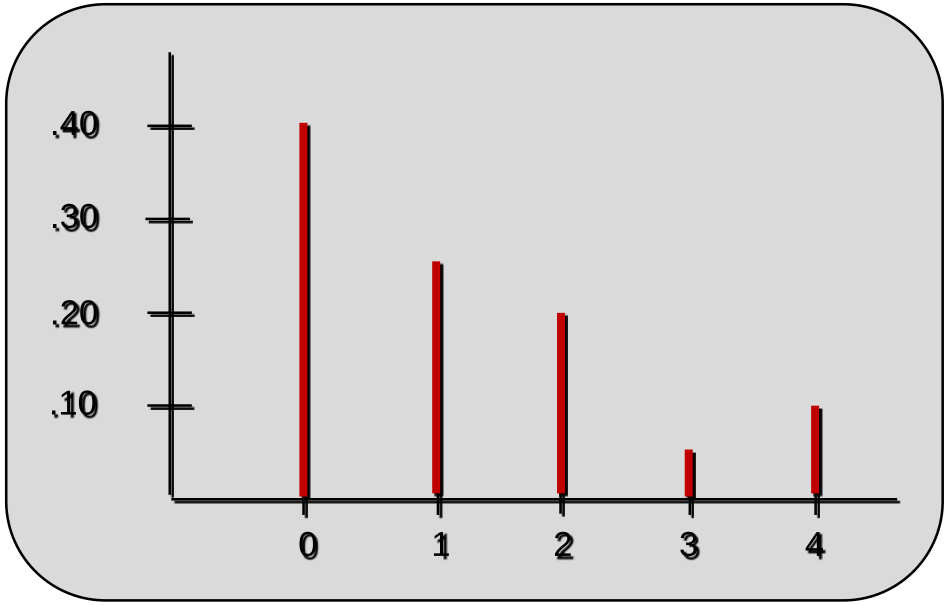
Chapter 5: Discrete Probability Distributions (離散機率分配)

- Random Variables (隨機變數)
- Developing Discrete Probability Distributions
- Expected Value and Variance
- Bivariate Distribution
- Binomial Probability Distribution (二項機率分配)
- Geometric (幾何) Probability Distribution
- Poisson (卜瓦松) Probability Distribution
- Hypergeometric (超幾何) Probability Distribution



Random Variables

- A random variable is a numerical description of the outcome of an experiment.
- A discrete random variable (離散隨機變數) may assume either a finite number of values or an infinite sequence of values.
- A continuous random variable (連續隨機變數) may assume any numerical value in an interval or collection of intervals.

Discrete Random Variable with a Finite Number of Values

- Example: JSL Appliances

Let x = number of TVs sold at the store in one day,

where x can take on 5 values (0, 1, 2, 3, 4)

We can count the TVs sold, and there is a finite upper limit on the number that might be sold (which is the number of TVs in stock).

Discrete Random Variable with an Infinite Sequence of Values

- Example: JSL Appliances

Let x = number of customers arriving in one day,

where x can take on the values 0, 1, 2, . . .

We can count the customers arriving, but there is
no finite upper limit on the number that might arrive.

Random Variables

Illustration	**Random Variable *x***	**Type**
Family size	*x* = Number of dependents reported on tax return	Discrete
Distance from home to stores on a highway	*x* = Distance in miles from home to the store site	Continuous
Own dog or cat	*x* = 1 if own no pet; = 2 if own dog(s) only; = 3 if own cat(s) only; = 4 if own dog(s) and cat(s)	Discrete

Discrete Probability Distributions

- The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.
- We can describe a discrete probability distribution with a table, graph, or formula.

Discrete Probability Distributions

Types of discrete probability distributions:

- First type: uses the rules of assigning probabilities to experimental outcomes to determine probabilities for each value of the random variable.
- Second type: uses a special mathematical formula to compute the probabilities for each value of the random variable.

Discrete Probability Distributions

- The probability distribution is defined by a probability function, denoted by $f(x)$, that provides the probability for each value of the random variable.
- The required conditions for a discrete probability function are:

$$f(x) \geq 0 \quad \text{and} \quad \sum f(x) = 1$$

Discrete Probability Distributions

- There are three methods for assigning probabilities to random variables: classical method, subjective method, and relative frequency method.
- The use of the relative frequency method to develop discrete probability distributions leads to what is called an empirical discrete distribution (實證離散分配).

Discrete Probability Distributions

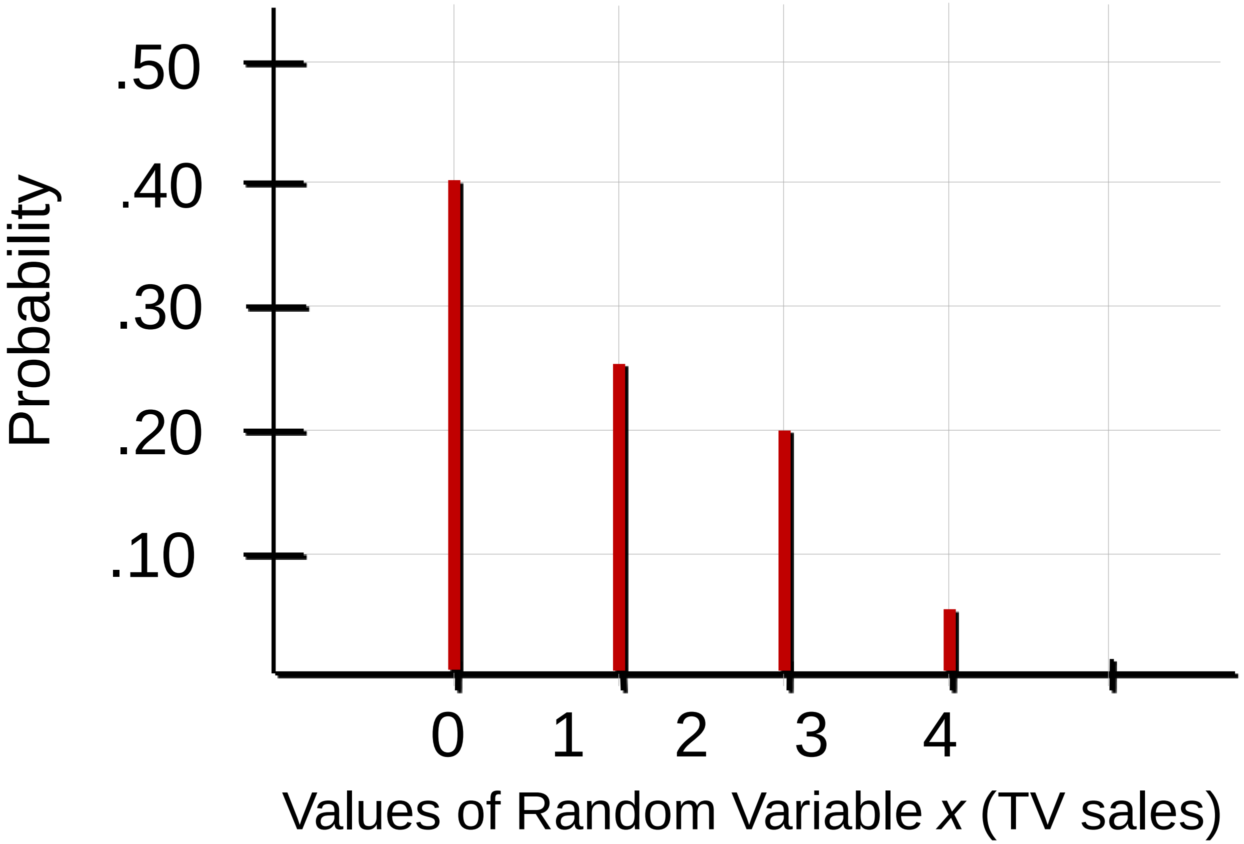
- Example: JSL Appliances

Using past data on TV sales, a <u>tabular representation</u> of the probability distribution for sales was developed.

