Convexity, Smoothness and the Gradient Method

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Solving the Finite Sum Training Problem

Optimization Sum of Terms

A Datum Function

$$f_i(w) := \ell\left(h_w(x^i), y^i\right) + \lambda R(w)$$

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} \ell\left(h_w(x^i), y^i\right) + \lambda R(w) &= \frac{1}{n}\sum_{i=1}^{n}\left(\ell\left(h_w(x^i), y^i\right) + \lambda R(w)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} f_i(w)\end{aligned}$$

Finite Sum Training Problem

$$\min_{w\in\mathbf{R}^d} \frac{1}{n}\sum_{i=1}^{n} f_i(w) =: f(w)$$

The Training Problem

Solving the *training problem*:

$$\min_{w \in \mathbf{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(w)$$

Reference method: Gradient descent

$$\nabla \left(\frac{1}{n} \sum_{i=1}^{n} f_i(w) \right) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w)$$

Gradient Descent Algorithm

Set $w^0 = 0$, choose $\alpha > 0$.

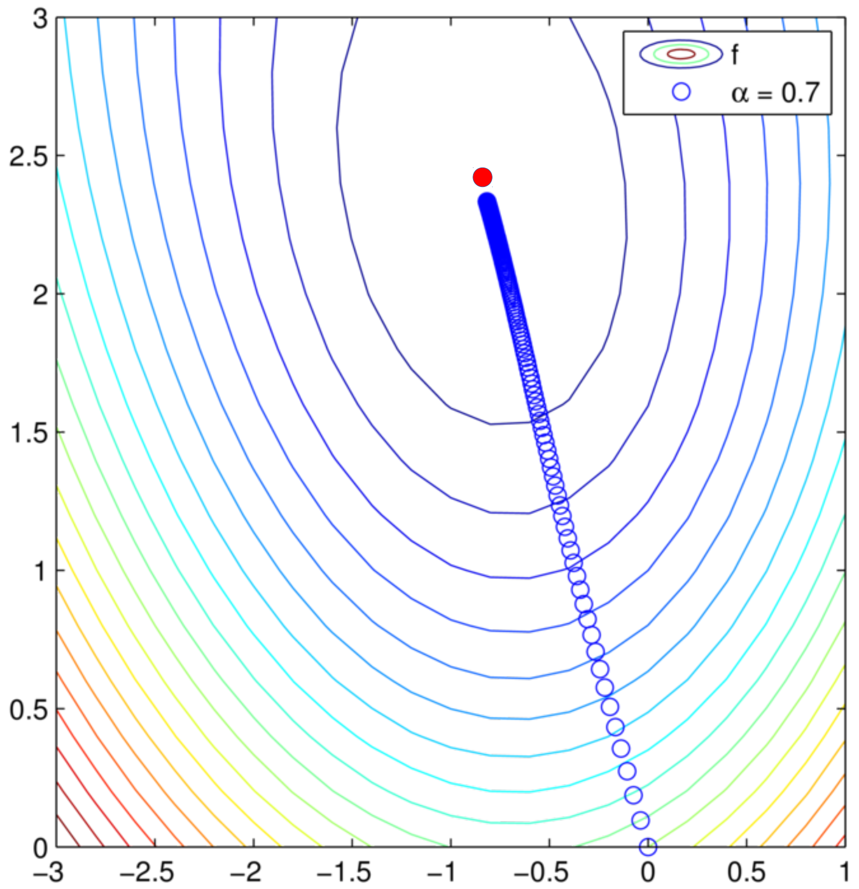
for $t = 1, 2, 3, \ldots, T$

$$w^{t+1} = w^t - \frac{\alpha}{n} \sum_{i=1}^{n} \nabla f_i(w^t)$$

Output w^{T+1}

Gradient Descent Example

A Logistic Regression problem using the fourclass labelled data from LIBSVM
$(n,\ d) = (862{,}2)$

