

Xiangying Chen

On log-concavity of sequences

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Institut für Algebra und Geometrie
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DFG-Graduiertenkolleg
**MATHEMATISCHE
KOMPLEXITÄTSREDUKTION**

Log-concavity



Log-concavity

A sequence of nonnegative numbers a_0, \dots, a_n is called **log-concave** if

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i = 1, \dots, n-1.$$



Binomial coefficients $\binom{n}{k}$

$\binom{n}{k}$ = the number of k -element subsets of an n -element set

$n = 0$								1
$n = 1$					1		1	
$n = 2$			1		2		1	
$n = 3$		1		3		3	1	
$n = 4$		1	4		6	4	1	
$n = 5$	1	5	10		10	5	1	
$n = 6$	1	6	15	20		15	6	1



Stirling numbers of the first kind $[n]_k$

$[n]_k$ = the number of permutations of $\{1, \dots, n\}$ with exactly k cycles

$n = 0$				0							
$n = 1$				0		1					
$n = 2$			0		1		1				
$n = 3$		0		2		3		1			
$n = 4$	0		6		11		6		1		
$n = 5$	0	24		50		35		10		1	
$n = 6$	0	120	274		225		85		15		1



Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$

$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ = the number of partitions of $\{1, \dots, n\}$ into exactly k blocks

$n = 0$				0						
$n = 1$				0		1				
$n = 2$			0		1		1			
$n = 3$		0		1		3		1		
$n = 4$	0		1		7		6		1	
$n = 5$	0	1		15		25		10		1
$n = 6$	0	1	31		90		65		15	1



Eulerian numbers $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$

$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle =$ the number of permutations of $\{1, \dots, n\}$ with exactly k descents

$n = 1$					1						
$n = 2$				1		1					
$n = 3$			1		4		1				
$n = 4$		1		11		11		1			
$n = 5$		1	26		66		26		1		
$n = 6$	1	57		302		302		57		1	
$n = 7$	120		1191		2146		1191	120		1	



Chromatic polynomials



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How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?



Chromatic polynomials



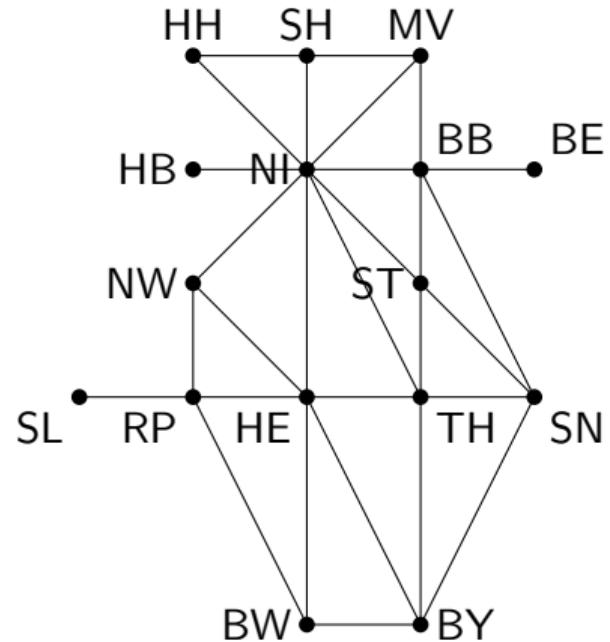
How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?

Answer:

$$\begin{aligned} & x^{16} - 29x^{15} + 392x^{14} - 3275x^{13} + 18903x^{12} \\ & - 79796x^{11} + 254315x^{10} - 622594x^9 + 1179280x^8 \\ & - 1726363x^7 + 1934949x^6 - 1628575x^5 \\ & + 995088x^4 - 416184x^3 + 106416x^2 - 12528x. \end{aligned}$$



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Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every pair of adjacent vertices are colored differently.



