4. Pólya's Theory of Counting

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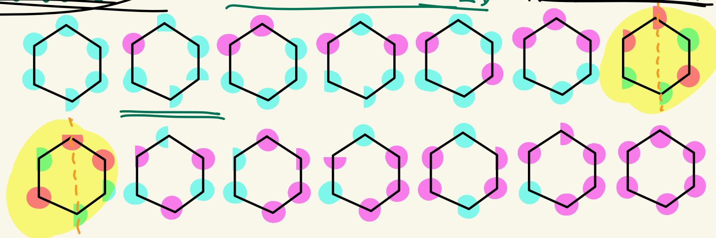
Counting with Symmetry → 等价类(equivalence class) : 对等价类计数

Rotation:
旋转

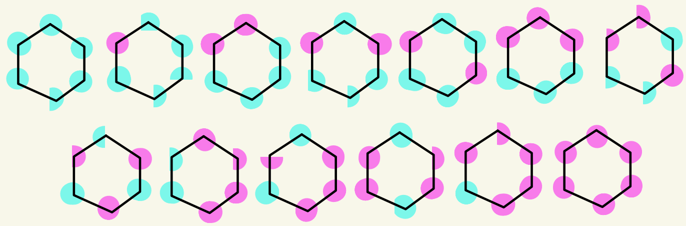


$X = [2]^{[6]}$

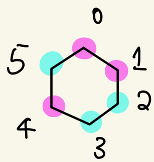
$G = C_n$

$j = X/G$

Rotation & Reflection:
旋转.翻转



$G = D_n$



配置: configuration $x: [n] \to [m]$ $X = [m]^{[n]}$

position ↑ [n], colors ↑ [m]

置换 permutation $\pi: [n] \xrightarrow{1-1} [n]$ group G

e.g. $(0,1,2,3,4,5)$, $(1,2,3,4,5,0)$

Permutation Groups 置换群

群 group $(G, \cdot)$ operator $\cdot : G \times G \to G$

① closuer: $\pi, \sigma \in G \Rightarrow \pi \cdot \sigma \in G$

② associativity: $(\pi \cdot \sigma) \cdot \tau = \pi \cdot (\sigma \cdot \tau)$

③ identity: $\exists\, e \in G$ s.t. $\forall \pi \in G$. $\pi \cdot e = e \pi = \pi$

④ inverse: $\forall \pi \in G$. $\exists \sigma \in G$ s.t. $\pi \cdot \sigma = \sigma \cdot \pi = e$ $\Rightarrow \sigma = \pi^{-1}$

abelian group : ① ~ ④ + commutativity : $\pi \cdot \sigma = \sigma \cdot \pi$

1) **Symmetric group** S_n : all permutation $\quad \pi : [n] \xrightarrow{1\text{-}1} [n]$

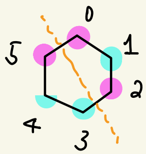
2) **cyclic group** C_n : all rotations 旋转

$$\pi = (0\ 1\ 2 \cdots n\text{-}1) \qquad \pi(i) = (i+1) \bmod n$$

$$\Rightarrow \langle (0\ 1\ 2 \cdots n\text{-}1) \rangle$$

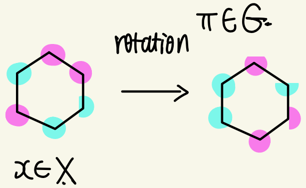
generator 生成元

3) **dihedral group** D_n : rotations & reflections 旋转 + 反射



$\rho(i) = (n-1) - i$ 镜像变换

generated by $(0\ 1\ 2 \cdots n\text{-}1)$ and ρ



operator 群作用 group action → 两个配置之间的变换

$\circ : G \times X \longrightarrow X$

$(\pi \circ x)(i) = x(\pi(i))$

- associativity : $(\pi \cdot \sigma) \circ x = \pi \circ (\sigma \circ x)$
- identity : $e \circ x = x$.

