8. Extremal Graph Theory

极值

Extremal Combinatorics: how large or how small a collection of finite objects can be, if it has to satisfy certain restrictions?

graph / set-family

e.g. What is the largest # of edges that an n-vertex cycle-free graph can have?

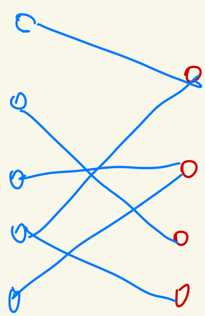
restrictions $\quad |E| \le n-1 \quad$ "spanning tree"

extremal graph.

① Triangle-Free Graph

e.g. bipartite graph 二分图

balanced complete. $|E| = \frac{n^2}{4}$.



Theorem (Mantel 1907) $G=(V,E)$

$|V|=n$, triangle-free: $|E| \le \frac{n^2}{4}$

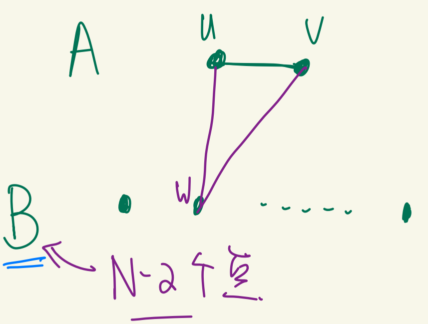
First Proof. (Induction)

$n=1,2$: trivial.

I.H (Induction Hypothesis): for any $n < N$: Δ-free $\Rightarrow |E| \le \frac{n^2}{4}$

$\Longleftrightarrow$ $\boxed{|E| > \frac{n^2}{4} \Rightarrow G \ni \Delta}$

for $n = N$: 一定存在一条边 $(u,v) \in E$.



$|A| = 2$. $|B| = N-2$

by I.H: $|E(B)| \le \frac{(N-2)^2}{4}$

$|E(A)| = 1$.

$$\underline{|E|} = \underline{|E(A)|} + \underline{|E(B)|} + \underbrace{|E(A,B)|}_{\text{crossing edge}}$$

$$\le 1 + \frac{(N-2)^2}{4} + |E(A,B)|$$

$$\Rightarrow |E(A,B)| \ge |E| - \frac{N^2}{4} + N - 2.$$

归纳: $|E| > \frac{N^2}{4}$: $\underline{|E(A,B)| > N-2}$.

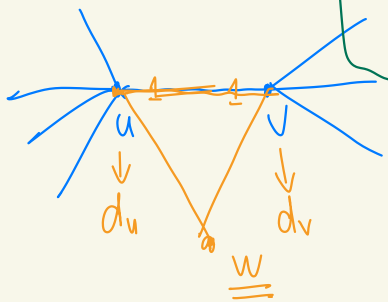
by Pigeonhole principle. $\exists\, w \in B$.

s.t. $(u,w), (v,w) \in E$. $\Rightarrow G \ni \Delta$ $\square$

Second Proof:

△-free

$\Rightarrow \forall (u,v) \in E:\ \underset{\text{degree}}{d_u} + d_v \le n.$



Double Counting:

对每一条边(u,v)赋予(d_u+d_v)的权重

$$\sum_{u\in V} d_u^2 = \sum_{(u,v)\in E} (d_u+d_v) \le n\cdot|E|$$

By handshaking lemma: $\sum_{u\in V} d_u = 2\cdot|E|$

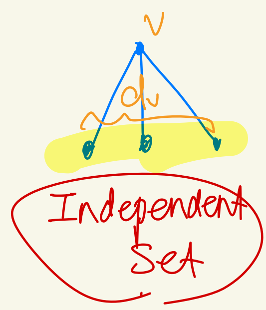
Cauchy-Schwarz:

$$\underbrace{\sum_{u\in V} d_u^2}_{n\cdot|E|} \cdot \underbrace{\sum_{u\in V} 1^2}_{n} \ge \left(\sum_{u\in V} d_u\right)^2 = 4\cdot|E|^2.$$

$$\Rightarrow n^2\cdot|E| \ge 4\cdot|E|^2 \Rightarrow |E| \le \frac{n^2}{4} \qquad \square$$

Third Proof:

△-free



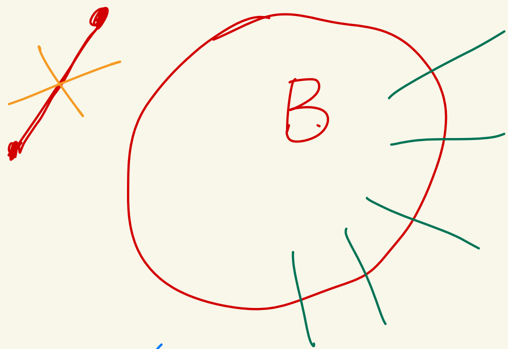
Intuition: 任一顶点的邻居不能相邻.

