Xiangying Chen

On log-concavity of sequences

Magdeburg, 16.11.2022

Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Log-concavity

Log-concavity

A sequence of nonnegative numbers $a_0, \ldots, a_n$ is called log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i = 1, \ldots, n-1.$$

Binomial coefficients $\binom{n}{k}$

$$\binom{n}{k} = \text{the number of } k\text{-element subsets of an } n\text{-element set}$$

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Stirling numbers of the first kind $\left[{n \atop k}\right]$

$$\left[{n \atop k}\right] = \text{the number of permutations of } \{1, \ldots, n\} \text{ with exactly } k \text{ cycles}$$

$n = 0$							0							
$n = 1$						0		1						
$n = 2$					0		1		1					
$n = 3$				0		2		3		1				
$n = 4$			0		6		11		6		1			
$n = 5$		0		24		50		35		10		1		
$n = 6$	0		120		274		225		85		15		1	

Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$

$$\left\{ {n \atop k} \right\} = \text{ the number of partitions of } \{1, \ldots, n\} \text{ with exactly } k \text{ blocks}$$

$n = 0$							0						
$n = 1$						0		1					
$n = 2$					0		1		1				
$n = 3$				0		1		3		1			
$n = 4$			0		1		7		6		1		
$n = 5$		0		1		15		25		10		1	
$n = 6$	0		1		31		90		65		15		1

Eulerian numbers $\left\langle {n \atop k} \right\rangle$

$$\left\langle {n \atop k} \right\rangle = \text{ the number of permutations of } \{1, \ldots, n\} \text{ into exactly } k \text{ descents}$$

$n = 1$							1						
$n = 2$						1		1					
$n = 3$					1		4		1				
$n = 4$				1		11		11		1			
$n = 5$			1		26		66		26		1		
$n = 6$		1		57		302		302		57		1	
$n = 7$	1		120		1191		2146		1191		120		1

Chromatic polynomials

Chromatic polynomials

Chromatic polynomials

How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?

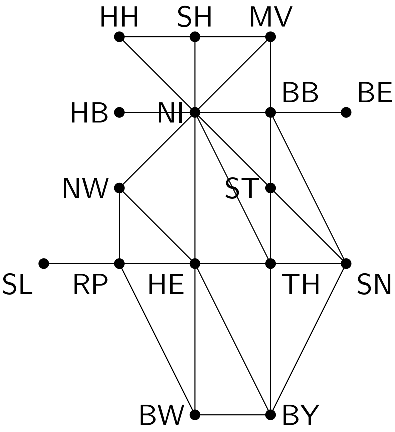
Chromatic polynomials

How many ways can you color this map with x colors such that every pair of adjacent Bundesländer are colored differently?

Answer:

$$
\begin{aligned}
&x^{16} - 29x^{15} + 392x^{14} - 3275x^{13} + 18903x^{12} \\
&- 79796x^{11} + 254315x^{10} - 622594x^{9} + 1179280x^{8} \\
&- 1726363x^{7} + 1934949x^{6} - 1628575x^{5} \\
&+ 995088x^{4} - 416184x^{3} + 106416x^{2} - 12528x.
\end{aligned}
$$

Chromatic polynomials



Chromatic polynomials

Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every adjacent vertices are colored differently.

Chromatic polynomials

Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every adjacent vertices are colored differently.
Facts:

1. $\chi_G(x)$ is a polynomial function in x called the chromatic polynomial of G.

Chromatic polynomials

Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every adjacent vertices are colored differently.
Facts:

1. $\chi_G(x)$ is a polynomial function in x called the chromatic polynomial of G.
2. The coefficients are integers, and are alternating in sign.

Chromatic polynomials

Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every adjacent vertices are colored differently.
Facts:

1. $\chi_G(x)$ is a polynomial function in x called the chromatic polynomial of G.
2. The coefficients are integers, and are alternating in sign.

Conjecture (Read 1968, Hoggar 1974)

The (absolute values of) coefficients of a chromatic polynomial of any graph form a log-concave sequence.

Chromatic polynomials

Let G be a graph. Let $\chi_G(x)$ be the number of colorings of vertices of G by at most x colors such that every adjacent vertices are colored differently.

Facts:

1. $\chi_G(x)$ is a polynomial function in x called the chromatic polynomial of G.
2. The coefficients are integers, and are alternating in sign.

