

Asymptotic Divergences and Strong Dichotomy

Xiang Huang^{1b}, Jack H. Lutz^{2b}, Elvira Mayordomo^{3b}, *Member, IEEE*, and Donald M. Stull

Abstract—The *Schnorr-Stimm dichotomy theorem* (Schnorr and Stimm, 1972) concerns finite-state gamblers that bet on infinite sequences of symbols taken from a finite alphabet Σ . The theorem asserts that, for any such sequence S , the following two things are true. (1) If S is not normal in the sense of Borel (meaning that every two strings of equal length appear with equal asymptotic frequency in S), then there is a finite-state gambler that wins money at an infinitely-often exponential rate betting on S . (2) If S is normal, then any finite-state gambler loses money at an exponential rate betting on S . In this paper we use the Kullback-Leibler divergence to formulate the *lower asymptotic divergence* $\text{div}(S||\alpha)$ of a probability measure α on Σ from a sequence S over Σ and the *upper asymptotic divergence* $\text{Div}(S||\alpha)$ of α from S in such a way that a sequence S is α -normal (meaning that every string w has asymptotic frequency $\alpha(w)$ in S) if and only if $\text{Div}(S||\alpha) = 0$. We also use the Kullback-Leibler divergence to quantify the *total risk* $\text{Risk}_G(w)$ that a finite-state gambler G takes when betting along a prefix w of S . Our main theorem is a *strong dichotomy theorem* that uses the above notions to *quantify* the exponential rates of winning and losing on the two sides of the Schnorr-Stimm dichotomy theorem (with the latter routinely extended from normality to α -normality). Modulo asymptotic caveats in the paper, our strong dichotomy theorem says that the following two things hold for prefixes w of S . (1') The infinitely-often exponential rate of winning in 1 is $2^{\text{Div}(S||\alpha)|w|}$. (2') The exponential rate of loss in 2 is $2^{-\text{Risk}_G(w)}$. We also use (1') to show that $1 - \text{Div}(S||\alpha)/c$, where $c = \log(1/\min_{a \in \Sigma} \alpha(a))$, is an upper bound on the finite-state α -dimension of S and prove the dual fact that $1 - \text{div}(S||\alpha)/c$ is an upper bound on the finite-state strong α -dimension of S .

Index Terms—Finite-state dimension, finite-state gambler, Kullback-Leibler divergence, normal sequences.

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Xiang Huang is with the Department of Computer Science, University of Illinois at Springfield, Springfield, IL 62703 USA (e-mail: xhuan5@uis.edu).

Jack H. Lutz is with the Department of Computer Science, Iowa State University, Ames, IA 50011 USA (e-mail: lutz@iastate.edu).

Elvira Mayordomo is with the Departamento de Informática e Ingeniería de Sistemas, Instituto de Investigación en Ingeniería de Aragón, Universidad de Zaragoza, 50018 Zaragoza, Spain (e-mail: elvira@unizar.es).

Donald M. Stull is with the Department of Computer Science, Northwestern University, Evanston, IL 60208 USA (e-mail: dstull30@gmail.com).

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I. INTRODUCTION

AN INFINITE sequence S over a finite alphabet is *normal* in the 1909 sense of Borel [7] if every two strings of equal length appear with equal asymptotic frequency in S . Borel normality played a central role in the origins of measure-theoretic probability theory [6] and is intuitively regarded as a weak notion of randomness. For a masterful discussion of this intuition, see section 3.5 of [22].

The theory of computing was used to make this intuition precise. This took place in three steps in the 1960s and 1970s. First, Martin-Löf [28] used constructive measure theory to give the first successful formulation of the randomness of individual infinite binary sequences. Second, Schnorr [30] gave an equivalent, and more flexible, formulation of Martin-Löf's notion in terms of gambling strategies called *martingales*. In this formulation, an infinite binary sequence S is random if no lower semicomputable martingale can make unbounded money betting on the successive bits of S . Third, Schnorr and Stimm [31] proved that an infinite binary sequence S is normal if and only if no martingale that is computed by a finite-state automaton can make unbounded money betting on the successive bits of S . That is, *normality is finite-state randomness*.

This equivalence was a breakthrough that has already had many consequences (discussed later in this introduction), but the Schnorr-Stimm result said more. It is a *dichotomy theorem* asserting that, for any infinite binary sequence S , the following two things are true.

- 1) If S is not normal, then there is a finite-state gambler that makes money at an infinitely-often exponential rate when betting on S .
- 2) If S is normal, then every finite-state gambler that bets infinitely many times on S loses money at an exponential rate.

The main contribution of this paper is to *quantify* the exponential rates of winning and losing on the two sides (1 and 2 above) of the Schnorr-Stimm dichotomy.

To describe our main theorem in some detail, let Σ be a finite alphabet. It is routine to extend the above notion of normality to an arbitrary positive probability measure α on Σ . Specifically, an infinite sequence S over Σ is α -normal if every finite string w over Σ appears with asymptotic frequency $\alpha^{|w|}(w)$ in S , where α^ℓ is the natural (product) extension of α to strings of length ℓ . Schnorr and Stimm [31] correctly noted that their dichotomy theorem extends to α -normal sequences in a straightforward manner, and it is this extension whose exponential rates we quantify here. To be very clear, our contribution is *not* the extension of Schnorr and Stimm's

results to arbitrary positive probability measures on Σ . That, as they wrote, is routine. Our contribution, the *quantification* of the exponential rates of winning and losing on the two sides of their dichotomy, is new for *all* probability measures α on S , including the uniform probability measure. Our approach, which we describe next, is a fresh start, not the approach of [31].

The quantitative tool that drives our approach is the Kullback-Leibler divergence [23], also known as the relative entropy [12]. If α and β are positive probability measures on Σ , then the *Kullback-Leibler divergence of β from α* is

$$D(\alpha||\beta) = E_\alpha \log \frac{\alpha}{\beta},$$

i.e., the expectation with respect to α of the random variable $\log \frac{\alpha}{\beta} : \Sigma \rightarrow \mathbb{R}$, where the logarithm is base-2. Although the Kullback-Leibler divergence is not a metric on the space of probability measures on Σ , it does quantify “how different” β is from α , and it has the crucial property that $D(\alpha||\beta) \geq 0$, with equality if and only if $\alpha = \beta$.

Here we use the empirical frequencies of symbols in S to define the *asymptotic lower divergence* $\text{div}(S||\alpha)$ of α from S and the *asymptotic upper divergence* $\text{Div}(S||\alpha)$ of α from S in a natural way, so that S is α -normal if and only if $\text{Div}(S||\alpha) = 0$.

The first part of our *strong dichotomy theorem* says that the infinitely-often exponential rate that can be achieved in 1 above is essentially at least $2^{\text{Div}(S||\alpha)|w|}$, where w is the prefix of S on which the finite-state gambler has bet so far. More precisely, it says the following.

- 1'. If S is not α -normal, then, for every $\gamma < 1$, there is a finite-state gambler G such that, when G bets on S with payoffs according to α , there are infinitely many prefixes w of S after which G 's capital exceeds $2^{\gamma \text{Div}(S||\alpha)|w|}$.

The second part of our strong dichotomy theorem, like the second part of the Schnorr-Stimm dichotomy theorem, is complicated by the fact that a finite-state gambler may, in some states, decline to bet. In this case, its capital after a bet is the same as it was before the bet, regardless of what symbol actually appears in S . Once again, however, it is the Kullback-Leibler divergence that clarifies the situation. As explained in section 3 below, in any particular state q , a finite-state gambler's betting strategy is a probability measure $B(q)$ on Σ . If $B(q) = \alpha$, then the gambler does not bet in state q . We thus define the *risk* that the gambler G takes in state q to be

$$\text{risk}_G(q) = D(\alpha||B(q)),$$

i.e., the divergence of $B(q)$ from not betting. We then define the *total risk* that the gambler takes along a prefix w of the sequence S on which it is betting to be the sum $\text{Risk}_G(w)$ of the risks $\text{risk}_G(q)$ in the states that G traverses along w . The second part of our strong dichotomy theorem says that, if S is α -normal and G is a finite-state gambler betting on S , then after each prefix w of S , the capital of G on prefixes w of S is essentially bounded above by $2^{-\text{Risk}_G(w)}$. In some sense, then, G loses all that it risks. More precisely, the second part of our strong dichotomy says the following.

