ds^2} + \alpha\left(\frac{dt}{ds}\right)^2 = 0 \ .$$

To do so, take a nontrivial solution $t(s)$ of this differential equation and use it to (implicitly) define s for each value of t.

Solution of (b)

Let T^μ and S^μ be as in item (a). Suppose this time that

$$\sum_\sigma T^\sigma \nabla_\sigma T^\mu = 0 \ ,$$
$$\sum_\sigma S^\sigma \nabla_\sigma S^\mu = 0 \ .$$

As before, we compute

$$0 = \sum_\sigma S^\sigma \nabla_\sigma S^\mu = T^\mu \frac{d^2t}{ds^2} + \left(\frac{dt}{ds}\right)^2 \cancelto{0}{\sum_\sigma T^\sigma \nabla_\sigma T^\mu} \ .$$

It follows that $d^2t/ds^2 = 0$, so $t = as + b$ for some constants a and b.

Problem 6

The metric of Euclidean $\mathbb{R}^3$ in spherical coordinates is $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\; d\phi^2)$ (see problem 8 of chapter 2).

(a) Calculate the Christoffel components $\Gamma^\sigma{}_{\mu\nu}$ in this coordinate system.

(b) Write down the components of the geodesic equation in this coordinate system and verify that the solutions correspond to straight lines in Cartesian coordinates.

Solution of (a)

Plugging equation (3.1.30) into Mathematica, we find that the nonzero components of the Christoffel symbol are

$$
\begin{aligned}
&\Gamma^r{}_{\theta\theta} = -r\ , \\
&\Gamma^r{}_{\phi\phi} = -r\sin^2\theta\ , \\
&\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r\ , \\
&\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\ , \\
&\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta\ .
\end{aligned}
$$

The code used in Mathematica follows.

```
g = ConstantArray[0, {3, 3}];

g[[1, 1]] = 1;
g[[2, 2]] = r^2;
g[[3, 3]] = r^2*Sin[\[Theta]]^2;

v = ConstantArray[0, {3}];

v[[1]] = r;
v[[2]] = \[Theta];
v[[3]] = \[Phi];

\[CapitalGamma] = ConstantArray[0, {3, 3, 3}];

For[\[Rho] = 1, \[Rho] <= 3, \[Rho]++,
 For[\[Mu] = 1, \[Mu] <= 3, \[Mu]++,
  For[\[Nu] = 1, \[Nu] <= 3, \[Nu]++,
   {
    \[CapitalGamma][[\[Rho], \[Mu], \[Nu]]] =
      1/(2*g[[\[Rho], \[Rho]]])*(D[g[[\[Nu], \[Rho]]], v[[\[Mu]]]] +
         D[g[[\[Mu], \[Rho]]], v[[\[Nu]]]] -
         D[g[[\[Mu], \[Nu]]], v[[\[Rho]]]]);
```

```
    }
   ]
  ]
 ]
```

Solution of (b)

The components of the geodesic equation are obtained by substitution of the Christoffel components of item (a) into equation (3.3.5)

$$\frac{d^2x^\mu}{dt^2} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} \frac{\partial x^\sigma}{dt} \frac{\partial x^\nu}{dt} = 0 \ .$$

The result is the following system.

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta \ \dot{\phi}^2 = 0 \\ \ddot{\theta} + (2/r)\dot{r}\dot{\theta} - \sin\theta\cos\theta \ \dot{\phi}^2 = 0 \\ \ddot{\phi} + (2/r)\dot{r}\dot{\theta} + 2\cot\theta \ \dot{\theta}\dot{\phi} = 0 \end{cases} \tag{3.10}$$

The solutions of (3.10) represent curves in Cartesian coordinates. Let C be one of these curves. By construction, the tangent vector T^a of C satisfies $T^a\nabla_a T^b = 0$. Writing the cartesian components of this equation using (3.3.5) yields $d^2x^\mu/dt^2 = 0$, because $\Gamma^\mu{}_{\sigma\nu} = 0$ in these coordinates. This shows that C is a straight line in cartesian coordinates.

Problem 7

As shown in problem 2, an arbitrary Lorentz metric on a two-dimensional manifold locally can always be put in the form $ds^2 = \Omega^2(x,t)[-dt^2 + dx^2]$. Calculate the Riemann curvature tensor of this metric (a) by the coordinate basis methods of section 3.4a and (b) by the tetrad methods of section 3.4b.

Solution of (a)

I will use in this problem $\partial_x f$ as shorthand for $\partial f/\partial x$.

Step 1. We compute the Christoffel components via equation (3.1.30).

$$\Gamma^t{}_{tt} = \Gamma^t{}_{xx} = \Gamma^x{}_{tx} = \Gamma^x{}_{xt} = \Omega^{-1}\partial_t\Omega$$
$$\Gamma^t{}_{tx} = \Gamma^t{}_{xt} = \Gamma^x{}_{tt} = \Gamma^x{}_{xx} = \Omega^{-1}\partial_x\Omega$$

Step 2. We compute the component $R_{txt}{}^x$ using equation (3.4.4).

$$R_{txt}{}^x = \Omega^{-2}([\partial_t\Omega]^2 - [\partial_x\Omega]^2) + \Omega^{-1}(\partial_{xx}\Omega - \partial_{tt}\Omega) \tag{3.11}$$

The other components can be found from this one using the symmetries

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$$

of the Riemann tensor.

Solution of (b)

Step 1. We choose an orthonormal basis. The natural one here is $\{e_1, e_2\}$, where

$$(e_1)^a = \Omega^{-1}(\partial/\partial t)^a \ , \qquad\qquad (e_2)^a = \Omega^{-1}(\partial/\partial x)^a \ .$$

Step 2. We compute the left hand side of

$$\partial_{[a}(e_\sigma)_{b]} = \sum_{\mu,\nu} \eta^{\mu\nu}(e_\mu)_{[a}\omega_{b]\sigma\nu} \ . \tag{3.12}$$

First we lower the indices of the vectors in the orthonormal basis.

$$(e_1)_b = g_{ab}(e_1)^a = -\Omega(dt)_b \qquad\qquad (e_2)_a = g_{ab}(e_2)^a = \Omega(dx)_b$$

From the definition of the ordinary derivative operator, it follows

$$\begin{aligned}\partial_{[a}(e_1)_{b]} &= -\partial_t\Omega(dt)_{[a}(dt)_{b]} - \partial_x\Omega(dx)_{[a}(dt)_{b]} \\ &= \partial_x\Omega(dt)_{[a}(dx)_{b]} \ ,\end{aligned}$$

$$\begin{aligned}\partial_{[a}(e_2)_{b]} &= \partial_t\Omega(dt)_{[a}(dx)_{b]} + \partial_x\Omega(dx)_{[a}(dx)_{b]} \\ &= \partial_t\Omega(dt)_{[a}(dx)_{b]} \ .\end{aligned}$$

Step 3. We solve equation (3.12) for $\omega_{b\sigma\nu}$. Letting $\sigma = 1$, we have

$$\partial_{[a}(e_1)_{b]} = -(e_1)_{[a}\omega_{b]11} + (e_2)_{[a}\omega_{b]12} \ .$$

The term with ω_{b11} vanishes because the connection one-forms $\omega_{b\sigma\nu}$ are antisymmetric in $\sigma\nu$. Contracting $(e_1)^a\partial_{[a}(e_1)_{b]}$, we have in one hand

$$\begin{aligned}(e_1)^a\partial_{[a}(e_1)_{b]} &= (e_1)^a(e_2)_{[a}\omega_{b]12} \\ &= -(1/2)(e_2)_b\omega_{112} \ ,\end{aligned}$$

and on the other

$$\begin{aligned}(e_1)^a\partial_{[a}(e_1)_{b]} &= (e_1)^a\partial_x\Omega(dt)_{[a}(dx)_{b]} \\ &= (1/2)\Omega^{-2}(\partial_x\Omega)(e_2)_b \ .\end{aligned}$$

It follows that $\omega_{112} = -\Omega^{-2}\partial_x\Omega$. Letting $\sigma = 2$ and contracting $(e_2)^a\partial_{[a}(e_2)_{b]}$, we similarly find $\omega_{221} = \Omega^{-2}\partial_t\Omega$. In two dimensions, these two coefficients determine all others.

Step 4. We compute R_{1212} using equation (3.4.21). This equation reduces in the present case to

$$R_{1212} = \Omega^{-1}(\partial_t\omega_{212} - \partial_x\omega_{112}) + \omega_{112}^2 - \omega_{212}^2 \ .$$

Plugging here the expressions for ω_{112} and $\omega_{221} = -\omega_{212}$ obtained in the preceding step, we find

$$R_{1212} = \Omega^{-4}([\partial_t\Omega]^2 - [\partial_x\Omega]^2) + \Omega^{-3}(\partial_{xx}\Omega - \partial_{tt}\Omega) \ . \tag{3.13}$$

It wouldn't be very nice not to check that (3.13) and expression (3.11) found in item (a) agree. Because of the simple form of the metric, it is clear that lowering the raised index x in (3.11) amounts to multiplying the right hand side by Ω^2. We compute R_{1212} from (3.11).

$$\begin{aligned} R_{1212} &= R_{abcd}(e_1)^a(e_2)^b(e_1)^a(e_2)^b \\ &= R_{abcd}[\Omega^{-1}(\partial/\partial t)^a][\Omega^{-1}(\partial/\partial x)^a][\Omega^{-1}(\partial/\partial t)^a][\Omega^{-1}(\partial/\partial x)^a] \\ &= \Omega^{-4}R_{txtx} \\ &= \Omega^{-4}\{[\partial_t\Omega]^2 - [\partial_x\Omega]^2 + \Omega(\partial_{xx}\Omega - \partial_{tt}\Omega)\} \end{aligned}$$

This is precisely (3.13).

