# Stochastic Variance Reduced Gradient Methods

Master 2 Data Science, Univ. Paris Saclay

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# References for this class

Section 6.3:

Sébastien Bubeck (2015)
Foundations and Trends
**Convex Optimization: Algorithms and Complexity**

M. Schmidt, N. Le Roux, F. Bach (2016), Mathematical Programming **Minimizing Finite Sums with the Stochastic Average Gradient.**

How to transform convergence results into iteration complexity

Section 1.3.5, R.M. Gower, Ph.d thesis: Sketch and Project: Randomized Iterative Methods for Linear Systems and Inverting Matrices University of Edinburgh, 2016

# Solving the Finite Sum Training Problem

# Optimization Sum of Terms

**A Datum Function**

$$f_i(w) := \ell\left(h_w(x^i), y^i\right) + \lambda R(w)$$

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\ell\left(h_w(x^i), y^i\right) + \lambda R(w) &= \frac{1}{n}\sum_{i=1}^{n}\left(\ell\left(h_w(x^i), y^i\right) + \lambda R(w)\right)\\ &= \frac{1}{n}\sum_{i=1}^{n} f_i(w)\end{aligned}$$

**Finite Sum Training Problem**

$$\min_{w\in\mathbf{R}^d} \frac{1}{n}\sum_{i=1}^{n} f_i(w) =: f(w)$$

# SGD shrinking stepsize

**SGD 1.0: Descreasing stepsize**

Set $w^0 = 0$, choose $\alpha > 0$, $\alpha_t = \frac{\alpha}{\sqrt{t+1}}$,

for $t = 0, 1, 2, \ldots, T-1$

sample $j \in \{1, \ldots, n\}$

$w^{t+1} = w^t - \alpha_t \nabla f_j(w^t)$

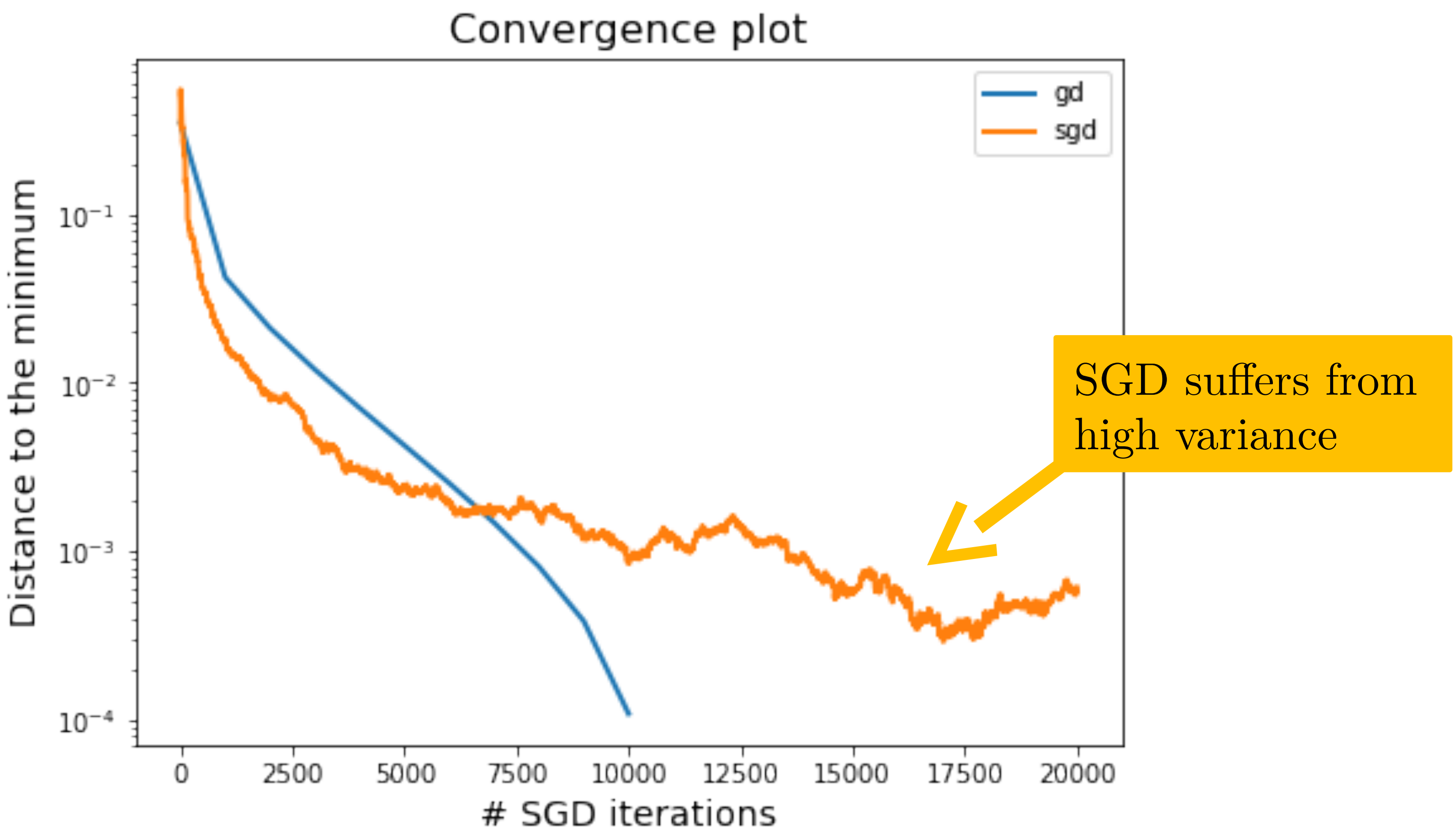
Output $w^T$

**Convergence for Strongly Convex**

- $f(w)$ is $\lambda$ - strongly convex
- Subgradients bounded

$$\alpha_t = O\left(\frac{1}{\lambda t}\right) \quad \Rightarrow \quad \mathbb{E}[f(w^T)] - f(w^*) = O\left(\frac{1}{\lambda T}\right)$$

# SGD initially fast, slow later



# Variance reduced methods through Sketching

# Build an Estimate of the Gradient

Instead of using directly $\nabla f_j(w^t) \approx \nabla f(w^t)$