- 2'. If S is α -normal, then, for every finite-state gambler G and every $\gamma < 1$, after all but finitely many prefixes w of S , the gambler G 's capital is less than $2^{-\gamma \text{Risk}_G(w)}$.

A routine ergodic argument, already present in [31], shows that, if a finite-state gambler G bets on an α -normal sequence S , then every state of G that occurs infinitely often along S occurs with positive frequency along S . Hence 2 above follows from 2' above.

Our strong dichotomy theorem has implications for finite-state dimensions. For each probability measure α on Σ and each sequence S over Σ , the *finite-state α -dimension* $\text{dim}_{\text{FS}}^\alpha(S)$ and the *finite-state strong α -dimension* $\text{Dim}_{\text{FS}}^\alpha(S)$ (defined in section 4 below) are finite-state versions of Billingsley dimension [5], [10] introduced in [26]. When α is the uniform probability measure on Σ , these are the finite dimension $\text{dim}_{\text{FS}}(S)$, introduced in [14] as a finite-state version of Hausdorff dimension [17], [20], and the finite-state strong dimension $\text{Dim}_{\text{FS}}(S)$, introduced in [2] as a finite-state version of packing dimension [17], [34], [35]. The finite-state strong dimension $\text{Dim}_{\text{FS}}(s)$ has been shown [2] to coincide with the upper asymptotic compression ratio $R_{\text{FS}}(S)$ investigated by Ziv [37], and the finite-state dimension $\text{dim}_{\text{FS}}(S)$ was earlier shown [14] to coincide with the dual lower asymptotic compression ratio $\rho_{\text{FS}}(S)$. These two finite-state dimensions have also been characterized in terms of asymptotic entropy rates [8]. Intuitively, $\text{dim}_{\text{FS}}^\alpha(S)$ and $\text{Dim}_{\text{FS}}^\alpha(S)$ measure the lower and upper asymptotic α -densities of the finite-state information in S .

Here we use part 1' of our strong dichotomy theorem to prove that, for every positive probability measure α on Σ and every sequence S over Σ ,

$$\text{dim}_{\text{FS}}^\alpha(S) \leq 1 - \text{Div}(S||\alpha)/c,$$

where $c = \log(1/\min_{a \in \Sigma} \alpha(a))$. We also establish the dual result that, for all such α and S ,

$$\text{Dim}_{\text{FS}}^\alpha(S) \leq 1 - \text{div}(S||\alpha)/c.$$

We also prove a divergence inequality relating finite-state dimension with divergence and asymptotic Shannon entropy. Lutz has proven in [26] a divergence equality that relates the α -finite-state dimension of a β -normal sequence S to the divergence of positive probability distribution α from positive probability distribution β and the Shannon entropy of β , namely

$$\text{dim}_{\text{FS}}^\alpha(S) = \frac{\mathcal{H}(\beta)}{\mathcal{H}(\beta) + \text{div}(\beta||\alpha)}.$$

Our inequality holds instead for any sequence S , and bounds $\text{dim}_{\text{FS}}^\alpha(S)$ in terms of the asymptotic Shannon entropy of S and the divergence of α from S .

Research on normal sequences and normal numbers (real numbers whose base- b expansions are normal sequences for various choices of b) connected the theory of normal numbers directly to the theory of computing. Further work along these lines has been continued in [3], [21], [29], [33]. After the discovery of algorithmic dimensions in the present century [2], [14], [24], [25], the Schnorr-Stimm dichotomy led to the

realization [8] that the finite-state world, unlike any other known to date, is one in which maximum dimension is not only necessary, but also sufficient, for randomness. This in turn led to the discovery of nontrivial extensions of classical theorems on normal numbers [11], [36] to new quantitative theorems on finite-state dimensions [16], [19], a line of inquiry that will certainly continue. It has also led to a polynomial-time algorithm [4] that computes real numbers that are provably absolutely normal (normal in every base) and, via Lempel-Ziv methods, to a nearly linear time algorithm for this [27]. In parallel with these developments, connections among normality, Weyl equidistribution theorems, and Diophantine approximation have led to a great deal of progress surveyed in the books [9], [15]. This paragraph does not begin to do justice to the breadth and depth of recent and ongoing research on normal numbers and their growing involvement with the theory of computing. It is to be hoped that our strong dichotomy theorem and the quantitative methods implicit in it will further accelerate these discoveries.

II. DIVERGENCE AND NORMALITY

This section reviews the discrete Kullback-Leibler divergence, introduces asymptotic extensions of this divergence, and uses these to give useful characterizations of Borel normal sequences.

A. The Kullback-Leibler Divergence

We work in a finite alphabet Σ with $2 \leq |\Sigma| < \infty$. We write Σ^ℓ for the set of strings of length ℓ over Σ , $\Sigma^* = \bigcup_{\ell=0}^{\infty} \Sigma^\ell$ for the set of (finite) strings over Σ , Σ^ω for the set of (infinite) sequences over Σ , and $\Sigma^{\leq \omega} = \Sigma^* \cup \Sigma^\omega$. We write λ for the empty string, $|w|$ for the length of a string $w \in \Sigma^*$, and $|S| = \omega$ for the length of a sequence $S \in \Sigma^\omega$. For $x \in \Sigma^{\leq \omega}$ and $0 \leq i < |x|$, we write $x[i]$ for the i -th symbol in x , noting that $x[0]$ is the leftmost symbol in x . For $x \in \Sigma^{\leq \omega}$ and $0 \leq i \leq j < |x|$, we write $x[i..j]$ for the string consisting of the i -th through j -th symbols in x . Specially, we write $x \upharpoonright n$ to mean $x[0..n-1]$. For $x, y \in \Sigma^{\leq \omega}$, we write $x \sqsubseteq y$ if x is a prefix of y . We write $x \sqsubset y$ to denote x being a strict prefix of y , which excludes the case $x = y$.

A (discrete) probability measure on a nonempty finite set Ω is a function $\pi : \Omega \rightarrow [0, 1]$ satisfying

$$\sum_{\omega \in \Omega} \pi(\omega) = 1. \quad (1)$$

We write $\Delta(\Omega)$ for the set of all probability measures on Ω , $\Delta^+(\Omega)$ for the set of all $\pi \in \Delta(\Omega)$ that are strictly positive (i.e., $\pi(\omega) > 0$ for all $\omega \in \Omega$), $\Delta_{\mathbb{Q}}(\Omega)$ for the set of all $\pi \in \Delta(\Omega)$ that are rational-valued, and $\Delta_{\mathbb{Q}}^+(\Omega) = \Delta^+(\Omega) \cap \Delta_{\mathbb{Q}}(\Omega)$. In this paper we are most interested in the case where $\Omega = \Sigma^\ell$ for some $\ell \in \mathbb{Z}^+$.

Intuitively, we identify each probability measure $\pi \in \Delta(\Omega)$ with the point in $\mathbb{R}^{|\Omega|}$ whose coordinates are the probabilities $\pi(\omega)$ for $\omega \in \Omega$. By (1) this implies that $\Delta(\Omega)$ is the $(|\Omega| - 1)$ -dimensional simplex in $\mathbb{R}^{|\Omega|}$ whose vertices are the points at 1 on each of the coordinate axes. For each $\omega \in \Omega$, the vertex on axis ω is the degenerate probability measures

π_ω with $\pi_\omega(\omega) = 1$. The centroid of the simplex $\Delta(\Omega)$ is the uniform probability measure on Ω , and the (topological) interior of $\Delta(\Omega)$ is $\Delta^+(\Omega)$. We write $\partial\Delta(\Omega) = \Delta(\Omega) \setminus \Delta^+(\Omega)$ for the boundary of $\Delta(\Omega)$.