A: Maximum Independent Set. $|A| = \alpha$

$\forall v \in V:\ d_v \le \alpha$

$B = V \setminus A$. $|B| = \beta$

(dominating set 支配集)



→ 每条边至少有一个顶点在B中

$$|E| \le \sum_{v\in B} d_v \le \alpha \cdot \beta \le \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{n^2}{4}$$

均值不等式

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② Clique-Free Graph

团

(K_r: complete graph with r vertices)

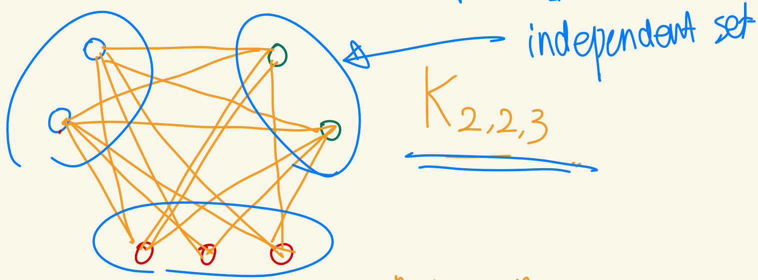
Theorem (Turán 1941) If $G=(V,E)$, $|V|=n$,

K_r-free: $|E| \le \dfrac{r-2}{2(r-1)} n^2$.

* $r=3$: Δ-free (Mantel Thm) $\dfrac{n^2}{4}$.

r-partite graph 多分图 (balanced. complete)

$K_{n_1,n_2,\dots,n_r}$ extremal graph



Turán's Graph $T(n,r) = K_{n_1,n_2,\dots,n_r}$ $\begin{cases} n_1+\dots+n_r = n \\ n_i \in \left\{ \lfloor \frac{n}{r} \rfloor, \lceil \frac{n}{r} \rceil \right\} \end{cases}$

$\boxed{T(n,r-1) \text{ has no } r\text{-clique}}$

$$|T(n,r-1)| \le \binom{r-1}{2}\underbrace{\left(\frac{n}{r-1}\right)^2}_{\text{两个独立集之间的边}} = \frac{(r-1)(r-2)}{2}\cdot\frac{n^2}{(r-1)^2} = \frac{(r-2)}{2(r-1)}\cdot n^2.$$

First Proof (Induction)

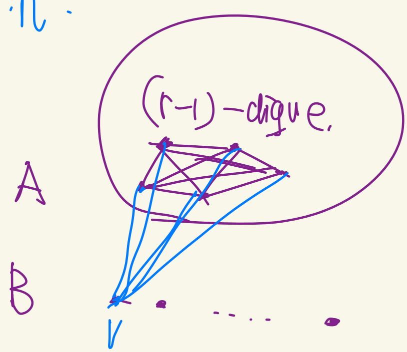
Basic Case: $n = 1, 2, \dots, r-1$

I.H: $\forall n < N$: $|E| \le \frac{r-2}{2(r-1)}\cdot n^2$.

Induction Case: $n = N$.

Suppose G is maximal K_r-free:

$\exists$ $(r-1)$-clique A.



by I.H.: $|E(B)| \le \frac{r-2}{2(r-1)} \cdot (n-r+1)^2$

K_r-free: $|E(A,B)| \le \underbrace{(n-r+1)}_{\text{B中顶点数}} \cdot (r-2)$ by pigeonhole

$$|E(A)| = \binom{r-1}{2} = \frac{(r-1)(r-2)}{2}.$$

$$|E| = |E(A)| + |E(B)| + |E(A,B)|$$

$$\le \frac{(r-1)(r-2)}{2} + (n-r+1) \cdot (r-2) + \frac{r-2}{2(r-1)} \cdot (n-r+1)^2$$

$$= \frac{r-2}{2(r-1)} \cdot \left((r-1)^2 + 2(r-1)(n-r+1) + (n-r+1)^2\right)$$

$$= \frac{r-2}{2(r-1)} \cdot n^2 \qquad \square$$

Extremal: 离散情况
的最优化

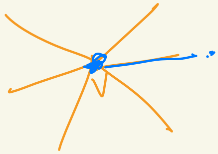
Second Proof (weight shifting)

potential analysis.