Use $\nabla f_j(w^t)$ to update estimate $g_t \approx \nabla f(w^t)$

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$$w^{t+1} = w^t - \alpha g^t$$

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$$w^{t+1} = w^t - \alpha g^t$$

We would like gradient estimate such that:

**Unbiased**

$$\mathbb{E}[g^t] = \nabla f(w^t)$$

**Converges in $L2$**

$$\mathbb{E}||g^t||_2^2 \underset{w^t \rightarrow w^*}{\longrightarrow} 0$$

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We would like gradient estimate such that:

**Unbiased**

$$\mathbb{E}[g^t] = \nabla f(w^t)$$

Solves problem of $||\nabla f_j(w)||_2^2 \leq B^2$

**Converges in *L2***

$$\mathbb{E}||g^t||_2^2 \underset{w^t \rightarrow w^*}{\longrightarrow} 0$$

# Covariates

Let $x$ and $z$ be random variables. We say that $x$ and $z$ are covariates if:

$$\mathrm{cov}(x, z) \geq 0$$

Variance Reduced Estimate:

$$x_z = x - z + \mathbb{E}[z]$$

# Covariates

$$\operatorname{cov}(x,z) := \mathbb{E}[(x-\mathbb{E}[x])(z-\mathbb{E}[z])]$$

Let $x$ and $z$ be random variables. We say that $x$ and $z$ are covariates if:

$$\operatorname{cov}(x,z) \geq 0$$

Variance Reduced Estimate:

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**EXE**:

1. Show that $\mathbb{E}[x_z] = \mathbb{E}[x]$
2. $\mathbb{VAR}[x_z] = \mathbb{E}[(x_z - \mathbb{E}[x_z])^2] = ?$
3. When is $\mathbb{VAR}[x_z] \leq \mathbb{VAR}[x]$

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$$\begin{aligned}
\mathbb{E}[(x_z-\mathbb{E}[x_z])^2] &= \mathbb{E}[(x-\mathbb{E}[x]-(z-\mathbb{E}[z]))^2] \\
&= \mathbb{E}[(x-\mathbb{E}[x])^2] - 2\mathbb{E}[(x-\mathbb{E}[x])(z-\mathbb{E}[z])] \\
&\quad +\mathbb{E}[(z-\mathbb{E}[z])^2] \\
&= \mathbb{VAR}[x] - 2\text{cov}(x,z) + \mathbb{VAR}[z]
\end{aligned}$$