Definition ([23]): Let $\alpha, \beta \in \Delta(\Omega)$, where Ω is a nonempty finite set. The Kullback-Leibler divergence (or KL-divergence) of β from α is

$$D(\alpha||\beta) = E_\alpha \log \frac{\alpha}{\beta}, \quad (2)$$

where the logarithm is base-2.

Note that the right-hand side of (2) is the α -expectation of the random variable

$$\log \frac{\alpha}{\beta} : \Omega \longrightarrow \mathbb{R}$$

defined by

$$\left(\log \frac{\alpha}{\beta} \right) (\omega) = \log \frac{\alpha(\omega)}{\beta(\omega)}$$

for each $\omega \in \Omega$. Hence (2) is a convenient shorthand for

$$D(\alpha||\beta) = \sum_{\omega \in \Omega} \alpha(\omega) \log \frac{\alpha(\omega)}{\beta(\omega)}.$$

Note also that $D(\alpha||\beta)$ is infinite if and only if $\alpha(\omega) > 0 = \beta(\omega)$ the some $\omega \in \Omega$.

The Kullback-Leibler divergence $D(\alpha||\beta)$ is a useful measure of how different β is from α . It is not a metric, because it is not symmetric and does not satisfy the triangle inequality, but it has the crucial property that $D(\alpha||\beta) \geq 0$, with equality if and only if $\alpha = \beta$. The two most central quantities in Shannon information theory, entropy and mutual information, can both be defined in terms of divergence as follows.

- 1) *Entropy is divergence from certainty.* The entropy of a probability measure $\alpha \in \Delta(\Omega)$, conceived by Shannon [32] as a measure of the uncertainty of α , is

$$H(\alpha) = \sum_{\omega \in \Omega} \alpha(\omega) D(\pi_\omega||\alpha), \quad (3)$$

i.e., the α -average of the divergences of α from the “certainties” π_ω .

- 2) *Mutual information is divergence from independence.* If $\alpha, \beta \in \Delta(\Omega)$ have a joint probability measure $\gamma \in \Delta(\Omega \times \Omega)$ (i.e., are the marginal probability measures of γ), then the mutual information between α and β , conceived by Shannon [32] as a measure of the information shared by α and β , is

$$I(\alpha; \beta) = D(\alpha\beta||\gamma), \quad (4)$$

i.e., the divergence of γ from the probability measure in which α and β are independent.

Two additional properties of the Kullback-Leibler divergence are useful for our asymptotic concerns. First, the divergence $D(\alpha||\beta)$ is continuous on $\Delta(\Omega)^2$ (as a function into $[0, \infty]$). Hence, if $\alpha_n \in \Delta(\Omega)$ for each $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ in the sense of the Euclidean metric on the simplex $\Delta(\Omega)$, then $\lim_{n \rightarrow \infty} D(\alpha_n||\alpha) = \lim_{n \rightarrow \infty} D(\alpha||\alpha_n) = 0$. Second, the converse holds. It is well known [12] that

$$D(\alpha||\beta) \geq \frac{1}{2 \ln 2} \|\alpha - \beta\|_1^2,$$

where $\|\alpha - \beta\|_1 = \sum_{\omega \in \Sigma} |\alpha(\omega) - \beta(\omega)|$ is the \mathcal{L}_1 -norm. Hence, if either $\lim_{n \rightarrow \infty} D(\alpha_n || \alpha) = 0$ or $\lim_{n \rightarrow \infty} D(\alpha || \alpha_n) = 0$, then $\lim_{n \rightarrow \infty} \alpha_n = \alpha$.

More extensive discussions of the Kullback-Leibler divergence appear in [12], [13].

B. Asymptotic Divergences

For nonempty strings $w, x \in \Sigma^*$, we write

$$\#_{\square}(w, x) = \left| \left\{ m \leq \frac{|x|}{|w|} - 1 \mid x[m|w|..(m+1)|w|-1] = w \right\} \right|$$

for the number of block occurrences of w in x . Note that $0 \leq \#_{\square}(w, x) \leq \frac{|x|}{|w|}$.

For each $S \in \Sigma^{\omega}$, $n \in \mathbb{Z}^+$, and $\lambda \neq w \in \Sigma^*$, the n -th block frequency of w in S is

$$\pi_{S,n}(w) = \frac{\#_{\square}(w, S[0..n|w|-1])}{n} \quad (5)$$

Note that (5) defines, for each $S \in \Sigma^{\omega}$ and $n \in \mathbb{Z}^+$, a function

$$\pi_{S,n} : \Sigma^* \setminus \{\lambda\} \longrightarrow \mathbb{Q}.$$

For each such S and n and each $\ell \in \mathbb{Z}^+$, let $\pi_{S,n}^{(\ell)} = \pi_{S,n} \upharpoonright \Sigma^{\ell}$ be the restriction of the function $\pi_{S,n}$ to the set Σ^{ℓ} of strings of length ℓ .

Observation 1: For each $S \in \Sigma^{\omega}$ and $n, \ell \in \mathbb{Z}^+$,

$$\pi_{S,n}^{(\ell)} \in \Delta_{\mathbb{Q}}(\Sigma^{\ell}),$$

i.e., $\pi_{S,n}^{(\ell)}$ is a rational-valued probability measure on Σ^{ℓ} .

We call $\pi_{S,n}^{(\ell)}$ the n -th empirical probability measure on Σ^{ℓ} given by S .

A probability measure $\alpha \in \Delta(\Sigma)$ naturally induces, for each $\ell \in \mathbb{Z}^+$, a probability measure $\alpha^{(\ell)} \in \Delta(\Sigma^{\ell})$ defined by

$$\alpha^{(\ell)}(w) = \prod_{i=0}^{|w|-1} \alpha(w[i]). \quad (6)$$

The empirical probability measures $\pi_{S,n}^{(\ell)}$ provide a natural way to define useful empirical divergences of probability measures from sequences.

Definition: Let $\ell \in \mathbb{Z}^+$, $S \in \Sigma^{\omega}$, and $\alpha \in \Delta(\Sigma)$.

- 1) The lower ℓ -divergence of α from S is $\text{div}_{\ell}(S||\alpha) = \liminf_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)})$.
- 2) The upper ℓ -divergence of α from S is $\text{Div}_{\ell}(S||\alpha) = \limsup_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)})$.
- 3) The lower divergence of α from S is $\text{div}(S||\alpha) = \sup_{\ell \in \mathbb{Z}^+} \text{div}_{\ell}(S||\alpha)/\ell$.
- 4) The upper divergence of α from S is $\text{Div}(S||\alpha) = \sup_{\ell \in \mathbb{Z}^+} \text{Div}_{\ell}(S||\alpha)/\ell$.

A similar approach gives useful empirical divergences of one sequence from another.

Definition: Let $\ell \in \mathbb{Z}^+$ and $S, T \in \Sigma^{\omega}$.

- 1) The lower ℓ -divergence of T from S is $\text{div}_{\ell}(S||T) = \liminf_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \pi_{T,n}^{(\ell)})$.

- 2) The upper ℓ -divergence of T from S is $\text{Div}_{\ell}(S||T) = \limsup_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \pi_{T,n}^{(\ell)})$.
- 3) The lower divergence of T from S is $\text{div}(S||T) = \sup_{\ell \in \mathbb{Z}^+} \text{div}_{\ell}(S||T)/\ell$.
- 4) The upper divergence of T from S is $\text{Div}(S||T) = \sup_{\ell \in \mathbb{Z}^+} \text{Div}_{\ell}(S||T)/\ell$.

C. Normality

The following notions are essentially due to Borel [7].

