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Problem 3

Since $A \succ 0$, let $A = U^T U$.

let $q = Up$, Then the constraint become

$d^2 \geq p^T A p = p^T U^T U p = q^T q = \|q\|_2^2$.

Plug in $p = U^{-1} q$ into the objective we have:

$$\underset{q}{\text{minimize}} \quad g^T U^{-1} q + \frac{1}{2} (U^{-1} q)^T B (U^{-1} q)$$

$$\Leftrightarrow (U^{-T} g)^T q + \frac{1}{2} q^T (U^{-T} B U^{-1}) q$$

s.t. $\|q\| \leq d$.

Here $U^{-T} = (U^{-1})^T = (U^T)^{-1}$.

solving the above optimization we have:

$$\begin{cases} (U^{-T} B U^{-1} + \lambda I) q^* = -U^{-T} g & ① \\ \lambda (\|q^*\| - d) = 0 & ② \\ U^{-T} B U^{-1} + \lambda I \succeq 0 & ③ \end{cases}$$

Left Multiply by U^T in ① we have

$$U^T (U^{-T} B U^{-1} + \lambda I) U p^* = -U^T U^{-T} g$$

$\Leftrightarrow (B + \lambda U^T U) p^* = -g \Leftrightarrow (B + \lambda A) p^* = -g$

② $\Leftrightarrow \lambda (\sqrt{p^{*T} A p^*} - d) = 0$

③ $\Leftrightarrow B + \lambda U^T U \succeq 0 \Leftrightarrow B + \lambda A \succeq 0$

so:
$$\begin{cases} (B + \lambda A) p^* = -g & \text{for } \lambda \geq 0 \\ \lambda (\sqrt{p^{*T} A p^*} - d) = 0 \\ B + \lambda A \succeq 0 \end{cases}$$