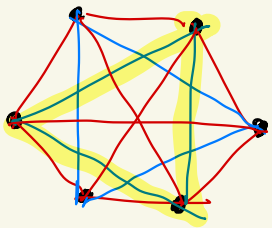
7. The Probabilistic Method

概率法

Example 1 Ramsey Number

"六个人派对上 至少有三个人 相互认识 / 相互不认识"



构造 ~~object~~

probability space of object (event)

$\Pr(A) > 0$

↳ A 有可能会发生 (某种 configuration)

染色 (chromating)

$\forall$ edge-2-coloring K_6

$\exists$ a monochromatic K_3

$R(3,3) = 6$

$\Downarrow$

Ramsey Theorem: If $n \geq R(k,k)$ for any edge-2-coloring K_n, there is a monochromatic K_k.

* $R(k,l)$

红色 K_k or 蓝色 K_l

$\binom{n}{k} \geq 2^{\binom{k}{2}-1}$

lower bound

Thm (Erdős 1947) If $\boxed{\binom{n}{k}\cdot 2^{1-\binom{k}{2}}} < 1$

$\exists$ 2-coloring of edges of K_n: no monochromatic K_k subgraph.

Proof: 等概率 (uniform)

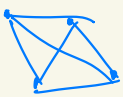
构造概率空间

each edge $e \in K_n$: $\begin{cases} \text{red} & \text{w.p. } \frac{1}{2} \\ \text{blue} & \text{w.p. } \frac{1}{2} \end{cases}$

$\downarrow$

probability space.

单个 for any K_k subgraph:



$$\Pr(\text{the } K_k \text{ is monochromatic})$$

$$= \underset{\substack{\uparrow \\ \text{红蓝}}}{2} \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

多个: totally $\binom{n}{k}$ K_k subgraphs

求事件(反事件)的概率

$$\Rightarrow \Pr(\exists \text{ a } K_k \text{ is monochromatic})$$

$$= \bigcup_{K_k} \Pr(\text{the } K_k \text{ is monochromatic})$$

Union Bound

$\Pr(\bigcup_i A_i) \le \sum_i \Pr(A_i)$

$$\le \sum_{K_k} \Pr(\text{the } K_k \text{ is monochromatic})$$

$$= \binom{n}{k}\cdot 2^{1-\binom{k}{2}} < 1$$

$\Rightarrow \Pr(\text{no } K_k \text{ is monochromatic}) > 0$

应用概率法

$\Rightarrow \exists$ a 2-coloring of edges of K_n such that no K_k is monochromatic $\square$

e.g. $k \geq 3$, $n = \lfloor 2^{k/2} \rfloor$

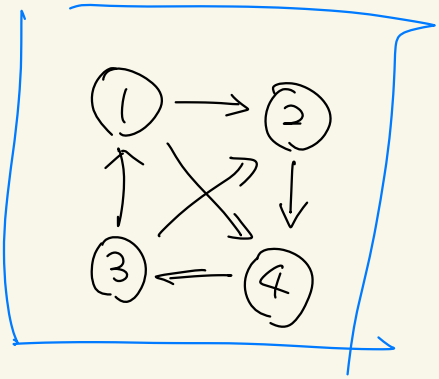
$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{k^2/2}}$$

$$\leq \frac{2^{k^2/2}}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{k^2/2}} = \frac{2^{1+\frac{k}{2}}}{k!} < 1. \quad (k \geq 3)$$

$\Rightarrow$ lower bound: Ramsey number $R(k,k) > \lfloor 2^{k/2} \rfloor$

Example 2. Tournament.

锦标赛（竞赛图）



n players, each pair has a match

$u \to v$ iff u beats v.

k-paradoxical: for any k-subset S of V:

$\exists$ a player in $V \setminus S$ who beats all players in S.

Does there exist a k-paradoxical tournament for every finite k?

