Lagrangian duality

Let us consider the following optimisation problem with constraints :

$$\begin{aligned}&\text{minimize}_{x\in\mathbb{R}^n} f(x)\\ &\text{subject to: } g_j(x) \le 0, \quad \forall j \in \{1,\dots,p\}\end{aligned}$$

We assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, $g_j : \mathbb{R}^n \to \mathbb{R}$ is convex and finite-valued for all j and that there exists $x_0 \in \operatorname{dom} f$ such that $g_j(x_0) < 0$ for all j.
We define the Lagrangian of the optimization problem as

$$\begin{aligned}L :\ &\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty, +\infty\}\\ &(x, \phi) \mapsto f(x) + \langle \phi, g(x)\rangle - \iota_{\mathbb{R}^p_+}(\phi)\end{aligned}$$

Question 1. Show that

$$\inf_{x\in\mathbb{R}^n} \Big\{ f(x) \text{ subject to: } g_j(x) \le 0, \quad \forall j \in \{1,\dots,p\}\Big\} = \inf_{x\in\mathbb{R}^n} f(x) + \iota_{\mathbb{R}^p_-}(g(x))$$

and that $h : x \mapsto f(x) + \iota_{\mathbb{R}^p_-}(g(x))$ is convex.

Question 2. Show that

$$f(x) + \iota_{\mathbb{R}^p_-}(g(x)) = \sup_{\phi\in\mathbb{R}^p} L(x, \phi)$$

Question 3. Show that

$$\inf_{x\in\mathbb{R}^n} \sup_{\phi\in\mathbb{R}^p} L(x, \phi) \ge \sup_{\phi\in\mathbb{R}^p} \inf_{x\in\mathbb{R}^n} L(x, \phi)$$

This property is called the *weak duality* property. The problem $\text{minimize}_{x\in\mathbb{R}^n} \big(\sup_{\phi\in\mathbb{R}^p} L(x, \phi)\big)$ is the primal problem, the problem $\text{maximize}_{\phi\in\mathbb{R}^p} \big(\inf_{x\in\mathbb{R}^n} L(x, \phi)\big)$ is the dual problem.

We say that (x^*, ϕ^*) is a *saddle point* of L if

$$\forall x \in \mathbb{R}^n, \forall \phi \in \mathbb{R}^p, \quad L(x^*, \phi) \le L(x^*, \phi^*) \le L(x, \phi^*).$$

Question 4. Show that if (x^*, ϕ^*) is a saddle point of L then

$$L(x^*, \phi^*) = \sup_{\phi\in\mathbb{R}^p} \inf_{x\in\mathbb{R}^n} L(x, \phi) = \inf_{x\in\mathbb{R}^n} \sup_{\phi\in\mathbb{R}^p} L(x, \phi)$$

In that case, the *duality gap*

$$G = \inf_{x\in\mathbb{R}^n} \sup_{\phi\in\mathbb{R}^p} L(x, \phi) - \sup_{\phi\in\mathbb{R}^p} \inf_{x\in\mathbb{R}^n} L(x, \phi)$$

is equal to 0.

Question 5. Show that (x^*, ϕ^*) is a saddle point of L if and only if

$$\begin{cases} 0 \in \partial_x L(x^*, \phi^*) \\ 0 \in \partial_\phi(-L)(x^*, \phi^*) \end{cases}$$

where $\partial_\phi(-L)(x^*, \phi^*)$ means the sub-differential of the function $\big(\phi \mapsto -L(x^*, \phi)\big)$ at ϕ^*.

Question 6. Calculate $\partial_\phi(-L)$ and deduce that (x^*, λ) is a saddle point of L if and only if

$$\begin{cases} \phi^* \geq 0 \\ g(x^*) \leq 0 \\ \phi_j^* g_j(x^*) = 0, \forall j \\ 0 \in \partial f(x^*) + \sum_{j=1}^p \phi_j^* \partial g_j(x^*) \end{cases}$$

You may use the results of Exercises 6 and 7 in Exercise sheet # 1.

These conditions are called the Karush-Kuhn-Tucker (KKT) conditions.

Question 7. We consider the following basis pursuit problem:

$$\begin{aligned} &\text{minimize}_{x \in \mathbb{R}^n} \|x\|_1 \\ &\text{subject to: } \|Ax - b\|_2^2 \leq \delta^2 \end{aligned}$$

(a) Write the KKT conditions for this problem.

(b) We take $n = 2$, $\delta = 0.5$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Solve the KKT system. It may be useful to distinguish between cases depending on the sign of the dual variable and eliminate the impossible alternatives.