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Automatica xx (xxxx) xxx–xxx

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Technical Communique

Q1 Parameter-memorized Lyapunov functions for discrete-time systems with time-varying parametric uncertainties☆

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ARTICLE INFO

Article history:
Received 29 March 2017
Received in revised form 11 August 2017
Accepted 12 September 2017
Available online xxxx

Keywords:
Discrete-time systems
Parameter-memorized Lyapunov function
Time-varying parameters

ABSTRACT

In this paper, a novel parameter-memorized Lyapunov function is proposed for stability analysis of discrete-time linear systems with time-varying parametric uncertainties. The parameter-memorized Lyapunov function depends on the uncertain parameters with a memory of a certain interval. It is shown that the previous parameter-dependent Lyapunov function is a special memoryless case of the parameter-memorized Lyapunov function, and as a result, the parameter-memorized Lyapunov function approach leads to less conservative results. Furthermore, if the length of the interval for parameter memory is sufficiently long, a nonconservative stability analysis result can be achieved. Numerical examples are provided to evaluate the obtained theoretical results.

1. Introduction

In this brief note, we consider an uncertain linear discrete-time system in the form of

$$x(k+1) = A(\xi(k))x(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, time-varying matrices $A(\xi(k))$ are given in the following form $A(\xi(k)) \in \{A(\xi(k)) : A(\xi(k)) = \sum_{i=1}^{N}\xi_i(k)A_i\}$, where $A_1, \ldots, A_N \in \mathbb{R}^{n\times n}$ are given constant matrices and the time-varying uncertain parameter vector $\xi(k) = [\xi_1(k) \cdots \xi_N(k)]^\top \in \Xi \triangleq \{\xi : \sum_{i=1}^{N}\xi_i = 1,\ \xi_i \geq 0\}$.

One of the fundamental problems for uncertain system (1) is stability analysis, which has been extensively studied in the literature (Chesi, Garulli, Tesi, & Vicino, 2009). A variety of Lyapunov function based approaches have been proved to be powerful tools for stability analysis of uncertain systems. In some early works, a single common Lyapunov function (CLF) in the quadratic form was widely used as a straightforward extension of quadratic Lyapunov function approach of nominal linear systems, see (Dorato, Tempo, & Muscato, 1993) and references therein. However, despite its simple structure, CLF usually yields overly conservative results in stability analysis. For the sake of further improvements, several advanced Lyapunov functions are employed such as common nonquadratic Lyapunov functions (Geromel & Colaneri, 2006; Hu & Blanchini, 2010; Hu & Lin, 2003; Johansson & Rantzer, 1998; Xie, Shishkin, & Fu, 1997) where necessary and sufficient conditions were derived in Hu and Blanchini (2010), piecewise quadratic Lyapunov functions (Blanchini, 1995; Blanchini & Miani, 1999; Brayton & Tong, 1979), homogeneous polynomial Lyapunov functions (Chesi, 2008, 2011, 2013; Chesi, Garulli, Tesi, & Vicino, 2007; Xiang, 2016; Zelentsovsky, 1994), etc. Among these significant improvements, a parameter-dependent idea has been widely employed to construct parameter-dependent Lyapunov functions (PDLF), which can greatly reduce the conservativeness for the stability analysis results (Chesi, Garulli, Tesi, & Vicino, 2005; Daafouz & Bernussou, 2001; Geromel & Colaneri, 2006). One of the important results for discrete-time systems with time-varying parametric uncertainties was proposed in Daafouz and Bernussou (2001), where a quadratic PDLF is constructed and the stability analysis can be performed by solving a set of linear matrix inequalities (LMIs). Then, by generalizing the quadratic PDLF to polynomial PDLF, the conservativeness could be further reduced, even reach a nonconservative result (Chesi, 2008, 2011). In summary, the parameter-dependent idea plays an overriding role in stability analysis for uncertain systems in the past decades.

In this paper, we focus on reducing the conservatism over the proposed PDLF in Daafouz and Bernussou (2001). Two basic questions are the main concerns within the paper: (1) *How to develop less conservative results in the framework of parameter-dependent idea;* (2) *Can the newly developed method achieve a nonconservative result?* Rather than answering these two questions by increasing the degree of Lyapunov functions like using polynomial

☆ This work was supported by the National Natural Science Foundation of China (61603312, 51477146, 51707167). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Tingshu Hu under the direction of Editor André L. Tits.

