$) + \nabla g(x)^T(z-x) + \frac{1}{2t}$

$$\mathrm{dom}(V) = \{y : \exists x_0,\ f(x_0) < +\infty \text{ and } g(x_0 - y) < +\infty\}$$
$$= \{y : \exists x_0 \in \mathrm{dom} f,\ x_0 - y \in \mathrm{dom} g\}$$
$$= \{y : \exists x_0 \in \mathrm{dom} f,\ \exists z_0 \in \mathrm{dom} g : x_0 - y = z_0\}$$
$$= \mathrm{dom} f - \mathrm{dom} g$$

F is continous so :

$0 \in ri(\mathrm{dom} f - \mathrm{dom} g) \Rightarrow 0 \in ri(\mathrm{dom} V)$

$$\Rightarrow \begin{cases} p = d \\ \text{the infimum in } V^{**}(0) \text{ is attained} \end{cases}$$

Conclude

$0 \in ri(\mathrm{dom} f - \mathrm{dom} g) \Rightarrow f+g$ proper $\Rightarrow (f+g)^*$ proper

存疑

$0 \in \mathrm{dom} f - \mathrm{dom} g$ so $\exists x_0 \in \mathrm{dom} f,\ x_0 \in \mathrm{dom} g$

$$= \sup_x \left(\langle x, \phi \rangle - f(x) \right) + \sup_y \left(\langle y, \phi \rangle - g(y) \right)$$

$$= f^*(\phi) + g^*(\phi)$$

$\square$ f and g convex, proper

Theorem: if $0 \in \operatorname{ri}(\operatorname{dom} f - \operatorname{dom} g)$ then $(f+g)^* = f^* \square g^*$

moreover $(f+g)^*$ is proper and $f^* \square g^*(y) = \min_x f^*(x) + g^*(y-x)$

proof: Let $\phi \in \mathcal{X}$

~~Def~~ define $F : \mathcal{X} \times \mathcal{X} \to \bar{\mathbb{R}}$

$$(x, y) \mapsto f(x) + g(x-y) - \langle \varphi, x \rangle$$

$$p = \inf_x F(x, 0) = \inf_x \left[f(x) + g(x) - \langle \varphi, x \rangle \right]$$

$$= - \sup_x \left[\langle \varphi, x \rangle - f(x) - g(x) \right]$$

$$= - \left(f(x) + g(x) \right)^*$$

$$d = v^{**}(0)$$

$$v^*(\lambda) = \sup_y \left[\langle \lambda, y \rangle - v(y) \right]$$

$$= \sup_y \left[\langle \lambda, y \rangle - \inf_x F(x, y) \right]$$

$$= \sup_y \left[\langle \lambda, y \rangle - \inf_x \left(f(x) + g(x-y) - \langle \varphi, x \rangle \right) \right]$$

$$= \sup_y \left[\langle \lambda, y \rangle + \sup_x \left(\langle \varphi, x \rangle - f(x) - g(x-y) \right) \right]$$

$$= \sup_{x,y} \left(\langle \lambda, y \rangle + \langle \varphi, x \rangle - f(x) - g(x-y) \right)$$

$$\cancel{= \sup_x \left[\langle \varphi, x \rangle - f(x) \right]}$$

$$= \sup_{z = x-y} \left(\langle -\lambda, x-y \rangle - g(x-y) + \sup_x \left(\langle \varphi + \lambda, x \rangle - f(x) \right) \right)$$

$$= g^*(-\lambda) + f^*(\varphi + \lambda)$$

$$v^{**}(0) = \sup_\lambda -v^*(\lambda) = -\inf_\lambda \left(g^*(-\lambda) + f^*(\varphi + \lambda) \right)$$

$$= -\inf_{\lambda'} \left(g^*(\lambda') + f^*(\varphi - \lambda') \right)$$

$$= - g^* \square f^*(\varphi)$$

$$\operatorname{dom}(V) = \{ y : \exists x \text{ s.t. } f(x) + g(x-y) - \langle \varphi, x \rangle < +\infty \}$$

Definition: the infimal convolution of two functions f and g is the function $f\square g$ defined by

$\forall y \in X$, $(f\square g)$ ~~defined by~~ $(y) = \inf_x [f(x) + g(y-x)]$

usual convolution: $(f*g)(y) = \int f(x)g(y-x)dx$

Proposition:
- $f\square g = g\square f$
- if f and g have an affine minorant with slope ϕ, then $f\square g$ has an affine minorant with slope ϕ.
- if f and g are convexe, then $f\square g$ is convexe.

proof:
- $(f\square g)(y) = \inf_x [f(x)+g(y-x)]$

$(g\square f)(y) = \inf_x [g(x) + f(y-x)]$

$= \inf_{\bar{x}=y-x} [g(y-\bar{x}) + f(\bar{x})]$

- we assume that $f(x) \geq \langle\phi, x\rangle$ and $g(x) \geq \langle\phi, x\rangle$

$f\square g(y) = \inf_x [f(x) + g(y-x)]$

$\geq \inf_x [\langle\phi, x\rangle + \langle\phi, y-x\rangle]$

$= \inf_x [\langle\phi, y\rangle]$

$= \langle\phi, y\rangle$

- $h : (x,y) \mapsto f(x) + g(y-x)$. convex

$\therefore (f\square g)(y) = \inf_x (h(x,y))$ convex

proposition: $(f\square g)^* = f^* + g^*$

proof: $(f\square g)^*(\phi) = \sup_y (\langle y,\phi\rangle - (f\square g)(y))$

$= \sup_y [\langle y,\phi\rangle - \inf_x (f(x) + g(y-x))]$

$= \sup_y [\langle y,\phi\rangle + \sup_x (-f(x) - g(y-x))]$

$= \sup_{y,x} [\langle y,\phi\rangle - f(x) - g(y-x)]$

$= \sup_{y,x} [\langle x,\phi\rangle - f(x) + \langle y-x,\phi\rangle - g(y-x)]$

KOKUYO