2. Generating Function 生成函数

Compositions by 1 and 2

a_n: # of $(x_1, x_2, \cdots, x_k)$ for some $k \le n$ s.t. $\begin{cases} x_1 + \cdots + x_k = n \\ x_i \in \{1, 2\} \end{cases}$

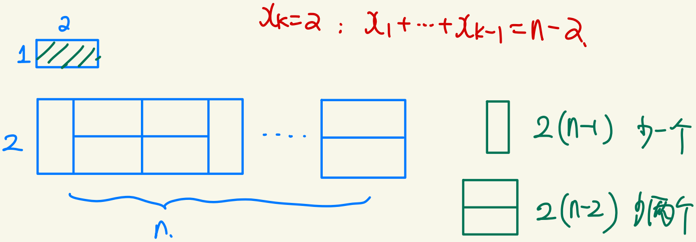
$x_k = 1$: $x_1 + \cdots + x_{k-1} = n-1$

$x_k = 2$: $x_1 + \cdots + x_{k-1} = n-2$

$a_n = a_{n-1} + a_{n-2}$

$a_0 = 1,\ a_1 = 1.$

Dominos



# of tilings

$\Rightarrow a_n = a_{n-1} + a_{n-2}.$

$a_0 = 1,\ a_1 = 1.$

Fibnacci $F_n = \begin{cases} F_{n-1} + F_{n-2} & n \ge 2 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases}$

Full parenthesization of (n+1) factors

1-1 $\updownarrow$

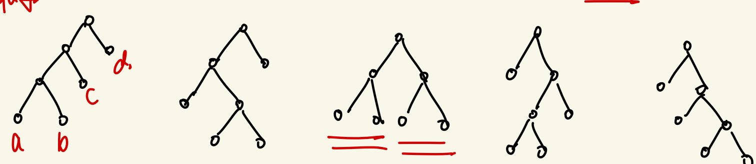
$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$

Full binary tree with (n+1) leaves.

满二叉树

C_n: # of full binary tree with $(n+1)$ leaves.

Catalan Number



$C_0 = 1.$

$C_k \quad C_{n-k-1}$

$\downarrow \quad \downarrow$

$k+1 \quad n-k$

$$\Rightarrow C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

卷积 (convolution)

Generate

~~Enumerate~~ all subsets of $\{\bigcirc\ \bigcirc\ \bigcirc\}$

$$(x^0 + x^1)\cdot(x^0 + x^1)(x^0 + x^1)$$

$$= x^0x^0x^0 + x^0x^1x^0 + x^0x^0x^1 + x^0x^1x^1$$

$$+ x^1x^0x^0 + x^1x^1x^0 + x^1x^0x^1 + x^1x^1x^1$$

K-subsets

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$ polynomial

$\boxed{\text{coefficient of } x^k : \# \text{ of } k\text{-subsets}}$ → combinatorics

$\{ \circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ \}$

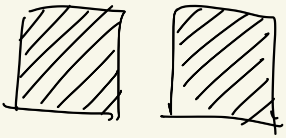
$(1+x+x^2+x^3)\cdot(1+x+x^2+x^3+x^4)\cdot(1+x+x^2+x^3+x^4+x^5)$

$= 1 + 3x + 6x^2 + \cdots$

m-subset

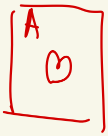
$\boxed{\text{Reduction}}$ 归约

Double Deck



choose m cards from 2 decks of n cards

$\binom{2n}{m}$ × 有重复!



不出现
出现1次
出现2次

$(x_1^0+x_1^1+x_1^2)\cdot(x_2^0+x_2^1+x_2^2)\cdots(x_n^0+x_n^1+x_n^2)$

$\#$ of m-subset $\Leftrightarrow$ $\boxed{\text{coefficient of } x^m \text{ in } (1+x+x^2)^n}$

$$(1+(x+x^2))^n = \sum_k \binom{n}{k}(x+x^2)^k$$

$$= \sum_k \binom{n}{k} x^k \cdot \sum_l \binom{k}{l} x^l$$

$$= \sum_k \sum_l \binom{n}{k}\binom{k}{l} x^{k+l} \quad \xrightarrow{} m$$

$$= \sum_m \boxed{\sum_l \binom{n}{m-l}\binom{m-l}{l}} \cdot x^m$$

Multisets

$S = \{x_1, \cdots, x_n\}$

multiplicity 重数

M s.t. $|M| = \sum_{i=1}^{n} m(x_i)$

$$\begin{cases} m(x_1)+\cdots+m(x_n)=k, \\ m(x_i)\geq 0, \quad i\in[n] \end{cases} \Rightarrow \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

