Some Thoughts on Normality, Algorithmic Randomness, and Analog Computing

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Computer Science Department, Iowa State University

A lunch talk

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A repeated theme in this talk is to demonstrate research areas on how we take care of the "**how fast**" and "**what resources**" concerns.

Outline

1. Normality and Automata

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2. Algorithmic Randomness

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Normality and Automata

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That is, *the frequency of 0's* $=$ *the frequency of 1's* $= 1/2$.

We call such a property **simply normal**.

Beyond simply normal

Consider the sequence x=01 01 01 01 ...

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You won't believe someone get it by tossing a fair coin, because the blocks "00" and "11" never occur!

Let's apply our previous idea again to rule out this sequence. That is to say, we require a "normal" sequence must have

$$\text{Freq}(00) \approx \text{Freq}\ (01) \approx \text{Freq}\ (10) \approx \text{Freq}\ (11) \approx 1/4 = 1/2^2.$$

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But why do we stop at considering length 2 blocks? We can do

$$\text{Freq}(000) \approx \text{Freq}\ (001) \approx \cdots \approx \text{Freq}\ (111) \approx 1/8 = 1/2^3.$$

Beyond simply normal

Consider the sequence x=$\underline{01}\ \underline{01}\ \underline{01}\ \underline{01}$ ...

You won't believe someone get it by tossing a fair coin, because the blocks "00" and "11" never occur!

Let's apply our previous idea again to rule out this sequence. That is to say, we require a "normal" sequence must have

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$$\text{Freq}(0000) \approx \text{Freq}\ (0001) \approx \cdots \approx \text{Freq}\ (1111) \approx 1/16 = 1/2^4.$$

and so on. Note that 2, 3, 4 are the **length**s of the blocks.

Normal sequences/numbers

Definition (Normality[1])
A sequence x is normal, if for any length ℓ, and any block of string s of length ℓ, Freq(s, x)$=\frac{1}{2^\ell}$.

Remark
If you want to talk about normality of base-10 numbers, just change the right-hand side of the above to $\frac{1}{10^\ell}$*, and only focus on the* ***fraction part*** *of the number.*

[1] Borel, E. (1909), "Les probabilités dénombrables et leurs applications arithmétiques".

Some facts about normal numbers

- Champernowne constant[2]: 0.12345678910111213141516... is normal. Or similarly in binary it is 0.$\underline{0}$ $\underline{1}$ $\underline{00}$ $\underline{01}$ $\underline{10}$ $\underline{11}$...

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    - the name of every person you will ever love
    - the date, time and manner of your death
    - the answers to all the great questions of the universe

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  - All these just simply say that, π is normal. But we don't really know that, yet.

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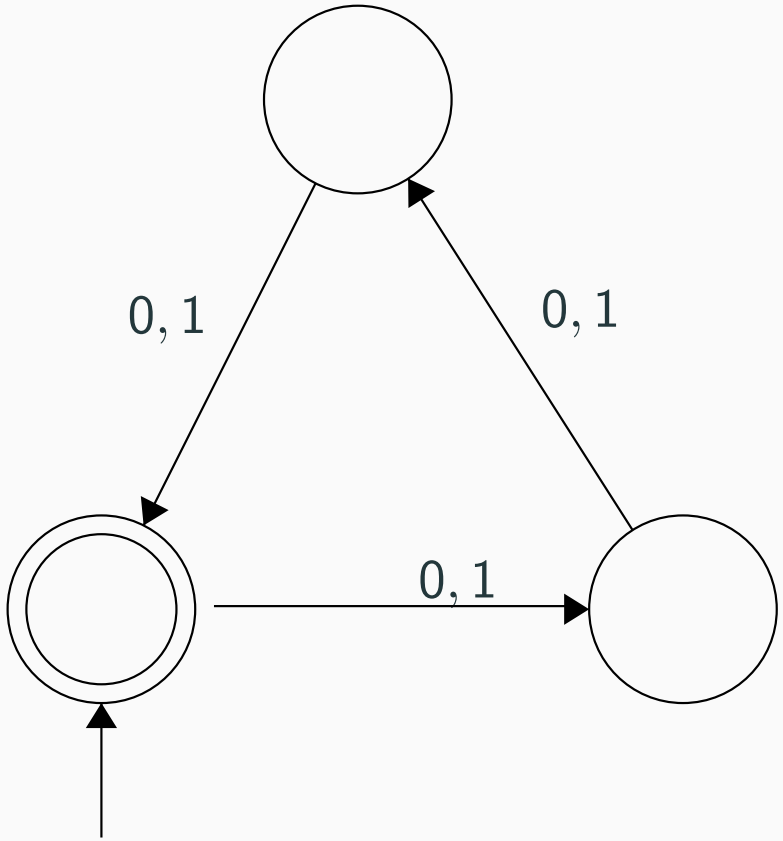
Finite State Selector

Rule of the game: hit green (accepting state), spit next bit.



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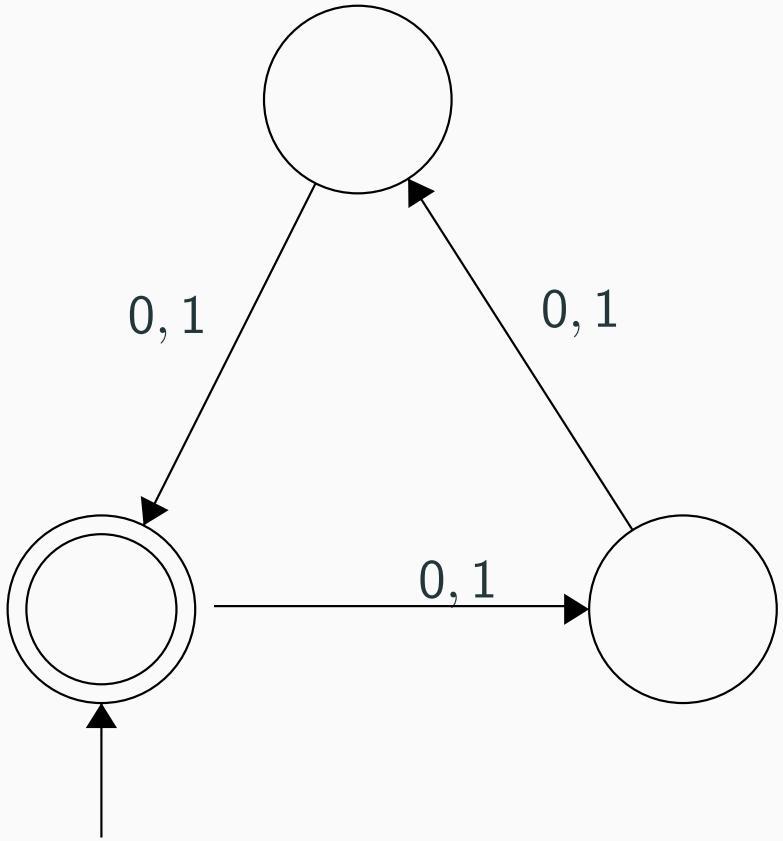

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Question: Is the resulting sequence normal?

Agafonov's Theorem

Theorem (Agafonov[4])
Finite state automata preserves normality.

[4] Agafonov, V. N. (1968), "Normal sequences and finite automata".

Selection: Cont.

Definition (Selection Rule)
A selection rule S is a subset of $\{0,1\}^*$.

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- We only discuss normality of the resulting subsequence when it has infinitely many digits.
- Agafonov theorem says that **regular** selection rules preserve normality.