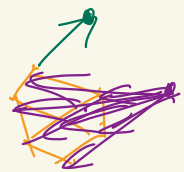
Thm (Erdős, 1963) If $\boxed{\binom{n}{k}\cdot\left(1-\underbrace{2^{-k}}_{(\frac{1}{2})^k}\right)^{n-k} < 1}$? CSP

then there is a k-paradoxical tournament of n players

Proof: random tournament

$\forall (u,v)$: $\begin{cases} \text{w.p. } \frac{1}{2}: & u \to v. \\ \text{w.p. } \frac{1}{2}: & v \to u. \end{cases}$

Fixed any $S \in \binom{[n]}{k}$:



坏事件

$\Pr(A_S) = \Pr(\text{no player in } V\setminus S \text{ who beats all players in } S)$

$= \left(1-\left(\frac{1}{2}\right)^k\right)^{n-k}$

任一个 $V\setminus S$ 而人没有打败 S 而人

至少有一个 A_S 发生

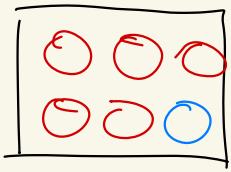
$\Pr\left(\bigcup_S A_S\right)$

$\leq \sum_S \Pr(A_S) = \binom{n}{k}\cdot(1-2^{-k})^{n-k} < 1.$

$\Rightarrow \Pr\left(\bigcap_S A_S^c\right) > 0$. i.e. $\Pr(T \text{ is k-paradoxical}) > 0$ $\square$

Summary:

① Pick random ball from a box:



$\Pr(\text{the ball is blue}) > 0$

$\Rightarrow$ there is a blue ball in the box.

counting
Lebesgue
⋮

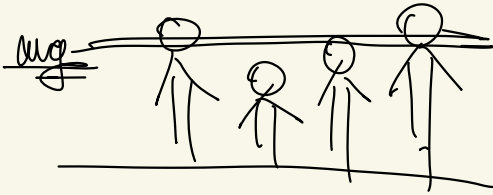
Define a probability space $\Omega = (X, \mathcal{F}, \mathbb{P})$ and a property P.

$\Pr_x(P(x)) > 0$

$\Rightarrow \exists$ a sample $x \in \Omega$ with property P.

② average height of the students in class is ℓ.

$\Rightarrow$ there is a student of height $\geq \ell$ ($\leq \ell$)



for a random variable X.

$\Pr(X \leq \mathbb{E}[X]) > 0$

$\Pr(X \geq \mathbb{E}[X]) > 0$

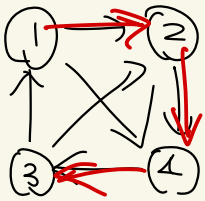
averaging principle

pigeon hole

通常:
本可造 indicator r.v.
再用 linearity

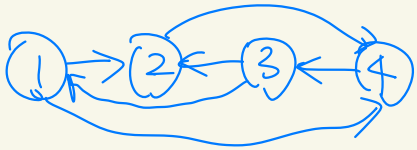
Example 3. Hamiltonian Path: 哈密顿路径.

a path visiting every vertex exactly once



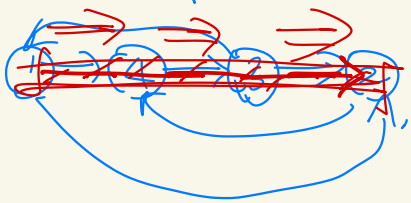
Thm (Szele, 1943) There is a tournament on n players with at least $\boxed{n!}\cdot 2^{-(n-1)}$ Hamiltonian paths.

Proof: uniformly random tournament on $[n]$



linearization

topological sort



For any permutation π on $[n]$

r.v. indicator

$$X_\pi = \begin{cases} 1, & \text{if } \boxed{\pi \text{ is HP}} \\ 0, & \text{if } \pi \text{ is NOT HP} \end{cases}$$

$$\begin{cases} X_{(1,2,3,4)} = 0 \\ X_{(1,4,3,2)} = 1 \end{cases}$$

# of HPs. $X = \sum_\pi X_\pi$ counting

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_\pi X_\pi\right] = \sum_\pi \mathbb{E}[X_\pi]$$

linearity of expectation

$$\mathbb{E}[X_\pi] = \Pr(X_\pi = 1)$$
$$= \left(\frac{1}{2}\right)^{n-1} = 2^{-(n-1)}.$$

$$\Rightarrow \mathbb{E}[X] = n!\cdot 2^{-(n-1)}$$

By Averaging Principle: $\exists$ tournament s.t. $\geq n!\cdot 2^{-(n-1)}$ HPs. $\square$

So far: 古典概型 < uniform prob. space / counting measure

Example 4. Large Independent Set

Graph $G=(V,E)$

独立集 independent set $S \subseteq V$: $\forall u,v \in S.\ (u,v) \notin E$.

