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The geometry of conditional independence structures and their Coxeter friends

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MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Overview

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linear independence

algebraic independence, subforests,

matchings in bipartite graphs, ...

matroids

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"Combinatorial Erlangen Program"

"Conditional Erlangen Program"

Separation and connection

in topological spaces

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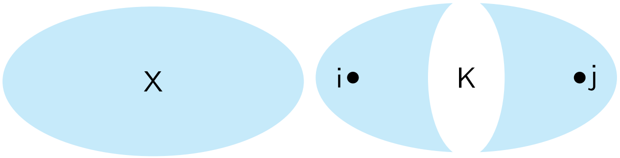
Let X be a topological space.
$\mathcal{G} := \{(ij|K) : K \subseteq X, i \neq j \in X \backslash K$ such that i and j are separated by $K\}$.



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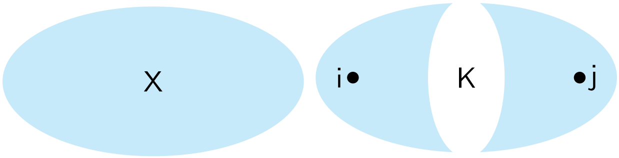

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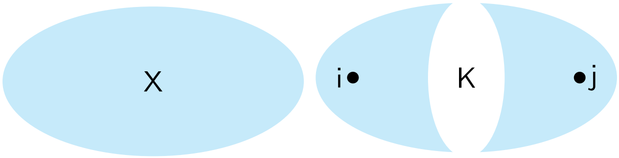

Then $\mathcal{G}$ satisfies

(Ascension) $(ij|L) \in \mathcal{G} \Rightarrow (ij|kL) \in \mathcal{G}$,

(Intersection) $(ij|kL), (ik|jL) \in \mathcal{G} \Rightarrow (ij|L) \in \mathcal{G}$,

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Let G be a graph with vertex set V and $K \subseteq V$, $i \neq j \in V \setminus K$.
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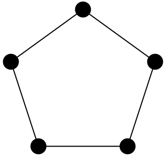

The cycle graph C_5, which is not topologizable.

Graphical models

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Remark: A regular Gaussian random vector $X_{[n]}$ satisfies the Markov property iff its covariance matrix Σ satisfies

$$(\Sigma^{-1})_{ij} = 0 \qquad \text{for all } i \neq j \text{ and } ij \notin E.$$

Gaussians

Let $X_{[n]} = (X_1, \ldots, X_n)$ be an n-dimensional Gaussian distributed random vector with positive definite covariance matrix Σ.

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Lemma ([Mat02])

$$\det(\Sigma_{kL}) \det(\Sigma_{ij|L}) = \det(\Sigma_L) \det(\Sigma_{ij|kL}) + \det(\Sigma_{ik|L}) \det(\Sigma_{jk|L}).$$

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Therefore $\mathcal{G}$ satisfies

(Semigraphoid) $\{(ij|L), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L), (ij|kL)\} \subseteq \mathcal{G}$,

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(Composition) $\{(ij|L), (ik|L)\} \subseteq \mathcal{G} \Rightarrow \{(ij|kL), (ik|jL)\} \subseteq \mathcal{G}$,

(Weak transition) $\{(ij|L), (ij|kL)\} \subseteq \mathcal{G} \Rightarrow (ik|L)$ or $(jk|L) \in \mathcal{G}$.

Semigraphoids, graphoids, gaussoids

Let $\mathcal{A}_n$ be the set $\{(ij|K) : K \subseteq [n], i \neq j \in [n] \backslash K\}$ of conditional independence statements.
A conditional independence structure is a subset of $\mathcal{A}_n$.

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A CI-structure $\mathcal{G} \subseteq \mathcal{A}_n$ is

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- and is a graphoid if it satisfies additionally (Intersection),
- and is a gaussoid if it satisfies additionally (Composition) and (Weak transition).

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$[[X]] := \{(ij|K) : i \perp\!\!\!\perp j|K\}$ is a semigraphoid for any random vector X.
$[[X]]$ is a graphoid for any random vector X with positive density function.
$[[X]]$ is a gaussoid for any regular Gaussian random vector X.