Definition: Let $\alpha \in \Delta(\Sigma)$, $S \in \Sigma^{\omega}$, and $\ell \in \mathbb{Z}^+$.

- 1) S is α - ℓ -normal if, for all $w \in \Sigma^{\ell}$,

$$\lim_{n \rightarrow \infty} \pi_{S,n}(w) = \alpha^{(\ell)}(w).$$

- 2) S is α -normal if, for all $\ell \in \mathbb{Z}^+$, S is α - ℓ -normal.
- 3) S is ℓ -normal if S is μ - ℓ -normal, where μ is the uniform probability measure on Σ .
- 4) S is normal if, for all $\ell \in \mathbb{Z}^+$, S is ℓ -normal.

Lemma II.1: For all $\alpha \in \Delta(\Sigma)$, $S \in \Sigma^{\omega}$, and $\ell \in \mathbb{Z}^+$, the following four conditions are equivalent.

- (1) S is α - ℓ -normal.
- (2) $\text{Div}_{\ell}(S||\alpha) = 0$.
- (3) For every α - ℓ -normal sequence $T \in \Sigma^{\omega}$, $\text{Div}_{\ell}(S||T) = 0$.
- (4) There exists an α - ℓ -normal sequence $T \in \Sigma^{\omega}$ such that $\text{Div}_{\ell}(S||T) = 0$.

Proof: Let α , S , and ℓ be as given.

To see that (1) implies (2), assume (1). Then $\lim_{n \rightarrow \infty} \pi_{S,n}^{(\ell)} = \alpha^{(\ell)}$, so the continuity of KL-divergence tells us that

$$\text{Div}_{\ell}(S||\alpha) = \lim_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)}) = 0,$$

i.e., that (2) holds.

To see that (2) implies (3), assume (2). Then $\lim_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)}) = \text{Div}_{\ell}(S||\alpha) = 0$, whence the \mathcal{L}_1 bound in section II-A tells us that $\lim_{n \rightarrow \infty} \pi_{S,n}^{(\ell)} = \alpha^{(\ell)}$. For any α - ℓ -normal sequence $T \in \Sigma^{\omega}$, we have $\lim_{n \rightarrow \infty} \pi_{T,n}^{(\ell)} = \alpha^{(\ell)}$, whence the continuity of KL-divergence tells us that

$$\text{Div}_{\ell}(S||T) = \lim_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \pi_{T,n}^{(\ell)}) = D(\alpha^{(\ell)} || \alpha^{(\ell)}) = 0,$$

i.e., that (3) holds.

Since α - ℓ -normal sequences exist, it is trivial that (3) implies (4).

Finally, to see that (4) implies (1), assume that (4) holds. Then we have

$$\lim_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)} || \pi_{T,n}^{(\ell)}) = \text{Div}_{\ell}(S||T) = 0,$$

whence the \mathcal{L}_1 bound in section II-A tells us that

$$\lim_{n \rightarrow \infty} \|\pi_{S,n}^{(\ell)} - \pi_{T,n}^{(\ell)}\|_1 = 0. \quad (7)$$

We also have

$$\lim_{n \rightarrow \infty} \pi_{T,n}^{(\ell)} = \alpha^{(\ell)},$$

whence

$$\lim_{n \rightarrow \infty} \|\pi_{T,n}^{(\ell)} - \alpha^{(\ell)}\|_1 = 0. \quad (8)$$

By (7), (8), and the triangle inequality for the \mathcal{L}_1 -norm, we have

$$\lim_{n \rightarrow \infty} \|\pi_{S,n}^{(\ell)} - \alpha^{(\ell)}\|_1 = 0,$$

whence

$$\lim_{n \rightarrow \infty} \pi_{S,n}^{(\ell)} = \alpha^{(\ell)},$$

i.e., (1) holds. \square

Lemma II.1 immediately implies the following.

Theorem II.2 (Divergence Characterization of Normality):

For all $\alpha \in \Delta(\Sigma)$ and $S \in \Sigma^\omega$, the following conditions are equivalent.

- (1) S is α -normal.
- (2) $\text{Div}(S||\alpha) = 0$.
- (3) For every α -normal sequence $T \in \Sigma^\omega$, $\text{Div}(S||T) = 0$.
- (4) There exists an α -normal sequence $T \in \Sigma^\omega$ such that $\text{Div}(S||T) = 0$.

III. STRONG DICHOTOMY

This section presents our main theorem, the strong dichotomy theorem for finite-state gambling. We first review finite-state gamblers.

Fix a finite alphabet Σ with $|\Sigma| \geq 2$.

Definition ([14], [18], [31]): A *finite-state gambler (FSG)* is a 4-tuple

$$G = (Q, \delta, s, B),$$

where Q is a finite set of *states*, $\delta : Q \times \Sigma \rightarrow Q$ is the *transition function*, $s \in Q$ is the *start state*, and $B : Q \rightarrow \Delta_Q(\Sigma)$ is the *betting function*.

The transition structure (Q, δ, s) here works as in any deterministic finite-state automaton. For $w \in \Sigma^*$, we write $\delta(w)$ for the state reached from s by processing w .

Intuitively, a gambler $G = (Q, \delta, s, B)$ bets on the successive symbols of a sequence $S \in \Sigma^\omega$. The payoffs in the betting are determined by a *payoff probability measure* $\alpha \in \Delta(\Sigma)$. (We regard α and S as external to the gambler G .) We write $d_{G,\alpha}(w)$ for the gambler G 's capital (amount of money) after betting on the successive symbols of a prefix $w \sqsubseteq S$, and we assume that the initial capital is $d_{G,\alpha}(\lambda) = 1$.

The meaning of the betting function B is as follows. After betting on a prefix $w \sqsubseteq S$, the gambler is in state $\delta(w) \in Q$. The betting function B says that, for each $a \in \Sigma$, the gambler bets the fraction $B(\delta(w))(a)$ of its current capital $d_{G,\alpha}(w)$ that $wa \sqsubseteq S$, i.e., that the next symbol of S is an a . If it then turns out to be the case that $wa \sqsubseteq S$, the gambler's capital will be

$$d_{G,\alpha}(wa) = d_{G,\alpha}(w) \frac{B(\delta(w))(a)}{\alpha(a)}. \quad (9)$$

(Note: If $\alpha(a) = 0$ here, we may define $d_{G,\alpha}(wa)$ however we wish.)

The payoffs in (9) are fair with respect to α , which means that the conditional α -expectation

$$\sum_{a \in \Sigma} \alpha(a) d_{G,\alpha}(wa)$$

of $d_{G,\alpha}(wa)$, given that $w \sqsubseteq S$, is exactly $d_{G,\alpha}(w)$. This says that the function $d_{G,\alpha}$ is an α -martingale.

If $\delta(w) = q$ is a state in which $B(q) = \alpha$, then (9) says that, for each $a \in \Sigma$, $d_{G,\alpha}(wa) = d_{G,\alpha}(w)$. That is, the condition $B(q) = \alpha$ means that G *does not bet* in state q . Accordingly, we define the *risk* that G takes in a state $q \in Q$ to be

$$\text{risk}_G(q) = D(\alpha||B(q)).$$

i.e., the divergence of $B(q)$ from not betting. We also define the *total risk* that G takes *along* a string $w \in \Sigma^*$ to be

$$\text{Risk}_G(w) = \sum_{u \sqsubseteq w} \text{risk}_G(\delta(u)).$$

We now prove our main theorem.

Theorem III.1 (Strong Dichotomy Theorem): Let $\alpha \in \Delta^+(\Sigma)$, $S \in \Sigma^\omega$, and $\gamma < 1$.