# SVRG: Stochastic Variance Reduced Gradients

$$w^{t+1} = w^t - \alpha g^t$$

Reference point $\tilde{w} \in \mathbb{R}^d$

Sample $\nabla f_i(w^t), \quad i \in \{1, \ldots, n\}$ uniformly

grad estimate $g^t = \nabla f_i(w^t) - \nabla f_i(\tilde{w}) + \nabla f(\tilde{w})$

$$x_z = x - z + \mathbb{E}[z]$$

# SVRG: Stochastic Variance Reduced Gradients

Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$

$\tilde{w}^0 = w^0$

for $t = 0, 1, 2, \ldots, T-1$

  calculate $\nabla f(\tilde{w}^t)$

  $w^0 = \tilde{w}^t$

  for $k = 0, 1, 2, \ldots, m-1$

    sample $j \in \{1, \ldots, n\}$

    $g^k = \nabla f_j(w^k) - \nabla f_j(\tilde{w}^t) + \nabla f(\tilde{w}^t)$

    $w^{k+1} = w^k - \alpha g^k$

  Option I: $\tilde{w}^{t+1} = w^m$

  Option II: $\tilde{w}^{t+1} = \frac{1}{m}\sum_{i=0}^{m-1} w^i$

Output $\tilde{w}^T$

Freeze reference point for $m$ iterations

# SAGA: Stochastic Average Gradient

$$w^{t+1} = w^t - \alpha g^t$$

Sample

$\nabla f_i(w^t), \quad i \in \{1, \ldots, n\}$ uniformly

Reference points

if $i$ is sampled store $w^{t_i} = w^t$

grad estimate

$$g^t = \nabla f_i(w^t) - \nabla f_i(w^{t_i}) + \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(w^{t_j})$$

$$x_z = \quad x \quad - \quad z \quad + \quad \mathbb{E}[z]$$

# SVRG: Stochastic Variance Reduced Gradients

Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$
$\tilde{w}^0 = w^0$
for $t = 0, 1, 2, \ldots, T-1$
    calculate $\nabla f(\tilde{w}^t)$
    $w^0 = \tilde{w}^t$
    for $k = 0, 1, 2, \ldots, m-1$
        sample $j \in \{1, \ldots, n\}$
        $g^k = \nabla f_j(w^k) - \nabla f_j(\tilde{w}^t) + \nabla f(\tilde{w}^t)$
        $w^{k+1} = w^k - \alpha g^k$
    Option I: $\tilde{w}^{t+1} = w^m$
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Output $\tilde{w}^T$

Freeze reference point for $m$ iterations

# SAG: Stochastic Average Gradient (Biased version)

$$w^{t+1} = w^t - \alpha g^t$$

Sample $\nabla f_i(w^t), \quad i \in \{1, \ldots, n\}$ uniformly

Reference points: if $i$ is sampled store $w^{t_i} = w^t$

grad estimate

$$g^t = \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{t_j})$$

$$\mathbb{E}[g^t] \neq \nabla f(w^t)$$

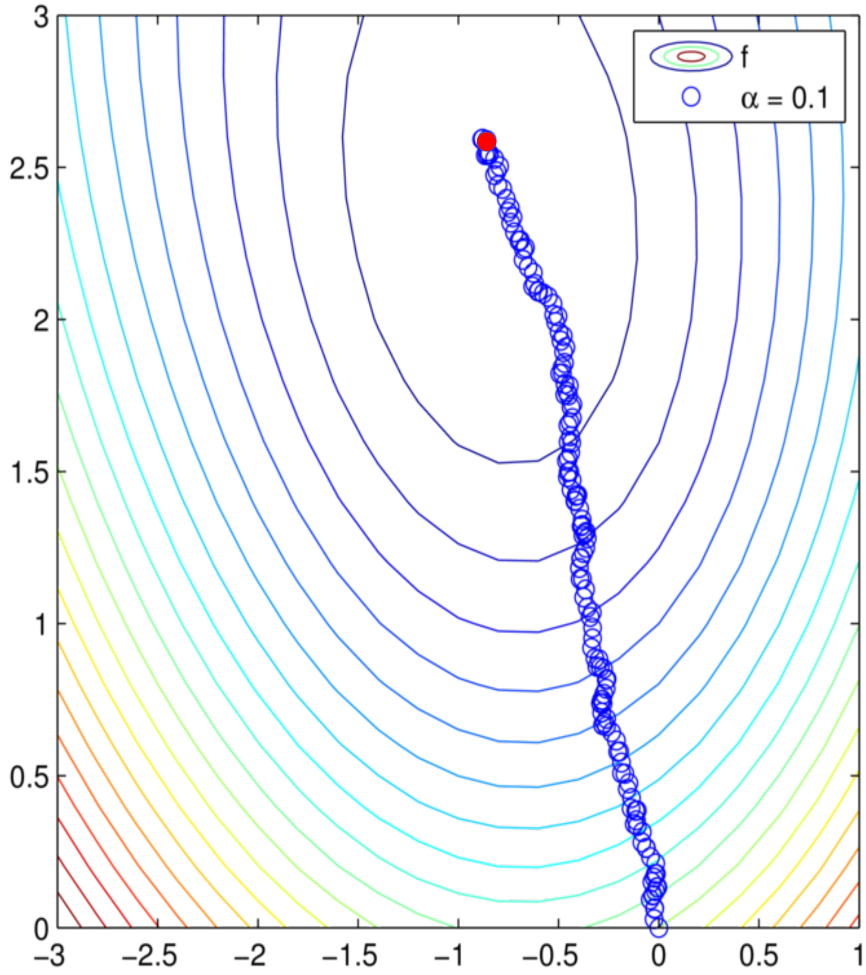
$$\cancel{x_z = x - z + \mathbb{E}[z]}$$

