Convex functions, subdifferential, normal cones

Exercise 1. Derive the subdifferential of $x \mapsto |x|$ on $\mathbb{R} \to \mathbb{R}$.

Exercise 2. Let ι_C denote the indicator function of a non empty closed convex set C in $\mathcal{X}$. We set $N_C(x) = \{\varphi \in \mathcal{X} : \forall y \in C, \langle \varphi, y - x \rangle \leq 0\}$ for all $x \in C$, and $N_C(x) = \emptyset$ otherwise.

1. Show that for all $x \in C$, $N_C(x)$ is a cone, *i.e.* $\forall \varphi \in N_C(x)$, $\forall \lambda \geq 0$, $\lambda\varphi \in N_C(x)$.
2. Derive $N_C(x)$ when $x \in \mathrm{int}(C)$.
3. Derive $N_C(x)$ in the following cases: $C = (-\infty, 0]$, $C = (-\infty, 1]$, $C = \mathbb{R}_- \times \mathbb{R}_-$, $C = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$.

Exercise 3. For every $x \in \mathbb{R}^n$, denote $x = (x_1, \dots, x_n)$. Consider a function $f : \mathbb{R}^n \to (-\infty, +\infty]$ s.t. $f(x) = \sum_{i=1}^n f_i(x_i)$ where $f_i : \mathbb{R}^n \to (-\infty, +\infty]$ for all i.

1. Express ∂f as a function of $\partial f_1, \dots, \partial f_n$.
2. Application: $f(x) = \|x\|_1$.

Exercise 4. Let $f : \mathcal{X} \to [-\infty, +\infty]$ satisfy: $\exists x_0 \in \mathrm{relint}(\mathrm{dom}\, f)$, $f(x_0) \in \mathbb{R}$. Show that $-\infty \notin f(\mathcal{X})$.

Operations that preserve convexity

② $\forall x, f_x$ 闭 l.s.c. $\Updownarrow$ ~~f 闭 l.s.c.~~ epi f_x 闭 ~~f closed~~ $\Updownarrow$ $\bigcap$ epi f_x 闭 $\Updownarrow$ f_x l.s.c.

Exercise 5. Let $(f_\alpha)_{\alpha \in I}$ represent an arbitrary collection of functions on $\mathcal{X} \to (-\infty, +\infty]$. Set $f = \sup_\alpha f_\alpha$. Show that:

① $\binom{y}{t} \in \mathrm{epi}\, f \Leftrightarrow t \geq f(y) \Leftrightarrow t \geq \sup_x f_x(y) \Leftrightarrow \forall x, t \geq f_x(y) \Leftrightarrow \forall x \binom{y}{t} \in \mathrm{epi}\, f_x \Leftrightarrow \binom{y}{t} \in \bigcap \mathrm{epi}\, f_x$

1. If f_α is convex for all α, then f is convex. $\mathrm{epi}(f_\alpha) = \bigcap_\alpha \mathrm{epi}(f_\alpha)$
2. If f_α is closed for all α, then f is closed.
3. Let S denote the set of all symetric matrices in $\mathbb{R}^n$. Prove the function defined on S by $f(X) = \lambda_{max}(X)$ is convex (*Hint: write $f(X)$ as a supremum*).

Exercise 6. Let $F : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]$ be a convex function. Show that the mapping $y \mapsto \inf_{x \in \mathcal{X}} F(x,y)$ is convex.

Exercise 7. 1. Show that a nonnegative weighted sum of convex functions is convex.

2. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be convex. Let $A \in \mathbb{R}^{n \times m}$ be a matrix and b a vector. Show that $x \mapsto f(Ax + b)$ is convex.

Operations on multivalued functions

Let $A : \mathcal{X} \to 2^{\mathcal{X}}$ be a set-valued mapping, where $\mathcal{X} = \mathbb{R}^n$. We define the graph of A by $\mathrm{gr}A = \{(x,y) : y \in A(x)\}$. We define $A^{-1} : \mathcal{X} \to 2^{\mathcal{X}}$ as $A^{-1}(x) = \{y : x \in A(y)\}$ that is

$$\mathrm{gr}(A^{-1}) = \{(x,y) : (y,x) \in \mathrm{gr}A\}.$$

Its domain $\mathrm{dom}(A)$ is the set of $x \in \mathcal{X}$ such that $A(x) \neq \emptyset$.

Exercise 8. Let C be a non empty closed convex set of $\mathcal{X}$ and let $P_C(b)$ be the projection of a point $b \in \mathcal{X}$ onto C.

