Xiangying Chen

On log-concavity of sequences

Magdeburg, 16.11.2022

Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Log-concavity

Log-concavity

A sequence of nonnegative numbers $a_0, \ldots, a_n$ is called log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i = 1, \ldots, n-1.$$

Binomial coefficients $\binom{n}{k}$

$$\binom{n}{k} = \text{the number of } k\text{-element subsets of an } n\text{-element set}$$

$n=0$							1							
$n=1$						1		1						
$n=2$					1		2		1					
$n=3$				1		3		3		1				
$n=4$			1		4		6		4		1			
$n=5$		1		5		10		10		5		1		
$n=6$	1		6		15		20		15		6		1	

Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \text{the number of permutations of } \{1, \ldots, n\} \text{ with exactly } k \text{ cycles}$$

$$\begin{array}{lccccccccccccc}
n = 0 & & & & & & & 0 & & & & & & \\
n = 1 & & & & & & 0 & & 1 & & & & & \\
n = 2 & & & & & 0 & & 1 & & 1 & & & & \\
n = 3 & & & & 0 & & 2 & & 3 & & 1 & & & \\
n = 4 & & & 0 & & 6 & & 11 & & 6 & & 1 & & \\
n = 5 & & 0 & & 24 & & 50 & & 35 & & 10 & & 1 & \\
n = 6 & 0 & & 120 & & 274 & & 225 & & 85 & & 15 & & 1
\end{array}$$

Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$

$$\left\{ {n \atop k} \right\} = \text{the number of partitions of } \{1, \dots, n\} \text{ into exactly } k \text{ blocks}$$

$n = 0$							0						
$n = 1$						0		1					
$n = 2$					0		1		1				
$n = 3$				0		1		3		1			
$n = 4$			0		1		7		6		1		
$n = 5$		0		1		15		25		10		1	
$n = 6$	0		1		31		90		65		15		1

Eulerian numbers $\left\langle {n \atop k} \right\rangle$

$$\left\langle {n \atop k} \right\rangle = \text{ the number of permutations of } \{1, \dots, n\} \text{ with exactly } k \text{ descents}$$

$n=1$							1						
$n=2$						1		1					
$n=3$					1		4		1				
$n=4$				1		11		11		1			
$n=5$			1		26		66		26		1		
$n=6$		1		57		302		302		57		1	
$n=7$	1		120		1191		2146		1191		120		1

Chromatic polynomials

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How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?

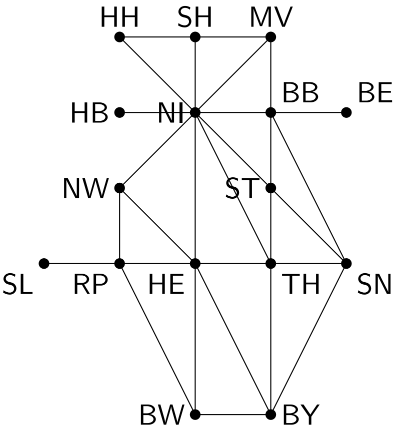
Chromatic polynomials

How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?

Answer:

$$\begin{aligned}&x^{16} - 29x^{15} + 392x^{14} - 3275x^{13} + 18903x^{12}\\ &-79796x^{11} + 254315x^{10} - 622594x^{9} + 1179280x^{8}\\ &-1726363x^{7} + 1934949x^{6} - 1628575x^{5}\\ &+995088x^{4} - 416184x^{3} + 106416x^{2} - 12528x.\end{aligned}$$

Chromatic polynomials



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Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every pair of adjacent vertices are colored differently.

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Conjecture (Read 1968, Hoggar 1974)

The (absolute values of) coefficients of a chromatic polynomial of any graph form a log-concave sequence.