$\boxed{\text{max IS} \to \text{NP hard}}$ → approximation algorithm α

max IS in tree → DP

Thm G has n vertices, m edges

$\exists$ an independent set of S of size $\frac{n^2}{4m}$. lower bound NOT optimal

Proof: Draw a random independent set $S \subseteq V$:

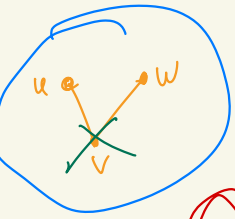
1) (先加点) $\forall v \in V$: add it to R independently with probability $p \in (0,1)$ ← TBD

$\mathbb{E}[|R|] = np$ (binomial distribution)

2) (再减点) $\forall (u,v) \in E$:

delete one of u, v from R if $u, v \in R$

$\Rightarrow$ independent set S.



删去的边数 Y # of edges in R

$\mathbb{V}$

删去的点数

$(|R| - |S|)$

删前 删后

$Y = \sum_{(u,v)\in E} Y_{uv}$. $\boxed{Y_{u,v}} = \begin{cases} 1, & \text{if } u,v \in S \\ 0, & \text{if o.w.} \end{cases}$

$$\mathbb{E}[Y] \overset{\text{linearity}}{=} \sum_{(u,v)\in E} \mathbb{E}[Y_{u,v}]$$

$$= \sum_{(u,v)\in E} \Pr(u,v \in S) = m \cdot p^2$$

$Y \geq |R| - |S|$

$\Rightarrow |S| \geq |R| - Y$

$\Rightarrow \mathbb{E}[|S|] \geq \mathbb{E}[|R|] - \mathbb{E}[Y] = np - mp^2$ $\forall p \in (0,1)$

关于p的二次函数

推论 当 $p = \frac{n}{2m}$ 时, $\mathbb{E}[|S|] \geq \frac{n^2}{4m}$

$\Rightarrow$ By Averaging Principle. .... $\square$

average degree $d = \frac{2m}{n}$.

Refined Version: $\exists$ independent set S of size $\boxed{\frac{n^2}{2m+n}}$

optimal

Turan's Theorem

Luby Algorithm 计算极大独立集:

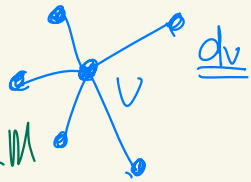
- Draw a random IS $S \subseteq V$.
  - each $v \in V$: $r_v \sim [0,1]$ u.a.r. independently
  - each $v \in V$ joins S iff r_v is local maximal within the neighborhood of v.

$\Downarrow$

S

$$\forall v \in V: \Pr(\underbrace{v \in S}_{X_v}) = \frac{1}{d_v + 1}$$

$\sum_v d_v = 2m$



$$\Rightarrow \mathbb{E}[|S|] = \sum_{v \in V} \mathbb{E}[X_v]$$

$$= \sum_{v \in V} \Pr(v \in S) = \sum_{v \in V} \frac{1}{d_v + 1}.$$

Cauchy-Schwarz: $\langle \vec{x}, \vec{y} \rangle^2 \le \|\vec{x}\|_2^2 \cdot \|\vec{y}\|_2^2$

$$n^2 = \left\langle \left(\frac{1}{\sqrt{d_v+1}}\right)_{v\in V}, \left(\sqrt{d_v+1}\right)_{v\in V} \right\rangle^2$$

$$\le \left(\sum_v \frac{1}{d_v+1}\right) \cdot \left(\sum_v (d_v+1)\right)$$

$$= \mathbb{E}[|S|] \cdot (2m+n)$$

$$\Rightarrow \mathbb{E}[|S|] \ge \frac{n^2}{n+2m} \quad \square$$

Example 5. Coloring large-girth graph.