Problem 8

Using the antisymmetry of $\omega_{a\mu\nu}$ in μ and ν, equation (3.4.15), show that

$$\omega_{\lambda\mu\nu} = 3\omega_{[\lambda\mu\nu]} - 2\omega_{[\mu\nu]\lambda} \ .$$

Use this formula together with equation (3.4.23) to solve for $\omega_{\lambda\mu\nu}$ in terms of commutators (or antisymmetrized derivatives) of the orthonormal basis vectors.

Solution

Instead of labeling indices with Greek letters, I will use numbers.

Because of the antisymmetry (3.4.15), we have

$$\begin{aligned} 3\omega_{[123]} &= \omega_{123} + \omega_{231} + \omega_{312} \ , \\ 2\omega_{[23]1} &= \omega_{231} + \omega_{312} \ . \end{aligned}$$

It is clear that formula $\omega_{123} = 3\omega_{[123]} - 2\omega_{[23]1}$ holds.

The following three equations are (3.4.23) with the roles of the indices interchanged.

$$\omega_{123} + \omega_{312} = (e_2)_a[e_1, e_3]^a \tag{3.14}$$

$$\omega_{231} + \omega_{123} = (e_3)_a[e_2, e_1]^a \tag{3.15}$$

$$\omega_{312} + \omega_{231} = (e_1)_a[e_3, e_2]^a \tag{3.16}$$

Adding equations (3.14) and (3.15) and subtracting equation (3.16) from the result yields

$$\omega_{123} = \frac{1}{2}\{(e_2)_a[e_1, e_3]^a + (e_3)_a[e_2, e_1]^a - (e_1)_a[e_3, e_2]^a\} \ .$$

Chapter 4

Einstein's Equation

Problem 1

Show that Maxwell's equation (4.3.12) implies strict charge conservation, $\nabla_a j^a = 0$.

Solution

To do this I applied the derivative operator to both sides of Maxwell's equation and looked at the right hand side from different points of view until I could show it vanishes. The idea that worked was using equation (3.2.12) which relates the Riemann tensor to the commutator of two derivative operators.

$$\begin{aligned}
-8\pi\nabla^b j_b &= 2\nabla^b\nabla^a F_{ab} \\
&= \nabla^b\nabla^a F_{ab} - \nabla^b\nabla^a F_{ba} \\
&= -g^{ae}g^{bf}(\nabla_a\nabla_b - \nabla_b\nabla_a)F_{ef} \\
&= -g^{ae}g^{bf}(R_{abe}{}^d F_{df} + R_{abf}{}^d F_{ed})
\end{aligned}$$

Playing with the right hand side a little bit, one can show that it equals zero.

$$\begin{aligned}
-8\pi\nabla^b j_b &= -g^{bf}R_{bd}F^d{}_f + g^{ae}g^{bf}R_{bafd}F_e{}^d \\
&= -R_{bd}F^{db} + g^{ae}R_{ad}F_e{}^d \\
&= -R_{db}F^{db} + R_{ad}F^{ad} = 0
\end{aligned}$$

Problem 2

(a) Let $\boldsymbol{\alpha}$ be a p-form on an n-dimensional oriented manifold with metric g_{ab}, i.e., $\alpha_{a_1\ldots a_p}$ is a totally antisymmetric tensor field (see appendix B).

We define the *dual*, $*\boldsymbol{\alpha}$, of $\boldsymbol{\alpha}$ by

$$*\alpha_{b_1 \dots b_{n-p}} = \frac{1}{p!} \alpha^{a_1 \dots a_p} \epsilon_{a_1 \dots a_p b_1 \dots b_{n-p}} \ ,$$

where $\epsilon_{a_1 \dots a_n}$ is the natural volume element on M, i.e., the totally antisymmetric tensor field determined up to sign by equation (B.2.9). Show that $**\boldsymbol{\alpha} = (-1)^{s+p(n-p)}\boldsymbol{\alpha}$, where s is the number of minuses occurring the signature of the g_{ab}.

(b) Show that in differential forms notation (see appendix B), Maxwell's equations (4.3.12) and (4.3.13) can be written as

$$d * \boldsymbol{F} = 4\pi * j \ ,$$
$$d\boldsymbol{F} = 0 \ .$$

Note that if we apply Stokes's theorem (see appendix B) to the first equation, we obtain $\int_\Sigma *j = (1/4\pi)\int_S *F$, where Σ is a three-dimensional hypersurface with two-dimensional boundary S. But $-\int_\Sigma *j = \int_\Sigma j^a t_a d\Sigma$ is just the total electric charge e in the volume Σ, where t^a is the unit normal to Σ, and $-\int_S *F = \int_S E^a n_a dA$ is just the integral of the normal component of $E_a = F_{ab}t^b$ on S. Thus, Gauss's law of electromagnetism continues to hold in curved spacetime.

(c) Define for each $\beta \in [0, 2\pi]$ the tensor field $\tilde{F}_{ab} = F_{ab}\cos\beta + *F_{ab}\sin\beta$. We call $\tilde{F}_{ab}$ a *duality rotation* of F_{ab} by "angle" β. It follows immediately from part (b) that if F_{ab} satisfies the source-free Maxwell's equations ($j^a = 0$), then so does $\tilde{F}_{ab}$. Show that the stress-energy, T_{ab}, of the solution $\tilde{F}_{ab}$ is the same as that of F_{ab}.

Solution of (a)

By the definition of the $*$ operation,

$$\begin{aligned} **\alpha_{c_1 \dots c_p} &= \frac{1}{(n-p)!} * \alpha^{b_1 \dots b_{n-p}} \epsilon_{b_1 \dots b_{n-p} c_1 \dots c_p} \\ &= \frac{1}{p!(n-p)!} \alpha_{a_1 \dots a_p} \epsilon^{a_1 \dots a_p b_1 \dots b_{n-p}} \epsilon_{b_1 \dots b_{n-p} c_1 \dots c_p} \ . \end{aligned}$$

In order to use equation (B.2.13), we permute the raised indices. The permutation

$$a_1 \dots a_p b_1 \dots b_{n-p} \rightsquigarrow b_1 a_1 \dots a_p b_2 \dots b_{n-p}$$

has sign $(-1)^p$, and from this it isn't hard to see that the sign of

$$a_1 \dots a_p b_1 \dots b_{n-p} \rightsquigarrow b_1 \dots b_{n-p} a_1 \dots a_p$$

is $(-1)^{p(n-p)}$. It follows that

$$\begin{aligned}
**\alpha_{c_1\ldots c_p} &= \frac{(-1)^{p(n-p)}}{p!(n-p)!}\alpha_{a_1\ldots a_p}\epsilon^{b_1\ldots b_{n-p}a_1\ldots a_p}\epsilon_{b_1\ldots b_{n-p}c_1\ldots c_p} \\
&= (-1)^{s+p(n-p)}\alpha_{a_1\ldots a_p}\delta^{[a_1}{}_{c_1}\ldots\delta^{a_p]}{}_{c_p} \\
&= (-1)^{s+p(n-p)}\alpha_{[a_1\ldots a_p]}\delta^{a_1}{}_{c_1}\ldots\delta^{a_p}{}_{c_p} \\
&= (-1)^{s+p(n-p)}\alpha_{c_1\ldots c_p}\ .
\end{aligned}$$

The second equality follows from (B.2.13); the fourth is the desired result.

Solution of (b)

By the definition (B.1.4) of the exterior derivative operator, $(dF)_{abc} = 3\nabla_{[a}F_{bc]}$, so it is clear that (4.3.13) and $d\boldsymbol{F} = 0$ are equivalent.

Below is a (somewhat sketchy) proof of the fact that equation (4.3.12) implies its version in differential forms notation, $d*\boldsymbol{F} = 4\pi * j$. I did not show that the converse holds, though this would be necessary for one to conclude that both equations are indeed equivalent.

By the definition of the $*$ operation,

$$\begin{aligned}
(d*F)_{ecd} &= \nabla_e(*F)_{cd} + \nabla_c(*F)_{de} + \nabla_d(*F)_{ec} \\
&= \frac{1}{2}\{\nabla_e(F^{ab}\epsilon_{abcd}) + \nabla_c(F^{ab}\epsilon_{abde}) + \nabla_d(F^{ec}\epsilon_{abcd})\} \\
&= \frac{1}{2}\{\nabla_e F^{ab}\epsilon_{abcd} + \nabla_c F^{ab}\epsilon_{abde} + \nabla_d F^{ab}\epsilon_{abec}\}\ . \qquad (4.1)
\end{aligned}$$

Equation (B.2.11), $\nabla_a\epsilon_{b_1\ldots b_n} = 0$, was used in the last line. Using Maxwell's equation (4.3.12) and again the definition of $*$,

$$4\pi(*j)_{ecd} = (*\nabla^a F_{ba})_{ecd} = \nabla_a F^{ba}\epsilon_{becd}\ . \qquad (4.2)$$

For what follows, we fix some coordinate system and adhere to the Einstein summation convention.