# SAGA: Stochastic Average Gradient

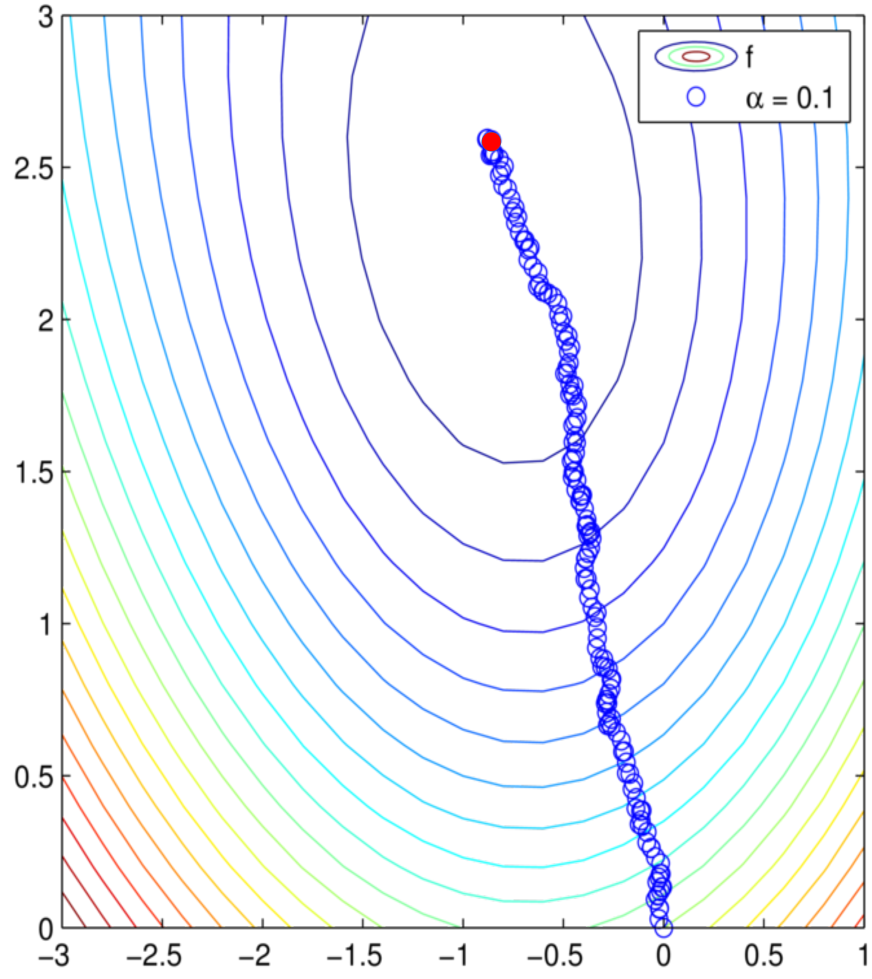
Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$
$z_i = \nabla f_i(w^0)$, for $i = 1, \ldots, n$
for $t = 0, 1, 2, \ldots, T-1$
    sample $j \in \{1, \ldots, n\}$
    $g^t = \nabla f_j(w^t) - z_j + \frac{1}{n}\sum_{i=1}^{n} z_i$
    $w^{t+1} = w^t - \alpha g^t$
    $z_j = \nabla f_j(w^t)$
Output $w^T$