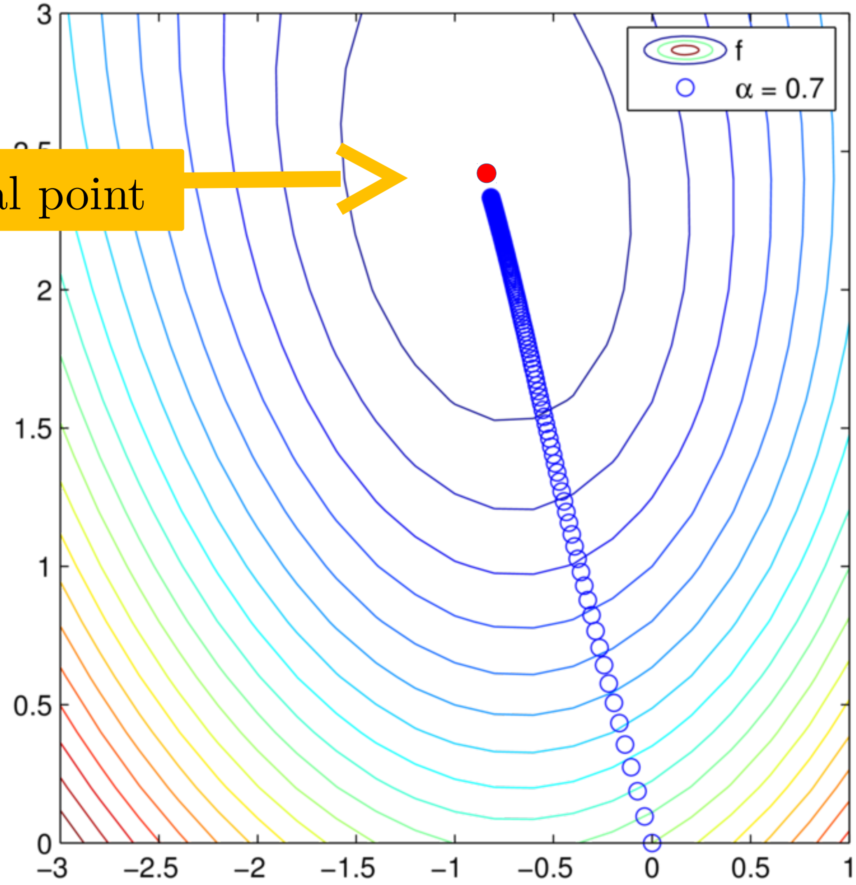


Can we prove that this always works!?