$$(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)$$

$$= \sum_{m:S\to\mathbb{N}} \prod_{x_i\in S} x_i^{m(x_i)}$$

k-Multiset : $(1+x+x^2+\cdots)^n$ 中 x^k 前面系数就是 k-Multiset 的数量

① 几何级数 $= \left(\frac{1}{1-x}\right)^n = (1-x)^{-n}$

② 泰勒级数 (二项级数)

$$= \sum_{k\geq 0} \frac{(-n)(-n-1)\cdots(-n-k+1)(-1)^k}{k!} x^k$$

$$= \sum_{k\geq 0} \frac{n(n+1)\cdots(n+k-1)}{k!} x^k = \binom{n+k-1}{k}$$

$$(1+\alpha x)^\beta = \sum_{k\geq 0} \frac{\beta(\beta-1)\cdots(\beta-k+1)(\alpha x)^k}{k!}$$

Matching 匹配

???

OGF (Ordinary Generating Function)

常生成函数

to be solved.

Goal: $\{a_n\}$ $a_0, a_1, a_2, \cdots$

generating function: $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$[x^n]G(x) \triangleq a_n$$

Fibonacci Number

① $F_n = \begin{cases} F_{n-1} + F_{n-2}, & n \geqslant 2 \\ 1, & n = 1 \\ 0, & n = 0 \end{cases}$

0, 1, 1, 2, 3, 5, 8, 13, 21 …

Closed-form: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

$\phi \qquad \hat{\phi}$

generating function: $G(x) = \sum_{n \geqslant 0} F_n \cdot x^n$

① $G(x) = F_0 + F_1 x + \sum_{n \geqslant 2} F_n x^n$

$= x + \sum_{n \geqslant 2} F_{n-1} x^n + \sum_{n \geqslant 2} F_{n-2} x^n$ ← recurrence

$\sum_{n \geqslant 2} F_{n-1} x^n = F_1 x^2 + F_2 x^3 + \cdots$

$= x \cdot (F_1 x + F_2 x^2 + \cdots) = x \cdot G(x)$

$\sum_{n \geqslant 2} F_{n-2} x^n = F_0 \cdot x^2 + F_1 \cdot x^3 + F_2 x^4 + \cdots$

$= x^2 \cdot (F_1 x + F_2 x^2 + \cdots) = x^2 G(x)$

$\Rightarrow \boxed{G(x) = x + (x + x^2)\, G(x)}$ identity 宏观平衡

② $G(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\hat{\phi} x}$

令 $\phi = \frac{1+\sqrt{5}}{2} \quad \hat{\phi} = \frac{1-\sqrt{5}}{2}$

$1 - x - x^2 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$

$\phi = \frac{1+\sqrt{5}}{2} = \frac{2}{-1+\sqrt{5}}$

$\hat{\phi} = \frac{1-\sqrt{5}}{2} = \frac{2}{-1-\sqrt{5}}$

$\Rightarrow 1 - x - x^2 = (1-\phi x)(1-\hat{\phi} x)$

③ $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$$\Rightarrow G(x) = \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left((\phi x)^n - (\hat{\phi} x)^n \right)$$

$$= \sum_{n\geq 0} \boxed{\frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)} \, x^n$$

$\uparrow$ F_n

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)$$

$$\frac{x}{1-x-x^2} = \frac{x}{(1-\phi x)(1-\hat{\phi} x)}$$

$$= \frac{\alpha}{1-\phi x} + \frac{\beta}{1-\hat{\phi} x}$$

$$\alpha(1-\hat{\phi} x) + \beta(1-\phi x) = x.$$

$$\Rightarrow \begin{cases} \alpha + \beta = 0 \\ \alpha\hat{\phi} + \beta\phi = -1. \end{cases}$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{5}}. \quad \beta = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow G(x) = \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\hat{\phi} x}$$

OGF

$\{a_n\}$ $a_0, a_1, a_2, \cdots\cdots$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$\frac{1}{1-\alpha x} = \sum_{n\geq 0} (\alpha x)^n$$

formal power series 形式幂级数

$$G(x) = \sum_{n\geq 0} a_n x^n.$$

$\mathbb{C}[[x]]$: ring of formal power series with complex coefficient.

