Homework 4 Theory Portion

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November 2, 2017

Separating Hyperplane Problem:

In my Methods Applied Mathematics I proved the following theorem which I will here take for granted:

Theorem 1. *(Hahn-Banach Theorem for Real Vector Spaces). Suppose that X is a vector space over $\mathbb{R}$, Y a linear subspace, and p is **sublinear** on X. If f is a linear functional on Y such that*

$$f(x) \leqslant p(x)$$

for all $x \in Y$, then there is a linear functional F on X such that

$$F|_Y = f$$

(i.e. F is a linear extension of f) and

$$-p(-x) \leqslant F(x) \leqslant p(x)$$

for all $x \in X$.

Now I am in a position to state and prove the Separating Hyperplane Theorem. This is a gnerealization of the homework problem when it is well posed (i.e. one of the sets needs to be open)

Lemma 1. *(Separating Hyperplane Theorem) Let A and B be disjoint, nonempty, convex sets in an NLS X.*

(a) If A is open, there there is an $f \in X^*$ and a $\gamma \in \mathbb{R}$ such that

$$Ref(x) \leqslant \gamma \leqslant Ref(y) \quad \forall x \in A \quad y \in B$$

(b) If both A and B are open, there there is an $f \in X^*$ and a $\gamma \in \mathbb{R}$ such that

$$Ref(x) < \gamma < Ref(y) \quad \forall x \in A \quad y \in B$$

(c) If A is compact and B is closed, there there is an $f \in X^*$ and a $\gamma \in \mathbb{R}$ such that

$$Ref(x) < \gamma < Ref(y) \quad \forall x \in A \quad y \in B$$

Notice that when the ground field is $\mathbb{F} = \mathbb{C}$ it is the real part of f that separates A and B.

Proof. It is sufficient to prove the result when the ground field $\mathbb{F} = \mathbb{R}$. For if $\mathbb{F} = \mathbb{C}$, first view X as a real Banach space and infer existence of a continuous, real-linear functional g satisfying the separation result. Then, construct $f \in X^*$ as follows:

$$f(x) = g(x) - ig(ix)$$

Thus attention is restricted to the case $\mathbb{F} = \mathbb{R}$.

For (a) fix $-w \in A - B = \{x - y \mid (x, y) \in A \times B\}$, and let

$$C = A - B + w$$

an open, convex neighborhood of 0 in X. Then $w \notin C$ wince A and B are disjoint. Define the subspace $Y = \mathbb{R}w$ and the linear functional $g : Y \to \mathbb{R}$ by

$$g(tw) \quad = \quad t$$

Now let $p : X \to [0, \infty)$ be the Minkowski functional for C,

$$p(x) \quad = \quad \inf \left\{ t > 0 \;\middle|\; \frac{x}{t} \in C \right\}$$

Since $w \notin C$, $p(w) \geqslant 1$, and so $g(y) \leqslant p(y)$ for $y \in Y$. Use the Hahn-Banach Theorem for real-linear functionals to extend g toa linear mapping on all of X which is still bounded by p. Now $g \leqslant 1$ on C, so also $g \leqslant -1$ on $-C$, and therefore $|g| \leqslant 1$ on $C \cap (-C)$, which is a neighborhood of 0. Thus g is bounded, and so continuous.

If $a \in A$ and $b \in B$, then $a - b + w \in C$, so

$$1 \quad \geqslant \quad g(a - b + w) \quad = \quad g(a) - g(b) + g(w) \quad = \quad g(a) - g(b) + 1$$

which gives that $g(a) \leqslant g(b)$ and the result follows with $\gamma = \sup\limits_{a \in A} g(a)$

(b) and (c) use similar constructions $\square$