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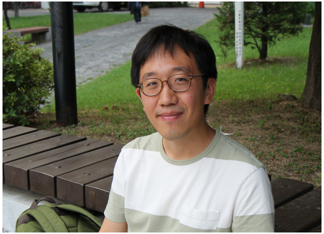
The same question can also be asked for chromatic polynomials of

$$\{\text{planar graphs}\} \subsetneq \{\text{graphs}\} \subsetneq \{\text{vector configurations}\} \subsetneq \{\text{matroids}\}.$$

Chromatic polynomials



The 2022 Fields Medalist
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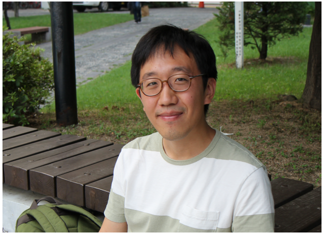

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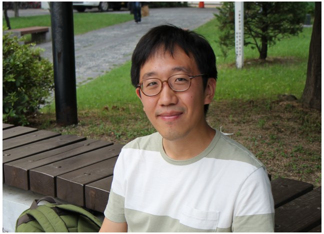

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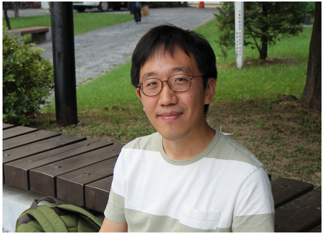

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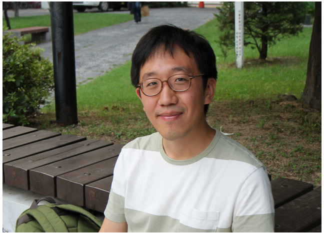

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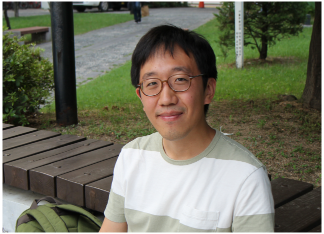

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- (2020, with Brändén) Lorentzian polynomials
- and log-concavity of matroid h-vectors, normalized Schur polynomials, different evaluations of Tutte polynomials...

Numbers of independent vectors

Let E be a set of vectors.
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Conjecture (Mason 1972)

(i) $I_k^2 \geq I_{k-1} I_{k+1}$ *(i.e.* $(I_k)_k$ *is log-concave),*

(ii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1} I_{k+1}$ *(i.e.* $(k! I_k)_k$ *is log-concave),*

(iii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k-1} I_{k+1}$ *(i.e.* $\left(I_k / \binom{n}{k}\right)_k$ *is log-concave).*

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As a consequence of log-concavity of chromatic polynomials, Mason's conjecture (i) is first proven for matroids in 2018 by Adiprasito, Huh and Katz.

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Mason's conjecture (iii) is proven independently by Anari, Liu, Gharan and Vinzant (2018) and Brändén-Huh (2020).

Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

INTRODUCTION TO THE COMBINATORIAL ATLAS

SWEE HONG CHAN* AND IGOR PAK*

ABSTRACT. We give elementary self-contained proofs of the *strong Mason conjecture* recently proved by Anari et. al [ALOV18] and Brändén–Huh [BH20], and of the classical *Alexandrov–Fenchel inequality*. Both proofs use the *combinatorial atlas* technology recently introduced by the authors [CP21]. We also give a formal relationship between combinatorial atlases and *Lorentzian polynomials*.

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1. INTRODUCTION

In this paper we tell three interrelated but largely independent stories. While we realize that this sounds self-contradictory, we insist on this description. We prove no new results, nor do we claim to give new proofs of known results. Instead, we give a *new presentation* of the existing proofs.

Our goal is explain the *combinatorial atlas* technology from [CP21] in three different contexts. The idea is to both give a more accessible introduction to our approach and connect it to other approaches in the area. Although one can use this paper as a companion to [CP21], it is written completely independently and aimed at a general audience.

(1) *Strong Mason conjecture* claims ultra-log-concavity of the number of independent sets of matroid according to its size. This is perhaps the most celebrated problem recently resolved in a series of paper culminating with independent proofs by Anari et. al [ALOV18] and Brändén–Huh [BH20]. These proofs use the technology of *Lorentzian polynomials*, which in turn substantially simplify earlier heavily algebraic tools.