It isn't hard to show that for each particular multi-index $\lambda\mu\nu$, the $\lambda\mu\nu$-component of the right hand sides of equations (4.1) and (4.2) are the same. This is what is necessary to establish $d*\boldsymbol{F} = 4\pi * j$. Here we make the case $\lambda\mu\nu = 123$.

$$\begin{aligned}
4\pi(*j)_{123} &= \nabla_\mu F^{\nu\mu}\epsilon_{\nu 123} = \nabla_\mu F^{4\mu}\epsilon_{4123} \\
&= (\nabla_1 F^{14} + \nabla_2 F^{24} + \nabla_3 F^{34})\epsilon_{1234}
\end{aligned}$$

$$\begin{aligned}
(d*F)_{123} &= \frac{1}{2}\{\nabla_1 F^{\mu\nu}\epsilon_{\mu\nu 23} + \nabla_2 F^{\mu\nu}\epsilon_{\mu\nu 31} + \nabla_3 F^{\mu\nu}\epsilon_{\mu\nu 12}\} \\
&= \nabla_1 F^{14}\epsilon_{1423} + \nabla_2 F^{24}\epsilon_{2431} + \nabla_3 F^{34}\epsilon_{3412} \\
&= (\nabla_1 F^{14} + \nabla_2 F^{24} + \nabla_3 F^{34})\epsilon_{1234}
\end{aligned}$$

Solution of (c)

I had some trouble with the messy algebra here, so I went to the oracle[1] for help. I then found the following (symmetric in F_{ab} and $*F_{ab}$) expression for the electromagnetic stress-energy tensor T_{ab}:

$$T_{ab} = \frac{1}{8\pi}\{F_{ac}F_b{}^c + *F_{ac} * F_b{}^c\} \ . \tag{4.3}$$

It is easy to check using (4.3) and $** F_{ab} = -F_{ab}$ (see item (a)) that F_{ab} and $\tilde{F}_{ab}$ yield the same T_{ab}. Thus it only remains to prove that (4.3) agrees with the expression for the electromagnetic stress-energy tensor given in Wald's book. This amounts to showing

$$F_{ac}F_b{}^c - \frac{1}{2}g_{ab}F_{de}F^{de} - *F_{ac} * F_b{}^c = 0 \ .$$

We compute $*F_{ac} * F_b{}^c$. By definition, $*F_{ab} = (1/2)\epsilon_{abcd}F^{cd}$, so

$$*F_{ac} * F_b{}^c = \frac{1}{4}\epsilon_{acij}\epsilon_b{}^c{}_{kl}F^{ij}F^{kl} \ . \tag{4.4}$$

We disentangle the ϵ terms using equation (B.2.13).

$$\begin{aligned}
\epsilon_{acij}\epsilon_b{}^c{}_{kl} &= g_{bd}g_{ke}g_{lf}\epsilon_{caij}\epsilon^{cdef} \\
&= -3! g_{bd}g_{ke}g_{lf}\delta^{[d}{}_a\delta^e{}_i\delta^{f]}{}_j \\
&= g_{ba}g_{kj}g_{li} - g_{ba}g_{ki}g_{lj} + g_{bi}g_{ka}g_{lj} - \\
&\quad - g_{bj}g_{ka}g_{li} + g_{bj}g_{ki}g_{la} - g_{bi}g_{kj}g_{la}
\end{aligned}$$

Plugging this into equation (4.4) yields after some algebra

$$*F_{ac} * F_b{}^c = F_{ac}F_b{}^c - \frac{1}{2}g_{ab}F_{de}F^{de} \ .$$

This is the desired result.

Problem 3

(a) Derive equation (4.4.24).

(b) Show that the "gravitational electric and magnetic fields" $\vec{E}$ and $\vec{B}$ inside a spherical shell of mass M and radius R (with $M \ll R$) slowly rotating with angular velocity $\vec{\omega}$ are

$$\vec{E} = 0 \quad , \quad \vec{B} = \frac{2}{3}\frac{M}{R}\vec{\omega} \ .$$

[1] http://www.google.com/

(c) An observer at rest at the center of the shell of part (b) parallelly propagates along his (geodesic) world line a vector S^a with $S^a u_a = 0$, where u^a is the tangent to his world line. Show that the inertial components, $\vec{S}$, precess according to $d\vec{S}/dt = \vec{\Omega} \times \vec{S}$, where $\vec{\Omega} = 2\vec{B} = \frac{4}{3}(M/R)\vec{\omega}$. This effect, first analyzed by Thirring and Lense (1918) and discussed further by Brill and Cohen (1966), may be interpreted as a "dragging of inertial frames" caused by the rotating shell. At the center of the shell the local standard of "nonrotating," defined by parallel propagation alon a geodesic, is changed from what it would be without the shell, in a manner in accord with Mach's principle.

Solution of (a)

Below is a proof that

$$a^1 = -E^1 - 4(v^2 B^3 - v^3 B^2) \ . \tag{4.5}$$

The verification of the other components of equation (4.4.24) is done in an analogous manner.

The following formulas will be used in this problem; they are reproduced here for reference.

$$A_\mu = -\tfrac{1}{4}\overline{\gamma}_{0\mu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$E_\mu = F_{\mu 0} \qquad B_\mu = -\tfrac{1}{2}\sum_{\sigma,\nu} \epsilon_{\mu 0}{}^{\sigma\nu} F_{\sigma\nu}$$

We start with geodesic equation,

$$a^1 = -\sum_{\sigma,\nu} \Gamma^1{}_{\sigma\nu} v^\sigma v^\nu \ .$$

The relevant Christoffel components are as usual computed using equation (3.1.30). To relate expressions found with to A_μ, we use the equalities $(\sigma = 1,2,3)$

$$\gamma_{00} = \tfrac{1}{2}\overline{\gamma}_{00} \ ,$$
$$\gamma_{0\sigma} = \overline{\gamma}_{0\sigma} \ ,$$

which follow from

$$\gamma_{\mu\nu} = \overline{\gamma}_{\mu\nu} - \frac{1}{2}\overline{\gamma} = \overline{\gamma}_{\mu\nu} - \frac{1}{2}\overline{\gamma}_{00} \ .$$

We also use below the relation $(\mu,\nu = 1,2,3)$

$$\overline{\gamma}_{\mu\nu} = 0 \ .$$

This follows from fact that in Linearized Theory, if $T_{\mu\nu} = 0$ for some multi-index $\mu\nu$ and $\partial_0 T_{\mu\nu} = 0$, then one may take $\overline{\gamma}_{\mu\nu} = 0$. This fact is discussed and used in the text after equation (4.4.14). In the present case, we take $\overline{\gamma}_{\mu\nu} = 0$ for $\mu, \nu = 1, 2, 3$ because $T_{\mu\nu}$ is assumed to have the form

$$\begin{bmatrix} \rho & -J_1 & -J_2 & -J_3 \\ -J_1 & & & \\ -J_2 & & 0 & \\ -J_3 & & & \end{bmatrix} . \tag{4.6}$$

The Christoffel components then read

$$\Gamma^1{}_{\rho\sigma} = \tfrac{1}{2} \textstyle\sum_\alpha \eta^{1\alpha}(\partial_\rho \gamma_{\sigma\alpha} + \partial_\sigma \gamma_{\rho\alpha} - \partial_\alpha \gamma_{\rho\sigma}) = 0 \ ,$$

$$\Gamma^1{}_{0\sigma} = \tfrac{1}{2}(\partial_\sigma \gamma_{01} - \partial_1 \gamma_{0\sigma}) = 2(\partial_1 A_\sigma - \partial_\sigma A_1) = 2F_{1\sigma} \ ,$$

$$\Gamma^1{}_{00} = \tfrac{1}{2}(-\partial_1 \gamma_{00}) = -\tfrac{1}{4}\partial_1 \overline{\gamma}_{00} = \partial_1 A_0 = \partial_1 A_0 - \partial_0 A_1 = F_{10} = E_1 \ ,$$

where ρ and σ are nonzero indices.

The geodesic equation now looks like

$$\begin{aligned} a^1 &= -E_1 v^0 v^0 - \sum_{\sigma=1}^{3} \Gamma^1{}_{0\sigma} v^0 v^\sigma - \sum_{\sigma=1}^{3} \Gamma^1{}_{\sigma 0} v^0 v^\sigma \\ &= -E^1 - 4\sum_{\sigma=1}^{3} F_{1\sigma} v^\sigma \ . \end{aligned}$$

Here it was used that $v^0 = 1$ (approximately) and $\Gamma^1{}_{0\sigma} = \Gamma^1{}_{\sigma 0}$. Now, by definition,

$$B_\mu = \epsilon_{0\mu}{}^{12} F_{12} + \epsilon_{0\mu}{}^{13} F_{13} + \epsilon_{0\mu}{}^{23} F_{23} \ ,$$

so that

$$\begin{aligned} v^2 B^3 - v^3 B^2 &= v^2 \epsilon_{03}{}^{12} F_{12} - v^3 \epsilon_{02}{}^{13} F_{13} \\ &= v^2 \epsilon_{0312} F_{12} - v^3 \epsilon_{0213} F_{13} \\ &= v^2 F_{12} + v^3 F_{13} = \sum_{\sigma=1}^{3} F_{1\sigma} v^\sigma \ . \end{aligned}$$

This completes the proof of (4.5).

Solution of (b)

The stress-energy tensor for this configuration has the form (4.6), where ρ is the mass-energy density and, for $\mu = 1, 2, 3$, J_μ is the x^μ-component of the momentum density. Note that the vector $\vec{J} = (J_1, J_2, J_3)$ also represents the current density of a charged sphere of radius R and charge[2] M rotating with angular velocity $\vec{\omega}$.

