Nonconvex Sparse Optimization and Applications

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Outline

Introduction

Optimization

Applications

Future Work

Sparse Optimization

- Sparse optimization considers

$$\min_{x\in\mathbb{R}^n} \ f(x) + \lambda\|x\|_0$$

$$\Updownarrow$$

$$\min_{x\in\mathbb{R}^n} \ f(x) \quad \text{s.t.} \quad \|x\|_0 \le s$$

  - x can be extended to matrices and tensors
  - $f(x)$ may be nonsmooth even nonconvex
  - $\|x\|_0$ counts the number of nonzeros
  - λ and s are parameters
- Also called compressed sensing and variable selection
- Broad applications in machine learning, pattern recognition and engineering
- `https://github.com/xianchaoxiu/Sparse-Optimization`

Methods

- Convex methods
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Introduction

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$\ell_1 - \ell_p$ Minimization

- Xiu-Kong-Li-Qi, Computational Optimization and Applications, 2018

$$\min_{x \in \mathbb{R}^n} \quad \|Ax - b\|_1 + \lambda \|x\|_p^p \; (0 < p < 1) \tag{1}$$

- Consider the following ϵ-approximations

$$\min_{x \in \mathbb{R}^n} F_{\alpha,\epsilon}(x) = \|Ax - b\|_1 + \lambda \sum_{i=1}^{n} (|x_i|^\alpha + \epsilon_i)^{\frac{p}{\alpha}}$$

$$\Downarrow$$

$$\min_{x \in \mathbb{R}^n} F_\epsilon(x) = \|Ax - b\|_1 + \lambda \sum_{i=1}^{n} h_{u_\epsilon}(x_i) \tag{2}$$

where

$$h_{u_\epsilon}(x_i) = \min_{0 \le s \le u_\epsilon} p\left(|x_i| s - \frac{p-1}{p} s^{\frac{p}{p-1}}\right), \quad u_\epsilon = \left(\frac{\epsilon}{\lambda n}\right)^{\frac{p-1}{p}}$$

$\ell_1 - \ell_p$ Minimization

- (Definition) We say that $x^* \in \mathbb{R}^n$ is a generalized first-order stationary point of (1) if

$$0 \in (A^\top \mathrm{sgn}(Ax^* - b))_i x_i^* + \lambda p |x_i^*|^p, \quad i = 1, 2, \cdots, n$$

Furthermore, the following statement holds

$$|x_i^*| \ \geq \ (\frac{\lambda p}{\|A_i\|_1})^{\frac{1}{1-p}}, \quad \forall i \in T \tag{3}$$

- (Lower Bound) Let ϵ be a constant such that

$$0 < \epsilon < \lambda n \left(\frac{\|A_i\|_1}{\lambda p} \right)^{\frac{p}{p-1}} \tag{4}$$

Suppose that x^* is a generalized first-order stationary point of (2). Then, x^* is also a generalized first-order stationary point of (1). Moreover, the nonzero entries of x^* satisfy the lower bound property (3).

$\ell_1 - \ell_p$ Minimization

- (Convergent Theorem) Assume that ϵ satisfies (4) and set q as $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that x^* is an accumulation point of $\{x^k\}$. Then x^* is a generalized first-order stationary point of (1). Moreover, the nonzero entries of x^* satisfy the lower bound (3).

Choose an arbitrary $x^0 \in \mathbb{R}^n$ and ϵ such that (4) holds. Set $k = 0$

1) Solve the weighted ℓ_1 minimization problem

$$x^{k+1} \in \mathrm{argmin}_x \left\{\|Ax - b\|_1 + \lambda p \sum_{i=1}^n s_i^k |x_i|\right\}$$

where $s_i^k = \min\left\{\left(\frac{\epsilon}{\lambda n}\right)^{\frac{1}{q}}, |x_i^k|^{\frac{1}{q-1}}\right\}$ for all i

2) Set $k \leftarrow k + 1$ and go to step 1)