Store all $n$ vectors $z_i \in \mathbb{R}^d$

# The Stochastic Average Gradient



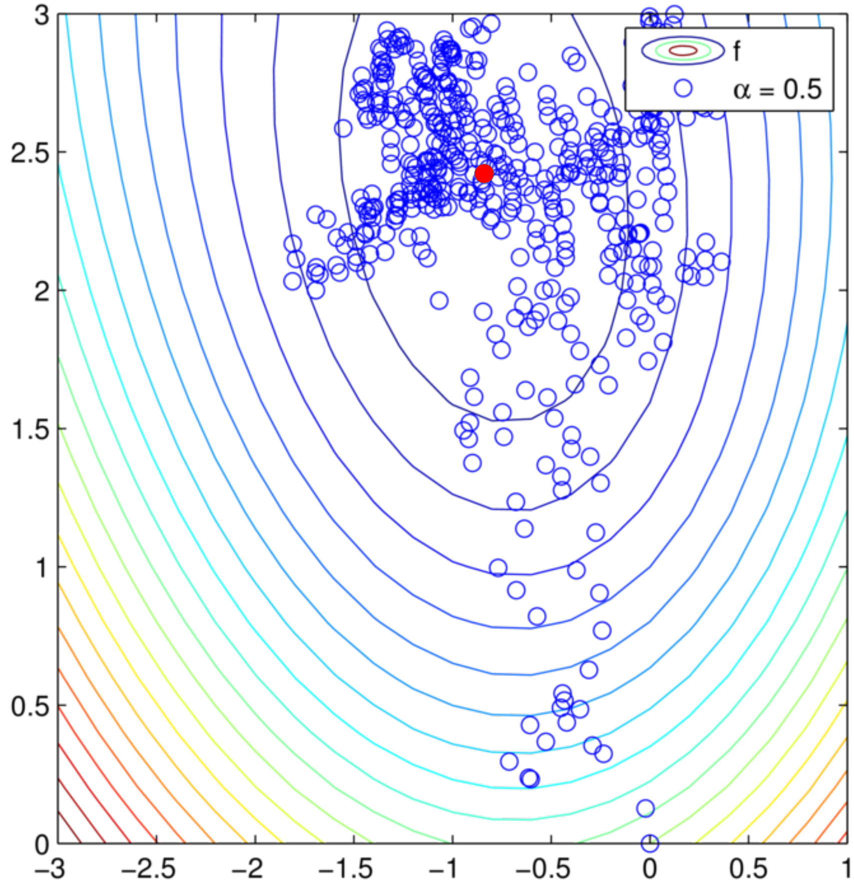
# The Stochastic Average Gradient




How to prove this converges? Is this the only option?

# Stochastic Gradient Descent

## α =0.5



# Proving Convergence

# Assumptions for Convergence

**Strong Convexity**

$$f(w) \geq f(y) + \langle \nabla f(y), w - y \rangle + \frac{\lambda}{2}||w - y||_2^2$$

**Smoothness**

$$f_i(w) \leq f_i(y) + \langle \nabla f_i(y), w - y \rangle + \frac{L_i}{2}||w - y||_2^2, \quad \text{for } i = 1, \ldots, n$$

**EXE:** Calculate $L_i$ and $L_{\max} := \max_{i=1,\ldots,n} L_i$ for

1. $f(w) = \frac{1}{2}||Aw - y||_2^2 + \frac{\lambda}{2}||w||_2^2, \quad \text{where } A \in \mathbb{R}^{n \times d}$
2. $f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + e^{-y_i \langle w, a_i \rangle}) + \frac{\lambda}{2}||w||_2^2$

# Assumptions for Convergence

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$$= \frac{1}{n}\sum_{i=1}^{n} f_i(w)$$

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$$= \frac{1}{n}\sum_{i=1}^{n} f_i(w)$$

$$\nabla^2 f_i(w) = nA_{i:}A_{i:}^\top + \lambda \quad \preceq \quad (n||A_{i:}||_2^2 + \lambda)I \quad = \quad L_i \; I$$

# Assumptions for Convergence

**EXE:** Calculate $L_{\mathrm{i}}$ for

2. $$f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + e^{-y_i \langle w, a_i \rangle}) + \frac{\lambda}{2}||w||_2^2$$