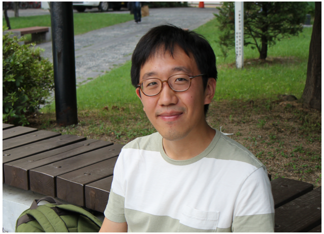
Conjecture (Read 1968, Hoggar 1974)

The (absolute values of) coefficients of a chromatic polynomial of any graph form a log-concave sequence.

The same question can also be asked for chromatic polynomials of

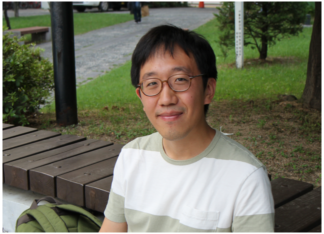
$$\{\text{planar graphs}\} \subsetneq \{\text{graphs}\} \subsetneq \{\text{vector configurations}\} \subsetneq \{\text{matroids}\}.$$

Chromatic polynomials



The 2022 Fields Medalist
June Huh

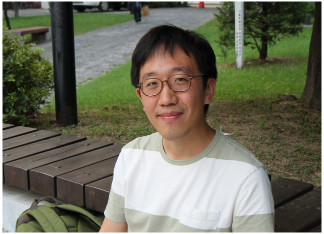
Chromatic polynomials



The 2022 Fields Medalist June Huh

- (2012) Proved the Read-Hoggar conjecture in his PhD thesis.

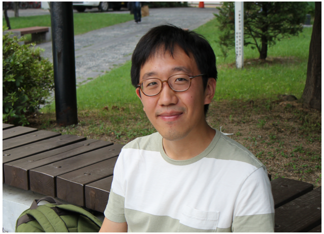
Chromatic polynomials



The 2022 Fields Medalist June Huh

- (2012) Proved the Read-Hoggar conjecture in his PhD thesis.
- (2014, with Katz) Proved the log-concavity of coefficients of chromatic polynomials of vector configurations.

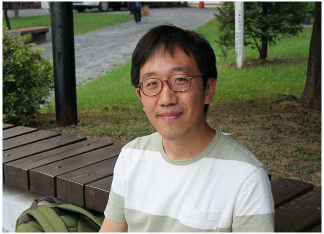
Chromatic polynomials



The 2022 Fields Medalist June Huh

- (2012) Proved the Read-Hoggar conjecture in his PhD thesis.
- (2014, with Katz) Proved the log-concavity of coefficients of chromatic polynomials of vector configurations.
- (2018, with Adiprasito and Katz) Proved the log-concavity of coefficients of chromatic polynomials of Matroids.

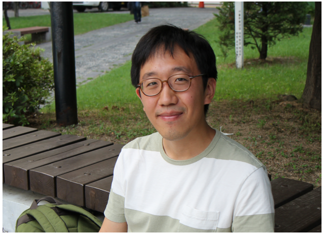
Chromatic polynomials



The 2022 Fields Medalist
June Huh

- (2012) Proved the Read-Hoggar conjecture in his PhD thesis.
- (2014, with Katz) Proved the log-concavity of coefficients of chromatic polynomials of vector configurations.
- (2018, with Adiprasito and Katz) Proved the log-concavity of coefficients of chromatic polynomials of Matroids.
- (2020, with Brändén) Lorentzian polynomials

Chromatic polynomials



The 2022 Fields Medalist June Huh

- (2012) Proved the Read-Hoggar conjecture in his PhD thesis.
- (2014, with Katz) Proved the log-concavity of coefficients of chromatic polynomials of vector configurations.
- (2018, with Adiprasito and Katz) Proved the log-concavity of coefficients of chromatic polynomials of Matroids.
- (2020, with Brändén) Lorentzian polynomials
- and log-concavity of matroid h-vectors, normalized Schur polynomials, different evaluations of Tutte polynomials...

