1. Basic Enumeration

The Twelvefold Way

$f: N \to M$. $|N| = n$ $|M| = m$

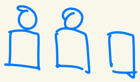
$\#f$

elements of N	elements of M	any f	injection 单射	surjection 满射
distinct (distinguishable)	distinct	m^n		
identical	distinct			
distinct	identical			
identical	identical			

Balls-into-Bins

n balls are put into m bins

balls / bin	unrestricted	≤ 1 (injection)	≥ 1
n distinct balls, m distinct bins.	m^n.	$\dfrac{m!}{(m-n)!}$	
n identical balls (unordered), m distinct bins	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins		$\mathbb{1}\{m \geq n\}$	
n identical balls, m identical bins		$\mathbb{1}\{m \geq n\}$	



↑ 鴿巢原理 (pigeon hole)

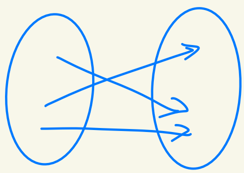
★ **sum rule**: $|S \cup T| = |S| + |T|$. for disjoint set S, T. $S \cap T = \emptyset$

product rule: $|S \times T| = |S| \cdot |T|$. for any set S, T Catersian Project.

bijection rule: $|S| = |T|$. if $\exists$ bijection $f: S \to T$

Tuple: $[m] = \{1, 2, \cdots, m\}$ 投一个球 $|[m]| = m$.

$[m]^n = [m] \times [m] \times \cdots \times [m]$. 投n个distinct球: $|[m]^n| \underset{\text{product rule}}{=} |[m]|^n = m^n$.

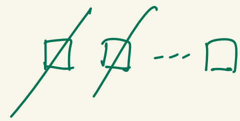


$[n]$ $[m]$

$\boxed{\# f: [n] \to [m].}$ 1-1 correspondence.

$(f(1), f(2), \cdots, f(n)) \in [m]^n$.

$|[n] \to [m]| = |[m]^n| = m^n$.



Injection: $\pi = (f(1), f(2), \cdots, f(n))$

$\pi \in [m]^n$ of distinct elements (没有两个球投到同一个框中)

$(m)_n = m(m-1) \cdots\cdots (m-n+1) = \frac{m!}{(m-n)!}$ n-permutation

A_n^m. P_n^m.

Subsets

$[n] = \{1, 2, \cdots, n\}$

power set 幂集 $2^{[n]} = \{S \mid S \subseteq [n]\}$ $|2^{[n]}| = 2^n$

① $\{1, 2, \cdots, n\}$

$S \subseteq [n]$

$\Updownarrow$

a string of n bits $\mathbb{1}\{i \in S\} = \chi_S$

$\Rightarrow |2^{[n]}| = |[2]^n| = |[2]|^n = 2^n$.

② $f(n) = |2^{[n]}|$,

$\{1, 2, \cdots, n-1, n\}$

递归 (recursion)

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \cup \{S \subseteq [n] \mid n \in S\}$$

disjoint

$$|2^{[n]}| = |2^{[n-1]}| + |2^{[n-1]}|$$

$$\Rightarrow f(n) = f(n-1) + f(n-1) = 2f(n-1).$$

$$f(0) = |2^{\phi}| = 1.$$

$$\Rightarrow f(n) = 2^n$$

$\downarrow$

Subset of fixed size

$$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$$

$\{1, 2, 3\}$

2-subset:

$\{1,2\}, \{1,3\}, \{2,3\}$

$$\Rightarrow n \text{ choose } k: \binom{n}{k} = \left|\binom{[n]}{k}\right|$$

如何求解?

① # of ordered k-subsets of $[n]$, (n元集中有序取出k个数)

$$n(n-1)\cdots(n-k+1)$$

有序

② # of permutations of a k-set:

无序

$$k(k-1)\cdots 1.$$

$$\Rightarrow \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}$$

binomial coefficient!

$\{1, \cdots, n-1, n\}$

Properties:

$$\begin{cases} \binom{n}{k} = \binom{n}{n-k} \\ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \\ 2^n = \sum_{k=0}^{n} \binom{n}{k} \end{cases}$$

二项式定理 Binomial Theorem: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ ☆

generating function

$(1+x)^n = \underbrace{(1+x)\cdots\cdots(1+x)}_{n个}$

\# of x^k: choose k factors out of n.

\# of n-ordered subset of m: $\frac{m!}{(m-n)!}$

\# of n-subset of m: $\binom{m}{n} = \frac{m!}{(m-n)!\,n!}$

Composition of an Integer

n块钱分给k个人

balls　　bins

① k个人 每人至少有一块钱 (满射)

② k个人 有人可以空手而归 (unrestricted)

① k-composition of n.

a k-tuple $(x_1, \cdots, x_k)$ s.t. $x_1+\cdots+x_k = n$. $x_i \in \mathbb{Z}^+$.