# Assumptions for Convergence

**EXE:** Calculate $L_i$ for

$$2. \quad f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + e^{-y_i \langle w, a_i \rangle}) + \frac{\lambda}{2}||w||_2^2$$

$$2. \quad f_i(w) = \ln(1 + e^{-y_i \langle w, a_i \rangle}) + \frac{\lambda}{2}||w||_2^2,$$

# Assumptions for Convergence

**EXE:** Calculate $L_i$ for

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$$2. \quad f_i(w) = \ln(1+e^{-y_i\langle w, a_i\rangle}) + \frac{\lambda}{2}||w||_2^2,$$

$$\nabla f_i(w) = \frac{-y_i a_i e^{-y_i\langle w, a_i\rangle}}{1+e^{-y_i\langle w, a_i\rangle}} + \lambda w$$

$$\nabla^2 f_i(w) = a_i a_i^\top \left( \frac{(1+e^{-y_i\langle w, a_i\rangle})e^{-y_i\langle w, a_i\rangle}}{(1+e^{-y_i\langle w, a_i\rangle})^2} - \frac{e^{-2y_i\langle w, a_i\rangle}}{(1+e^{-y_i\langle w, a_i\rangle})^2} \right) + \lambda I$$

$$= a_i a_i^\top \frac{e^{-y_i\langle w, a_i\rangle}}{(1+e^{-y_i\langle w, a_i\rangle})^2} + \lambda I \quad \preceq \quad \left( \frac{||a_i||_2^2}{4} + \lambda \right) I = L_i \; I$$

# Assumptions for Convergence

**EXE:** Let $f(w)$ be $L$–smooth and $f_i(w)$ be $L_i$–smooth for $i = 1, \ldots, n$.

Show that

$$L \quad \leq \quad \frac{1}{n}\sum_{i=1}^{n} L_i \quad \leq \quad L_{\max} := \max_{i=1,\ldots,n} L_i$$

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**Proof:** From definition of $f_i(w)$ smoothness

$$\frac{1}{n}\sum_{i=1}^{n} f_i(w) \leq \frac{1}{n}\sum_{i=1}^{n} f_i(y) + \langle \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(y), x - y\rangle + \frac{1}{2n}\sum_{i=1}^{n} L_i ||w-y||_2^2$$

$$= f(y) + \langle \nabla f(y), x-y\rangle + \frac{1}{2n}\sum_{i=1}^{n} L_i ||w-y||_2^2$$

# Convergence SVRG

**Theorem**

If $\alpha = 1/10L_{\max}$ and $m = 20L_{\max}/\lambda$ then

$$\mathbb{E}[f(\tilde{w}^t)] - f(w^*) \quad \leq \quad 0.9^t(f(\tilde{w}^0) - f(w^*))$$

Need $O(L_{\max}/\lambda)$ inner iterations to have linear convergence

In practice use $\alpha = 1/L_{\max},\ m = n$

Johnson, R. & Zhang, T. **Accelerating Stochastic Gradient Descent using Predictive Variance Reduction,** NIPS 2013

**Proof:**

$$\begin{aligned} ||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\ &= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2||g^k||_2^2. \end{aligned}$$

Taking expectation with respect to $j$

Unbiased estimator

$$\begin{aligned} \mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2\mathbb{E}_j\left[||g^k||_2^2\right] \\ &\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2\mathbb{E}_j\left[||g^k||_2^2\right] \end{aligned}$$

Must
control this!

$$\boxed{\mathbb{E}_j\left[||g^k||_2^2\right]}$$

# Smoothness Consequences I

**Smoothness**

$$f(w) \leq f(y) + \langle \nabla f(y), w - y \rangle + \frac{L}{2}||w - y||_2^2, \quad \text{for } i = 1, \ldots, n$$

**EXE: Lemma 1**

$$f(y - \frac{1}{L}\nabla f(y)) - f(y) \leq -\frac{1}{2L}\nabla f(y), \quad \forall y.$$

**Proof:**

Substituting $w = y - \dfrac{1}{L}\nabla f(y)$ into the smoothness inequality gives

$$\begin{aligned} f(y - \tfrac{1}{L}\nabla f(y)) - f(y) \quad &\leq \quad \langle \nabla f(y), -\frac{1}{L}\nabla f(y) \rangle + \frac{L}{2}|| - \frac{1}{L}\nabla f(y)||_2^2 \\ &= \quad -\tfrac{1}{2L}\nabla f(y). \quad \blacksquare \end{aligned}$$

