6. Existence Problems

存在性

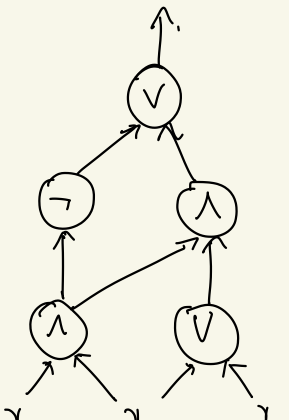
有 or 没有

Counting Argument

1. Circuit Complexity

\# of f: $\left|\{0,1\}^n \to \{0,1\}\right| = \underbrace{2\cdot 2 \cdots 2}_{2^n} = 2^{2^n}$ doubly exponential

Boolean circuit. $f: \{0,1\}^n \to \{0,1\}$ discriminative



DAG (directed acyclic graph)

↳ Nodes
- inputs: $x_1, \cdots, x_n$
- gates: $\wedge, \vee, \neg$

Complexity: # of gates

- P: polynomial.
- NP: non-deterministic polynomial

⇓

是否存在难题.

Thm (Shannon, 1949)

$\exists$ boolean function $f: \{0,1\}^n \to \{0,1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.

⇓

Almost all

☆ 计算确定性: One circuits computes one function

Given t gates: wlog $\wedge$ or $\vee$

$\neg(A \vee B) = (\neg A) \wedge (\neg B)$

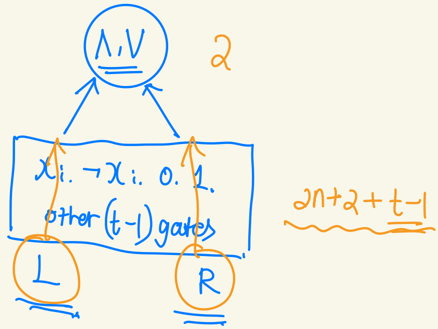
$\neg(A \wedge B) = (\neg A) \vee (\neg B)$

Inputs: $x_1, \cdots, x_n, \neg x_1, \cdots, \neg x_n, 0, 1$ ↪ $(2n+2)$个数

\# of circuits with t gates:

$$\le \Big(\underbrace{2 \cdot (2n+2+t-1)^2}\Big)^t$$

$$= 2^t \cdot (2n+t+1)^{2t}$$



$\Rightarrow$ \# of funtions by t gates

$\le$ \# of circuits with t gates $\le 2^t \cdot (2n+t+1)^{2t} \ll 2^{2^n} = $ \#f

for $t \le \frac{2^n}{3n}$

$\Rightarrow$ 算法的数量 $\ll$ 问题的数量

$\Rightarrow$ 能高效解决的问题仅仅是沧海一粟.

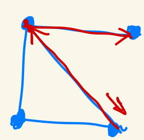
2. Handshaking Lemma: the number of people who shake an odd number of other people's hands is even.

$$\boxed{\sum_{v \in V} d(v) = 2 \cdot |E|}$$

n people $\Leftrightarrow$ n vertices

handshaking $\Leftrightarrow$ edge

\# of handshaking $\Leftrightarrow$ degree



double counting

↳ Count the \# of edge orientations

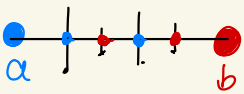
- Count by vertex: $\forall v \in V$. $d(v)$ directed edges
- Count by edge: $\forall e \in E$. (u,v) or (v,u) 2 orientations

$\Downarrow$

\# of odd-degree vertices is even.

Sperner's Lemma

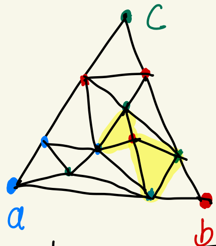
线段 ab → divided into small segments.



each endpoint: red / blue.

⇒ ∃ a small segment •—• (#: odd)

三角形 abc → triangulation



proper coloring: { 3 color • • •

a b c : tricolors.

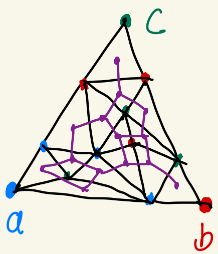
lines ab. bc. ca are 2-colors

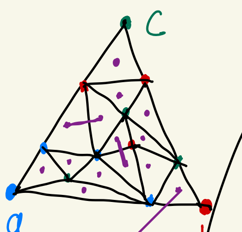
Lemma: ∀ properly colored triangulation of a triangle,

∃ a properly colored small triangle ☆

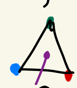
dual graph 对偶图

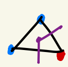
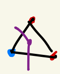
⇒ partial dual graph





add an edge if 2 Δ share a •—• edge

 degree 1. → odd

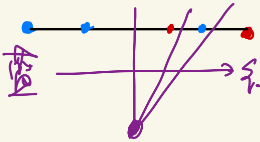
  degree 2

other cases: degree 0. } → even

by handshaking Lemma.