End

Distributed Optimization

- Qu-Chen-Xiu-Liu, Neurocomputing, 2024

$$\min_{Y \in \mathbb{R}^{n\times p}} \quad \sum_{i=1}^{d} f_i(Y) \qquad (5)$$
$$\text{s.t.} \quad \|Y\|_{2,0} \leq s, \;\; Y^\top Y = I_p$$

$$\Downarrow$$

$$\min_{Y \in \mathbb{R}^{n\times p}} \quad \sum_{i=1}^{d} f_i(Y) + \frac{\mu}{4}\|Y^\top Y - I_p\|_F^2$$
$$\text{s.t.} \quad \|Y\|_{2,0} \leq s$$

$$\Downarrow$$

$$\min_{Y,\{X_i\} \in \mathbb{R}^{n\times p}} \quad \sum_{i=1}^{d} f_i(X_i) + \frac{\mu}{4}\|Y^\top Y - I_p\|_F^2 \qquad (6)$$
$$\text{s.t.} \quad X_i = Y, \; \forall i \in [d], \;\; \|Y\|_{2,0} \leq s$$

Distributed Optimization

- (Lemma) Let $(\widetilde{Y}^*, \{\widetilde{X}_i^*\})$ be the (local) minimizer of (6). Then there exists $\mu_\epsilon > 0$ such that $\widetilde{Y}^*$ is an ϵ-(local) minimizer of (5) for any $\mu \geq \mu_\epsilon$.
- (Definition) We say $(Y^*, \{X_i^*\}, \{\Lambda_i^*\})$ is a KKT point of (6) if it satisfies

$$\begin{cases} 0 \in \nabla g(Y^*) + \sum_{i=1}^{d} \Lambda_i^* + \mathcal{N}_{\mathcal{S}}(Y^*) \\ 0 = \nabla f_i(X_i^*) - \Lambda_i^*, \ \forall i \in [d] \\ 0 = X_i^* - Y^*, \ \forall i \in [d] \end{cases}$$

- (Definition) We say $(Y^*, \{X_i^*\}, \{\Lambda_i^*\})$ is a stationary point of (6) if there exists $\alpha > 0$ such that

$$\begin{cases} Y^* = \mathcal{P}_{\mathcal{S}}(Y^* - \alpha(\nabla g(Y^*) + \sum_{i=1}^{d} \Lambda_i^*)) \\ 0 = \nabla f_i(X_i^*) - \Lambda_i^*, \ \forall i \in [d] \\ 0 = X_i^* - Y^*, \ \forall i \in [d] \end{cases}$$

- (Optimal Conditions) Suppose that $(Y^*, \{X_i^*\})$ is a local minimizer of (6). Then, there exists Λ_i^* $(i \in [d])$ such that $(Y^*, \{X_i^*\}, \{\Lambda_i^*\})$ is a KKT point of (6).

Distributed Optimization

- (Nonincreasing Lemma) Let $\{(Y^k, \{X_i^k\}, \{\Lambda_i^k\})\}$ be the generated sequence and $\beta \geq \sqrt{2}r$. Then the generated augmented Lagrangian sequence is nonincreasing, i.e.,

$$\mathcal{L}_{\beta}(Y^{k+1}, \{X_i^{k+1}\}; \{\Lambda_i^{k+1}\}) \leq \mathcal{L}_{\beta}(Y^k, \{X_i^k\}; \{\Lambda_i^k\})$$

- (Bounded Lemma) Suppose that $\beta \geq 2r$ holds. Then the sequence $\{(Y^k, \{X_i^k\}, \{\Lambda_i^k\})\}$ is bounded. Moreover, it satisfies

$$\begin{cases} \lim\limits_{k\to\infty} \|Y^{k+1} - Y^k\|_F = 0 \\ \lim\limits_{k\to\infty} \|X_i^{k+1} - X_i^k\|_F = 0, \ \forall i \in [d] \\ \lim\limits_{k\to\infty} \|\Lambda_i^{k+1} - \Lambda_i^k\|_F = 0, \ \forall i \in [d] \end{cases}$$

- (Convergent Theorem) Let $\{(Y^k, \{X_i^k\}, \{\Lambda_i^k\})\}$ be the generated sequence and $\beta \geq 2r$. Then, any accumulation point $(Y^*, \{X_i^*\}, \{\Lambda_i^*\})$ is a stationary point of (6).

