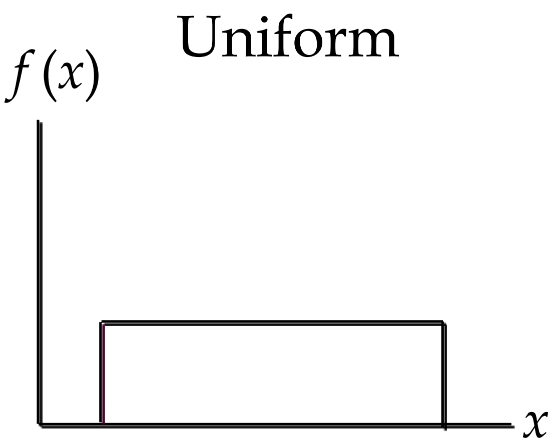
Chapter 6
Continuous Probability Distributions

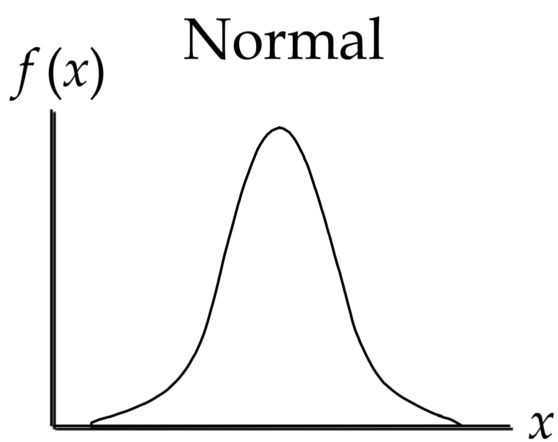
Uniform Probability Distribution (均勻機率分配)

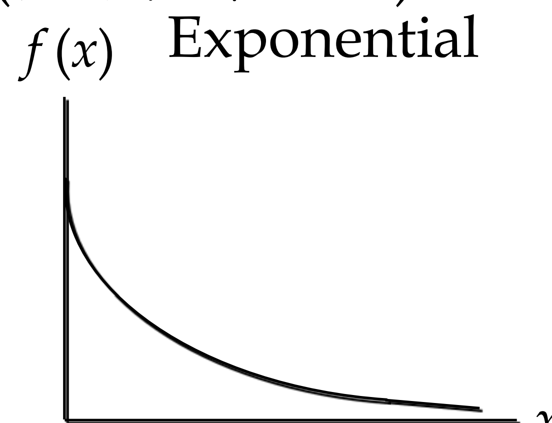
Normal Probability Distribution (常態機率分配)

Normal Approximation of Binomial Probabilities (二項機率的常態分配近似值)

Exponential Probability Distribution (指數機率分配)







Continuous Probability Distributions

A <u>continuous random variable</u> can assume any value in an interval on the real line or in a collection of intervals.

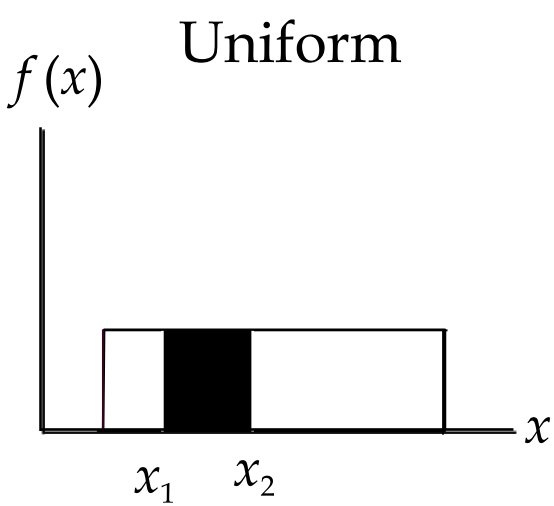
It is not possible to talk about the probability of the random variable assuming a particular value.

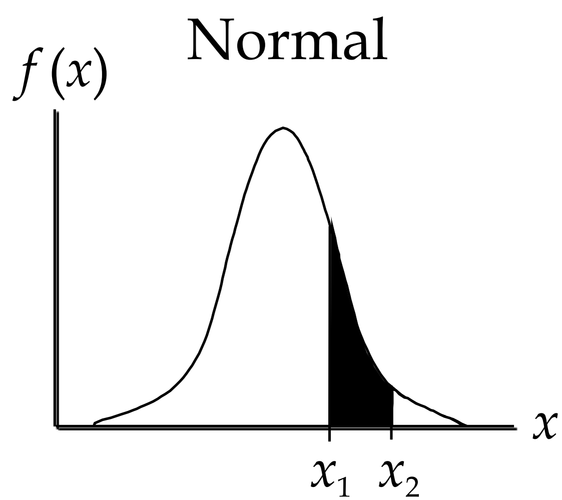
The probability of any single point is zero.

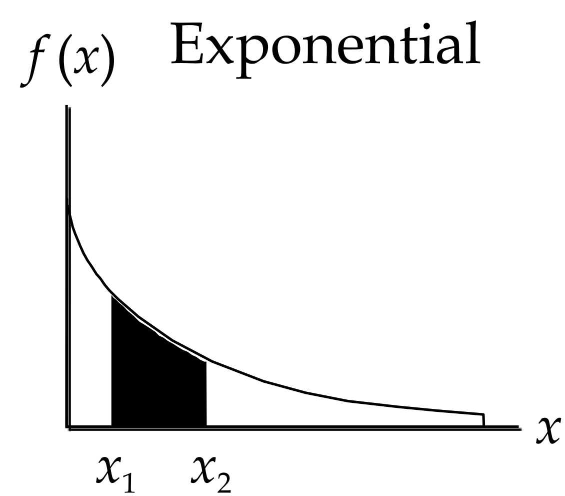
Instead, we talk about the probability of the random variable assuming a value within a given interval.

Continuous Probability Distributions

The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function (機率密度函數) between x_1 and x_2.







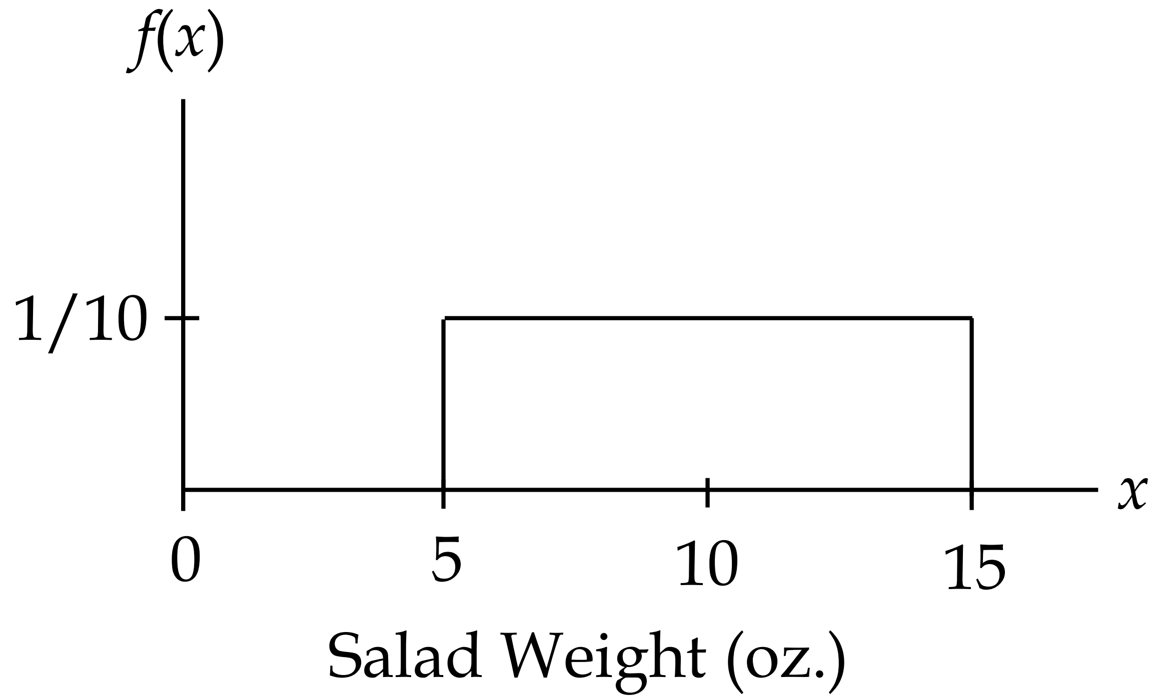
Uniform Probability Distribution

Example: Slater's Buffet

Slater customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces.

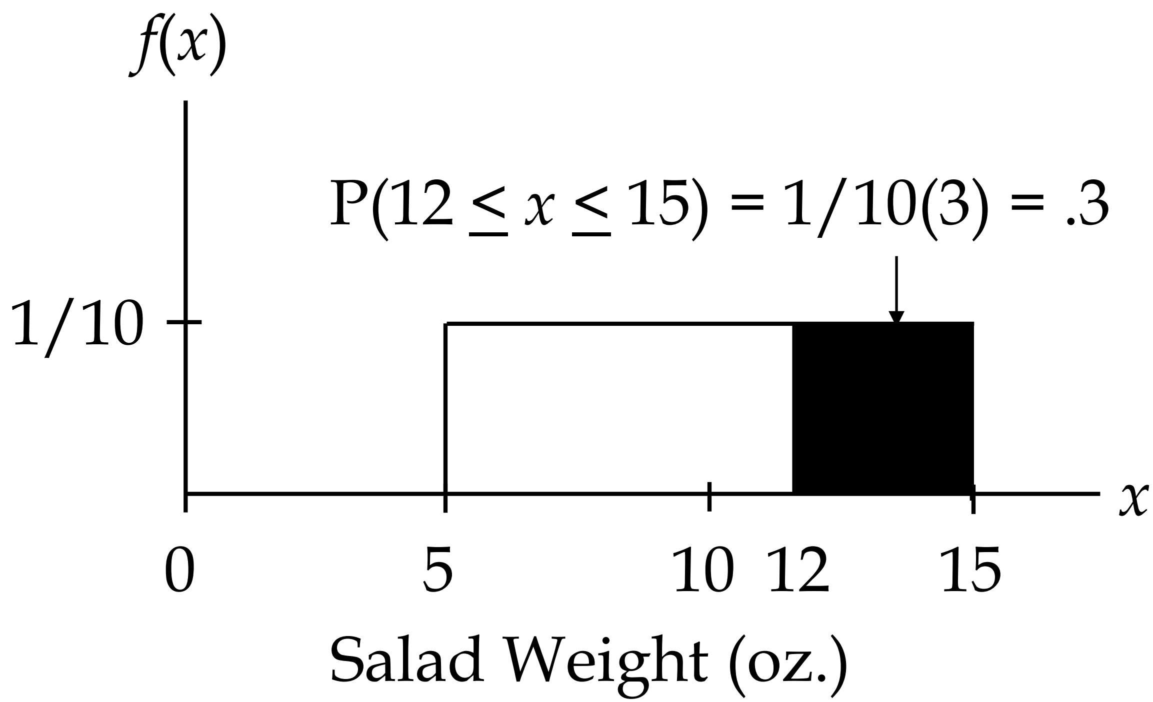
Example: Slater's Buffet

Uniform Probability Distribution for Salad Plate Filling Weight



Example: Slater's Buffet

What is the probability that a customer will take between 12 and 15 ounces of salad?



Example: Slater's Buffet

Uniform Probability Density Function (均勻機率密度函數)

$$f(x) = 1/10 \quad \text{for } 5 \leq x \leq 15$$
$$= 0 \quad \text{elsewhere}$$

where:

x = salad plate filling weight

Continuous Random Variable, X

Probability density function (*pdf*) :

$$f(x) \geq 0; \int_a^b f(x)dx = 1\,, (a \leq X \leq b)$$

Cumulative probability function:

$$F(x) = P(X \leq x) = \int_a^x f(t)dt$$

Expected value:

$$E(X) = \int_a^b xf(x)dx = \mu$$

Variance:

$$V(X) = \sigma^2 = \int_a^b (x-\mu)^2 \cdot f(x)dx$$

Uniform Probability Distribution

A random variable is uniformly distributed whenever the probability is proportional to the interval's length.