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A set function $\omega\colon 2^{[n]} \to \mathbb{R}$ is called submodular if for all $A, B \subseteq [n]$,

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Figure: Nonsubmodular price

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Therefore,

$$\begin{aligned}
[[\Sigma]] &= \{(ij|K) \in \mathcal{A}_n : X_i \perp\!\!\!\perp X_j | X_K\} = \{(ij|K) \in \mathcal{A}_n : \det(\Sigma_{ij|K}) = 0\} \\
&= \{(ij|K) \in \mathcal{A}_n : \log\det(\Sigma_{iK}) + \log\det(\Sigma_{jK}) = \log\det(\Sigma_{ijK}) + \log\det(\Sigma_K)\}.
\end{aligned}$$

Semimatroid

Definition

A semigraphoid $\mathcal{G} \subseteq \mathcal{A}_n$ is a semimatroid if there is a submodular function $\omega : 2^{[n]} \to \mathbb{R}$ with $\omega(\emptyset) = 0$ such that

$$\mathcal{G} = [[\omega]] := \{(ij|K) \in \mathcal{A}_n : \omega(iK) + \omega(jK) = \omega(ijK) + \omega(K)\}.$$

The permutohedral fan

The permutohedral fan $\Sigma_{A_{n-1}}$ is the normal fan of the permutohedron

$$\Pi_{n-1} = \mathrm{conv}\left\{(\delta^{-1}(n), \ldots, \delta^{-1}(1))^\top : \delta \in \mathfrak{S}_n\right\} = \frac{n}{2}\mathbf{1} + \sum_{1\leq i<j\leq n}\left[-\frac{\mathbf{e}_j - \mathbf{e}_i}{2}, \frac{\mathbf{e}_j - \mathbf{e}_i}{2}\right]$$

It can also be defined by the hyperplane arrangement $\{\{x_i = x_j\} : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$.

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We associate this wall to a CI statement $\delta(i) \perp\!\!\!\perp \delta(i+1)|\delta(1)\cdots\delta(i-1)$ in $\mathcal{A}_n$.

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We associate this wall to a CI statement $\delta(i) \perp\!\!\!\perp \delta(i+1)|\delta(1)\cdots\delta(i-1)$ in $\mathcal{A}_n$. Every CI statement $(ij|K) \in \mathcal{A}_n$ corresponds to $|K|!(n-|K|-2)!$ walls of $\Sigma_{A_{n-1}}$.

The permutohedral fan

- Ridges: Let s_ℓ be the reflection across the hyperplane $\{x_\ell = x_{\ell+1}\}$, $\ell = 1, \ldots, n-1$. Then $\langle s_1, \ldots, s_{n-1} \rangle = \mathfrak{S}_n$, and a 2-face of Π_{n-1} is a coset $\delta \cdot \langle s_\ell, s_{\ell'} \rangle$.
  Case 1: $\ell' > \ell + 1$

$$
\begin{tikzcd}
(L_1|j|i|L_2|i'|j'|L_3) \arrow[r, dash, "i \perp\!\!\!\perp j|L_1"] \arrow[d, dash, "i' \perp\!\!\!\perp j'|L_1 ijL_2"'] & (L_1|i|j|L_2|i'|j'|L_3) \arrow[d, dash, "i' \perp\!\!\!\perp j'|L_1 ijL_2"] \\
(L_1|j|i|L_2|j'|i'|L_3) \arrow[r, dash, "i \perp\!\!\!\perp j|L_1"'] & (L_1|i|j|L_2|j'|i'|L_3)
\end{tikzcd}
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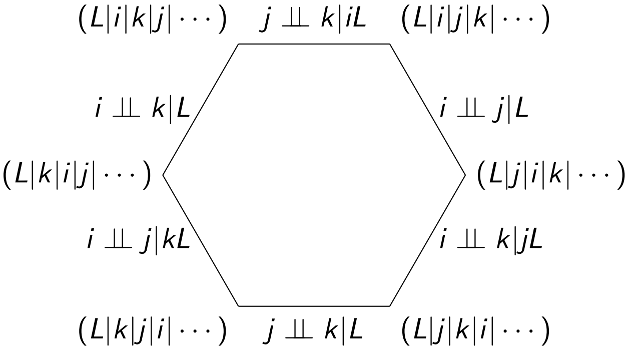
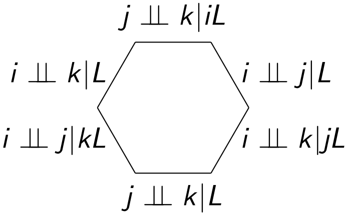
  Case 2: $\ell' = \ell + 1$