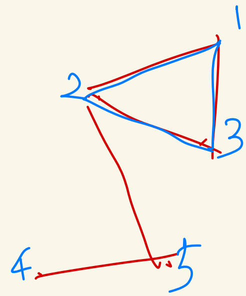
Assign each vertex v a weight $w_v > 0$ s.t. $\sum_{v \in V} w_v = 1$,

Goal: maximize $S(\vec{w}) = \sum_{(u,v) \in E} w_u w_v = \frac{1}{2} \cdot \sum_{v \in V} w_v \cdot W_v$

(double counting)



$$W_u = \sum_{v:(u,v)\in E} w_v$$ ：所有u的邻居的权重之和.



If $(u,v)\notin E$: Suppose $W_u \geq W_v$.

保证 $W_u.W_v$ 不变 ⇒

$$(w_u+\varepsilon)\cdot W_u + (w_v-\varepsilon)\cdot W_v$$
$$= w_u\cdot W_u + w_v\cdot W_v + \varepsilon\cdot(W_u - W_v)$$
$$\geq w_u\cdot W_u + w_v\cdot W_v$$

shifting all weight of v to u.

$\Longrightarrow$ $S(\vec{w})$ non-decreasing.

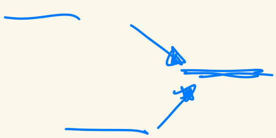
$S(\vec{w})$ is maximized

$\Rightarrow$ all weights on a clique.

If $W_u > W_v > 0$: choose ε s.t. $0 < \varepsilon < W_u - W_v$

$$W_u' = W_u - \varepsilon.$$
$$W_v' = W_v + \varepsilon.$$

$\Rightarrow$ $S' = S + \varepsilon(W_u - W_v) - \varepsilon^2 > S$



让差值减小 ⇒ 能够 improve!

Optimum: uniform weights on a clique.

G: K_r-free $\Rightarrow$ weights concentrate on a $(r-1)$-clique

$$\Rightarrow \quad S \le \binom{r-1}{2} \cdot \frac{1}{(r-1)^2} = \frac{r-2}{2(r-1)}$$

uniform weight

Any feasible solution $\vec{w}$:

$$S(\vec{w}) \le \frac{r-2}{2(r-1)}$$

最优解

Choosing $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$;

$$S(\vec{w}) = \sum_{(u,v)\in E} w_u w_v = \frac{|E|}{n^2}.$$

可行解

$$\Rightarrow \quad |E| \le \frac{r-2}{2(r-1)} \cdot n^2. \qquad \square$$

Third Proof. (The Probabilistic Method)

Clique number: $\omega(G)$: size of the largest clique.

Claim: $\omega(G) \geqslant \boxed{\sum_{v\in V} \frac{1}{n-d_v}}$

Proof: random permutation π of V:

$$S=\{v \mid \underbrace{\pi_u<\pi_v \Longrightarrow (u,v)\in E}\}$$

所有排在顶点v之前的点均与其相邻

S is a clique. (random clique)

$$\mathbb{E}[|S|] = \sum_{v\in V} \Pr(v\in S)$$

linearity of expectation

$$= \sum_{v\in V} \Pr(\text{所有}v\text{之前的点都和}v\text{相邻})$$

$$= \sum_{v\in V} \Pr(\text{所有非邻居的点都在}v\text{之后})$$

$$= \sum_{v\in V} \Pr(\underbrace{v\text{和其所有非邻居}}_{(n-d_v)\text{个点}},\ v\text{在第一个})$$

$$= \sum_{v\in V} \frac{1}{n-d_v}$$

In K_r-free graph: $\omega(G) \leq r-1$.

$$\Rightarrow r-1 \geq \sum_{v\in V} \frac{1}{n-d_v}.$$

Cauchy-Schwarz:

$$\left(\sum_{v\in V}\frac{1}{n-d_v}\right)\left(\sum_{v\in V}(n-d_v)\right) \geq \left(\sum_{v\in V} 1\right)^2$$

$$n^2 \leq (r-1)\cdot \underbrace{\sum_{v\in V}(n-d_v)}$$

$$= (r-1)\cdot\left(n^2-2|E|\right)$$

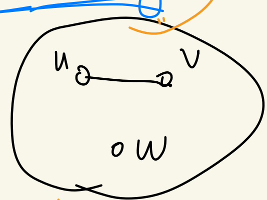
handshaking

$$\Rightarrow |E| \leq \frac{r-2}{2(r-1)}\cdot n^2 \qquad \square$$

Fourth Proof:

G is K_r-free with maximum edges. 极值图

Claim: G does NOT have.