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https://doi.org/10.1016/j.automatica.2017.10.001
0005-1098/

PDLF in Chesi (2008, 2011, 2013) and Chesi et al. (2007), especially in contrast to the non-conservative result for discrete-time systems (Chesi, 2014), we resort to a novel class of Lyapunov functions called parameter-memorized Lyapunov functions (PMLF), which still use quadratic Lyapunov functions but make use of uncertain parameters in a certain past time interval. It will be shown in the rest of paper that the PMLF is able to further reduce the conservatism as PDLF is only a special memoryless case of PMLF. Moreover, as the length of the interval for memory is sufficiently long, the non-conservativeness can be obtained in the framework of PMLF.

2. Preliminaries and problem formulation

Let $\mathbb{N}$ represent the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers, $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. $\|\cdot\|$ stands for Euclidean norm. The notation $A \succ 0$ means A is real symmetric and positive definite. In addition, in symmetric block matrices, we use $*$ as an ellipsis for the terms that are induced by symmetry. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. It will be a class $\mathcal{K}_\infty$ function if $\alpha(s) \to \infty$ as $s \to \infty$. Moreover, a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{KL}$ function if for each fixed s the function $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r, and for each fixed r the function $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. For two integers k_1 and k_2, $k_1 \le k_2$, we define $\mathcal{I}[k_1, k_2] \triangleq \{k_1, k_1 + 1, \ldots, k_2\}$.

The state transition matrix $\Phi(k, \ell) \in \mathbb{R}^{n\times n}$, $k \ge \ell$ for system (1) is introduced as

$$x(k) = \Phi(k, \ell)x(\ell),\ k \ge \ell. \tag{2}$$

The form of the state transition matrix $\Phi(k, \ell)$ follows as

$$\Phi(k, \ell) = \begin{cases} I, & k = \ell \\ \prod_{s=\ell}^{k-1} A(\xi(s)), & k > \ell \ge 0 \end{cases} \tag{3}$$

where $\prod_{s=\ell}^{k-1} A(\xi(s)) = A(\xi(k-1))A(\xi(k-2)) \cdots A(\xi(\ell))$.

The definition of global uniform asymptotic stability (GUAS) for system (1) is given below.

Definition 1 (*Vidyasagar, 2002*). The equilibrium $x = 0$ of system (1) is GUAS if, for any initial condition $x(0)$, there exists a class $\mathcal{KL}$ function β such that $\|x(k)\| \le \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{N}$ holds.

In Vidyasagar (2002), a result is proposed that the GUAS of system (1) can be established by the existence of Lyapunov function $V(x(k), \xi(k)) = x^\top(k)P(\xi(k))x(k)$ satisfying

$$\alpha_1(\|x(k)\|) \le V(x(k), \xi(k)) \le \alpha_2(\|x(k)\|) \tag{4}$$

$$\Delta V(x(k), \xi(k)) \le -\alpha_3(\|x(k)\|) \tag{5}$$

where $\Delta V(x(k), \xi(k)) = V(x(k+1), \xi(k+1)) - V(x(k), \xi(k))$, $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ are class $\mathcal{K}_\infty$ functions. Due to the presence of uncertain varying parameter $\xi(k)$, the construction of Lyapunov function $V(x(k), \xi(k))$ represents challenges in practical use. One straightforward but conservative choice is to let $P(\xi(k)) = P \succ 0$ to construct a common Lyapunov function (CLF). To reduce the overly conservativeness of CLP, a parameter-dependent Lyapunov function (PDLF) is proposed in Daafouz and Bernussou (2001), where PDLF is $V(x(k), \xi(k)) = x^\top(k)P(\xi(k))x(k)$ with $P(\xi(k))$ defined by

$$P(\xi(k)) = \sum_{i=1}^{N} \xi_i(k)P_i,\ P_i \succ 0,\ \forall i \in \mathcal{I}[1, N]. \tag{6}$$