Semigraphoids



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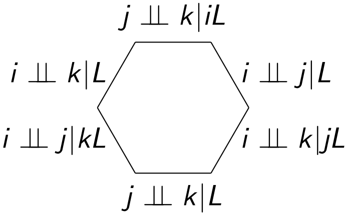

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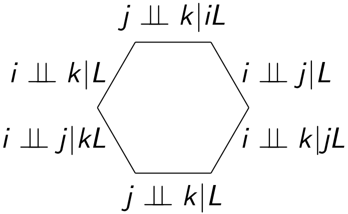

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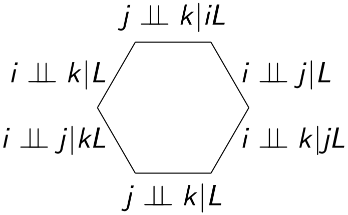
(Square) if an edge of a square is in M, then the opposite edge is also in M.

(Hexagon) if two adjacent edges of a hexagon are in M, then the two opposite edges are also in M.

Semigraphoids



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Theorem ([Mor+09])

A set of walls of the permutohedral fan $\Sigma_{A_{n-1}}$ is a semigraphoid iff removing them from $\Sigma_{A_{n-1}}$ results in a fan.

Semimatroids

Semimatroids

A polytope $P \subseteq \mathbb{R}^n$ is a generalized permutohedron if its normal fan coarsens $\Sigma_{A_{n-1}}$. Equivalently, there exists a submodular function $\omega : 2^{[n]} \to \mathbb{R}$ with $\omega(\emptyset) = 0$ such that

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i \in I} x_i \leq \omega(I) \ \forall \emptyset \neq I \subseteq [n], \sum_{i \in [n]} x_i = \omega([n]) \right\}. \tag{1}$$

Theorem ([Mor+09])

A semigraphoid is a semimatroid iff the corresponding coarsening of the permutohedral fan is polytopal. In particular, it is the normal fan of the generalized permutohedron (1) defined by the submodular function.

Semimatroid

$$\{\text{generalized permutohedra}\} \overset{1:1}{\longleftrightarrow} \{\text{submodular functions}\}$$

$$\left\{\begin{array}{c}\text{combinatorial types of}\\ \text{generalized permutohedra}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \{\text{faces of submodularity cone}\} \overset{1:1}{\longleftrightarrow} \{\text{semimatroids}\}$$

$$\{\text{facets of submodularity cone}\} \overset{1:1}{\longleftrightarrow} \{\text{CI-statements}\}$$

Root systems

Root systems

A root system $\Phi \subset V$ is a finite set of vectors, called roots, which satisfies

(R0) $\operatorname{span}(\Phi) = V$,

(R1) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for any $\alpha \in \Phi$,

(R2) $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.

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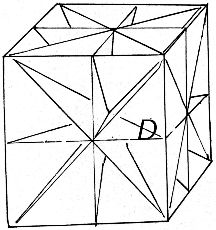
Theorem

The irreducible root systems can be completely classified into four infinite families A_d, B_d, C_d, D_d, the exceptional types $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ in the dimensions indicated by their subscripts, and $I_2(m)$ for $m \geq 3$.

Root systems

$\Phi = C_3 = \{\pm\mathbf{e}_1 \pm \mathbf{e}_2, \pm\mathbf{e}_1 \pm \mathbf{e}_3, \pm\mathbf{e}_2 \pm \mathbf{e}_3, \pm 2\mathbf{e}_1, \pm 2\mathbf{e}_2, \pm 2\mathbf{e}_3\}$



Let Φ be a root system.

The Coxeter complex Σ_Φ is the simplicial fan defined by the hyperplane arrangements with the roots as normal vectors.

Fix a chamber D of Σ_Φ called the fundamental domain.

Root systems

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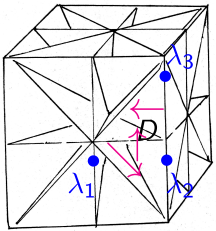
$\Delta_{C_3} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, 2\mathbf{e}_3\}$

$\lambda_1 = \mathbf{e}_1, \lambda_2 = \mathbf{e}_1 + \mathbf{e}_2,$
$\lambda_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$



The simple roots $\Delta = \{\alpha_1, \ldots, \alpha_d\} \subseteq \Phi$ are the roots in Φ which are the inner normals of the walls in D.
The fundamental weights $(\lambda_1, \ldots, \lambda_d)$ is the basis of V dual to the simple coroots $(\alpha_1^\vee, \ldots, \alpha_d^\vee)$, that is, $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, where $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$.