Convex training problems

Gradient Descent Example

Optimal point



A Logistic Regression problem using the fourclass labelled data from LIBSVM

$(n,\ d) = (862{,}2)$

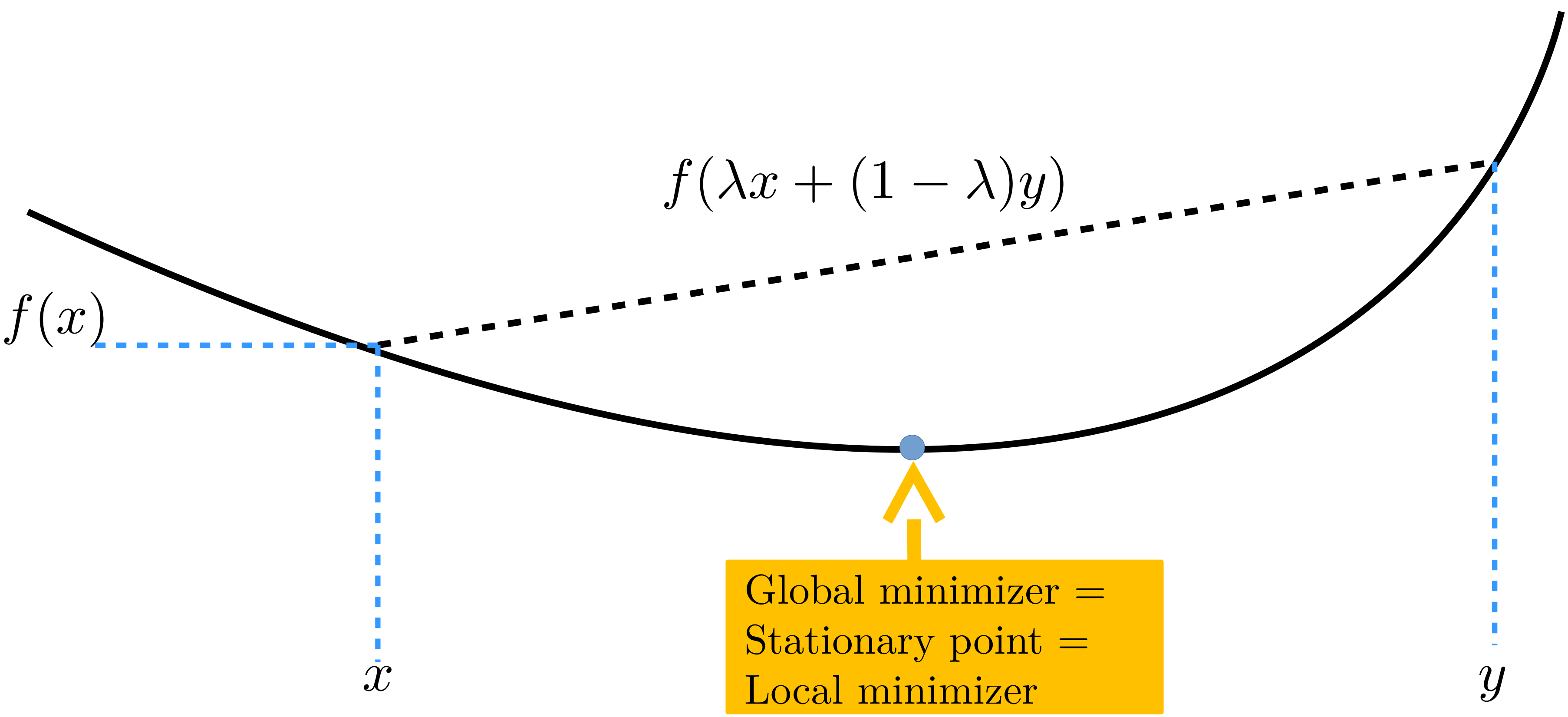
Can we prove that this always works!?

Convex training problems

Convexity

We say $f : \mathrm{dom}(f) \subset \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is convex and