1. Show that $P_C(b)$ is the unique minimizer of the problem $\min_x \|x-b\|^2 + g(x)$ where $g(x)$ is a function to be determined.
2. Show that $x = P_C(b)$ iff (if and only if) $0 \in x - b + N_C(x)$..
3. Deduce that $P_C = (I + N_C)^{-1}$.
4. Application: Set $C = [-1, 1]$.
   (a) Calculate algebrically $P_C(b)$ as a function of b.
   (b) Plot the graph of the multifunction N_C.
   (c) Plot the graph of $I + N_C$.
   (d) Plot the graph of $(I + N_C)^{-1}$ and conclude.

Exercise 9. Let $\gamma > 0$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.

1. Draw the graph of ∂f and the graph of $\gamma\partial f$.
2. Draw the graph of $I + \gamma\partial f$ where $I : x \mapsto x$ is the identity function.
3. Draw the graph of $(I + \gamma\partial f)^{-1}$. Whose function is it the graph ?

Exercise 10. Let $f : \mathcal{X} \to (-\infty, +\infty]$ be a proper l.s.c. convex function. Let $x \in \mathcal{X}$ and $\gamma > 0$. Define

$$\mathrm{prox}_{\gamma f}(x) = \arg\min_{y \in \mathcal{X}} f(y) + \frac{\|y-x\|^2}{2\gamma}.$$

1. Show that the mapping $\mathrm{prox}_{\gamma f} : \mathcal{X} \to \mathcal{X}$ is well defined.
2. Show that $p = \mathrm{prox}_{\gamma f}(x)$ iff $p \in x + \gamma\partial f(x)$.
3. Deduce the identity $\mathrm{prox}_{\gamma f} = (I + \gamma\partial f)^{-1}$.
4. What are the fixed points of $\mathrm{prox}_{\gamma f}$?

Exercise 11. Evaluate $\mathrm{prox}_{\gamma f}$ when $f(x)$ coincides with $\iota_C(x)$ (C closed convex and non empty), $|x|$, $\|x\|_1$, $\|x\|^2$, $\|x\|$.

Def. f is coercive if $\lim_{\|x\|\to\infty} f(x) = +\infty$

Prop. f l.s.c. and coercive admits a minimizer

Proof: We must find x^* s.t. $f(x^*) = \inf f$.

Choose $(x_n)_n$ s.t. $f(x_n) \to \inf f$.

The sequence (x_n) is bounded.

Proof: 不 bound $f(x_n) \to \infty$

There exists x^* and a subsequence (x_{φ_n}) s.t. $x_{\varphi_n} \to x^*$

$\inf f = \lim f(x_n) = \liminf f(x_{\varphi_n}) \geq f(x^*)$

$\therefore \inf f = f(x^*)$

Def. strictly convex

Prop. If $f: \mathcal{X} \mapsto (-\infty, +\infty]$ is strictly convex, it admits at most one minimizer. $[\, e^{-x}$ 无 min $]$

Proof: Assume by contradiction that, $\exists\, x^* \neq y^*$, $f(x^*) = f(y^*) = \inf f$.

$f\left(\frac{x^*+y^*}{2}\right) < \frac{1}{2} f(x^*) + \frac{1}{2} f(y^*)$

$f\left(\frac{x^*+y^*}{2}\right) < \inf f$ impossible

Def: Let $\mu > 0$. $f: \mathcal{X} \to (-\infty, +\infty]$ is said μ-strongly convex if the mapping $x \mapsto f(x) - \frac{\mu\|x\|^2}{2}$ is convex

Prop. If f is strongly convex, proper, it is strictly convex and coercive

Corollary: If f is strongly convex, proper and l.s.c, it admits a unique minimizer.

Proof: Let $g = f - \frac{\mu\|\cdot\|^2}{2}$, convex.

Thus it admits an affine minorant: $\exists\, \vec{a}, b$ s.t. $\forall x$,

$g(x) \geq \langle a, x \rangle + b + \frac{\mu\|x\|^2}{2} \Rightarrow f$ coercive

Let $x \neq y$ and $t \in (0,1)$.

$f(tx + (1-t)y) = g(tx + (1-t)y) + \frac{\mu}{2}\|tx + (1-t)y\|^2$

$\leq t g(x) + (1-t) g(y) + \frac{\mu}{2}\|\cdots\|^2 < t g(x) + (1-t) g(y) + \frac{\mu}{2}(t\|x\|^2 + (1-t)\|y\|^2)$

$= t f(x) + (1-t) g(y)$.

~~Remind~~ Collary : Any convex function $f: \mathcal{X} \to (-\infty, +\infty]$ admits an affine minorant.

- $h: \mathcal{X} \to \mathbb{R}$ is affine if it has the form $h(x) = \langle a, x \rangle + b$.
- h is said to be a minorant of f if $\forall x$, $h(x) \le f(x)$.