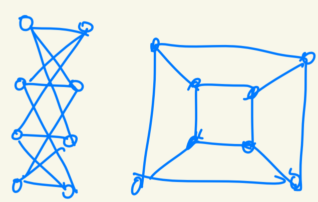
Example. Graph-Isomorphism. (GI)

图的同构

input : 2 undirected graph G, H

output : $G \cong H$?

$O(n!)$



GI is NP

but is UNKNOWN to be in P or NPC

Babai-Luks '83: $2^{O(\sqrt{n \log n})}$ time

Babai '2017: quasi-polynomial $\underline{\underline{n^{\log n}}}$

* String Isomorphism (SI)

input: 2 strings $x, y: [n] \to [m]$,

a permutation group $G \subseteq S_n$

output: $x \cong_G y$? (i.e. $\exists\, \sigma \in G$, s.t. $\sigma \circ x = y$?)

Graph $X(V,E)$ is a string: $x: \underbrace{\binom{V}{2}}_{\text{vertex pairs}} \to \{0,1\}$ $\begin{cases} 1: \text{edge} \\ 0: \text{no-edge} \end{cases}$

Johnson group. $S_V^{(2)} \subset S_{\binom{V}{2}}$ on vertex pairs

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group action $\circ$: $G \times X \to X$

$(\pi \circ x)(i) = x(\pi(i))$

Orbit 轨迹(道) of x: $Gx = \{\pi \circ x \mid \pi \in G\}$

和x等价的所有配置

⇓

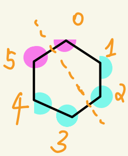
商集 $X/G = \{Gx \mid x \in X\}$ equivalence class 等价类

目标：对所有的等价类进行计数. 也即 count $|X/G|$

$\boxed{\text{Burnside's Lemma}}$ $|X/G| = \frac{1}{|G|} \cdot \sum_{\pi \in G} |X_\pi|$

不变集 invariant set of π: $X_\pi = \{x \in X \mid \pi \circ x = x\}$

e.g.



π: 反射变换

$|X_\pi| = 2^3 = 8$ 种

double counting 双重计数

Proof: $\Rightarrow \boxed{\sum_{\pi\in G} |X_\pi|} = \boxed{|X/G|\cdot|G|}$

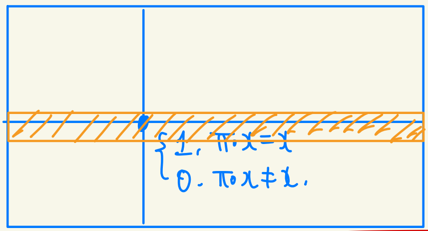
对同一个目标有两种不同的计数方式

其中一种会有目标式子

另一种中的元素是已知的

配置 $x\in X$ 行和 Goal: 列和

变换 $\pi\in G$



行和 $|X_\pi|$ → invariant set

矩阵 A → 行和 / 列和

$$\text{目标: } |X/G|\cdot|G| = \sum_{x\in X} \boxed{|G_x|}$$

Duality 对偶性

stabilizer

Invariant set of π: $X_\pi = \{x\in X \mid \pi\circ x = x\}$ → 给定 π（行）

Stabilizer of x: $G_x = \{\pi\in G \mid \pi\circ x = x\}$ → 给定 x（列）

$$\boxed{\text{Lemma: } \forall x\in X,\ |G_x|\cdot|Gx| = |G|}$$

列和: $\sum_{x\in X} |G_x| = \sum_{x\in X} \frac{|G|}{|Gx|}$ ← x 所在等价类中元素的数量

$$= |G|\cdot \boxed{\sum_{x\in X} \frac{1}{|Gx|}}$$

$$= |G|\cdot|X/G|$$

设所有 $X_1, X_2, \cdots, X_{|X/G|}$ 个等价类

→ 构成对 X 的一个划分.

$$\sum_{x\in X} \frac{1}{|Gx|} = \sum_{i=1}^{|X/G|} \underbrace{\sum_{x\in X_i} \frac{1}{|X_i|}}_{1} = |X/G|.$$