The idea of PDLF making use of uncertain parameter $\xi(k)$ to construct Lyapunov function is able to significantly reduce the conservativeness in stability analysis in comparison with CLF. However, despite the improvement made by PDLF, the stability criterion proposed in Daafouz and Bernussou (2001) is a sufficient condition for uncertain system (1), that means there is still a room for the employment of uncertain parameter $\xi(k)$ to further reduce the conservativeness till a nonconservative condition, namely a necessary and sufficient condition, can be achieved. In this paper, the main task is to answer the following questions:

Question 1. *Can we find a parameter-dependent stability criterion that is less conservative than the one based on PDLF?*

Question 2. *Can we find a nonconservative stability criterion in the parameter-dependent framework?*

In order to answer the above two questions, a novel parameter-memorized Lyapunov function (PMLF), which not only depends on the $\xi(k)$ at the current time instant k but also relies on the memory of past $\xi(s)$, $s \in [\ell, k]$ with $\ell \le k$, is presented in the next section.

3. Main results

3.1. Parameter-memorized Lyapunov functions

First, a new variable is introduced as $\theta(k_2, k_1) = [\theta_1(k_2, k_1), \ldots, \theta_{N^{k_2-k_1}}(k_2, k_1)]$, which is defined as

$$\theta_j(k_2, k_1) = \prod_{k=k_1}^{k_2-1} \xi_{i_k}(k),\ i_k \in \mathcal{I}[1, N] \tag{7}$$

where $j \in \mathcal{I}[1, N^{k_2-k_1}]$. Note that $\theta(k_2, k_1)$ records the information of evolution of uncertain parameter $\xi(k)$ at each step during the interval $[k_1, k_2]$, thus we view that variable $\theta(k_2, k_1)$ possesses a memory of uncertain parameter $\xi(k)$, $k \in [k_1, k_2]$. For the new variable $\theta(k_1, k_2)$, it has the following property:

$$\theta_j(k_2, k_1) \ge 0,\ \forall j \in \mathcal{I}[1, N^{k_2-k_1}] \tag{8}$$

and

$$\begin{aligned} \sum_{j=1}^{N^{k_2-k_1}} \theta_j(k_2, k_1) &= \sum_{j=1}^{N^{k_2-k_1}} \left(\prod_{k=k_1}^{k_2-1} \xi_{i_k}(k)\right) \\ &= \prod_{k=k_1}^{k_2-1} \left(\sum_{i=1}^{N} \xi_i(k)\right) \\ &= 1. \end{aligned} \tag{9}$$

Then, we also define matrices $\mathscr{A}_j(k_2 - k_1)$ with a memory of all system matrices A_i, $i \in \mathcal{I}[1, N]$ undergoing during $[k_1, k_2]$ as follows:

$$\mathscr{A}_j(k_2 - k_1) = \prod_{k=k_1}^{k_2-1} A_{i_k},\ i_k \in \mathcal{I}[1, N] \tag{10}$$

where $j \in \mathcal{I}[1, N^{k_2-k_1}]$. With the $\theta(k_2, k_1)$ and $\mathscr{A}_j(k_2 - k_1)$ defined above, the state transition matrix $\Phi(k_2, k_1)$ can be expressed by

$$\begin{aligned} \Phi(k_2, k_1) &= \prod_{k=k_1}^{k_2-1} A(\xi(k)) \\ &= \prod_{k=k_1}^{k_2-1} \left(\sum_{i=1}^{N} \xi_i(k)A_i\right) \\ &= \sum_{j=1}^{N^{k_2-k_1}} \theta_j(k_2, k_1)\mathscr{A}_j(k_2 - k_1). \end{aligned} \tag{11}$$

Construct a time sequence $\mathcal{S} \triangleq \{k_n\}_{n\in\mathbb{N}}$, where k_n, $n \in \mathbb{N}$ are particularly chosen as $k_{n+1} = k_n + K$, $n \in \mathbb{N}$. For the sake of simplicity for notations, we denote $\theta(n) \triangleq \theta(k_{n+1}, k_n)$, $\Phi(n+1, n)$

Please cite this article in press as: Xiang, W., Parameter-memorized Lyapunov functions for discrete-time systems with time-varying parametric uncertainties. Automatica (2017), https://doi.org/10.1016/j.automatica.2017.10.001.

