CS395T Numerical Optimization for Graphics & AI Homework 4

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November 14th, 2017

1 Programming

Problems 1 and 2

We are interested in solving

$$\min_{x_i, 1 \le i \le m} \sum_{i=1}^{m} \|A_i x_i - b_i\|^2 + \lambda \sum_{1 \le i < j \le m} \|x_i - x_j\|_1$$

where $\|a\|_1 = \sum_{i=1}^{n} |a_i|$ stands for the L1-norm of a vector. Here A_i and b_i are constant matrices. Please apply proximal gradient descent to solve (1). An example dataset can be downloaded from https://www.cs.utexas.edu/~huangqx/hw4_data.mat.

Solution

In order to minimize the objective function using the proximal gradient method, we can reformulate it as

$$f(x) = g(x) + h(x)$$

where

$$g(x) = \sum_{i=1}^{m} \|A_i x_i - b_i\|^2 \qquad h(x) = \lambda \sum_{1 \le i < j \le m} \|x_i - x_j\|_1$$

At each iteration, we perform an update using the proximal functio of $h(x)$,

$$\begin{aligned} x^{k+1} &= \mathbf{prox}_{th}(x - t\nabla g(x)) \\ &= \min_u (h(u) + \frac{1}{2t}\|u - x + t\nabla g(x)\|_2^2) \\ &= \min_u (h(u) + g(x) + \nabla g(x)^T (u - x) + \frac{1}{2t}\|u - x\|_2^2) \end{aligned}$$

We can solve for the min u by setting the derivative equal to 0,

$$\begin{aligned} \frac{d}{du}(h(u) + g(x) + \nabla g(x)^T (u - x) + \frac{1}{2t}\|u - x\|_2^2) = 0 \\ \Rightarrow \frac{d}{du}h(u) + \frac{d}{du}g(x) + \frac{d}{du}\nabla g(x)^T (u - x) + \frac{d}{du}\frac{1}{2t}\|u - x\|_2^2 = 0 \\ \Rightarrow \frac{d}{du}h(u) + \nabla g(x)^T + \frac{1}{t}(u - x) = 0 \end{aligned}$$

2 Theory

Problem 3

Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exists a nonzero vector $\mathbf{v}$ and a real number c such that

$$\langle \mathbf{x}, \mathbf{v} \rangle \geq c \text{ and } \langle \mathbf{y}, \mathbf{v} \rangle \leq c$$

for all $\mathbf{x} \in \mathcal{A}$ and $\mathbf{y} \in \mathcal{B}$, i.e, the hyperplane $\langle \cdot, \mathbf{v} \rangle = c, \mathbf{v}$ the normal vector, separates $\mathcal{A}$ and $\mathcal{B}$.

Solution

Given the disjoint, nonempty convex sets A, B, let

$$S = A + (-B) = \{x - y | x \in A, y \in B\}$$

Because B is convex, $-B$ is also convex and thus the sum of the convex sets A and $-B$ is also convex. We can define $\hat{S}$ to be the closure of S and we know $\hat{S}$ is convex. Because $\hat{S}$ is convex, we know that there exists a vector v of minimum norm. For any $z \in S$, we know that

$$v + t(z - v), 0 \leq t \leq 1$$

lies within the closure of S, $\hat{S}$. Therefore, we know that for $0 \leq t \leq 1$,

$$\begin{aligned} \|v\|^2 &\leq \|v - t(z - v)\|^2 \\ \Rightarrow \|v\|^2 &\leq \|v\|^2 - 2t\langle v, z - v \rangle + t^2 \|z - v\|^2 \\ \Rightarrow 0 &\leq 2\langle v, z \rangle - 2\|v\|^2 + t\|v, z\|^2. \end{aligned}$$

As $t \to 0$, then $\|v\|^2 \leq \langle v, z \rangle$. From earlier, we can represent z as $x - y$ where $x \in A, y \in B$ and thus,

$$\|v\|^2 \leq \langle x - y, v \rangle$$

Therefore for any $x \in A$ and $y \in B$,

$$\langle x, v \rangle \geq \langle y, v \rangle + \|v\|^2$$