行和: $\sum_{\pi\in G} |X_\pi|$

$$\Rightarrow |X/G|\cdot|G| = \sum_{\pi\in G} |X_\pi|.$$

□

Proof of Lemma: $Gx = \{\pi\circ x \mid \pi\in G\} \triangleq \{x_1, \cdots, x_t\}$

$\exists$ set $P = \{\pi_1, \cdots, \pi_t\}$ s.t. $\pi_i\circ x = x_i$

目标：$|G_x|\cdot|Gx| = |G|$

$\downarrow$

P

构建从 $P\times G_x$ 到 G 的一个双射！

$\forall \pi_i \in P,\ \sigma\in G_x.\ \pi = \pi_i\cdot\sigma \in G.$ 1-1 mapping !

Burnside's Lemma : $|X/G| = \frac{1}{|G|}\cdot\sum_{\pi\in G}|X_\pi|$

permutation $\pi : [n] \xrightarrow{1-1} [n]$

两种写法，
$$\begin{pmatrix} 0 & 1 & 2 & \cdots & n-1 \\ \pi(0) & \pi(1) & \pi(2) & \cdots & \pi(n-1)\end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 0 & 1 & 3\end{pmatrix} = (0\ 2)(1\ 4\ 3)$$

invariant set

$\pi\circ x = x.$

$x(\pi(i)) = x(i)$

以 cycle 为单位进行染色 ☆

$\Rightarrow$ 同一个 cycle 染同一个颜色

不同的 cycle 随便

$X = [m]^{[n]}$

m-coloring of n positions

Given $\pi = \underbrace{(\cdots)(\cdots)(\cdots)(\cdots)}_{k \text{ cycles}}$

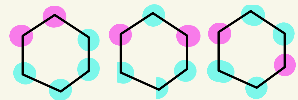
$|X_\pi| = |\{x\in X \mid \pi\circ x = x\}| = m^k$ ☆

Burnside's Lemma: $|X/G| = \frac{1}{|G|}\sum_{\pi\in G}|X_\pi| = \frac{1}{|G|}\sum_{\pi\in G} m^{\#cycle(\pi)}$

Motivation: 想要更细分的信息！ e.g. 2红4蓝

pattern:

$\vec{v} = (n_1, n_2, \cdots, n_m)$ s.t. $n_1 + n_2 + \cdots + n_m = n$.



$a_{(2,4)} = 3$

n_i: 第i种颜色有 n_i 个 position

goal! $a_{\vec{v}}$: # of configurations (up to symmetry) with n_i many color i

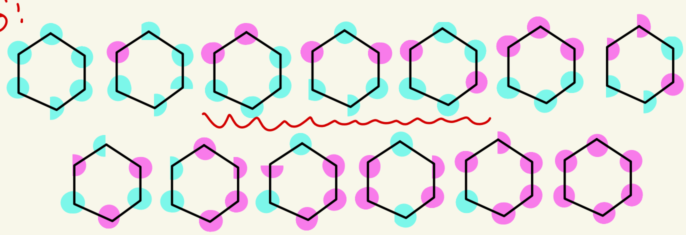
Pattern inventory ~~工具箱~~ / 答案纸

$$F_G(y_1, y_2, \cdots, y_m) = \sum_{\substack{\vec{v}=(n_1,\cdots,n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}} \, y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

多元生成函数

系数即代表了对应 # of config.

X/G:



$X = [2]^{[6]}$

$G = D_6$

$$F_{D_6}(y_1, y_2) = y_1^6 + y_1^5 y_2 + 3y_1^4 y_2^2 + 3y_1^3 y_2^3 + 3y_1^2 y_2^4 + y_1 y_2^5 + y_2^6.$$

怎么求? Polya's Theory of Counting:

☆ Polya's enumeration formula:

pattern inventory

$$F_G(y_1, y_2, \cdots, y_m) = P_G\left(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n\right)$$

cycle index

$$P_G(t_1, \cdots, t_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(t_1, t_2, \cdots, t_n)$$

$$\pi = \underbrace{(\overset{l_1}{\cdots})(\overset{l_2}{\cdot\cdot})\cdots(\overset{l_k}{\cdots})}_{k \text{ cycle}}$$

$$M_\pi(t_1, t_2, \cdots, t_n) = \prod_{i=1}^{k} t_{l_i}$$

$$= t_{l_1} \cdot t_{l_2} \cdots t_{l_k}$$

△ Burnside's Lemma:

$$|X/G| = \frac{1}{|G|} \sum_{\pi \in G} m^{\#\text{of cycle}(\pi) \to k}$$

$$P_G(\underbrace{m, m, \cdots, m}_{n\text{个}}) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(\underbrace{m, m, \cdots, m}_{n\text{个}})$$

$$\Rightarrow \sum_{i=1}^{m} y_i = m. \quad \sum_{i=1}^{m} y_i^2 = m \quad \cdots \quad \sum_{i=1}^{m} y_i^n = m. \quad \Rightarrow y_1 = y_2 = \cdots = y_m = 1.$$

$$\Rightarrow F_G(1, 1, \cdots, 1) = \sum_{\substack{\vec{v}=(n_1, \cdots, n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}} = |X/G|$$

即为所有等价类的个数

Proof: 在子空间 (refined space) 上运用 Burnside.

$\vec{v} = (n_1, n_2, \cdots, n_m)$ s.t. $n_1 + n_2 + \cdots + n_m = n$.

$X = [m]^{[n]}$.

$$X^{\vec{v}} = \{ x \in [m]^{[n]} \mid \forall i \in [m] : x^{-1}(i) = n_i \}$$

$X^{\vec{v}}$ ↳ 满足要求v的所有配置的集合

$x^{-1}(i)$ ↳ 原像集(颜色为i的球子数量)

invariant set: $X_{\pi}^{\vec{v}} = \{ x \in X^{\vec{v}} \mid \pi \cdot x = x \}$

Given a permutation π, $\pi = \underbrace{\overbrace{(\cdots)}^{l_1}\overbrace{(\cdots)}^{l_2}\cdots\overbrace{(\cdots)}^{l_k}}_{k\text{ cycles}}$

By Burnside Lemma: $a_{\vec{v}} = \frac{1}{|G|} \sum_{\pi \in G} |X_{\pi}^{\vec{v}}|$

y_i:第i个颜色对应的变量　以cycle为单位进行染色!求出对应的pattern v即可:

$$(y_1^{l_1} + y_2^{l_1} + \cdots + y_m^{l_1})(y_1^{l_2} + y_2^{l_2} + \cdots + y_m^{l_2}) \cdots (y_1^{l_k} + y_2^{l_k} + \cdots + y_m^{l_k})$$

第i个cycle染什么颜色

$$= \sum_{\substack{\vec{v}=(n_1,\cdots,n_m) \\ n_1+\cdots+n_m=n}} |X_{\pi}^{\vec{v}}|\, y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}.$$

→ 如何表示不变集中元素数量

$$\prod_{i=1}^{k} (y_1^{l_i} + y_2^{l_i} + \cdots + y_m^{l_i}) = M_{\pi}\left(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \cdots, \sum_{i=1}^{m} y_i^n\right)$$

Recall: $M_{\pi}(t_1, t_2, \cdots, t_n) = \prod_{i=1}^{k} t_{l_i}$

pattern inventory: $F_G(y_1, y_2, \cdots, y_m)$

$$= \sum_{\substack{\vec{v}=(n_1,\cdots,n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}}\, y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m},$$

$$= \sum_{\substack{\vec{v}=(n_1,\cdots,n_m) \\ n_1+\cdots+n_m=n}} \left(\frac{1}{|G|} \sum_{\pi \in G} |X_{\pi}^{\vec{v}}| \right) y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}.$$

$$= \frac{1}{|G|} \sum_{\pi \in G} \sum_{\vec{v}} |X_{\pi}^{\vec{v}}| \cdot y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} = \frac{1}{|G|} \sum_{\pi \in G} M_{\pi}\left(\sum_{i=1}^{m} y_i, \cdots, \sum_{i=1}^{m} y_i^n\right)$$