Root systems

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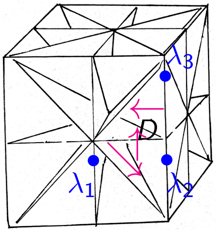

$W_{C_3} \cong \mathbb{Z}_2 \rtimes \mathfrak{S}_3$

s_1 = exchange the 1st and 2nd coord.

s_2 = exchange the 2nd and 3rd coord.

s_3 = change the sign of the 3rd coord.

$\langle s_1, s_2 \rangle \cong \mathfrak{S}_3 \cong D_6$

$\langle s_1, s_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_4$

$\langle s_2, s_3 \rangle \cong \mathbb{Z}_2 \rtimes \mathfrak{S}_2 \cong D_8$

The Weyl group of Φ is $W_\Phi := \langle s_\alpha : \alpha \in \Phi \rangle = \langle s_\alpha : \alpha \in \Delta \rangle \subseteq \mathrm{GL}(V)$.
The parabolic subgroups of W_Φ are the subgroups

$$(W_\Phi)_I := \langle s_\alpha : \alpha \in I \rangle \subseteq W_\Phi \text{ for } I \subseteq \Delta.$$

Φ-permutohedra

The Coxeter complex Σ_Φ is the normal fan of the Φ-permutohedron

$$\Pi_\Phi := \sum_{\alpha \in \Phi_+} [-\alpha/2, \alpha/2] = \mathrm{conv}\{w \cdot \rho : w \in W\},$$

where $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi_+} \alpha) = \lambda_1 + \cdots + \lambda_d$ is the sum of fundamental weights.

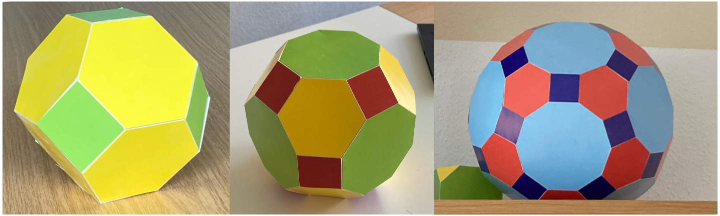

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

Φ-permutohedra

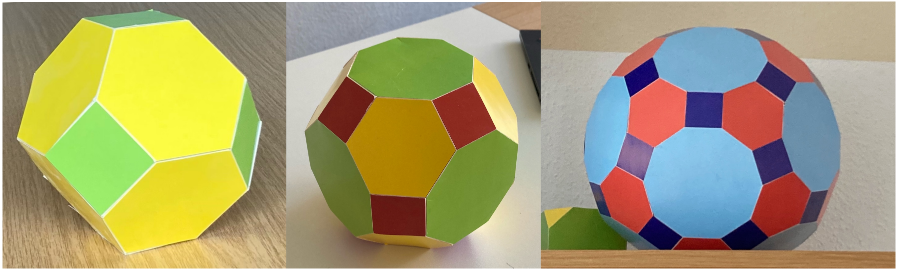

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

polytope	fan	toric variety
truncation	stellar subdivision	blow-up
omnitruncation	Coxeter complex	wonderful compactification

Φ-permutohedra

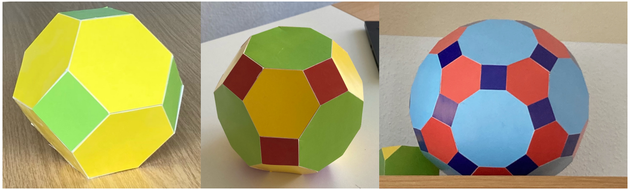

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

Coxeter complex Σ_Φ	Φ-permutohedron Π_Φ	parabolic cosets
chambers	vertices	$w \cdot D : w \in W$
walls	edges	$\{w, ws_\alpha\} = w\langle s_\alpha \rangle$ for $w \in W$ and $\alpha \in \Delta$
ridges	2-faces	$w\langle s_{\alpha_1}, s_{\alpha_2} \rangle$ for $w \in W$, $\alpha_1 \neq \alpha_2 \in \Delta$
rays	facets	$W\{\lambda_1, \dots, \lambda_d\} =: \mathcal{R}_\Phi$

Φ-semigraphoid

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Definition

Let Φ be a root system. A Φ-semigraphoid is a fan which is a coarsening of the Coxeter complex Σ_Φ.