$L_{1/2}$ Reglarization

- Fan-Yan-Xiu-Liu, under review

$$\min_{x \in \mathbb{H}^p} F(x) := f(x) + \lambda \|x\|_{1/2}^{1/2} \; (\mathbb{H} = \mathbb{R} \text{ or } \mathbb{C}) \tag{7}$$

where $f(x) := \frac{1}{n} \sum_{i=1}^{n} h_\alpha(|\langle a_i, x\rangle|^2 - b_i)$ and

$$h_\alpha(u) = \begin{cases} \frac{1}{2}u^2, & \text{if } |u| \leq \alpha \\ \alpha|u| - \frac{1}{2}\alpha^2, & \text{if } |u| > \alpha \end{cases}$$

- For ease of expression, define

$$g(x) := \frac{1}{n} \sum_{i=1}^{n} h'_\alpha(|\langle a_i, x\rangle|^2 - b_i)\langle a_i, \rangle \bar{a}_i$$

which implies that $\nabla f(x) = 2g(x)$ for $\mathbb{H} = \mathbb{R}$ and $\nabla_x f(x) = g(x)$ for $\mathbb{H} = \mathbb{C}$

- Wirtinger derivative

$L_{1/2}$ Reglarization

- (Optimal Conditions) There exists a global minimizer $\hat{x}$ which lies in the level set $S = \{x \in \mathbb{H}^p : F(x) \leq F(x^0)\}$, and further satisfies the fixed point inclusion

$$\hat{x} \in \mathcal{H}_{\lambda\tau}(\hat{x} - 2\tau g(\hat{x}))$$

Initialize spectral point x^0, let $k = 0,\ j = 0,\ \tau_0 = \beta$

1) Do

$$x^{k+1} = \mathcal{H}_{\lambda\tau_k}(x^k - 2\tau_k \nabla f(x^k))$$

where $\tau_k = \gamma\beta^{j_k}$ and j_k is the smallest nonnegative integer such that

$$F(x^k) - F(x^{k+1}) \geq \delta\|x^{k+1} - x^k\|^2$$

2) Check convergence: if

$$\|x^{k+1} - x^k\| \leq \epsilon \max\{1, \|x^k\|\}$$

otherwise, set $k \leftarrow k+1$, and go back to Step 1)

$L_{1/2}$ Reglarization

- (Subsequence Convergence) Assume that $\{x^k\}$ is the generated sequence. Then the following conclusions hold
  (a) $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$.
  (b) Every accumulation point of $\{x^k\}$ satisfies the following fixed point equation
  $$x = \mathcal{H}_{\lambda\tau}(x - 2\tau g(x)) \tag{8}$$
  for $\tau \in [\gamma\beta^{\bar{j}}, \gamma]$ when $\gamma \leq \lambda^2/(64\ell^3)$ with $\ell = \alpha \sum_{i=1}^{n} \|a_i\|^2 \sup_{F(x)\leq F(x^0)} \|x\|/n$.
  (c) $\{F(x^k)\}$ decreasingly converges to $F(x^*)$, where x^* is any accumulation point of $\{x^k\}$.
- (Whole Sequence Convergence) Assume that $\{x^k\}$ is the generated sequence. Then the whole sequence $\{x^k\}$ is convergent.
- (Convergence Rate) Under mild conditions, the whole sequence $\{x^k\}$ is convergent and converges at least sublinearly to a vector x^* satisfying (8).

Outline

Data Analysis

- Xiu-Liu-Li-Kong, Computational Statistics & Data Analysis, 2019

$$\min_{\beta} \quad \frac{1}{2}\|y - X\beta\|^2 + \Phi_{\tau_1}(\beta) + \sum_{i=1}^{p} \Phi_{\tau_2}(\beta_{i+1} - \beta_i)$$

  - Φ_{τ_1} and Φ_{τ_2} can be the same or different
  - Nonconvex penalty functions: ℓ_p, SCAD, MCP, capped ℓ_1
- For notational simplicity, define

$$\min_{\beta} \quad \frac{1}{2}\|y - X\beta\|^2 + \Phi_{\tau_1}(\beta) + \Phi_{\tau_2}(D\beta)$$