$G = (V, E)$

围长 girth $g(G)$: length of the shortest cycle

染色数 chromatic number $\chi(G)$: minimum # of colors to properly color the vertices of G.

coloring q 真染色 ($\forall (u,v) \in E$, $q(u) \ne q(v)$)

CSP (Constraint-satisfaction Problem)
约束满足问题

Intuition: large cycles (girth) are easy to color.

Thm (Erdős 1959) for all k, l.

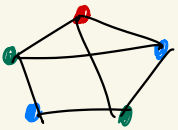
$\exists$ a finite graph G s.t. $\begin{cases} \chi(G) \geq k \\ g(G) \geq l \end{cases}$

染色 coloring $\rightarrow$ equivalence classes

$\Downarrow$

proper coloring $\rightarrow$ independent set

independence number $\alpha(G)$:



size of the largest independent set in G.

$\chi(G)$: optimal coloring

n vertices $\rightarrow$ $\chi(G)$ ISs

by pigeonhole, $\exists$ a color class (i.e. IS)

of size $\geq \frac{n}{\chi(G)}$ $\Rightarrow$ $\alpha(G) \geq \frac{n}{\chi(G)}$

$\geq k$.

最大的色集,

It suffices to show:

$\alpha(G) \leq \frac{n}{k}$ $\quad$ $g(G) \geq l$.

随机图: Random Graph < Paul Erdős, Alfréd Rényi

$G=(V,E)$

$|V|=n$. $\forall \{u,v\} \in \binom{V}{2}$ $\boxed{G(n,p)}$

☆ independently $\Pr((u,v)\in E)=p$

每条边通过抛硬币来决定是否加入

Plan: fix k, l.

→ random graph $G \sim G(n,p)$ | uniform random graph: $G(n,1/2)$

TBD

$$\Pr\left(\alpha(G) \le \frac{n}{k} \cap g(G) \ge l\right) > 0$$

$$\Longleftrightarrow \Pr\left(\alpha(G) > \frac{n}{k} \cup g(G) < l\right) < 1$$

$\Longleftrightarrow$ Union Bound:

① $\boxed{\Pr\left(\alpha(G) > \frac{n}{k}\right) < \frac{1}{2}}$

$\Pr(g(G) < l) < \frac{1}{2}$

① Independence Number

$$\Pr\left(\alpha(G) > \frac{n}{k}\right) \le \Pr\left(\exists \text{ IS of size } \frac{n}{k}\right)$$

IS定义

$$= \Pr\left(\exists S \in \binom{[n]}{n/k}, \forall \{u,v\} \in \binom{S}{2}, (u,v) \notin G\right)$$

Union Bound

$$\le \sum_{S \in \binom{[n]}{n/k}} \Pr\left(\forall \{u,v\} \in \binom{S}{2},\ \overset{\text{边}}{(u,v) \notin G}\right)$$

independence

$$= \sum_{S \in \binom{[n]}{n/k}} \prod_{\{u,v\} \in \binom{S}{2}} \Pr\left((u,v) \notin G\right)$$

$1-p$

$$= \binom{n}{n/k} \cdot (1-p)^{\binom{n/k}{2}}$$

$$\le n^{n/k} \cdot (1-p)^{\binom{n/k}{2}}$$

② $g(G)$: length of smallest cycle.

3-cycle

↓

l-cycle.

for each i-cycle σ: $u_1 \to u_2 \to \cdots \to u_i \to u_1$

$$\Pr(\sigma \text{ is a cycle of } G) = p^i$$

$$X_\sigma = \mathbb{1}\{\sigma \text{ is a cycle of } G\}$$

short cycle

$\boxed{\#\text{ of length} \le l \text{ cycles in } G:}$

长度 $\le l$ 的 cycle

的数量

希望!点控制的

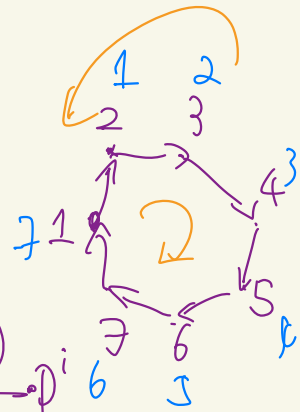
$$X = \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} X_\sigma :$$