- 1) If S is not α -normal, then there is a finite-state gambler G such that, for infinitely many prefixes $w \sqsubseteq S$,

$$d_{G,\alpha}(w) > 2^{\gamma \text{Div}(S||\alpha)|w|}.$$

- 2) If S is α -normal, then, for every finite-state gambler G , for all but finitely many prefixes $w \sqsubseteq S$,

$$d_{G,\alpha}(w) < 2^{-\gamma \text{Risk}_G(w)}.$$

Proof: To prove the first part, let S be a non-normal sequence. Then by Theorem II.2, we know that $\text{Div}(S||\alpha) > 0$. Let $r < 1$ and let $\epsilon > 0$. By the definition of $\text{Div}(S||\alpha)$ there must exist ℓ such that

$$\text{Div}_\ell(S||\alpha)/\ell > r \text{Div}(S||\alpha). \quad (10)$$

That is

$$\text{Div}_\ell(S||\alpha) = \limsup_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)}||\alpha^{(\ell)}) > \ell r \text{Div}(S||\alpha).$$

We can pick a subsequence of indices n_k 's, such that $\lim_{k \rightarrow \infty} D(\pi_{S,n_k}^{(\ell)}||\alpha^{(\ell)}) = \text{Div}_\ell(S||\alpha)$. Therefore by inequality (3.2)

$$D(\pi_{S,n_k}^{(\ell)}||\alpha^{(\ell)}) > \ell r \text{Div}(S||\alpha) \quad (11)$$

for sufficiently large k . In particular, by compactness of $[0, 1]^{|\Sigma|}$ equipped with the \mathcal{L}_1 -norm, we can further require that

$$\lim_{k \rightarrow \infty} \pi_{S,n_k}^{(\ell)} \text{ exists.} \quad (12)$$

Let $\pi_0 = \pi_0(r, m) \in \Delta_Q(\Sigma^\ell)$ be the m -th $\pi_{S,n_k}^{(\ell)}$ that satisfies (11), indexed by k . By the way we define π_0 , we have

$$D(\pi_0||\alpha^{(\ell)}) > \ell r \text{Div}(S||\alpha), \quad \text{and} \quad (13)$$

$$\|\pi_0 - \pi_{S,n_k}^{(\ell)}\| \rightarrow 0, \quad \text{as } m \rightarrow \infty \text{ and } k \rightarrow \infty, \quad (14)$$

whence D 's continuity in section II-A tells us that

$$D(\pi_{S,n_k}^{(\ell)}||\pi_0) \rightarrow 0, \quad \text{as } m \rightarrow \infty \text{ and } k \rightarrow \infty. \quad (15)$$

Also note that,

$$\lim_{m \rightarrow \infty} D(\pi_0||\alpha^{(\ell)}) = \lim_{k \rightarrow \infty} D(\pi_{S,n_k}^{(\ell)}||\alpha^{(\ell)}) = \text{Div}_\ell(S||\alpha) > 0. \quad (16)$$

Fix m such that for $\pi_0 = \pi_0(r, m)$, for any n_k sufficiently large, we have

$$D(\pi_{S,n_k}^{(\ell)}||\pi_0) < \epsilon/(1 + \epsilon) \text{Div}_\ell(S||\alpha) < \epsilon D(\pi_0||\alpha^{(\ell)}), \quad (17)$$

and

$$\begin{aligned} D(\pi_{S,n_k}^{(\ell)} || \alpha^{(\ell)}) &> \text{Div}_\ell(S || \alpha)(1 - \epsilon^2) \\ &> D(\pi_0 || \alpha^{(\ell)})(1 - \epsilon) > 0. \end{aligned} \quad (18)$$

By doing the above we pick a probability measure π_0 that is “far” away from $\alpha^{(\ell)}$, we now hard code π_0 in a gambler $G = (Q, \delta, s, B)$, where

$$Q = \Sigma^{\leq \ell-1}, \quad \delta(w, a) = \begin{cases} wa & \text{if } |wa| < \ell \\ \lambda & \text{if } |wa| = \ell \end{cases},$$

$$s = \lambda, \quad \text{and } B(w)(a) = \pi_0(a|w),$$

where $\pi_0(a|w)$ describes the conditional probability (induced by π_0) of occurrence of an a after $w \in Q$, and is defined by $\pi_0(a|w) = \pi_0(wa)/\pi_0(w)$, where for $u \in Q$, the notation $\pi_0(u)$ is defined recursively by $\pi_0(w) = \sum_{a \in \Sigma} \pi_0(wa)$.

Let $u = a_0 \cdots a_{\ell-1}$ be in Σ^ℓ . The following observation captures the above intuition:

$$\frac{B(\lambda)(a_0) \cdots B(u[0..\ell-2])(a_{\ell-1})}{\alpha(a_0) \cdots \alpha(a_{\ell-1})} = \frac{\pi_0(u)}{\alpha^{(\ell)}(u)}.$$

Now let $w = S \upharpoonright n_k$ for some k . We can view w as $w = u_0 u_1 \cdots u_{n-1} u_n$, where $|u_i| = \ell$ for $0 \leq i \leq n-1$ and $u_n = a_0 \cdots a_m$ with $m < \ell$. Then we have

$$\begin{aligned} d_{G,\alpha}(w) &= \left(\prod_0^{n-1} \frac{\pi_0(u_i)}{\alpha^{(\ell)}(u_i)} \right) \frac{B(\lambda)(a_0) \cdots B(u_n[0..m-1])(a_m)}{\alpha(a_0) \cdots \alpha(a_m)} \\ &\geq C_0 \prod_0^{n-1} \frac{\pi_0(u_i)}{\alpha^{(\ell)}(u_i)}, \end{aligned} \quad (19)$$

where C_0 is the minimum value of

$$\frac{B(\lambda)(a_0) \cdots B(u_n[0..m-1])(a_m)}{\alpha(a_0) \cdots \alpha(a_{\ell-1})},$$

where $u_n = a_0 \cdots a_m$ ranges over $\Sigma^{<\ell}$. Taking log on both sides of (19) we get

$$\begin{aligned} \log d_{G,\alpha}(w) - \log C_0 &\geq \sum_{i=0}^{n-1} \log \frac{\pi_0(u_i)}{\alpha^{(\ell)}(u_i)} = \sum_{|u|=\ell} \#_{\square}(u, w) \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \frac{\#_{\square}(u, w)}{n} \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \pi_{S,n}^{(\ell)}(u) \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \left[\pi_{S,n}^{(\ell)}(u) \log \frac{\pi_{S,n}^{(\ell)}(u)}{\alpha^{(\ell)}(u)} - \pi_{S,n}^{(\ell)}(u) \log \frac{\pi_{S,n}^{(\ell)}(u)}{\pi_0(u)} \right] \\ &= n \left(D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)}) - D(\pi_{S,n}^{(\ell)} || \pi_0) \right) \end{aligned} \quad (20)$$

Then by (18), for $w = S \upharpoonright n_k$ long enough, we have

$$\begin{aligned} \log d_{G,\alpha}(w) - \log C_0 &\geq n \left(D(\pi_{S,n}^{(\ell)} || \alpha^{(\ell)}) - D(\pi_{S,n}^{(\ell)} || \pi_0) \right) \\ &\geq n \left(D(\pi_0 || \alpha^{(\ell)})(1 - \epsilon) - D(\pi_{S,n}^{(\ell)} || \pi_0) \right) \\ &\geq \frac{|w|}{\ell} D(\pi_0 || \alpha^{(\ell)})(1 - 2\epsilon). \end{aligned}$$

Therefore, by (13) we have

$$d_{G,\alpha}(w) \geq C_0 2^{\frac{|w|(1-2\epsilon)}{\ell} D(\pi_0 || \alpha^{(\ell)})} \geq 2^{|w|r(1-2\epsilon) \text{Div}(S || \alpha)}.$$

Since r and $1 - 2\epsilon$ can be picked arbitrarily close to 1, take $r(1 - 2\epsilon) > \gamma$, then there is a G such that

$$d_{G,\alpha}(w) \geq 2^{\gamma \text{Div}(S || \alpha) |w|}$$

for $w = S \upharpoonright n_k$ long enough.

We now prove the second part of the main theorem.