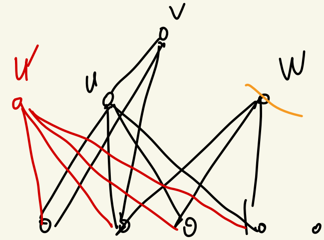
Proof: (Case 1). $d_w < d_u$ or $d_w < d_v$.

suppose

delete w

replaced by u'.

$\rightarrow$ still K_r-free



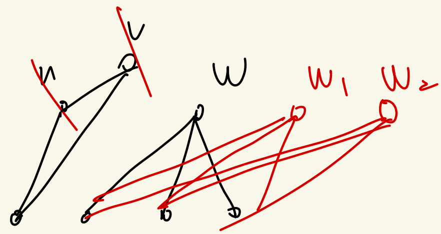
$$|E'| = |E| + d_u - d_w > |E|.$$ contradiction!

(Case 2) $d_w \geq d_u$ and $d_w \geq d_v$

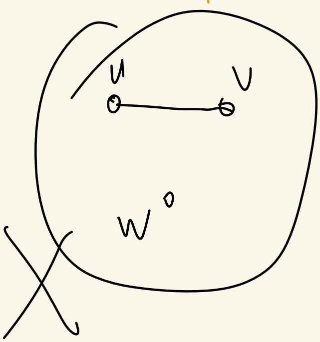
delete u. v.

replaced by $w_1 . w_2$

$\rightarrow$ still K_r-free.



$$|E'| = |E| + 2d_w - (d_u + d_v - 1) > |E|.$$ contradiction!



自反

对称

传递

$\Longrightarrow$ $u \nsim v$ $((u,v) \notin E)$ is an equivalence.

不相邻关系是一种等价关系

$\rightarrow$ optimize a complete multi-partite graph opt. $K_{n_1, n_2, \cdots, n_{r-1}}$ $\Rightarrow$ $\boxed{\text{Turán graph } T(n, r-1)}$

s.t. $n_1 + n_2 + \cdots + n_{r-1} = n$

balanced

Recall: Turán's Theorem.

(clique) If $G = (V, E)$, $|V| = n$, K_r-free

then $|E| \le \frac{r-2}{2(r-1)} n^2$

(independent set) If $G = (V, E)$, $|V| = n$, $|E| = m$,

then G has an independent set of size $\ge \frac{n^2}{2m+n}$.

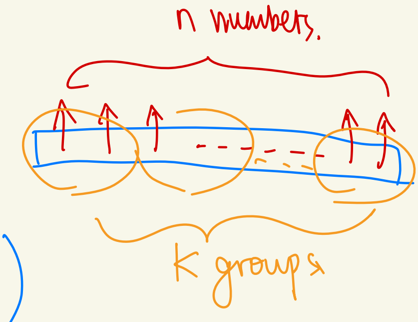
一个 Turán 定理的应用：

Goal: compute max of n distinct numbers.

↳ Comparison-based + $\boxed{\text{parallel 并行计算}}$

e.g. 1-round : $\binom{n}{2}$ comparisons of all pairs

$\Omega(n^2)$ worst case.



2-round：将 n 个数均分成 k 份：

1st：find max in each group $\quad k \cdot \binom{n/k}{2}$

2nd：find max of the k maxes. $\quad \binom{k}{2}$

→ total comparison：$k \cdot \binom{n/k}{2} + \binom{k}{2} = O(n^{\frac{4}{3}})$ for $k = n^{\frac{2}{3}}$.

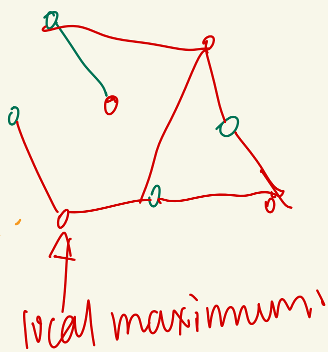
Optimality：$\Omega(n^{\frac{4}{3}})$ by Turán theorem.

Adversary Argument

Choose an independent set of size $\geq \frac{n^2}{2m+n}$

independent set → local maximal.