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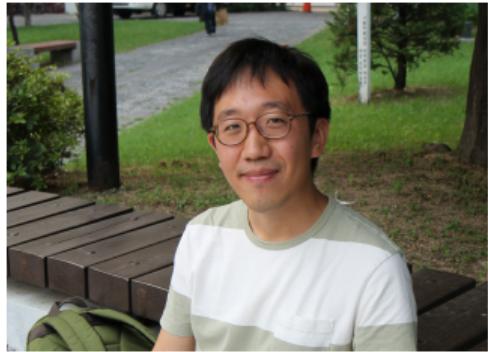
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The same question can also be asked for chromatic polynomials of

$\{\text{planar graphs}\} \subsetneq \{\text{graphs}\} \subsetneq \{\text{vector configurations}\} \subsetneq \{\text{matroids}\}.$



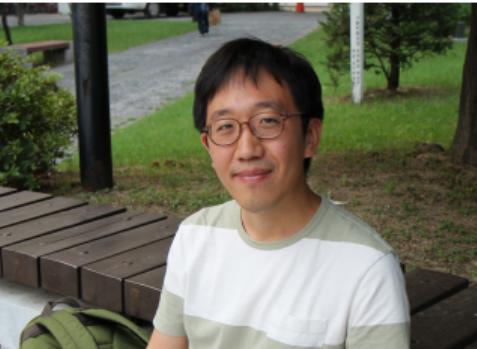
Chromatic polynomials



The 2022 Fields Medalist
June Huh



Chromatic polynomials



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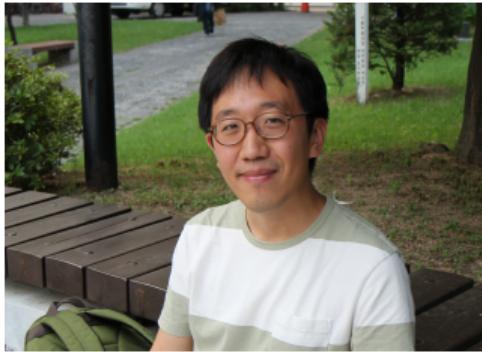


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- (2020, with Brändén) Lorentzian polynomials
- and log-concavity of matroid h -vectors, normalized Schur polynomials, different evaluations of Tutte polynomials...



Numbers of independent vectors

Let E be a set of vectors.

$I_k :=$ number of linearly independent subsets of E of cardinality k



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Conjecture (Mason 1972)

(i) $I_k^2 \geq I_{k-1}I_{k+1}$ (i.e. $(I_k)_k$ is log-concave),

(ii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1}I_{k+1}$ (i.e. $(k!I_k)_k$ is log-concave),

(iii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k-1}I_{k+1}$ (i.e. $\left(I_k/\binom{n}{k}\right)_k$ is log-concave).



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As a consequence of log-concavity of chromatic polynomials, Mason's conjecture (i) is first proven for matroids in 2018 by Adiprasito, Huh and Katz.



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Mason's conjecture (iii) is proven independently by Anari, Liu, Gharan and Vinzant (2018) and Brändén-Huh (2020).



Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

INTRODUCTION TO THE COMBINATORIAL ATLAS

SWEE HONG CHAN* AND IGOR PAK*

ABSTRACT. We give elementary self-contained proofs of the *strong Mason conjecture* recently proved by Anari et. al [ALOV18] and Brändén–Huh [BH20], and of the classical *Alexandrov–Fenchel inequality*. Both proofs use the *combinatorial atlas* technology recently introduced by the authors [CP21]. We also give a formal relationship between combinatorial atlases and *Lorentzian polynomials*.

1. INTRODUCTION

In this paper we tell three interrelated but largely independent stories. While we realize that this sounds self-contradictory, we insist on this description. We prove no new results, nor do we claim to give new proofs of known results. Instead, we give a *new presentation* of the existing proofs.