⇒ # of odd-degree vertices is even. < 红蓝绿 / outer vertex (1)

Outer Vertex:
degree is odd!



蓝 → 红 (支氏等着论)

$\Rightarrow$ # $\triangle$: odd !

- high-dimensional < triangle → simplex; triangulation → simplitical subdivision.

Brouwer's fixed point theorem : $\forall$ continuous function $f: B \to B$. of an n-dimensional ball B, $\exists$ fixed point $x = f(x)$

Averaging Principle

- $n+1$ pigeons cannot sit in n holes so that every pigeon is alone on its own.
- If $> mn$ objects are partitioned into n classes, then some class receives $> m$ objects.

(r.v. X: $\Pr(X \geq \mathbb{E}[X]) > 0$
$\Pr(X \leq \mathbb{E}[X]) > 0$)

< drawer principle; pigeonhole principle

① Approximate any irrational x by a rational with bounded denominator.

Thm (Dirichlet, 1879) $\forall$ irrational x. $n \in \mathbb{N}$, $\exists$ rational $\frac{p}{q}$.

s.t. $1 \leq q \leq n$ and $\left|x - \frac{p}{q}\right| < \frac{1}{nq}$.

Proof: "注意到" $\Leftrightarrow$ $|qx - p| < \frac{1}{n}$

$x \begin{cases} \text{integer part: } \lfloor x \rfloor \\ \text{fractional part: } \{x\} = x - \lfloor x \rfloor \end{cases}$

$(0,1) \to \underbrace{(0,\tfrac{1}{n}), (\tfrac{1}{n},\tfrac{2}{n}), \cdots, (\tfrac{n-1}{n},1)}_{n \text{ holes}}$

$\underbrace{\{kx\} \text{ for } k = 1, 2, \cdots, n+1}_{n+1 \text{ pigeons}}$

$\Rightarrow$ By pigeonhole principle, $\exists\ 1 \le b < a \le n+1$ s.t. $\{ax\}, \{bx\}$ in same hole.

$\Rightarrow \left| \{ax\} - \{bx\} \right| < \frac{1}{n}$

$\Rightarrow \left| \underbrace{(a-b)}_{q \in [n]} x - \underbrace{(\lfloor ax \rfloor - \lfloor bx \rfloor)}_{x} \right| < \frac{1}{n} \Rightarrow |qx - p| < \frac{1}{n}$ $\square$

② Paul Erdős

$\boxed{\text{Claim: } \forall S \subseteq \{1, 2, \cdots, 2n\} \text{ that } |S| > n,\ \exists a, b \in S \text{ s.t. } a \mid b.}$

Proof: $\forall a \in \{1, 2, \cdots, 2n\}$, a 二进制展开

$a = 2^k \cdot m$ for an odd m.

$\boxed{1011000}$

$\Rightarrow m$ is the quotient of $a / 2^k$.

$64 + 16 + 8 = 88$

$C_m = \{2^k m \mid k \ge 0,\ 2^k m \le 2n\}$

$2^3 \cdot m \in C_{11}$

n pigeonholes: $C_1, C_3, C_5, \cdots, C_{2n-1}$

$> n$ pigeons: S.

$\Rightarrow \exists\, a<b.\ a.b \in C_m \Rightarrow a \mid b$ □

$2^k m.\ 2^{k'} m$

③ Monotonic Subsequences

Sequence $(a_1, \cdots, a_n)$: n different numbers

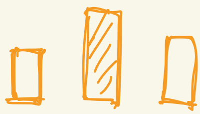
↳ $1 \le i_1 < i_2 < \cdots < i_k \le n \Rightarrow$ Subsequence $(a_{i_1}, a_{i_2}, \cdots, a_{i_k})$

- increasing: $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$
- decreasing: $a_{i_1} > a_{i_2} > \cdots > a_{i_k}$

Thm (Erdős–Szekeres, 1935) A sequence of $>mn$ different numbers must contain

- an ↗ subsequence of length $m+1$
- or
- an ↘ subsequence of length $n+1$

$a_i \neq a_j$



Proof: $(a_1, \cdots, a_N)$. $N > mn$

$a_i \Leftrightarrow (x_i, y_i)$

- x_i: length of longest ↗ subsequence ending at a_i
- y_i: length of longest ↘ subsequence starting at a_i

$\forall i \neq j$ $(x_i, y_i) \neq (x_j, y_j)$ (assume $i<j$)

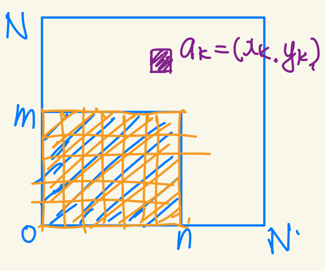
- $a_i < a_j$: $x_i < x_j$
- $a_i > a_j$: $y_i > y_j$

⇓

$m \cdot n$ holes

One pigeon per one holes!

$\Rightarrow$ No way to put N pigeons into mn holes.



▨ (x_k, y_k): $x_k > m$ or $y_k > n$. □