$2m+n$: to be tuned



1st：m comparisons

↓ 给第二轮留下一个尽可能坏的结果

2nd：$\geq \binom{n^2/(2m+n)}{2}$ comparisons

$\Rightarrow$ total comparisons：$\geq m + \binom{\frac{n^2}{2m+n}}{2} = \Omega(n^{\frac{4}{3}})$

Mantel Theorem

$\triangle$-free → r-clique free (Turán's Theorem)

$\triangle$-free → $\boxed{\text{forbidden cycles}}$ ← open problem

girth 围长 $g(G)$: length of the shortest cycle

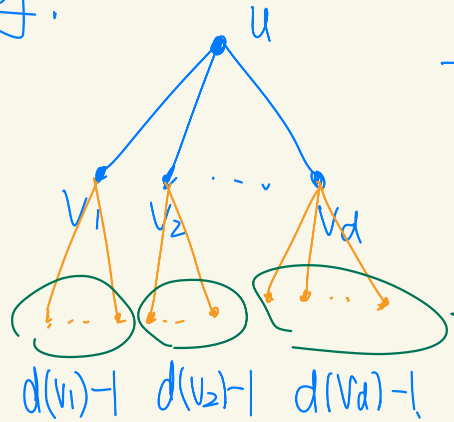
$\triangle$-free: $g(G) \geq 4$

Theorem If $G=(V,E)$, $|V|=n$, $g(G) \geq 5$

then $|E| \leq \frac{1}{2} n \cdot \sqrt{n-1}$. (loose)

Proof:

$d = \text{degree}(u)$

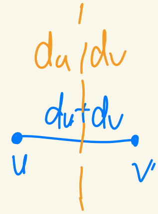


$(g(G) \geq 5)$

→ disjoint sets

$$\Rightarrow \quad 1 + d + (d(v_1)-1) + \cdots + (d(v_d)-1) \leq n.$$

$$\Rightarrow \sum_{v:(u,v)\in E} d(v) \le n-1.$$



By Cauchy-Schwarz,

$$n\cdot(n-1) \ge \sum_{u\in V}\sum_{v\in \partial(u)} d(v) = \sum_{u\in V} d(u)^2 \ge \frac{\left(\sum_{u\in V} d(u)\right)^2}{n} = \frac{4\cdot|E|^2}{n}$$

$$\Rightarrow |E| \le \frac{1}{2}\cdot n\cdot\sqrt{n-1}. \qquad \square$$

Extremal Graph Theory

fix a graph H: ← 禁止H作为子图
(forbidden subgraph)

$$ex(n,H) = \max_{\substack{G\not\supseteq H \\ |V(G)|=n}} |E(G)| :$$

largest # of edges of $G \not\supseteq H$ on n vertices.

Turán's Theorem: $ex(n, K_r) = |T(n, r-1)| \le \frac{r-2}{2(r-1)}\cdot n^2$,

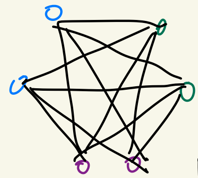
Fundamental Theorem of Extremal Graph Theory

极值图论基本定理. (Erdös-Stone)

$K_s^r = K_{\underbrace{s,s,\ldots,s}_{r}} = T(rs, r)$:

complete r-partite graph with s vertices in each part.

$ex(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1)\right) \cdot n^2$.



K_2^3.

For $r \geq 2$, $s \geq 1$: $\forall\, \varepsilon > 0$. if n sufficiently large.

$\Rightarrow$ for any graph on n vertices with $\geq \left(\frac{r-2}{2(r-1)} + \varepsilon\right) \cdot n^2$ edges: it must contains K_s^r as a subgraph.

Corollary for every non-empty graph H:

$$\lim_{n \to \infty} \boxed{\frac{ex(n, H)}{\binom{n}{2}}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

extremal density of subgraph H.

Chromatic number 染色数

Proof: let $r = \chi(H)$

$H \not\subseteq \underline{T(n, r-1)}$ for any n.

Turán graph 也禁止了 H 这个子图

lower bound $ex(n, H) \geq \underline{|T(n, r-1)|}$

$$|T(n,r-1)| \geq \binom{r-1}{2} \cdot \left\lfloor \frac{n}{r-1} \right\rfloor^2 \geq \binom{r-1}{2} \cdot \left(\frac{n}{r-1} - 1\right)^2$$

$$= \left(\frac{r-2}{2(r-1)} - o(1)\right) \cdot n^2.$$

$\boxed{H \subseteq K_s^r}$ for sufficiently large s:

$$ex(n, H) \leq ex(n, K_s^r) \rightarrow \text{upper bound}$$

$$= \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2.$$

$$\Rightarrow \frac{r-2}{r-1} - o(1) \leq \frac{ex(n,H)}{\binom{n}{2}} \leq \frac{r-2}{r-1} + o(1) \quad \square$$