The uniform probability density function is:

$$
\begin{aligned}
f(x) &= 1/(b - a) \quad \text{for } a \leq x \leq b \\
&= 0 \qquad\qquad\quad \text{elsewhere}
\end{aligned}
$$

where: a = smallest value the variable can assume

b = largest value the variable can assume

Uniform Probability Distribution

Expected Value of x

$$E(x) = (a + b)/2$$

Variance of x

$$Var(x) = (b - a)^2/12$$

$$f(x) = \frac{1}{b-a}, \mu = \frac{a+b}{2}$$

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x) dx$$

$$= \int_a^b (x-\mu)^2 \frac{1}{b-a} dx$$

$$= \left(\frac{1}{b-a}\right) \frac{1}{3} (x-\mu)^3 \Big|_a^b$$

$$= \frac{1}{3(b-a)} \left[\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3 \right]$$

$$= \frac{1}{3(b-a)} \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3 \right]$$

$$= \frac{1}{24(b-a)} [(b-a)^3 - (a-b)^3]$$

$$= \frac{1}{24(b-a)} [(b-a)^3 + (b-a)^3]$$

$$= \frac{1}{24(b-a)} [2(b-a)^3] = \frac{1}{12} (b-a)^2$$

Example: Slater's Buffet

Expected Value of x

$$
\begin{aligned}
E(x) &= (a + b)/2 \\
&= (5 + 15)/2 \\
&= 10
\end{aligned}
$$

Variance of x

$$
\begin{aligned}
Var(x) &= (b - a)^2/12 \\
&= (15 – 5)^2/12 \\
&= 8.33
\end{aligned}
$$

Area as a Measure of Probability

The area under the graph of $f(x)$ and probability are identical.

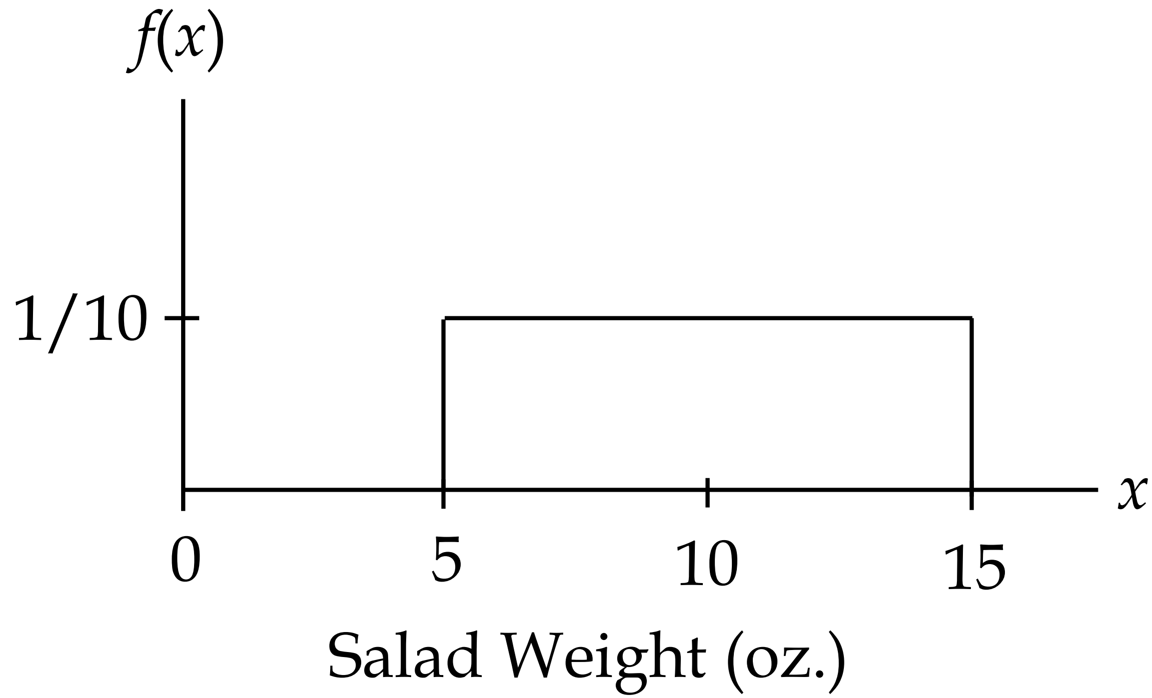
This is valid for all continuous random variables.

The probability that x takes on a value between some lower value x_1 and some higher value x_2 can be found by computing the area under the graph of $f(x)$ over the interval from x_1 to x_2.

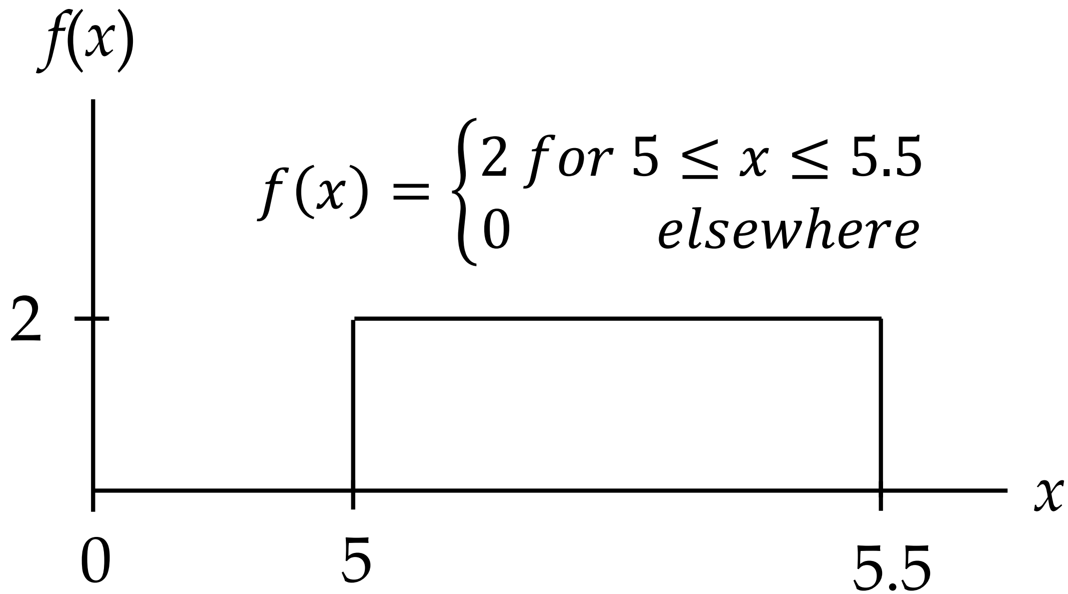
The probability of any single point is zero.

Compute the probability of a single point, $x = 10$

$P(x = 10) = P(10 \leq x \leq 10) = 0(1/10) = 0$



The height of a probability density function is not a probability.



Normal Probability Distribution

The normal probability distribution is the most important distribution for describing a continuous random variable.

It is widely used in statistical inference.

It has been used in a wide variety of applications including: heights of people, rainfall amounts, test scores, scientific measurements

Abraham de Moivre, a French mathematician, published *The Doctrine of Chances* in 1733.

He derived the normal distribution.

Normal Probability Distribution

Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where:

μ = mean

σ = standard deviation

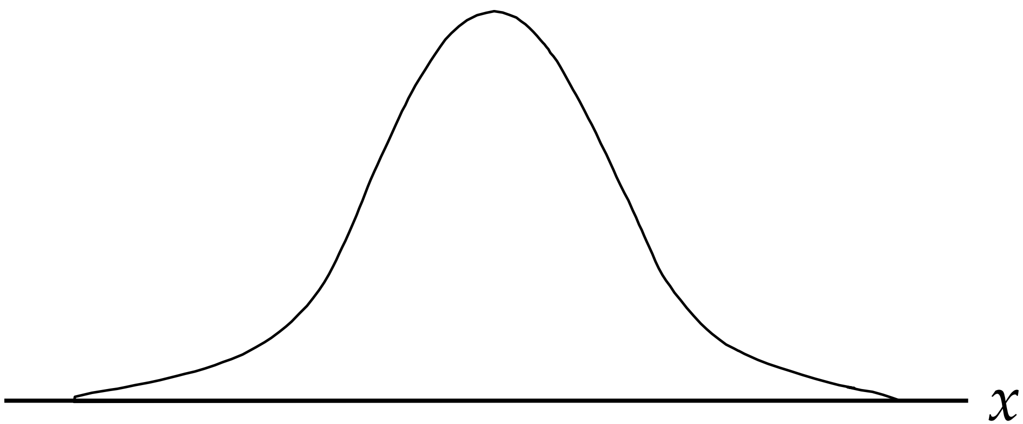
π = 3.14159

e = 2.71828

Normal Probability Distribution

Characteristics

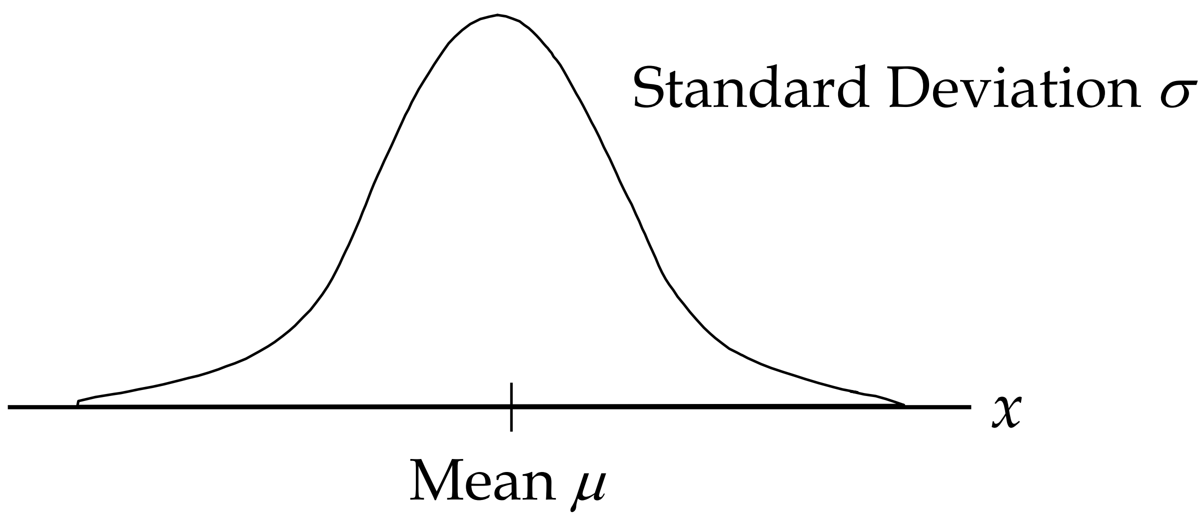
The distribution is symmetric; its skewness measure is zero.



Normal Probability Distribution

Characteristics

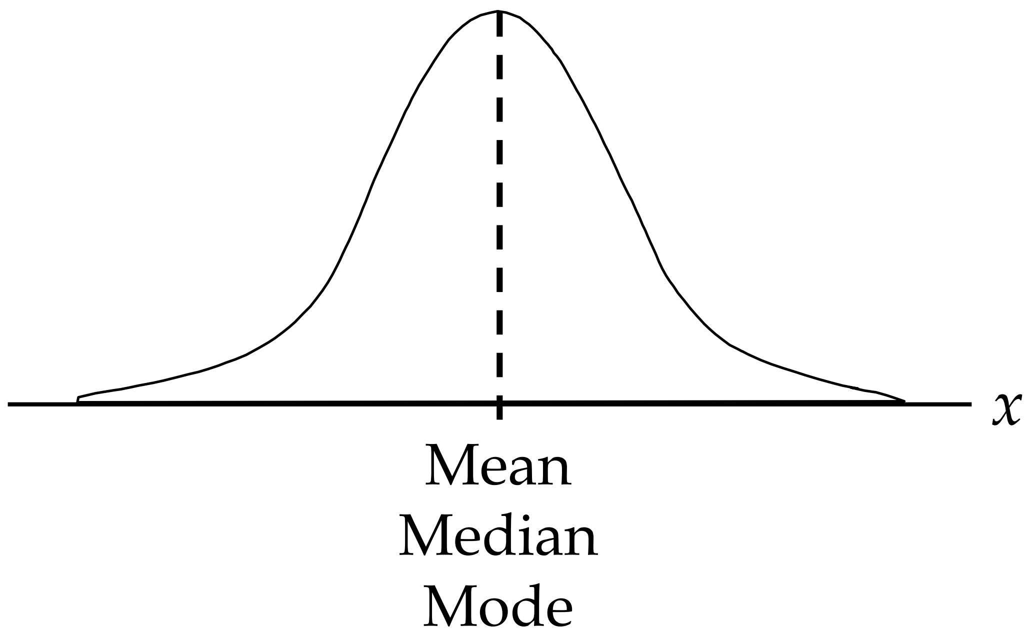
The entire family of normal probability distributions is defined by its mean μ and its standard deviation σ.



