Reading: Section 2.3 and 2.5 of the reference book *Discrete Mathematics and Its Applications (7th Edition).*

Extra video: the video link of a proof of Cantor-Schröder-Berbstein Theorem

Next lecture we will talk about logic.

Notation 1. *Recall that we adopt the following convention on notations.*

1. *The notation* $\mathbb{N}$ *= the set of all natural numbers.*
2. *The notation* $\mathbb{Z}$ *= the set of all whole numbers.*
3. *The notation* $\mathbb{Q}$ *= the set of all rational numbers.*
4. *The notation* $\mathbb{R}$ *= the set of all real numbers.*

1 Cardinality

1: $|\mathbb{N}| = |\mathbb{Z}|$ **(10 pts)**

Define a bijection $f : \mathbb{Z} \to \mathbb{N}$.Explain why the function you define is a bijection.

2: Diagonal Argument (10 pts *)**

This question prepares you to answer the next question. Suppose there exist an onto function between the set $A = \{1, 2, 3, 4\}$ and the powerset of A, i.e., $\mathcal{P}(A)$. For example, the following table represent a function $f : A \to \mathcal{P}(A)$

n \ $f(n)$	1	2	3	4	$f(n)$
1	1	0	1	1	$\{1, 3, 4\}$
2	0	1	1	0	$\{2, 3\}$
3	1	1	0	1	$\{1, 2, 4\}$
4	0	0	0	0	$\emptyset$

Note that in the above table, we use 0 to present an element NOT in a set and 1 for in. So we can see $1 \mapsto \{1, 3, 4\}$ and the representation for the image is 1011; and $4 \mapsto \emptyset$ and the representation is 0000.

Your task: Show the above function f is not an onto function by Diagonal Argument. (Surely you will say this is too obvious since we are talking about a finite set, but don't forget this problem is for demonstration purpose.) You will need to construct a set T that is in $\mathcal{P}(A)$ but not the the range of f. Let $d_1d_2d_3d_4$ be the 4-bit representation of T, the diagonal argument will tell us to do

$$d_n = \begin{cases} 0, & \text{if } n \in f(n). \\ 1, & \text{if } n \notin f(n) \end{cases}$$

Simply put it, you will just do the opposite on the diagonal. A conciser way to write it

$$T = \{n \mid n \notin f(n)\}.$$

Or you will only include an element n in T if it is not in $f(n)$.

Question: What is the resulting T out of the above table/function f?

3: Larger Cardinality Always Exists(10 pts ****)

Show that $|A| < |\mathcal{P}(A)|$, for any set A, where $\mathcal{P}(A)$ denotes the powerset of A.

[Hint: Suppose an onto (or bijection) function f existed between A and $\mathcal{P}(A)$. Let

$$T = \{a \in A | a \notin f(a)\}$$

and show that no element a can exist for which $f(a) = T$. If that happens, will T contains a?]

Fun fact: The argument goes somewhat like the The Paradox of the Barber(https://www.youtube.com/watch?v=nI-MMJiFpqo).

4: One to One Correspondence

Determine whether each of these functions is a one-to-one correspondence (a.k.a. "bijection") from $\mathbb{R}$ to $\mathbb{R}$.

1. $f(x) = 2x + 1$.
2. $f(x) = x^2 + 1$.
3. $f(x) = x^3 + 1$.
4. $f(x) = \frac{x^2+1}{x^2+2}$.
5. $f(x) = \frac{1}{x} + \frac{1}{x-1}$.

5: Reading and Writing: Schröder-Berbstein Theorem

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1. Define a notion of $|A| \leq |B|$ through functions.
2. Define a notion of $|A| \geq |B|$ through functions.
3. State Schröder-Berbstein Theorem and give an application example. Why is Schröder-Berbstein Theorem useful?

6: Extra:Application of Schröder-Berbstein Theorem (10 pts ***)

Use the theorem to show that $(1, 2)$ and $[1, 2]$ has the same cardinality.