Units Sold	Number of Days	x	$f(x)$
0	80	0	.40 = 80/200
1	50	1	.25
2	40	2	.20
3	10	3	.05
4	20	4	.10
	200		1.00

Discrete Probability Distributions

- Example: JSL Appliances



Graphical representation of probability distribution

Discrete Probability Distributions

- In addition to tables and graphs, a formula that gives the probability function, $f(x)$, for every value of x is often used to describe the probability distributions.
- Several discrete probability distributions specified by formulas are the discrete-uniform (離散均勻), binomial (二項), negative binomial (負二項), geometric (幾何), Poisson (卜瓦松), and hypergeometric (超幾何) distributions.

Expected Value

- The expected value, or mean, of a random variable is a measure of its central location.

$$E(x) = \mu = \sum xf(x)$$

- The expected value is a weighted average of the values the random variable may assume. The weights are the probabilities.
- The expected value does not have to be a value the random variable can assume.

Variance and Standard Deviation

- The variance summarizes the variability in the values of a random variable.

$$Var(x) = \sigma^2 = \Sigma(x - \mu)^2 f(x)$$

- The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities.
- The standard deviation, σ, is defined as the positive square root of the variance.

Expected Value

- Example: JSL Appliances

x	$f(x)$	$xf(x)$
0	.40	.00
1	.25	.25
2	.20	.40
3	.05	.15
4	.10	.40

$E(x) =$ 1.20 = expected number of TVs sold in a day

Variance

- Example: JSL Appliances

x	$x - \mu$	$(x - \mu)^2$	$f(x)$	$(x - \mu)^2 f(x)$
0	-1.2	1.44	.40	.576
1	-0.2	0.04	.25	.010
2	0.8	0.64	.20	.128
3	1.8	3.24	.05	.162
4	2.8	7.84	.10	.784

Variance of daily sales = σ^2 = 1.660

Standard deviation of daily sales = 1.2884 TVs

Example: Carolina Industries

The demand for a product of Carolina Industries varies greatly from month to month. The probability distribution in the following table, based on the past two years of data, shows the company's monthly demand.

Unit Demand	Probability
300	.20
400	.30
500	.35
600	.15

Assume that each unit demanded generates $70 in revenue and that each unit ordered costs $50. How much will the company gain or lose in a month if it places an order based expected value of the monthly demand and the actual demand for the item is 300 units?

Example: Housing and Vacancy Survey

The New York City Housing and Vacancy Survey showed a total of 59,324 rent-controlled housing units and 236,263 rent-stabilized units built in 1947 or later. For these rental units, the probability distributions for the number of persons living in the unit are given (U.S. Census Bureau website, January 12, 2004).

a. What is the expected value of the number of persons living in each type of unit?

b. What is the variance of the number of persons living in each type of unit?

c. Make some comparisons between the number of persons living in rent-controlled units and the number of persons living in rent-stabilized units.

Number of Persons	Rent-Controlled	Rent-Stabilized
1	.61	.41
2	.27	.30
3	.07	.14
4	.04	.11
5	.01	.03
6	.00	.01

Discrete Uniform Probability Distribution

- The discrete uniform probability distribution is the simplest example of a discrete probability distribution given by a formula.

- The discrete uniform probability function is

$$f(x) = 1/n$$

where: n = the number of values the random variable may assume

- The values of the random variable are equally likely.

Number Obtained x	Probability of x $f(x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Bivariate Distributions (二元分配)

A probability distribution involving two random variables is called a bivariate probability distribution.

Each outcome of a bivariate experiment consists of two values, one for each random variable.

When dealing with bivariate probability distributions, we are often interested in the relationship between the random variables.

A Bivariate Discrete Probability Distribution

A company asked 200 of its employees how they rated their benefit package and job satisfaction. The crosstabulation below shows the ratings data.

Benefits Package (x)	Job Satisfaction (y) 1	2	3	Total
1	28	26	4	58
2	22	42	34	98
3	2	10	32	44
Total	52	78	70	200

The bivariate empirical discrete probabilities for benefits rating and job satisfaction are shown below.

Benefits Package (x)	Job Satisfaction (y) 1	2	3	Total
1	.14	.13	.02	.29
2	.11	.21	.17	.49
3	.01	.05	.16	.22
Total	.26	.39	.35	1.00

Expected Value and Variance for Benefits Package, x

(1)	(2)	(3) = (1)*(2)	(4)	(5) = $(4)^2$	(6) = (5)*(2)
x	$f(x)$	$xf(x)$	$x - E(x)$	$(x - E(x))^2$	$(x - E(x))^2 f(x)$
1	0.29	0.29	-0.93	0.8649	0.250821
2	0.49	0.98	0.07	0.0049	0.002401
3	0.22	0.66	1.07	1.1449	0.251878
	$E(x)$ =	1.93		$Var(x)$ =	0.505100

Expected Value and Variance for Job Satisfaction, y

(1)	(2)	(3) = (1)*(2)	(4)	(5) = (4)²	(6) = (5)*(2)
y	$f(y)$	$yf(y)$	$y - E(y)$	$(y - E(y))^2$	$(y - E(y))^2 f(y)$
1	0.26	0.26	-1.09	1.1881	0.308906
2	0.39	0.78	-0.09	0.0081	0.003159
3	0.35	1.05	0.91	0.8281	0.289835
	$E(y)$ =	2.09		$Var(y)$ =	0.601900

Expected Value and Variance for Bivariate Distrib.