Normal Probability Distribution

Characteristics

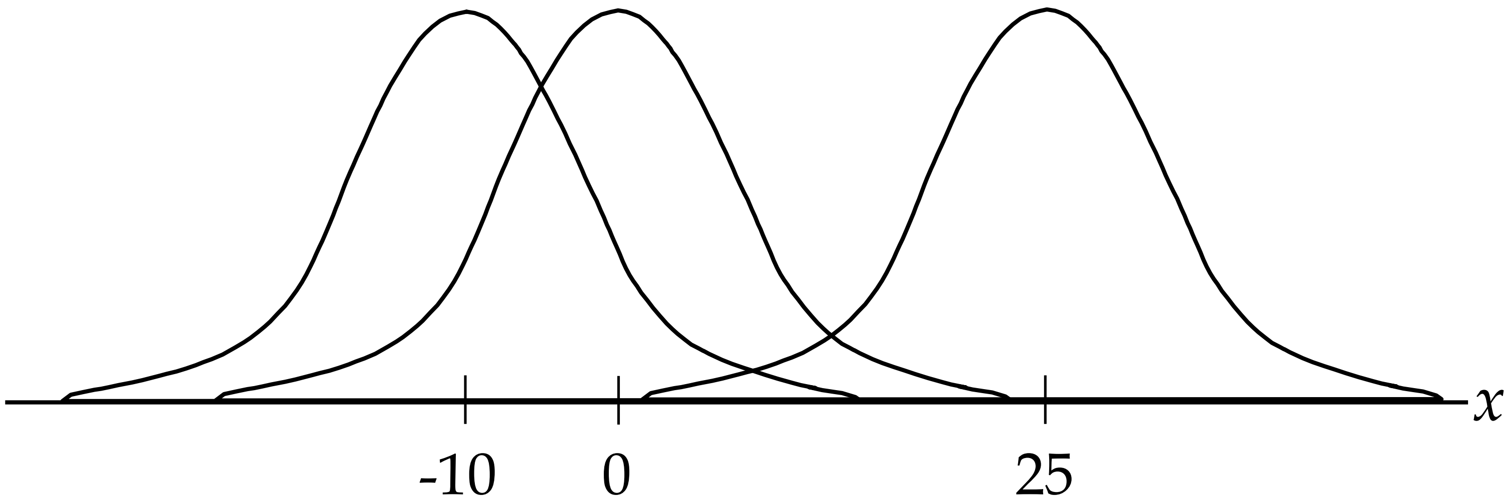
The highest point on the normal curve is at the mean, which is also the median and mode.



Normal Probability Distribution

Characteristics

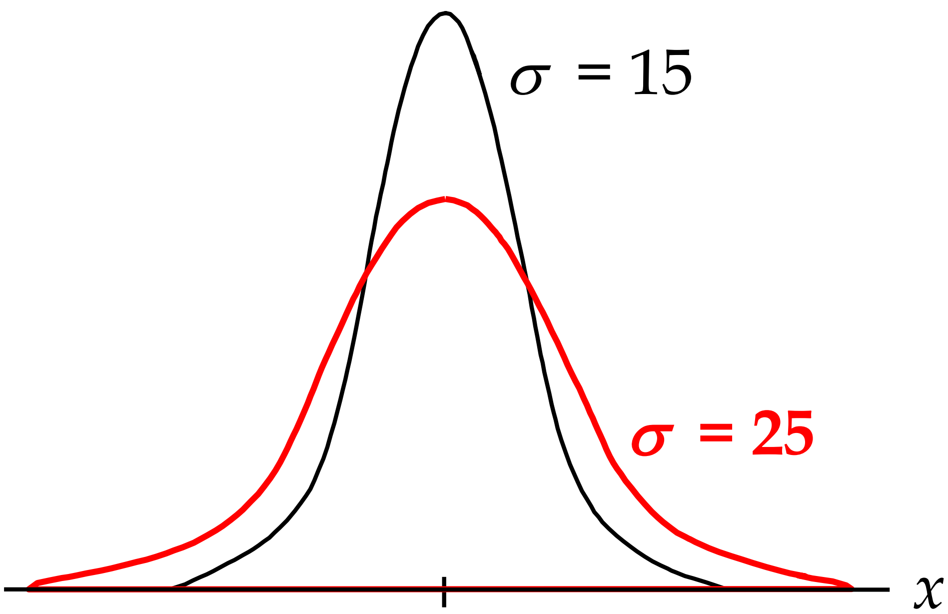
The mean can be any numerical value: negative, zero, or positive.



Normal Probability Distribution

Characteristics

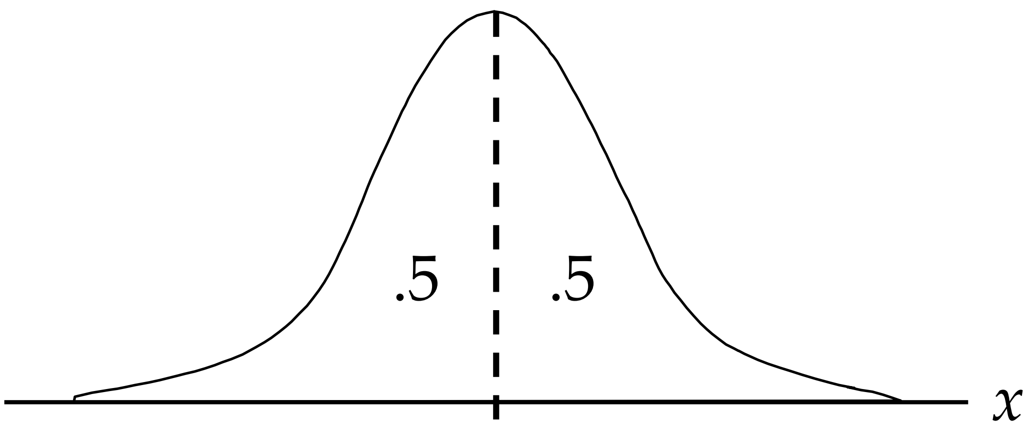
The standard deviation determines the width of the curve: larger values result in wider, flatter curves.



Normal Probability Distribution

Characteristics

Probabilities for the normal random variable are given by areas under the curve. The total area under the curve is 1 (.5 to the left of the mean and .5 to the right).



Normal Probability Distribution

Characteristics (basis for the empirical rule)

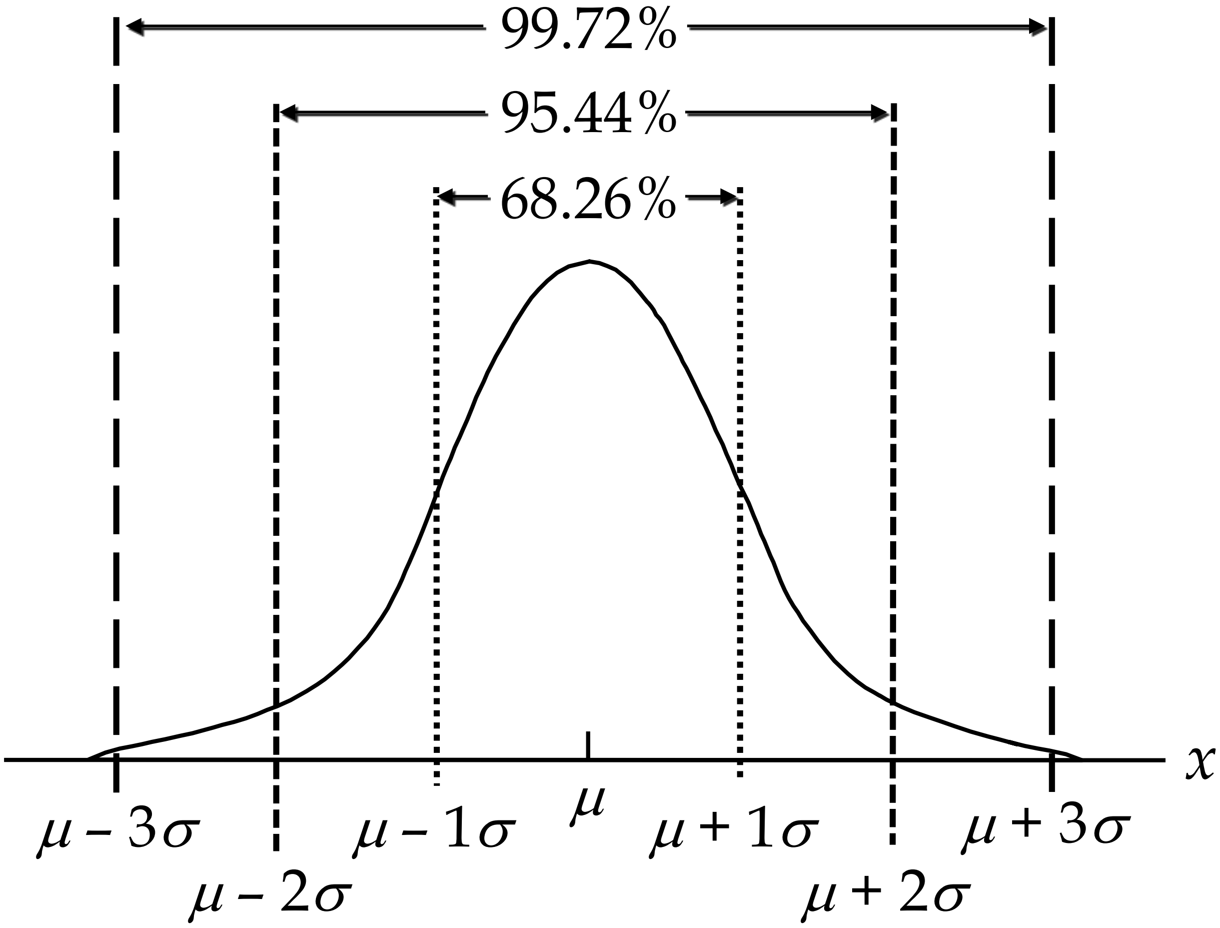
68.26% of values of a normal random variable are within +/- 1 standard deviation of its mean.

95.44% of values of a normal random variable are within +/- 2 standard deviations of its mean.

99.72% of values of a normal random variable are within +/- 3 standard deviations of its mean.

Normal Probability Distribution

Characteristics (basis for the empirical rule)



Standard Normal Probability Distribution

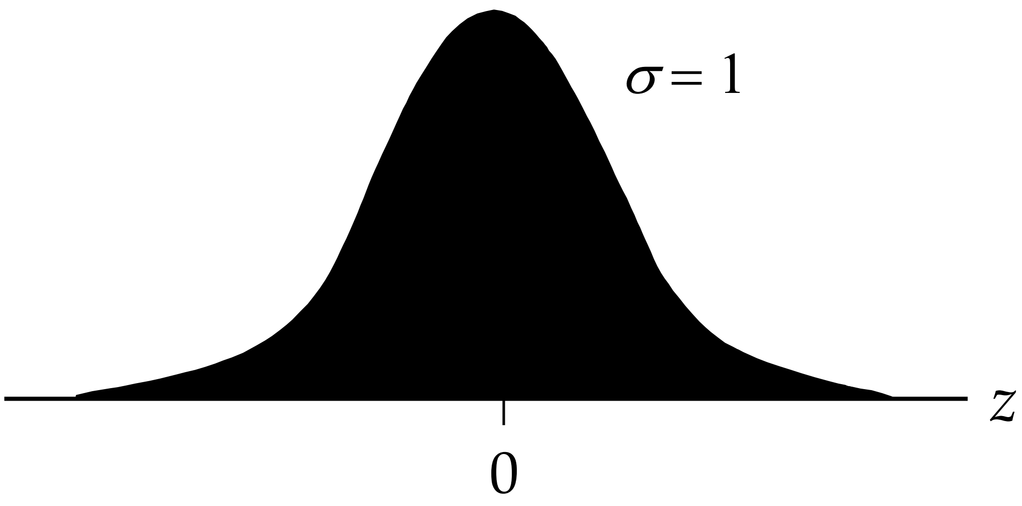
Characteristics

A random variable having a normal distribution with a mean of 0 and a standard deviation of 1 is said to have a standard normal probability distribution (標準常態機率分配).

Standard Normal Probability Distribution

Characteristics

The letter *z* is used to designate the standard normal random variable.



Standard Normal Probability Distribution

Converting to the Standard Normal Distribution

$$z = \frac{x - \mu}{\sigma}$$

We can think of z as a measure of the number of standard deviations x is from μ.

Example

Given that z is a standard normal random variable, compute the following probabilities.