$\triangleq \Phi(k_{n+1}, k_n)$ and $\mathscr{A}_j \triangleq \mathscr{A}_j(k_{n+1} - k_n) = \mathscr{A}_j(K)$, then using (11), it implies

$$\Phi(n+1, n) = \mathscr{A}(\theta(n)) \tag{12}$$

where

$$\mathscr{A}(\theta(n)) \in \{\mathscr{A}(\theta(n)) : \mathscr{A}(\theta(n)) = \sum_{j=1}^{N^K} \theta_j(n)\mathscr{A}_j\}. \tag{13}$$

Matrices $\mathscr{A}_1, \ldots, \mathscr{A}_{N^K} \in \mathbb{R}^{n\times n}$ are constant matrices given by $\mathscr{A}_j = \prod_{k=1}^{K} A_{i_k}$, $i_k \in \mathcal{I}[1, N]$, $j \in \mathcal{I}[1, N^K]$. Moreover, by (8) and (9), the time-varying uncertain parameter vector $\theta(n) = [\theta_1(n), \ldots, \theta_{N^K}(n)]^\top \in \Theta \triangleq \{\theta : \sum_{i=1}^{N^K}\theta_j = 1,\ \theta_j \geq 0\}$.

Furthermore, letting $\chi(n) \triangleq x(k_n)$, the definition of state transition matrix of (2) leads to $\chi(n+1) = \Phi(n+1, n)\chi(n)$. Along with (12), the update of $\chi(n)$ can be obtained by the following dynamics

$$\chi(n+1) = \mathscr{A}(\theta(n))\chi(n). \tag{14}$$

Based on the above system with parameter-memorized variable $\theta(n)$ and matrices $\mathscr{A}_j$, a novel parameter-memorized Lyapunov function (PMLF) is proposed as

$$V(\chi(n), \theta(n)) = \chi^\top(n)P(\theta(n))\chi(n) \tag{15}$$

where $P(\theta(n))$ is defined by

$$P(\theta(n)) = \sum_{j=1}^{N^K} \theta_j(n)P_j,\ P_j \succ 0,\ \forall j \in \mathcal{I}[1, N^K]. \tag{16}$$

In particular, if we select $K = 1$, the PMLF is exactly reduced to the PDLF with parameter-dependent $P(\xi(k))$ defined by (6). Thus, PDLF is only a special memoryless case of PMLF, since PDLF is only relevant to the current value of uncertain parameters. In fact, PMLF generalizes the parameter-dependent conception from present time instant to a past time window of K-step length. This generalization is able to reduce the conservativeness of PDLF in stability analysis, and finally to achieve the non-conservativeness, as shown in the next subsection.

3.2. Nonconservative stability criterion

In this section, a novel stability criterion based on PMLF will be presented. First, the following lemma is proposed.

Lemma 1. *System (1) is GUAS if and only if system (14) is GUAS.*

Proof. If system (1) is GUAS, it implies that there exists a class $\mathcal{KL}$ function β such that $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{N}$. Obviously, given any $K \in \mathbb{N}$, it holds for any $x(nK)$, $n \in \mathbb{N}$, namely, $\|x(nK)\| \leq \beta(\|x(0)\|, nK)$. Due to $\chi(n) = x(k_n) = x(nK)$, it directly leads to $\|\chi(n)\| \leq \beta(\|\chi(0)\|, n)$, $\forall n \in \mathbb{N}$, thus the GUAS of system (14) can be guaranteed.

On the other hand, if system (14) is GUAS, which implies $\|\chi(n)\| \leq \beta(\|\chi(0)\|, n)$, $\forall n \in \mathbb{N}$. In addition, let $\gamma = \max_{j\in\mathcal{I}[1,N^K]} \left\|\mathscr{A}_j\right\|$, we have $\|x(k)\| \leq \gamma\,\|\chi(n)\|$, $k \in [k_n, k_{n+1}]$. Thus, due to $\chi(0) = x(0)$, it arrives $\|x(k)\| \leq \tilde{\beta}(\|x(0)\|, k)$, where $\tilde{\beta} = \gamma\beta$ is a class $\mathcal{KL}$ function. Thus, the GUAS of system (1) can be ensured. □

Now, we are going to answer Question 1, namely deriving a less conservative stability criterion for uncertain discrete-time linear system (1).