$\mathbb{C}[x]$: 多项式环

$\mathbb{C}[[x]]$: 形式幂级数环

$$F(x) \cdot G(x) = 1.$$

乘法幺元

$$F(x) = G(x)^{-1} = \frac{1}{G(x)}.$$

$$(1-\alpha x) \cdot \left(\sum_{n\geq 0} \alpha^n x^n \right) = 1 \Rightarrow \sum_{n\geq 0} \alpha^n x^n = \frac{1}{1-\alpha x}$$

Generating Function Algebra:

$G(x)=\sum_{n\geq 0} g_n x^n$. $F(x)=\sum_{n\geq 0} f_n x^n$.

① shift: $x^k G(x) = \sum_{n\geq k} g_{n-k} x^n$.

② addition: $F(x)+G(x) = \sum_{n\geq 0} (f_n+g_n)\cdot x^n$.

③ convolution 卷积: $F(x)\cdot G(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k g_{n-k}\right) x^n$

④ differentiation: $G'(x) = \sum_{n\geq 0} (n+1) g_{n+1} x^n$.

"Generating functionology" ← Recurrence. Manipulation. Solving & Expending.

Expansion:

- Taylor: $G(x)=\sum_{n\geq 0} \frac{G^{(n)}(0)}{n!} x^n$.
- Geometric sequence: $\frac{a}{1-bx} = a\cdot\sum_{n\geq 0}(bx)^n$.
- Binomial: $(1+x)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n} x^n$.

Newton's formula.

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

$$G(x)= \frac{a_1}{1-b_1x} + \frac{a_2}{1-b_2x} + \cdots + \frac{a_k}{1-b_kx},$$

$$\Rightarrow [x^n]G(x) = a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n.$$

Changing Money

壹. 伍

- 壹$_n$: 用$\boxed{\text{壹圆}}$来换n元钱的方式 $(1,1,1,1,1,1,\cdots)$
- 伍$_n$: 用$\boxed{\text{伍圆}}$来换n元钱的方式 $(1,0,0,0,0,1,\cdots)$

$$(\text{壹},\text{伍})_n = \underbrace{\sum_{k=0}^{n} \text{壹}_k \text{伍}_{n-k}} = \text{壹}_n \circ \text{伍}_n$$

generating function
$$\begin{cases} \sum_{n\geq 0} \text{壹}_n x^n = 1 + x + x^2 + \cdots = \dfrac{1}{1-x} \\ \sum_{n\geq 0} \text{伍}_n x^n = 1 + x^5 + x^{10} + \cdots = \dfrac{1}{1-x^5} \end{cases}$$

$$\Rightarrow \sum_{n\geq 0} (\text{壹},\text{伍})_n x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^5}$$

类似地，$\sum_{n\geq 0} (\text{壹},\text{伍},\text{拾},\text{贰拾},\text{伍拾},\text{壹佰})_n x^n$

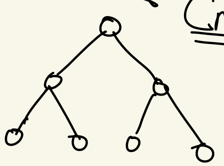
$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$= \frac{1}{(1-x^{100})^6} \cdot (1+x^{50}) \cdot (1+x^{20}+x^{40}+x^{60}+x^{80}) \cdots\cdots (1+x+x^2+\cdots+x^{99})$$

$$\frac{1}{(1-x^{100})^6} = \sum_{n\geq 0} \binom{-6}{n} (-x^{100})^n$$

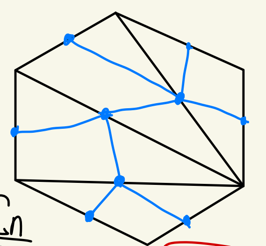
Catalan Number

n+1 leaves

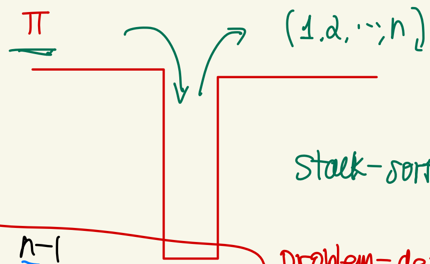


C_k C_{n-k-1}

$k+1$ $n-k$



$\underline{\underline{C_n}}$



stack-sortable

$$\boxed{C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}}$$

problem-dependent

卷积

$$\Downarrow$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$C_0 = 1$

generating function $G(x) = \sum_{n\geq 0} C_n x^n = C_0 + \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$

$F(x)\cdot G(x) = \sum_{n\geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n.$