Prf : If $f \equiv +\infty$, trivial

If f is prope, then since $\mathrm{dom} f$ is convex, $ri(\mathrm{dom} f) \neq \phi$

There exists $x_0 \in ri(\mathrm{dom} f) \neq \phi$

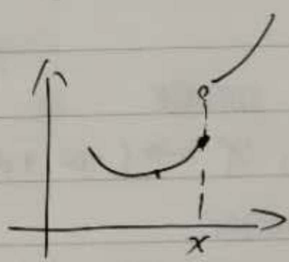
There exists $\varphi \in \partial f(x_0)$

$\forall y \in \mathcal{X}$, $f(y) \ge f(x_0) + \langle \varphi, y - x_0 \rangle$

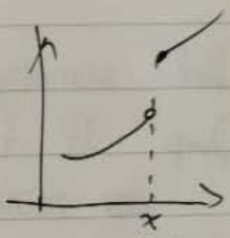
III. Minimizers, l.s.c. functions

1) l.s.c. Def: $f: \mathcal{X} \to [-\infty, +\infty]$ is l.s.c. at a point x if for every sequence (x_n) s.t. $x_n \to x$.

$\liminf_{n\to\infty} f(x_n) \ge f(x)$.



l.s.c



不是l.s.c.

Def. f is l.s.c if it is l.s.c. at every point $x \in \mathcal{X}$

Prop. f is l.s.c $\Leftrightarrow$ $\mathrm{epi}(f)$ is closed

Pf : Assume f l.s.c.

$(x_n, t_n) \in \mathrm{epi} f$ s.t. $(x_n, t_n) \to (x, t)$

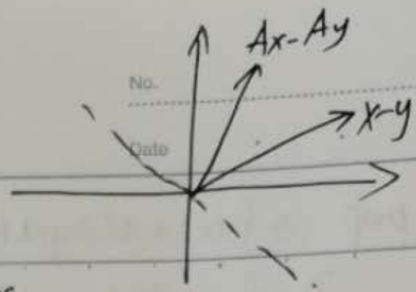
Let us prove that $(x, t) \in \mathrm{epi} f$.

$t_n \ge f(x_n)$

$\liminf t_n \ge \liminf f(x_n)$

$t \ge f(x)$

$\therefore (x, t) \in \mathrm{epi} f$.



IV. Properties of subdifferential

$\partial f : \mathcal{X} \to 2^{\mathcal{X}}$ Multifunction / operator

Def : $A : \mathcal{X} \to 2^{\mathcal{X}}$ is said monotone if $\forall x, y$, $\forall u \in A(x)$, $v \in A(y)$, $\langle u-v, x-y \rangle \geq 0$.

if $A(x)$ is a singleton $\forall x$, A is a function. $A : \mathcal{X} \to \mathcal{X}$.

The function A is monotone if $\forall x, y$, $\langle A(x) - A(y), x-y \rangle \geq 0$.

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Prop : ∂f is monotone

proof : $u \in \partial f(x)$, $v \in \partial f(y)$.

$f(y) \geq f(x) + \langle u, y-x \rangle$

$f(x) \geq f(y) + \langle v, y-x \rangle$

$0 \geq \langle u-v, y-x \rangle$ #

Minkowsky's sum

$A+B = \{x+y : x \in A, y \in B\}$

Let M be a matrix. $M \cdot B = \{Mb : b \in B\}$

Inverse of an operation $A : \mathcal{X} \to 2^{\mathcal{X}}$.

Def : $A^{-1} : \mathcal{X} \to 2^{\mathcal{X}}$ defined $A^{-1}(x) = \{y : x \in A(y)\}$

otherwise stated, $y \in A^{-1}(x) \Leftrightarrow x \in A(y)$

Ex : $A(x) = \{e^x\}$ $\forall x$ $\quad A^{-1}(x) = \begin{cases} \{\ln x\} & \forall x > 0 \\ \phi & \forall x \leq 0 \end{cases}$

Subdifferentiation rules

$f : \mathcal{X} \to (-\infty, +\infty]$ $\quad g : \mathcal{Y} \to (-\infty, +\infty]$ where $\mathcal{Y} = \mathbb{R}^m$

$M : \mathcal{X} \to \mathcal{Y}$ linear operator (a matrix).

Consider $f + g \circ M : \mathcal{X} \to (-\infty, +\infty]$

$x \to f(x) + g(Mx)$

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Prop. $\partial f(x) + M^T \partial g(Mx) \subset \partial(f + g\circ M)(x)$

Proof: Take $\varphi \in \partial f(x)$ $\psi \in \partial g(Mx)$.