Theorem 1. *Consider uncertain discrete-time linear system (1), if there exist a $K \in \mathbb{N} \setminus \{0\}$ and N^K symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N^K]$, such that the following inequalities hold*

$$\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i \prec 0,\ \forall i, j \in \mathcal{I}[1, N^K] \tag{17}$$

then system (1) is GUAS.

Proof. By Schur complement formula, (17) is equivalent to the following inequality

$$Q_{ij} = \begin{bmatrix} -P_i & * \\ P_j\mathscr{A}_i & -P_j \end{bmatrix} \prec 0 \tag{18}$$

holds for any $i, j \in \mathcal{I}[1, N^K]$. Then, we have the following derivation:

$$\begin{aligned} &\sum\nolimits_{i=1}^{N^K} \theta_i(n)\left(\sum\nolimits_{j=1}^{N^K} \theta_j(n+1)Q_{ij}\right) \\ &= \sum\nolimits_{i=1}^{N^K} \theta_i(n)\begin{bmatrix} -P_i & * \\ P(\theta(n+1))\mathscr{A}_i & -P(\theta(n+1)) \end{bmatrix} \\ &= \begin{bmatrix} -P(\theta(n)) & * \\ P(\theta(n+1))\mathscr{A}(\theta(n)) & -P(\theta(n+1)) \end{bmatrix} \prec 0. \end{aligned} \tag{19}$$

Again, using Schur complement formula, it arrives

$$\mathscr{A}^\top(\theta(n))P(\theta(n+1))\mathscr{A}(\theta(n)) - P(\theta(n)) \prec 0. \tag{20}$$

Construct PMLF (15), (20) ensures that $\Delta V(x(n), \theta(n)) = V(x(n+1), \theta(n+1)) - V(x(n), \theta(n)) < 0$. Thus, the GUAS of system (14) can be guaranteed by standard Lyapunov function theorem. Finally, the GUAS of system (1) can be established by Lemma 1. □

As a special case of Theorem 1, the PDLF result in Daafouz and Bernussou (2001) can be recovered by letting $K = 1$ in (17) as below:

Corollary 1. *Consider uncertain discrete-time linear system (1), if there exist N symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N]$, such that the following inequalities hold*

$$A_i^\top P_j A_i - P_i \prec 0,\ \forall i, j \in \mathcal{I}[1, N] \tag{21}$$

then system (1) is GUAS.

Proof. It can be proved by letting $K = 1$ in (17). □

Question 1 can be answered by comparing Theorem 1 and Corollary 1. Since Corollary 1 holds absolutely ensures Theorem 1 holds with $K = 1$, namely Corollary 1 is only a special case of Theorem 1, PMLF method certainly has less conservativeness over PDLF method for uncertain systems. The computational complexity, however, increases as K grows in (17), since the number of variables and the size of LMIs in are $n(n+1)N^K/2$ and nN^{2K}, respectively. The number of variables and the size of LMIs of PDLF method can be obtained by letting $K = 1$.

Example 1. Let us consider uncertain discrete-time linear system (1) with the following vertices matrices:

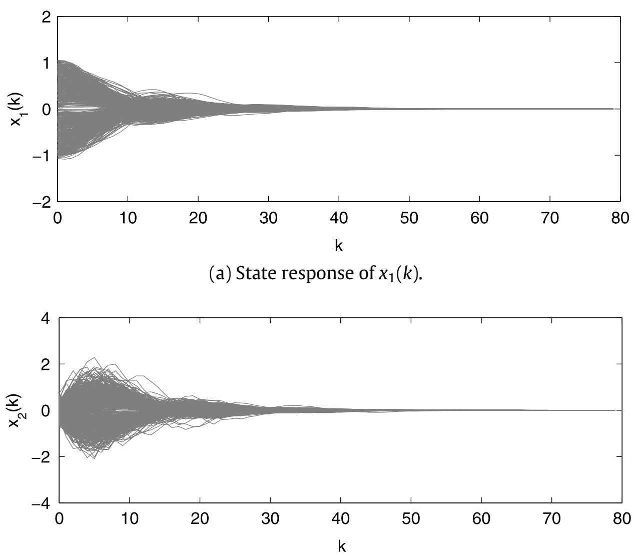
$$A_1 = \begin{bmatrix} 0.9520 & 0.0936 \\ -0.9358 & 0.8584 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.9996 & 0.0824 \\ -0.0082 & 0.6699 \end{bmatrix}.$$