$G(x)^2 = \sum_{n\geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n.$

$x\cdot G(x)^2 = \sum_{n\geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^{n+1}$

$= \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n.$

$\Rightarrow \underline{G(x) = 1 + x\, G(x)^2},$

$\Rightarrow G(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$

$\lim_{x\to 0} G(x) = C_0 = 1.$

$\Rightarrow G(x) = \dfrac{1-(1-4x)^{\frac{1}{2}}}{2x} = \sum_{n\geq 0} C_n x^n$

$(1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n = 1 + \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-4x)^n.$

$= 1 + \sum_{n\geq 0} \binom{\frac{1}{2}}{n+1} (-4x)^{n+1} = 1 - 4x\cdot \sum_{n\geq 0} \binom{\frac{1}{2}}{n+1} (-4x)^n.$

$\Rightarrow G(x) = 2\cdot \sum_{n\geq 0} \binom{\frac{1}{2}}{n+1} (-4x)^n = \sum_{n\geq 0} \boxed{\binom{\frac{1}{2}}{n+1}\cdot 2\cdot (-4)^n} x^n$

C_n : Catalan Number

$C_n = 2\cdot \binom{\frac{1}{2}}{n+1}\cdot(-4)^n$

$= \cancel{2}\cdot \frac{1}{\cancel{2}}\cdot \frac{-1}{2}\cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2}\cdot \frac{1}{(n+1)!}\cdot(-4)^n.$

$= \frac{2^n}{(n+1)!}\cdot (1\times 3\times \cdots \times (2n-1))$

$1\times 2\times \cdots \times 2n = (2n)!$

$2\times 4\times \cdots \times 2n = 2^n\cdot (1\times 2\times \cdots \times n) = 2^n\cdot n!$

$\Rightarrow 1\times 3\times 5\times 7\times \cdots \times (2n-1) = \dfrac{(2n)!}{2^n\cdot n!}$

$= \frac{2^n}{(n+1)!}\cdot \frac{(2n)!}{2^n\cdot n!} = \frac{(2n)!}{n!\cdot n!}\cdot \frac{1}{n+1}$

$= \binom{2n}{n}\cdot \frac{1}{n+1} = C_n$

Average-Case Analysis of Quicksort

① 期望的线性性质

② 生成函数来求解 recursion

③ ......

Input: an array A of n numbers

$QSort(A)$:

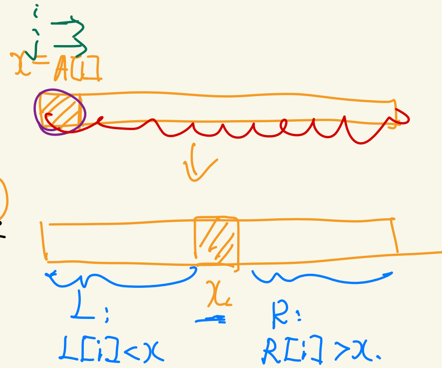
$A[k]$ $\quad k \sim [n]$, u.a.r.

choose a pivot $x = A[1]$;

partition A into L with $L[i] < x$,

R with $R[i] > x$. $\quad O(n)$

$QSort(L)$, $\quad QSort(R)$



Complexity: # of comparisons

$T(n) \doteq 2T(\frac{n}{2})$

$\Rightarrow T(n) = \Theta(n\log n)$

$T(n) \doteq T(\frac{9n}{10}) + T(\frac{1}{10}n)$

$\Rightarrow T(n) = \Theta(n\log n)$

$T(n) = T(n-1) + T(1)$

$\Rightarrow T(n) = \Theta(n^2)$

worst-case: $\Theta(n^2)$

average-case: $\Theta(n\log n)$

T_n: average # of comparison used by $Qsort$,

taken over all $n!$ total orders of A

pivot: the k-th smallest number in A.