In our paper [CP21], we introduce the *combinatorial atlas* technology motivated by geometric considerations of the *Alexandrov–Fenchel inequality*. This allowed us, among other things, to

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Hodge theory for combinatorial geometries

By KARIM ADIPRASITO, JUNE HUH, and ERIC KATZ

Abstract

We prove the hard Lefschetz theorem and the Hodge-Riemann relations for a commutative ring associated to an arbitrary matroid M. We use the Hodge-Riemann relations to resolve a conjecture of Heron, Rota, and Welsh that postulates the log-concavity of the coefficients of the characteristic polynomial of M. We furthermore conclude that the f-vector of the independence complex of a matroid forms a log-concave sequence, proving a conjecture of Mason and Welsh for general matroids.

1. Introduction

The combinatorial theory of matroids starts with Whitney [Whi35], who introduced matroids as models for independence in vector spaces and graphs. See [Kun86, Ch. I] for an excellent historical overview. By definition, a *matroid* M is given by a closure operator defined on all subsets of a finite set E satisfying the Steinitz-Mac Lane exchange property:

For every subset I of E and every element a not in the closure of I, if a is in the closure of $I \cup \{b\}$, then b is in the closure of $I \cup \{a\}$.

Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

- The whole proof can be written down in 1-2 pages instead of 72
- Basic linear-algebraic proof instead of combinatorial Hodge theory
- Generalized to morphisms of matroids/antimatroids/interval greedoids
- Refined inequalities for special classes
- Full classification of equality cases

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- Full classification of equality cases
- The TRUE Mathematical Complexity Reduction

Log-concavity in linear algebra

Let A be a $n \times n$ real symmetric matrix.

- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \leq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

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- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y} \rangle \geq 0$$

  iff A has at most one positive eigenvalue.

Log-concavity in analysis

Theorem (Newton's Theorem)

For any real-rooted polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ *with* $a_0, \ldots, a_n \geq 0$, *we have*

$$\left(\frac{a_i}{\binom{n}{i}}\right)^2 \geq \left(\frac{a_{i-1}}{\binom{n}{i-1}}\right)\left(\frac{a_{i+1}}{\binom{n}{i+1}}\right).$$

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Examples:

1. $\sum_{i=0}^{n} \binom{n}{k} x^k = (x+1)^n$.
2. $\sum_{i=0}^{n} \left[{n \atop k}\right] x^k = x(x+1)\cdots(x+n-1)$.
3. The polynomials $\sum_{i=0}^{n} \left\{{n \atop k}\right\} x^k$ and $\sum_{i=0}^{n} \left\langle{n \atop k}\right\rangle x^k$ satisfy certain recurrence relations, from which we can deduce their real-rootedness.

Log-concavity in analysis

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is called strictly Lorentzian if

- the support consists of all monomials of degree d,
- all coefficients are positive,
- and for any choice of $i_1, \ldots, i_{d-2} \in \{1, \ldots, n\}$, the quadratic form $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalues.

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The limits of strictly Lorentzian polynomials are called Lorentzian.

1. **Lorentzian polynomials and real-rooted polynomials**
   $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the open upper half of the complex plane
   A polynomial $f \in \mathbb{C}[x_1, \cdots, x_m]$ is stable if $f \equiv 0$ or $f(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \mathcal{H}^m$.
   Stable polynomials in $\mathbb{R}[x_1, \cdots, x_m]$ are Lorentzian.

Log-concavity in analysis

2. **Lorentzian polynomials and eigenvalues of Hessian**
    - For any Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ has exactly one positive eigenvalue.
    - For any strictly Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ is nonsingular.

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3. **Lorentzian polynomials and log-concave functions**
   f is log-concave on $\mathbb{R}^n_{>0}$ iff the Hessian of f has exactly one positive eigenvalue on $\mathbb{R}^n_{>0}$.

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   f is log-concave on $\mathbb{R}^n_{>0}$ iff the Hessian of f has exactly one positive eigenvalue on $\mathbb{R}^n_{>0}$.
4. **Lorentzian polynomials and discrete convexity**
   The support of any Lorentzian polynomial is M-convex, that is, the lattice point set of a polytope whose edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$.