Preservation of Normality: Kamae-Weiss language

Let $S \subseteq \{0,1\}^*$ be a selection rule. We define an equivalence relation (Myhill-Nerode Relation) $\equiv_S$ on $\{0,1\}^*$ as follows:

$$u \equiv_S v \quad \text{if} \quad \{w \mid uw \in S\} = \{w \mid vw \in S\}.$$

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Remark
Regular Languages $\subsetneq$ Kamae-Weiss Languages

Kamae-Weiss languages preserves normality

Theorem (Kamae and Weiss 1975[5])
Let S be a selection rule. If $S/\equiv_S$ is a finite set, then for any normal number x, if digits in x get selected "frequently", then the resulting subsequence, x_S, is also normal.

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- There is a "combinatorial" proof[6] for Agafonov theorem, adapt that?

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Selection rule that does not preserve normality

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Remark
Note that S can be recognized by a stack machine (pushdown automata). Hence this says stack machines do not preserve normality.

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An open problem

What is the largest class of languages (as selectors) that preserves normality?

- No weaker than finite state machines
- It contains some Kamae-Weiss languages.
- Weaker than stack machine.

Remark
This is a question asking what is the right amount of resources we need to preserve normality.

Finite-state gambler and strong dichotomy

Finite-state Gambler

Definition
A *finite-state gambler (FSG)* is a 4-tuple

$$G = (Q, \delta, s, B),$$

where Q is a finite set of *states*, $\delta : Q \times \Sigma \to Q$ is the *transition function*, $s \in Q$ is the *start state*, and $B : Q \to \Delta_{\mathbb{Q}}(\Sigma)$ is the *betting function*.

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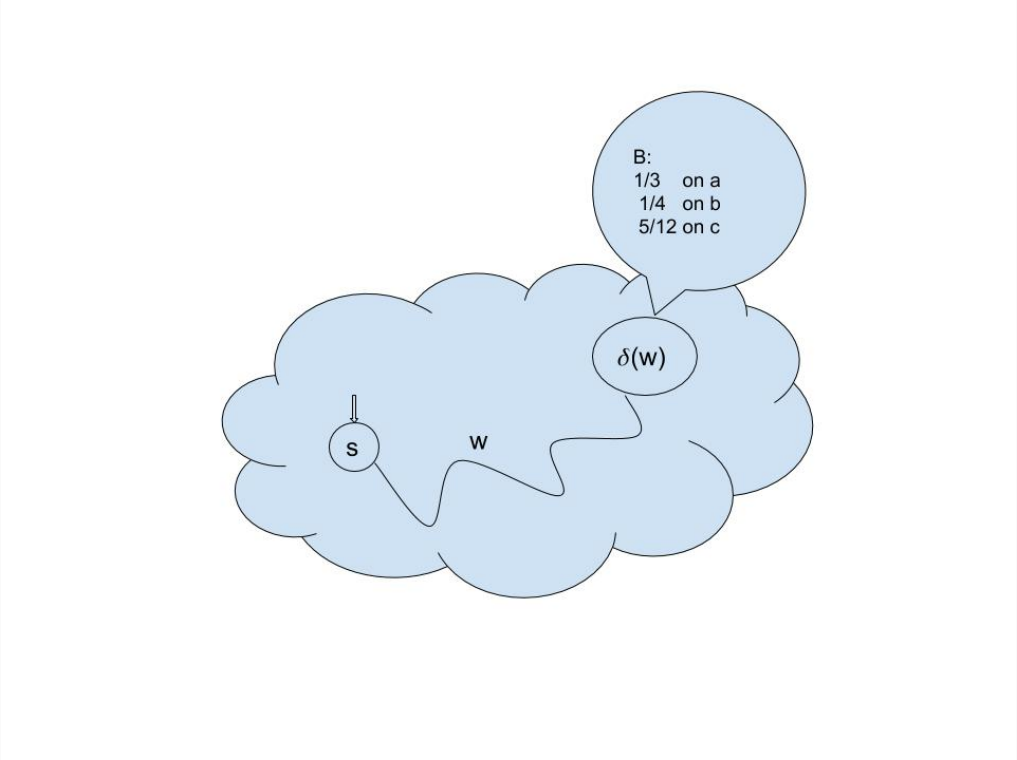

Let our capital after seeing $w \sqsubseteq S$ be $d_{G,\alpha}(w)$. The gambler's bets $B(\delta(w))(a)$ of the money that the next symbol of S is an a. If it then turns out to be the case, the gambler's capital will be

$$d_{G,\alpha}(wa) = d_{G,\alpha}(w)\frac{B\big(\delta(w)\big)(a)}{\alpha(a)}. \tag{1}$$

Schnorr-Stimm Dichotomy Theorem

Theorem (Schnorr and Stimm 72')

1. *If S is not normal, then there is a finite-state gambler that wins money at an infinitely-often exponential rate betting on S.*
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Question: What are these exponential rates? How to interpret them?

Information Theory: a Short Introduction

Self-information

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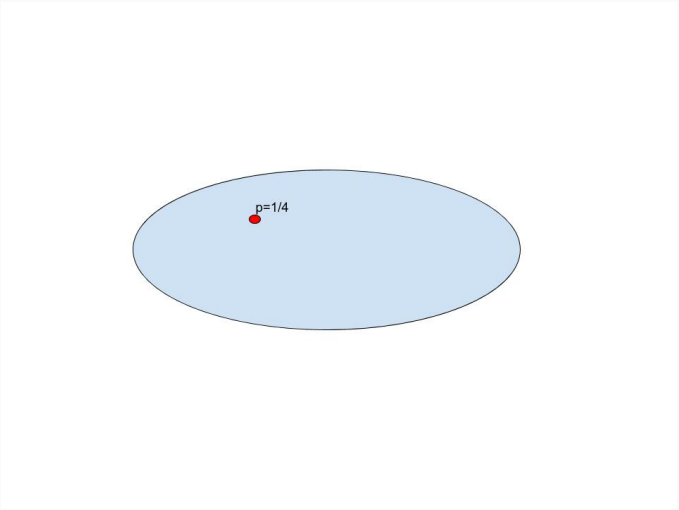

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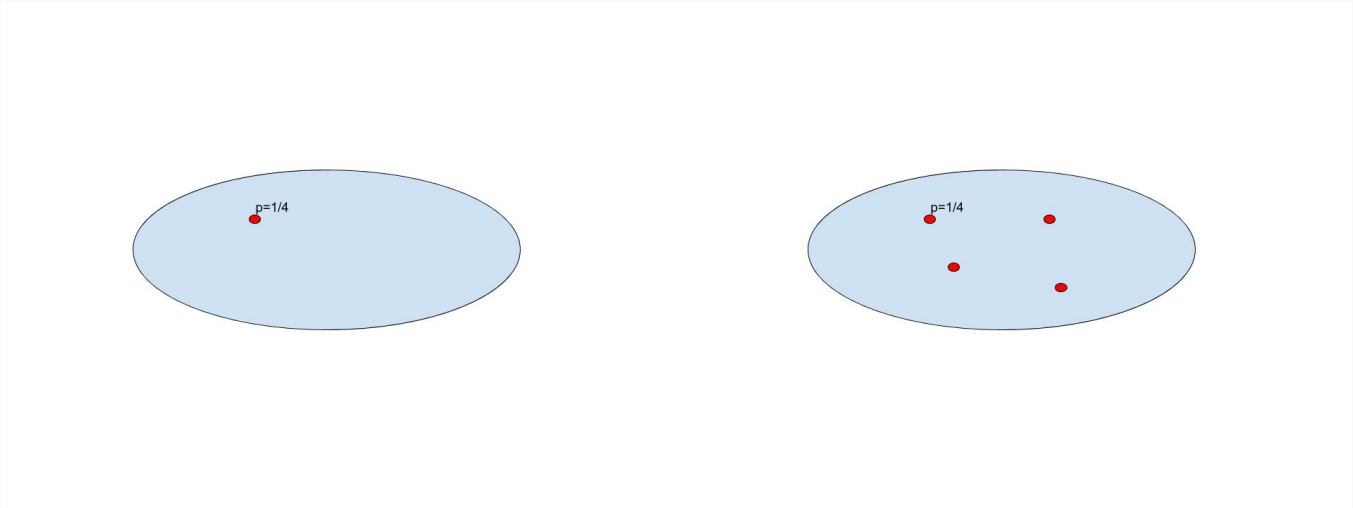