$$\Rightarrow \mathbb{E}[X] = \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} \mathbb{E}[X_\sigma]$$

$$= \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} p^i$$

$$= \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2 \cdot i} p^i$$

$$\leq \sum_{i=3}^{\ell} p^i \cdot \frac{n^i}{2i}$$



$$\boxed{\Pr(X \geq \cdots) \leq \cdots}$$

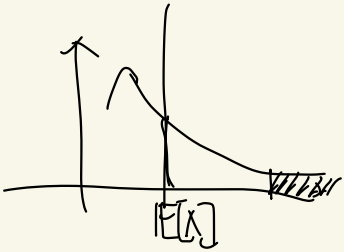
$$p = n^{\theta - 1}$$

$$\theta < \frac{1}{\ell}$$

Markov Inequality

$$\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

$$\Leftrightarrow \Pr(X \geq t \cdot \mathbb{E}[X]) \leq \frac{1}{t}$$



(first-moment method)

Proof: $\mathbb{E}[X] = \sum_x x \cdot \Pr(X=x)$

$$= \sum_{x<t} x \cdot \Pr(X=x) + \sum_{x \geq t} \underset{t}{x} \cdot \Pr(X=x)$$

$$\geq t \cdot \sum_{x \geq t} \Pr(X=x)$$

$$= t \cdot \Pr(X \geq t) \qquad \square$$

$$\Rightarrow \mathbb{E}[X] \leq \sum_{i=3}^{\ell} \frac{(np)^i}{2i} = \sum_{i=3}^{\ell} \frac{n^{\theta i}}{2i} = o(n)$$

$$\Rightarrow \Pr\left(X \geq \frac{n}{2}\right) \leq \frac{\mathbb{E}[X]}{\frac{n}{2}} = o(1)$$

$\Rightarrow$ w.h.p : random graph G has less than $n/2$ cycles

① $\Pr\left(\alpha(G) \geq \frac{n}{k}\right) \leq n^{\frac{n}{k}} \cdot (1-p)^{\binom{n/k}{2}}$

"坏事件"

$$\leq n^{\frac{n}{k}} \cdot e^{-p\binom{n/k}{2}} \rightarrow \frac{\frac{n}{k}\left(\frac{n}{k}-1\right)}{2}$$

$$= \left(n e^{-\frac{p}{2}\left(\frac{n}{k}-1\right)}\right)^{\boxed{\frac{n}{k}}} = o(1)$$

let $\frac{n}{k} = \frac{3\ln n}{p}$

$$\boxed{p = n^{\theta-1},\quad \theta < \frac{1}{2l},\quad k = \frac{np}{3\ln n} = \frac{n^{\frac{1}{2l}}}{3\ln n}}$$

$\boxed{\text{fixed } k, l}$ $\Rightarrow$ choose n, p to satisfy

$$\Rightarrow \begin{cases} \Pr\left(\alpha(G) \geq \frac{n}{k}\right) = o(1) \quad \checkmark \\ \Pr\left(\#\text{ of length} \leq l \text{ cycles} \geq \frac{n}{2}\right) = o(1) \end{cases}$$

被

$\Downarrow$?

$$\Pr\left(\boxed{g(G) \geq l}\right) = o(1)$$

length of shortest cycle

$\exists G: \alpha(G) < \frac{n}{k}$

$$\boxed{\#\text{ of length} \leq l \text{ cycles in } G} < \frac{n}{2}$$

随机事件 $\Longrightarrow$ 确定性事件

删点破圈: delete 1 vertex per each length $\le \ell$ cycle in G $\Rightarrow$ G'

$$\begin{cases} g(G') > \ell \\ \alpha(G') \le \alpha(G) < \frac{n}{k}. \end{cases} \quad \square$$

2021 Abel

Lovász Local Lemma (LLL)

Recall: Ramsey Number $R(k,k)$

K_n monochromatic K_k.

$\Updownarrow$

$\exists$ a 2-coloring of K_n with no monochromatic K_k

概率法: a random 2-coloring of K_n

$\forall S \in \binom{[n]}{k}$

"坏事件" event A_S: S is a monochromatic K_k.