Numbers of independent vectors

Let E be a set of vectors.
$I_k :=$ number of linearly independent subsets of E of cardinality k

Numbers of independent vectors

Let E be a set of vectors.
$I_k :=$ number of linearly independent subsets of E of cardinality k

Conjecture (Mason 1972)

(i) $I_k^2 \geq I_{k-1} I_{k+1}$ *(i.e. $(I_k)_k$ is log-concave),*

(ii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1} I_{k+1}$ *(i.e. $(k! I_k)_k$ is log-concave),*

(iii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k-1} I_{k+1}$ *(i.e. $\left(I_k / \binom{n}{k}\right)_k$ is log-concave).*

Numbers of independent vectors

Let E be a set of vectors.
$I_k :=$ number of linearly independent subsets of E of cardinality k

Conjecture (Mason 1972)

(i) $I_k^2 \geq I_{k-1} I_{k+1}$ *(i.e. $(I_k)_k$ is log-concave),*

(ii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1} I_{k+1}$ *(i.e. $(k! I_k)_k$ is log-concave),*

(iii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k-1} I_{k+1}$ *(i.e. $\left(I_k / \binom{n}{k}\right)_k$ is log-concave).*

As a consequence of log-concavity of chromatic polynomials, Mason's conjecture (i) is first proven for matroids in 2018 by Adiprasito, Huh and Katz.

Numbers of independent vectors

Let E be a set of vectors.
$I_k :=$ number of linearly independent subsets of E of cardinality k

Conjecture (Mason 1972)

(i) $I_k^2 \geq I_{k-1} I_{k+1}$ *(i.e. $(I_k)_k$ is log-concave),*

(ii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1} I_{k+1}$ *(i.e. $(k! I_k)_k$ is log-concave),*

(iii) $I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k-1} I_{k+1}$ *(i.e. $\left(I_k / \binom{n}{k}\right)_k$ is log-concave).*

As a consequence of log-concavity of chromatic polynomials, Mason's conjecture (i) is first proven for matroids in 2018 by Adiprasito, Huh and Katz.

Mason's conjecture (iii) is proven independently by Anari, Liu, Gharan and Vinzant (2018) and Brändén-Huh (2020).

Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

INTRODUCTION TO THE COMBINATORIAL ATLAS

SWEE HONG CHAN* AND IGOR PAK*

ABSTRACT. We give elementary self-contained proofs of the *strong Mason conjecture* recently proved by Anari et. al [ALOV18] and Brändén–Huh [BH20], and of the classical *Alexandrov–Fenchel inequality*. Both proofs use the *combinatorial atlas* technology recently introduced by the authors [CP21]. We also give a formal relationship between combinatorial atlases and *Lorentzian polynomials*.

1. INTRODUCTION

In this paper we tell three interrelated but largely independent stories. While we realize that this sounds self-contradictory, we insist on this description. We prove no new results, nor do we claim to give new proofs of known results. Instead, we give a *new presentation* of the existing proofs.

Our goal is explain the *combinatorial atlas* technology from [CP21] in three different contexts. The idea is to both give a more accessible introduction to our approach and connect it to other approaches in the area. Although one can use this paper as a companion to [CP21], it is written completely independently and aimed at a general audience.

(1) *Strong Mason conjecture* claims ultra-log-concavity of the number of independent sets of matroid according to its size. This is perhaps the most celebrated problem recently resolved in a series of paper culminating with independent proofs by Anari et. al [ALOV18] and Brändén–Huh [BH20]. These proofs use the technology of *Lorentzian polynomials*, which in turn substantially simplify earlier heavily algebraic tools.

In our paper [CP21], we introduce the *combinatorial atlas* technology motivated by geometric considerations of the *Alexandrov–Fenchel inequality*. This allowed us, among other things, to

Annals of Mathematics **188** (2018), 381–452
https://doi.org/10.4007/annals.2018.188.2.1

Hodge theory for combinatorial geometries

By KARIM ADIPRASITO, JUNE HUH, and ERIC KATZ

Abstract

We prove the hard Lefschetz theorem and the Hodge-Riemann relations for a commutative ring associated to an arbitrary matroid M. We use the Hodge-Riemann relations to resolve a conjecture of Heron, Rota, and Welsh that postulates the log-concavity of the coefficients of the characteristic polynomial of M. We furthermore conclude that the f-vector of the independence complex of a matroid forms a log-concave sequence, proving a conjecture of Mason and Welsh for general matroids.

1. **Introduction**

The combinatorial theory of matroids starts with Whitney [Whi35], who introduced matroids as models for independence in vector spaces and graphs. See [Kun86, Ch. I] for an excellent historical overview. By definition, a *matroid* M is given by a closure operator defined on all subsets of a finite set E satisfying the Steinitz-Mac Lane exchange property:

For every subset I of E and every element a not in the closure of I, if a is in the closure of $I \cup \{b\}$, then b is in the closure of $I \cup \{a\}$.

Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

- The whole proof can be written down in 1-2 pages instead of 72
- Basic linear-algebraic proof instead of combinatorial Hodge theory
- Generalized to morphisms of matroids/antimatroids/interval greedoids
- Refined inequalities for special classes
- Full classification of equality cases

Numbers of independent vectors

A new proof of Mason's conjecture (iii) (and much more) is given by Chan and Pak (2021 and 2022):

- The whole proof can be written down in 1-2 pages instead of 72
- Basic linear-algebraic proof instead of combinatorial Hodge theory
- Generalized to morphisms of matroids/antimatroids/interval greedoids
- Refined inequalities for special classes
- Full classification of equality cases
- The TRUE Mathematical Complexity Reduction

Log-concavity in linear algebra

Let A be a $n \times n$ real symmetric matrix.

- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \le \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

  iff A is

Log-concavity in linear algebra

Let A be a $n \times n$ real symmetric matrix.

- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \leq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

  iff A is positive semidefinite or negative semidefinite.
- The matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, A\mathbf{x} \rangle \langle \mathbf{y}, A\mathbf{y} \rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y} \rangle \geq 0$$

  iff A has at most one positive eigenvalue.

Log-concavity in analysis

Theorem (Newton's Theorem)

For any real-rooted polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ *with* $a_0, \ldots, a_n \geq 0$, *we have*

$$\left(\frac{a_i}{\binom{n}{i}} \right)^2 \geq \left(\frac{a_{i-1}}{\binom{n}{i-1}} \right) \left(\frac{a_{i+1}}{\binom{n}{i+1}} \right).$$

Log-concavity in analysis

Theorem (Newton's Theorem)

For any real-rooted polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ *with* $a_0, \dots, a_n \geq 0$, *we have*

$$\left(\frac{a_i}{\binom{n}{i}}\right)^2 \geq \left(\frac{a_{i-1}}{\binom{n}{i-1}}\right)\left(\frac{a_{i+1}}{\binom{n}{i+1}}\right).$$

Examples:

1. $\sum_{i=0}^{n} \binom{n}{k} x^k = (x+1)^n$.
2. $\sum_{i=0}^{n} \left[{n \atop k}\right] x^k = x(x+1)\cdots(x+n-1)$.
3. The polynomials $\sum_{i=0}^{n} \left\{{n \atop k}\right\} x^k$ and $\sum_{i=0}^{n} \left\langle{n \atop k}\right\rangle x^k$ satisfy certain recurrence relations, from which we can deduce their real-rootedness.

Log-concavity in analysis

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is called strictly Lorentzian if

- the support consists of all monomials of degree d,
- all coefficients are positive,
- and for any choice of $i_1, \ldots, i_{d-2} \in \{1, \ldots, n\}$, the quadratic form $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalues.

The limits of strictly Lorentzian polynomials are called Lorentzian.

Log-concavity in analysis

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is called strictly Lorentzian if

- the support consists of all monomials of degree d,
- all coefficients are positive,
- and for any choice of $i_1, \ldots, i_{d-2} \in \{1, \ldots, n\}$, the quadratic form $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalues.

The limits of strictly Lorentzian polynomials are called Lorentzian.

1. **Lorentzian polynomials and real-rooted polynomials**
   $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the open upper half of the complex plane
   A polynomial $f \in \mathbb{C}[x_1, \cdots, x_m]$ is stable if $f \equiv 0$ or $f(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \mathcal{H}^m$.
   Stable polynomials in $\mathbb{R}[x_1, \cdots, x_m]$ are Lorentzian.

Log-concavity in analysis

2. **Lorentzian polynomials and eigenvalues of Hessian**

    - For any Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ has exactly one positive eigenvalue.
    - For any strictly Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ is nonsingular.

Log-concavity in analysis

2. **Lorentzian polynomials and eigenvalues of Hessian**
   - For any Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ has exactly one positive eigenvalue.
   - For any strictly Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ is nonsingular.
3. **Lorentzian polynomials and log-concave functions**
   f is log-concave on $\mathbb{R}^n_{>0}$ iff the Hessian of f has exactly one positive eigenvalue on $\mathbb{R}^n_0$.