Log-concavity in geometry

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Let $A, B \subseteq \mathbb{R}^2$ be two convex sets. The mixed volume of A and B is

$$V(A, B) = \frac{1}{2}\left(\mathrm{Vol}(A + B) - \mathrm{Vol}(A) - \mathrm{Vol}(B)\right).$$

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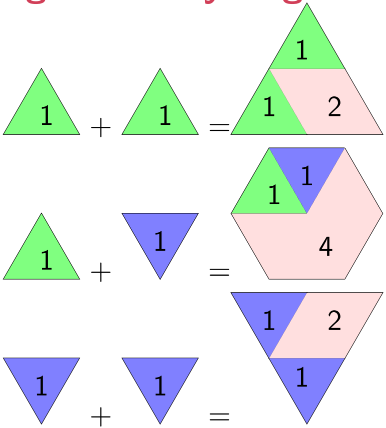
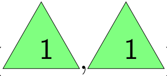
Theorem (Alexandrov-Fenchel inequality in dimension 2)

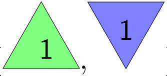
$$V(A, B)^2 \geq V(A, A) \cdot V(B, B).$$

Log-concavity in geometry

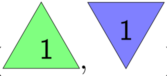




$V(\ \ ,\ \ ) = 1 = \binom{2}{0}$



$V(\ \ ,\ \ ) = 2 = \binom{2}{1}$



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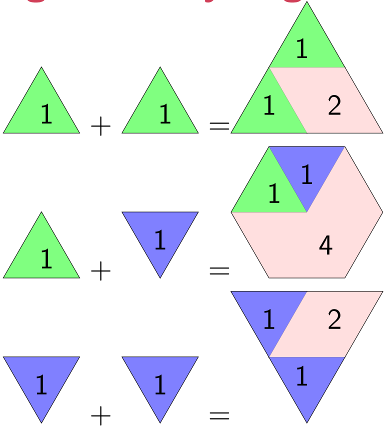
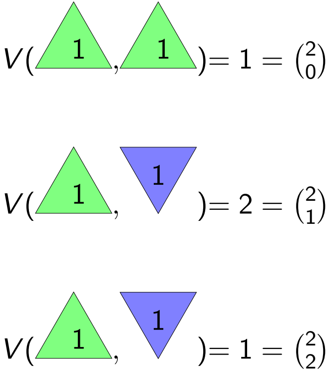
Log-concavity in geometry





Corollary

$$\binom{2}{1}^2 \geq \binom{2}{0} \cdot \binom{2}{2}.$$

Log-concavity in geometry

The mixed volume of n convex bodies $K_1, \ldots, K_n \subset \mathbb{R}^n$ is

$$V(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{1 \le r_1 < \cdots < r_k \le n} \mathrm{Vol}(K_{r_1} + \cdots + K_{r_k}).$$

Theorem (Alexandrov-Fenchel inequality)

Let $K_1, \ldots, K_{n-2}, A, B \subset \mathbb{R}^n$ be convex bodies in $\mathbb{R}^n$, then

$$V(K_1, \ldots, K_{n-2}, A, B)^2 \ge V(K_1, \ldots, K_{n-2}, A, A) \cdot V(K_1, \ldots, K_{n-2}, B, B).$$

Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because

$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \ldots, \Delta_n}_{k}, \underbrace{-\Delta_n, \ldots, -\Delta_n}_{n-k})$$

where $\Delta_n = \mathrm{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.

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$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \ldots, \Delta_n}_{k}, \underbrace{-\Delta_n, \ldots, -\Delta_n}_{n-k})$$

   where $\Delta_n = \operatorname{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.

2. If $a_0, \ldots, a_n \geq 0$ are coefficients of a real-rooted polynomial $p(t) \in \mathbb{R}[t]$, then $\left(a_k/\binom{n}{k}\right)_k$ is log-concave because

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Log-concavity in geometry

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$$= \sum_{k=0}^{n} \binom{n}{k} V\left(\underbrace{[0,1]^n, \dots, [0,1]^n}_{k}, \underbrace{\prod_{i=1}^{n} [0, \lambda_i], \dots, \prod_{i=1}^{n} [0, \lambda_i]}_{n-k} \right) x^k.$$

Log-concavity in geometry

Examples:

3. (Shephard 1960) A sequence $(a_0, \ldots, a_n)$ of nonnegative numbers is log-concave iff there exist convex compact sets $A, B \subseteq \mathbb{R}^n$ such that for $k = 0, \ldots, n$,

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4. There are Alexandrov-Fenchel type inequalities for the intersection numbers of nef divisors on a variety.
5. A far-reaching combinatorial abstraction leads to the "combinatorial Hodge theory".