Our goal is explain the *combinatorial atlas* technology from [CP21] in three different contexts. The idea is to both give a more accessible introduction to our approach and connect it to other approaches in the area. Although one can use this paper as a companion to [CP21], it is written completely independently and aimed at a general audience.

(1) *Strong Mason conjecture* claims ultra-log-concavity of the number of independent sets of matroid according to its size. This is perhaps the most celebrated problem recently resolved in a series of paper culminating with independent proofs by Anari et. al [ALOV18] and Brändén–Huh [BH20]. These proofs use the technology of *Lorentzian polynomials*, which in turn substantially simplify earlier heavily algebraic tools.

In our paper [CP21], we introduce the *combinatorial atlas* technology motivated by geometric considerations of the *Alexandrov–Fenchel inequality*. This allowed us, among other things, to

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<https://doi.org/10.4007/annals.2018.188.2.1>

Hodge theory for combinatorial geometries

BY KARIM ADIPRASITO, JUNE HUH, and ERIC KATZ

Abstract

We prove the hard Lefschetz theorem and the Hodge-Riemann relations for a commutative ring associated to an arbitrary matroid M . We use the Hodge-Riemann relations to resolve a conjecture of Heron, Rota, and Welsh that postulates the log-concavity of the coefficients of the characteristic polynomial of M . We furthermore conclude that the f -vector of the independence complex of a matroid forms a log-concave sequence, proving a conjecture of Mason and Welsh for general matroids.

1. Introduction

The combinatorial theory of matroids starts with Whitney [Whi35], who introduced matroids as models for independence in vector spaces and graphs. See [Kun86, Ch. I] for an excellent historical overview. By definition, a *matroid* M is given by a closure operator defined on all subsets of a finite set E satisfying the Steinitz-Mac Lane exchange property:

For every subset I of E and every element a not in the closure of I , if a is in the closure of $I \cup \{b\}$, then b is in the closure of $I \cup \{a\}$.



Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

- The whole proof can be written down in 1-2 pages instead of 72
- Basic linear-algebraic proof instead of combinatorial Hodge theory
- Generalized to morphisms of matroids/antimatroids/interval greedoids
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- Refined inequalities for special classes
- Full classification of equality cases
- **The TRUE Mathematical Complexity Reduction**



Log-concavity in linear algebra

Let A be a $n \times n$ real symmetric matrix.

- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \leq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

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iff A has at most one positive eigenvalue.



Log-concavity in analysis

Theorem (Newton's Theorem)

For any real-rooted polynomial $p(t) = \sum_{i=0}^n a_i t^i \in \mathbb{R}[t]$ with $a_0, \dots, a_n \geq 0$, we have

$$\left(\frac{a_i}{\binom{n}{i}} \right)^2 \geq \left(\frac{a_{i-1}}{\binom{n}{i-1}} \right) \left(\frac{a_{i+1}}{\binom{n}{i+1}} \right).$$



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Examples:

1. $\sum_{i=0}^n \binom{n}{k} x^k = (x+1)^n$.
2. $\sum_{i=0}^n [n]_k x^k = x(x+1)\cdots(x+n-1)$.
3. The polynomials $\sum_{i=0}^n \{n\}_k x^k$ and $\sum_{i=0}^n \langle n \rangle_k x^k$ satisfy certain recurrence relations, from which we can deduce their real-rootedness.



Log-concavity in analysis

A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is called **strictly Lorentzian** if

- the support consists of all monomials of degree d ,
- all coefficients are positive,
- and for any choice of $i_1, \dots, i_{d-2} \in \{1, \dots, n\}$, the quadratic form $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalues.

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The limits of strictly Lorentzian polynomials are called **Lorentzian**.

1. Lorentzian polynomials and real-rooted polynomials

$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ the open upper half of the complex plane

A polynomial $f \in \mathbb{C}[x_1, \dots, x_m]$ is **stable** if $f \equiv 0$ or $f(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \mathcal{H}^m$.

