Xiangying Chen

What is also a matroid and a delta-matroid?

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MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Pricing

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Correct pricing:

The pricing function $p\colon E \to \mathbb{R}$ should be

- monotonic $p(A) \leq p(B)$ for all $A \subseteq B \subseteq E$,
- submodular $p(A) + p(B) \geq p(A \cap B) + p(A \cup B)$ for all $A, B \subseteq E$ and
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A polymatroid is a monotonic submodular normalized set function.
Examples: correct pricing, dimensions of subspaces, entropy functions, ...

Matroids

Let E be a finite ground set.

A set function $r\colon E \to \mathbb{N}$ is (the rank function of) a matroid if it satisfies

(monotonicity) $r(A) \leq r(B)$ for all $A \subseteq B \subseteq E$,

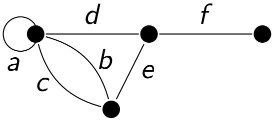
(submodularity) $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$ for all $A, B \subseteq E$,

(subcardinality) $r(A) \leq |A|$ for all $A \subseteq E$.

Examples

$E = \{a, b, c, d, e, f\}, A \subseteq E$

Graph $G = (V, E)$
$r(A) = |V| - k(G\langle A \rangle)$



vectors in $\mathbb{R}^3$
$r(A) = \dim(\mathrm{span}(A))$

$a = \mathbf{0},\ b = 2\mathbf{e}_1,\ c = \mathbf{e}_1,$
$d = \mathbf{e}_2,\ e = \mathbf{e}_1 + \mathbf{e}_2,\ f = \mathbf{e}_3$

rational functions in $\mathbb{C}(x, y, z)$
$r(A) = \mathrm{trdeg}_{\mathbb{C}}(\mathbb{C}(A))$

$a = 3, b = x, c = x^2,$
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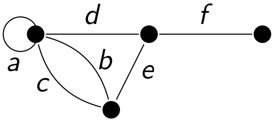

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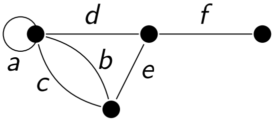

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independent sets: trees, linearly independent sets, algebraically independent sets

$$bdf, bef, cdf, cef, def, bd, be, bf, cd, ce, cf, de, df, ef, b, c, d, e, f, \emptyset$$

Bases and independent sets

Let $r\colon 2^E \to \mathbb{N}$ be the rank function of a matroid on E.

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A simplicial complex $\mathcal{I} \subseteq 2^E$ is the set of independent sets of a matroid iff

(AUG) $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists x \in I_2 \backslash I_1 : I_1 \cup x \in \mathcal{I}$.

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These are also two *equivalent definitions* of matroids!

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Partition and equivalent relations

reflexive, symmetric and transitive relations

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Boolean ring: a ring E such that $x^2 = x$ for any $x \in E$

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Topology: open sets, closed sets, neighborhoods, closure, interior, exterior, boundary,...

...

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- (Brändén, Huh 2020) supports of multiaffine Lorentzian polynomials
- (Chan, Pak 2022) hyperbolic combinatorial atlases
- (C. 2023+) semigraphoids satisfying $i \not\perp\!\!\!\perp j | K \Rightarrow i \perp\!\!\!\perp \ell | jKL$

Conditional independence

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Let $\{X_e : e \in E\}$ be a set of random variables indexed by E.
For $K \subseteq E$ and $i \neq j \in E \backslash K$ we write

$$i \perp\!\!\!\perp j | K$$

iff the random variables X_i and X_j are independent under the condition $\{X_k : k \in K\}$.

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Theorem (Dawid 1979)

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In particular, if the random variables take finite numbers of values, the entropy (surprise) function $h \colon 2^E \to \mathbb{R}$ is a polymatroid.

$$h(iK) + h(jK) \geq h(ijK) + h(K)$$

$$h(iK) + h(jK) = h(ijK) + h(K) \quad \Leftrightarrow \quad i \perp\!\!\!\perp j | K$$

Matroid and conditional independence

Theorem

A CI-structure is defined by a loopless matroid M iff it satisfies

(MCI) $i \not\perp\!\!\!\perp j | K \Rightarrow i \perp\!\!\!\perp \ell | jKL$,

(SG) $i \perp\!\!\!\perp j | K \wedge i \perp\!\!\!\perp \ell | jK \Rightarrow i \perp\!\!\!\perp \ell | K \wedge i \perp\!\!\!\perp j | \ell K$.