To prove: $\Pr\left(\bigcap_{S \in \binom{[n]}{k}} \overline{A_S}\right) > 0$

$\exists$ $g(G) \ge \ell$

$\chi(G) \ge k$

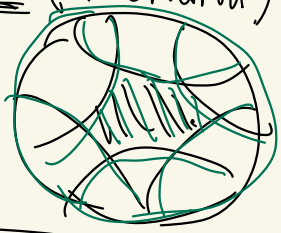
$\Downarrow$

"坏事件"

$\Pr(g(G) < \ell) = o(1)$

$\Pr(\chi(G) < k) = o(1)$

"筛法" (Sieve Method)



As dependency

坏事件一个都不发生

$$1-\Pr\left(\bigcup_{S\in\binom{[n]}{k}} A_S\right) \le 1-\sum_{S\in\binom{[n]}{k}}\Pr(A_S)$$

坏事件: $A_1, A_2, \cdots, A_n$

考虑 $\Pr\left(\bigcap_{i=1}^{n}\overline{A_i}\right)>0$ being good is possible.

概率法.

① $\overline{A_i}$ 独立: $\boxed{\Pr\left(\bigcap_{i=1}^{n}\overline{A_i}\right)} = \prod_{i=1}^{n}\Pr(\overline{A_i})$

② $\overline{A_i}$ 全相关: $\Pr\left(\bigcap_{i=1}^{n}\overline{A_i}\right) = 1-\Pr\left(\bigcup_{i=1}^{n}A_i\right)$

Union Bound

$\le 1-\boxed{\sum_{i=1}^{n}\Pr(A_i)}$

Intermediate: $\overline{A_i}$ 部分相关

What if $\boxed{\sum_{i=1}^{n}\Pr(A_i)} \ge 1$.

events: $A_1, A_2, \cdots, A_n$

dependency graph. $D=(V,E)$

相关图.

$V=\{1,2,\cdots,n\}$

$(i,j)\in E \iff$ A_i 和 A_j "相关"

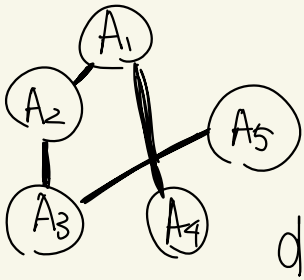
d: max degree of $D(V,E)$

e.g. mutually independent

X_1, X_2, X_3, X_4

$A_1(X_1, X_4)$, $A_2(X_1, X_2)$,

$A_3(X_2, X_3)$, $A_4(X_4)$, $A_5(X_3)$



More formally.

A_i is mutually independant of A_j for all j such that $(i,j) \notin E$.

$$\Pr(A_i) = \Pr(A_i \mid A_j)$$

$d = 2$ → 刻画了"最坏情况下"事件的依赖关系

"局部引理"

Lovász Local Lemma (Symmetric LLL)

Thm

$$\forall i: \Pr(A_i) \le p$$

$$e \cdot p(d+1) \le 1$$

global

$$\Rightarrow \Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) > 0$$

trivial

General Lovász Local Lemma

Thm

$$\exists\, x_1, \cdots, x_n \in [0,1)$$

$$\forall i \in [n]: \Pr(A_i) \le x_i \cdot \prod_{j \sim i} (1 - x_j)$$

$$\Rightarrow \Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) \ge \prod_{i=1}^{n} (1 - x_i)$$

Set $x_j = \frac{1}{d+1}$.

j is a neighbor of i

只关心有个点的邻部

$$x_i \cdot \prod_{j\sim i}(1-x_j) = \frac{1}{d+1}\left(1-\frac{1}{d+1}\right)^{d_i}$$

max-degree

$$\geq \frac{1}{d+1}\cdot\left(1-\frac{1}{d+1}\right)^{d+1} \geq \frac{1}{(d+1)e}.$$

$$\Pr(A_i) \leq p \leq \frac{1}{e(d+1)} \leq x_i \cdot \prod_{j\sim i}(1-x_j)$$

已知: $\exists\, x_1, \cdots, x_n \in [0,1)$, s.t. $\Pr(A_i) \leq x_i \cdot \prod_{j\sim i}(1-x_j)$