1. $P(z \leq 1.00)$
2. $P(-.50 \leq z \leq 1.25)$
3. $P(-1.00 \leq z \leq 1.00)$
4. $P(z \geq 1.58)$

$$P(z \leq 1.00) = P(z < 1.00)$$

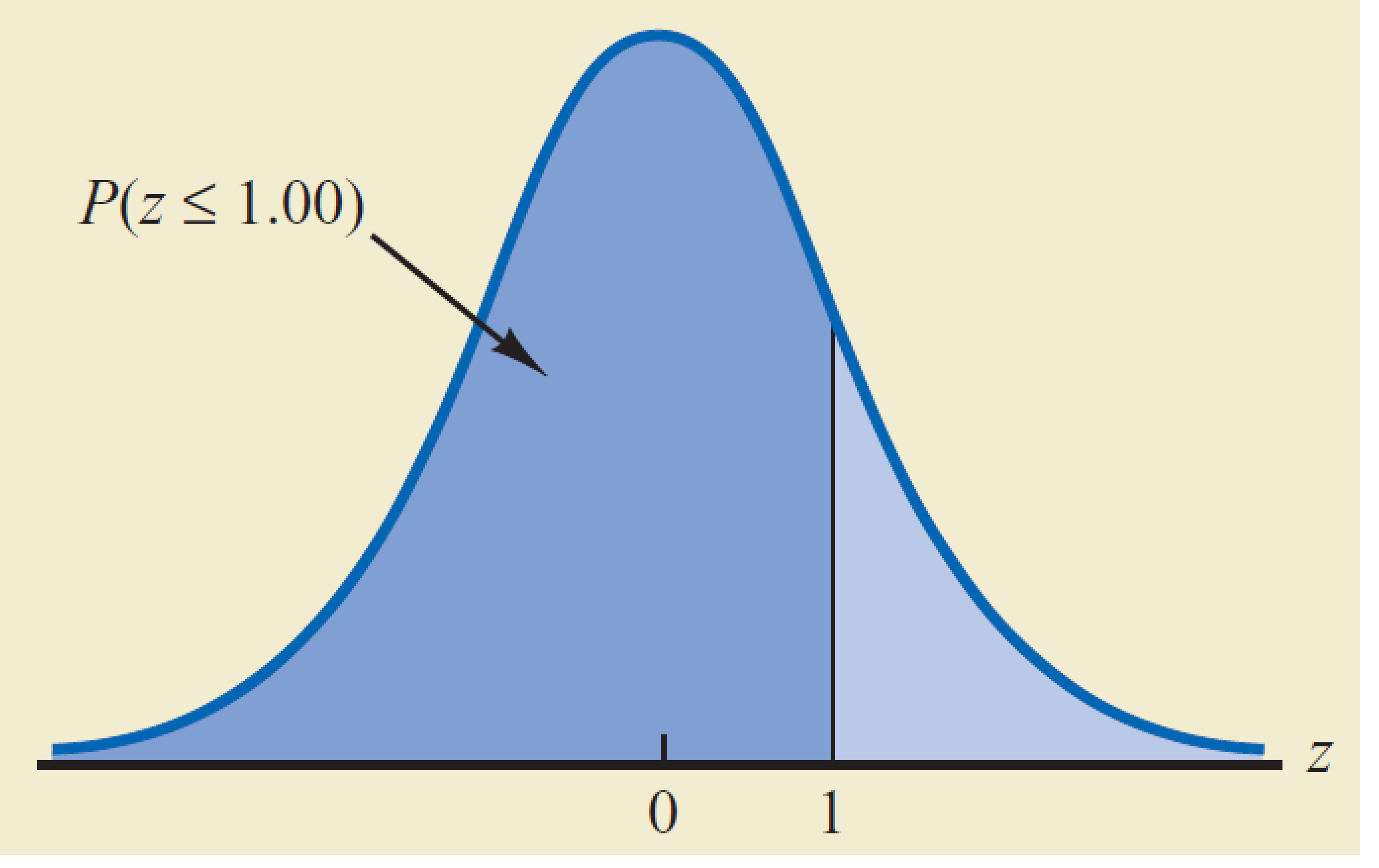
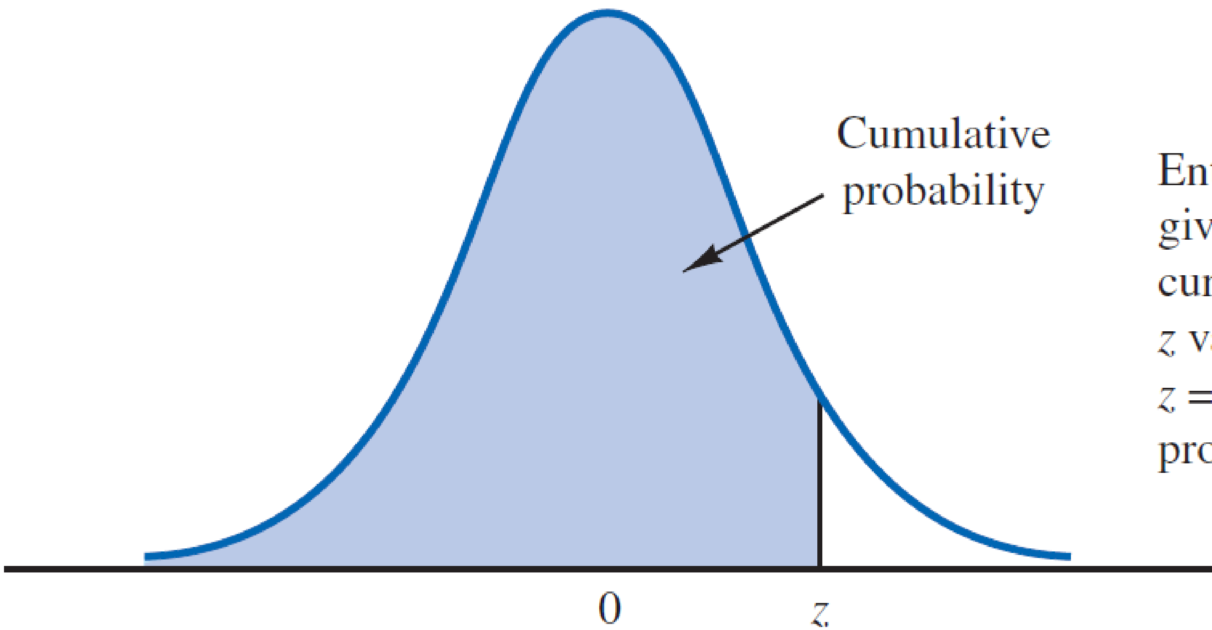


TABLE 1 CUMULATIVE PROBABILITIES FOR THE STANDARD NORMAL DISTRIBUTION (*Continued*)



Entries in the table give the area under the curve to the left of the z value. For example, for $z = 1.25$, the cumulative probability is .8944.

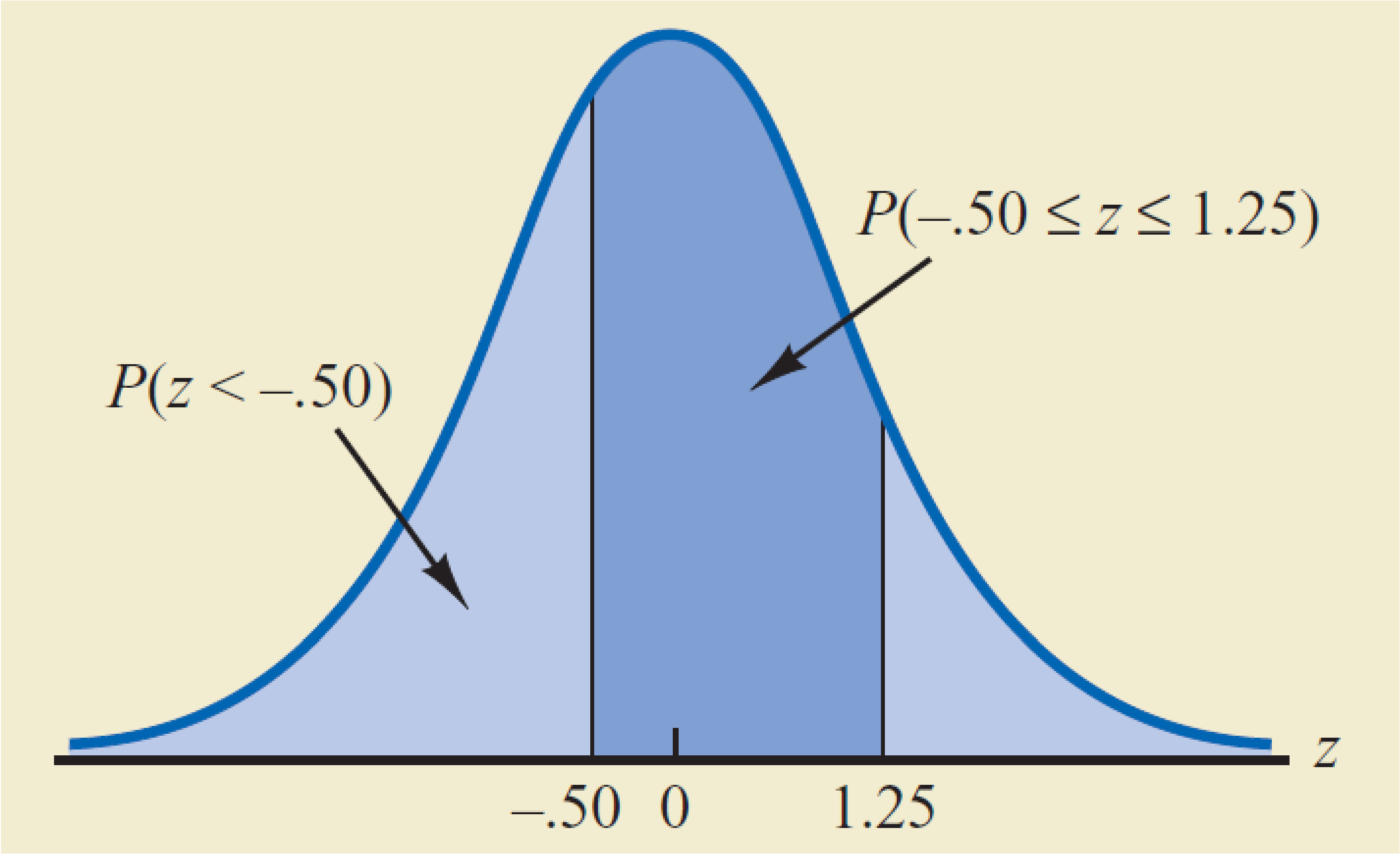
$$P(z \le 1.00) = .8413$$

z	.00	.01	.02
.			
.			
.			
.9	.8159	.8186	.8212
1.0	$\boxed{\mathbf{.8413}}$	.8438	.8461
1.1	.8643	.8665	.8686
1.2	.8849	.8869	.8888
.			
.			
.			