○ ○ | ○ | ○ ○ ○ | ○　　n identical balls

插板法　x_1　x_2　- - -　x_k

\# of k-compositions of n: $\binom{n-1}{k-1}$

$\phi((x_1, x_2, \cdots, x_k)) = \{x_1, x_1+x_2, x_1+x_2+x_3, \cdots, x_1+x_2+\cdots+x_{k-1}\}$

x_1　x_2　x_3

前缀和

$\{\text{k-composition of } n\} \xrightarrow[\phi]{\text{1-1 correspondence}} \binom{\{1,2,\cdots,n-1\}}{k-1}$

$x_i \in \mathbb{Z}^+$

$x_k \geq 1$

② weak k-composition of n:

a k-tuple $(x_1, \cdots, x_k)$ s.t. $x_1 + x_2 + \cdots + x_k = n$, $x_i \in \mathbb{N}$

$x_i + 1 \in \mathbb{Z}^+$

$\Downarrow$

$$(x_1+1) + (x_2+1) + \cdots + (x_k+1) = n+k.$$

\# of weak k-composition of n

$= \#$ of k-composition of $n+k$ $= \binom{n+k-1}{k-1}$

1-1 correspondence

Multisets 多元集

k-subset of S: "k-combination of S without repetition"

$\{1, 2, 3\}$,

2-subset
- w/o repetition: $\{1,2\}, \{1,3\}, \{2,3\}$
- w repetition: $\{1,1\}, \{2,2\}, \{3,3\}$, $\{1,2\}, \{1,3\}, \{2,3\}$

multiset M of S

$m: S \to \mathbb{N}$ multiplicity (重数) of $x \in S$.

$m(x)$: # of repetitions of x in M

$$k \triangleq |M| = \sum_{x \in S} m(x).$$

- set: bit vector $(0, 1, 1, 0, \cdots)$
- multiset: int vector $(0, 2, 4, 1, \cdots)$

k-multiset of S

$\overset{1\text{-}1}{\Longleftrightarrow}$ k-combination of S with repetition

$\left(\!\binom{n}{k}\!\right)$: # of k multisets on an n-set

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

weak k-composition

k-multiset on $S=\{x_1,\cdots,x_n\}$

$$m(x_1)+m(x_2)+\dots+m(x_n)=k. \quad m(x_i)\geq 0$$

Binomial → Multinomial

二项 多项

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{n}{m_1,\cdots,m_k} = \frac{n!}{m_1!\,m_2!\cdots m_k!}$$

: # of permutations of a multiset on k elements of size n with multiplicities $m_1,\cdots,m_k$. $(m_1+\dots+m_k=n)$

⇕

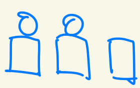
# of assignment of n distinct balls to k distinct bins with the i-th bin receiving m_i balls

$$\underbrace{(x_1+x_2+\dots+x_k)\cdots(x_1+x_2+\dots+x_k)}_{n\text{ 项}}$$

Balls-into-Bins

n balls are put into m bins

balls / bin	unrestricted	≤ 1 (injection)	≥ 1 (surjection)
n distinct balls, m distinct bins	m^n	$\frac{m!}{(m-n)!}$	$m!\left\{ {n \atop m} \right\}$
n identical balls, m distinct bins (unordered)	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m}\left\{ {n \atop k} \right\}$	$1\{m\geq n\}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins	$\sum_{k=1}^{m}p_k(n)$	$1\{m\geq n\}$	$p_m(n)$



↑ 鸽巢原理 (pigeon hole)

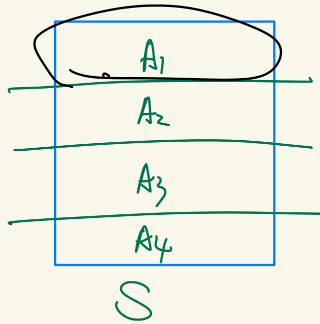
n块钱分给k个人 : composition

n个人上k条船 : partition of a set > △

Partition of a Set

$P = \{A_1, \cdots, A_k\}$ is a partition of S:

- $A_i \neq \phi$ 非空
- $A_i \cap A_j = \phi$ (disjoint)
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$



$\genfrac\{\}{0pt}{}{n}{k}$: # of k-partition of an n-set.

第二型Stirling数 → 无闭式解

total # of partitions. $\sum_{k=1}^{n} \genfrac\{\}{0pt}{}{n}{k} \triangleq B_n$ Bell number

无序 n个人上m条船: unrestricted $\sum_{k=1}^{m} \genfrac\{\}{0pt}{}{n}{k}$

$\{1, 2, \cdots, n-1, n\}$

Recall: $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$

$\genfrac\{\}{0pt}{}{n}{k}$:
- $\{n\}$ is not a partition block. $\genfrac\{\}{0pt}{}{n-1}{k} \cdot k$
- $\{n\}$ is a partition block $\genfrac\{\}{0pt}{}{n-1}{k-1}$

$$\Rightarrow \boxed{\genfrac\{\}{0pt}{}{n}{k} = \genfrac\{\}{0pt}{}{n-1}{k-1} + k \cdot \genfrac\{\}{0pt}{}{n-1}{k}}$$

有序 (surjection) $f: [n] \xrightarrow{\text{on-to}} [m]$. i.e. $\forall i \in [m]$. $f^{-1}(\{i\}) \neq \phi$

$$\left(f^{-1}(\{1\}), f^{-1}(\{2\}), \cdots, f^{-1}(\{m\})\right)$$

ordered m-partition of n.