Stable polynomials in $\mathbb{R}[x_1, \dots, x_m]$ are Lorentzian.



Log-concavity in analysis

2. Lorentzian polynomials and eigenvalues of Hessian

- For any Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}_{>0}^n$, the Hessian $\mathbf{H}_f(\mathbf{w})$ has exactly one positive eigenvalue.
- For any strictly Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}_{>0}^n$, the Hessian $\mathbf{H}_f(\mathbf{w})$ is nonsingular.



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3. Lorentzian polynomials and log-concave functions

f is log-concave on $\mathbb{R}_{>0}^n$ iff the Hessian of f has exactly one positive eigenvalue on $\mathbb{R}_{>0}^n$.



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4. Lorentzian polynomials and discrete convexity

The support of any Lorentzian polynomial is **M-convex**, that is, the lattice point set of a polytope whose edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$.



Log-concavity in geometry

The **Minkowski sum** of two sets $A, B \subseteq \mathbb{R}^n$ is

$$A + B := \{a + b : a \in A, b \in B\}.$$



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Let $A, B \subseteq \mathbb{R}^2$ be two convex sets. The **mixed volume** of A and B is

$$V(A, B) = \frac{1}{2} (\text{Vol}(A + B) - \text{Vol}(A) - \text{Vol}(B)).$$



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Theorem (Alexandrov-Fenchel inequality in dimension 2)

$$V(A, B)^2 \geq V(A, A) \cdot V(B, B).$$



Log-concavity in geometry

$$\begin{array}{c} \text{green triangle} \\ 1 \end{array} + \begin{array}{c} \text{green triangle} \\ 1 \end{array} = \begin{array}{c} \text{green triangle} \\ 1 \end{array} \begin{array}{c} \text{pink triangle} \\ 2 \end{array}$$

$$V(\begin{array}{c} \text{green triangle} \\ 1 \end{array}, \begin{array}{c} \text{green triangle} \\ 1 \end{array}) = 1 = \binom{2}{0}$$

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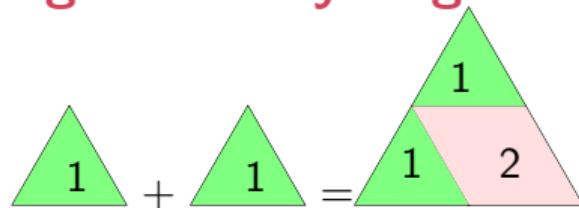
$$V(\begin{array}{c} \text{green triangle} \\ 1 \end{array}, \begin{array}{c} \text{blue triangle} \\ 1 \end{array}) = 2 = \binom{2}{1}$$

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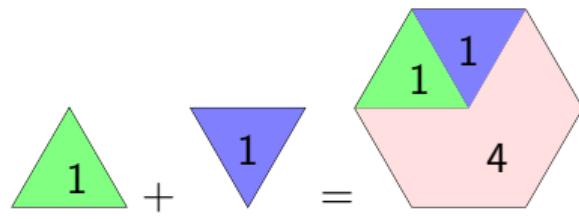
$$V(\begin{array}{c} \text{blue triangle} \\ 1 \end{array}, \begin{array}{c} \text{blue triangle} \\ 1 \end{array}) = 1 = \binom{2}{2}$$



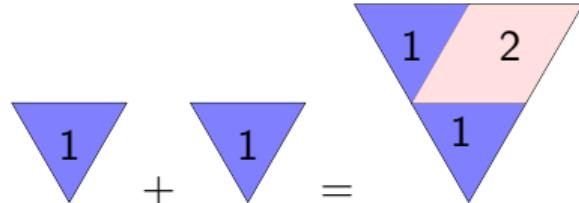
Log-concavity in geometry



$$V(\text{green triangle}, \text{green triangle}) = 1 = \binom{2}{0}$$



$$V(\text{green triangle}, \text{blue triangle}) = 2 = \binom{2}{1}$$



$$V(\text{blue triangle}, \text{blue triangle}) = 1 = \binom{2}{2}$$

Corollary

$$\binom{2}{1}^2 \geq \binom{2}{0} \cdot \binom{2}{2}.$$

Log-concavity in geometry

The **mixed volume** of n convex bodies $K_1, \dots, K_n \subset \mathbb{R}^n$ is