First, we use PDLF approach (also viewed as $K = 1$ in our PMLF approach), but no feasible solution can be found for the LMIs, so that the GUAS cannot be determined by the approach of PDLF. However, this stability analysis result by PDLF is conservative since GUAS can be established by our PMLF approach just with a parameter memory of past 2 steps, that is $K = 2$ in Theorem 1. The feasibility of the corresponding LMI problems can be established with the following matrices P_j, $j \in \mathcal{I}[1, 4]$:

$$P_1 = \begin{bmatrix} 2.7102 & 0.1384 \\ 0.1384 & 0.3700 \end{bmatrix},\ P_2 = \begin{bmatrix} 2.6706 & 0.2763 \\ 0.2763 & 0.3715 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 2.4393 & 0.3507 \\ 0.3507 & 0.3965 \end{bmatrix},\ P_4 = \begin{bmatrix} 2.8212 & 0.5910 \\ 0.5910 & 0.3576 \end{bmatrix}$$

which are sufficient to guarantee the GUAS.



(a) State response of $x_1(k)$.

(b) State response of $x_2(k)$.

Fig. 1. 1000 randomly generated state trajectories.

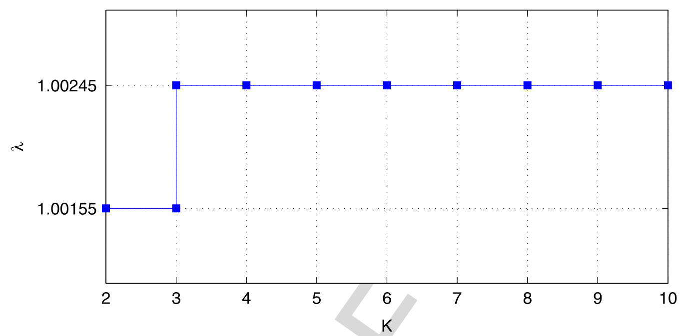


Fig. 2. Variation of λ along with K.

The convergent state evolutions are shown by the simulation Q3 results in Fig. 1, where 1000 state trajectories are randomly generated. All these state trajectories converge to the origin to show the GUAS of the system.

In Example 1, it has been shown that the conservativeness deceases as K, the length of memory intervals, increases from 1 to 2, that means more memories of past evolution of uncertain parameter $\xi(k)$ will lead to less conservative results by PMLF approach. Then, Question 2 arises here: *Can a nonconservative result be finally achieved if the K for PMLF method increases to a sufficiently large value?* In the following, an answer to this critical question for PMLF approach is presented.

Theorem 2. *Uncertain discrete-time linear system (1) is GUAS if and only if there exist a sufficiently large $K \in \mathbb{N} \setminus \{0\}$ and N^K symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N^K]$, such that (17) holds.*

Proof. The sufficiency has been proved by Theorem 1, thus in the rest of proof, we focus on the necessity.

If system (1) is GUAS, there exists a class $\mathcal{KL}$ function β such that $\|x(k)\| \leq \beta(\|x(0)\|, k)$ holds. Using the definition of state transition matrix $x(k) = \Phi(k, 0)x(0)$, we have $\|\Phi(k, 0)x(0)\| \leq \beta(\|x(0)\|, k)$. Because β is a class $\mathcal{KL}$ function, it implies that $\lim_{k\to\infty}\beta(\|x(0)\|, k) = 0$ and as a result, one has $\lim_{k\to\infty}\|\Phi(k, 0)x(0)\| = 0$, leading to $\lim_{k\to\infty}\Phi(k, 0) = 0$. Thus, arbitrarily choosing M (M could be arbitrary) matrices $P_i \succ 0$, $i \in \mathcal{I}[1, M]$, which are independent with k, and any $\theta = [\theta_1 \cdots \theta_M] \in \Theta \triangleq \{\theta : \sum_{i=1}^{M}\theta_i = 1,\ \theta_i \geq 0\}$, we have

$$\lim_{k\to\infty} \Phi^\top(k, 0)P_i(\theta)\Phi(k, 0) = 0,\ \forall i \in \mathcal{I}[1, M] \tag{22}$$

where $P_i(\theta) = \sum_{i=1}^{M}\theta_i P_i$.

