1: Combinatorial Proof via Bijection

We discuss the idea of proving the following equation

$$\sum_{k\ odd} \binom{n}{k} = \sum_{k\ even} \binom{n}{k}, \tag{1}$$

in class. Roughly, let $[n] = \{1, 2, \cdots, n\}$ be the universal set. We want to construct a bijection between the subsets of even sizes (E) and those of odd sizes (O). We define a function $f : E \to O$ as follows.

$$f(s) = \begin{cases} s \setminus \{1\}, & \text{if } 1 \in s. \\ s \cup \{1\}, & \text{ohterwise.} \end{cases}$$

Argue that f is a bijection and proof Equation (1).

2: The selected and the unselected

Prove the binomial identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

3: Half-and-half spilit

Prove the binomial identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots \binom{n}{n}^2 = \binom{2n}{n}.$$

Hint: The right-hand side ask you to do $2n$ choose n. You split the $2n$ element into two group, each with n element. To pick choose n elements, you will need to pick k element from one group and $n-k$ elements from another group, where $k \in \{0, 1, \cdots, n\}$. You need to apply the result of question 2.

4: Generazation of Question 3

Prove the identity

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{m+n}{r}.$$

5: Generating Function

Please deduct the ordinary generating function (OGF) of the following sequences.

(a) $a_n = 2$, for $n \in \mathbb{N}$.

(b) $a_n = n$, for $n \in \mathbb{N}$.

(c) $a_n = 3^n$, if $n \in \mathbb{N}$.