From the Linearized Einstein's Equation, it follows that $A_\mu = \frac{1}{4}\bar{\gamma}_{0\mu}$ satisfies Maxwell's equation $\sum_\nu \partial^\nu \partial_\nu A_\mu = -4\pi J_\mu$, where $J_0 := -\rho$. Using the approximation $\partial_0 \gamma_{\mu\nu} = 0$, we find that A_μ satisfies

$$\nabla^2 A_\mu = -4\pi J_\mu \ .$$

This is, essentially, equations (5.62) and (2.24) from Griffiths' book "Introduction to Electrodynamics". So $-A_0$ and $\vec{A} = (A_1, A_2, A_3)$ are the electrostatic and vector potentials determined by the charge and current densities ρ and $\vec{J}$. This analogy with Electromagnetism allows one to use Gauss' Law and conclude that $\vec{E} = 0$. The problem of finding $\vec{B}$ is solved in example (5.11) in Griffiths' book, and the answer agrees with $\vec{B} = (2M/3R)\vec{\omega}$.

Solution of (c)

The components of the tangent u^a to the observer's world line relative to the global inertial coordinates read $(1, 0, 0, 0)$. The components of the propagated vector S^μ satisfy

$$\begin{aligned} \frac{dS^\mu}{dt} + \sum_{\sigma,\nu} u^\sigma {\Gamma^\mu}_{\sigma\nu} S^\nu &= 0, \text{ or} \\ \frac{dS^\mu}{dt} &= -\sum_\nu {\Gamma^\mu}_{0\nu} S^\nu \ . \end{aligned} \tag{4.7}$$

This is just equation (3.1.19). We use this equation to show

$$\frac{dS^1}{dt} = 2(B^2 S^3 - B^3 S^2) \ . \tag{4.8}$$

The other components of $d\vec{S}/dt = 2\vec{B} \times \vec{S}$ are verified by proceding similarly. By computations made in items (a) and (b),

$$\begin{aligned} {\Gamma^1}_{00} &= E_1 = 0 \ , \\ {\Gamma^1}_{0\sigma} &= 2F_{1\sigma} \ . \end{aligned}$$

Plugging this into (4.7), we find

$$\begin{aligned} \frac{dS^1}{dt} &= -2 \sum_{\nu=1}^{3} F_{1\nu} S^\mu \\ &= -2(S^2 B^3 - S^3 B^2) \ , \end{aligned}$$

where in the second line a computation made in item (b) was used. This completes the proof of (4.8).

[2] Am I missing something like 4π or ϵ_0 in front of M?

Problem 4

Starting with equation (3.4.5) for R_{ab}, derive the formula, equation (4.4.51), for $R^{(2)}_{ab}$ by substituting $\eta_{ab} + \gamma_{ab}$ for g_{ab} and keeping precisely the terms quadratic in γ_{ab}.

Solution

This is a straightforward computation. Should anyone really need the solution to this problem, please let me know via email.

Problem 5

Let T_{ab} be a symmetric, converved tensor field (i.e., $T_{ab} = T_{ba}$, $\partial^a T_{ab} = 0$) in Minkowsky spacetime. Show that there existsa tensor field U_{acbd} with the symmetries $U_{acbd} = U_{[ac]bd} = U_{ac[bd]} = U_{bdac}$ such that $T_{ab} = \partial^c \partial^d U_{acbd}$. (Hint: For any vector field v^a in Minkowsky spacetime satisfying $\partial_a v^a = 0$, there exists a tensor field $s^{ab} = -s^{ba}$ such that $v^a = \partial_b s^{ab}$. [This follows from applying the converse of the Poincaré lemma (see the end of section B.1 in appendix B) to the 3-form $\epsilon_{abcd} v^d$.] Use this fact to show that $T_{ab} = \partial^c W_{cab}$ where $W_{cab} = W_{[ca]b}$. Then use the fact that $\partial^c W_{c[ab]} = 0$ to derive the desired result.)

Problem 6

As discussed in the text, in general relativity no meaningful expression is known for the local stress-energy of the gravitational field. However, a four-index tensor T_{abcd} can be constructed out of the curvature in a manner closely analogous to the way in which the stress tensor of the electromagnetic field is constructed out of F_{ab} (eq. [4.2.27]). We define the *Bel-Robinson tensor* in terms of the Weyl tensor by

$$\begin{aligned} T_{abcd} &= C_{aecf} C_b{}^e{}_d{}^f + \frac{1}{4} \epsilon_{ae}{}^{hi} \epsilon_b{}^{ej}{}_k C_{hicf} C_j{}^k{}_d{}^f \\ &= C_{aecf} C_b{}^e{}_d{}^f - \frac{3}{2} g_{a[b} C_{jk]cf} C^{jk}{}_d{}^f \ , \end{aligned}$$

where ϵ_{abcd} is defined in appendix B and equation (B.2.13) was used. It follows that $T_{abcd} = T_{(abcd)}$. (This is established most easily from the spinor decomposition of the Weyl tensor given in chapter 13.)

(a) Show that $T^a{}_{acd} = 0$.

(b) Using the Bianchi identity (3.2.16), show that in vacuum, $R_{ab} = 0$, we have $\nabla^a T_{abcd} = 0$.

Solution of (a)

We begin expanding the anti-commutator in the second expression for T_{abcd} and contracting a with b.

$$\begin{aligned}\frac{3}{2}g^a{}_{[a}C_{jk]cf}C^{jk}{}_d{}^f &= \frac{1}{2}[g^a{}_aC_{jkcf} + g^a{}_jC_{kacf} + g^a{}_kC_{ajcf}]C^{jk}{}_d{}^f \\ &= \frac{1}{2}[4C_{jkcf} + C_{kjcf} + C_{kjcf}]C^{jk}{}_d{}^f \\ &= \frac{1}{2}2C_{jkcf}C^{jk}{}_d{}^f\end{aligned}$$

This implies

$$T^a{}_{acd} = C_{aecf}C^{ae}{}_d{}^f - C_{jkcf}C^{jk}{}_d{}^f = 0 \ .$$

Problem 7

(a) Show that the total energy E, equation (4.4.55), is time independent, i.e., the value of E is unchanged if the integral is performed over a time translate, Σ', of Σ.

(b) Let ξ_a be a gauge transformation which vanishes outside a bounded region of space. Show that $E[\gamma_{ab}] = E[\gamma_{ab} + 2\partial_{(a}\xi_{b)}]$ by comparing them with $E[\gamma_{ab} + 2\partial_{(a}\xi'_{b)}]$ where ξ'_a is a new gauge transformation which agrees with ξ_a in a neighborhood of the hyperplane Σ but vanishes in a neighborhood of another hyperplane Σ'.

Solution of (a)

Let Σ_t be the collection of points (x^0, x^1, x^2, x^3) in spacetime such that $x^0 = t$. Then the total energy associated with γ_{ab} computed at Σ_t is

$$\begin{aligned}E(t) &= \int_{\Sigma_t} t_{00} d^3x \\ &= \int_{\mathbb{R}^3} t_{00}(t, x^1, x^2, x^3) dx^1 dx^2 dx^3 \ .\end{aligned}$$

We compute the derivative dE/dt.

$$\begin{aligned}\frac{dE}{dt} &= \int_{\mathbb{R}^3} \frac{\partial t_{00}}{\partial t} d^3x \\ &= \int_{\mathbb{R}^3} \frac{\partial t_{10}}{\partial x^1} + \frac{\partial t_{20}}{\partial x^2} + \frac{\partial t_{30}}{\partial x^3} d^3x = 0\end{aligned}$$

In the second equality it was used that $\partial^a t_{ab} = 0$. The integrand in the second line is the divergence of the vector field (t_{10}, t_{20}, t_{30}). The decay hypothesis on γ_{ab} described in the text after figure 4.2 imply that this field goes to zero sufficiently rapidly, so that the divergence theorem gives the third equality.

Solution of (b)

$$\begin{aligned}
E[\gamma_{ab}] &= \int_{\Sigma'} t_{00}[\gamma_{ab}] d^3x \\
&= \int_{\Sigma'} t_{00}[\gamma_{ab} + 2\partial_{(a}\xi'_{b)}] d^3x \\
&= \int_{\Sigma} t_{00}[\gamma_{ab} + 2\partial_{(a}\xi'_{b)}] d^3x \\
&= \int_{\Sigma} t_{00}[\gamma_{ab} + 2\partial_{(a}\xi_{b)}] d^3x \\
&= E[\gamma_{ab} + 2\partial_{(a}\xi_{b)}]
\end{aligned}$$

Problem 8

Two point masses of mass M are attached to the ends of a spring of spring constant K. The spring is set into oscillation. In the quadrupole approximation, equation (4.4.58), what fraction of the energy of oscillation of the spring is radiated away during one cycle of oscillation?

Solution

Let L denote the distance between the two masses. By elementary mechanics,

$$L - L_0 = A \sin \omega t \ ,$$

where L_0 is the equilibrium length of the spring and $\omega = \sqrt{2K/M}$. Here the origin of time is chosen so that $L = L_0$ at $t = 0$.