Combinatorial Hodge theory

$A^\bullet(X) = \bigoplus_{k=0}^{d} A^k(X)$ graded $\mathbb{R}$-algebra associated to "an object X of dimension d"

K = a convex cone in the space of linear operators on $A^\bullet(X)$

The Kähler package: For every $0 \le k \le \lfloor \frac{d}{2} \rfloor$:

(PD) The bilinear pairing $P : A^k(X) \times A^{d-k}(X) \to \mathbb{R}$ is non-degenerate.

(HL) For every $L \in K$, the composition $L^{d-2k} : A^k(X) \to A^{d-k}(X)$ is bijective.

(HR) For every $L \in K$, the bilinear form

$$A^k(X) \times A^k(X) \to \mathbb{R}, \quad (x_1, x_2) \mapsto (-1)^k P(x_1, L^{d-2k} x_2)$$

is symmetric, and is positive definite on the kernel of $L^{d-2k+1} : A^k(X) \to A^{d-k+1}(X)$.

Combinatorial atlas

Recall that a real symmetric matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y}\rangle^2 \geq \langle \mathbf{x}, A\mathbf{x}\rangle\langle \mathbf{y}, A\mathbf{y}\rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y}\rangle \geq 0$$

iff A has at most one positive eigenvalue. We call such a matrix A hyperbolic.

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Goal: To prove the hyperbolicity of

$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & n & \epsilon \end{array} \\ \begin{pmatrix} 0 & & & & \\ & 0 & M_{ij} & & \\ & & \ddots & & M_{i\epsilon} \\ & & & 0 & \\ & & & & M_{\epsilon\epsilon} \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{array}$$

where

$$M_{ij} := (k-1)!\{I \text{ indep.}, |I| = k+1, i, j \in I\}$$
$$M_{i\epsilon} := (k-1)!\{I \text{ indep.}, |I| = k, i \in I\}$$
$$M_{\epsilon\epsilon} := (k-1)!\{I \text{ indep.}, |I| = k-1\}$$

Define $\mathbf{v} := (1 \quad \cdots \quad 1 \quad 0)^\top$, $\mathbf{w} := (0 \quad \cdots \quad 0 \quad 1)^\top$.

The hyperbolicity of M implies Mason's conjecture (ii):

$$(k!I_k)^2 = \langle \mathbf{v}, M\mathbf{w}\rangle^2 \geq \langle \mathbf{v}, M\mathbf{v}\rangle\langle \mathbf{w}, M\mathbf{w}\rangle = (k+1)!I_{k+1} \cdot (k-1)!I_{k-1}.$$

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by induction on the already prescribed elements of I.
Induction steps based on controlling the eigenvalues by the Perron-Frobenius theorem.

Challenge on Mathematical Complexity Reduction

Combinatorial atlas is a super easy and powerful tool.
Can we prove more log-concavities using the combinatorial atlases?
Can we find easy proofs for known log-concavities?

Need more hints on hyperbolic matrices!

Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?