Log-concavity in analysis

2. **Lorentzian polynomials and eigenvalues of Hessian**
    - For any Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ has exactly one positive eigenvalue.
    - For any strictly Lorentzian polynomial f and any $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Hessian $\mathbf{H}_f(\mathbf{w})$ is nonsingular.
3. **Lorentzian polynomials and log-concave functions**
   f is log-concave on $\mathbb{R}^n_{>0}$ iff the Hessian of f has exactly one positive eigenvalue on $\mathbb{R}^n_0$.
4. **Lorentzian polynomials and discrete convexity**
   The support of any Lorentzian polynomial is M-convex, that is, the lattice point set of a polytope whose edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$.

Log-concavity in geometry

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is

$$A + B := \{a + b : a \in A, b \in B\}.$$

Log-concavity in geometry

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is

$$A + B := \{a + b : a \in A, b \in B\}.$$

Let $A, B \subseteq \mathbb{R}^2$ be two convex sets. The mixed volume of A and B is

$$V(A, B) = \frac{1}{2} \left(\mathrm{Vol}(A + B) - \mathrm{Vol}(A) - \mathrm{Vol}(B) \right).$$

Log-concavity in geometry

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is

$$A + B := \{a + b : a \in A, b \in B\}.$$

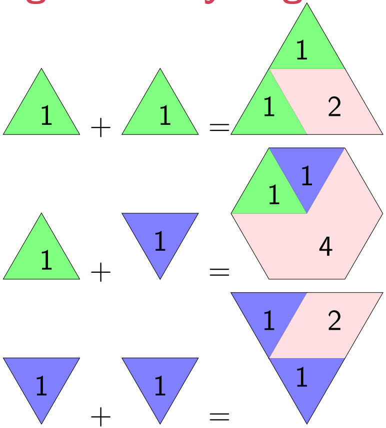
Let $A, B \subseteq \mathbb{R}^2$ be two convex sets. The mixed volume of A and B is

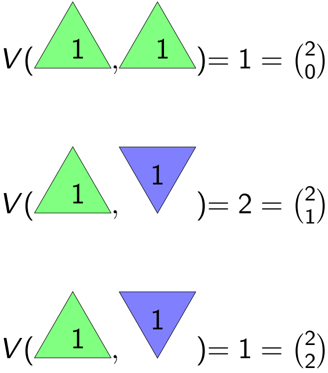
$$V(A, B) = \frac{1}{2}\left(\mathrm{Vol}(A + B) - \mathrm{Vol}(A) - \mathrm{Vol}(B)\right).$$

Theorem (Alexandrov-Fenchel inequality in dimension 2)

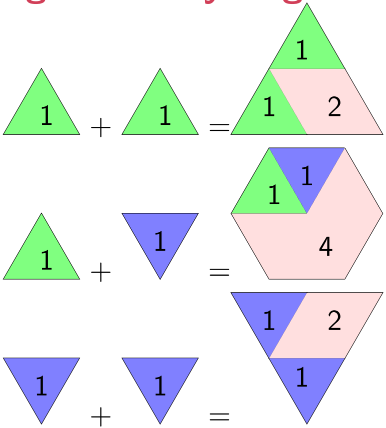
$$V(A, B)^2 \geq V(A, A) \cdot V(B, B).$$

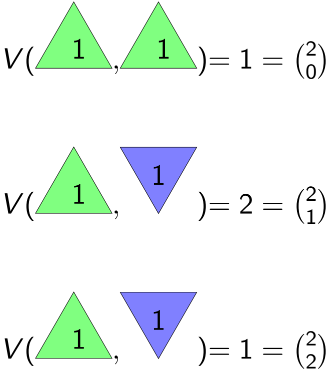
Log-concavity in geometry





Log-concavity in geometry





Corollary

$$\binom{2}{1}^2 \geq \binom{2}{0} \cdot \binom{2}{2}.$$

Log-concavity in geometry

The mixed volume of n convex bodies $K_1, \dots, K_n \subset \mathbb{R}^n$ is

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{1 \le r_1 < \dots < r_k \le n} \mathrm{Vol}(K_{r_1} + \dots + K_{r_k}).$$

Theorem (Alexandrov-Fenchel inequality)

Let $K_1, \dots, K_{n-2}, A, B \subset \mathbb{R}^n$ be convex bodies in $\mathbb{R}^n$, then

$$V(K_1, \dots, K_{n-2}, A, B)^2 \ge V(K_1, \dots, K_{n-2}, A, A) \cdot V(K_1, \dots, K_{n-2}, B, B).$$

Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because

$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \ldots, \Delta_n}_{k}, \underbrace{-\Delta_n, \ldots, -\Delta_n}_{n-k})$$

where $\Delta_n = \mathrm{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.

Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because

$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \ldots, \Delta_n}_{k}, \underbrace{-\Delta_n, \ldots, -\Delta_n}_{n-k})$$

   where $\Delta_n = \operatorname{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.

2. If $a_0, \ldots, a_n \geq 0$ are coefficients of a real-rooted polynomial $p(t) \in \mathbb{R}[t]$, then $\left(a_k/\binom{n}{k}\right)_k$ is log-concave because

$$p(t) = \sum_{i=0}^{n} a_i t^i = (x + \lambda_1) \cdots (x + \lambda_n)$$

Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because
$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \dots, \Delta_n}_{k}, \underbrace{-\Delta_n, \dots, -\Delta_n}_{n-k})$$
where $\Delta_n = \operatorname{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.
2. If $a_0, \dots, a_n \geq 0$ are coefficients of a real-rooted polynomial $p(t) \in \mathbb{R}[t]$, then $\left(a_k/\binom{n}{k}\right)_k$ is log-concave because
$$p(t) = \sum_{i=0}^{n} a_i t^i = (x + \lambda_1) \cdots (x + \lambda_n) = \sum_{k=0}^{n} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right) x^{n-k}$$

Log-concavity in geometry

Examples:

1. Binomial coefficients are log-concave because
$$\binom{n}{k} = n!\,V(\underbrace{\Delta_n, \dots, \Delta_n}_{k}, \underbrace{-\Delta_n, \dots, -\Delta_n}_{n-k})$$
where $\Delta_n = \operatorname{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the standard n-dimensional simplex.
2. If $a_0, \dots, a_n \geq 0$ are coefficients of a real-rooted polynomial $p(t) \in \mathbb{R}[t]$, then $\left(a_k/\binom{n}{k}\right)_k$ is log-concave because
$$p(t) = \sum_{i=0}^{n} a_i t^i = (x+\lambda_1)\cdots(x+\lambda_n) = \sum_{k=0}^{n} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right) x^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} V\left(\underbrace{[0,1]^n, \dots, [0,1]^n}_{k}, \underbrace{\prod_{i=1}^{n} [0,\lambda_i], \dots, \prod_{i=1}^{n} [0,\lambda_i]}_{n-k} \right) x^k.$$

Log-concavity in geometry

Examples:

3. (Shephard 1960) A sequence $(a_0, \ldots, a_n)$ of nonnegative numbers is log-concave iff there exist convex compact sets $A, B \subseteq \mathbb{R}^n$ such that for $k = 0, \ldots, n$,

$$a_k = V(\underbrace{A, \ldots, A}_{k}, \underbrace{B, \ldots, B}_{n-k}).$$

Log-concavity in geometry

Examples:

3. (Shephard 1960) A sequence $(a_0, \ldots, a_n)$ of nonnegative numbers is log-concave iff there exist convex compact sets $A, B \subseteq \mathbb{R}^n$ such that for $k = 0, \ldots, n$,

$$a_k = V(\underbrace{A, \ldots, A}_{k}, \underbrace{B, \ldots, B}_{n-k}).$$

4. There are Alexandrov-Fenchel type inequalities for the intersection numbers of nef divisors on a variety.

Log-concavity in geometry

Examples:

3. (Shephard 1960) A sequence $(a_0, \ldots, a_n)$ of nonnegative numbers is log-concave iff there exist convex compact sets $A, B \subseteq \mathbb{R}^n$ such that for $k = 0, \ldots, n$,

$$a_k = V(\underbrace{A, \ldots, A}_{k}, \underbrace{B, \ldots, B}_{n-k}).$$

4. There are Alexandrov-Fenchel type inequalities for the intersection numbers of nef divisors on a variety.
5. A far-reaching combinatorial abstraction leads to the "combinatorial Hodge theory".

Combinatorial Hodge theory

$A^\bullet(X) = \bigoplus_{k=0}^{d} A^k(X)$ graded $\mathbb{R}$-algebra associated to "an object X of dimension d"

K = a convex cone in the space of linear operators on $A^\bullet(X)$

The Kähler package: For every $0 \le k \le \lfloor \frac{d}{2} \rfloor$:

(PD) The bilinear pairing $P : A^k(X) \times A^{d-k}(X) \to \mathbb{R}$ is non-degenerate.

