Q 3.1

$$\nabla \left(\frac{1}{2} \| X w_{:,e} - y_{:,e} \|^2 + \frac{1}{2} \| X w_{:,m} - y_{:,m} \|^2 \right) = 0$$

$$\Leftrightarrow \begin{cases} X^T \left(X \cdot w_{:,e} - y_{:,e} \right) = 0 \\ X^T \left(X \cdot w_{:,m} - y_{:,m} \right) = 0 \end{cases}$$

Q 4.1

$$\operatorname{prox}_{\gamma f_2}(v) = \underset{w \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2}(w-v)^2 + \gamma \cdot \max\{-(y-w)+\tau(y-w), \tau(y-w)\}$$

1° $y - w > 0$

$$\operatorname{prox}_{\gamma f_2}(v) = \underset{w \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2}(w-v)^2 + \gamma \cdot \tau(y-w) = v + \gamma\tau$$

2° $y - w \leq 0$

$$\operatorname{prox}_{\gamma f_2}(v) = \operatorname{argmin} \frac{1}{2}(w-v)^2 + \gamma \cdot (-(y-w)+\tau(y-w)) = v - \gamma + \tau\gamma$$

$\Rightarrow$

$$\begin{cases} \operatorname{prox}_{\gamma f_2}(v) = v + \gamma\tau, & v + \gamma\tau \leq y \\ \operatorname{prox}_{\gamma f_2}(v) = v - \gamma + \tau\gamma, & v - \gamma + \tau\gamma \geq y \\ \operatorname{prox}_{\gamma f_2}(v) = y, & \text{else} \end{cases}$$

Q 4.2

$$\min_{\tilde{w}} f_1(\tilde{w}) + f_2(\tilde{X} \cdot \tilde{w})$$

where $\tilde{w} = \begin{pmatrix} w \\ w_0 \end{pmatrix}$ $\tilde{X} = (X \mid 1)$

$$f_1(\tilde{w}) = \frac{\alpha}{2} \|w\|_F^2 \qquad f_2(\tilde{X} \cdot \tilde{w}) = \sum_g \sum_i L_\tau (y_{ig} - \tilde{X} \cdot \tilde{w}_g)$$

$$= \frac{\alpha}{2} \left\| \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 & \\ & & & 0 \end{bmatrix} \tilde{w} \right\|_F^2$$

Q 4.3

ADMM

$$\tilde{\omega}^{k+1} = \underset{\tilde{\omega}}{\operatorname{argmin}} \{ f(\tilde{\omega}) + \langle \tilde{X}\tilde{\omega}, \phi^k \rangle + \frac{1}{2r} \| \hat{X}\tilde{\omega} - z^k \|^2 \}$$

$$z^{k+1} = \underset{v}{\operatorname{argmin}} \{ \sum_i L_\tau (y_{ig} - v_i) - \langle v, \phi^k \rangle + \frac{1}{2r} \| \tilde{X} \hat{\omega}^{k+1} - v \|^2$$

$$\phi^{k+1} = \phi^k + \frac{1}{r} (\tilde{X} \tilde{\omega}^{k+1} - z^{k+1})$$

$\phi \in \mathbb{R}^{n \times 2}$ $\quad z \in \mathbb{R}^{n \times 2}$

Q 4.4

$$\underline{\{ \alpha \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \; 0 \end{bmatrix} + \frac{1}{r} \tilde{X}^T \tilde{X} \} \tilde{\omega}^{k+1} = \frac{1}{r} \tilde{X}^T z^k - \tilde{X}^T \phi^k}$$

$$0 \in r \partial L_\tau (y_{ig} - v_{ig}) - r \phi^k_{ig} + (v_{ig} - \tilde{X}_i \tilde{\omega}^{k+1}_g)$$

$$\Rightarrow r (+ \phi^k_{ig} + \frac{\tilde{X}_i \tilde{W}_{k+1g}}{r}) \in v_{ig} + \partial r L_\tau (y_{ig} - v_{ig})$$

$$\Rightarrow \underline{ z^{k+1}_{ig} = \operatorname{prox}_{rf_2} (+ r \phi^k_{ig} + \tilde{X}_i \tilde{\omega}_{k+1,g}) }$$

$$\underline{ \phi^{k+1} = \phi^k + \frac{1}{r} (\tilde{X} \tilde{\omega}^{k+1} - z^{k+1}) }$$