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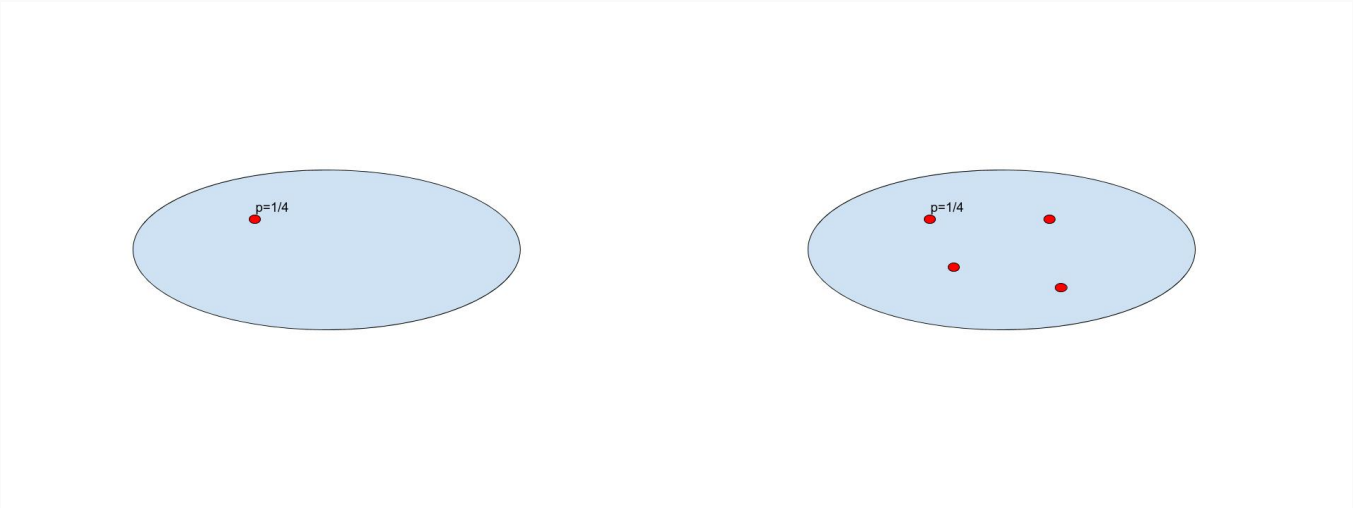

So he therefore believes he needs to have $\log \frac{1}{1/4} = 2$ bits to encode the objects in his galaxy. He can encode the four points “00”, “01”, “10” and “11”. And that is only **his** opinion.

Self-information

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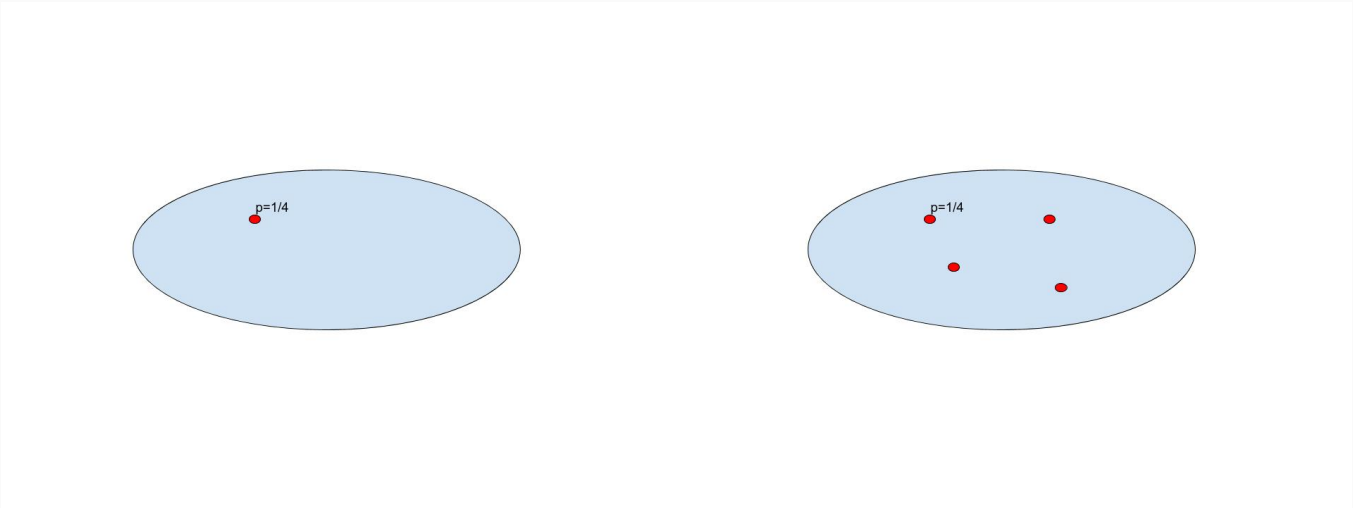

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We call $\log \frac{1}{p}$ this point's *self-information*.

Entropy

A *(discrete) probability measure* on a nonempty finite set Ω is a function $\pi : \Omega \to [0, 1]$ satisfying