# Smoothness Consequences II

**Smoothness**

$$f_i(w) \leq f_i(y) + \langle \nabla f_i(y), w - y \rangle + \frac{L_i}{2}||w-y||_2^2, \quad \text{for } i = 1, \ldots, n$$

**EXE: Lemma 2**

$$\mathbb{E}[||\nabla f_i(w) - \nabla f_i(w^*)||_2^2] \leq 2L_{\max}(f(w) - f(w^*))$$

**Proof:** Let $g_i(w) = f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$ which is $L_i$–smooth.

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**Proof:** Let $g_i(w) = f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$ which is $L_i$–smooth.

Convexity of $f_i(w) \Rightarrow g_i(w) \geq 0$ for all $w$. From Lemma 1 we have

$$g_i(w) \geq g_i(w) - g_i(w - \frac{1}{L_i}\nabla g_i(w)) \underbrace{\geq}_{\text{Lemma 1}} \frac{1}{2L_i}||\nabla g_i(w)||_2^2 \geq \frac{1}{2L_{\max}}||\nabla g_i(w)||_2^2$$

Inserting definition of $g_i(w)$ we have

$$\frac{1}{2L_{\max}}||\nabla f_i(w) - \nabla f_i(w^*)||_2^2 \leq f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$$

Result follows by taking expectation of $i$. ■

# Bounding gradient estimate

**EXE: Lemma 3**

$$\mathbb{E}[||g^k||_2^2] \leq 4L_{\max}(f(w^k) - f(w^*)) + 4L_{\max}(f(\tilde{w}^t) - f(w^*))$$

**Proof:** Hint: use $||a+b||_2^2 \leq 2||a||_2^2 + 2||b||_2^2$ and Lemma 2

Where we used in the first inequality that $\mathbb{E}[||X - \mathbb{E}X||_2^2] \leq \mathbb{E}[||X||_2^2]$ with $X = \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)$ thus $\mathbb{E}[X] = -\nabla f(\tilde{w}^t)$

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**Proof:** Hint: use $||a+b||_2^2 \leq 2||a||_2^2 + 2||b||_2^2$ and Lemma 2

$$\begin{aligned}
\mathbb{E}_j[||g^k||_2^2] &= \mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*) + \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t) + \nabla f(\tilde{w}^t)||_2^2] \\
&\leq 2\mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*)||_2^2] + 2\mathbb{E}_j[||\nabla f_i(w^*) - \nabla f_i(\tilde{w}^t) + \nabla f(\tilde{w}^t)||_2^2] \\
&\leq 2\mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*)||_2^2] + 2\mathbb{E}_j[||\nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)||_2^2] \\
&\overset{\text{Lemma 2}}{=} 4L_{\max}\left(f(w^k) - f(w^*) + f(\tilde{w}^t) - f(w^*)\right) \qquad \blacksquare
\end{aligned}$$

Where we used in the first inequality that $\mathbb{E}[||X - \mathbb{E}X||_2^2] \leq \mathbb{E}[||X||_2^2]$ with $X = \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)$ thus $\mathbb{E}[X] = -\nabla f(\tilde{w}^t)$

**Proof:**

$$||w^{k+1} - w^*||_2^2 = ||w^k - w^* - \alpha g^k)||_2^2$$
$$= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2||g^k||_2^2.$$

Taking expectation with respect to $j$

Unbiased estimator

$$\mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] = ||w^k - w^*||_2^2 - 2\alpha\langle\nabla f(w^k), w^k - w^*\rangle + \alpha^2\mathbb{E}_j\left[||g^k||_2^2\right]$$
$$\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2\mathbb{E}_j\left[||g^k||_2^2\right]$$

Must control this!

$$\boxed{\mathbb{E}_j\left[||g^k||_2^2\right]}$$

$$\mathbb{E}[||g^k||_2^2] \leq 4L_{\max}(f(w^k) - f(w^*)) + 4L_{\max}(f(\tilde{w}^t) - f(w^*))$$

**Proof (continued I):**

$$
\begin{aligned}
||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\
&= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2.
\end{aligned}
$$