$s = x + y$					
(1)	(2)	(3) = (1)*(2)	(4)	(5) = (4)^2	(6) = (5)*(2)
s	$f(s)$	$sf(s)$	$s - E(s)$	$(s - E(s))^2$	$(s - E(s))^2 f(s)$
2	0.14	0.28	-2.02	4.0804	0.571256
3	0.24	0.72	-1.02	1.0404	0.249696
4	0.24	0.96	-0.02	0.0004	0.000960
5	0.22	1.10	0.98	0.9604	0.211376
6	0.16	0.96	1.98	3.9204	0.627264
	$E(s)$ =	4.02		$Var(s)$ =	1.660552

Benefits Package (x)	Job Satisfaction (y) 1	2	3	Total
1	0.14	0.13	0.02	0.29
2	0.11	0.21	0.17	0.49
3	0.01	0.05	0.16	0.22
Total	0.26	0.39	0.35	1.00

A Bivariate Discrete Probability Distribution

Covariance for Random Variables x and y

$$\sigma_{xy} = [Var(x + y) - Var(x) - Var(y)]/2$$

$$\sigma_{xy} = [1.660552 - 0.5051 - 0.6019]/2 = 0.276776$$

$$or\ \sigma_{xy} = \sum_{i.j}[x_i - E(\,x_i)][y_j - E(y_j)]f(x_i, y_j)$$

$$\sigma_{xy} = \sum_{i.j}[x_i - E(\, x_i)][y_j - E(y_j)]f\,(x_i, y_j)$$

Benefits Package (x)	Job Satisfaction (y) 1	2	3	Total
1	.14	.13	.02	.29
2	.11	.21	.17	.49
3	.01	.05	.16	.22
Total	.26	.39	.35	1.00

A Bivariate Discrete Probability Distribution

Correlation Between Variables x and y

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{0.5051} = 0.7107038$$

$$\sigma_y = \sqrt{0.6019} = 0.7758221$$

$$\sigma_{xy} = 0.276776$$

$$\rho_{xy} = \frac{0.276776}{0.7107038 \times 0.7758221} = 0.50197$$

Financial applications

$E(x)$ = .10(-40) + .25(5) + .5(15) + .15(30) = 9.25

$E(y)$ = .10(30) + .25(5) + .5(4) + .15(2) = 6.55

$Var(x) = .1(-40-9.25)^2 + .25(5-9.25)^2 + .5(15-9.25)^2 + .15(30-9.25)^2 = 328.1875$

$Var(y)$ = 61.9475

Economic scenario	Probability $f(x, y)$	Large-Cap Stock Fund (x)	Long-Term Government Bond Fund (y)
Recession	.10	-40	30
Weak growth	.25	5	5
Stable growth	.50	15	4
Strong growth	.15	30	2

Financial application

- Expected value of a linear combination of random variables x and y

  $E(ax + by) = aE(x) + bE(y)$

  $E(.5x + .5y) = .5E(x) + .5E(y) = .5(9.25) + .5(6.55) = 7.9$

- Variance of linear combination of two random variables

  $Var(ax + by) = a^2 Var(x) + b^2 Var(y) + 2ab\sigma_{xy}$

  where σ_{xy} is the covariance of x and y

  $Var(.5x + .5y) = .5^2(328.1875) + .5^2\,(61.9475) + 2(.5)(.5)(-135.3375) = 29.865$

  $\sigma_{.5x+.5y} = \sqrt{29.865} = 5.4650\%$

Covariance

Economic scenario	Probability $f(x, y)$	$(x + y)$	Large-Cap Stock Fund (x)	Long-Term Government Bond Fund (y)
Recession	.10	-10	-40	30
Weak growth	.25	10	5	5
Stable growth	.50	19	15	4
Strong growth	.15	32	30	2

$E(x + y) = 9.25 + 6.55 = 15.8$

$Var(x + y) = .1(-10-15.8)^2 + .25(10-15.8)^2 + .5(19-15.8)^2 + .15(32-15.8)^2 = 119.46$

$\sigma_{xy} = [Var(x + y) - Var(x) - Var(y)]/2$

$= [119.46 - 328.1875 - 61.9475]/2$

$= -135.3375$

Investment alternatives

Investment alternative	Expected Return (%)	Variance of Return	Standard Deviation of Return (%)
100% in Stock Fund	9.25	328.1875	18.1159
100% in Bond Fund	6.55	61.9475	7.8707
Portfolio (50% in Stock fund, 50% in Bond fund)	7.90	29.865	5.4650

Portfolio risk

	B	S
B	$(w_B\sigma_B)(w_B\sigma_B)$	$(w_B\sigma_B)(w_S\sigma_S)\ \rho_{BS}$
S	$(w_S\sigma_S)(w_B\sigma_B)\rho_{SB}$	$(w_S\sigma_S)(w_S\sigma_S)$

	B	S
B	$(w_B\sigma_B)^2$	$w_B w_S cov(B,S)$
S	$w_S w_B cov(S,B)$	$(w_S\sigma_S)^2$

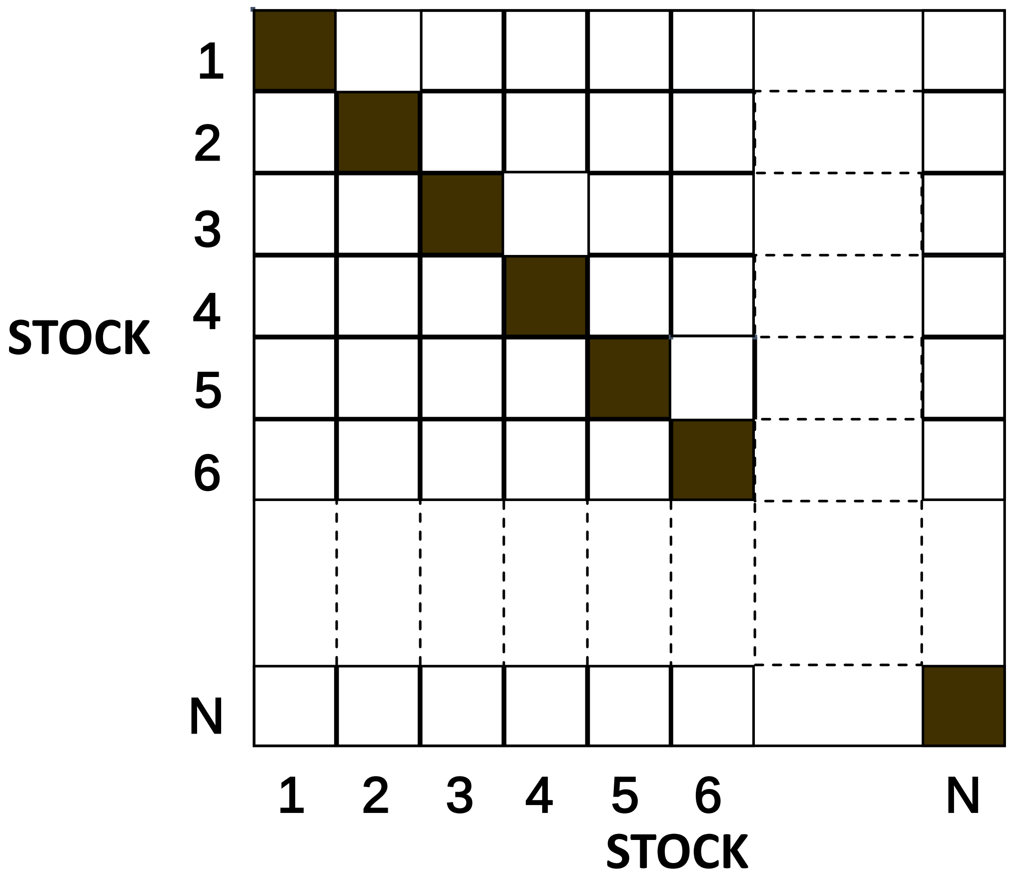
Portfolio risk

S1	$(w_{S1}\sigma_{S1})^2$	$w_{S1}w_{S2}cov(S1,S2)$	$w_{S1}w_{S3}cov(S1,S3)$
S2	$w_{S2}w_{S1}cov(S2,S1)$	$(w_{S2}\sigma_{S2})^2$	$w_{S2}w_{S3}cov(S2,S3)$
S3	$w_{S3}w_{S1}cov(S3,S1)$	$w_{S3}w_{S2}cov(S3,S2)$	$(w_{S3}\sigma_{S3})^2$
	S1	**S2**	**S3**