$|L| = k-1$, $\quad |R| = n-k$

$$T_n = \mathbb{E}[T_{k-1} + T_{n-k} + n - 1]$$

$$\boxed{T_n = \frac{1}{n}\sum_{k=1}^{n}(T_{k-1} + T_{n-k} + n - 1)} \rightarrow \text{recursion.}$$

$$\boxed{T_0 = T_1 = 0}$$

generating function: $G(x) = \sum_{n \geq 0} T_n x^n$

$$n \cdot T_n = \sum_{k=1}^{n} \left(T_{k-1} + T_{n-k} + n - 1 \right)$$

$T_1 + T_2 + \cdots + T_{n-1}$

$$\boxed{\sum_{n \geq 0} (n T_n) x^n} = \sum_{n \geq 0} \sum_{k=1}^{n} \left(T_{k-1} + T_{n-k} + n - 1 \right) \cdot x^n$$

$$= \boxed{\sum_{n \geq 0} n(n-1) \cdot x^n} + 2 \boxed{\sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n}$$

convolution

$$\sum_{n \geq 0} n T_n x^n = x \cdot \sum_{n \geq 1} n T_n x^{n-1} = x \cdot \underbrace{\sum_{n \geq 1} (n+1) T_{n+1} x^n}_{} = x\, G'(x)$$

$$G(x) = \sum_{n \geq 0} T_n x^n$$

differentiation

$$G'(x) = \sum_{n \geq 1} n T_n x^{n-1} = \sum_{n \geq 1} (n+1) T_{n+1}$$

$$\sum_{n \geq 0} x^n = \frac{1}{1-x}$$

$$\boxed{\sum_{n \geq 1} n x^{n-1} = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}}$$

$$\sum_{n \geq 2} n(n-1) x^{n-2} = \left(\frac{1}{1-x} \right)'' = \frac{2}{(1-x)^3}$$

$$\Rightarrow \sum_{n \geq 0} n(n-1) x^n = x^2 \cdot \sum_{n \geq 2} n(n-1) x^{n-2} = x^2 \cdot \frac{2}{(1-x)^3}$$

$T_k \cdot 1$

$$\sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n = x \cdot \sum_{n \geq 0} \left(\sum_{k=0}^{n} T_k \right) x^n$$

$$= x \cdot \underbrace{\left(\sum_{n \geq 0} x^n \right)}_{1/(1-x)} \underbrace{\left(\sum_{n \geq 0} T_n x^n \right)}_{G(x)} = \frac{x G(x)}{1-x}$$

$\Rightarrow$ identity:

$$x G'(x) = \frac{2x^2}{(1-x)^3} + \frac{2x G(x)}{1-x}$$

$$\Rightarrow \underbrace{G'(x) = \frac{2x}{(1-x)^3} + \frac{2G(x)}{1-x}}_{}$$

一阶线性常微分方程

$$y' + P(x)\cdot y = Q(x) \Rightarrow y(x) = \frac{1}{u(x)}\int u(x)Q(x)\,dx.$$

$$u(x) = e^{\int p(x)\,dx}.$$

$$\Rightarrow G(x) = e^{\int \frac{2}{1-x}dx}\cdot\int e^{-\int\frac{2}{1-x}dx}\cdot\frac{2x}{(1-x)^3}\,dx.$$

$$\overset{\dots}{=} \frac{2}{(1-x)^2}\cdot\ln\frac{1}{1-x} - \frac{2x}{(1-x)^2}.$$

无穷多个解

令 $x=0$: $G(0)=T_0=0$

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n. \Rightarrow \frac{2x}{(1-x)^2} = 2\sum_{n\geq 0}(n+1)x^{n+1} = 2\sum_{n\geq 0} n x^n.$$

$$\ln\frac{1}{1-x} = \sum_{n\geq 1}\frac{x^n}{n}$$

$$\Rightarrow G(x) = 2\left(\sum_{n\geq 0}(n+1)x^n\right)\left(\sum_{n\geq 1}\frac{x^n}{n}\right) - 2\cdot\sum_{n\geq 0} n x^n$$

$$= 2\cdot\sum_{n\geq 1}\sum_{k=1}^{n}\frac{1}{k}(n+1-k)x^n - 2\cdot\sum_{n\geq 0} n x^n.$$

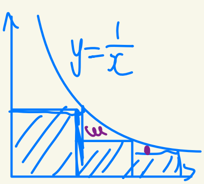
$$T_n = [x^n]G(x) = -2n + 2\cdot\sum_{k=1}^{n}\frac{n+1-k}{k}$$

$$= -2n + 2\cdot\sum_{k=1}^{n}\frac{n+1}{k} - 2n.$$

$$= 2(n+1)\cdot\sum_{k=1}^{n}\frac{1}{k} - 4n.$$

$\ln n \leq H(n) \leq \ln n + 1$

Harmonic number 调和数 $H(n)$



$$= 2(n+1)H(n) - 4n.$$

$$H(n) = \ln n + O(n) \Rightarrow T_n = 2n\ln n + O(n)$$