$P(z \le 1.00)$

$P(-.50 \le z \le 1.25)$

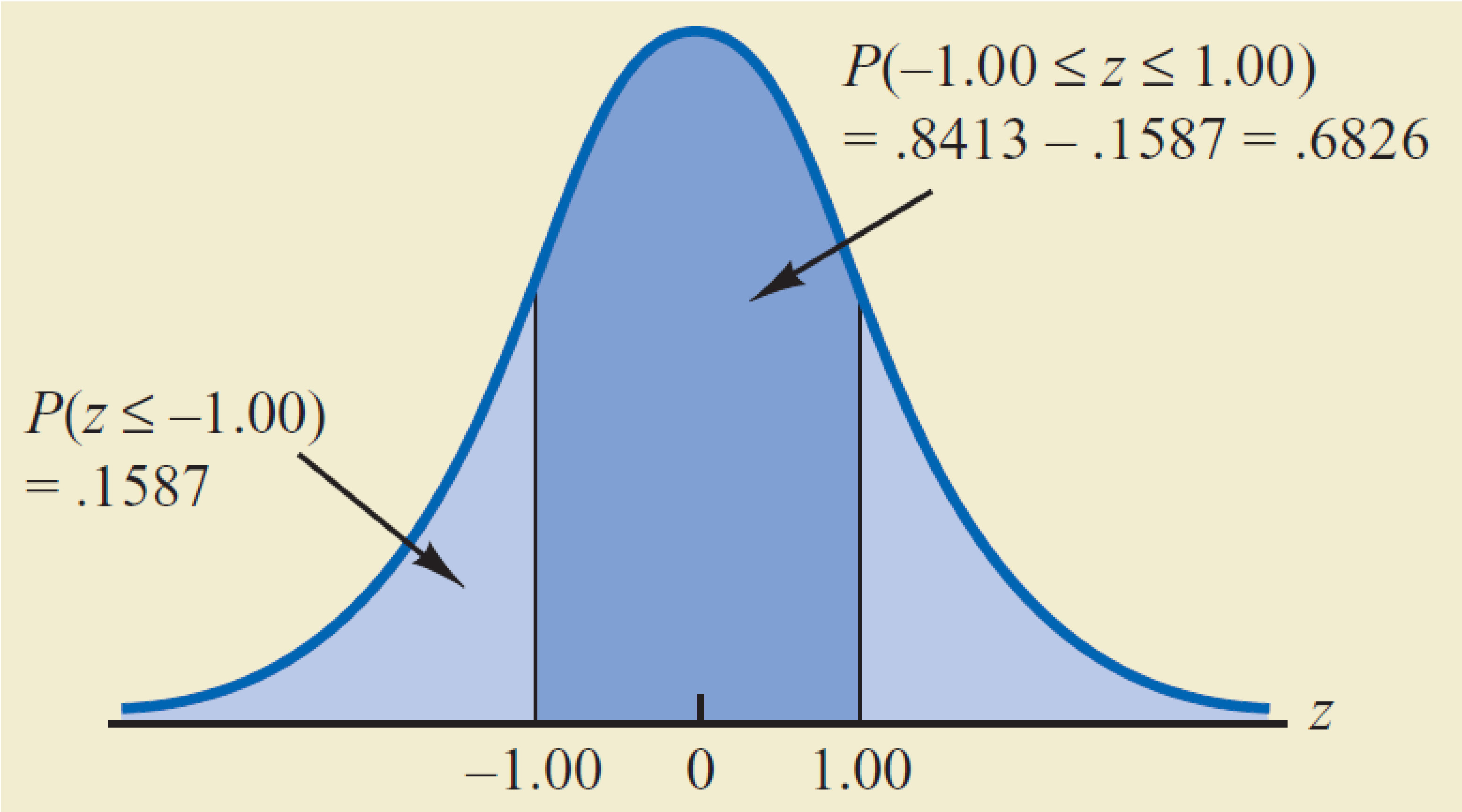
$P(-.50 \le z \le 1.25) = P(z \le 1.25) - P(z \le -.50)$
$= .8944 - .3085 = .5859$



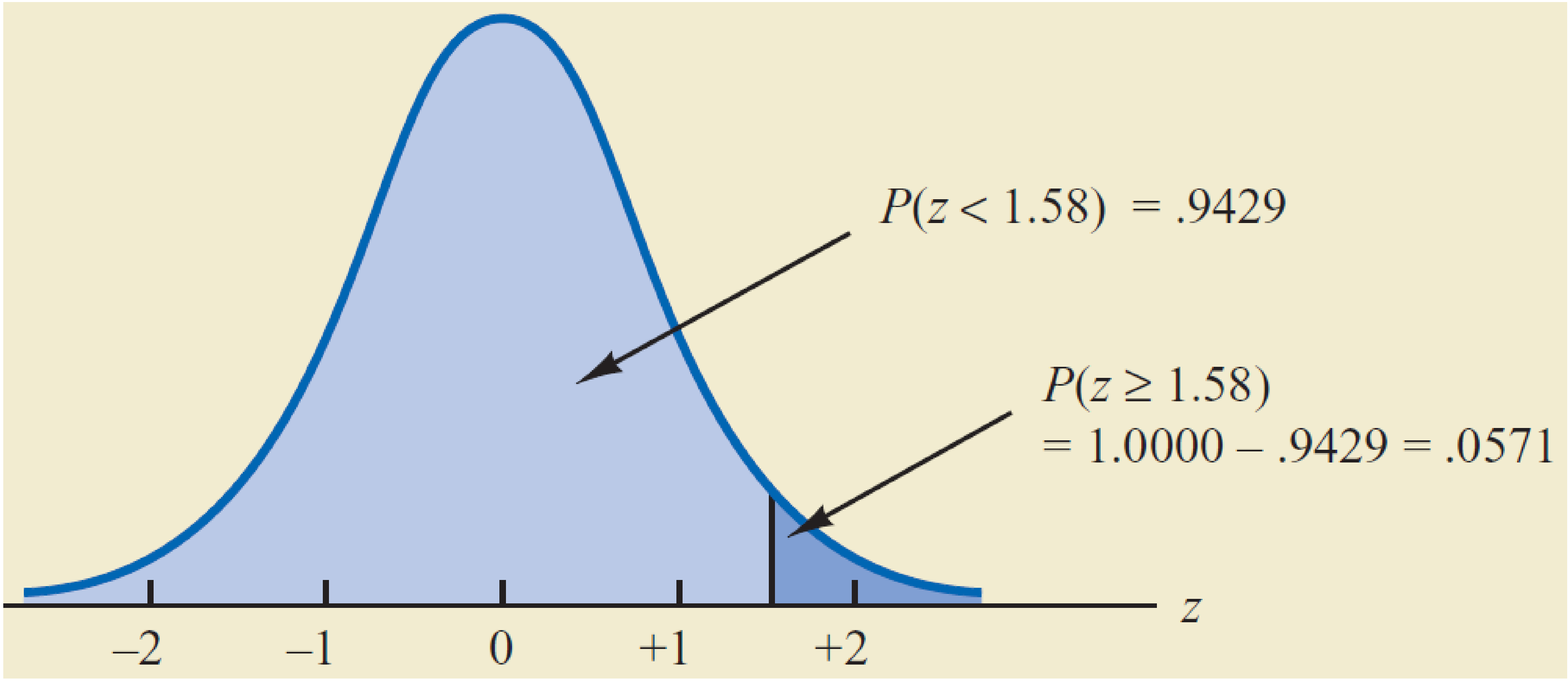
$P(-1.00 \le z \le 1.00)$

$P(-1.00 \le z \le 1.00) = P(z \le 1.00) - P(z \le -1.00)$

$P(z \le 1.00) = .8413$



$$P(z \geq 1.58)$$



Standard Normal Probability Distribution

Example: Pep Zone

Pep Zone sells auto parts and supplies including a popular multi-grade motor oil. When the stock of this oil drops to 20 gallons, a replenishment order is placed.

The store manager is concerned that sales are being lost due to stockouts while waiting for a replenishment order.

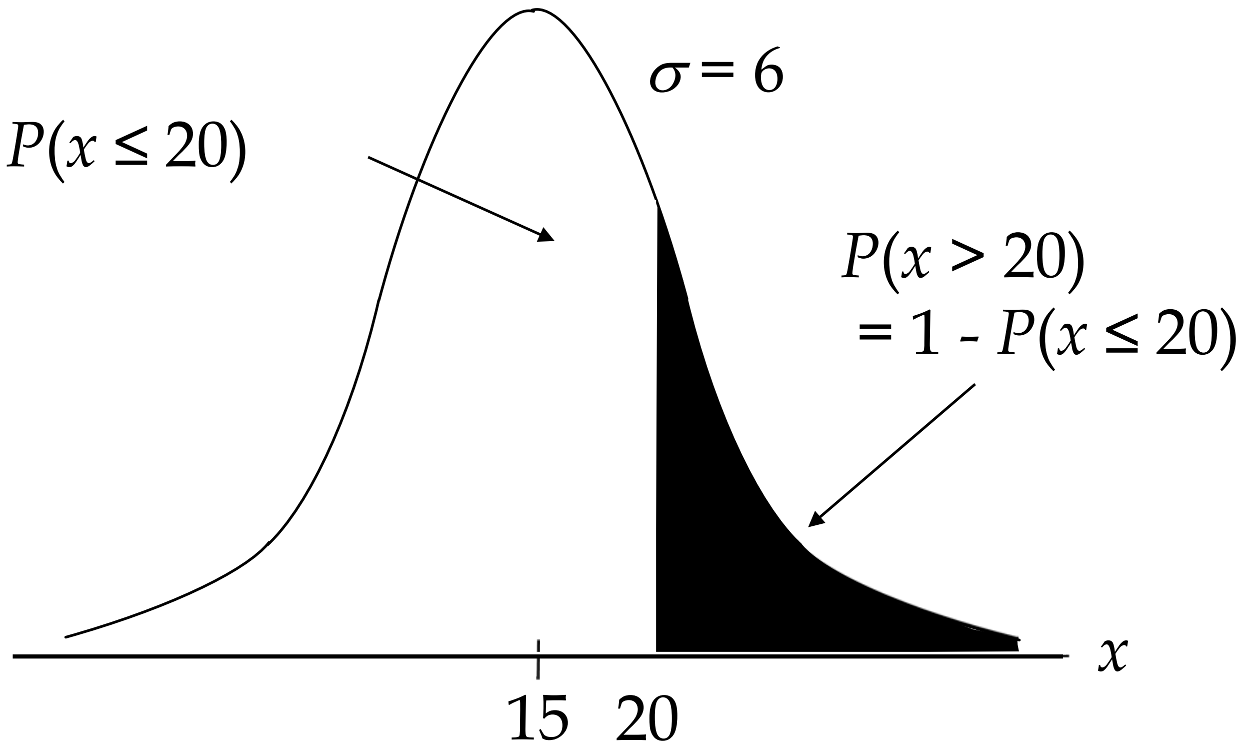
Example: Pep Zone

It has been determined that demand during replenishment lead-time is normally distributed with a mean of 15 gallons and a standard deviation of 6 gallons.

The manager would like to know the probability of a stockout during replenishment lead-time. In other words, what is the probability that demand during lead-time will exceed 20 gallons?

$$P(x > 20) = ?$$

Example: Pep Zone



Example: Pep Zone

Solving for the Stockout Probability

Step 1: Convert x to the standard normal distribution.

$$
\begin{aligned}
z &= (x - \mu)/\sigma \\
&= (20 - 15)/6 \\
&= .83
\end{aligned}
$$

Step 2: Find the area under the standard normal curve to the left of $z = .83$.

Example: Pep Zone

Cumulative Probability Table for the Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
.	.	.	.	.	.	.	.	.	.	.

$P(z \leq .83)$

Example: Pep Zone

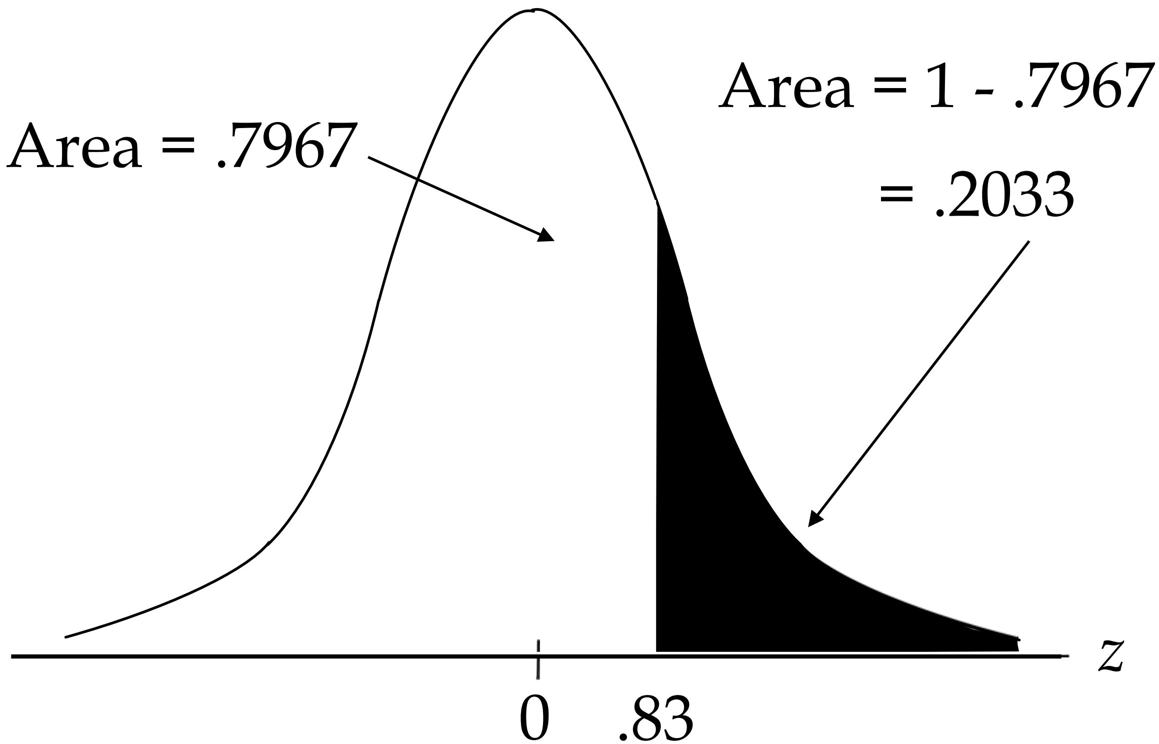
Step 3: Compute the area under the standard normal curve to the right of $z = .83$.