D_{20}

$$
\begin{aligned}
F_{D_{20}}(r,q,l) = {} & r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}q^2l \\
& + 90r^{17}ql^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^3l + 774r^{16}q^2l^2 + 489r^{16}ql^3 + 145r^{16}l^4 \\
& + 406r^{15}q^5 + 1956r^{15}q^4l + 3912r^{15}q^3l^2 + 3912r^{15}q^2l^3 + 1956r^{15}ql^4 + 406r^{15}l^5 \\
& + 1032r^{14}q^6 + 5832r^{14}q^5l + 14724r^{14}q^4l^2 + 19416r^{14}q^3l^3 + 14724r^{14}q^2l^4 \\
& + 5832r^{14}ql^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 13608r^{13}q^6l + 40824r^{13}q^5l^2 \\
& + 67956r^{13}q^4l^3 + 67956r^{13}q^3l^4 + 40824r^{13}q^2l^5 + 13608r^{13}ql^6 + 1980r^{13}l^7 \\
& + 3260r^{12}q^8 + 25236r^{12}q^7l + 88620r^{12}q^6l^2 + 176484r^{12}q^5l^3 + 221110r^{12}q^4l^4 \\
& + 176484r^{12}q^3l^5 + 88620r^{12}q^2l^6 + 25236r^{12}ql^7 + 3260r^{12}l^8 + 4262r^{11}q^9 \\
& + 37854r^{11}q^8l + 151416r^{11}q^7l^2 + 352968r^{11}q^6l^3 + 529452r^{11}q^5l^4 + 529452r^{11}q^4l^5 \\
& + 352968r^{11}q^3l^6 + 151416r^{11}q^2l^7 + 37854r^{11}ql^8 + 4262r^{11}l^9 + 4752r^{10}q^{10} \\
& + 46252r^{10}q^9l + 208512r^{10}q^8l^2 + 554520r^{10}q^7l^3 + 971292r^{10}q^6l^4 + 1164342r^{10}q^5l^5 \\
& + 971292r^{10}q^4l^6 + 554520r^{10}q^3l^7 + 208512r^{10}q^2l^8 + 46252r^{10}ql^9 + 4752r^{10}l^{10} \\
& + 4262r^9q^{11} + 46252r^9q^{10}l + 231260r^9q^9l^2 + 693150r^9q^8l^3 + 1386300r^9q^7l^4 \\
& + 1940568r^9q^6l^5 + 1940568r^9q^5l^6 + 1386300r^9q^4l^7 + 693150r^9q^3l^8 + 231260r^9q^2l^9 \\
& + 46252r^9ql^{10} + 4262r^9l^{11} + 3260r^8q^{12} + 37854r^8q^{11}l + 208512r^8q^{10}l^2 \\
& + 693150r^8q^9l^3 + 1560534r^8q^8l^4 + 2494836r^8q^7l^5 + 2912112r^8q^6l^6 + 2494836r^8q^5l^7 \\
& + 1560534r^8q^4l^8 + 693150r^8q^3l^9 + 208512r^8q^2l^{10} + 37854r^8ql^{11} + 3260r^8l^{12} \\
& + 1980r^7q^{13} + 25236r^7q^{12}l + 151416r^7q^{11}l^2 + 554520r^7q^{10}l^3 + 1386300r^7q^9l^4 \\
& + 2494836r^7q^8l^5 + 3326448r^7q^7l^6 + 3326448r^7q^6l^7 + 2494836r^7q^5l^8 + 1386300r^7q^4l^9 \\
& + 554520r^7q^3l^{10} + 151416r^7q^2l^{11} + 25236r^7ql^{12} + 1980r^7l^{13} + 1032r^6q^{14} \\
& + 13608r^6q^{13}l + 88620r^6q^{12}l^2 + 352968r^6q^{11}l^3 + 971292r^6q^{10}l^4 + 1940568r^6q^9l^5 \\