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By the main theorem from [Rea12], we have

Corollary

A set of edges $\mathcal{G}$ of Π_Φ is a Φ-semigraphoid iff it satisfies

> *For every $2k$-gonal 2-face F of Π_Φ, whenever $\mathcal{G}$ contains any $k-1$ consecutive edges of F, then $\mathcal{G}$ also contains the opposite $k-1$ consecutive edges of F.*

Φ-semimatroid

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Definition

A Φ-semigraphoid, regarded as a fan, is a Φ-semimatroid if it is a polytopal fan. That is, a fan coarsening Σ_Φ which is the normal fan of a polytope Q. Such a polytope Q is called a generalized Φ-permutohedron.

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A function $h \colon \mathcal{R}_\Phi \to \mathbb{R}$ is Φ-submodular if it is convex when regarded as a piecewise linear function $|\Sigma_\Phi| \to \mathbb{R}$. Equivalently, it is the support function of a generalized Φ-permutohedron.

Φ-semimatroid

Theorem ([Ard+20])

A function $h\colon \mathcal{R}_\Phi \to \mathbb{R}$ is Φ-submodular iff for every $w \in W_\Phi$ and every simple reflection s_i and corresponding fundamental weight λ_i, the following local Φ-submodularity inequalities hold:

$$h(w\lambda_i) + h(ws_i\lambda_i) \geq \sum_{j \in N(i)} -2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} h(w\lambda_j),$$

where $N(i)$ is the set of neighbors of i in the Dynkin diagram and α_i the simple root corresponding to s_i.

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where $N(i)$ is the set of neighbors of i in the Dynkin diagram and α_i the simple root corresponding to s_i.

A Φ-semigraphoid is a Φ-semimatroid iff there is a Φ-submodular function $h : \mathcal{R}_\Phi \to \mathbb{R}$ such that the equality is attended in the local Φ-submodularity inequalities exactly at its elements.

Φ-CI-statements

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The cone SF_Φ of Φ-submodular functions: parameter space of generalized Φ-permutohedra

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Theorem ([Ard+20])

Each local Φ-submodularity inequality associated with a pair (w, s_i), for $w \in W$ and $i \in [d]$, gives a facet of the Φ-submodular cone SF_Φ. Two pairs (w, s_i) and $(w', s_{i'})$ define the same facet iff $i = i'$ and $w^{-1}w' \in W_{[d]\setminus N(i)}$.

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The Φ-CI statements are exactly the orbits

$$\mathcal{A}_\Phi = \{W_{[d]\setminus N(i)} \cdot wW_{\{i\}} : w \in W, i \in [d]\}.$$

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The Φ-CI statements are exactly the orbits

$$\mathcal{A}_\Phi = \{W_{[d]\setminus N(i)} \cdot wW_{\{i\}} : w \in W, i \in [d]\}.$$

A subset $\mathcal{G}$ of $\mathcal{A}_\Phi$ is called a semigraphoid resp. semimatroid if $\bigcup \mathcal{G}$ is a semigraphoid resp. semimatroid as a set of edges of Π_Φ.

Type B and C

Write $[\pm n] := \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$, and $S \sqsubseteq [\pm n]$ if $\emptyset \neq S \subseteq [\pm n]$ and $j \in S \Rightarrow \bar{j} \notin S$.
We define the set of C-conditional independence (CI) statements to be

$$\begin{aligned}\mathcal{C}_n :=& \{(ij|K) : K \sqsubseteq [\pm n], \{i,j\} \sqsubseteq [\pm n] \backslash (K \cup \bar{K}), i \neq j\} \cup \\ &\cup \{(i\bar{i}|K) : K \sqsubseteq [\pm n], |K| = n-1, i \in [\pm n] \backslash (K \cup \bar{K})\}.\end{aligned}$$

A C-semigraphoid on $[n]$ is a subset $\mathcal{G} \subseteq \mathcal{C}_n$ which satisfies (Semigraphoid), and for every $L \sqsubseteq [\pm n]$, $|L| = n-2$, $\{i,j\} \sqsubseteq [\pm n] \backslash (L \cup \bar{L})$, $i \neq j$:

(CSG1) $\{(ij|L), (j\bar{j}|iL), (i\bar{j}|L)\} \subseteq \mathcal{G} \Rightarrow \{(\bar{i}\bar{j}|L), (j\bar{j}|\bar{i}L), (\bar{i}j|L)\} \subseteq \mathcal{G}$,

(CSG2) $\{(i\bar{i}|jL), (\bar{i}j|L), (j\bar{j}|\bar{i}L)\} \subseteq \mathcal{G} \Rightarrow \{(i\bar{i}|\bar{j}L), (i\bar{j}|L), (j\bar{j}|iL)\} \subseteq \mathcal{G}$.

Type B and C

For $f : \mathcal{R}_{C_n} = \{\mathbf{e}_S : S \sqsubseteq [\pm n]\} \to \mathbb{R}$, write $f(S) = f(\mathbf{e}_S)$ for any $S \sqsubseteq [\pm n]$.
The function f is bisubmodular if it satisfies the local C_n-submodularity inequalities

$$\begin{cases} f(Sa) + f(Sb) \geq f(S) + f(Sab) & S \sqsubseteq [\pm n], |S| \leq n-2, ab \sqsubseteq [\pm n] \backslash (S\bar{S}), \\ f(Sa) + f(S\bar{a}) \geq 2f(S) & S \sqsubseteq [\pm n], |S| = n-1, a \in [\pm n] \backslash (S\bar{S}). \end{cases}$$

A C-semigraphoid $\mathcal{G}$ on $[n]$ is a C-semimatroid if there is a bisubmodular function $f : \{S \sqsubseteq [\pm n]\} \to \mathbb{R}$ such that the equality is attended in the local C_n-bisubmodularity inequalities exactly at the triples $(ij|K) \in \mathcal{G}$.

Type D

Let

$$\tilde{\mathcal{D}}_n := \{(ij|K) : K \sqsubseteq [\pm n], |K| \leq n-2, \{i,j\} \sqsubseteq [\pm n] \backslash K\bar{K}, i \neq j\} \subseteq \mathcal{C}_n.$$

The set of D-CI-statements is

$$\mathcal{D}_n := \tilde{\mathcal{D}}_n / \sim$$

where $\sim$ is the equivalence relation in $\mathcal{D}_n$ defined by

$$\sim := \{((ij|K), (\overline{ij}|K)) \in \tilde{\mathcal{D}}_n \times \tilde{\mathcal{D}}_n : |K| = n-2\}.$$

By abusing of notations, we write an element of $\tilde{\mathcal{D}}_n$ for its class in $\mathcal{D}_n$. In other words, we identify $(ij|K)$ with $(\overline{ij}|K)$ for $|K| = n-2$.

Type D

A D-semigraphoid on $[n]$ is a subset $\mathcal{G} \subseteq \mathcal{D}_n$ satisfying

(DSG1) $\{(ij|L),(ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L),(ij|kL)\} \subseteq \mathcal{G}$.

A function $f : \mathcal{R}_{D_n} \to \mathbb{R}$ is disubmodular if

$$\begin{cases} f(Sa)+f(Sb) \geq f(S)+f(Sab) & S \sqsubseteq [\pm n], |S| \leq n-4, ab \sqsubseteq [\pm n] \backslash S\bar{S}, \\ f(Sa)+f(Sb) \geq f(S)+g(Sabc)+g(Sab\bar{c}) & S \sqsubseteq [\pm n], |S| = n-3, abc \sqsubseteq [\pm n] \backslash S\bar{S}, \\ g(Sab)+g(S\bar{a}\bar{b}) \geq f(S) & S \sqsubseteq [\pm n], |S| = n-2, ab \sqsubseteq [\pm n] \backslash S\bar{S}, \end{cases}$$

where $f(S) := f(\mathbf{e}_S)$ and $g(S) := f(\frac{1}{2}\mathbf{e}_S)$.
A D-semigraphoid $\mathcal{G}$ is a D-semimatroid if there is a disubmodular function $f : \mathcal{R}_{D_n} \to \mathbb{R}$ such that the equality in the local D_n-submodularity inequalities is attained exactly at the elements of $\mathcal{G}$.

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Thank you!