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{1 \leq r_1 < \dots < r_k \leq n} \text{Vol}(K_{r_1} + \dots + K_{r_k}).$$

Theorem (Alexandrov-Fenchel inequality)

Let $K_1, \dots, K_{n-2}, A, B \subset \mathbb{R}^n$ be convex bodies in \mathbb{R}^n , then

$$V(K_1, \dots, K_{n-2}, A, B)^2 \geq V(K_1, \dots, K_{n-2}, A, A) \cdot V(K_1, \dots, K_{n-2}, B, B).$$



Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because

$$\binom{n}{k} = n! V(\underbrace{\Delta_n, \dots, \Delta_n}_k, \underbrace{-\Delta_n, \dots, -\Delta_n}_{n-k})$$

where $\Delta_n = \text{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n -dimensional simplex.



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2. If $a_0, \dots, a_n \geq 0$ are coefficients of a real-rooted polynomial $p(t) \in \mathbb{R}[t]$, then $(a_k / \binom{n}{k})_k$ is log-concave because

$$p(t) = \sum_{i=0}^n a_i t^i = (x + \lambda_1) \cdots (x + \lambda_n)$$



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$$p(t) = \sum_{i=0}^n a_i t^i = (x + \lambda_1) \cdots (x + \lambda_n) = \sum_{k=0}^n \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right) x^{n-k}$$



Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because

$$\binom{n}{k} = n! V(\underbrace{\Delta_n, \dots, \Delta_n}_k, \underbrace{-\Delta_n, \dots, -\Delta_n}_{n-k})$$

where $\Delta_n = \text{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n -dimensional simplex.

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$$\begin{aligned} p(t) &= \sum_{i=0}^n a_i t^i = (x + \lambda_1) \cdots (x + \lambda_n) = \sum_{k=0}^n \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right) x^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} V\left(\underbrace{[0, 1]^n, \dots, [0, 1]^n}_k, \underbrace{\prod_{i=1}^n [0, \lambda_i], \dots, \prod_{i=1}^n [0, \lambda_i]}_{n-k}\right) x^k. \end{aligned}$$



Log-concavity in geometry

Examples:

3. (Shephard 1960) A sequence (a_0, \dots, a_n) of nonnegative numbers is log-concave iff there exist convex compact sets $A, B \subseteq \mathbb{R}^n$ such that for $k = 0, \dots, n$,

$$a_k = V(\underbrace{A, \dots, A}_k, \underbrace{B, \dots, B}_{n-k}).$$



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4. There are Alexandrov-Fenchel type inequalities for the intersection numbers of nef divisors on a variety.
5. A far-reaching combinatorial abstraction leads to the "combinatorial Hodge theory".



Combinatorial Hodge theory

$A^\bullet(X) = \bigoplus_{k=0}^d A^k(X)$ graded \mathbb{R} -algebra associated to “an object X of dimension d ”

K = a convex cone in the space of linear operators on $A^\bullet(X)$

The Kähler package: For every $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$:

(PD) The bilinear pairing $P : A^k(X) \times A^{d-k}(X) \rightarrow \mathbb{R}$ is non-degenerate.

(HL) For every $L \in K$, the composition $L^{d-2k} : A^k(X) \rightarrow A^{d-k}(X)$ is bijective.

(HR) For every $L \in K$, the bilinear form

$$A^k(X) \times A^k(X) \rightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto (-1)^k P(x_1, L^{d-2k} x_2)$$

is symmetric, and is positive definite on the kernel of $L^{d-2k+1} : A^k(X) \rightarrow A^{d-k+1}(X)$.



