nfluence(effect,impact)
difference(distinction,gap)
communicate(exchange,associate)
haveaccessto(makecontactwith,keepintouchwith)
giveprioritytoput...intofirstplace)
economize(conserve,cherish)
participatein(takepartin,engagein)
measure(step,action)

ambitious. aggressive. aspirant.

No.

Date

We have: ① $\inf_x [f(x) + C_{\mathbb{R}^p_-}(g(x))] = \sup_{\phi \in \mathbb{R}^p} \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \phi)$

② $\exists \phi^*$ s.t. $\inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \phi^*) = \sup_\phi \inf_x \mathcal{L}(x, \phi)$

③ if x^* is a solution of the primal problem, then (x^*, ϕ^*) is a saddle point of $\mathcal{L}$.

proof: we introduce the value function

$$v(y) = \inf_x f(x) + C_{\mathbb{R}^p_-}(g(x) - y)$$

$$v(0) = \inf_x f(x) + C_{\mathbb{R}^p_-}(g(x)) \quad : \text{primal pb.}$$

$$v^*(\phi) = \sup_y (\langle \phi, y\rangle - v(y))$$

$$= \sup_y [\langle \phi, y\rangle - \inf_x (f(x) + C_{\mathbb{R}^p_-}(g(x) - y))]$$

$$= \sup_y [\langle \phi, y\rangle + \sup_x (-f(x) - C_{\mathbb{R}^p_-}(g(x) - y))]$$

$$= \sup_y \sup_x [\langle \phi, y\rangle - f(x) - C_{\mathbb{R}^p_-}(g(x) - y)]$$

$$= \sup_x \sup_y [\quad \dots \quad]$$

$$= \sup_x \left(\sup_y \sum_i [\phi_i y_i - C_{\mathbb{R}_-}(g_i(x) - y_i)] - f(x)\right)$$

$$= \sup_x \left(\sum_i \sup_{y_i} (\phi_i y_i - C_{\mathbb{R}_-}(g_i(x) - y_i)) - f(x)\right) \quad (*)$$

1° $\phi_j \le 0$ 2° $\phi_j > 0$

$\sup(\quad) = \phi_j g_j(x)$ $\sup_{y_j}(\;) = +\infty$

$$\therefore (*) = \sup_x \left[\left(\sum_i \phi_j g_j(x) + C_{\mathbb{R}_-}(\phi_j)\right) - f(x)\right]$$

$$= -\inf_x [f(x) - \langle \phi, g(x)\rangle - C_{\mathbb{R}^p_-}(\phi)]$$

$$= -\inf_x \mathcal{L}(-\phi) := -D(-\phi) \quad \text{dual function}$$

KOKUYO

For the moment, assume $\exists\, \lambda \in \partial v(0)$

~~By Fenchel-Young~~ $v(0) \geq v^{**}(0) = \sup_{\phi} \langle 0, \phi \rangle - v^{*}(\phi) = \sup_{\phi} D(-\phi)$

FY: $v(0) + v^{*}(\lambda) = \langle 0, \lambda \rangle$

$v(0) = -v^{*}(\lambda) = D(-\lambda) \geq D(\phi) \;\forall \phi$

$-\lambda \in \operatorname{argmin} D$ and the minimal value is equal to the dual value.

- If V is convex and $0 \in ri(\operatorname{dom} V)$, then $\partial v(0) \neq \phi$

Lemma: if $F(x,y) \mapsto \underset{F(x,y)}{\cancel{\mathbb{R}}}$ is jointly convex, then $(y \mapsto \inf_{x} F(x,y))$ convex.

$\operatorname{dom} V = \{ y : V(y) < +\infty \} = \{ y : \exists x_0 \in \operatorname{dom} f : g(x_0) \leq y \}$

$ri(\operatorname{dom}(V)) = int(\operatorname{dom} V)$ because $\operatorname{span}(\operatorname{dom} V) = \mathbb{R}^p$

$int(\operatorname{dom} V) = \{ y : \exists x_0 \in \operatorname{dom} f : g(x_0) < y \}$

$0 \in ri(\operatorname{dom} V) \iff \exists\, x_0 \in \operatorname{dom} f : g(x_0) < 0$