You should read Section 5.2 and 5.3 of Discrete Math and Its Application and Chapter 6 of the text to make sure you are familiar with the basic definitions before you start working on this assignment.

1 Mathematical Induction

1: Fibonacci Number properties (Text P47 Problem 6.5, 30 pts)

Prove the following identities.

(a) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

(b) $F_0 - F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$.

(c) $F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$.

2 Strong Induction

2: Binary Expression (10 pts) **

Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on.

[Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is odd, note that $\frac{k+1}{2}$ is an integer.]

3 Recursive function

3: A self-generating sequence (10 pts) ****

The sequence we are going to discuss is a non-decreasing integer sequence where a_n is the number of times that n occurs in the sequence, starting with $a_1 = 1$, and with the property that for $n > 1$ each a_n is the smallest unique integer which makes it possible to satisfy the condition. For example, $a_1 = 1$ says that 1 only occurs once in the sequence, so a_2 cannot be 1 too, but it can be, and therefore must be, 2. The first few values are

1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 10, 10, 10, 10, 10, 11, 11, 11, 11, 11, 12, 12, 12, 12, 12, 12

Suppose now we know a formula to predict what a_n is is a function $a(n)$ such that

$$\begin{cases} a(1) = 1; \\ a(n+1) = 1 + a(n + 1 - a(a(n))). \end{cases}$$

From the introduction we know the first 9 number in the sequence if we start with $a_1 = 1$ is 1, 2, 2, 3, 3, 4, 4, 4, 5; that is $a_1 = 1$, $a_2 = 2$, $a_3 = 2$,..., $a_9 = 5$. Your tasks:

- Please use the above formula the compute a_{10}, or $a(10)$, demonstrate in detail how you get the result.

(I put this problem here just for fun. There is a way to deduct the recursive formula. I will let you have fun doing some research on it.)

4 Counting with Recursion

4: Sets (10 pts) ***

Let $f(n)$ be the number of subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain two consecutive integers. For example, for $n = 4$ we have the subset $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}$, so $f(4) = 8$. Can you come up with a recurrence relation for function f(n)?