Taking expectation with respect to $j$

Unbiased estimator

$$
\begin{aligned}
\mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\
&\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\
&\leq ||w^k - w^*||_2^2 - 2\alpha(1 - 2\alpha L_{\max})(f(w^k) - f(w^*)) \\
&\quad + 4\alpha^2 L_{\max}(f(\tilde{w}^t) - f(w^*))
\end{aligned}
$$

## **Proof (continued I):**

$$\begin{aligned} ||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\ &= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2. \end{aligned}$$

Taking expectation with respect to $j$

Unbiased estimator

$$\begin{aligned} \mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\leq ||w^k - w^*||_2^2 - 2\alpha(1 - 2\alpha L_{\max})(f(w^k) - f(w^*)) \\ &\quad + 4\alpha^2 L_{\max}(f(\tilde{w}^t) - f(w^*)) \end{aligned}$$

Taking expectation and summing from $k = 0, \ldots, m-1$ gives

$$\begin{aligned} \mathbb{E}\left[||w^m - w^*||_2^2\right] &\leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \\ &\quad + 4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right] \end{aligned}$$

**Proof (continued II):**

$$\mathbb{E}\left[||w^m - w^*||_2^2\right] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

$$2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right]$$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

$$\leq 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

**Proof (continued II):**

$$\mathbb{E}\left[||w^m - w^*||_2^2\right] \quad \leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Re-arranging and using strong convexity $f(\tilde{w}^t) - f(w^*) \geq \frac{\lambda}{2}||\tilde{w}^t - w^*||_2^2$

$$2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \quad \leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right]$$

$w^0 = \tilde{w}^t$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

$$\leq \quad 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

## **Proof (continued II):**

$$\mathbb{E}\left[||w^m - w^*||_2^2\right] \quad \leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Re-arranging and using strong convexity $f(\tilde{w}^t) - f(w^*) \geq \frac{\lambda}{2}||\tilde{w}^t - w^*||_2^2$

$$\begin{aligned} 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \quad &\leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right] \\ &\quad +4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right] \\ &\leq \quad 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right] \end{aligned}$$

$w^0 = \tilde{w}^t$

Re-arranging again

$$\begin{aligned} \mathbb{E}[(f(\sum_{k=0}^{m-1}\frac{w^k}{m}) - f(w^*))] \quad &\leq \quad \mathbb{E}[\frac{1}{m}\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \\ &\leq \quad \left(\frac{2\alpha L_{\max}}{1 - 2\alpha L_{\max}} + \frac{1}{\lambda\alpha(1 - 2\alpha L_{\max})m}\right)\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right] \end{aligned}$$

Jensen's inequality

Now plug in values $\alpha = 1/(10L_{\max})$ and $m = 20L_{\max}/\lambda$ ■

# Convergence SAGA

**Theorem SAGA**

If $\alpha = 1/3L_{\max}$ then

$$\mathbb{E}\left[||w^t - w^*||_2^2\right] \leq \left(1 - \min\left\{\frac{1}{4n}, \frac{\lambda}{3L_{\max}}\right\}\right)^t ||w^0 - w^*||_2^2$$

A practical convergence result!

M. Schmidt, N. Le Roux, F. Bach (2016)
Mathematical Programming
**Minimizing Finite Sums with the Stochastic Average Gradient.**

# Comparisons in complexity for strongly convex

**Approximate solution**

$$\mathbb{E}[f(w^T)] - f(w^*) \leq \epsilon$$

**SGD**

$$O\left(\frac{1}{\lambda\epsilon}\right)$$

**Gradient descent**

$$O\left(\frac{nL}{\lambda}\log\left(\frac{1}{\epsilon}\right)\right)$$

**SVRG/SAGA**

$$O\left(\left(n + \frac{L_{\max}}{\lambda}\right)\log\left(\frac{1}{\epsilon}\right)\right)$$

Variance reduction faster than GD when $L \geq \lambda + L_{\max}/n$

How did I get these complexity results from the convergence results?

Section 1.3.5, R.M. Gower, Ph.d thesis: Sketch and Project: Randomized Iterative Methods for Linear Systems and Inverting Matrices University of Edinburgh, 2016

# Take for home Variance Reduction

- Variance reduced methods use only **one stochastic gradient per iteration** and converge linearly on strongly convex functions

- Choice of **fixed stepsize** possible

- **SAGA** only needs to know the smoothness parameter to work, but requires storing $n$ past stochastic gradients

- **SVRG** only has $O(d)$ stored, but requires full gradient computations every so often