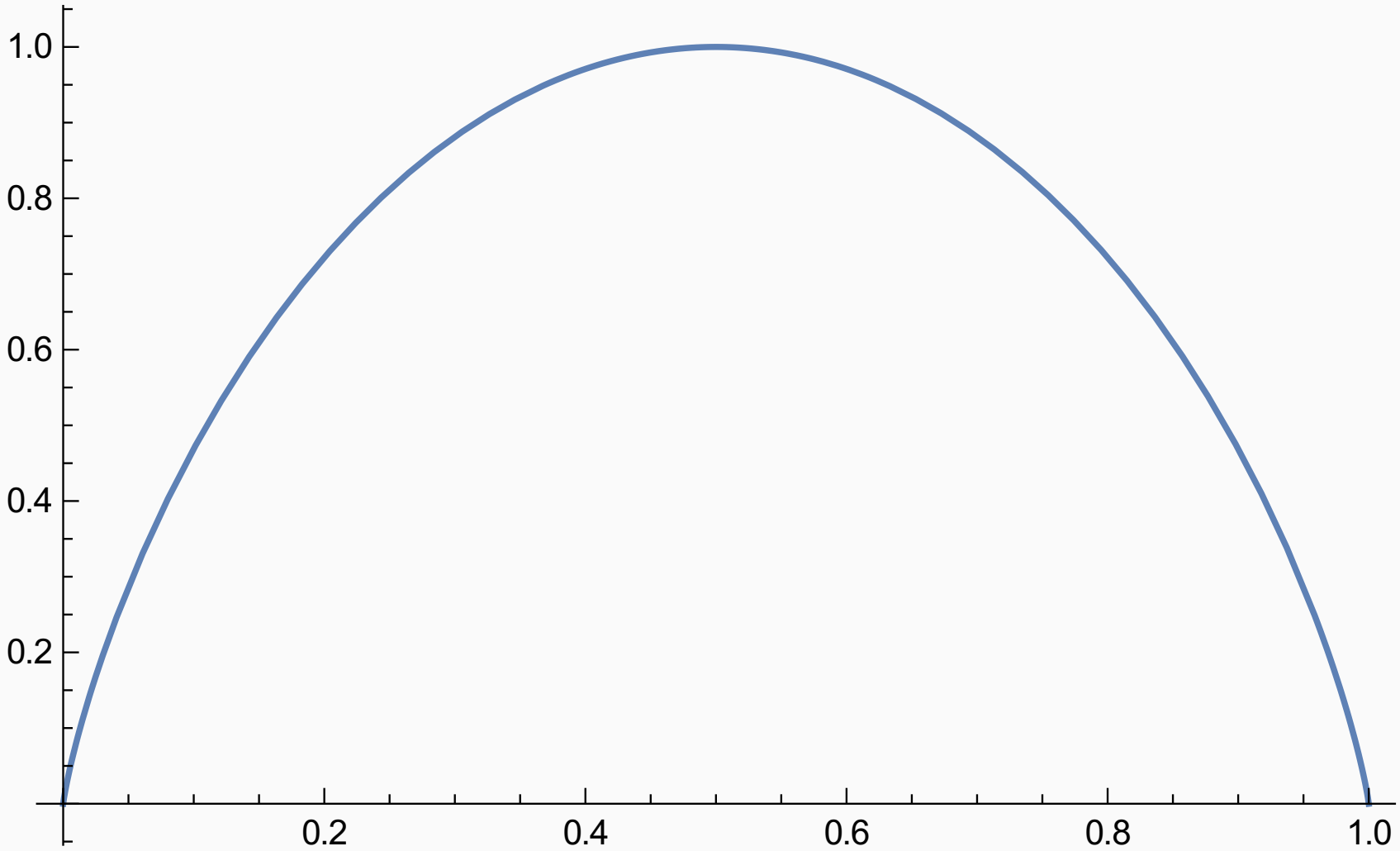
$$\sum_{\omega \in \Omega} \pi(\omega) = 1. \tag{2}$$

We write $\Delta(\Omega)$ for the set of all probability measures on Ω. Then the Shannon entropy of π is define as:

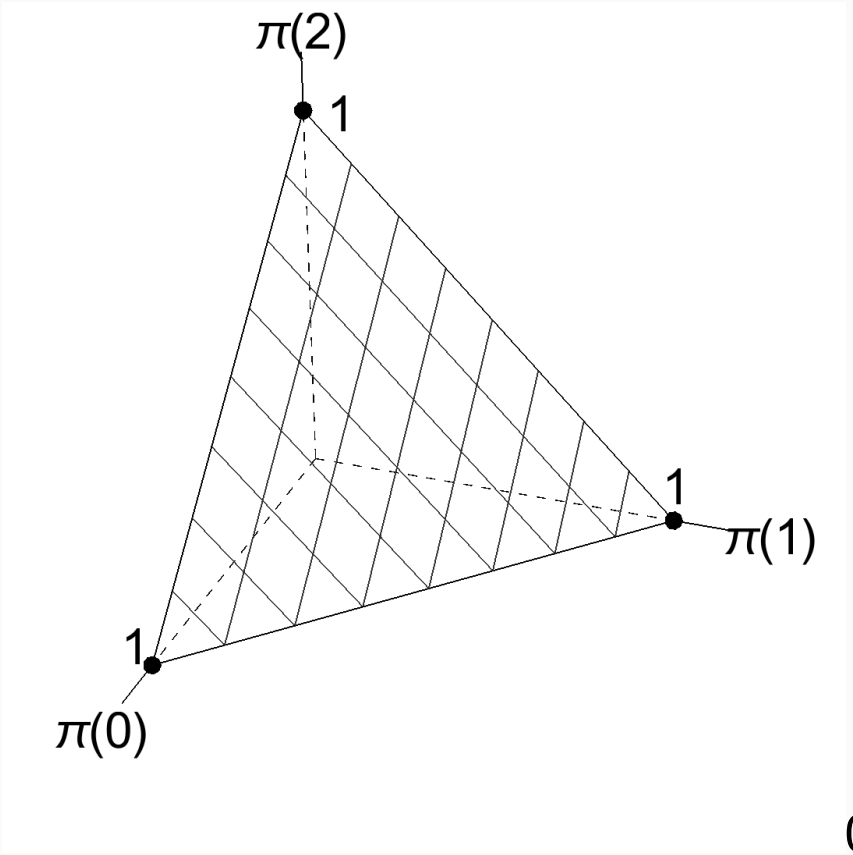
$$H(\pi) = \sum_{\omega \in \Omega} \pi(\omega) \log \frac{1}{\pi(\omega)} \tag{3}$$

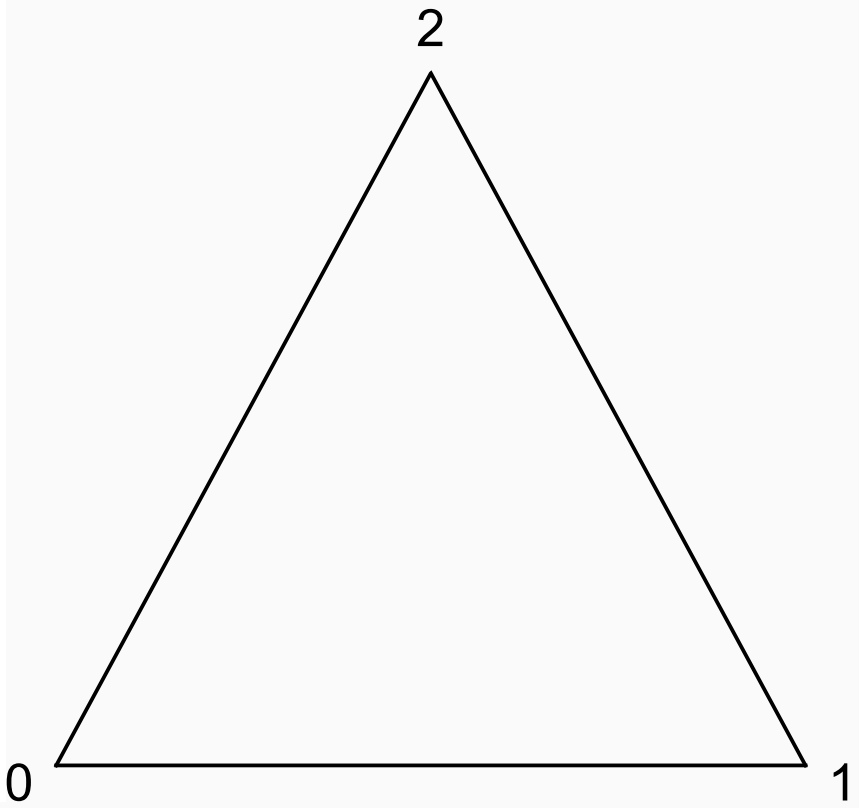
That is, a (weighted) average of the self-information of all points.