It implies that there exists a sufficiently large K^* such that, for any $K \geq K^*$, the following inequality holds

$$\Phi^\top(K, 0)P_j(\hat{\theta})\Phi(K, 0) \prec P_i(\theta),\ \forall i, j \in \mathcal{I}[1, M] \tag{23}$$

where $\hat{\theta}, \theta \in \Theta$. Then, by Schur complement formula, it is equal to

$$Q_{ij}(K) = \begin{bmatrix} -P_i(\theta) & * \\ P_j(\hat{\theta})\Phi(K, 0) & -P_j(\hat{\theta}) \end{bmatrix} \prec 0. \tag{24}$$

Since M could be arbitrarily chosen, it can be particularly selected as $M = N^K$. Moreover, from (11), it implies

$$\Phi(K, 0) = \sum_{i=1}^{N^K} \theta_i(K, 0)\mathscr{A}_i(K). \tag{25}$$

Since $\hat{\theta}, \theta$ can be any values belonging to Θ, we let $\theta = \theta(K, 0)$ and substitute (25) into (24) to obtain

$$Q_{ij}(K) = \sum_{i=1}^{N^K} \theta_i(K, 0) \left(\sum_{j=1}^{N^K} \hat{\theta}_j Q_{ij} \right) \prec 0 \tag{26}$$

where, by denoting $\mathscr{A}_i = \mathscr{A}_i(K)$, Q_{ij} is

$$Q_{ij} = \begin{bmatrix} -P_i & * \\ P_j\mathscr{A}_i & -P_j \end{bmatrix} \tag{27}$$

and therefore, (26) ensures that $Q_{ij} \prec 0$. Using Schur complement formula to obtain $\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i \prec 0$, $\forall i, j \in \mathcal{I}[1, N^K]$, that is (17) holds. The proof is complete. □

Theorem 2 shows that the PMLF method not only can reduce the conservativeness in stability analysis for uncertain discrete-time linear systems, it also can reach a nonconservative result if a sufficiently long memory of past parameters is considered in constructing PMLF. Therefore, Question 2 can be answered that the nonconservative stability criterion can be achieved in the framework of PMLF. Moreover, Question 1 is also answered by Theorem 2 here, since PDLF only derives a sufficient condition, but on the other hand, PMLF gives out a necessary and sufficient condition.

Example 2. We consider the following vertices matrices for the uncertain system

$$A_1 = \begin{bmatrix} 0.9520 & 0.0936 \\ -0.9358 & 0.8584 \end{bmatrix},\ A_2 = \lambda \begin{bmatrix} 0.9996 & 0.0824 \\ -0.0082 & 0.6699 \end{bmatrix}$$

where $\lambda \geq 1$ is a scaling parameter. It has been shown in Example 1 that GUAS can be guaranteed by PMLF with $K = 2$ when $\lambda = 1$. Then, we keep increasing K to determine the maximal λ such that the system is GUAS. Fig. 2 shows the variation of λ along with K. With a precision of 0.00001 for λ, it can be seen that we can obtain $\lambda = 1.00245$ after $K \geq 3$ from Fig. 2. The value of $\lambda = 1.00245$ is nonconservative with precision of 0.00001, since A_2 has an eigenvalue equals 1 when $\lambda = 1.00246$. Thus, in this example, the sufficiently large length of memory for PMLF claimed by Theorem 2 is $K = 3$ to reach the non-conservativeness.

4. Conclusions

A novel parameter-memorized Lyapunov function is proposed for stability analysis of uncertain discrete-time linear systems. Based on PMLF, a new stability criterion is proposed to check

Please cite this article in press as: Xiang, W., Parameter-memorized Lyapunov functions for discrete-time systems with time-varying parametric uncertainties. Automatica (2017), https://doi.org/10.1016/j.automatica.2017.10.001.

the stability of uncertain discrete-time linear systems. It has been shown that the proposed PMLF method is less conservative than PDLF method. Moreover, a nonconservative stability condition can be obtained if the length of the interval for parameter memory is large enough. The future study should be carried for the estimation on the length of the parameter memorizing interval that assures the non-conservativeness and moreover, for applications in controller design.

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