证明: $\Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) \geq \boxed{\prod_{i=1}^{n}(1-x_i)}$

Proof: $\Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right)$

$$= \prod_{i=1}^{n} \Pr\left(\bar{A}_i \,\middle|\, \bigcap_{j=1}^{i-1} \bar{A}_j\right)$$

$$= \prod_{i=1}^{n}\left(1-\Pr\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} \bar{A}_j\right)\right)$$

只需证: $\Pr\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} \bar{A}_j\right) \leq x_i$

Chain Rule

events $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n$

$$\Pr\left(\bigcap_{i=1}^{n} \mathcal{E}_i\right) = \prod_{k=1}^{n} \Pr\left(\mathcal{E}_k \,\middle|\, \bigcap_{i<k} \mathcal{E}_i\right)$$

$$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$$

$$\cancel{\Pr(\mathcal{E}_1)} \cdot \frac{\cancel{\Pr(\mathcal{E}_1,\mathcal{E}_2)}}{\cancel{\Pr(\mathcal{E}_1)}} \cdot \frac{\cancel{\Pr(\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3)}}{\cancel{\Pr(\mathcal{E}_1,\mathcal{E}_2)}} \cdots \frac{\Pr(\mathcal{E}_{1:k})}{\cancel{\Pr(\mathcal{E}_{1:k-1})}}$$

$$= \Pr\left(\bigcap_{i=1}^{n} \mathcal{E}_i\right)$$

telescoping

更general: $\boxed{\Pr\left(\boxed{A_{i_1}} \,\middle|\, \bar{A}_{i_2}, \cdots, \bar{A}_{i_m}\right) \leq x_{i_1}}$ for any $\{i_1, \cdots, i_m\}$

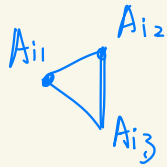
I.H.

Induction on m: $m=1$: $\Pr(A_{i_1}) \leq x_{i_1} \cdot \prod_{j\sim i_1}(1-x_j) \leq x_{i_1}$

$$\{\underset{A}{i_1},\ \underbrace{i_2,\cdots,i_k}_{\text{和 } i_1 \text{ 相邻}},\ \underbrace{i_{k+1},\cdots,i_m}_{\text{和 } i_1 \text{ 不相邻}}\}$$

A $\quad$ B $\quad$ C

无关!



$$\Pr(A_{i_1} \mid \bar{A}_{i_2},\cdots,\bar{A}_{i_m}) = \Pr(A \mid B, C)$$

$$= \frac{\Pr(A, B \mid C)}{\Pr(B \mid C)} = \frac{\Pr(A_{i_1}, \bar{A}_{i_2},\cdots,\bar{A}_{i_k} \mid \bar{A}_{i_{k+1}},\cdots,\bar{A}_{i_m})}{\Pr(\bar{A}_{i_2},\cdots,\bar{A}_{i_k} \mid \bar{A}_{i_{k+1}},\cdots,\bar{A}_{i_m})}$$

$$\le \Pr(\underset{A}{A_{i_1}} \mid \underset{C}{\bar{A}_{i_{k+1}},\cdots,\bar{A}_{i_m}}) = \Pr(A_{i_1}) \le x_{i_1} \underbrace{\prod_{j=2}^{k}(1-x_{i_j})}_{j \sim i_1}$$

链式法则

$$= \prod_{j=2}^{k} \Pr(\bar{A}_{i_j} \mid \bar{A}_{i_{j+1}},\cdots,\bar{A}_{i_k}, \bar{A}_{i_{k+1}},\cdots,\bar{A}_{i_m})$$

$$= \prod_{j=2}^{k}\Big(1 - \underbrace{\Pr(A_{i_j} \mid \bar{A}_{i_{j+1}},\cdots,\bar{A}_{i_m})}_{\text{I.H.}}\Big) \ge \prod_{j=2}^{k}(1-x_{i_j})$$