& + 2912112r^6q^8l^6 + 3326448r^6q^7l^7 + 2912112r^6q^6l^8 + 1940568r^6q^5l^9 + 971292r^6q^4l^{10} \\
& + 352968r^6q^3l^{11} + 88620r^6q^2l^{12} + 13608r^6ql^{13} + 1032r^6l^{14} + 406r^5q^{15} + 5832r^5q^{14}l \\
& + 40824r^5q^{13}l^2 + 176484r^5q^{12}l^3 + 529452r^5q^{11}l^4 + 1164342r^5q^{10}l^5 + 1940568r^5q^9l^6 \\
& + 2494836r^5q^8l^7 + 2494836r^5q^7l^8 + 1940568r^5q^6l^9 + 1164342r^5q^5l^{10} + 529452r^5q^4l^{11} \\
& + 176484r^5q^3l^{12} + 40824r^5q^2l^{13} + 5832r^5ql^{14} + 406r^5l^{15} + 145r^4q^{16} + 1956r^4q^{15}l \\
& + 14724r^4q^{14}l^2 + 67956r^4q^{13}l^3 + 221110r^4q^{12}l^4 + 529452r^4q^{11}l^5 + 971292r^4q^{10}l^6 \\
& + 1386300r^4q^9l^7 + 1560534r^4q^8l^8 + 1386300r^4q^7l^9 + 971292r^4q^6l^{10} + 529452r^4q^5l^{11} \\
& + 221110r^4q^4l^{12} + 67956r^4q^3l^{13} + 14724r^4q^2l^{14} + 1956r^4ql^{15} + 145r^4l^{16} + 33r^3q^{17} \\
& + 489r^3q^{16}l + 3912r^3q^{15}l^2 + 19416r^3q^{14}l^3 + 67956r^3q^{13}l^4 + 176484r^3q^{12}l^5 \\
& + 352968r^3q^{11}l^6 + 554520r^3q^{10}l^7 + 693150r^3q^9l^8 + 693150r^3q^8l^9 + 554520r^3q^7l^{10} \\
& + 352968r^3q^6l^{11} + 176484r^3q^5l^{12} + 67956r^3q^4l^{13} + 19416r^3q^3l^{14} + 3912r^3q^2l^{15} \\
& + 489r^3ql^{16} + 33r^3l^{17} + 10r^2q^{18} + 90r^2q^{17}l + 774r^2q^{16}l^2 + 3912r^2q^{15}l^3 \\
& + 14724r^2q^{14}l^4 + 40824r^2q^{13}l^5 + 88620r^2q^{12}l^6 + 151416r^2q^{11}l^7 + 208512r^2q^{10}l^8 \\
& + \mathbf{231260r^2q^9l^9} + 208512r^2q^8l^{10} + 151416r^2q^7l^{11} + 88620r^2q^6l^{12} + 40824r^2q^5l^{13} \\
& + 14724r^2q^4l^{14} + 3912r^2q^3l^{15} + 774r^2q^2l^{16} + 90r^2ql^{17} + 10r^2l^{18} + rq^{19} + 10rq^{18}l \\
& + 90rq^{17}l^2 + 489rq^{16}l^3 + 1956rq^{15}l^4 + 5832rq^{14}l^5 + 13608rq^{13}l^6 + 25236rq^{12}l^7 \\
& + 37854rq^{11}l^8 + 46252rq^{10}l^9 + 46252rq^9l^{10} + 37854rq^8l^{11} + 25236rq^7l^{12} \\
& + 13608rq^6l^{13} + 5832rq^5l^{14} + 1956rq^4l^{15} + 489rq^3l^{16} + 90rq^2l^{17} + 10rql^{18} + rl^{19} \\
& + q^{20} + q^{19}l + 10q^{18}l^2 + 33q^{17}l^3 + 145q^{16}l^4 + 406q^{15}l^5 + 1032q^{14}l^6 + 1980q^{13}l^7 \\
& + 3260q^{12}l^8 + 4262q^{11}l^9 + 4752q^{10}l^{10} + 4262q^9l^{11} + 3260q^8l^{12} + 1980q^7l^{13} \\
& + 1032q^6l^{14} + 406q^5l^{15} + 145q^4l^{16} + 33q^3l^{17} + 10q^2l^{18} + ql^{19} + l^{20}
\end{aligned}
$$