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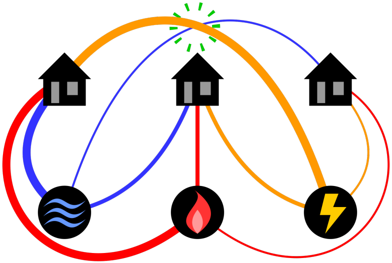

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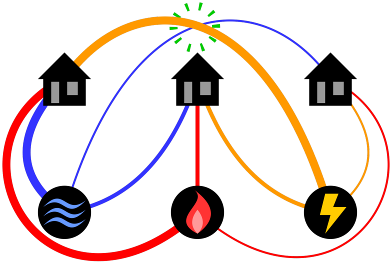

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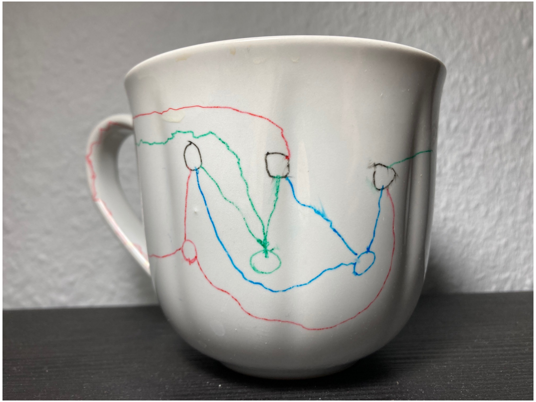

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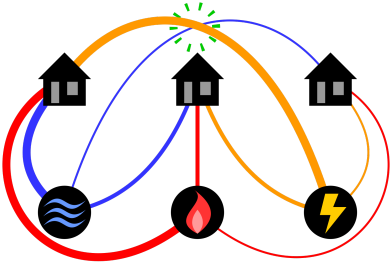

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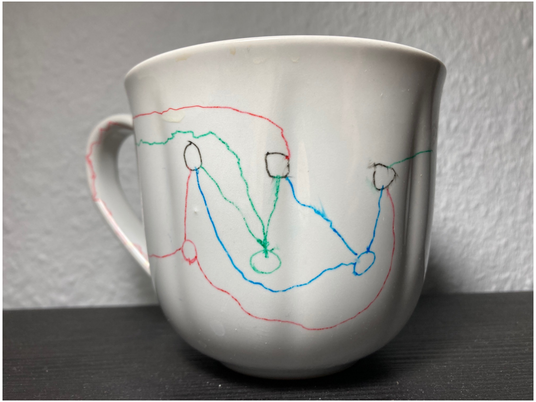


The graph $K_{3,3}$ cannot be embedded on a **plane**, but can be embedded on a **torus**.

Hyperbolic matrices in topological graph theory

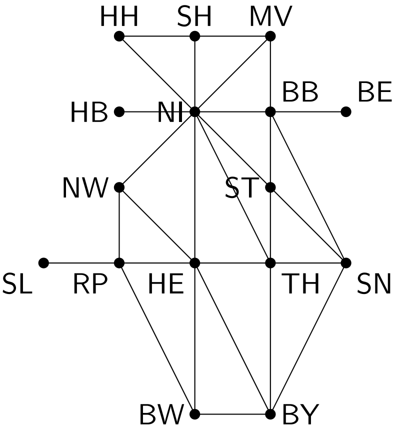
Let G graph with vertex set $\{1, \ldots, n\}$ and edge set E, the Colin de Verdière number $\mu(G)$ is the largest corank of any $n \times n$ real symmetric matrix $M = (M_{ij})_{ij}$ such that

- $\forall i \neq j : M_{ij} < 0$ if $ij \in E$, $M_{ij} = 0$ otherwise.
- M has exactly one negative eigenvalue.
- If a real symmetric matrix X satisfies $MX = \mathbf{0}$ and ($X_{ij} = 0$ if $i = j$ or $ij \in E$), then $X = 0$.

$\mu(G)$ is a very strong topological invariant of graphs:

$\mu(G) \leq 0$ iff G is edgeless.
$\mu(G) \leq 1$ iff G is disj. union of paths.
$\mu(G) \leq 2$ iff G is outerplanar.
$\mu(G) \leq 3$ iff G is planar.

$\mu(G) \leq 4$ iff G is linklessly embeddable in $\mathbb{R}^3$.
$\mu(G) \leq 5$ if G is embeddable on $\mathbb{RP}^2$ or Klein bottle.
$\mu(G) \leq 6$ if G is embeddable on torus.
$\cdots$



(1,29,392,3275,18903,79796,254315,622594,1179280,1726363,1934949,1628575,995088,416184,106416,12528)

Thank you!