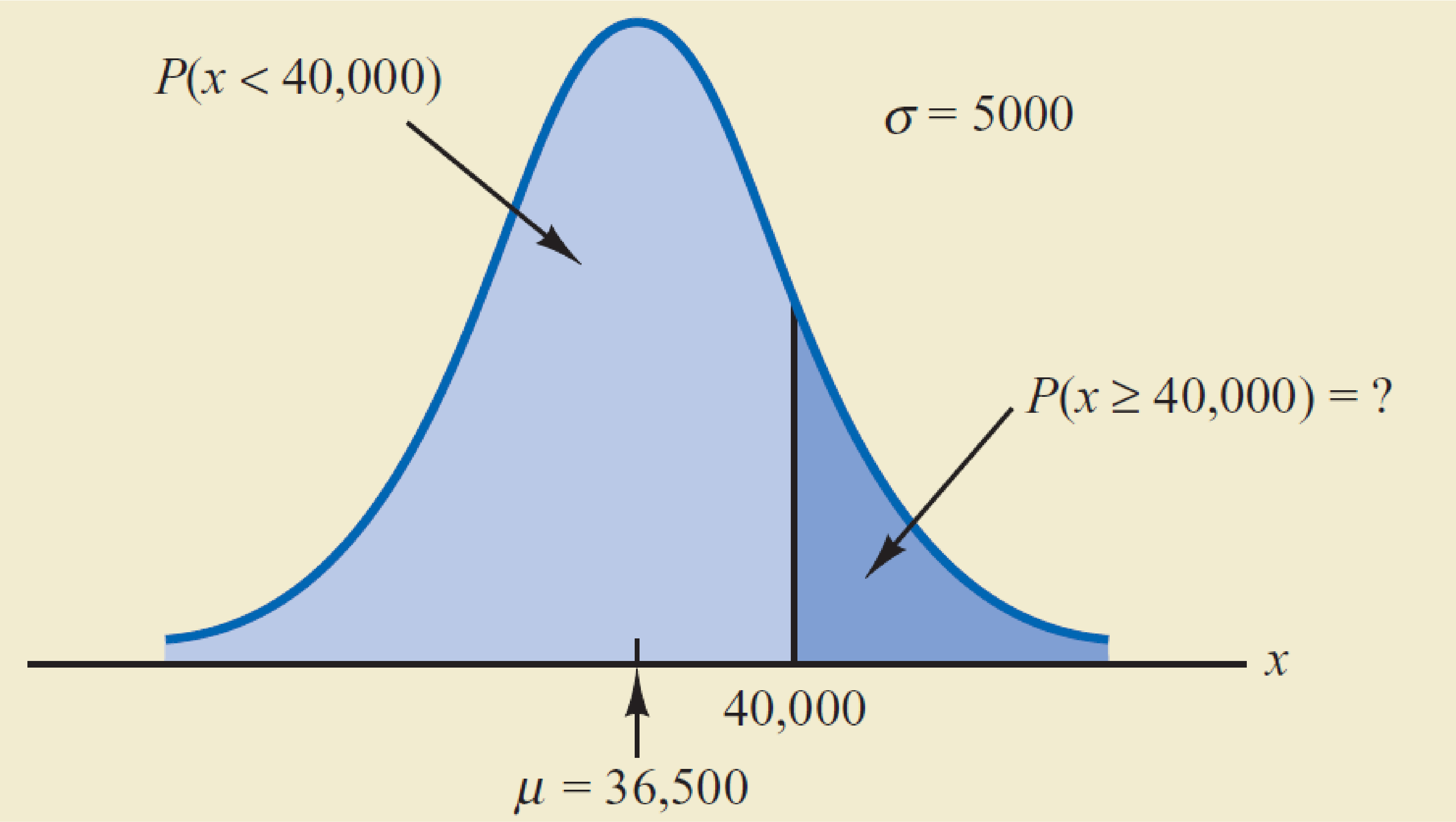
$$
\begin{aligned}
P(z > .83) &= 1 - P(z \leq .83) \\
&= 1 - .7967 \\
&= .2033
\end{aligned}
$$

Probability of a stockout

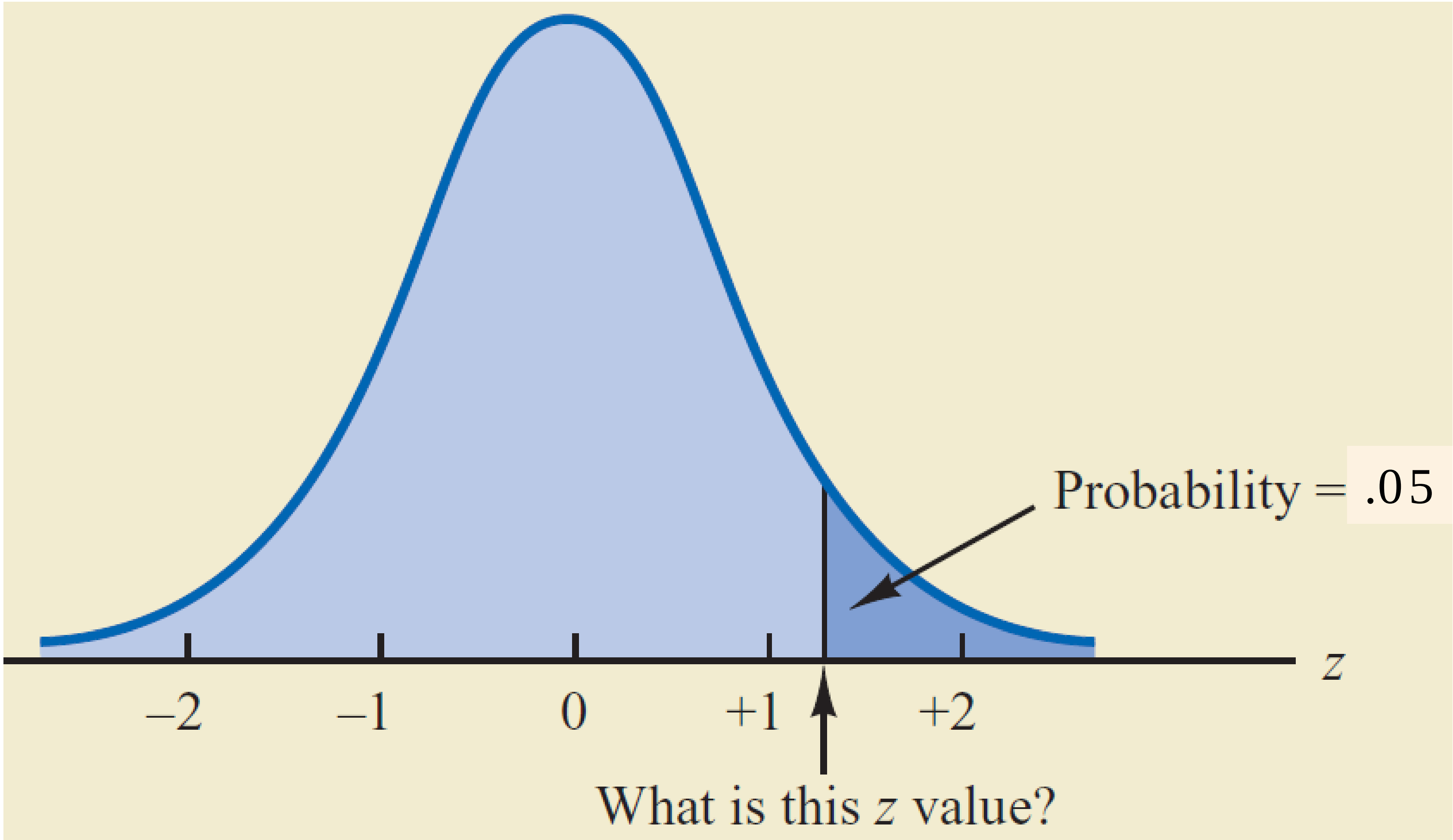
$P(x > 20)$

Example: Pep Zone





P(x < 40,000)
σ = 5000
P(x ≥ 40,000) = ?
x
40,000
μ = 36,500



Probability = .05
z
–2
–1
0
+1
+2
What is this z value?

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	[illegible]	[illegible]	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	[illegible]	[illegible]	[illegible]	[illegible]
1.9	.9713	.9719	.9726	.9732	.9738	.9744	[illegible]	[illegible]	[illegible]	[illegible]
.	.	.	.	.	.	.				

We look up
the complement
of the tail area
(1 - .05 = .95)

$$z_{.05} = 1.645$$

Standard Normal Probability Distribution

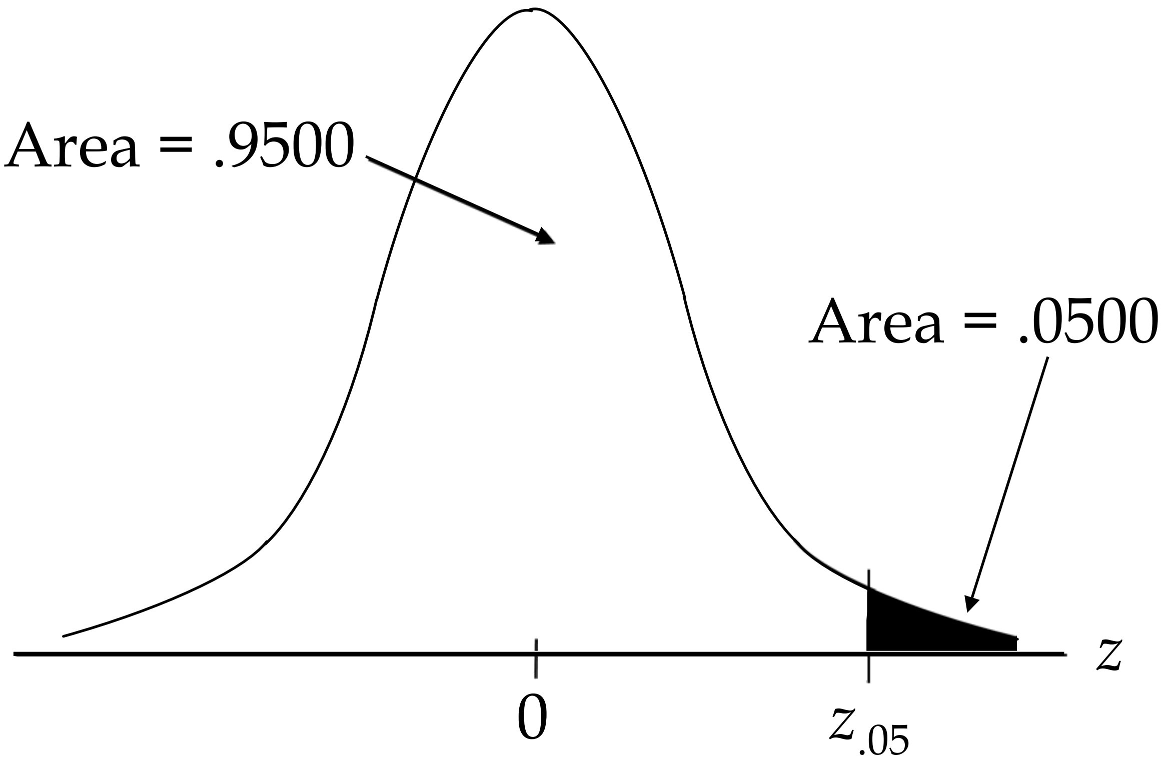
Standard Normal Probability Distribution

If the manager of Pep Zone wants the probability of a stockout during replenishment lead-time to be no more than .05, what should the reorder point be?

(Hint: Given a probability, we can use the standard normal table in an inverse fashion to find the corresponding *z* value.)

Standard Normal Probability Distribution

Solving for the Reorder Point



Standard Normal Probability Distribution

Solving for the Reorder Point

Step 1: Find the z-value that cuts off an area of .05 in the right tail of the standard normal distribution. $z_{.05} = 1.645$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	[illegible]	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	[illegible]	[illegible]	[illegible]	[illegible]
1.9	.9713	.9719	.9726	.9732	.9738	.9744	[illegible]	[illegible]	[illegible]	[illegible]
.	.	.	.	.	.	.				

We look up the complement of the tail area (1 - .05 = .95)

Standard Normal Probability Distribution

Solving for the Reorder Point

Step 2: Convert $z_{.05}$ to the corresponding value of x.