Moreover, the correspondence between matroids and CI-structures satisfying (MCI) *and* (SG) *is one-to-one.*

Delta-matroids

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- bisubmodular

$$r(\text{something}, 🍎, 🍌) + r(\text{something}, 🍌) \geq r(\text{something}) + r(\text{something}, 🍎),$$

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$$r(\text{nothing}) = 0, \qquad r(🍎), r(🍌), \ldots \leq 1.$$

Delta-matroid

Recall that a nonempty family $\mathcal{B} \subseteq 2^E$ is the set of bases of a matroid iff

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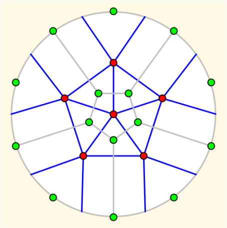
A nonempty family $\mathcal{B} \subseteq 2^E$ is the set of bases of a delta-matroid iff

(Δ-EXC) $B_1, B_2 \in \mathcal{B}, x \in B_1 \Delta B_2 \Rightarrow \exists y \in B_1 \Delta B_2 : B_1 \Delta \{x, y\} \in \mathcal{B}$.

Matroids and delta-matroids



The Petersen graph and K_6 are not planar, but they are dual to each other when embedded on $\mathbb{RP}^2$.

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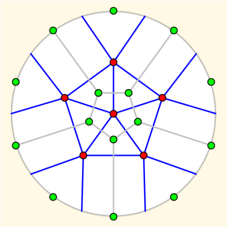

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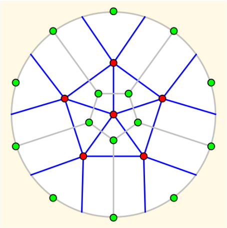

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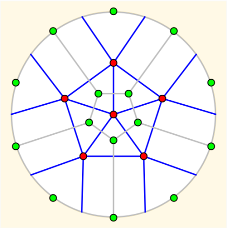

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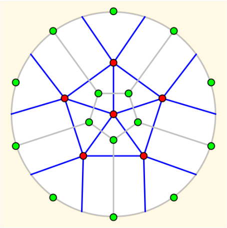

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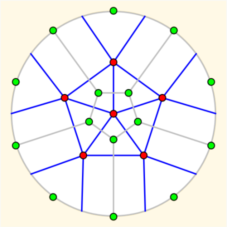

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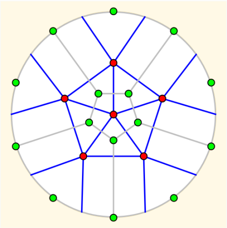

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Delta-matroids as CI-structures of type B

Write $[\pm n] := \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, and $S \sqsubseteq [\pm n]$ if $\emptyset \neq S \subseteq [\pm n]$ and $j \in S \Rightarrow \bar{j} \notin S$.

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The set of B-CI statements is

$$\begin{aligned}\mathcal{B}_n :=& \{(ij|K) : K \sqsubseteq [\pm n], \{i,j\} \sqsubseteq [\pm n] \setminus (K \cup \bar{K}), i \neq j\} \cup \\ & \cup \{(i\bar{i}|K) : K \sqsubseteq [\pm n], |K| = n-1, i \in [\pm n] \setminus (K \cup \bar{K})\}.\end{aligned}$$

Delta-matroids as CI-structures of type B

Write $[\pm n] := \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, and $S \sqsubseteq [\pm n]$ if $\emptyset \neq S \subseteq [\pm n]$ and $j \in S \Rightarrow \bar{j} \notin S$.
The set of *B*-CI statements is

$$\begin{aligned}\mathcal{B}_n :=& \{(ij|K) : K \sqsubseteq [\pm n], \{i,j\} \sqsubseteq [\pm n] \setminus (K \cup \bar{K}), i \neq j\} \cup \\ & \cup \{(i\bar{i}|K) : K \sqsubseteq [\pm n], |K| = n-1, i \in [\pm n] \setminus (K \cup \bar{K})\}.\end{aligned}$$

Theorem

A subset $\mathcal{G} \subseteq \mathcal{B}_n$ corresponds to a delta-matroid on $[n]$ iff

(CG1) $\{(ij|L), (j\bar{j}|iL), (i\bar{j}|L)\} \subseteq \mathcal{G} \Rightarrow \{(\bar{i}\bar{j}|L), (j\bar{j}|\bar{i}L), (\bar{i}j|L)\} \subseteq \mathcal{G}$,

(DMCI) $(ij|K) \in \mathcal{B}_n \setminus \mathcal{G} \Rightarrow (\bar{i}j|K), (i\ell|jKL), (\bar{i}\ell|jKL), (i\bar{i}|jKL') \in \mathcal{G}$.

Moreover, the correspondence between loopless delta-matroids and subsets of $\mathcal{B}_n$ satisfying (CG1) *and* (DMCI) *is one-to-one.*

"Anyone who has worked with matroids has come away with the conviction that matroids are one of the richest and most useful mathematical ideas of our day. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that any would a priori deem impossible, were it not for the fact that matroids do exist."

Indiscrete thoughts (2008), Gian-Carlo Rota

Thank you!