Let S be a normal number, G an arbitrary finite-state gambler. By Proposition 2.5 of [31], $G = (Q, \delta, s, B)$ will eventually reach a *bottom strongly connected component* (a component that has no path to leave) when processing S . A similar argument can also be found in [33]. Without loss of generality, we will therefore assume that every state in G is recurrent in processing S .

Let $w = a_0 \cdots a_{n-1} \sqsubseteq S$. Then

$$\begin{aligned} d_{G,\alpha}(w) &= \frac{B(\delta(\lambda))(a_0) \cdots B(\delta(w[a_0..a_{n-2}])(a_{n-1}))}{\alpha(a_0) \cdots \alpha(a_{n-1})} \\ &= \prod_{q \in Q} \prod_{a \in \Sigma} \left(\frac{B(q)(a)}{\alpha(a)} \right)^{\#_{G,w}(q,a)}, \end{aligned} \quad (21)$$

where the notation $\#_{G,w}(q, a)$ denotes the number of times G lands on state q and the next symbol is a while processing w . Similarly, we use the notation $\#_{G,w}(q)$ to denote the number of times G lands on q in the same process.

Taking the logarithm of both sides of (21), we have

$$\begin{aligned} \log d_{G,\alpha}(w) &= \sum_{q \in Q} \sum_{a \in \Sigma} \#_{G,w}(q, a) \log \frac{B(q)(a)}{\alpha(a)} \\ &= \sum_{q \in Q} \#_{G,w}(q) \sum_{a \in \Sigma} \frac{\#_{G,w}(q, a)}{\#_{G,w}(q)} \log \frac{B(q)(a)}{\alpha(a)} \end{aligned} \quad (22)$$

By a result of Agafonov [1], which extends easily to the arbitrary probability measures considered here, we have that, for every state q , the limit of $\frac{\#_{G,w}(q,a)}{\#_{G,w}(q)}$ along S exists and converges to $\alpha(a)$. That is

$$\lim_{w \rightarrow S} \frac{\#_{G,w}(q, a)}{\#_{G,w}(q)} = \alpha(a), \quad (23)$$

for every state q .

Therefore, by equations (22) and (23), and the fact that there are finitely many states, we have

$$\begin{aligned} \log d_{G,\alpha}(w) &\leq \sum_{q \in Q} \#_{G,w}(q) \sum_{a \in \Sigma} (\alpha(a) + o(1)) \log \frac{B(q)(a)}{\alpha(a)} \\ &= \sum_{q \in Q} -\text{risk}_G(q) \#_{G,w}(q) + \sum_{q \in Q} \#_{G,w}(q) \sum_{a \in \Sigma} o(1) \log \frac{B(q)(a)}{\alpha(a)} \\ &= -\text{Risk}_G(w) + \sum_{q \in Q} \#_{G,w}(q) \sum_{a \in \Sigma} o(1) \log \frac{B(q)(a)}{\alpha(a)} \\ &= \text{Risk}_G(w)(-1 + o(1)). \end{aligned}$$

It follows that

$$d_{G,\alpha}(w) \leq 2^{-(1+o(1)) \text{Risk}_B(w)},$$

so part 2 of the theorem holds. \square

IV. DIMENSION

Finite-state dimensions give a particularly sharp formulation of part 1 of the strong dichotomy theorem, along with a dual of this result.

Finite-state dimensions were introduced for the uniform probability measure on Σ in [2], [14] and extended to arbitrary probability measure on Σ in [26]. For each $\alpha \in \Delta(\Sigma)$ and each $S \in \Sigma^\omega$, define the sets

$$\mathfrak{G}^\alpha(S) = \left\{ s \in [0, \infty) \mid \left(\exists \text{FSG } G \right) \limsup_{w \rightarrow S} \alpha^{|w|}(w)^{1-s} d_{G,\alpha}(w) = \infty \right\}$$

and

$$\mathfrak{G}_{\text{str}}^\alpha(S) = \left\{ s \in [0, \infty) \mid \left(\exists \text{FSG } G \right) \liminf_{w \rightarrow S} \alpha^{|w|}(w)^{1-s} d_{G,\alpha}(w) = \infty \right\}$$

The limits superior and inferior here are taken for successively longer prefixes $w \sqsubseteq S$. The “strong” subscript of $\mathfrak{G}_{\text{str}}(S)$ refers to the fact that $\alpha^{|w|}(w)^{1-s} d_{G,\alpha}(w)$ is required to converge to infinity in a stronger sense than in $\mathfrak{G}^\alpha(S)$.

Definition ([26]): Let $\alpha \in \Delta(\Sigma)$ and $S \in \Sigma^\omega$.

- 1) The *finite-state α -dimension* of S is

$$\dim_{\text{FS}}^\alpha(S) = \inf \mathfrak{G}^\alpha(S).$$

- 2) The *finite-state strong α -dimension* of S is

$$\text{Dim}_{\text{FS}}^\alpha(S) = \inf \mathfrak{G}_{\text{str}}^\alpha(S).$$

It is easy to see that, for all $\alpha \in \Delta(\Sigma)$ and $S \in \Sigma^\omega$, $0 \leq \dim_{\text{FS}}^\alpha(S) \leq \text{Dim}_{\text{FS}}^\alpha(S) \leq 1$.

Theorem IV.1: For all $\alpha \in \Delta^+(\Sigma)$ and $S \in \Sigma^\omega$ let $c = \log(1/\min_{a \in \Sigma} \alpha(a))$. Then,

$$\dim_{\text{FS}}^\alpha(S) \leq 1 - \text{Div}(S||\alpha)/c$$

$$\text{Dim}_{\text{FS}}^\alpha(S) \leq 1 - \text{div}(S||\alpha)/c.$$

Proof: Let $t < \text{Div}(S||\alpha)/c$, and let $s = 1 - t$. Fix ℓ such that $\text{Div}_\ell(S||\alpha)/\ell > tc$, then for i.o. n , $D(\pi_{S,n}^{(\ell)}||\alpha^{(\ell)}) > \ell tc$. Note that $\alpha^{|w|}(w) \geq (1/2^c)^{|w|}$ for every $w \in \Sigma^*$.

Define the gambler G be $G = (Q, \delta, s_0, B_n)$, where

$$\begin{aligned} Q &= \Sigma^{\leq \ell-1}, \\ \delta(w, a) &= \begin{cases} wa & \text{if } |wa| < \ell \\ \lambda & \text{if } |wa| = \ell \end{cases}, \\ s_0 &= \lambda, \text{ and} \\ B_n(w)(a) &= \pi_{S,n}^{(\ell)}(a|w), \end{aligned}$$

where $\pi_{S,n}^{(\ell)}(a|w)$ describes the conditional probability (induced by $\pi_{S,n}^{(\ell)}$) of occurrence of an a after $w \in Q$.

Let $u = a_0 \cdots a_{\ell-1}$ be in Σ^ℓ .

$$\frac{B_n(\lambda)(a_0) \cdots B_n(u[0..\ell-2])(a_{\ell-1})}{\alpha(a_0) \cdots \alpha(a_{\ell-1})} = \frac{\pi_{S,n}^{(\ell)}(u)}{\alpha^{(\ell)}(u)}.$$

Then for $z \in \Sigma^*$ with $z \sqsubseteq S$ and $|z| = \ell n$, we have

$$\begin{aligned} \alpha^{|z|}(z)^{1-s} d_{G,\alpha}(z) &= \alpha^{|z|}(z)^t d_{G,\alpha}(z) \\ &= \alpha^{|z|}(z)^t \prod_{u \in \Sigma^\ell} \left(\frac{\pi_{S,n}^{(\ell)}(u)}{\alpha^{(\ell)}(u)} \right)^{n \pi_{S,n}^{(\ell)}(u)} \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha^{|z|}(z)^t d_{G,\alpha}(z) &\geq \left(\frac{1}{2^c} \right)^{|z|t} 2^{n D(\pi_{S,n}^{(\ell)}||\alpha^{(\ell)})} \\ &\geq 2^{-c|z|t+c|z|t} \end{aligned}$$

Since the number of states is fixed, this implies $\dim_{\text{FS}}^\alpha(S) \leq 1 - \text{Div}(S||\alpha)/c$.