Letting the x-axis coincide with the line through the centers of the masses and the origin of space be at the midpoint between them, we compute $q_{\mu\nu}$. Because mass is concentrated, the integral in formula (4.4.48) becomes a sum. It is easy to check that

$$q_{xx} = \tfrac{3}{2} M L^2$$

and that the other components $q_{\mu\nu}$ vanish. From this we compute $Q_{\mu\nu}$ and P using equations (4.4.58) and (4.4.59).

$$\begin{aligned}
&Q_{xx} = q_{xx} - \tfrac{1}{3} q_{xx} = ML^2 \\
&Q_{yy} = Q_{zz} = -\tfrac{1}{3} q_{xx} = -\tfrac{1}{2} ML^2 \\
&\text{other components} = 0
\end{aligned}$$

$$\begin{aligned}
P &= \frac{1}{45}\left(\frac{d^3Q_{xx}}{dt^3}\bigg|_{\text{ret}}\right)^2\left(1+\frac{1}{4}+\frac{1}{4}\right)\\
&= \frac{1}{30}\left(\frac{3}{2}M\frac{d^3}{dt^3}(L_0+A\sin\omega t)^2\bigg|_{\text{ret}}\right)\\
&= \frac{16}{15}\frac{A^2K^3}{M}\cos^2\omega t_{\text{ret}}(4A\sin\omega t_{\text{ret}}+L_0)^2
\end{aligned}$$

The energy radiated in one period is

$$E_{\text{rad}} = \int_0^{2\pi/\omega} P\ dt\ .$$

Computing the integral (using Maple), we find that the ratio E_{rad}/E, where $E=\frac{1}{2}KA^2$, is

$$\frac{16}{15}\sqrt{2}\pi K^{3/2}M^{-1/2}(4A^2+L_0^2)\ .$$

Problem 9

A binary star system consists of two star of mass M and of negligible size in a nearly Newtonian circular orbit of radius R around each other. Assuming the validity of equation (4.4.58), calculate the rate of increase of the orbital frequency due to emission of gravitational radiation.

Solution

To do this we proceed in three steps.

Step 1. We relate the time derivative of the orbital frequency, $d\omega/dt$, to the power P radiated in form of gravitational radiation.

Let the system execute its motion in the xy-plane with the origin of the coordinates coinciding with its center of mass. The elementary relation

$$MR\omega^2 = \frac{M^2}{4R^2}$$

(left hand side is mass times acceleration and right hand side is force) gives the orbital frequency of the system: $\omega = \sqrt{M/4R^3}$. The kinetic energy is $M^2/4R$, and the potential energy is $-M^2/2R$, so the energy of the system is

$$E = -\frac{M}{4R^3}\ .$$

These expressions allow us to write

$$\omega(E) = \sqrt{\frac{-8E^3}{M^5}}\ ,$$

so that

$$\frac{d\omega}{dt} = \frac{d\omega}{dE}\frac{dE}{dt} = 2^{-1/2}3(MR)^{-3/2}P \ .$$

Step 2. We compute P.

The position vector of one of the masses is

$$\vec{x_1} = R(\cos\omega t, \sin\omega t)$$

with the origin of time suitably chosen. The position vector of the second mass is $\vec{x_2} = -\vec{x_1}$. Using equation (4.4.8), we compute the components of the quadrupole moment tensor.

$$\begin{aligned}
&q_{xx} = 6M\cos^2\omega t \\
&q_{xy} = q_{yx} = 6M\sin\omega t\cos\omega t \\
&q_{xy} = 6M\sin^2\omega t \\
&\text{other components} = 0
\end{aligned}$$

The trace-free quadrupole moment tensor is then given by

$$\begin{aligned}
&Q_{xx} = 6M\cos^2\omega t - 2M \ , \\
&Q_{xy} = Q_{yx} = 6M\sin\omega t\cos\omega t - 2M \ , \\
&Q_{xy} = 6M\sin^2\omega t - 2M \ , \\
&\text{other components} = -2M \ .
\end{aligned}$$

Using Maple to compute P via (4.4.58), we find

$$P = \frac{2}{5}M^5R^9 \ .$$

Step 3. We put the results of the two previous steps together.

$$\frac{d\omega}{dt} = \frac{3\sqrt{2}}{5}M^{7/2}R^{15/2}$$

Chapter 5

Homogeneous, Isotropic Cosmology

Problem 1

Show that the Robertson-Walker metric, equation (5.1.11), can be expressed in the form

$$ds^2 = -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right] . \tag{5.1}$$

What portion of the 3-sphere (k=+1) can be covered by these coordinates?

Solution

The metric of Euclidean space, $k = 0$, can be put in the form above by use of spherical coordinates. Consider the case of 3-sphere geometry. The metric is

$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\ d\phi^2) .$$

To write ds^2 in the desired form, change the coordinates from (ψ, θ, ϕ) to (r, θ, ϕ), where $r = \sin\psi$. By the tensor transformation law (2.3.8),

$$\begin{aligned}(ds^2)_{rr} &= \sum_{\mu,\nu}(ds^2)_{\mu\nu}\frac{\partial x^\mu}{\partial r}\frac{\partial x^\nu}{\partial r} ,\\ &= (ds^2)_{\psi\psi}\left(\frac{d\psi}{dr}\right)^2 ,\\ &= \frac{1}{(\partial r/\partial\psi)^2} ,\\ &= \frac{1}{\cos^2\psi} = \frac{1}{1-r^2} .\end{aligned}$$

It is easy to see that computing the other coefficients of ds^2 in (r, θ, ϕ) coordinates by this method will yield formula (5.1).

To see how much of the 3-sphere the coordinates (r, θ, ϕ) cover, we recall the definition of (ψ, θ, ϕ).

$$\begin{aligned}
x &= \cos\phi\cos\theta\cos\psi \\
y &= \cos\phi\cos\theta\sin\psi \\
z &= \cos\phi\sin\theta \\
w &= \sin\phi
\end{aligned}$$

Using the tensor transformation law, we can verify expressions such as

$$dx = -\sin\phi\cos\theta\cos\psi \; d\psi - \cos\phi\sin\theta\cos\psi \; d\theta - \cos\phi\cos\theta\sin\psi \; d\phi$$

and use them to show that $dx^2 + dy^2 + dz^2 + dw^2$ indeed equals

$$d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta \; d\phi^2) \; .$$

This justifies the "definition" given above.

Now, by letting (ψ, θ, ϕ) range in $(0, 2\pi) \times (0, \pi) \times (0, \pi)$, we essentially cover the 3-sphere in the same way spherical coordinates essentially cover the 2-sphere in 3-space. To see this, we note that the points on the 3-sphere satisfy

$$\begin{aligned}
&w^2 \leq 1 \; , && w^2 + z^2 \leq 1 \; , \\
&w^2 + z^2 + y^2 \leq 1 \; , && w^2 + z^2 + y^2 + x^2 = 1 \; .
\end{aligned}$$

So it is clear that with the above range, w can take any possible value. If we fix some choice of w, it is clear that z can take any possible value. Similarly, x and y can also take take any values which would be possible for points on a 3-sphere.

Letting r range in $(-1, 1)$ while ψ ranges in $(\pi/2, 3\pi/2)$, half of the 3-sphere is (essentially) covered.

Problem 2

Derive Einstein's equations, (5.2.14) and (5.2.15), for the 3-sphere ($k = +1$) and hyperboloid ($k = -1$) cases.

Solution

We do this in two steps.

Step 1. We compute the Ricci tensor components using the coordinate component methods of section 3.4a.

We review the method here. First we choose a coordinate system. Then we compute the Christoffel symbol components using equation (3.1.30). Finally, we compute the Ricci tensor components using (3.4.5).

The best coordinate system to use is the one of the previous problem because it allows us to handle all three geometries at the same time. The computations (3.4.5) and (3.1.30) are straightforward and can easily be done using Mathematica or Maple.

The nonvanishing Christoffel and Ricci components follow.

$$\Gamma^{\tau}{}_{rr} = \frac{aa'}{1-kr^2} \qquad \Gamma^{\tau}{}_{\theta\theta} = r^2 a\dot{a}$$

$$\Gamma^{\tau}{}_{\phi\phi} = r^2 a\dot{a}\sin^2\theta \qquad \Gamma^{r}{}_{\tau r} = \Gamma^{r}{}_{r\tau} = \Gamma^{\theta}{}_{\tau\theta} = \Gamma^{\theta}{}_{\theta\tau} = \Gamma^{\phi}{}_{\tau\phi} = \Gamma^{\phi}{}_{\phi\tau} = \frac{\dot{a}}{a}$$

$$\Gamma^{r}{}_{rr} = \frac{kr}{1-kr^2} \qquad \Gamma^{r}{}_{\theta\theta} = -r(1-kr^2)$$

$$\Gamma^{r}{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta \qquad \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \Gamma^{\phi}{}_{r\phi} = \Gamma^{\theta}{}_{\phi r} = \frac{1}{r}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

$$R_{\tau\tau} = -\frac{3\dot{a}'}{a} \qquad R_{\theta\theta} = -1 + 2kr^2 - \cot^2\theta + \csc^2\theta + 2r^2\dot{a}^2 + r^2 a\ddot{a}$$

$$R_{rr} = \frac{2k + 2\dot{a}^2 + \dot{a}\ddot{a}}{1-kr^2} \qquad R_{\phi\phi} = r^2\sin^2\theta(2k + 2\dot{a}^2 + a\ddot{a})$$

Step 2. We derive equations (5.2.14) and (5.2.15) by writing

$$R_{\tau\tau} + \tfrac{1}{2}R = 8\pi\rho \ , \tag{5.2}$$
$$R_{**} - \tfrac{1}{2}R = 8\pi P \ , \tag{5.3}$$

in terms of ρ, P, k and a and its derivatives.