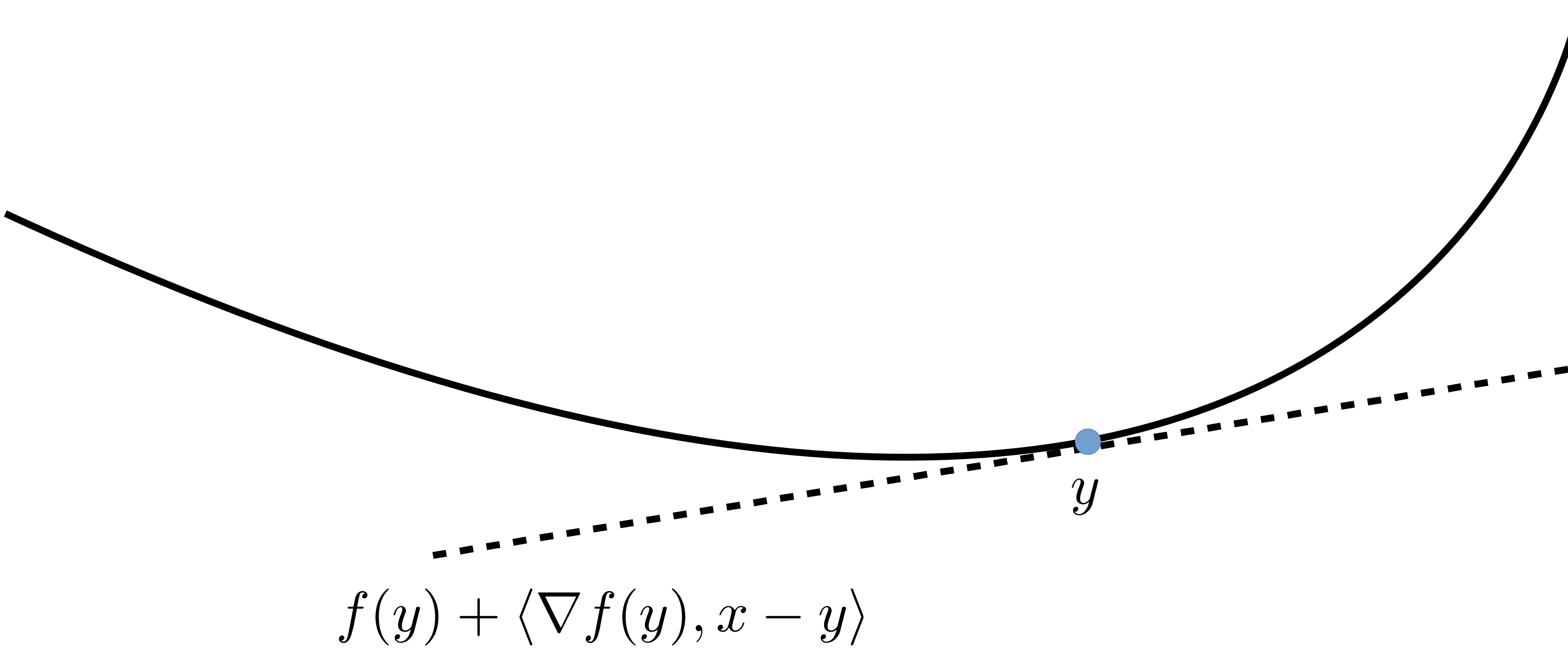
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in C, \lambda \in [0, 1]$$



Convexity: First derivative

A differential function $f : \mathrm{dom}(f) \subset \mathbb{R}^n \to \mathbb{R}$ is convex iff

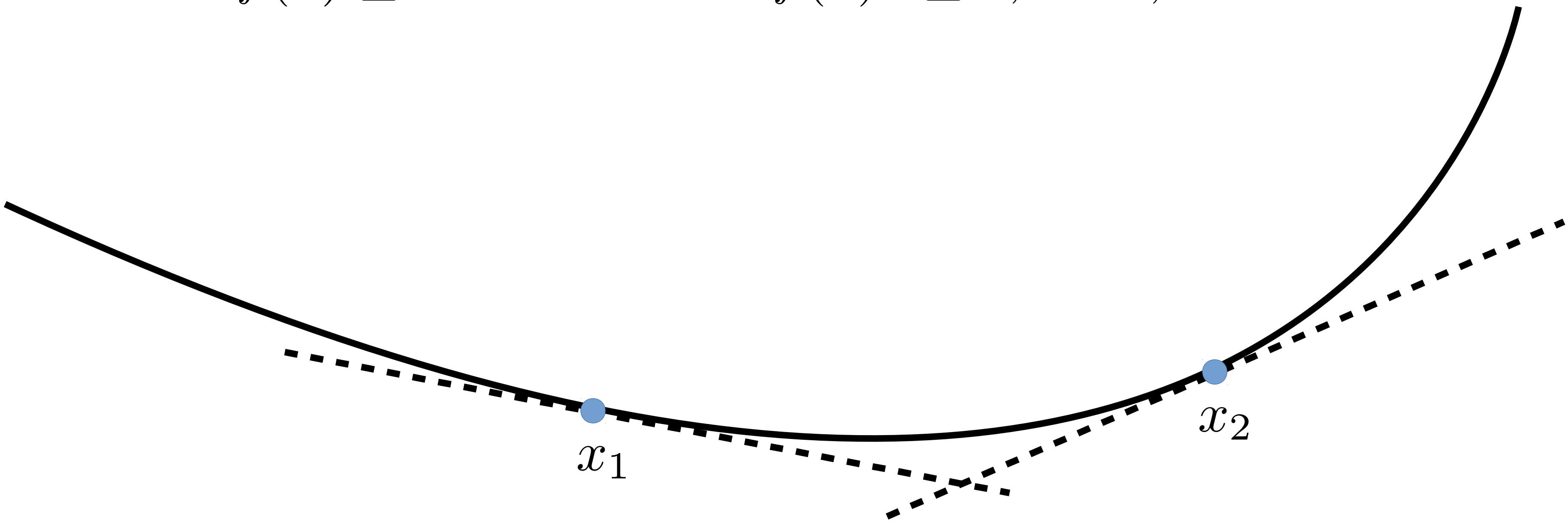
$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$$