The proof of the other case is similar, where we use the fact that, for a.e. n ,

$$D(\pi_{S,n}^{(\ell)}||\alpha^{(\ell)}) > \ell tc.$$

□

V. ASYMPTOTIC ENTROPY AND FINITE-STATE DIMENSION

In this last section we present a divergence inequality relating finite-state dimension to asymptotic divergence and Shannon entropy. A divergence equality exists for β -normal sequences S , for β a positive probability distribution, characterizing $\dim_{\text{FS}}^\alpha(S)$ in terms of the divergence of α from β and the Shannon entropy of β [26]. Our result here holds for any sequence S , and therefore uses instead the divergence of α from S and the asymptotic Shannon entropy of a sequence S , defined below.

Definition: Let $S \in \Sigma^\omega$. We define the following asymptotic entropy.

- 1) $H_+^{(\ell)}(S) = \frac{1}{\ell} \limsup_{n \rightarrow \infty} H(\pi_{S,n}^{(\ell)})$.
- 2) $H^+(S) = \inf_\ell H_+^{(\ell)}$.
- 3) $H_-^{(\ell)}(S) = \frac{1}{\ell} \liminf_{n \rightarrow \infty} H(\pi_{S,n}^{(\ell)})$.
- 4) $H^-(S) = \inf_\ell H_-^{(\ell)}$.

Our next lemma compares the condition $\text{div}(S||\alpha) > 0$ (not equivalent to $\text{Div}(S||\alpha) > 0$, characterising non-normality) to the success obtained by finite-state gamblers on S .

Lemma V.1: Let $\alpha \in \Delta^+(\Sigma)$ and $S \in \Sigma^\omega$. For every $\gamma < 1$, if $\text{div}(S||\alpha) > 0$ then there is a finite-state gambler G such that, for infinitely many prefixes $w \sqsubseteq S$,

$$d_{G,\alpha}(w) > 2^{\gamma \text{div}(S||\alpha)|w|}.$$

Proof: Let S be such that $\text{div}(S||\alpha) > 0$. Let $r < 1$, let $\epsilon > 0$. By the definition of $\text{div}(S||\alpha)$ there must exist ℓ such that

$$\text{div}_\ell(S||\alpha)/\ell > r \text{div}(S||\alpha) \quad (24)$$

That is

$$\liminf_{n \rightarrow \infty} D(\pi_{S,n}^{(\ell)}||\alpha^{(\ell)}) > \ell r \text{div}(S||\alpha).$$

We can pick a subsequence of indices n_k 's, such that $\lim_{k \rightarrow \infty} D(\pi_{S,n_k}^{(\ell)}||\alpha^{(\ell)}) = \text{div}_\ell(S||\alpha)$. Therefore

$$D(\pi_{S,n_k}^{(\ell)}||\alpha^{(\ell)}) > \ell r \text{div}(S||\alpha). \quad (25)$$

for sufficiently large k . In particular, by compactness of $\mathbb{R}^{|\Sigma^\ell|}$ equipped with \mathcal{L}_1 -norm, we can further request that

$$\lim_{k \rightarrow \infty} \pi_{S, n_k}^{(\ell)} \text{ exists.} \quad (26)$$

Let

$\pi_0 = \pi_0(r, m) \equiv$ “the m -th number in $\{n_k\}$ that satisfies (25)”

By the way we pick π_0 , we have

$$D(\pi_0 || \alpha^{(\ell)}) > \ell r \operatorname{div}(S || \alpha), \text{ and} \quad (27)$$

$$\|\pi_0 - \pi_{S, n_k}^{(\ell)}\| \rightarrow 0, \text{ as } m \rightarrow \infty \text{ and } k \rightarrow \infty, \quad (28)$$

whence the \mathcal{L}_1 bound in section II-A tells us that

$$D(\pi_{S, n_k}^{(\ell)} || \pi_0) \rightarrow 0, \text{ as } m \rightarrow \infty \text{ and } k \rightarrow \infty. \quad (29)$$

Also note that,

$$\lim_{m \rightarrow \infty} D(\pi_0 || \alpha^{(\ell)}) = \lim_{k \rightarrow \infty} D(\pi_{S, n_k}^{(\ell)} || \alpha^{(\ell)}) = \operatorname{div}_\ell(S || \alpha) > 0. \quad (30)$$

For a fixed $\pi_0 = \pi_0(r, m)$, by the definition, for any $n_k > m$'s, we have

$$D(\pi_{S, n_k}^{(\ell)} || \alpha^{(\ell)}) > D(\pi_0 || \alpha^{(\ell)}) > 0. \quad (31)$$

By doing the above we pick a probability measure π_0 that is “far” away from α , we now hard code π_0 in a gambler $G = (Q, \delta, s, B)$, where

$$\begin{aligned} Q &= \Sigma^{\leq \ell-1}, \\ \delta(w, a) &= \begin{cases} wa & \text{if } |wa| < \ell \\ \lambda & \text{if } |wa| = \ell \end{cases}, \\ s &= \lambda, \text{ and} \\ B(w)(a) &= \pi_0(a|w) \end{aligned}$$

where $\pi_0(a|w)$ describes the conditional probability (induced by π_0) of occurrence of an a after $w \in Q$, and is defined by $\pi_0(a|w) = \pi_0(wa)/\pi_0(w)$, where for $u \in Q$, the notation $\pi_0(u)$ is defined recursively by $\pi_0(w) = \sum_{a \in \Sigma} \pi_0(wa)$.

Note that G can be viewed as a gambler gambling on every ℓ symbols, in the way that he always “waits” until he sees the first $\ell - 1$ symbols of a string $u = wa$ of length ℓ , and then bets a fraction of $\pi_0(wa)$ of his capital on the next symbol being an a .

Let $u = a_0 \cdots a_{\ell-1}$ be in Σ^ℓ . The following observation captures the above intuition:

$$\frac{B(\lambda)(a_0) \cdots B(u[0..\ell-2])(a_{\ell-1})}{\alpha(a_0) \cdots \alpha(a_{\ell-1})} = \frac{\pi_0(u)}{\alpha^{(\ell)}(u)}.$$

Now let $\{n_k\}$ satisfy (25) and (26), and let $w = S \upharpoonright n_k$ for some k . We can view w as $w = u_0 u_1 \cdots u_{n-1} u_n$, where $|u_i| = \ell$ for $0 \leq i \leq n-1$ and $u_n = a_0 \cdots a_m$ with $m < \ell$. Then we have

$$\begin{aligned} d_{G, \alpha}(w) &= \left(\prod_0^{n-1} \frac{\pi_0(u_i)}{\alpha(u_i)} \right) \frac{B(\lambda)(a_0) \cdots B(u_n[0..m-1])(a_m)}{\alpha(a_0) \cdots \alpha(a_m)} \\ &\geq C_0 \prod_0^{n-1} \frac{\pi_0(u_i)}{\alpha(u_i)}, \end{aligned} \quad (32)$$

where C_0 is the minimum value of

$$\frac{B(\lambda)(a_0) \cdots B(u_n[0..m-1])(a_m)}{\alpha(a_0) \cdots \alpha(a_{\ell-1})},$$

where $u_n = a_0 \cdots a_m$ ranges over $\Sigma^{< \ell}$. Taking log on both sides of (32) we get

$$\begin{aligned} \log d_{G, \alpha}(w) - \log C_0 &\geq \sum_{i=0}^{n-1} \log \frac{\pi_0(u_i)}{\alpha^{(\ell)}(u_i)} \\ &= \sum_{|u|=\ell} \#_{\square}(u, w) \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \frac{\#_{\square}(u, w)}{n} \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \pi_{S, n}^{(\ell)}(u) \log \frac{\pi_0(u)}{\alpha^{(\ell)}(u)} \\ &= n \sum_{|u|=\ell} \left[\pi_{S, n}^{(\ell)}(u) \log \frac{\pi_{S, n}^{(\ell)}(u)}{\alpha^{(\ell)}(u)} - \pi_{S, n}^{(\ell)}(u) \log \frac{\pi_{S, n}^{(\ell)}(u)}{\pi_0(u)} \right] \\ &= n \left(D(\pi_{S, n}^{(\ell)} || \alpha^{(\ell)}) - D(\pi_{S, n}^{(\ell)} || \pi_0) \right) \end{aligned} \quad (33)$$

