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Semigraphoids and fans and their Coxeter friends

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MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Goals

1. Find a good geometric model for semigraphoids and gaussoids.
2. Generalize the definitions of semigraphoids, gaussoids and semimatroids to other Coxeter types.

Definition (conical and convex hull)

Let $S \subseteq \mathbb{R}^n$ be a set.

$$\operatorname{cone}(S) := \{\lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n : \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq S, \lambda_i \geq 0\}$$

$$\operatorname{conv}(S) := \left\{\lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n : \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq S, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\right\}$$

Definition (Minkowski sum)

For two sets $M, N \subseteq \mathbb{R}^n$, the Minkowski sum of M and N is

$$M + N := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in M, \mathbf{y} \in N\}.$$

Cones

Definition

A subset $P \subseteq \mathbb{R}^d$ is a polyhedral cone if it is the conical hull of a finite point set

$$P = \operatorname{cone}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \quad \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d.$$

Equivalently, it is an intersection of closed linear halfspaces

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{0}\}, \quad A \in \mathbb{R}^{m \times d}.$$

Polytopes

Definition

A subset $P \subseteq \mathbb{R}^d$ is a polytope if it is the convex hull of a finite point set

$$P = \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \quad \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d.$$

Equivalently, it is a bounded intersection of closed halfspaces

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{z}\}, \quad A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m.$$

Remark

- The equivalences are not trivial!
- ...But the proof is algorithmic (Fourier-Motzkin elimination and double description method).

Polytopes

Example

Regular polytopes are fully characterized by Schläfli (1852):

(1) simplices $\Delta_{d-1} := \mathrm{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \subseteq \mathbb{R}^d$,

(2) cubes $C_d := [-1, 1]^d$,

(3) crosspolytopes $C_d^\vee := \mathrm{conv}\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}$,

(4) dim. 2: regular n-gons,

(5) dim. 3: dodecahedron and icosahedron,

(6) dim. 4: 120-cell, 600-cell, 24-cell.

Polytopes

Definition

A face of a polytope $P \subseteq \mathbb{R}^d$ is a set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{c}^\top \mathbf{x} = c_0\},$$

where the inequality $\mathbf{c}^\top \mathbf{x} \leq c_0$ is satisfied by all $\mathbf{x} \in \mathbb{R}^d$.

Remark

- The faces of a polytope P are also polytopes. They form a lattice $\mathcal{L}(P)$ if ordered by inclusion, called the face lattice of P.
- Polytopes P and Q are combinatorially equivalent if $\mathcal{L}(P) = \mathcal{L}(Q)$.
- The same can be done for polyhedral cones and polyhedra.
- The dimension of a polytope (or cone) P is $\dim P = \dim \operatorname{aff}(P)$. The faces of dimensions $0, 1, \dim P - 1$ are called vertices, edges and facets, respectively.

Fans

Definition

A fan is a family $\mathcal{F} = \{C_1, \ldots, C_N\}$ of nonempty polyhedral cones such that

1. $C \in \mathcal{F}$ and $C' \neq \varnothing$ is a face of $C \Rightarrow C' \in \mathcal{F}$,
2. $C_1, C_2 \in \mathcal{F} \Rightarrow C_1 \cap C_2$ is a face of both C_1 and C_2.

The dimension of $\mathcal{F}$ is $\dim \mathcal{F} := \max_{C \in \mathcal{F}} \dim C$.
The cones in $\mathcal{F}$ of dimensions $\dim \mathcal{F}$, $\dim \mathcal{F} - 1$, $\dim \mathcal{F} - 2$, 1 are called chambers, walls, ridges and rays, respectively.

Fans

Example

1. Let $P \subseteq \mathbb{R}^n$ be a polytope. The normal fan $\mathcal{N}(P)$ of P consists of cones of those linear functions which are maximal on a fixed face of P.
2. Let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement, where $H_1, \dots, H_n$ are hyperplanes in $\mathbb{R}^d$. The arrangement $\mathcal{H}$ decomposes $\mathbb{R}^d$ into a fan $\mathcal{F}_{\mathcal{H}}$.
3. The Minkowski sum of line segments is called a zonotope.
   Let $\mathbf{v}_i \in \mathbb{R}^d$ be the normal vector of H_i. Then $\mathcal{F}_{\mathcal{H}} = \mathcal{N}(Z)$ where Z is the zonotope
   $$Z = [-\mathbf{v}_1, \mathbf{v}_1] + \cdots + [-\mathbf{v}_n, \mathbf{v}_n].$$
4. A fan is called polytopal if it is the normal fan of some polytope.

Permutohedra

The $(n-1)$-dimensional permutohedron (or permutahedron) is the zonotope

$$\begin{aligned}\Pi_{n-1} &= \operatorname{conv}\left\{(\delta^{-1}(n),\dots,\delta^{-1}(1)) : \delta \in \mathcal{S}_n\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}, \forall S \subseteq E : \sum_{i \in S} x_i \geq \frac{|S|(|S|+1)}{2}\right\} \\ &= \frac{n}{2}\mathbf{1} + \sum_{1 \leq i < j \leq n} \left[-\frac{\mathbf{e}_j - \mathbf{e}_i}{2}, \frac{\mathbf{e}_j - \mathbf{e}_i}{2}\right].\end{aligned}$$

The normal fan of Π_{n-1} is the permutohedral fan $\Sigma_{A_{n-1}}$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ whose chambers are $\{\mathbf{x} : x_{\delta(1)} \geq \cdots \geq x_{\delta(n)}\}$, $\delta \in \mathcal{S}_n$. We denote by $(\delta(1)|\cdots|\delta(n))$ the permutation $\delta \in \mathcal{S}_n$ corresponding to this chamber.

Questions

1. What are the groups of symmetries of Δ_{d-1}, C_d and $C_d^\vee$?
2. Let $C_3 = [-1,1]^3$ be the 3-dimensional cube. Is the fan $\mathcal{F}(C_3) := \{\mathrm{cone}(F) : F \in \mathcal{L}(C_3) \setminus \{C_3\}\}$ polytopal?
3. Construct a non-polytopal fan by modifying the fan $\mathcal{F}(C_3)$.
4. Convince yourself that if ξ is regular Gaussian with positive definite covariance matrix Σ, then $(ij|K) \in [[\xi]]$ iff $\det(\Sigma_{iK,jK}) = 0$.
   Deduce the gaussoid axioms from the "Master Lemma" (Matúš, 2005): for any positive definite matrix Σ,

$$\det(\Sigma_{iL,jL})\det(\Sigma_{kL}) - \det(\Sigma_{iL,kL})\det(\Sigma_{kL,jL}) = \pm\det(\Sigma_{ikL,jkL})\det(\Sigma_L).$$