Convexity: Second derivative

A twice differential function $f : \mathrm{dom}(f) \subset \mathbb{R}^n \to \mathbb{R}$ is convex iff

$$\nabla^2 f(x) \succeq 0 \quad \Leftrightarrow \quad v^\top \nabla^2 f(x) v \geq 0, \quad \forall x, v \in \mathbb{R}^n$$



$$x_1 \leq x_2 \quad \Rightarrow f'(x_1) \leq f'(x_2)$$

Convexity: Examples

Extended-value extension:

$$f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$$

$$f(x) = \infty, \quad \forall x \notin \mathrm{dom}(f)$$

Norms and squared norms:

$$x \mapsto ||x||$$

$$x \mapsto ||x||^2$$

Proof is an exercise!

Negative log and logisitc:

$$x \mapsto -\log(x)$$

$$x \mapsto \log\left(1 + e^{-y\langle a, x\rangle}\right)$$

Hinge loss

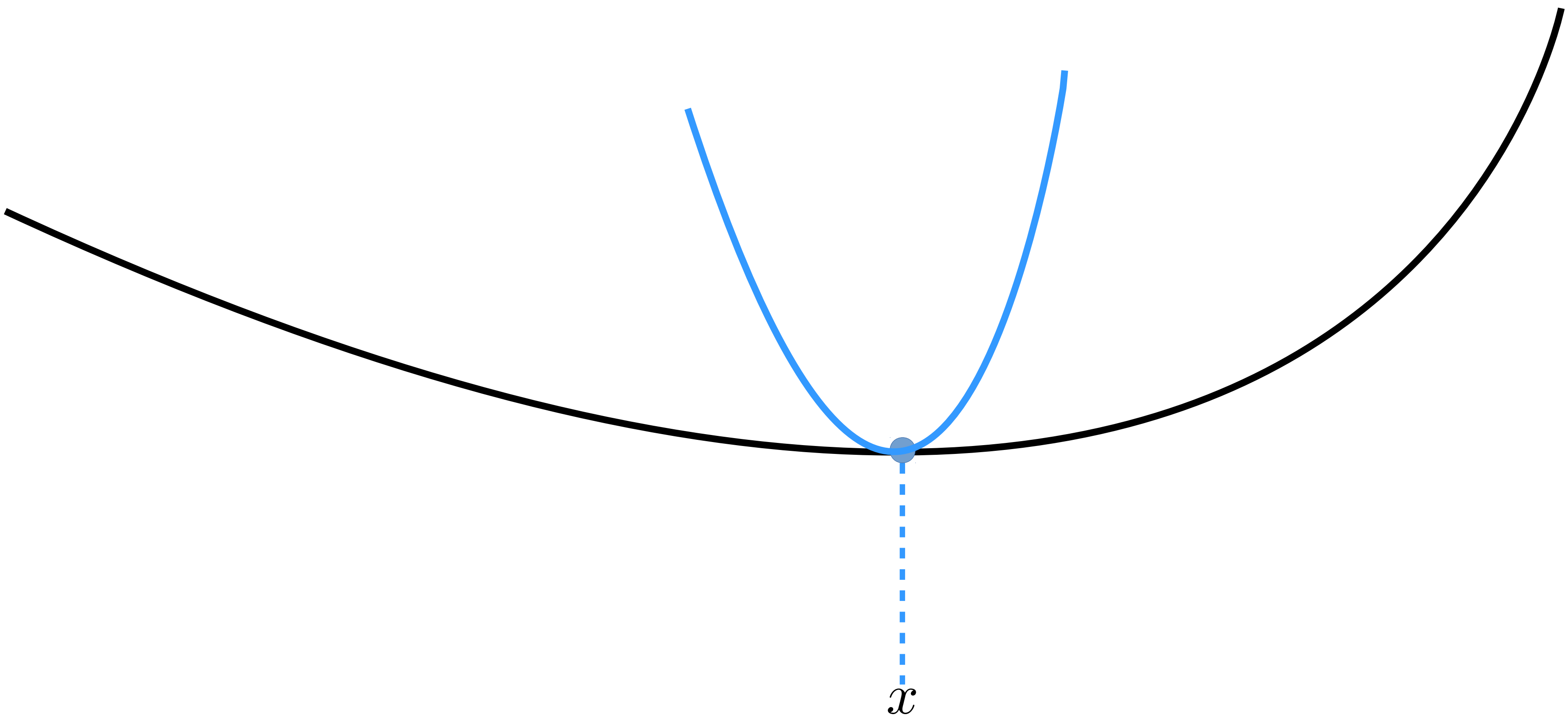
$$x \mapsto \max\{0, 1 - yx\}$$

Negatives log determinant, exponentiation ... etc

Smoothness

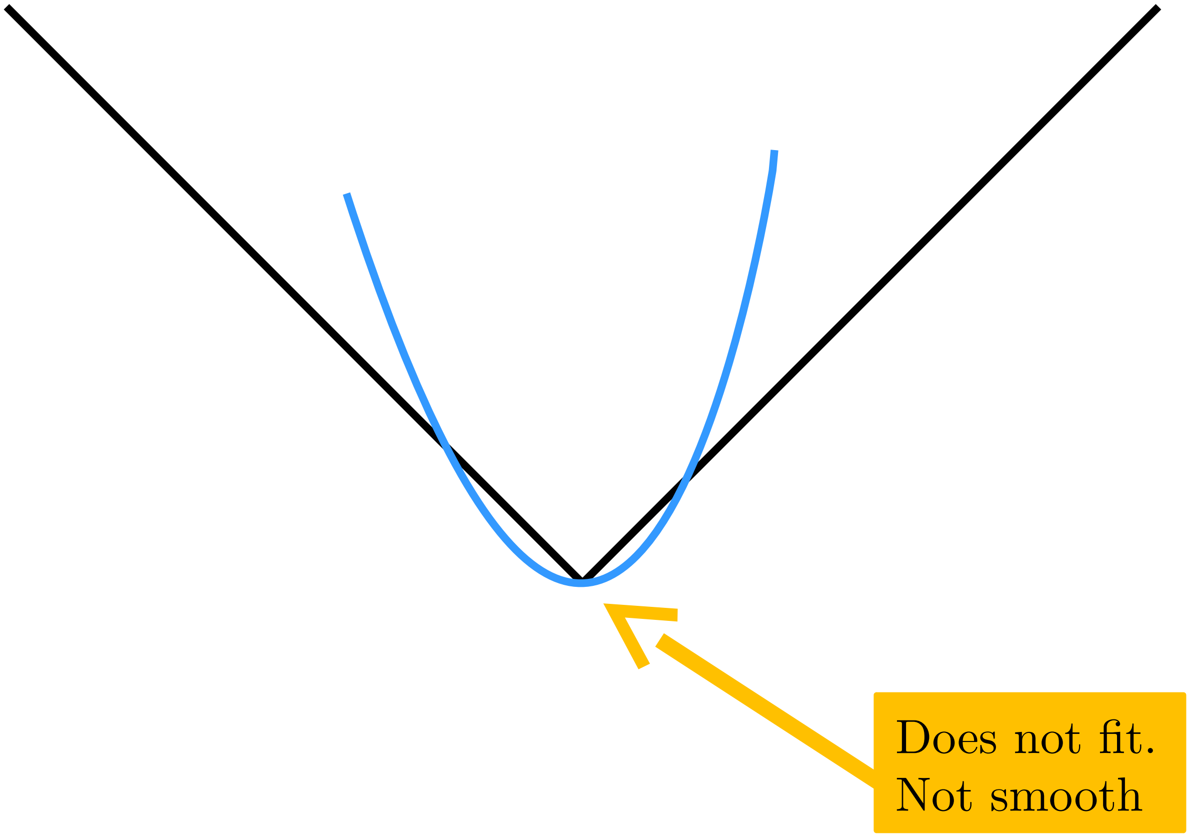
We say $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is smooth if