Entropy: 2D-plots



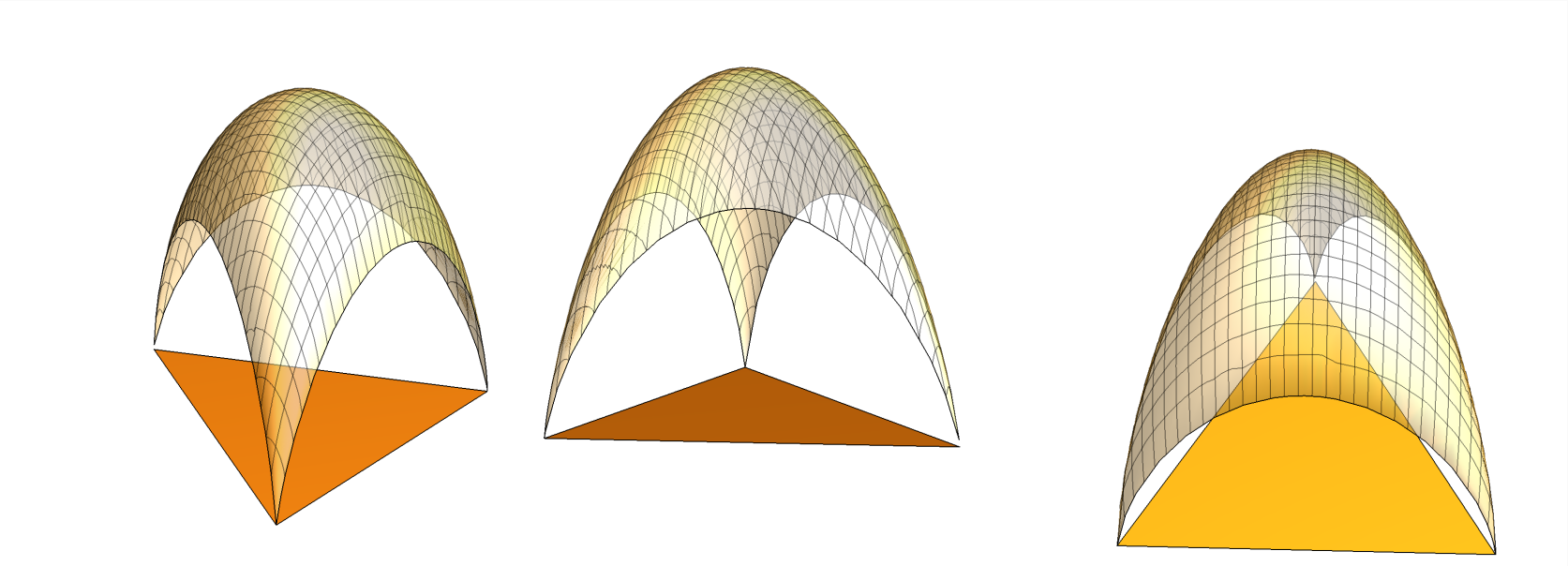
Entropy: 3D Probability Measures





Two views of the simplex $\Delta(\{0,1,2\})$. The simplex is part of the plane $x+y+z=1$.

Entropy: 3D-entropy



3D-entropy plotted on the previous equilateral triangle.

Kullback-Leibler Divergence

Definition
Let $\alpha, \beta \in \Delta(\Omega)$, where Ω is a nonempty finite set. The *Kullback-Leibler divergence* (or *KL-divergence*) *of* β *from* α is

$$D(\alpha||\beta) = \sum_{\omega \in \Omega} \alpha(\omega) \log \frac{\alpha(\omega)}{\beta(\omega)} = E_\alpha \log \frac{\alpha}{\beta}, \tag{4}$$

where the logarithm is base-2.

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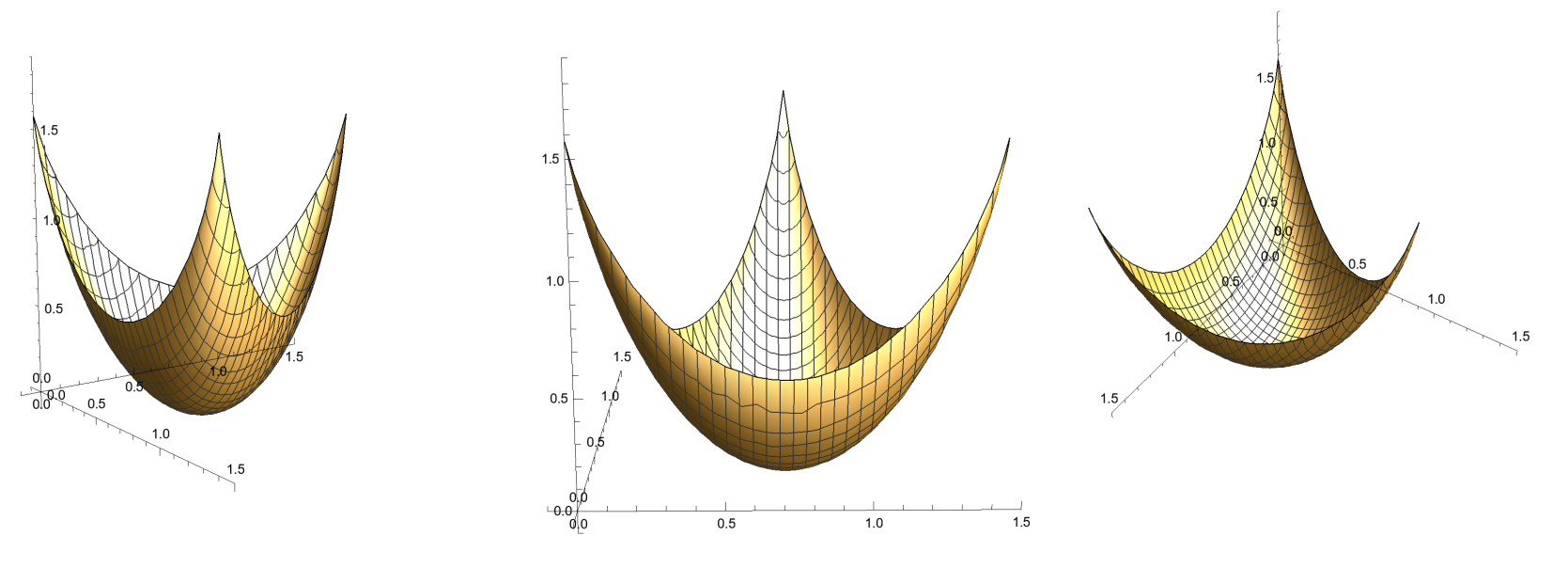
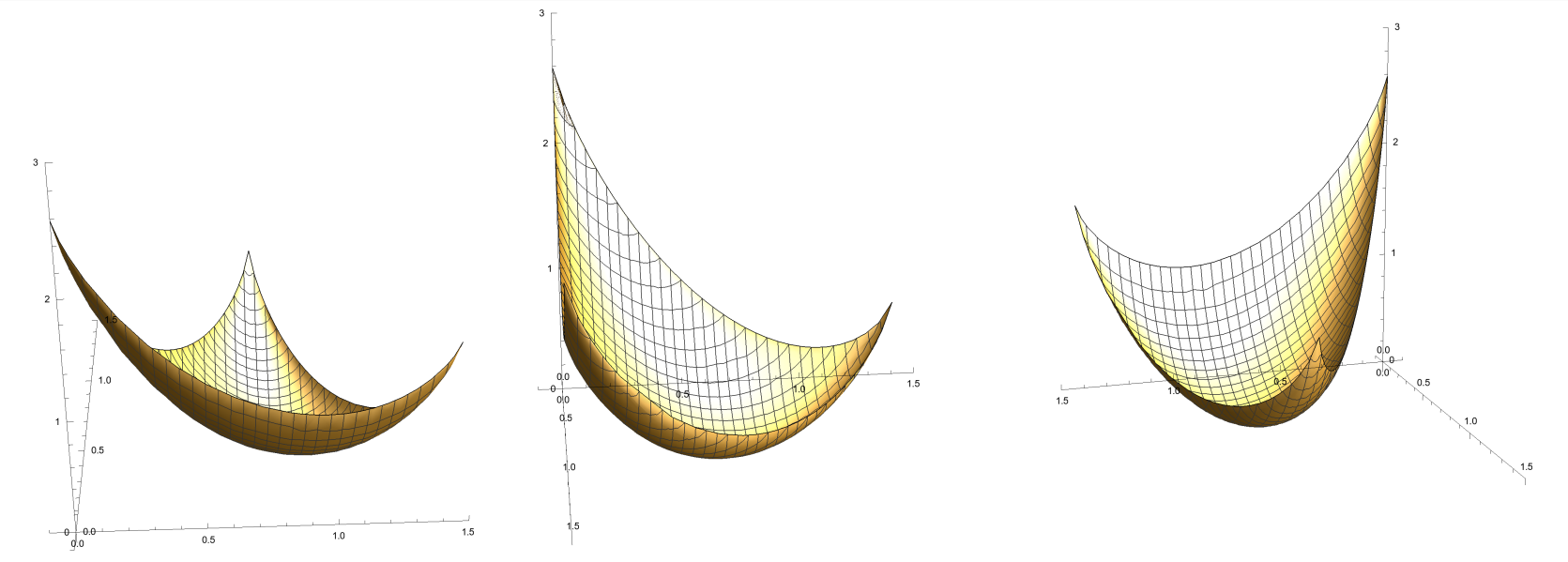
- $\beta(\omega)$ can be view as "subjective" probability, or a coding scheme.
- $\alpha(\omega)$ can be view as the "real" probability.
- $D(\alpha||\beta)$ measures the difference between the average code-length of a coding scheme and an optimal coding. So it is always ≥ 0.

