Dokl. Akad. Nauk SSSR
Tom 179 (1968), No. 2

Soviet Math. Dokl.
Vol. 9 (1968), No. 2

NORMAL SEQUENCES AND FINITE AUTOMATA

UDC 519.95:519.211:511

V. N. AGAFONOV

A series of works [1-6] contains studies of infinite binary sequences, which are "random" in one sense or another. The concept of a random sequence considered in [1-3] is that of a collective, introduced by von Mises [1] and made more precise by Wald [2]. In papers [5, 6] normal sequences are studied. The present note clarifies the connection between these concepts in terms of finite automata.

1. Collective. A predicate defined on the set of all finite binary sequences (including the empty sequence Λ) will be called a strategy. A strategy f defines the following procedure of singling out the (possibly finite) subsequence $y_1 y_2 y_3 \cdots$ from the infinite binary sequence $x_1 x_2 x_3 \cdots$. If $z_1 z_2 z_3 \cdots$ is the sequence of values of the strategy f corresponding to the sequence $x_1 x_2 x_3 \cdots$: $z_1 = f(\Lambda)$, $z_2 = f(x_1)$, $z_3 = f(x_1 x_2)$, $\cdots$ and if $i_1 < i_2 < i_3 < \cdots$ are the indices for which $z_{i_k} = 1$, then $y_1 = x_{i_1}$, $y_2 = x_{i_2}$, $y_3 = x_{i_3}$, $\cdots$. A sequence $x_1 x_2 x_3 \cdots$ is called a collective relative to a set of strategies F (containing the strategy identically equal to one), if, for all infinite subsequences singled out from $x_1 x_2 x_3 \cdots$ by the strategies of the set F, there exists a limit (the same for all these subsequences) of the relative frequency of the ones $p = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} x_i$ $(0 < p < 1)$.

2. Normal sequence. An infinite binary sequence $x_1 x_2 x_3 \cdots$ is called normal if there exists a number p $(0 < p < 1)$ such that for any positive integer n and any word $a_1 a_2 \cdots a_n$ of length n in the alphabet $\{0, 1\}$ the following condition is satisfied: in the sequence of words $x_1 x_2 \cdots x_n$, $x_{n+1} x_{n+2} \cdots x_{2n}$, $x_{2n+1} x_{2n+2} \cdots x_{3n}$, $\cdots$ the relative frequency of appearance of the word $a_1 a_2 \cdots a_n$ has a limit equal to $p^k(1-p)^{n-k}$, where $k = \sum_{i=1}^{n} a_i$ is the number of ones in the word $a_1 a_2 \cdots a_n$.

3. A normal sequence as a collective. In [7] it is established that a sequence is normal if and only if it is a collective relative to the set of strategies possessing the following property: the strategy singles out or does not single out a letter according to what the l preceding letters are, with l fixed for every given strategy.

It is obvious that for every such strategy f there exists a finite automaton $\langle \{0, 1\}, Q, Q^*, \phi, q \rangle$ with input alphabet $\{0, 1\}$, a set of states Q, a subset of singled out states $Q^* \subseteq Q$, a transition function $\phi: Q \times \{0, 1\} \to Q^*$ and an initial state $q \in Q$, which computes the strategy f in the following manner: $f(x_1 x_2 \cdots x_n) = 1 \Longleftrightarrow \phi(\phi(\cdots \phi(\phi(q, x_1), x_2) \cdots), x_n) \in Q^*$. Hence it follows that a collective relative to the set of strategies computed by finite automata is a normal sequence. The question naturally arises whether the converse assertion is valid. A positive answer to this question is given by the following

Theorem. *A normal sequence is a collective relative to the set of strategies computed by finite automata.*

The proof of the theorem utilizes the ergodic property of a finite automaton, whose input is a sequence of independent random variables and the following property of "nondecrease of measure", possessed by a transformation defined by strategies on words. We call the p-weight of the word $a_1 \cdots a_n$ the number $p^l(1-p)^{n-l}$, where l is the number of ones in the word $a_1 \cdots a_n$ and $0 < p < 1$;

the p-weight of a subset of words of fixed length is the sum of the p-weights of all the words of this set. Then for any strategy f and any word $a_1 \cdots a_k$ the set of words of length $n \geq k$, from which f singles out words beginning with the word $a_1 \cdots a_k$, has a p-weight not larger than the p-weight of the word $a_1 \cdots a_k$.

In conclusion I would like to express my gratitude to Professor B. A. Trahtenbrot, who suggested the problem and made a number of valuable remarks.

Institute of Mathematics
Siberian Branch, Academy of Sciences of the USSR

Received 14/MAY/67

BIBLIOGRAPHY

[1] R. von Mises, *Wahrscheinlichkeit, Statistik und Wahrheit*, Lecture notes, 1928, 3rd ed., Springer-Verlag, Berlin, 1951; Russian transl., GITTL, Moscow, 1930; English transl., 2nd rev. ed., Macmillan, New York, 1957. MR 12, 837; MR 19, 776.

[2] A. Wald, Colloq. consacré à la théorie des probabilités, 1938, p. 79.

[3] A. Church, Bull. Amer. Math. Soc. 46 (1940), 130. MR 1, 149.

[4] D. W. Loveland, Trans. Amer. Math. Soc. 125 (1966), 497. MR 34 #7377.

[5] A. Copeland, Amer. J. Math. 50 (1928), 535.

[6] A. G. Postnikov, Trudy Mat. Inst. Steklov. 57 (1960). MR 26 #6146.

[7] L. P. Postnikova, Teor. Verojatnost. i Primenen. 6 (1961), 232. MR 25 #3553.

Translated by:
A. Dvoretzky