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}||x - y||^2, \quad \forall x, y \in \mathbb{R}^n$$



Smoothness: Convex counter-example

$$f(x) = ||x||_1 = \sum_{i=1}^{n} |x_i|$$



Smoothness Equivalence

We say $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is L–smooth if

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \quad \forall x, y \in \mathbb{R}^n$$

$$\nabla^2 f(x) \preceq L \cdot I, \quad \forall x \in \mathbb{R}^n$$

$$\boxed{f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}||x - y||^2, \quad \forall x, y \in \mathbb{R}^n}$$

Proof on board

Insight into Gradient Descent

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}||x - y||^2, \quad \forall x, y \in \mathbb{R}^n$$

Minimizing the upper bound in x we get:

$$\nabla_x \left(f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}||x - y||^2 \right) = \nabla f(y) + L(x - y) = 0$$

$$x = y - \frac{1}{L}\nabla f(y)$$

A gradient descent step !

Smoothness: Examples

Convex quadratics: $x \mapsto x^T A x + b^\top x + c$

Logisitc: $x \mapsto \log\left(1 + e^{-y\langle a,x\rangle}\right)$

Trigonometric: $x \mapsto \cos(x), \sin(x)$

Proof is an exercise!

Smoothness Properties

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is smooth then

$$f(x - \tfrac{1}{L}\nabla f(x)) - f(x) \leq -\frac{1}{2L}||\nabla f(x)||_2^2, \quad \forall x \in \mathbb{R}^n$$

$$f(x^*) - f(x) \leq -\frac{1}{2L}||\nabla f(x)||_2^2, \quad \forall x \in \mathbb{R}^n$$

Proof on board

Convex and Smooth Properties

If $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ convex and L–smooth then

$$f(y) - f(x) \leq \langle \nabla f(y), y - x \rangle - \frac{1}{2L} ||\nabla f(y) - \nabla f(x)||_2^2$$

Co-coercivity

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \frac{1}{L} ||\nabla f(x) - \nabla f(y)||_2$$

change order x and y and take the sum

Proof on board

Convergence GD I

Theorem

Let f be convex and L-smooth.

$$f(x^T) - f(x^1) \leq \frac{2L||x^1 - x^*||_2^2}{T-1} = O\left(\frac{1}{T}\right).$$

Where

$$x^{t+1} = x^t - \frac{1}{L}\nabla f(x^t)$$

Proof on board

$$\Rightarrow \text{for } \frac{f(x^T) - f(x^*)}{||x^1 - x^*||_2^2} \leq \epsilon \text{ we need } T \geq \frac{2L}{\epsilon} = O\left(\frac{1}{\epsilon}\right)$$