Notice that $R_{\tau\tau}$ was found in step 1 and $R = -R_{\tau\tau} + R_{**}$, so all that is needed is to find a formula for R_{**}. By definition, $R_{**} = R_{ab}u^a u^b$, where u is some spacelike unit vector. Letting $X^a = (\partial/\partial\phi)^a$, we have, for instance,

$$R_{**} = \frac{R_{ab}X^a X^b}{g_{ab}X^a X^b} = \frac{R_{\phi\phi}}{g_{\phi\phi}} \ .$$

Using Mathematica or Maple to plug the results of step 1 into the left hand side of (5.2) and (5.3) and simplifying the resulting expressions yields

$$\frac{3k}{a^2} + \frac{3\dot{a}^2}{a^2} = 8\pi\rho \ , \tag{5.4}$$
$$-\frac{k}{a^2} - \frac{\dot{a}^2}{a^2} - \frac{2\ddot{a}}{a} = 8\pi P \ . \tag{5.5}$$

It is easy to check that the pair of equations (5.4) and (5.5) is equivalent to the pair (5.2.14) and (5.2.15). (We show the implication

$$\text{(5.4) and (5.5)} \Rightarrow \text{(5.2.14) and (5.2.15)}$$

in the general case of possibly nonzero cosmological constant in the next problem.)

Problem 3

(a) Consider the modified Einstein's equation (5.2.17) with cosmological constant Λ. Write out the analogs of equations (5.2.14) and (5.2.15) with the Λ-terms. Show that static solutions of these equations are possible if and only if $k = +1$ (3-sphere) and $\Lambda > 0$. (These solutions are called *Einstein static universes.*) For a dust filled Einstein universe ($P = 0$), relate the "radius of the universe" a to the density ρ. Examine small perturbations from the "equilibrium" value of a, and show that the Einstein static universe is unstable.

(b) Consider the modified Einstein equation with $\Lambda > 0$ and $T_{ab} = 0$. Obtain the spatially homogeneous and isotropic solution in the case $k = 0$. (The resulting space-time actually is spacetime homogeneous and isotropic and is known as the *de Sitter spacetime.* The solutions with $k = \pm 1$ correspond to different choices of spacelike hypersurfaces in this spacetime. See Hawking and Ellis 1973 for further details.)

Solution of (a)

The Einstein equation with cosmological constant reads

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \ .$$

So the analogs of equations (5.2.11) and (5.2.12) are

$$\begin{aligned} G_{\tau\tau} - \Lambda &= 8\pi\rho \ , \\ G_{**} + \Lambda &= 8\pi P \ . \end{aligned}$$

Using the expressions for $G_{\tau\tau}$ and G_{**} found in the solution of the preceding problem, these equations become

$$3\frac{\dot{a}^2}{a^2} = 8\pi\rho + \Lambda - \frac{3k}{a^2} \ , \tag{5.6}$$

$$-2\frac{\ddot{a}}{a} = 8\pi P - \Lambda + \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} \ . \tag{5.7}$$

This system is equivalent to

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = 8\pi\rho + \Lambda - \dfrac{3k}{a^2} \\ 3\dfrac{\ddot{a}}{a} = -4\pi(\rho + 3P) + \Lambda \ , \end{cases} \tag{5.8}$$

where the second equation is obtained by plugging (5.6) into (5.8). System (5.8) constitutes the analog of equations (5.2.14) and (5.2.15).

Suppose now that (5.8) admits a static ($\dot{a} = 0$) solution. The second equation implies (assuming $\rho + 3P > 0$) that $\Lambda > 0$. From this (and the assumption $\rho > 0$), it follows from the first equation that $k > 0$, i.e., $k = +1$. Conversely, if $k = 1$ and $\Lambda > 0$, then it is possible to choose physically plausible ρ and P such that (5.8) admits a static solution.

We now write a in terms of ρ in the case of a static, dust filled universe. From the second equation in (5.8), it follows that $\Lambda = 4\pi\rho$. Plugging this into the first equation (and using using $k = 1$), we find

$$a = \frac{1}{\sqrt{4\pi\rho}} \ .$$

Finally, we study the stability of the static solution for the dust filled universe. Consider equations (5.8) together with the conditions $\rho = 1/4\pi a$, $P = 0$, $k = 1$, $\Lambda > 0$. The system (5.8) becomes

$$\begin{cases} 3\dfrac{\dot{a}^2}{a^2} = \dfrac{2}{a^2} - \dfrac{3}{a^2} + \Lambda \\ 3\dfrac{\ddot{a}}{a} = -\dfrac{1}{a^2} + \Lambda \ . \end{cases} \tag{5.9}$$

Thus $\ddot{a} = \dot{a}^2/a$. The solutions of this differential equation are $a = C_1 e^{C_2\tau}$. Any deviation from $C_2 = 0$ results in a exponential evolution of the scale factor, i.e., the static solution is unstable.[1]

Solution of (b)

When $k = 0$, the metric has the form

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2) \ .$$

According to (5.8), the conditions $k = 0$ and $T_{ab} = 0$ imply that the scale factor a satisfies the equations

$$3\frac{\dot{a}^2}{a^2} = \Lambda$$
$$3\frac{\ddot{a}}{a} = \Lambda \ .$$

[1]Problem with the stability analysis: The function $a = C_1 e^{C_2\tau}$ does not exactly satisfy (5.9). Any suggestions?

In item (a), it was noted that $a = C_1 e^{C_2 \tau}$ solves these equations. So the metric takes the form

$$ds^2 = -d\tau^2 + e^{H\tau}(dx^2 + dy^2 + dz^2) \ ,$$

where $H = 2C_2$ and the coordinates were rescaled in such a way that the constant C_1 does not appear in this expression. Conversely, any metric of this form with $H = 2\sqrt{\Lambda/3}$ satisfies Einstein's equation.

Problem 4

Derive the cosmological redshift formula (5.3.6) by the following argument:

(a) Show that $\nabla_a u_b = (\dot{a}/a)h_{ab}$, where h_{ab} is defined by equation (5.1.10) and $\dot{a} = da/d\tau$.

(b) Show that along any null geodesic we have $d\omega/d\lambda = -k^a k^b \nabla_a u_b = -(\dot{a}/a)\omega^2$, where λ is the affine parameter along the geodesic.

(c) Show that the result of (b) yields equation (5.3.6).

Solution of (a)

This isn't really a "solution" since I got a sign wrong in the end. I would be very thankful if somebody sent me an email with a correct solution.

Using the defining equation for h_{ab}, we find

$$u^c h_{ca} = u^c g_{ca} + u^c u_c u_a = u_a - u_a = 0 \ .$$

It follows that

$$\nabla_b(u^c h_{ca}) = 0 = u^c \nabla_b h_{ca} + h_{ca} \nabla_b u^c \ . \tag{5.10}$$

The equality on the right translates (except for a minus sign) into the equation we are trying to prove. The term $h_{ca}\nabla_b u^c$ expands to

$$u_a \cancel{u_c \nabla_b u^c} + g_{ca} \nabla_b u^c = \nabla_b u_a \ .$$

To show $u^c \nabla_b h_{ca} = (\dot{a}/a)h_{ab}$ we work with the coordinates appearing on the Robertson-Walker metric as it is on the description of problem 1.

$$\begin{aligned}\sum_\sigma u^\sigma \nabla_\nu h_{\sigma\mu} &= -\nabla_\nu h_{\tau\mu} \\ &= -\partial_\nu \cancel{h_{\tau\mu}} + \sum_\sigma \Gamma^\sigma{}_{\nu\tau} h_{\sigma\mu} + \sum_\sigma \Gamma^\sigma{}_{\nu\mu} \cancel{h_{\tau\sigma}} \\ &= \Gamma^\nu{}_{\nu\tau} h_{\nu\mu} = (\dot{a}/a) h_{\nu\mu}\end{aligned}$$

The last two equalities are established by using the expressions for the Christoffel symbol components computed in the solution of problem 2. The components for the plane geometry ($k = 0$) can also be found on page 97 of the book.

Plugging the results of the preceding paragraph in equation (5.10), we get

$$(\dot{a}/a)h_{ba} + \nabla_b u_a = 0 \ ,$$

which is the equation we wished to prove, except for a minus sign.

Solution of (b)

One of the equalities is established by noticing that $\omega = -k_a u^a$ and, because the geodesic is null, $k^a k_a = 0$.

$$\begin{aligned} -k^a k^b \nabla_a u_b &= -(\dot{a}/a) k^a k^b h_{ab} \\ &= -(\dot{a}/a)(k^a k^b u_a u_b + k^a k^b g_{ab}) \\ &= -(\dot{a}/a)(\omega^2 + \cancel{k^a k_a}) \end{aligned}$$

The other equality follows from the very clever observation that a geodesic satisfies the geodesic equation: $k^a \nabla_a k^b = 0$.

$$\begin{aligned} d\omega/d\lambda &= -k^a \nabla_a (k_b u^b) \\ &= -(k^a \nabla_a k_b) u^b - (k^a \nabla_a u^b) k_b \\ &= -k^a k^b \nabla_a u_b \end{aligned}$$

Solution of (c)

We wish to show $\omega_2/\omega_1 = a(\tau_2)/a(\tau_1)$ or, equivalently, $a_1\omega_1 = a_2\omega_2$. To this effect we show that the derivative of $a\omega$ along the null geodesic vanishes.

$$\begin{aligned} \frac{d(a\omega)}{d\lambda} &= \omega \frac{da}{d\lambda} + a \frac{d\omega}{d\lambda} \\ &= \omega \frac{da}{d\lambda} - \omega^2 \frac{da}{d\tau} \end{aligned}$$

Now, the τ component of k^a is $d\tau/d\lambda$. We get this component by contracting $-u_c k^c$. The minus sign is there to account for the fact that $u^c = (1, 0, 0, 0)$ and therefore $u_c = (-1, 0, 0, 0)$. So the formula for ω gives $d\tau/d\lambda$. This implies

$$\frac{da}{d\lambda} = \omega \frac{da}{d\tau}$$

and consequently $d(a\omega)/d\lambda = 0$.

