第 3 問　複素解析

[1]

極は $z = i$ であり、そこでの留数は

$$\mathrm{Res}[f, z=i] = \lim_{z \to i}(z-i)f(z) = \lim_{z=i} \frac{\log(z+i)}{z+i} = \frac{\ln 2 + i\pi/2}{2i} = \frac{\pi}{4} - i\frac{\ln 2}{2} \tag{1.1}$$

となる。

[2]

留数定理より

$$\int_C f(z)dz = 2\pi i \,\mathrm{Res}[f, z=i] = \pi \ln 2 + i\frac{\pi^2}{2} \tag{2.1}$$

[3]

三角不等式を使うと

$$\begin{aligned} \left| \int_{C_1} \frac{\log(z+i)}{z^2+i} dz \right| &= \left| \int_{C_1} frac\log(z+i)z^2 + idz \right| \\ &\le \int_{C_1} \frac{|\log(z+i)|}{|z^2+i|} |dz| \end{aligned} \tag{3.1}$$

耐えられた不等式と

$$\left| z^2 + i \right| = \sqrt{1 + R^4 + 2R^2 \sin 2\theta} > \sqrt{R^4 - 2R^2 + 1} = R^2 - 1 \tag{3.2}$$

を使うと

$$\left| \int_{C_1} \frac{\log(z+i)}{z^2+i} dz \right| < \int_0^{\pi} \frac{\ln(R+1) + 3\pi/2}{R^2 - 1} |dz| = \pi R \frac{\ln(R+1) + 3\pi/2}{R^2 - 1} \tag{3.3}$$

[4]

$g(z)$ の極は $z = -i$ で留数は

$$\mathrm{Res}[g, z=-i] = \lim_{z \to -i}(z+i)g(z) = \lim_{z \to -i} \frac{\log(z-i)}{z-i} = \frac{\ln 2 - i\pi/2}{-2i} = \frac{\pi}{4} + i\frac{\ln 2}{2} \tag{4.1}$$

これより留数定理を使うと

$$\int_\Gamma g(z)dz = 2\pi i \,\mathrm{Res}[g, z=-i] = -\pi \ln 2 + i\frac{\pi^2}{2} \tag{4.2}$$

[5]

半径 R の経路を $C_1(R), \Gamma_1(R)$ のように書くことにすると、

$$
\begin{aligned}
\int_{C_2+C_1}\frac{\log(z+i)}{z^2+1}dz+\int_{-\Gamma_1-\Gamma_2}\frac{\log(z+i)}{z^2+1}dz &= \int_{-R}^{R}\frac{\ln(z+i)+\ln(z-i)}{z^2+1}dz+\int_{C_1(R)}\frac{\log(z+i)}{z^2+1}dz-\int_{\Gamma_1(R)}\frac{\log(z-i)}{z^2+1}dz \\
\pi\ln 2+i\frac{\pi^2}{2}-\left(-\pi\ln 2+i\frac{\pi^2}{2}\right) &= \int_{-\infty}^{\infty}\frac{\ln\left(x^2+1\right)}{x^2+1}dx+\int_{C_1(\infty)}\frac{\log(z+i)}{z^2+1}dz-\int_{\Gamma_1(\infty)}\frac{\log(z-i)}{z^2+1}dz \\
2\pi\ln 2 &= 2\int_0^{\infty}\frac{\ln\left(x^2+1\right)}{x^2+1}dx \\
\int_0^{\infty}\frac{\ln\left(x^2+1\right)}{x^2+1}dx &= \pi\ln 2
\end{aligned}
\tag{5.1}
$$

[6]

前問で得られた積分の変数を $x=\tan\theta$ に変えると、

$$
\begin{aligned}
\pi\ln 2 &= \int_0^{\pi/2}\frac{\ln\left(\tan^2\theta+1\right)}{\tan^2\theta+1}\frac{d\theta}{\cos^2\theta} \\
&= -2\int_0^{\pi/2}\ln(\cos\theta)d\theta=-2I \\
I &= -\frac{\pi}{2}\ln 2
\end{aligned}
\tag{6.1}
$$