Then by (29) and (31), we have

$$\begin{aligned} \log d_{G, \alpha}(w) - \log C_0 &= n \left(D(\pi_{S, n}^{(\ell)} || \alpha^{(\ell)}) - D(\pi_{S, n}^{(\ell)} || \pi_0) \right) \\ &\geq n \left(D(\pi_0 || \alpha^{(\ell)}) - D(\pi_{S, n}^{(\ell)} || \pi_0) \right) \\ &\geq \frac{|w|}{\ell} D(\pi_0 || \alpha^{(\ell)}) (1 - \epsilon). \end{aligned}$$

Therefore, by (27) we have

$$d_{G, \alpha}(w) \geq C_0 2^{\frac{|w|}{\ell} D(\pi_0 || \alpha^{(\ell)})} (1 - \epsilon) \geq 2^{|w|r(1-\epsilon) \operatorname{div}(S || \alpha)}.$$

Since r and $1 - \epsilon$ can be picked arbitrary close to 1, take $r(1 - \epsilon) > \gamma$, then

$$d_{G, \alpha}(w) \geq 2^{\gamma \operatorname{div}(S || \alpha) |w|}$$

for $w = S \upharpoonright n_k$ long enough. \square

We can now prove a divergence inequality for finite-state dimension that holds for any sequence.

Theorem V.2: Let $\alpha \in \Delta^+(\Sigma)$ and $S \in \Sigma^\omega$. Then for every $\gamma < 1$ there is an ℓ such that

$$\dim_{\text{FS}}^\alpha(S) \leq \frac{H_+^{(\ell)}(S) + (1 - \gamma) \operatorname{div}(S || \alpha)}{H^-(S) + \gamma \operatorname{div}(S || \alpha)}.$$

Proof: If $\operatorname{div}(S || \alpha) = 0$ the theorem is trivially true. So we assume $\operatorname{div}(S || \alpha) > 0$. Let $s \in (0, 1)$.

By Lemma V.1, there must be a FSG G such that

$$d_{G, \alpha}(w) > 2^{\gamma \operatorname{div}(S || \alpha) |w|},$$

for infinitely many prefixes $w \sqsubseteq S$. We also let ℓ be picked as in the proof of Lemma V.1. And let $w \sqsubseteq S$ and $|w| = n\ell$, where n is integer.

Then consider

$$\begin{aligned}
 & \alpha^{|w|}(w)^{1-s} d_{G,\alpha}(w) \\
 &= d_{G,\alpha}(w) 2^{(1-s) \log \alpha^{|w|}(w)} \\
 &= d_{G,\alpha}(w) 2^{(1-s) \frac{|w|}{\ell} \sum_{|u|=\ell} \frac{\#\square(u,w)}{|w|/\ell} \log \alpha(u)} \\
 &= d_{G,\alpha}(w) 2^{(1-s) \frac{|w|}{\ell} \sum_{|u|=\ell} \pi_{S,n}^{(\ell)}(u) \log \alpha(u)} \\
 &= d_{G,\alpha}(w) 2^{-(1-s) \frac{|w|}{\ell} \sum_{|u|=\ell} \pi_{S,n}^{(\ell)}(u) \left(\log \frac{\pi_{S,n}^{(\ell)}(u)}{\alpha(u)} - \log \pi_{S,n}^{(\ell)}(u) \right)} \\
 &= d_{G,\alpha}(w) 2^{-(1-s) \frac{|w|}{\ell} \left(D(\pi_{S,n}^{(\ell)} \| \alpha^{(\ell)}) + H(\pi_{S,n}^{(\ell)}) \right)} \\
 &> 2^{\left(\gamma \operatorname{div}(S \| \alpha) - \frac{(1-s)}{\ell} \left(D(\pi_{S,n}^{(\ell)} \| \alpha^{(\ell)}) + H(\pi_{S,n}^{(\ell)}) \right) \right) |w|}
 \end{aligned}$$

To make the left-hand side “hit infinity” infinitely often, we only need to make,

$$\limsup_{w \rightarrow S} \left\{ \gamma \operatorname{div}(S \| \alpha) - \frac{(1-s)}{\ell} \left(D(\pi_{S,n}^{(\ell)} \| \alpha^{(\ell)}) + H(\pi_{S,n}^{(\ell)}) \right) \right\} > 0 \quad (34)$$

Equation (34) holds whenever

$$s > \frac{\operatorname{div}_{\ell}(S \| \alpha) / \ell + H_{+}^{(\ell)}(S) - \gamma \operatorname{div}(S \| \alpha)}{\operatorname{div}_{\ell}(S \| \alpha) / \ell + H_{-}^{(\ell)}(S)} \quad (35)$$

From the proof of Lemma V.1

$$\operatorname{div}_{\ell}(S \| \alpha) / \ell > \gamma \operatorname{div}(S \| \alpha)$$

Replacing this in inequality (35) we have that equation (34) holds whenever

$$s > \frac{H_{+}^{(\ell)}(S) + (1 - \gamma) \operatorname{div}(S \| \alpha)}{H^{-}(S) + \gamma \operatorname{div}(S \| \alpha)}$$

We then have

$$\dim_{\text{FS}}^{\alpha}(S) \leq \frac{H_{+}^{(\ell)}(S) + (1 - \gamma) \operatorname{div}(S \| \alpha)}{H^{-}(S) + \gamma \operatorname{div}(S \| \alpha)}.$$

□

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Xiang Huang received the B.E. degree in software engineering from Nanjing University in 2009 and the Ph.D. degree in computer science from Iowa State University in 2020. He is currently an Assistant Professor of computer science with the University of Illinois at Springfield. His current research interests lie in algorithmic randomness and analog computing.

Jack H. Lutz received the B.G.S. and M.A. degrees in mathematics and the M.S. degree in computer science from the University of Kansas in 1976, 1979, and 1981, respectively, and the Ph.D. degree in mathematics from the California Institute of Technology in 1987. Since that time he has been with the Faculty of Iowa State University, where he is currently a Professor of computer science and a Professor of mathematics. He has held visiting positions at Rutgers University, Cornell University, the NEC Research Institute, the University of Wisconsin, the California Institute of Technology, and the University of Cambridge. His research interests include algorithmic information theory, algorithmic fractal dimensions, molecular programming, and computational complexity.

Elvira Mayordomo (Member, IEEE) received the M.S. degree in mathematics from the University of Zaragoza, Spain, in 1990, and the Ph.D. degree in mathematics from the Polytechnic University of Catalonia, Spain, in 1994. Since then she has been with the Faculty of the University of Zaragoza, where she is currently a Professor of computer science, with a collaborator appointment at Iowa State University, USA. Her research interests include algorithmic information theory, computational complexity, algorithmic fractal dimensions, computational phylogenetics, and computational genomics.

Donald M. Stull received the B.S. degree in computer science from The University of Texas in 2011 and the Ph.D. degree in computer science from Iowa State University in 2017. He was a Post-Doctoral Research Fellow with Inria Grand Est and a Lecturer in computer science with Iowa State University. He is currently an Esther Thornton Post-Doctoral Fellow in computer science with Northwestern University. His research is focused on the interaction between analysis and theoretical computer science.