Strong convexity

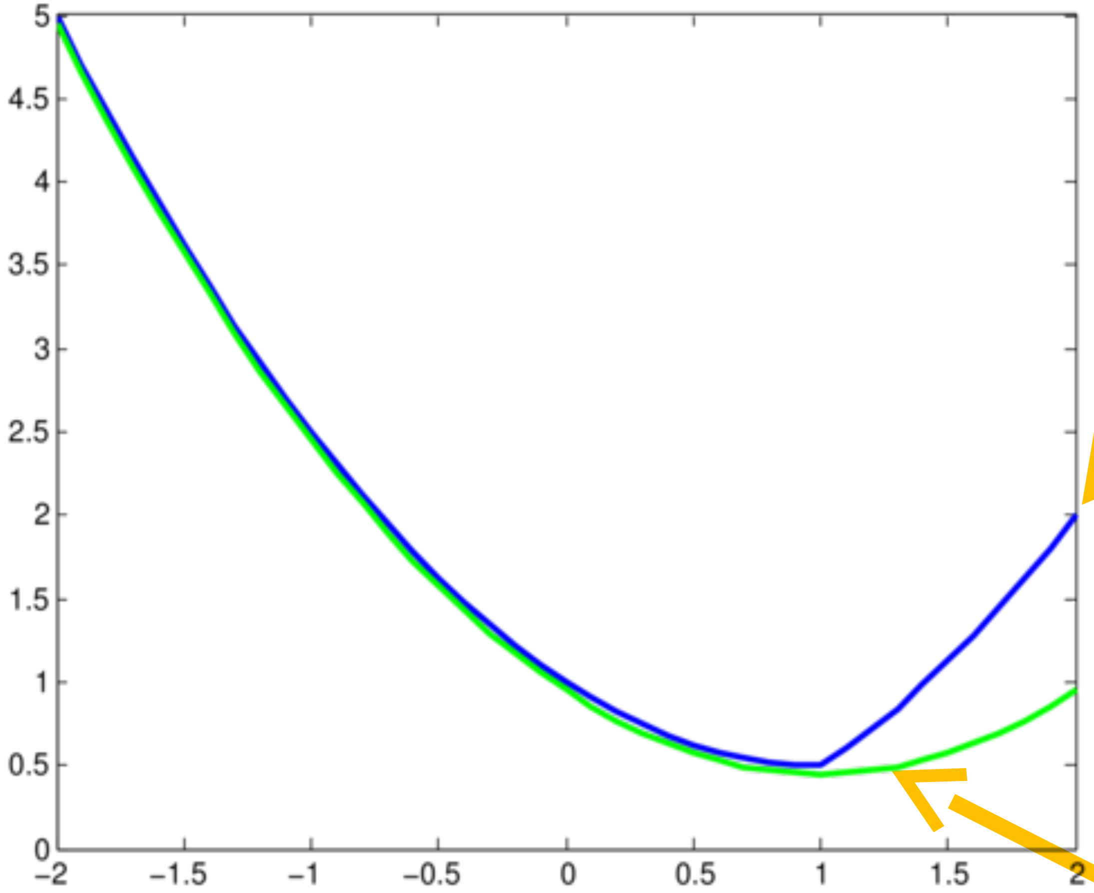
We say $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is μ–strongly convex if

$$||\nabla f(x) - \nabla f(y)|| \geq \mu ||x - y||, \quad \forall x, y \in \mathbb{R}^n$$

$$\nabla^2 f(x) \succeq \mu \cdot I, \quad \forall x \in \mathbb{R}^n$$

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} ||x - y||^2, \quad \forall x, y \in \mathbb{R}^n$$

Example of Strong Convexity



Hinge loss + L2

$$\max\{0, 1 - x\} + \frac{1}{2}||x||_2^2$$

Quadratic lower bound

Strong Convexity Properties

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is μ–strongly convex then

$$||\nabla f(x)||_2^2 \geq 2\mu(f(x) - f(x^*)), \quad \forall x \in \mathbb{R}^n$$

This is known as the *Polyak- Lojasiewicz* inequality.

Proof on board

Convergence GD II

Theorem

Let f be μ-convex and L-smooth.

$$||x^T - x^*||_2^2 \leq \left(1 - \frac{\mu}{L}\right)^T ||x^1 - x^*||_2^2$$

Where

$$x^{t+1} = x^t - \frac{1}{L}\nabla f(x^t)$$

Proof on board

$\Rightarrow$ for $\frac{||x^T - x^*||_2^2}{||x^1 - x^*||_2^2} \leq \epsilon$ we need $T \geq \frac{L}{\mu}\log\left(\frac{1}{\epsilon}\right) = O\left(\log\left(\frac{1}{\epsilon}\right)\right)$