S1	$(w_{S1}\sigma_{S1})^2$	$w_{S1}w_{S2}cov(S1,S2)$	$w_{S1}w_{S3}cov(S1,S3)$	**…**
S2	$w_{S2}w_{S1}cov(S2,S1)$	$(w_{S2}\sigma_{S2})^2$	$w_{S2}w_{S3}cov(S2,S3)$	
S3	$w_{S3}w_{S1}cov(S3,S1)$	$w_{S3}w_{S2}cov(S3,S2)$	$(w_{S3}\sigma_{S3})^2$	
…	**…**			**…**
	S1	**S2**	**S3**	**…**

Portfolio risk

- The shaded boxes contain variance terms; the remainder contain covariance terms.
- To calculate portfolio variance add up the boxes.
- The variance of the return on a portfolio with many securities is more dependent on the covariances between the individual securities than on the variances of the individual securities



Example: Diversified investment portfolio

The Knowles/Armitage (KA) group at Merrill Lynch advises clients on how to create a diversified investment portfolio. (All World Fund composed of global stocks with good dividend yields)

a. Which of the funds would be considered the more risky? Why?

	All World Fund	Treasury bond fund
Expected Return (%)	7.80	5.50
Standard Deviation of Return (%)	18.90	4.60
Covariance (σ_{xy})	-12.4	

Example: Diversified investment portfolio

b. If KA recommends that the client invest 75% in the All World Fund and 25% in the treasury bond fund, what is the expected percent return and standard deviation for such a portfolio? What would be the expected return and standard deviation, in dollars, for a client investing $10,000 in such a portfolio?

c. If KA recommends that the client invest 25% in the All World Fund and 75% in the treasury bond fund, what is the expected return and standard deviation for such a portfolio? What would be the expected return and standard deviation, in dollars, for a client investing $10,000 in such a portfolio?

d. Which of the portfolios in parts (b) and (c) would you recommend for an aggressive investor? Which would you recommend for a conservative investor? Why?

	All World Fund	Treasury bond fund
Expected Return (%)	7.80	5.50
SD (%)	18.90	4.60
σ_{xy}	-12.4	

Binomial Probability Distribution

- Four Properties of a Binomial Experiment

  1. The experiment consists of a sequence of n identical trials.
  2. Two outcomes, success and failure, are possible on each trial.
  3. The probability of a success, denoted by p, does not change from trial to trial. (This is referred to as the stationarity assumption.)
  4. The trials are independent.

Binomial Probability Distribution

- Our interest is in the number of successes occurring in the n trials.
- Let x denote the number of successes occurring in the n trials.

Binomial Probability Distribution

- Example: Evans Electronics

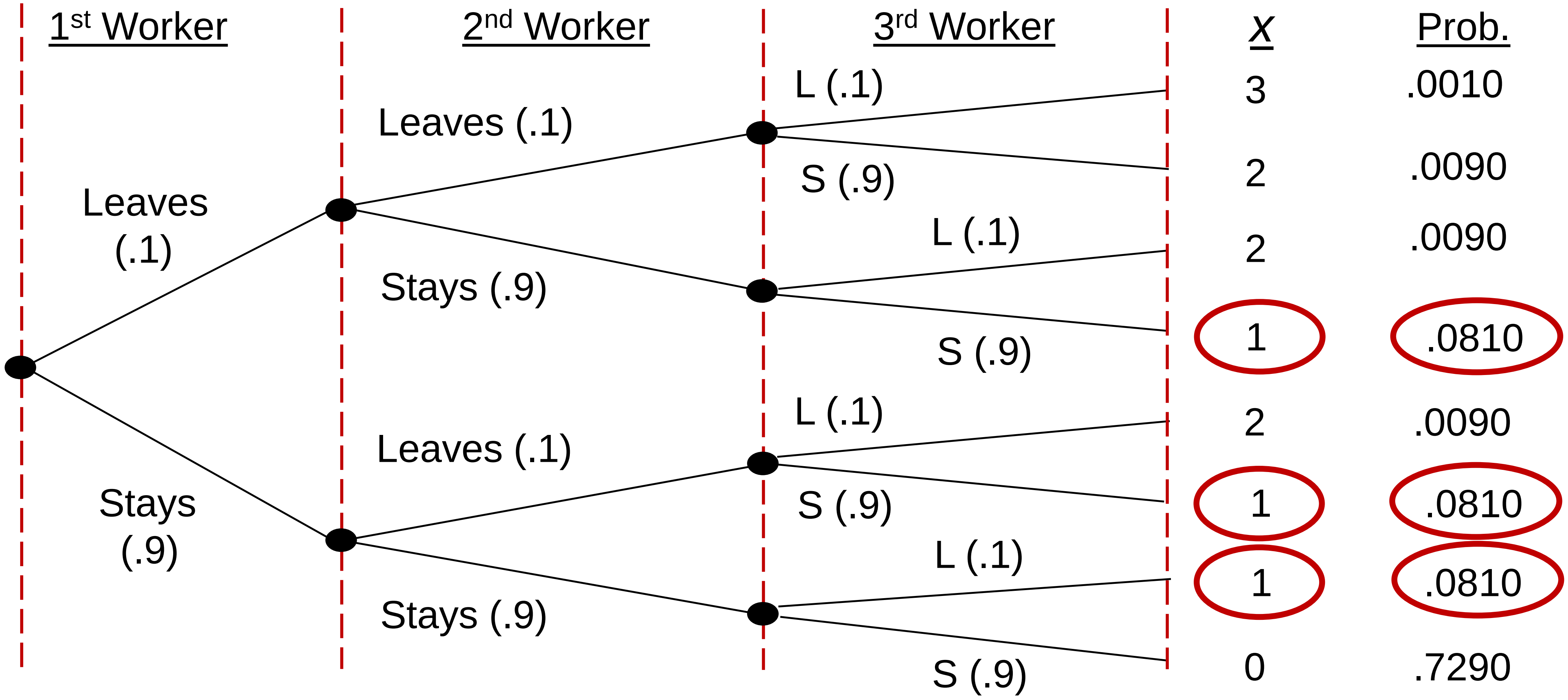
Evans Electronics is concerned about a low retention rate for its employees. In recent years, management has seen a turnover of 10% of the hourly employees annually.