$$\Rightarrow \Pr(A_{i_1} \mid \bar{A}_{i_2},\cdots,\bar{A}_{i_m}) \le x_{i_1} \text{ for any } \{i_1,\cdots,i_m\}$$

$$\Rightarrow \Pr(A_i \mid \bar{A}_1,\cdots,\bar{A}_{i-1}) \le x_i$$

$$\Rightarrow \Pr\Big(\bigcap_{i=1}^{n}\bar{A}_i\Big) = \prod_{i=1}^{n}\Big(1 - \underbrace{\Pr(A_i \mid \bar{A}_1,\cdots,\bar{A}_{i-1})}_{\le x_i}\Big) \ge \prod_{i=1}^{n}(1-x_i) \quad \square$$

oracle

Problem $\rightarrow$ LLL $\left\langle \begin{array}{l} \Pr(A_i) \le p \\ ep(d+1) \le 1 \end{array} \right.$ $\rightarrow$ ✓

oracle : subroutine

CSP (Constraints Satisfaction Problem)

Variables : $x_1, \cdots, x_n \in [q]$

(local) constraint : $C_1, \cdots, C_m$.

$C_i : [q]^{Vbl(C_i)} \rightarrow \{\text{True, False}\}$

Any $\vec{x} \in [q]^n$ is a CSP solution

if it satisfies all $C_1, \cdots, C_m$

K-CNF. coloring. set covering. matching. independent set

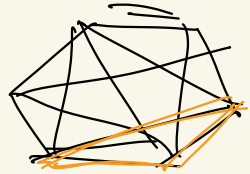
e.g.1. Ramsey Number

K_n : random. uniform 2-coloring

$\forall (u,v) \in E$ $\left\langle \begin{array}{l} \text{w.p. } \frac{1}{2}; \\ \text{w.p. } \frac{1}{2}; \end{array} \right.$

$\forall S \in \binom{[n]}{k}$ event $\boxed{A_S}$ S is monochromatic K_k.

$D(V,E)$: $|V| = \binom{n}{k}$



$$\Pr(A_S) = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent $\Longleftrightarrow$ $|S \cap T| \geq 2$

k个点

(S和T共用一条边)

max degree: $d \leq \binom{k}{2}\binom{n}{k-2}$

S.T共用两点时

$\Rightarrow ep(d+1)$

$$\leq e \cdot 2^{1-\binom{k}{2}} \cdot \left(\binom{k}{2}\binom{n}{k-2}+1\right) \leq 1.$$

for some $n = C \cdot k \cdot 2^{k/2}$ with some constant C.

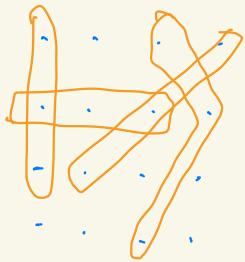
$\Rightarrow$ lower bound: $R(k,k) \geq n = \Omega(k \cdot 2^{\frac{k}{2}})$

e.g.2. Hypergraph Coloring 超图染色

k-uniform hypergraph $H=(V,E)$

V: vertex set.

$E \subseteq \binom{V}{k}$: set of hyperedge



degree of vertex $v \in V$: # of hyperedges $e \ni v$

3-uniform H.

Proper q-coloring of H:

$f: V \to [q]$ such that no hyperedge is monochromatic.

$$\forall e \in E: \quad |f(e)| > 1.$$

Thm For any k-uniform hypergraph H of max-degree Δ:

$$\Delta \le \frac{q^{k-1}}{ek} \Rightarrow H \text{ is } q\text{-colorable}$$

Proof: uniformly independent color each $v \in [V]$

a random color $q_v \in [q]$.

Bad event A_e: e is monochromatic.

$$\Pr(A_e) \le p = q \cdot \left(\frac{1}{q}\right)^k = q^{1-k}.$$

dependency graph $D(V, E)$

dependency degree d: 每条单色标边最多能与多少条单色标边相关?

$$d \le k \cdot (\Delta - 1)$$

$$\Delta \le \frac{q^{k-1}}{ek} \Rightarrow ep(d+1) \le e \cdot q^{1-k} \cdot k \cdot \frac{q^{k-1}}{ek} = 1 \text{ by LLL } \square.$$