Divergence: $D(\alpha||\beta)$, where $\beta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$



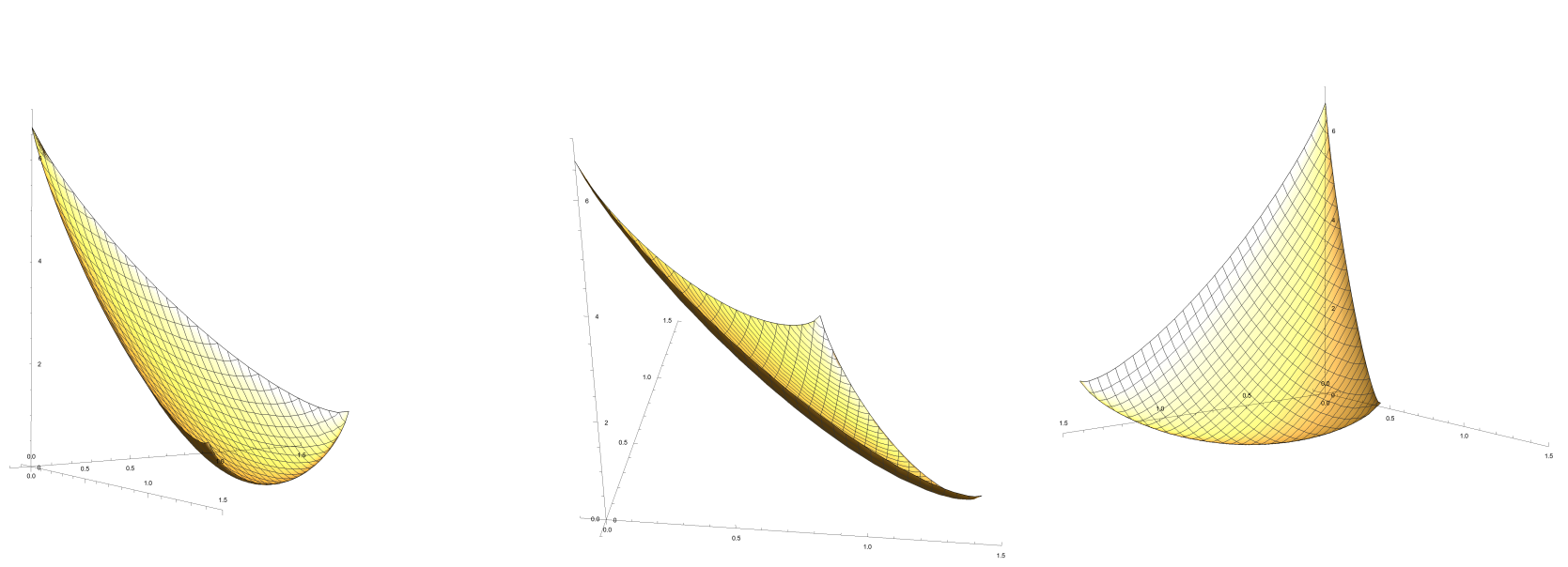
Three views of the function $D(\alpha||\beta)$, where $\beta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is the upside-down copy of the 3D-entropy function.

Divergence: $D(\alpha||\beta)$, where $\beta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$



Three views of the function $D(\alpha||\beta)$, where $\beta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$.

Divergence: $D(\alpha||\beta)$, where $\beta = (\frac{1}{100}, \frac{1}{2}, \frac{49}{100})$



Three views of the function $D(\alpha||\beta)$, where $\beta = (\frac{1}{100}, \frac{1}{2}, \frac{49}{100})$.

Strong Dichotomy Theorem

Modulo asymptotic caveats, We show that[8]

1'. The infinitely-often exponential rate of winning in 1 is $2^{\mathrm{Div}(S||\alpha)|w|}$.

2'. The exponential rate of loss in 2 is $2^{-\mathrm{Risk}_G(w)}$.

[8] Xiang Huang, Jack H. Lutz, Elvira Mayordomo and Donald M. Stull (2019), "Asymptotic Divergences and Strong Dichotomy."

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1'. The infinitely-often exponential rate of winning in 1 is $2^{\mathrm{Div}(S||\alpha)|w|}$.

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This is a **quantify** version of Schnorr-Stimm's dichotomy theorem.

Remark
$\mathrm{Div}(S||\alpha)$ *is an asymptotic version of divergence* D*, and* $\mathrm{Risk}_G(w)$ *is defined based on divergence* D*.*

[8] Xiang Huang, Jack H. Lutz, Elvira Mayordomo and Donald M. Stull (2019), "Asymptotic Divergences and Strong Dichotomy."

Algorithm Randomness

Algorithmic Randomness

We revisit the number

$$C_2 = 0.0\ 1\ 00\ 01\ 10\ 11\ 000\cdots$$

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- Normality is a **resource bounded** randomness.
- Is there a notion of randomness that corresponds to the max computing power we have, i.e., Turing Machine?

Walk along this path (computable selection rule) we can get to the von Mises-Church definition of algorithmic randomness. It is one of many ways to define algorithmic randomness.