Problem 5

Consider a radial ($d\theta/d\lambda = d\phi/d\lambda = 0$) null geodesic propagating in a Robertson-Walker cosmology, equation (5.1.11).

(a) Show that for all three spatial geometries the change in the coordinate ψ of the ray between times τ_1 and τ_2 is $\Delta\psi = \int_{\tau_1}^{\tau_2} d\tau/a(\tau)$. [Here, in the flat case ($k = 0$), ψ is defined to be the ordinary radial coordinate, $\psi = (x^2 + y^2 + z^2)^{1/2}$.]

(b) Show that in the dust filled spherical model, a light ray emitted at the big bang travels precisely all the way around the universe by the time of the "big crunch."

(c) Show that in the radiation filled spherical model, a light ray emitted at the big bang travels precisely halfway around the universe by the time of the "big crunch."

Solution of (a)

Let k^a be the tangent to a radial, null geodesic. Equation

$$0 = ds^2(k,k) = -\left(\frac{d\tau}{d\lambda}\right)^2 + a^2\left(\frac{d\psi}{d\lambda}\right)^2$$

holds for any k. It follows that

$$\frac{d\tau}{d\lambda} = \pm a \frac{d\psi}{d\lambda} \ .$$

We use this to compute $\Delta\psi$.

$$\Delta\psi = \int_{\lambda_1}^{\lambda_2} \frac{d\psi}{d\lambda} d\lambda = \pm \int_{\lambda_1}^{\lambda_2} \frac{1}{a}\frac{d\tau}{d\lambda} d\lambda = \pm \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)}$$

Here λ_i is the parameter value corresponding to τ_i.

Solution of (b)

We assume the light ray's world line to be a radial, null geodesic.

The solution to the spherical dust filled model is given in table 5.1:

$$a = \tfrac{1}{2}C(1 - \cos\eta) \ ,$$
$$\tau = \tfrac{1}{2}C(\eta - \sin\eta) \ .$$

According to item (a), the change in the angular coordinate ψ between two consecutive values of η such that $a(\eta) = 0$ is

$$\Delta\psi = \int_0^{2\pi} \frac{1 - \cos\eta}{1 - \cos\eta} d\eta = 2\pi.$$

Solution of (c)

We assume the light ray's world line to be a radial, null geodesic.

The solution to the spherical radiation filled model is given in table 5.1:

$$a(\tau) = B\sqrt{1-(1-\tau/B)^2}\ .$$

Here B corresponds to $\sqrt{C'}$ in the formula given in the table. According to item (a), the change in the angular coordinate ψ between two consecutive values of η such that $a(\eta) = 0$ is

$$\Delta\psi = \frac{1}{B}\int_0^{2B} \frac{d\tau}{\sqrt{1-(1-\tau/B)^2}} = -\pi.$$

The integral is easily computed by making the substitution $\sin t = 1-\tau/B$.

Chapter 6

The Schwarzschild Solution

Problem 1

Let M be a three-dimensional manifold possessing a spherically symmetric Riemannian metric with $\nabla_a r \neq 0$, where r is defined by equation (6.1.3).

(a) Show that a new "isotropic" radial coordinate $\tilde{r}$ can be introduced so that the metric takes the form $ds^2 = H(\tilde{r})[d\tilde{r}^2 + \tilde{r}^2\ d\Omega^2]$. (This shows that every spherically symmetric three-dimensional space is conformally flat.)

(b) Show that in isotropic coordinates the Schwarzschild metric is

$$ds^2 = -\frac{(1 - M/2\tilde{r})^2}{(1 + M/2\tilde{r})^2} dt^2 + \left(1 + \frac{M}{2\tilde{r}}\right)^4 [d\tilde{r}^2 + \tilde{r}^2\ d\Omega^2].$$

Solution to (a)

By spherical symmetry and the condition $\nabla_a r \neq 0$, the metric on M can be written as

$$ds^2 = h(r)^2\ dr^2 + r^2\ d\Omega^2\ ,$$

where, as usual, $d\Omega^2 = d\theta^2 + \sin^2\theta\ d\phi^2$. We wish to find $\tilde{r}$ so that

$$H(\tilde{r})\tilde{r}^2 = r^2\ ,$$
$$\sqrt{H(\tilde{r})}\frac{dr}{d\tilde{r}} = h(r)$$

for some H. Using the first equation to write $H(\tilde{r}) = r^2/\tilde{r}^2$ and plugging this in the second equation, we find

$$\frac{r}{\tilde{r}}\frac{dr}{d\tilde{r}} = h(r)\ , \tag{6.1}$$

or

$$\frac{d\tilde{r}}{dr} = \frac{\tilde{r}}{h(r)r}\ .$$

From the theory of ordinary differential equations, we know this has a solution. We use this solution as the new coordinate.

Solution to (b)

We do this by defining $\tilde{r}$ in a way that seems natural and checking that it yields the correct form for the metric.

In one hand we want $H(\tilde{r})$ to have the form $(1 + M/2\tilde{r})^4$. On the other we saw in the solution to item (a) that $H(\tilde{r}) = r^2/\tilde{r}^2$. Putting these two equalities together, one gets

$$r = \tilde{r}\left(1 + \frac{M}{2\tilde{r}}\right)^2 .$$

It is straightforward to check that with this relation between r and $\tilde{r}$, equations

$$\frac{(1 - M/2\tilde{r})^2}{(1 + M/2\tilde{r})^2} = \left(1 - \frac{2M}{r}\right)$$

and (6.1) with $h(r) = (1 - 2M/r)^{-1}$ hold. Thus the Schwarzschild metric takes the desired form in these coordinates.

Problem 2

Calculate the Ricci tensor, R_{ab}, for a static, spherically symmetric spacetime, equation (6.1.5), using the coordinate component method of section 3.4a. Compare the amount of labor involved with that of the tetrad method approach given in the text.

Solution

It took me 15 minutes to do this with Mathematica. The computer didn't get stuck trying to do any "dumb algebra" and everything went fine. Feel free to send me an email if you'd like a copy of the code.

Problem 3

Consider the source-free ($j^a = 0$) Maxwell's equations (4.3.12) and (4.3.13) in a static, spherically symmetric spacetime, equation (6.1.5).

(a) Argue that the general form of a Maxwell tensor which shares the static and spherical symmetries of the spacetime is $F_{ab} = 2A(r)(e_0)_{[a}(e_1)_{b]} + 2B(r)(e_2)_{[a}(e_3)_{b]}$, where the $(e_\mu)_a$ are defined by equation (6.1.6).

(b) Show that if $B(r) = 0$, the general solution of Maxwell's equations with the form of part (a) is $A(r) = -q/r^2$, where q may be interpreted as the total charge. [The solution obtained with $B(r) \neq 0$ is a "duality rotation" of this solution, representing the field of a magnetic monopole.]

(c) Write down and solve Einstein's equation, $G_{ab} = 8\pi T_{ab}$, with the electromagnetic stress-energy tensor corresponding to the solution of part (b). Show that the general solution is the *Reissner-Nordstrom metric*,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2 .$$

Solution to (a)

The most general form of a 2-form F_{ab} in Schwarzschild spacetime is

$$F_{ab} = A(e_0)_{[a}(e_1)_{b]} + B(e_0)_{[a}(e_2)_{b]} + C(e_0)_{[a}(e_3)_{b]} + \\ + D(e_1)_{[a}(e_2)_{b]} + E(e_1)_{[a}(e_3)_{b]} + G(e_2)_{[a}(e_3)_{b]} ,$$

where A, B, C, D, E and G are functions of (t, r, θ, ϕ).

Because F_{ab} is assumed to be static, the functions $A, \ldots, G$ do not depend on t. Also, they do not depend on θ of ϕ because of spherical symmetry. So it only remains to show that B, C,D and E vanish.

Let f be a spacetime symmetry (a diffeomorphism) which rotates θ by π. It is intuitively clear that under the action of f (a rotation by π), the differential $d\phi$ becomes $f^*d\phi = -d\phi$. This can also be checked using the definitions found in Appendix C of the book. The other differentials remain unchanged:

$$f^*dt = dt ,$$
$$f^*dr = dr ,$$
$$f^*d\theta = d\theta .$$

It follows that

$$f^*F_{ab} = A(e_0)_{[a}(e_1)_{b]} + B(e_0)_{[a}(e_2)_{b]} - C(e_0)_{[a}(e_3)_{b]} + \\ + D(e_1)_{[a}(e_2)_{b]} - E(e_1)_{[a}(e_3)_{b]} + G(e_2)_{[a}(e_3)_{b]} .$$

Because $f^*F_{ab} = F_{ab}$, it follows that $C = E = 0$. Similarly, one can show that $B = D = 0$.

Solution to (b)

In differential forms notation, $\boldsymbol{F} = A(r)\boldsymbol{e}_0 \wedge \boldsymbol{e}_1$, since $B(r) = 0$. Maxwell's equations for $\boldsymbol{F}$ read

$$d * \boldsymbol{F} = 0 , \qquad d\boldsymbol{F} = 0 ,$$

because there are no sources (see problem 2(b) in chapter 4). As will be discussed below,

$$*\{A(r)\boldsymbol{e}_0 \wedge \boldsymbol{e}_1\} = A(r)\boldsymbol{e}_2 \wedge \boldsymbol{e}_3. \tag{6.2}$$

Writing $\boldsymbol{e}_2 = r\ d\theta$ and $\boldsymbol{e}_3 = r\sin\theta\ d\phi$ and using a few properties of the d operator, we find

$$0 = d * \boldsymbol{F} = d\{A(r)r^2 \sin\theta\ d\theta \wedge d\phi\} \tag{6.3}$$

$$= d\{A(r)r^2 \sin\theta\}\ d\theta \wedge d\phi \tag{6.4}$$

$$= \frac{\partial}{\partial r}(A(r)r^2)\sin\theta\ dr \wedge d\theta \wedge d\phi\ . \tag{6.5}$$

From this we conclude that $A(r)r^2$ equals some constant $-q$, or $A(r) = -q/r^2$. This is the equality we wished to prove.