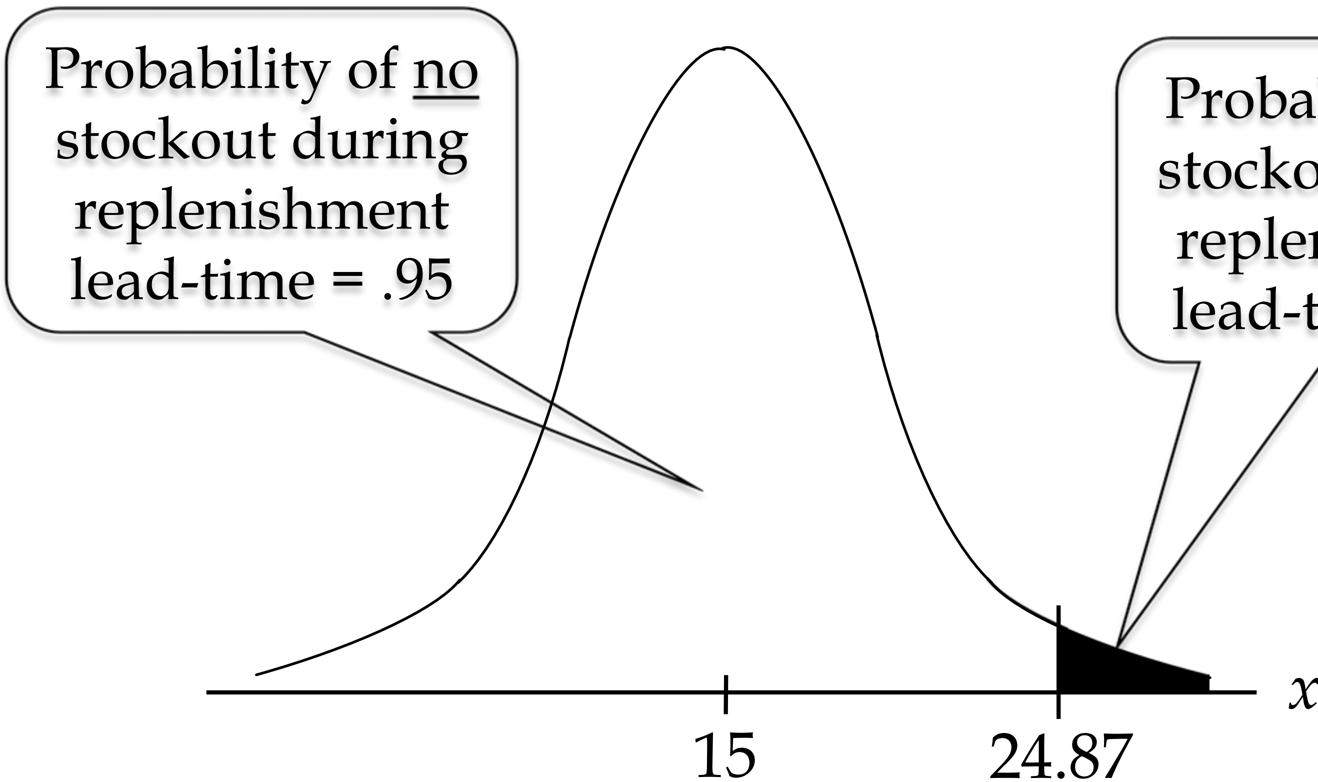
$$z = (x - \mu)/\sigma$$
$$\mu = 15$$
$$\sigma = 6$$

$$x = \mu + z_{.05}\sigma$$
$$= 15 + 1.645(6)$$
$$= 24.87 \text{ or } 25$$

A reorder point of 25 gallons will place the probability of a stockout during leadtime at (slightly less than) .05.

Normal Probability Distribution

Solving for the Reorder Point



Standard Normal Probability Distribution

Solving for the Reorder Point

By raising the reorder point from 20 gallons to 25 gallons on hand, the probability of a stockout decreases from about .20 to .05.

This is a significant decrease in the chance that Pep Zone will be out of stock and unable to meet a customer's desire to make a purchase.

Example: Final exam

The time needed to complete a final examination in a particular college course is normally distributed with a mean of 80 minutes and a standard deviation of 10 minutes.

a. What is the probability of completing the exam in one hour or less?

b. What is the probability that a student will complete the exam in more than 60 minutes but less than 75 minutes?

c. Assume that the class has 60 students and that the examination period is 90 minutes in length. How many students do you expect will be unable to complete the exam in the allotted time?

Example: NYSE

- Trading volume on the New York Stock Exchange is heaviest during the first half hour (early morning) and last half hour (late afternoon) of the trading day. The early morning trading volumes (millions of shares) for 13 days in January and February are shown here (*Barron's*, January 23, 2006; February 13, 2006; and February 27, 2006).
- The probability distribution of trading volume is approximately normal.

214	163	265	194	180
202	198	212	201	
174	171	211	211	

Example: NYSE

a. Compute the mean and standard deviation to use as estimates of the population mean and standard deviation
b. What is the probability that, on a randomly selected day, the early morning trading volume will be less than 180 million shares?
c. What is the probability that, on a randomly selected day, the early morning trading volume will exceed 230 million shares?
d. How many shares would have to be traded for the early morning trading volume on a particular day to be among the busiest 5% of days?

214	163	265	194	180
202	198	212	201	
174	171	211	211	

Normal Approximation of Binomial Probabilities

When the number of trials, n, becomes large, evaluating the binomial probability function by hand or with a calculator is difficult.

The normal probability distribution provides an easy-to-use approximation of binomial probabilities where $np \geq 5$ and $n(1 - p) \geq 5$.

In the definition of the normal curve,
set $\mu = np$ and $\sigma = \sqrt{np(1-p)}$

Normal Approximation of Binomial Probabilities

Add and subtract a continuity correction factor (連續校正因子) because a continuous distribution is being used to approximate a discrete distribution.

For example, $P(x = 12)$ for the discrete binomial probability distribution is approximated by $P(11.5 < x < 12.5)$ for the continuous normal distribution.

Normal Approximation of Binomial Probabilities

Example

Suppose that a company has a history of making errors in 10% of its invoices. A sample of 100 invoices has been taken, and we want to compute the probability that 12 invoices contain errors.

np = 100(.1) = 10 $\geq$ 5 and $n(1 - p)$ = 100(.9) = 90 $\geq$ 5

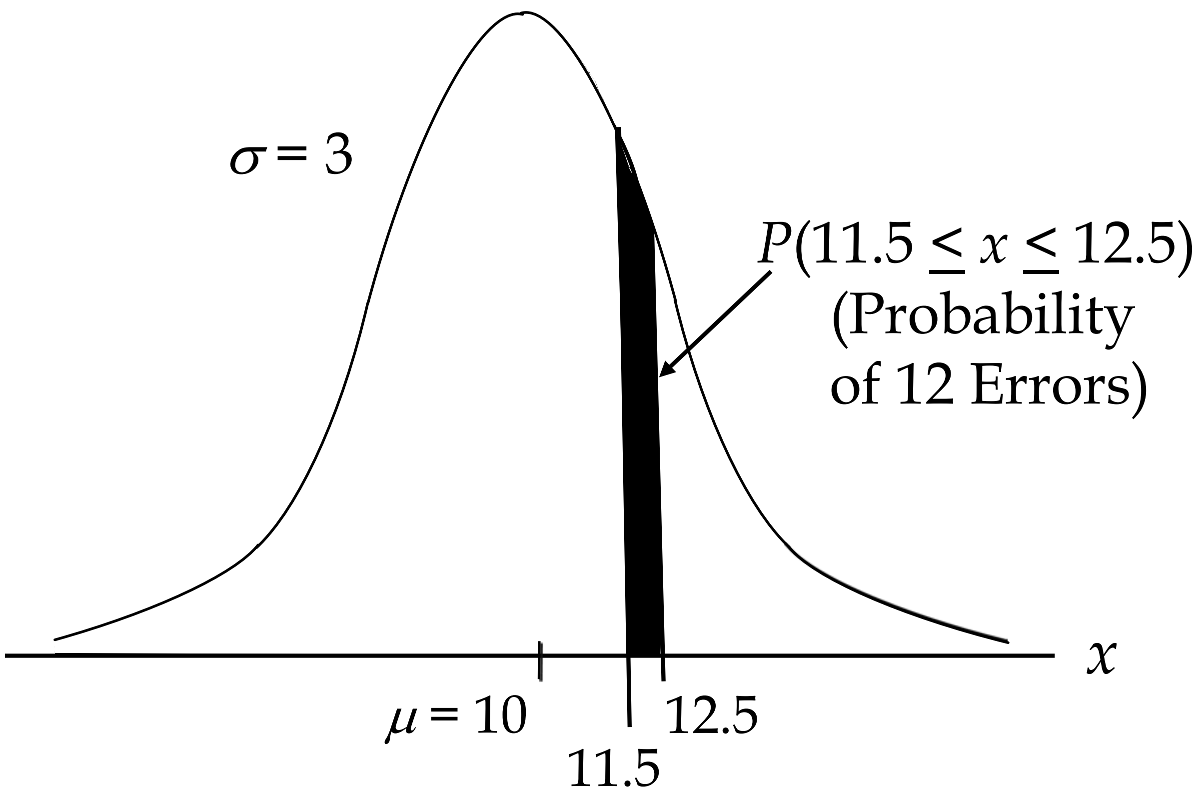
In this case, we want to find the binomial probability of 12 successes in 100 trials. So, we set:

$$\mu = np = 100(.1) = 10$$

$$\sigma = \sqrt{np(1-p)} = [100(.1)(.9)]^{½} = 3$$

Normal Approximation of Binomial Probabilities

Normal Approximation to a Binomial Probability Distribution with n = 100 and p = .1

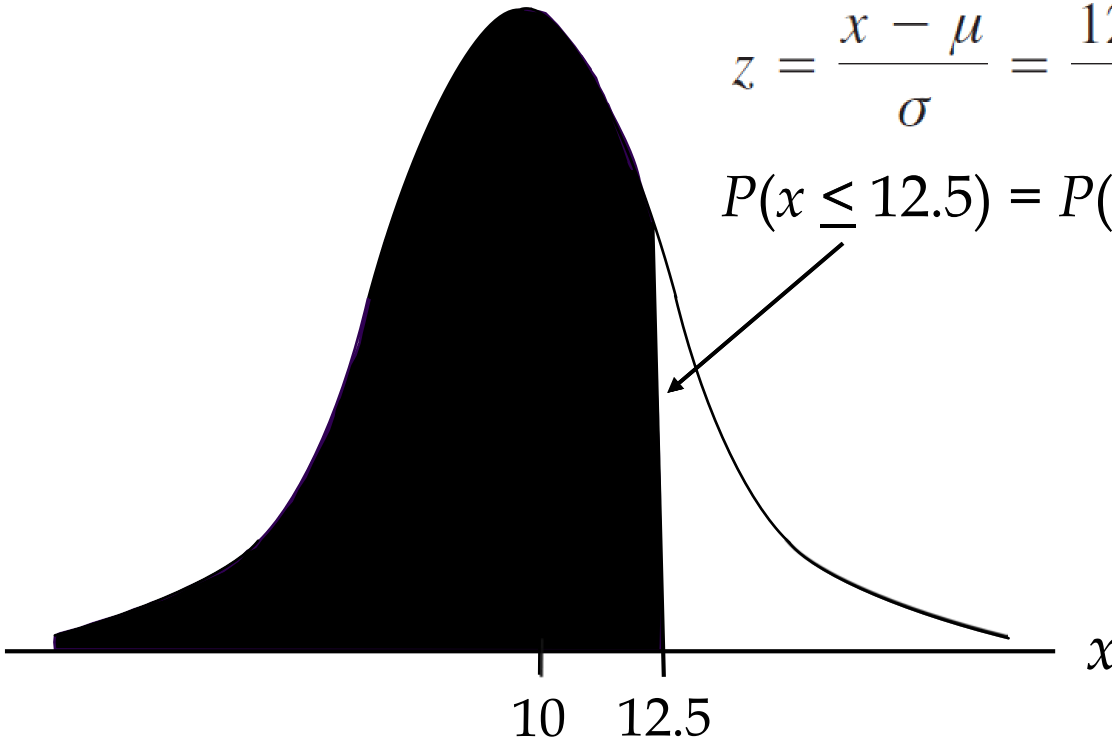


Normal Approximation of Binomial Probabilities

Normal Approximation to a Binomial Probability Distribution with n = 100 and p = .1