Thus, for any hourly employee chosen at random, management estimates a probability of 0.1 that the person will not be with the company next year.

Choosing 3 hourly employees at random, what is the probability that 1 of them will leave the company this year?

Binomial Probability Distribution

- Example: Evans Electronics



Binomial Probability Distribution

- Example: Evans Electronics
  - The probability of the first employee leaving and the second and third employees staying, denoted (S, F, F), is given by

$$p(1 - p)(1 - p)$$

  - With a .10 probability of an employee leaving on any one trial, the probability of an employee leaving on the first trial and not on the second and third trials is given by

$$(.10)(.90)(.90) = (.10)(.90)^2 = .081$$

Binomial Probability Distribution

- Example: Evans Electronics
  - Two other experimental outcomes result in one success and two failures. The probabilities for all three experimental outcomes involving one success follow.

Experimental Outcome	Probability of Experimental Outcome
(S, F, F)	$p(1 - p)(1 - p) = (.1)(.9)(.9) = .081$
(F, S, F)	$(1 - p)p(1 - p) = (.9)(.1)(.9) = .081$
(F, F, S)	$(1 - p)(1 - p)p = (.9)(.9)(.1) = \underline{.081}$
	Total = .243

Binomial Probability Distribution

- Example: Evans Electronics

Using the probability function:

Let: p = .10, n = 3, x = 1

$$f(x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{(n-x)}$$

$$f(1) = \frac{3!}{1!(3-1)!} (0.1)^1 (0.9)^2 = .243$$

$n!$: n factorial

Binomial Probability Distribution

- Binomial Probability Function

$$f(x) = \frac{n!}{x!\,(n-x)!}p^{x}(1-p)^{(n-x)}$$

where:

x = the number of successes

p = the probability of a success on one trial

n = the number of trials

$f(x)$ = the probability of x successes in n trials

$n! = n(n-1)(n-2) \ldots\ldots (2)(1)$

Binomial Probability Distribution

- Binomial Probability Function

$$f(x) = \frac{n!}{x!\,(n-x)!}\; p^x\,(1-p)^{(n-x)}$$

Number of experimental outcomes providing exactly *x* successes in *n* trials

Probability of a particular sequence of trial outcomes with *x* successes in *n* trials

Binomial Probabilities and Cumulative Probabilities

- Statisticians have developed tables that give probabilities and cumulative probabilities for a binomial experiment random variable .
- These tables can be found in some statistics textbooks.
- With modern calculators and the capability of statistical software packages, such tables are almost unnecessary.

Probability Distributions

Matthew Bognar Education

Binomial Probability Distribution

- Using Tables of Binomial Probabilities

		p									
n	x	0.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
3	0	.8574	.7290	.6141	.5120	.4219	.3430	.2746	.2160	.1664	.1250
	1	.1354	.2430	.3251	.3840	.4219	.4410	.4436	.4320	.4084	.3750
	2	.0071	.0270	.0574	.0960	.1406	.1890	.2389	.2880	.3341	.3750
	3	.0001	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250