When doing these computations, I didn't have in mind the formula for the d operation given in Appendix B of the book, but instead the following characterization of it:

Theorem. *Let M be an n-dimensional manifold. Denote by Λ^k the vector space of all k-forms on M. There exists a unique linear map*

$$d : \Lambda^k \longrightarrow \Lambda^{k+1}\ ,$$

defined for all k, such that:

(1) If f is a 0-form, that is, if it is a function, then df is the 1-form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i\ ,$$

where $\{x^1, \ldots, x^n\}$ can be any set of coordinates on M.

(2) If $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$ are forms of orders k and l, respectively, then

$$d(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = d\boldsymbol{\omega} \wedge \boldsymbol{\eta} + (-1)^k \boldsymbol{\omega} \wedge d\boldsymbol{\eta}\ .$$

(3) For every form $\boldsymbol{\omega}$,

$$d(d\boldsymbol{\omega}) = 0\ .$$

I wont prove this theorem here. It is a nice exercise to prove (6.3)-(6.5) using only properties (1), (2) and (3).

Now we discuss (6.2). The $*$ operator defined in the problem set for chapter 4, which is also called the "Hodge star operator", has a complicated formula, but what it really does is simple. If $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a positive basis of the space of 1-forms, then

$$*(\boldsymbol{e}_0) = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\ ,$$

$$*(\boldsymbol{e}_0 \wedge \boldsymbol{e}_1) = \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\ ,$$

$$*(\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \boldsymbol{e}_3\ .$$

In other contexts one defines the Hodge star operation by requiring it to be linear and that these equations hold for every even permutation of the indices $\{1, 2, 3, 4\}$.

This is all really nice, but we now turn to prove (6.2) using the regular abstract index notation. According to the definition given in chapter 4,

$$*(A(r)\ e_0 \wedge e_1)_{cd} = \frac{1}{2}(A(r)\ e_0 \wedge e_1)^{ab}\epsilon_{abcd}\ . \tag{6.6}$$

We must find what each term on the right hand side means.

According to (B.2.17),

$$\boldsymbol{\epsilon} = \sqrt{|g|}\ dt \wedge dr \wedge d\theta \wedge d\phi\ .$$

It is easy to see from formulas (6.1.6a-d) (which define the orthonormal basis $\{\boldsymbol{e}_0, \ldots, \boldsymbol{e}_3\}$) that $\sqrt{|g|} = \sqrt{fg}r^2 \sin\theta$. Looking at these formulas again, we find

$$\boldsymbol{\epsilon} = \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\ .$$

Now, from definition,

$$(e_0 \wedge e_1)_{ab} = 2(e_0)_{[a}(e_1)_{b]}\ ,$$
$$(e_0 \wedge e_1 \wedge e_2)_{abc} = 3(e_0 \wedge e_1)_{[ab}(e_2)_{c]}\ .$$

Plugging the first equation into the second and expanding, one finds that

$$(e_0 \wedge e_1 \wedge e_2)_{abc} = 6(e_0)_{[a}(e_1)_b(e_2)_{c]}\ .$$

Having proved this, it isn't hard to believe that

$$\begin{aligned}\boldsymbol{\epsilon} &= \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \\ &= 24(e_0)_{[a}(e_1)_b(e_2)_c(e_3)_{d]}\ .\end{aligned} \tag{6.7}$$

Now, one has

$$A(r)(e_0 \wedge e_1)_{ab} = A(r)[(e_0)_a(e_1)_b - (e_0)_b(e_1)_a]\ ,$$

and therefore

$$A(r)(e_0 \wedge e_1)^{ab} = -A(r)[(e_0)^a(e_1)^b - (e_0)^b(e_1)^a\ , \tag{6.8}$$

since using an orthonormal basis indices are raised and lowered as in Minkowsky spacetime: time components change sign, space components don't. Putting (6.7) and (6.8) into (6.6) and contracting (this isn't hard thanks to orthonormality) results in

$$\begin{aligned}*(A(r)\ e_0 \wedge e_1)_{cd} &= (1/2)A(r)4(e_2)_{[c}(e_3)_{d]} \\ &= A(r)(e_2 \wedge e_3)_{cd}\ .\end{aligned}$$

This is (6.2).

Solution of (c)

First we compute

$$T_{ab} = \frac{1}{4\pi}\left\{F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{de}F^{de}\right\} \ .$$

Remember that raising or lowering the index of $\boldsymbol{e}_0$ produces a minus sign, and that raising or lowering the index of the other basis elements has no effect on the components. So if $B_a = 3(e_0)_a + 11(e_1)_a$, then $B^a = -3(e_0)^a + 11(e_1)^a$.

$$\begin{aligned} F_{ac}F_b{}^c &= A^2[(e_0)_a(e_1)_c - (e_0)_c(e_1)_a][-(e_1)_b(e_0)^c + (e_0)_b(e_1)^c] \\ &= A^2[(e_0)_a(e_0)_b - (e_1)_a(e_1)_b] \ . \end{aligned}$$

Similarly, $F_{dc}F^{cd} = -2A^2$. Putting these results together yields

$$T_{ab} = \frac{A^2}{8\pi}[(e_0)_a(e_0)_b - (e_1)_a(e_1)_b + (e_2)_a(e_2)_b + (e_3)_a(e_3)_b] \ .$$

Einstein's equations then read

$$\begin{aligned} \frac{q^2}{r^4} &= \frac{h'}{rh^2} + \frac{1}{r^2}\left(1 - \frac{1}{h}\right) \ , && [8\pi T_{00} = G_{00}] \\ -\frac{q^2}{r^4} &= \frac{f'}{rfh} - \frac{1}{r^2}\left(1 - \frac{1}{h}\right) \ , && [8\pi T_{11} = G_{11}] \\ \frac{q^2}{r^4} &= \frac{1}{2\sqrt{fh}}\frac{d}{dr}\left(\frac{f'}{\sqrt{fh}}\right) + \frac{f'}{2rfh} - \frac{h'}{2rh^2} \ . && [8\pi T_{22} = G_{22}] \end{aligned}$$

(See [6.2.3]-[6.2.5] in book.)

The right hand side of the first equation can be seen to equal

$$\frac{1}{r^2}\frac{d}{dr}\left(r\left(1 - \frac{1}{h}\right)\right) \ .$$

Multiplying both sides of $[8\pi T_{00} = G_{00}]$ by r^2 and integrating, one obtains

$$r\left(1 - \frac{1}{h}\right) = -\frac{q^2}{r} + 2M \ ,$$

where $2M$ is the constant of integration. Solving this equation, one gets the correct formula for h (which is the factor multiplying dr^2 in the metric).

Adding the equations $[8\pi T_{ii} = G_{ii}]$ for $i = 1, 2$ and multiplying the result by rh, one gets

$$\frac{f'}{f} + \frac{h'}{h} = 0 \ .$$

This can be rewritten as $f'h + fh' = 0 = (fh)'$. It follows that $fh = K$ for some constant K. Changing the scale with which time is measured if necessary, one can assume $f = 1/h$. This is the correct formula for f (which is the factor multiplying $-dt^2$ in the metric). This completes the proof of the fact that the general solution of Einstein's equation in these circumstances is the Reissner-Nordstrom metric.

Problem 4

Let (M, g_{ab}) be a stationary spacetime with timelike Killing field ξ^a. Let $V^2 = -\xi^a \xi_a$.

(a) Show that the acceleration $a^b = u^a \nabla_a u^b$ of a stationary observer is given by $a^b = \nabla^b \ln V$.

(b) Suppose in addition that (M, g_{ab}) is asymptotically flat, i.e., that there exist coordinates t, x, y, z [with $\xi^a = (\partial/\partial t)^a$] such that the components of g_{ab} approach $\mathrm{diag}(-1, 1, 1, 1)$ as $r \to \infty$, where $r = (x^2 + y^2 + z^2)^{1/2}$. (See chapter 11 for further discussion of asymptotic flatness.) As in the case of the Schwarzschild metric, the "energy as measured at infinity" of a particle of mass m and 4-velocity u^a is $E = -m\xi^a u_a$. Suppose a particle of mass m is held stationary by a (massless) string, with the other end of the string held by a stationary observer at large r. Let F denote the force exerted by the string on the particle. According to part (a) we have $F = mV^{-1}[\nabla_a V \nabla^a V]^{1/2}$. Use conservation of energy arguments to show that the force exerted by the observer at infinity on the other end of the string is $F_\infty = VF$. Thus the magnitude of the force exerted at infinity differs from the force exerted locally by the redshift factor.

Problem 5

Derive the formula, equation (6.3.45), for the general relativistic time delay.

Problem 6

Show that any particle (not necessarily in geodesic motion) in region II ($r < 2M$) of the extended Schwarzschild spacetime, Figure 6.9, must decrease its radial coordinate at a rate given by $|dr/d\tau| \geq [2M/r - 1]^{1/2}$. Hence, show that the maximum lifetime of any observer in region II is $\tau = \pi M$ [$\sim 10^{-5}(M/M_\odot)$ s], i.e., any observer in region II will be pulled into the singularity at $r = 0$ within this proper time. Show that this maximum time is approached by freely falling (i.e., geodesic) motion from $r = 2M$ with $E \to 0$.