Numerical Optimization HW 3

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October 19, 2017

1 Problem 3

The sufficient and necessary condition is as follow, there exist a $\lambda \geq 0$

$$(B + \lambda A)\mathbf{p}^* = -g \tag{1}$$

$$\lambda(d^2 - \mathbf{p}^{*T} A\mathbf{p}^*) = 0 \tag{2}$$

$$(B + \lambda A)\ is\ PSD \tag{3}$$

Proof.
This proof use Lemma 3.1 in the lecture.
sufficient:
let

$$\hat{m}(\mathbf{p}) = \mathbf{g}^T\mathbf{p} + \frac{1}{2}\mathbf{p}^T(B + \lambda A)\mathbf{p} = m(\mathbf{p}) + \frac{1}{2}\lambda\mathbf{p}^T A\mathbf{p} \tag{4}$$

Since $(B + \lambda A)$ is PSD, $\mathbf{p}^*$ is a global minimum of $\hat{m}(\mathbf{p})$. That is

$$m(\mathbf{p}) \geq m(\mathbf{p}^*) + \frac{\lambda}{2}(\mathbf{p}^{*T} A\mathbf{p}^* - \mathbf{p}^T A\mathbf{p}) \tag{5}$$

Because $\lambda(d^2 - \mathbf{p}^{*T} A\mathbf{p}^*) = 0$, we have

$$m(\mathbf{p}) \geq m(\mathbf{p}^*) + \frac{\lambda}{2}(d^2 - \mathbf{p}^T A\mathbf{p}) \tag{6}$$

Thus, given the constrain of this problem, $\mathbf{p}^{*T} A\mathbf{p}^* \leq d^2$, $m(\mathbf{p}) \geq m(\mathbf{p}^*)$. That is p^* is a global minimum of the sub-problem.

necessary:
Assume p^* is a global solution of the sub-problem. When $\mathbf{p}^{*T} A\mathbf{p}^* < d^2$, the sub-problem is equivalent to the unconstrained problem such that B is PSD and $\mathbf{p}^* = -B^{-1}g$. This satifies the condition(1-3) above by $\lambda = 0$.
Otherwise $\mathbf{p}^{*T} A\mathbf{p}^* = d^2$, condition(2) naturally satisfied. we consider Lagrangian function

$$L(\mathbf{p}, \lambda) = m(\mathbf{p}) + \frac{\lambda}{2}(\mathbf{p}^T A\mathbf{p} - d^2) \tag{7}$$

Let the gradient equal to zero, we have

$$(B + \lambda A)\mathbf{p}^* = -g \tag{8}$$

Since $\mathbf{p}^*$ is a globle solution of $m(\mathbf{p})$,

$$m(\mathbf{p}) \geq m(\mathbf{p}^*) + \frac{\lambda}{2}(\mathbf{p}^{*T} A \mathbf{p}^* - \mathbf{p}^T A \mathbf{p}) \tag{9}$$

Conbining 8, we have

$$\frac{1}{2}(\mathbf{p} - \mathbf{p}^*)^T (B + \lambda A)(\mathbf{p} - \mathbf{p}^*) \geq 0 \tag{10}$$

Since $\mathbf{p}$ is dense we can conclude $(B + \lambda A)$ is PSD.
The last thing is to show there exist $\lambda \geq 0$. We firstly consider that all $\lambda < 0$, according to 5, if $\lambda < 0$, $m(\mathbf{p}) \geq m(\mathbf{p}^*)$ whenever $\mathbf{p}^{*T} A \mathbf{p}^* \leq \mathbf{p}^T A \mathbf{p}$. Since we know $\mathbf{p}^*$ is a global solution of the sub-problem, we know $\mathbf{p}^*$ is also the global solution of the unconstrained form of the origin sub-problem. So, condition (1)(3) also hold on $\lambda \geq 0$, contradicting to them $\lambda < 0$ assumption.