(HL) For every $L \in K$, the composition $L^{d-2k} : A^k(X) \to A^{d-k}(X)$ is bijective.

(HR) For every $L \in K$, the bilinear form

$$A^k(X) \times A^k(X) \to \mathbb{R}, \qquad (x_1, x_2) \mapsto (-1)^k P(x_1, L^{d-2k} x_2)$$

is symmetric, and is positive definite on the kernel of $L^{d-2k+1} : A^k(X) \to A^{d-k+1}(X)$.

Combinatorial atlas

Recall that a real symmetric matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y}\rangle^2 \geq \langle \mathbf{x}, A\mathbf{x}\rangle\langle \mathbf{y}, A\mathbf{y}\rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y}\rangle \geq 0$$

iff A has at most one positive eigenvalue. We call such a matrix A hyperbolic.

Combinatorial atlas

Recall that a real symmetric matrix A satisfies

$$\langle \mathbf{x}, A\mathbf{y}\rangle^2 \geq \langle \mathbf{x}, A\mathbf{x}\rangle\langle \mathbf{y}, A\mathbf{y}\rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \langle \mathbf{y}, A\mathbf{y}\rangle \geq 0$$

iff A has at most one positive eigenvalue. We call such a matrix A hyperbolic.

Goal: To prove the hyperbolicity of

$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & n & \epsilon \end{array} \\ \begin{pmatrix} 0 & & & & \\ & 0 & M_{ij} & & \\ & & \ddots & & M_{i\epsilon} \\ & & & 0 & \\ & & & & M_{\epsilon\epsilon} \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{array}$$

where

$$M_{ij} := (k-1)!\{I \text{ indep.}, |I| = k+1, i, j \in I\}$$
$$M_{i\epsilon} := (k-1)!\{I \text{ indep.}, |I| = k, i \in I\}$$
$$M_{\epsilon\epsilon} := (k-1)!\{I \text{ indep.}, |I| = k-1\}$$

Define $\mathbf{v} := (1 \quad \cdots \quad 1 \quad 0)^\top$, $\mathbf{w} := (0 \quad \cdots \quad 0 \quad 1)^\top$.

The hyperbolicity of M implies Mason's conjecture (ii):

$$(k!I_k)^2 = \langle \mathbf{v}, M\mathbf{w}\rangle^2 \geq \langle \mathbf{v}, M\mathbf{v}\rangle\langle \mathbf{w}, M\mathbf{w}\rangle = (k+1)!I_{k+1} \cdot (k-1)!I_{k-1}.$$

Combinatorial atlas

To prove the hyperbolicity of

$$M = \begin{array}{cc} \begin{matrix} 1 & 2 & \cdots & n & \epsilon \end{matrix} & \\ \begin{pmatrix} 0 & & & & \\ & 0 & M_{ij} & & \\ & & \ddots & & M_{i\epsilon} \\ & & & 0 & \\ & & & & M_{\epsilon\epsilon} \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{matrix} \end{array} \qquad \text{from} \qquad \begin{array}{cc} \begin{matrix} 1 & 2 & \cdots & n & \epsilon \end{matrix} & \\ \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ \vdots \\ n \\ \epsilon \end{matrix} \end{array}$$

by induction on the already prescribed elements of I.
Induction steps based on controlling the eigenvalues by the Perron-Frobenius theorem.

Challenge on Mathematical Complexity Reduction

Combinatorial atlas is a super easy and powerful tool.
Can we prove more log-concavities using the combinatorial atlases?
Can we find easy proofs for known log-concavities?

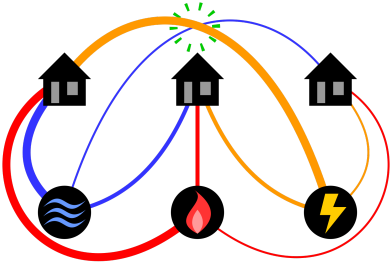
Need more hints on hyperbolic matrices!

Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?

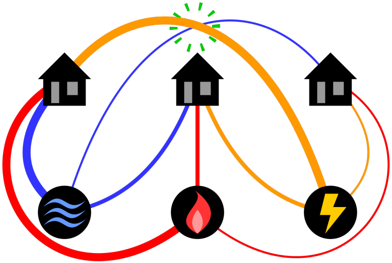
Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?