$$z = \frac{x - \mu}{\sigma} = \frac{12.5 - 10.0}{3} = .83$$

$P(x \leq 12.5) = P(z \leq .83) = .7967$

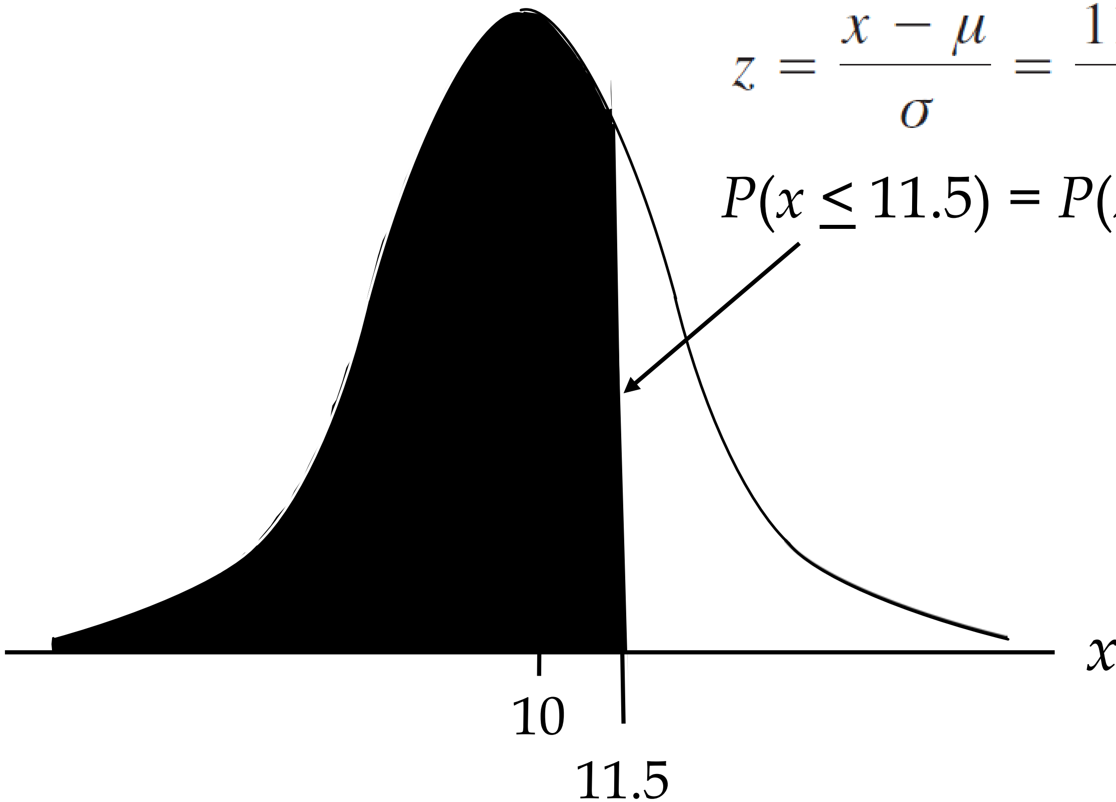


Normal Approximation of Binomial Probabilities

Normal Approximation to a Binomial Probability
Distribution with n = 100 and p = .1

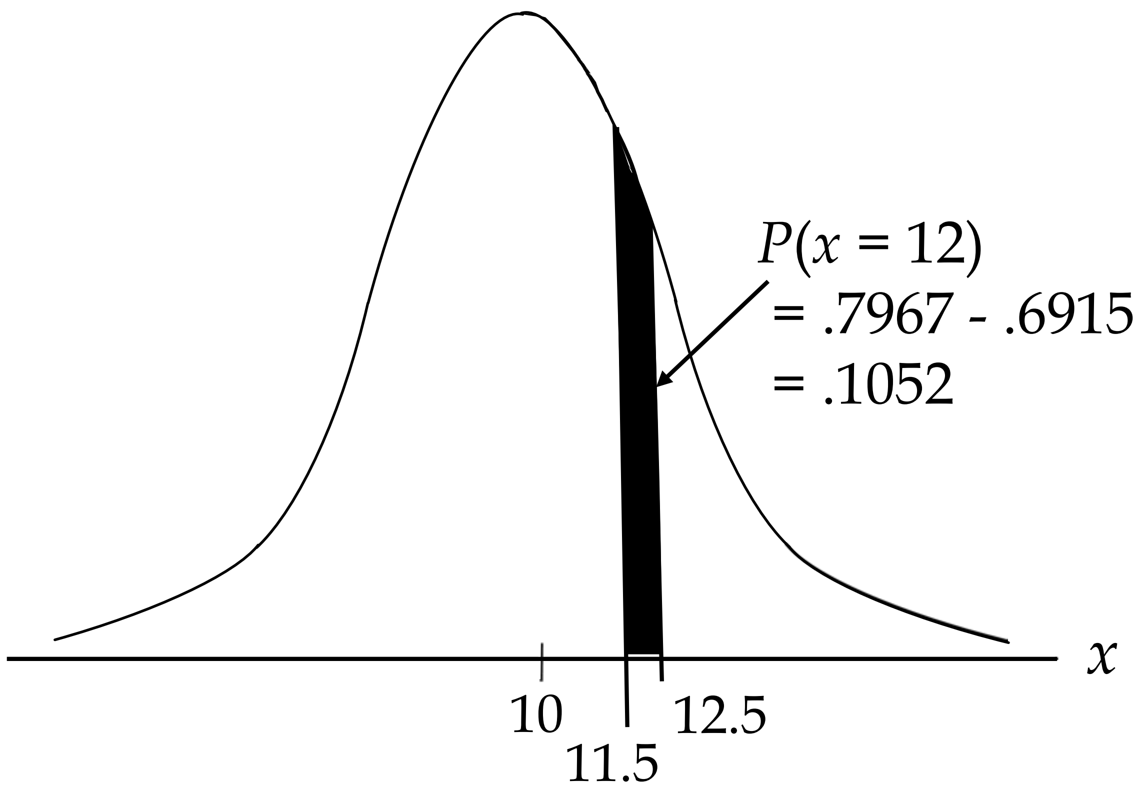
$$z = \frac{x - \mu}{\sigma} = \frac{11.5 - 10.0}{3} = .50$$

$P(x \le 11.5) = P(z \le .50) = .6915$



Normal Approximation of Binomial Probabilities

The Normal Approximation to the Probability of 12 Successes in 100 Trials is .1052



Example: Smoking

Although studies continue to show smoking leads to significant health problems, 20% of adults in the United States smoke. Consider a group of 250 adults.

a. The probability that fewer than 40 smoke

b. The probability that from 55 to 60 smoke

c. The probability that 70 or more smoke

Exponential Probability Distribution

The exponential probability distribution is useful in describing the time it takes to complete a task.

The exponential random variables can be used to describe:

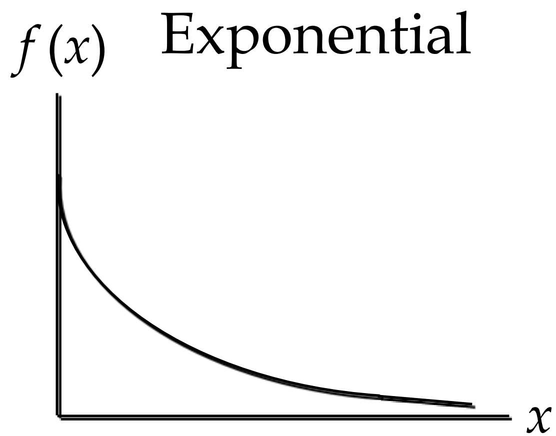
- Time between vehicle arrivals at a toll booth
- Time required to complete a questionnaire
- Distance between major defects in a highway

In waiting line applications, the exponential distribution is often used for service times.

Exponential Probability Distribution

A property of the exponential distribution is that the mean and standard deviation are equal.

The exponential distribution is skewed to the right. Its skewness measure is 2.



Exponential Probability Distribution

Density Function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0$$

where: μ = expected value or mean

e = 2.71828

$\frac{1}{\mu} = \lambda$

Cumulative Probabilities

$$P(x \leq x_0) = 1 - e^{-x_0/\mu}$$

where: x_0 = some specific value of x

Exponential Probability Distribution

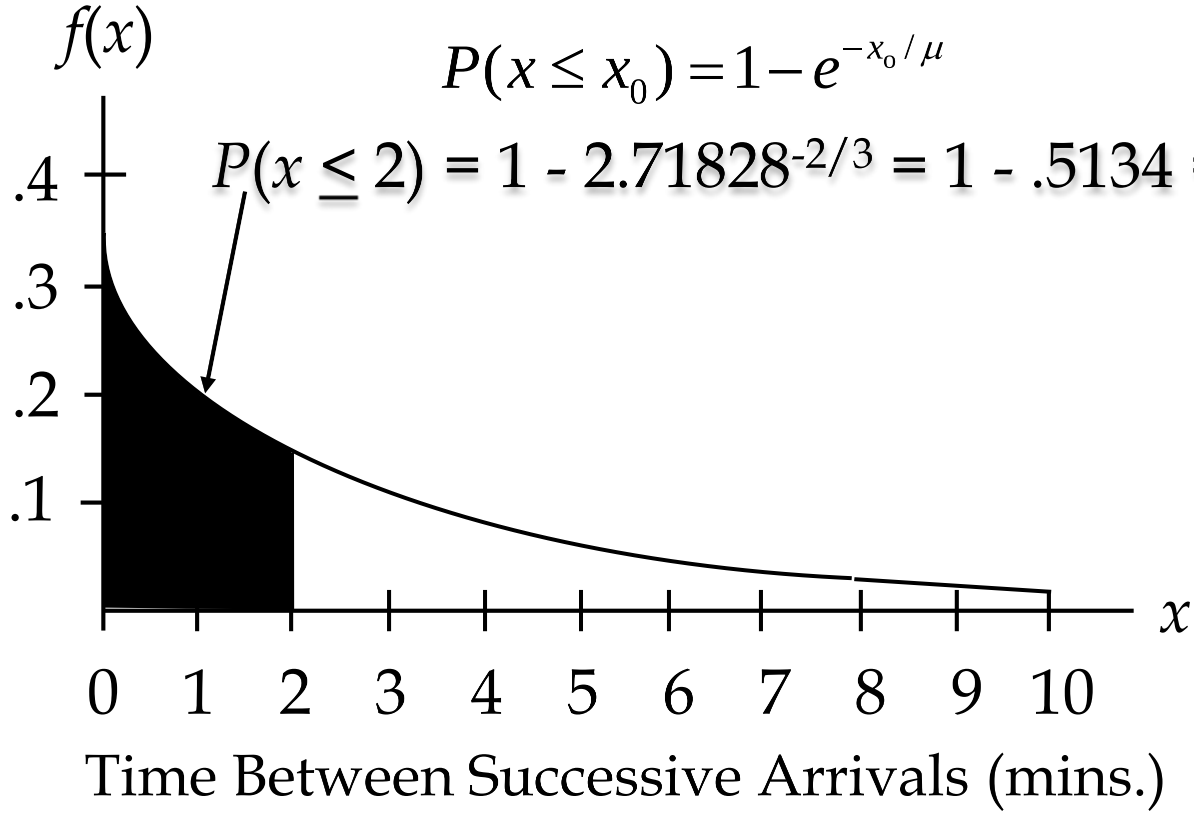
Example: Al's Full-Service Pump

The time between arrivals of cars at Al's full-service gas pump follows an exponential probability distribution with a mean time between arrivals of 3 minutes. Al would like to know the probability that the time between two successive arrivals will be 2 minutes or less.

Example: Al's Full-Service Pump

$$P(x \le x_0) = 1 - e^{-x_0/\mu}$$

$P(x \le 2) = 1 - 2.71828^{-2/3} = 1 - .5134 = .4866$



Relationship between the Poisson and Exponential Distributions

The Poisson distribution
provides an appropriate description
of the number of occurrences
per interval

↓ ↑

The exponential distribution
provides an appropriate description
of the length of the interval
between occurrences

Example: Car wash

Suppose the number of cars that arrive at a car wash during one hour is described by a Poisson probability distribution with a mean of 10 cars per hour.

Poisson probability function:

The average time between cars arriving is

The corresponding exponential probability density function is

Example: Interruptions

Do interruptions while you are working reduce your productivity? According to a University of California–Irvine study, businesspeople are interrupted at the rate of approximately 5.5 times per hour (*Fortune,* March 20, 2006). Suppose the number of interruptions follows a Poisson probability distribution.

a. Show the probability distribution for the time between interruptions.

b. What is the probability that the next interruption will occur within 10 minutes for a particular businessperson?

c. What is the probability a businessperson will have no interruptions during a 15-minute period?