① partition: $\genfrac\{\}{0pt}{}{n}{m}$

② permutation: $m!$

$\Rightarrow m! \genfrac\{\}{0pt}{}{n}{m}$: # of surjections $[n] \to [m]$

Partition of a Number

balls: identical
bins: identical

\# of partition of n into k parts. $p_k(n)$ — positive, unordered

$n=7$.

$\{7\}$.	$k=1$
$\{1,6\}$. $\{2,5\}$. $\{3,4\}$.	2
$\{1,1,5\}$. $\{1,2,4\}$. $\{1,3,3\}$. $\{2,2,3\}$	3
$\{1,1,1,4\}$. $\{1,1,2,3\}$. $\{1,2,2,2\}$	4
$\{1,1,1,1,3\}$. $\{1,1,1,2,2\}$.	5
$\{1,1,1,1,1,2\}$	6
$\{1,1,1,1,1,1,1\}$	7.

$p_k(n)$

Integral solutions to $\begin{cases} x_1+\cdots+x_k=n. \\ \boxed{x_1\geq x_2\geq\cdots\geq x_k\geq 1.} \end{cases}$ → wlog

Case 1: $x_k=1$ $\Rightarrow$ $(x_1,\dots,x_{k-1})$ 是 $(n-1)$ 的一个 $(k-1)$-partition.

Case 2: $x_k>1$. → $(x_1-1,\dots,x_k-1)$ 是 $(n-k)$ 的一个 k-partition.

$$\Rightarrow p_k(n)=p_{k-1}(n-1)+p_k(n-k)$$

- Composition ☆ $\binom{n}{k}$ △ → $\begin{cases} x_1+\cdots+x_k=n. \\ x_i\geq 1. \end{cases}$ (ordered)
- partition of set $[n]$. $\left\{ {n \atop k} \right\}$
- partition of number n. $p_k(n)$ → $\begin{cases} x_1+\cdots+x_k=n \\ x_1\geq x_2\geq\cdots\geq x_k\geq 1. \end{cases}$ (unordered)

① partition $\{x_1,\dots,x_k\}$ $\xrightarrow[\text{onto}]{\pi}$ composition. $(x_1,\dots,x_k)$

π是满射 (permutation)

$$k!\cdot p_k(n) \ge \binom{n-1}{k-1}$$

$\boxed{\text{某个 partition 是某些 composition 的代表}}$ ☆☆

permutation

② $x_1 \ge x_2 \ge \cdots \ge x_{k-2} \ge x_{k-1} \ge x_k \ge 1.$

$+k-1 \quad +k-2 \quad \cdots \quad +2 \quad +1.$

$y_i = x_i + k - i$

$$y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k \ge 1$$

partition $\{x_1,\cdots,x_k\} \xrightarrow{\ \pi\ }$ composition $(y_1,\cdots,y_k)$

$$y_1+\cdots+y_k = n+\frac{k(k-1)}{2}.$$

π是单射 (permutation)

$$k!\cdot p_k(n) \le \binom{n+\frac{(k-1)k}{2}-1}{k-1}$$

$$\Rightarrow \frac{\binom{n-1}{k-1}}{k!} \le p_k(n) \le \frac{\binom{n+\frac{k(k-1)}{2}-1}{k-1}}{k!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Stirling Formula

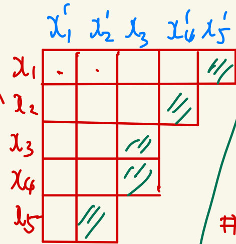
If k is fixed: $p_k(n) \sim \dfrac{n^{k-1}}{k!(k-1)!}$ as $n\to\infty$ (asymptotic)

Ramanujan: $p(n) = \sum_{k=1}^{n} p_k(n) \approx \dfrac{1}{4n\sqrt{3}}\cdot\exp\left(\pi\sqrt{\dfrac{2n}{3}}\right)$

对数 n 的所有 partition 的数量

$$\begin{cases} x_1+\cdots+x_k = n \\ x_1 \ge x_2 \ge \cdots \ge x_k \ge 1. \end{cases}$$

Ferrers Diagram



$$p_k(n) = \sum_{j=1}^{k} p_j(n-k)$$

Conjugate

# of partitions into k parts

(5, 4, 3, 3, 2)

⇕

(5, 5, 4, 2, 1)

# of partitions of n with largest part k.