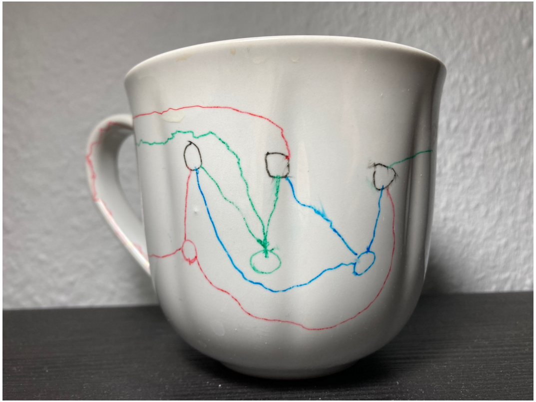
Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?



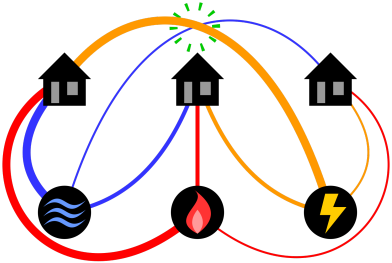
By Cmglee - Own work, CC BY-SA 4.0,

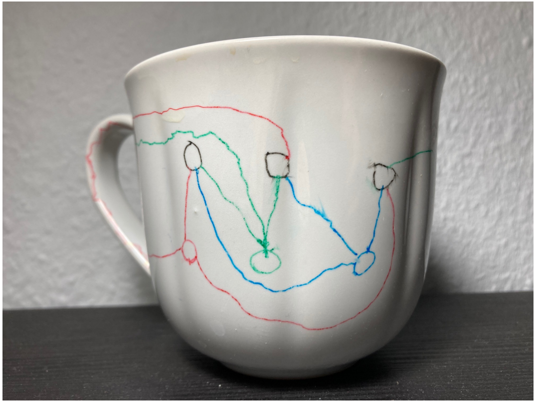
https://commons.wikimedia.org/w/index.php?curid=86717521



Three utilities problem

Three houses, three utility companies. Can each house be connected to each utility, with no connection lines crossing?





The graph $K_{3,3}$ cannot be embedded on a **plane**, but can be embedded on a **torus**.

Hyperbolic matrices in topological graph theory

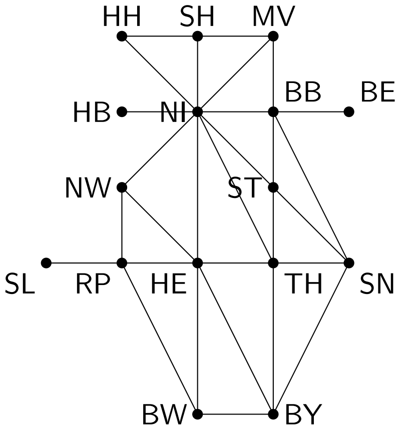
Let G graph with vertex set $\{1, \ldots, n\}$ and edge set E, the Colin de Verdière number $\mu(G)$ is the largest corank of any $n \times n$ real symmetric matrix $M = (M_{ij})_{ij}$ such that

- $\forall i \neq j : M_{ij} < 0$ if $ij \in E$, $M_{ij} = 0$ otherwise.
- M has exactly one negative eigenvalue.
- If a real symmetric matrix X satisfies $MX = \mathbf{0}$ and ($X_{ij} = 0$ if $i = j$ or $ij \in E$), then $X = 0$.

$\mu(G)$ is a very strong topological invariant of graphs:

$\mu(G) \leq 0$ iff G is edgeless.
$\mu(G) \leq 1$ iff G is disj. union of paths.
$\mu(G) \leq 2$ iff G is outerplanar.
$\mu(G) \leq 3$ iff G is planar.

$\mu(G) \leq 4$ iff G is linklessly embeddable in $\mathbb{R}^3$.
$\mu(G) \leq 5$ if G is embeddable on $\mathbb{RP}^2$ or Klein bottle.
$\mu(G) \leq 6$ if G is embeddable on torus.
$\cdots$



(1,29,392,3275,18903,79796,254315,622594,1179280,1726363,1934949,1628575,995088,416184,106416,12528)

Thank you!