Weakly PSPACE-randomness

Theorem (Weak PSPACE-randomness Characterization[9])
A point x is weakly PSPACE-random if and only if for every polynomial space L_1-computable $f \in L_1([0,1]^n)$, and every polynomial space computable sequence of simple function $\{f_m\}_{m\in\mathbb{N}}$ approximating f,

$$\lim_{m\to\infty} f_m(x) = \lim_{Q\to x} \frac{\int_Q f \mathrm{d}\mu}{\mu(Q)}$$

where the limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

Remark
This is an interaction between resource bounded (polynomial space) randomness and a fundamental result of real analysis, the Lebesgue differentiation theorem.

[9] Xiang Huang and Donald M. Stull (2016), "Polynomial Space Randomness in Analysis"

Analog Computing

General purpose analog computer

The General Purpose Analog Computer (GPAC) is a mathematical model of analog computers first introduced in 1941 by Claude Shannon.

For the purpose of this talk, we can view a GPAC as a polynomial initial value problem (PIVP) $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ with integral coefficients satisfying $y(0) = 0$ and the individual components of $\mathbf{y}$ obey the ODEs

$$\begin{aligned} y_1' &= p_1(y_1, \ldots, y_n), \\ y_2' &= p_2(y_1, \ldots, y_n), \\ &\vdots \\ y_n' &= p_n(y_1, \ldots, y_n). \end{aligned}$$

Real-time analog computability

We say that a real number α is *real-time analog computable*, and we write $\alpha \in \mathbb{R}_{RTA}$, if there exist a polynomial initial value problem (PIVP) $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ with integral coefficients satisfying $\mathbf{y}(0) = \mathbf{0}$ and

1. There is a constant $\beta > 0$ such that $|y_i(t)| \leq \beta$ for all $1 \leq i \leq n$ and $t \in [0, \infty)$, and
2. $|y_1(t) - \alpha| \leq 2^{-t}$ for all $t \in [0, \infty)$.

- "What resources:" zero initial values, integer coefficients, bounded values
- "How fast:" exponentially fast convergent rate.

Chemical reaction network: a restricted model

A **chemical reaction network (CRN)** is a GPAC with the ODE system being

$$\begin{aligned} y_1' &= p_1 - q_1 y_1, \\ y_2' &= p_2 - q_2 y_2, \\ &\vdots \\ y_n' &= p_n - q_n y_n \end{aligned}$$

where $p_i = p_i(y_1, \ldots, y_n)$, and $q_i = q_i(y_1, \ldots, y_n)$ are polynomial with **positive terms**.

That is, negative terms in y_i, must have occurrence of y_i.

Remark
We care about this model because it is an abstractions of chemical reactions. This model has become the language of choice for Molecular Programming.

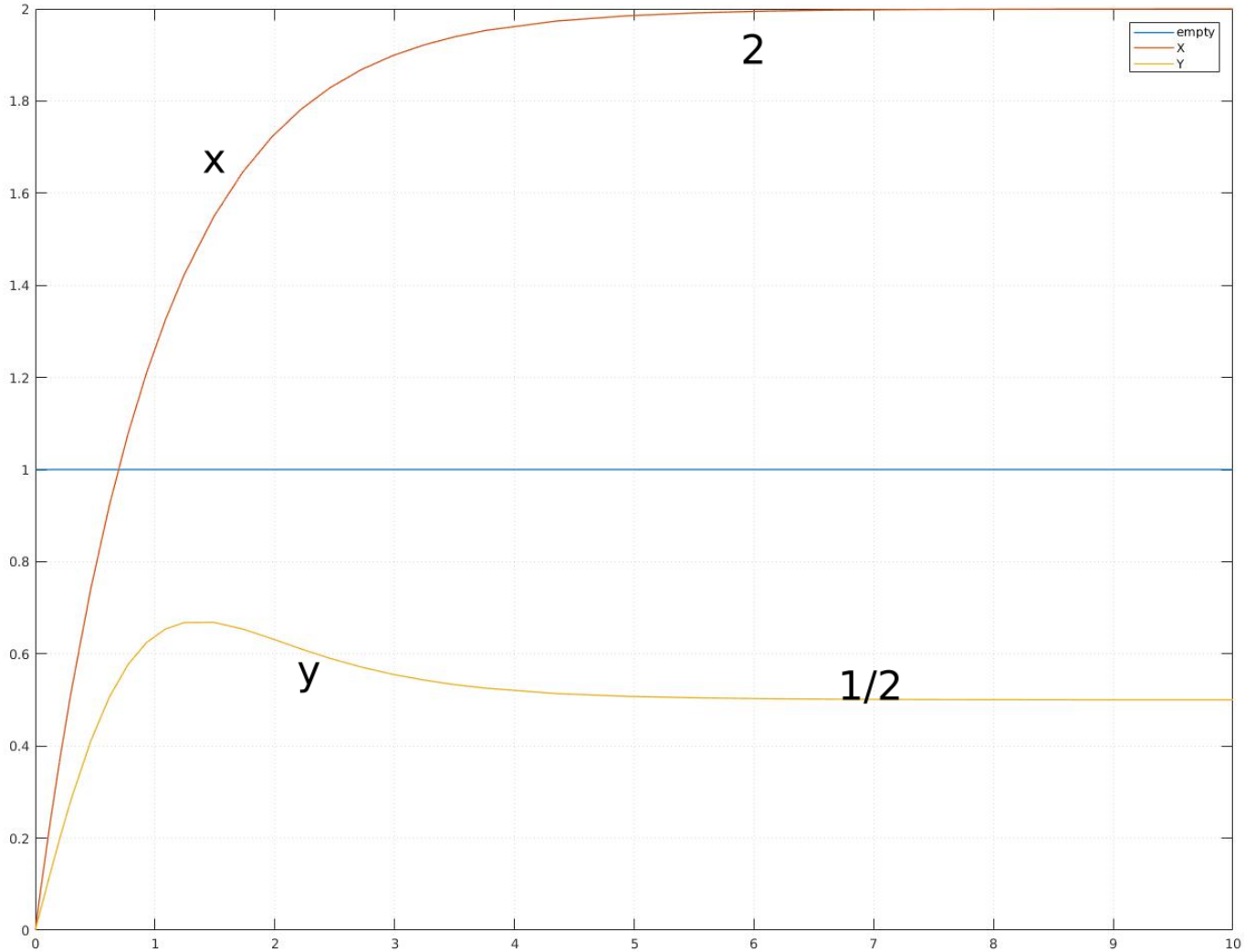
Real-time CRN computable numbers: an example

Example:

$$\begin{aligned} \emptyset &\xrightarrow{2} X \\ X &\xrightarrow{1} \emptyset \\ \emptyset &\xrightarrow{1} Y \\ X + Y &\xrightarrow{1} X \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} x' &= 2 - x \\ y' &= 1 - xy \end{aligned}$$

with initial value $x(0) = 0$, $y(0) = 0$.

Plot



RTCRN number: algebraic numbers

Theorem (Algebraics[10])
Algebraic numbers are real-time CRN-computable.

Remark
Algebraic numbers are roots of polynomials. For example, $\sqrt{2}$, $\frac{1+\sqrt{5}}{2}, \cdots$

Remark
We write the set of real-time CRN-computable number as $\mathbb{R}_{RTCRN}$.

[10] Xiang Huang, Titus H. Klinge, James I. Lathrop, Jack H. Lutz, and Xiaoyuan Li (2017), "Real-time computability of real numbers by chemical reaction networks."

Theorem (Transcendentals[11])
The number e, π are real-time CRN-computable.