Binomial Probability Distribution

- Expected Value

$$E(x) = \mu = np$$

- Variance

$$Var(x) = \sigma^2 = np(1 - p)$$

- Standard Deviation

$$\sigma = \sqrt{np(1-p)}$$

Binomial Probability Distribution

- Example: Evans Electronics
  - Expected Value

$$E(x) = np = 3(.1) = .3 \text{ employees out of } 3$$

  - Variance

$$Var(x) = np(1 - p) = 3(.1)(.9) = .27$$

  - Standard Deviation

$$\sigma = \sqrt{3(.1)(.9)} = .52 \text{ employees}$$

FIGURE 5.3 TREE DIAGRAM FOR THE MARTIN CLOTHING STORE PROBLEM

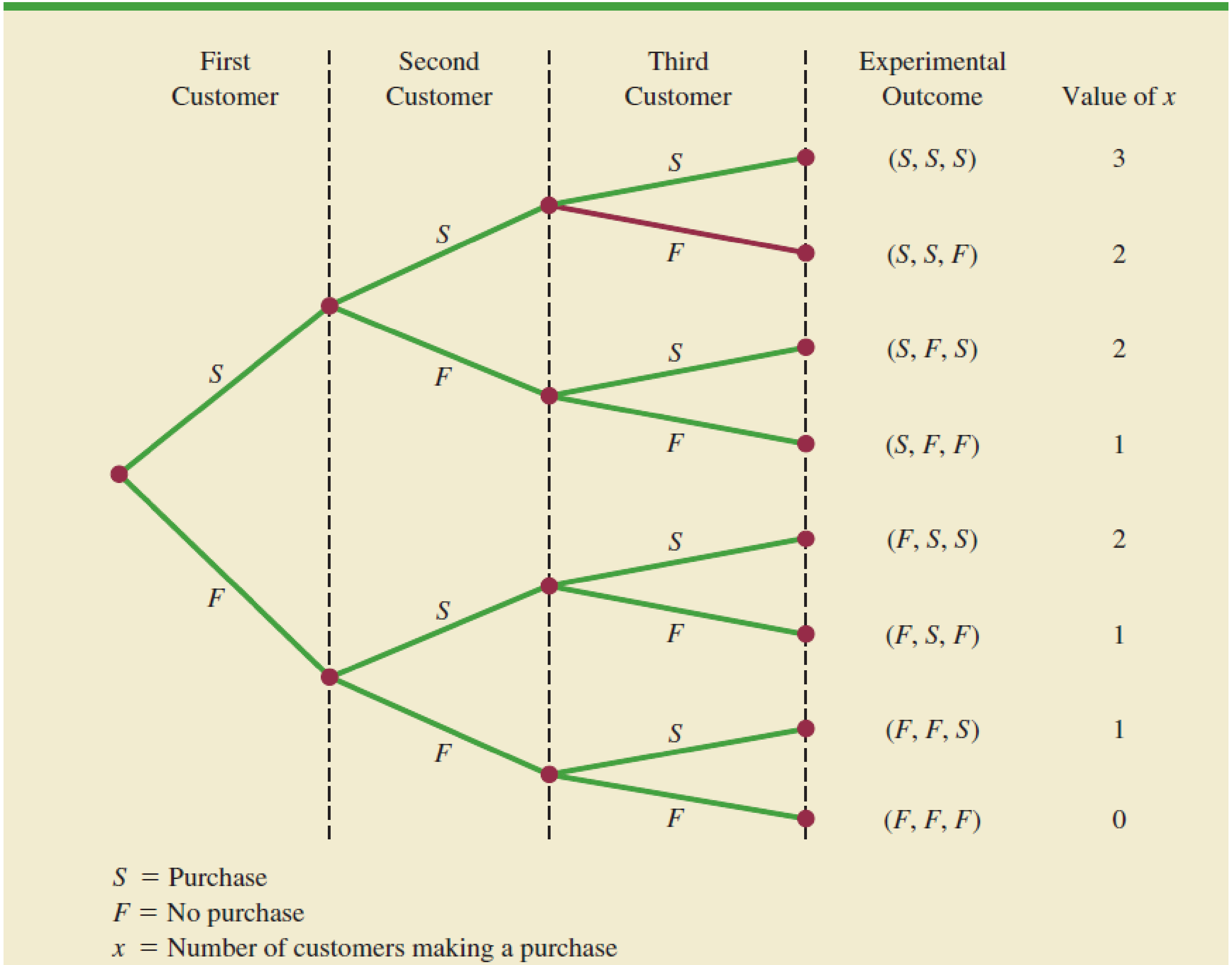
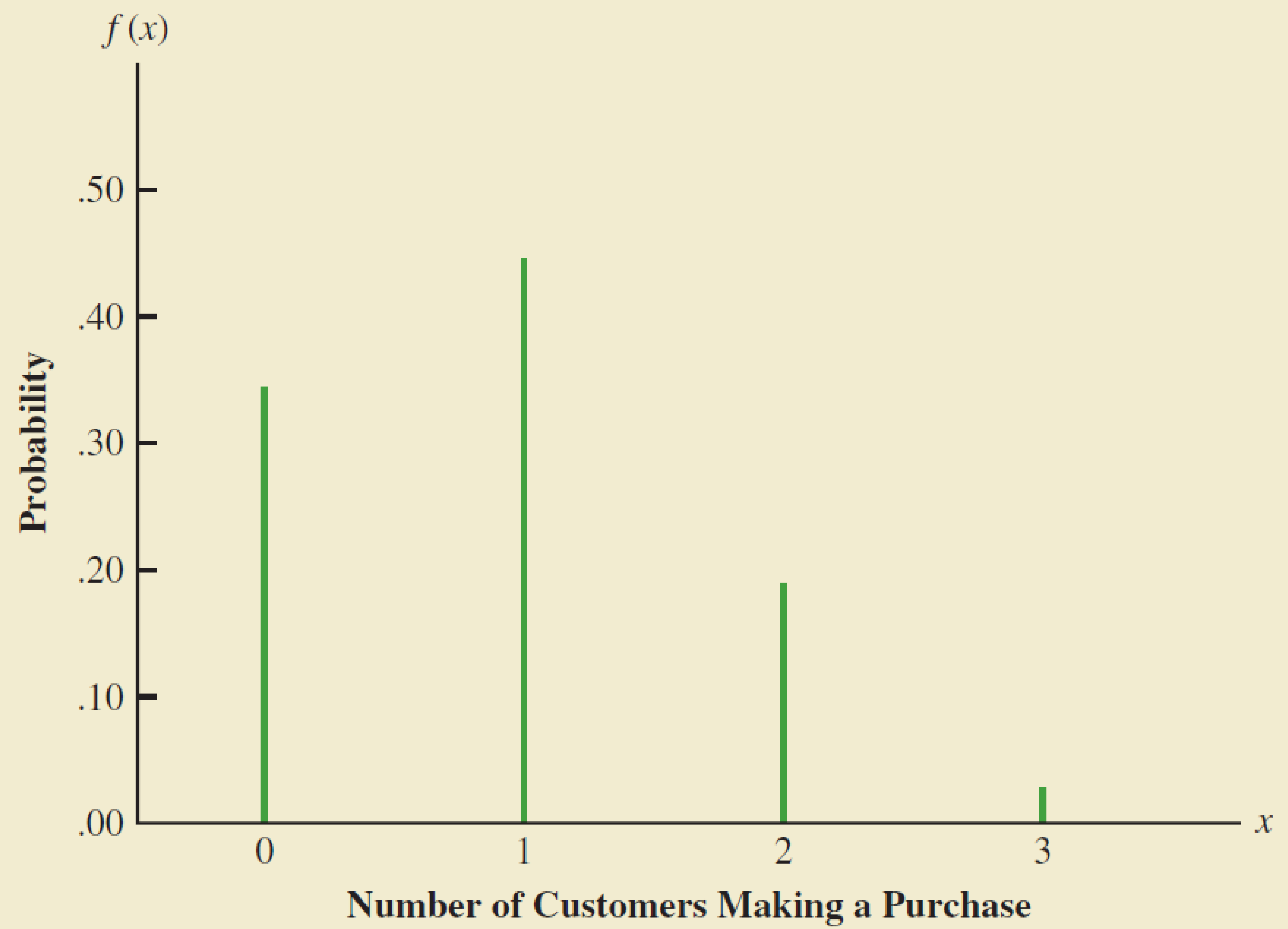


TABLE 5.13 PROBABILITY DISTRIBUTION FOR THE NUMBER OF CUSTOMERS MAKING A PURCHASE

x	$f(x)$
0	$\frac{3!}{0!3!}(.30)^0(.70)^3 = .343$
1	$\frac{3!}{1!2!}(.30)^1(.70)^2 = .441$
2	$\frac{3!}{2!1!}(.30)^2(.70)^1 = .189$
3	$\frac{3!}{3!0!}(.30)^3(.70)^0 = .027$
	1.000

FIGURE 5.4 GRAPHICAL REPRESENTATION OF THE PROBABILITY DISTRIBUTION FOR THE NUMBER OF CUSTOMERS MAKING A PURCHASE



Expected value and variance for Martin clothing store

$E(x) = np = 3(.30) = .9$

$\sigma^2 = np(1-p) = 3(.3)(.7) = .63$

$\sigma = .79$

Example: ETF

According to a survey conducted by TD Ameritrade, one out of four investors have exchange-traded funds in their portfolios (*USA Today,* January 11, 2007). Consider a sample of 20 investors.

a. The probability that exactly 4 investors have exchange-traded funds in their portfolios.

b. The probability that at least 2 of the investors have exchange-traded funds in their portfolios.

c. If you found that exactly 12 of the investors have exchange-traded funds in their portfolios, would you doubt the accuracy of the survey results?

d. The expected number of investors who have exchange-traded funds in their portfolios.

Negative Binomial Probability Distribution

- Four Properties of a Negative Binomial Experiment

   1. The experiment consists of a sequence of x identical trials.

   2. Two outcomes, success and failure, are possible on each trial.

   3. The probability of a success, denoted by p, does not change from trial to trial.

   4. The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

   5. The experiment continues until r successes are observed, where r is specified in advance.

Negative Binomial Probability Distribution

- The **negative binomial probability** refers to the probability that a negative binomial experiment results in *r* - 1 successes after trial *x* - 1 <u>and</u> *r* successes after trial *x*.