We must proove that:

$\varphi + M^T\psi \in \partial(f+g\circ M)(x)$.

$\forall y \in \mathcal{X},\ f(y) \geq f(x) + \langle \varphi, y-x\rangle$

$\forall z \in \mathcal{Y},\ g(z) \geq g(Mx) + \langle \psi, z-Mx\rangle$

令 $z = My$

则 $g(My) \geq g(Mx) + \langle M^T\psi, y-x\rangle$

$\Rightarrow f(y) + g(My) \geq f(x) + g(Mx) + \langle \varphi + M^T\psi, y-x\rangle$

$\Rightarrow \varphi + M\psi \in \partial(f+g\circ M)$

Theorem: Assume that f, g are closed convex functions

Assume $0 \in ri(M\operatorname{dom} f - \operatorname{dom} g)$

Then, $\partial(f+g\circ M) = \partial f(x) + M^T \partial g(Mx)$.

Fenchel-Legendre Transform

Def: $f: \mathcal{X} \to [-\infty, +\infty]$

$f^*(\varphi) = \sup\limits_{x\in X}(\langle \varphi, x\rangle - f(x))$

$f^*: \mathcal{X} \longmapsto [-\infty, +\infty]$ is called the Legendre transform or Fenchel conjugate.

Prop: $f \leq g$ than $f^* \geq g^*$

Proof: $\forall x,\ f(x) \leq g(x)$

$\forall x,\ \langle \varphi, x\rangle - f(x) \geq \langle \varphi, x\rangle - g(x)$

$\sup\limits_{x\in X}(\langle\ ,\ \rangle - \cdots) \geq \sup(\cdots)$

Prop: f^* is convex, l.s.c.

Def: $ri(C) = relint(C) \overset{def}{=} \{x : \exists V$ voisin of x, $V \cap Aff(C) \subset C\}$

Prop: $int(C) \subset ri(C)$

Prop: If C is convex, non empty, then $ri(C) \neq \phi$

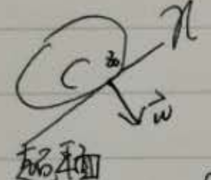
Existence of a subgradient

Prop: Let $f : \mathcal{X} \mapsto (-\infty, +\infty]$ convex. if $x \in ri(dom f)$, $\partial f(x) \neq \phi$

Proof: Consider the stronger assumption: $x \in int(dom f)$.

Reminder: $\exists$ C is a closed convex set (non ϕ) of $\mathcal{X}$

If $z_0 \in boundary(C)$ $\exists \vec{w} \in \mathcal{X}$, $\forall z \in C$, $\langle w, z - z_0 \rangle \leq 0$



$\mathcal{H} = \{ z : \langle w, z - z_0 \rangle = 0\}$ is a supporting hyperplane

$epi f = \{(x, y) : x \in \mathcal{X}, y \in \mathbb{R}, y \geq f(x)\} \subset \mathcal{X} \times \mathbb{R}$

$(x, f(x)) \in bdry(\overline{epi} f)$

Indeed, $\cdot$ $(x, f(x)) \in \overline{epi f}$

$\cdot$ $(x, f(x) - \frac{1}{n}) \in (epi f)^c$ and $\to$ to $(x, f(x))$

$\exists$ $w \in \mathcal{X} \times \mathbb{R}$, $\forall z \in \overline{epi f}$, $\langle w, z - \binom{x}{f(x)} \rangle \leq 0$

write $w = \binom{\varphi}{u}$ $\varphi \in \mathcal{X}$ $u \in \mathbb{R}$

$\forall \binom{y}{t}$ s.t. $t \geq f(y)$. $\langle \varphi, y - x \rangle + u(t - f(x)) \leq 0$

$\Rightarrow u \leq 0$

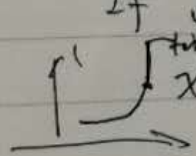
If $u < 0$ $\forall y \in \mathcal{X}$ $\forall t \geq f(y)$,

for $t = f(y)$, $\forall y \in \mathcal{X}$, $f(y) - f(x) + \langle \frac{\varphi}{u}, y - x \rangle \geq 0$

$\Rightarrow f(y) \geq f(x) + \langle -\frac{\varphi}{u}, y - x \rangle$

Thus, $-\frac{\varphi}{u} \in \partial f(x)$.

If $u = 0$



x must be at boundary 矛 $x \in int(dom f)$.

$\forall y \in dom f$, $\langle \varphi, y - x \rangle \leq 0$

$\forall \vec{d}$, $\langle \varphi, \vec{d} \rangle \leq 0$

Thus, $\forall d$, $\langle \varphi, d \rangle = 0$. Thus, $\varphi = 0$. 矛 $w \neq 0$