Combinatorial atlas

Recall that a real symmetric matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y} \rangle \geq 0$$

iff A has at most one positive eigenvalue. We call such a matrix A **hyperbolic**.



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Goal: To prove the hyperbolicity of

$$M = \begin{pmatrix} 1 & 2 & \cdots & n & \epsilon \\ 0 & 0 & M_{ij} & & \\ & & \ddots & M_{i\epsilon} & \\ & & & 0 & M_{\epsilon\epsilon} \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{matrix} \quad \text{where}$$
$$\begin{aligned} M_{ij} &:= (k-1)! \{I \text{ indep., } |I| = k+1, i, j \in I\} \\ M_{i\epsilon} &:= (k-1)! \{I \text{ indep., } |I| = k, i \in I\} \\ M_{\epsilon\epsilon} &:= (k-1)! \{I \text{ indep., } |I| = k-1\} \end{aligned}$$

Define $\mathbf{v} := (1 \ \dots \ 1 \ 0)^\top$, $\mathbf{w} := (0 \ \dots \ 0 \ 1)^\top$.

The hyperbolicity of M implies Mason's conjecture (ii):

$$(k! I_k)^2 = \langle \mathbf{v}, M\mathbf{w} \rangle^2 \geq \langle \mathbf{v}, M\mathbf{v} \rangle \langle \mathbf{w}, M\mathbf{w} \rangle = (k+1)! I_{k+1} \cdot (k-1)! I_{k-1}.$$



Combinatorial atlas

To prove the hyperbolicity of

$$M = \begin{pmatrix} 1 & 2 & \cdots & n & \epsilon \\ 0 & 0 & M_{ij} & & \\ & \ddots & & M_{i\epsilon} & \\ & & 0 & & M_{\epsilon\epsilon} \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{matrix} \quad \text{from} \quad \begin{pmatrix} 1 & 2 & \cdots & n & \epsilon \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{matrix}$$

by induction on the already prescribed elements of I .

Induction steps based on controlling the eigenvalues by the Perron-Frobenius theorem.



Challenge on Mathematical Complexity Reduction

Combinatorial atlas is a super easy and powerful tool.

Can we prove more log-concavities using the combinatorial atlases?

Can we find easy proofs for known log-concavities?

Need more hints on hyperbolic matrices!



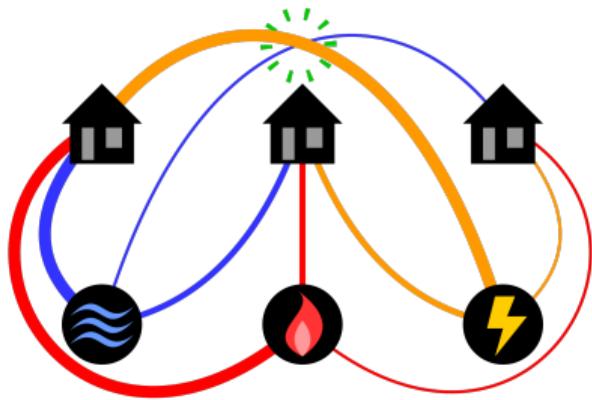
Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?