Theorem (Euler constant)
Euler constant γ is real-time CRN-computable.

Remark
The number $\gamma = \lim_{n\to\infty} 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n = 0.57721\cdots$

Theorem (Model Equivalence)
$\mathbb{R}_{RTA} = \mathbb{R}_{RTCRN}$

Remark
The weaker model (CRNs), turns out to have the same computing power as GPAC, in terms of computing real numbers in real time.

[11]Xiang Huang, Titus H. Klinge, and James I. Lathrop (2019), "Some Properties of Real-Time Computable Real Numbers by Chemical Reaction Networks", manuscript.

Open question: a possible tower?

$$\vdots$$

$$R_2 = CRN(R_1)$$

$$R_1 = CRN(R_0)$$

$$R_0 = CRN(Integers) = \mathbb{R}_{RTCRN}$$

Question: Do we really have a tower, or all layers collapse to $\mathbb{R}_{RTCRN}$?

Demo: compute π with CRN

Consider the PIVP defined by

$$w'(t) = -w(t), \tag{5}$$
$$x'(t) = -2w(t)x(t)y(t), \tag{6}$$
$$y'(t) = w(t)x(t)^2 - w(t)y(t)^2, \tag{7}$$
$$z'(t) = w(t)x(t), \tag{8}$$

with $w(0) = x(0) = 1$ and $y(0) = z(0) = 0$.

We know $z(t) = \arctan(1 - e^{-t})$ and it converges to $\frac{\pi}{4}$ in real-time.

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Summer Project: The conversion of an PIVP to CRNs is pretty mechanical. Write a program to do that?

Demo: sample of reactions

$$U_1 + V_2 + V_3 \xrightarrow{2} U_1 + 2V_2 + V_3, \qquad V_2 + V_3 \xrightarrow{2} 2V_2 + V_3,$$
$$V_1 + V_3 \xrightarrow{2} V_1 + V_2 + V_3, \qquad V_1 + U_2 + V_3 \xrightarrow{2} V_1 + U_2 + V_2 + V_3,$$
$$U_1 + U_2 + U_3 \xrightarrow{2} U_1 + U_2 + V_2 + U_3, \qquad U_1 + U_3 \xrightarrow{2} U_1 + V_2 + U_3,$$
$$U_2 + U_3 \xrightarrow{2} U_2 + V_2 + U_3, \qquad U_3 \xrightarrow{2} V_2 + U_3,$$
$$V_1 + V_2 + U_3 \xrightarrow{2} V_1 + 2V_2 + U_3.$$
$$U_2 + V_2 + V_3 \xrightarrow{2} V_3, \qquad U_2 + V_2 + V_3 + U_1 \xrightarrow{2} V_3 + U_1,$$
$$2U_2 + V_2 + V_3 \xrightarrow{2} U_2 + V_3, \qquad U_1 + 2U_2 + V_2 + V_3 \xrightarrow{2} U_1 + U_2 + V_3,$$
$$V_1 + U_2 + 2V_2 + V_3 \xrightarrow{2} V_1 + V_2 + V_3, \qquad U_1 + U_2 + 2V_2 + U_3 \xrightarrow{2} U_1 + V_2 + U_3,$$
$$U_2 + 2V_2 + U_3 \xrightarrow{2} V_2 + U_3, \qquad V_1 + U_2 + V_2 + U_3 \xrightarrow{2} V_1 + U_3,$$
$$V_1 + 2U_2 + V_2 + U_3 \xrightarrow{2} V_1 + U_2 + U_3. \qquad \dots$$

Remark

There are 120+ reactions in the network.

Plot

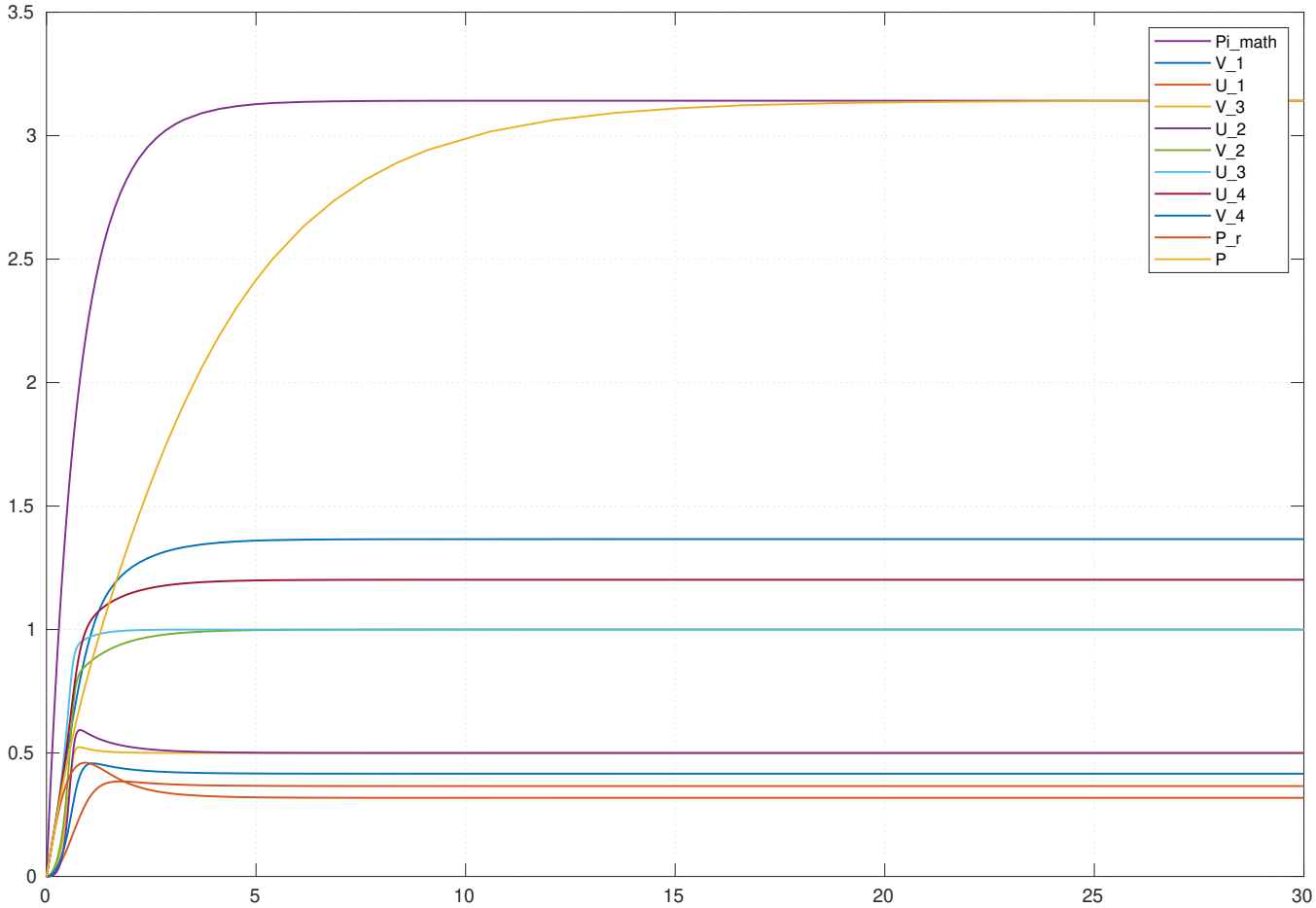

Figure 1: Compute π in real time by CRN

Thank you for your time!