Negative Binomial Probability Function:

$$f(x) = C_{x-r}^{x-1}(1-p)^{x-r}p^r, \qquad x = r, r+1, r+2, \ldots$$

where:

x = the number of trials

r = the number of successes

p = the probability of a success on one trial

Negative Binomial Probability Distribution

- Suppose we flip a coin repeatedly and count the number of heads (successes). If we continue flipping the coin until it has landed 2 times (r = 2) on heads.
- The number of coin flips is a random variable that can take on any integer value between 2 and plus infinity (x = 2, 3, 4,…).

$$f(2) = C_{2-2}^{2-1}(1-0.5)^{2-2}0.5^2 = 0.25$$

Number of heads (successes), r	Number of coin flips, x	$f(x)$
2	2	0.25
	3	0.25
	4	0.1875
	5	0.125
	6	0.078125
	7 or more	0.109375

Negative Binomial Probability Distribution

- Expected Value

$$E(x) = \mu = \frac{r}{p}$$

- Variance

$$Var(x) = \sigma^2 = \frac{r(1-p)}{p^2}$$

Geometric Probability Distribution

The three assumptions are:

- There are two possible outcomes for each trial (success or failure).
- The trials are independent.
- The probability of success is the same for each trial.

Geometric Probability Function:

$$f(x) = (1-p)^{x-1}p, \qquad x = \mathbf{1}, 2,3, \ldots$$

where:

x = the number of trials required until the first success occurs
p = the probability of a success on one trial

Geometric Probability Distribution

For example, suppose you want to flip a coin until the first heads turns up. What is the probability if it takes four flips for the first heads to occur (that is, three tails followed by one heads)?

In this example, x = 4 and p = 0.5:

$$f(4) = (1 - 0.5)^{4-1}(0.5) = (0.125)(0.5) = 0.0625$$

x	$f(x)$	$(1-p)^{x-1}p$
1	$f(1)$	$(1-0.5)^{1-1}(0.5) = 0.5$
2	$f(2)$	$(1-0.5)^{2-1}(0.5) = 0.25$
3	$f(3)$	$(1-0.5)^{3-1}(0.5) = 0.125$
4	$f(4)$	$(1-0.5)^{4-1}(0.5) = 0.0625$
5	$f(5)$	$(1-0.5)^{5-1}(0.5) = 0.03125$

Geometric Probability Distribution

- Expected Value

$$E(x) = \mu = \frac{1}{p}$$

- Variance

$$Var(x) = \sigma^2 = \frac{(1-p)}{p^2}$$

Negative Binomial Probability Function:

$$f(x) = C_{x-r}^{x-1}(1-p)^{x-r}p^r, \qquad x = r, r+1, r+2, \ldots$$

where:

x = the number of trials

r = the number of successes

p = the probability of a success on one trial

Geometric Probability Function:

$$f(x) = (1-p)^{x-1}p, \qquad x = 1,2,3, \ldots$$

where:

x = the number of trials required until the first success occurs

p = the probability of a success on one trial

Negative Binomial Probability Function:

$$f(x) = C_{x-1}^{x-1}(1-p)^{x-1}p^{1}, \qquad x = 1, 1+1, 1+2, \ldots$$

where:

x = the number of trials

r = 1, the number of successes

p = the probability of a success on one trial

Geometric Probability Function:

$$f(x) = (1-p)^{x-1}p, \qquad x = 1,2,3, \ldots$$

where:

x = the number of trials required until the first success occurs

p = the probability of a success on one trial

Geometric Probability Distribution

- The **geometric distribution** is a special case of the negative binomial distribution.
- It deals with the number of trials required for a single success.
- The geometric distribution is negative binomial distribution where the number of successes (*x*) is equal to 1.

Poisson Probability Distribution

- A Poisson distributed random variable is often useful in estimating the number of occurrences over a specified interval of time or space.

- It is a discrete random variable that may assume an infinite sequence of values ($x = 0, 1, 2, \ldots$).

- Examples of Poisson distributed random variables:

    - number of vehicles arriving at a toll booth in one hour

- Bell Labs used the Poisson distribution to model the arrival of phone calls.

Poisson Probability Distribution

- Two Properties of a Poisson Experiment

  1. The probability of an occurrence is the same for any two intervals of equal length.
  2. The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.

Poisson Probability Distribution

- Poisson Probability Function

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where:

x = the number of occurrences in an interval

$f(x)$ = the probability of x occurrences in an interval

μ = mean number of occurrences in an interval

e = 2.71828

$x! = x(x-1)(x-2)\ldots(2)(1)$

Poisson Probability Distribution

- Poisson Probability Function–
  - Since there is no stated upper limit for the number of occurrences, the probability function $f(x)$ is applicable for values $x = 0, 1, 2, \ldots$ without limit.
  - In practical applications, x will eventually become large enough so that $f(x)$ is approximately zero and the probability of any larger values of x becomes negligible.

Poisson Probability Distribution

- Example: Mercy Hospital

Patients arrive at the emergency room of Mercy Hospital at the average rate of 6 per hour on weekend evenings.

What is the probability of 4 arrivals in 30 minutes on a weekend evening?

Poisson Probability Distribution

- Example: Mercy Hospital

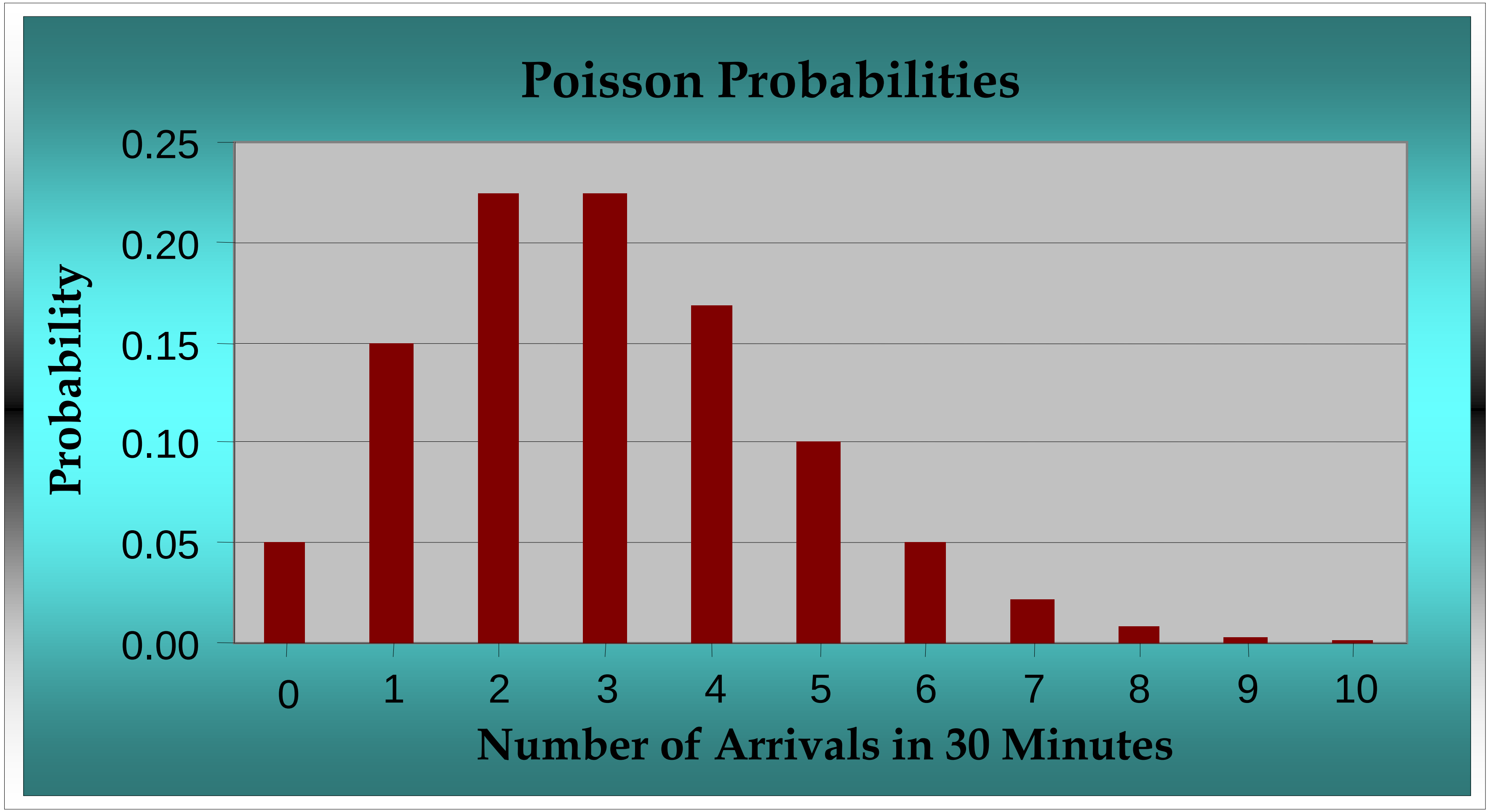
Using the probability function:

μ = 6/hour = 3/half-hour, x = 4

$$f(4) = \frac{3^4(2.71828)^{-3}}{4!} = .1680$$

Poisson Probability Distribution

- Example: Mercy Hospital



Poisson Probability Distribution

- A property of the Poisson distribution is that the mean and variance are equal.

$$\mu = \sigma^2$$

- Example: Mercy Hospital

Variance for Number of Arrivals during 30-Minute periods

$$\mu = \sigma^2 = 3$$

Example: Aircraft accidents

An average of 15 aircraft accidents occur each year (*The World Almanac and Book of Facts*, 2004).

a. The mean number of aircraft accidents per month.
b. The probability of no accidents during a month.
c. The probability of exactly one accident during a month.
d. The probability of more than one accident during a month.

Hypergeometric Probability Distribution

- The hypergeometric distribution is closely related to the binomial distribution.
- However, for the hypergeometric distribution:
    - the trials are not independent, and
    - the probability of success changes from trial to trial.

Hypergeometric Probability Distribution

- Hypergeometric Probability Function

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

where: *x* = number of successes

n = number of trials

f(*x*) = probability of *x* successes in *n* trials

N = number of elements in the population

r = number of elements in the population labeled success

Hypergeometric Probability Distribution

- Hypergeometric Probability Function

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \quad \text{for } 0 \le x \le r$$

number of ways
x successes can be selected
from a total of *r* successes
in the population

number of ways
n – *x* failures can be selected
from a total of *N* – *r* failures
in the population

number of ways
n elements can be selected
from a population of size *N*

Hypergeometric Probability Distribution

- Hypergeometric Probability Function

  - The probability function $f(x)$ is usually applicable for values of x = 0, 1, 2, … n.

  - However, only values of x where:
    1) $x \leq r$ and
    2) $n - x \leq N - r$ are valid.

  - If these two conditions do not hold for a value of x, the corresponding $f(x)$ equals 0.

Hypergeometric Probability Distribution

- Example: Neveready's Batteries

Bob Neveready has removed two dead batteries from a flashlight and inadvertently mingled them with the two good batteries he intended as replacements. The four batteries look identical.

Bob now randomly selects two of the four batteries. What is the probability he selects the two good batteries?

Hypergeometric Probability Distribution

- Example: Neveready's Batteries

Using the probability function:

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{\binom{2}{2}\binom{2}{0}}{\binom{4}{2}} = \frac{\left(\frac{2!}{2!0!}\right)\left(\frac{2!}{0!2!}\right)}{\left(\frac{4!}{2!2!}\right)} = \frac{1}{6} = .167$$

where: *x* = 2 = number of <u>good</u> batteries selected

n = 2 = number of batteries selected

N = 4 = number of batteries in total

r = 2 = number of <u>good</u> batteries in total

Hypergeometric Probability Distribution

- Mean

$$E(x) = \mu = n\left(\frac{r}{N}\right)$$

- Variance

$$Var(x) = \sigma^2 = n\left(\frac{r}{N}\right)\left(1 - \frac{r}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Hypergeometric Probability Distribution

- Example: Neveready's Batteries
  - Mean

$$\mu = n\left(\frac{r}{N}\right) = 2\left(\frac{2}{4}\right) = 1$$

  - Variance

$$\sigma^2 = 2\left(\frac{2}{4}\right)\left(1 - \frac{2}{4}\right)\left(\frac{4-2}{4-1}\right) = \frac{1}{3} = .333$$

Hypergeometric Probability Distribution

- Consider a hypergeometric distribution with n trials and let $p = (r/N)$ denote the probability of a success on the first trial.
- If the population size is large, the term $(N – n)/(N – 1)$ approaches 1.
- The expected value and variance can be written $E(x) = np$ and $Var(x) = np(1 – p)$.
- Note that these are the expressions for the expected value and variance of a binomial distribution.
- When the population size is large, a hypergeometric distribution can be approximated by a binomial distribution with n trials and a probability of success $p = (r/N)$.

Example: TARP

The Troubled Asset Relief Program (TARP, 問題(不良)資產紓困計畫), passed by the U.S. Congress in October 2008, provided $700 billion in assistance for the struggling U.S. economy. Over $200 billion was given to troubled financial institutions with the hope that there would be an increase in lending to help jump-start the economy. But three months later, a Federal Reserve survey found that two-thirds of the banks that had received TARP funds had tightened terms for business loans (*The Wall Street Journal,* February 3, 2009). Of the 10 banks that were the biggest recipients of TARP funds, only 3 had actually increased lending during this period.

Example: TARP

Increased Lending	Decreased Lending
BB&T Sun Trust Banks U.S. Bancorp	Bank of America Capital One Citigroup Fifth Third Bancorp J.P. Morgan Chase Regions Financial Wells Fargo

Example: TARP

Assume that you will randomly select 3 of these 10 banks for a study that will continue to monitor bank lending practices. Let x be a random variable indicating the number of banks in the study that had increased lending.

a. What is $f(0)$? What is your interpretation of this value?
b. What is $f(3)$? What is your interpretation of this value?
c. Compute $f(1)$ and $f(2)$. Show the probability distribution for the number of banks in the study that had increased lending. What value of x has the highest probability?
d. What is the probability that the study will have at least one bank that had increased lending?
e. Compute the expected value, variance, and standard deviation for the random variable.