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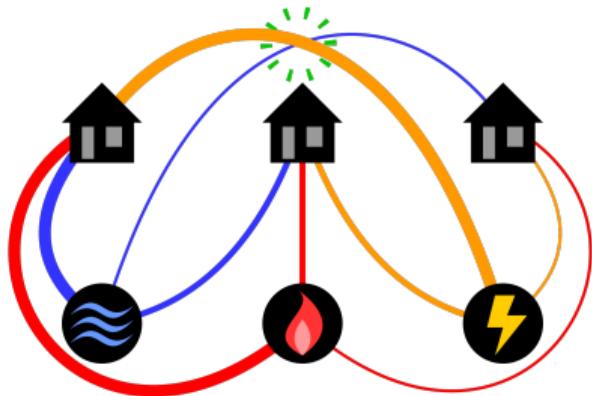
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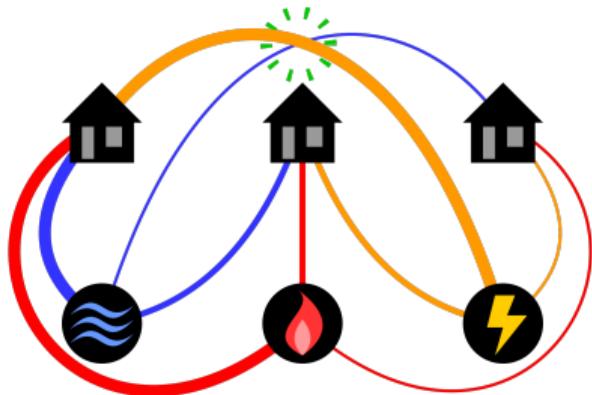
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The graph $K_{3,3}$ cannot be embedded on a **plane**, but can be embedded on a **torus**.



Hyperbolic matrices in topological graph theory

Let G graph with vertex set $\{1, \dots, n\}$ and edge set E , the **Colin de Verdière number** $\mu(G)$ is the largest corank of any $n \times n$ real symmetric matrix $M = (M_{ij})_{ij}$ such that

- $\forall i \neq j : M_{ij} < 0$ if $ij \in E$, $M_{ij} = 0$ otherwise.
- M has exactly one negative eigenvalue.
- If a real symmetric matrix X satisfies $MX = \mathbf{0}$ and ($X_{ij} = 0$ if $i = j$ or $ij \in E$), then $X = \mathbf{0}$.

$\mu(G)$ is a very strong topological invariant of graphs:

$$\mu(G) \leq 0 \text{ iff } G \text{ is edgeless.}$$

$$\mu(G) \leq 4 \text{ iff } G \text{ is linklessly embeddable in } \mathbb{R}^3.$$

$$\mu(G) \leq 1 \text{ iff } G \text{ is disj. union of paths.}$$

$$\mu(G) \leq 5 \text{ if } G \text{ is embeddable on } \mathbb{RP}^2 \text{ or Klein bottle.}$$

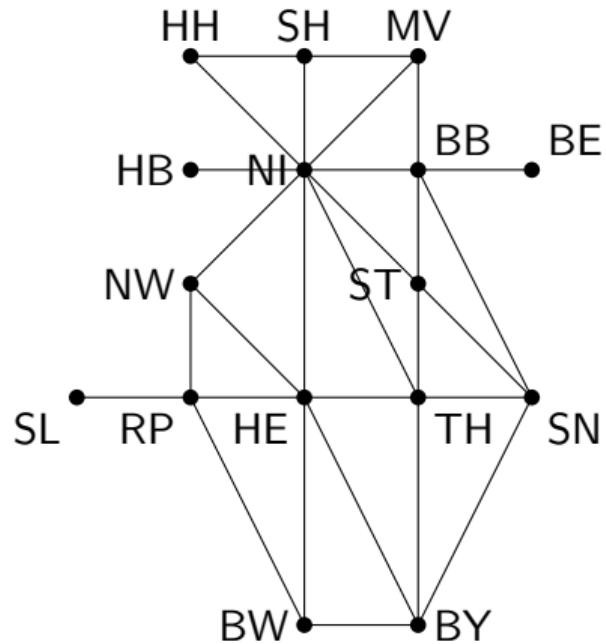
$$\mu(G) \leq 2 \text{ iff } G \text{ is outerplanar.}$$

$$\mu(G) \leq 6 \text{ if } G \text{ is embeddable on torus.}$$

$$\mu(G) \leq 3 \text{ iff } G \text{ is planar.}$$

...





(1,29,392,3275,18903,79796,254315,622594,1179280,1726363,1934949,1628575,995088,416184,106416,12528)

Thank you!