with

$$D = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(p-1)\times p}$$

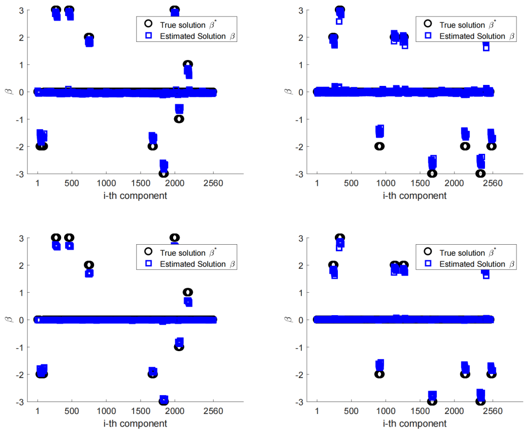
Data Analysis

- Alternating direction method of multipliers (ADMM)

$$\begin{aligned} \min_{\alpha,\gamma,\beta} \quad & \frac{1}{2}\|y - X\beta\|^2 + \Phi_{\tau_1}(\alpha) + \Phi_{\tau_2}(\gamma) \\ \text{s.t.} \quad & \alpha = \beta \\ & \gamma = D\beta \end{aligned}$$

- (Convergent Theorem) Suppose that $\{(\alpha^k, \gamma^k, \beta^k, w_1^k, w_2^k)\}$ is a generated sequence. Then the sequence converges to a stationary point.

- Recovery results



Signal Processing

- Li-Xiu-Liu-Miao, IEEE Signal Processing Letters, 2022

$$\begin{aligned} \min_{U,P_v} \quad & \sum_{v=1}^{M} \|U - X_v P_v\|_F^2 \\ \text{s.t.} \quad & U^\top U = I_d, \; \|P_v\|_{2,0} \le s_v \end{aligned}$$

- Alternating minimization algorithm (AMA): Update U, then update P_v
- Denote $f(P_v) := \|U^{k+1} - X_v P_v\|_F^2$. Then

$$\nabla f(P_v) = 2X_v^\top (X_v P_v - U^{k+1}) \in \mathbb{R}^{d_v \times d}$$

$$\nabla^2 f(P_v) = 2I_d \otimes X_v^\top X_v \in \mathbb{R}^{d_v d \times d_v d}$$

- Newton hard thresholding pursuit (NHTP)

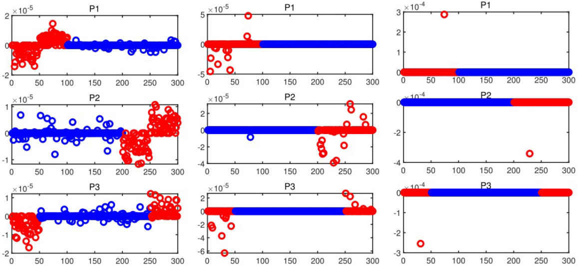
$$F(P_v; T_v) := \begin{pmatrix} \nabla_{T_v} f(P_v) \\ (P_v)_{\overline{T}_v} \end{pmatrix} = 0$$

Signal Processing

- Runtime comparison

Problem Scale	GCCA	SGCCA	SCGCCA
(1,000;300;300;300)	0.04	0.04	**0.01**
(5,000;300;300;300)	0.23	0.28	**0.03**
(10,000;300;300;300)	0.40	0.41	**0.07**
(50,000;300;300;300)	2.32	2.27	**0.34**
(100,000;300;300;300)	4.58	4.35	**0.66**
(1,000;1,500;1,500;1,500)	0.42	0.40	**0.02**
(5,000;1,500;1,500;1,500)	1.35	1.16	**0.12**
(10,000;1,500;1,500;1,500)	2.63	2.24	**0.24**
(50,000;1,500;1,500;1,500)	13.21	10.56	**1.18**
(100,000;1,500;1,500;1,500)	26.60	22.53	**2.35**
(1,000;3,000;3,000;3,000)	1.53	1.58	**0.17**
(5,000;3,000;3,000;3,000)	3.92	3.49	**0.23**
(10,000;3,000;3,000;3,000)	6.87	5.65	**0.45**
(50,000;3,000;3,000;3,000)	32.02	23.18	**2.29**
(100,000;3,000;3,000;3,000)	667.69	629.54	**4.91**

- Extracted feature comparison



Pattern Recognition

- Liu-Feng-Xiu-Liu, Pattern Recognition, 2024

$$\begin{aligned} \min_{Q} \quad & \mathrm{Tr}\left(Q^\top S Q\right) + \lambda \|Q\|_{2,1} \\ \text{s.t.} \quad & Q^\top Q = I \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \min_{P,Q,E} \quad & \mathrm{Tr}\left(Q^\top S Q\right) + \lambda_1 \|Q\|_{2,1} + \lambda_2 \|E\|_1 \\ \text{s.t.} \quad & X = PQ^\top X + E, \;\; P^\top P = I \end{aligned}$$

$$\Downarrow$$

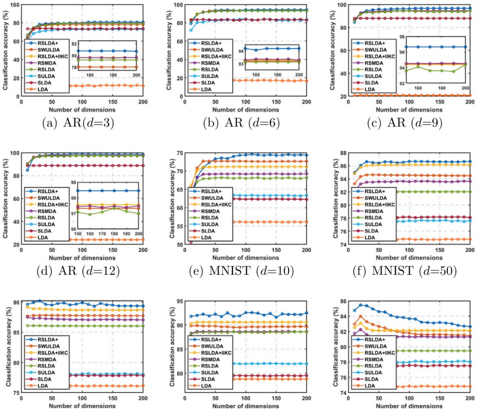
$$\begin{aligned} \min_{P,Q,E} \quad & \mathrm{Tr}\left(Q^\top S Q\right) + \lambda_1 \|Q\|_{2,0} + \lambda_2 \|E\|_0 \\ \text{s.t.} \quad & X = PQ^\top X + E, \;\; P^\top P = I \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \min_{P,Q,E} \quad & \mathrm{Tr}\left(Q^\top S Q\right) + \lambda_1 \|Q\|_{2,0} + \lambda_2 \|Q\|_0 + \lambda_3 \|E\|_0 \\ \text{s.t.} \quad & X = PQ^\top X + E, \;\; P^\top P = I \end{aligned}$$

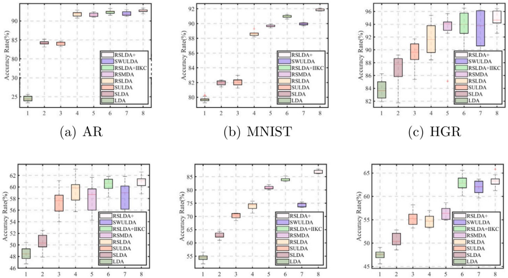
Pattern Recognition

▶ Classification accuracy



(a) AR(d=3)
(b) AR (d=6)
(c) AR (d=9)
(d) AR (d=12)
(e) MNIST (d=10)
(f) MNIST (d=50)
(g) MNIST (d=100)
(h) MNIST (d=200)
(i) HGR (d=4)

▶ Model stability



(a) AR
(b) MNIST
(c) HGR
(d) NEU
(e) Caltech
(f) CAR

Outline

Deep Unfolding Networks

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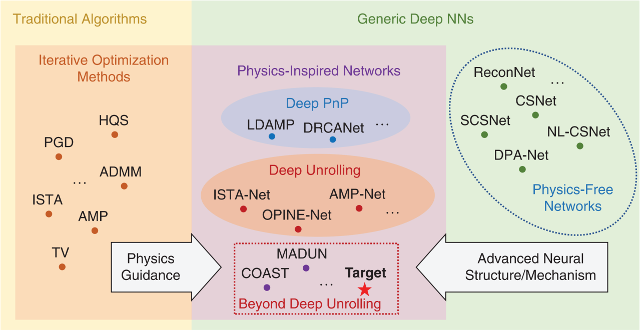

Large Language Models

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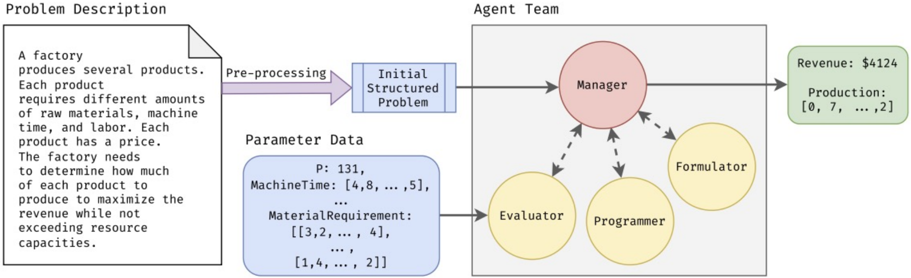

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