Xiangying Chen

The geometry of conditional independence structures and their Coxeter friends

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Overview

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linear independence

algebraic independence, subforests,
matchings in bipartite graphs, ...

matroids

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"Combinatorial Erlangen Program"

"Conditional Erlangen Program"

Outline

1. What is a conditional independence structure?
2. The geometry of conditional independence
3. The Conditional Erlangen Program
4. Their Coxeter friends

Separation and connection

in topological spaces

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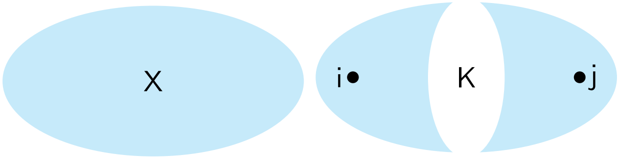
$\mathcal{G} := \{(ij|K) : i \text{ and } j \text{ are in different (path-)connected components in } X \backslash K\}$.



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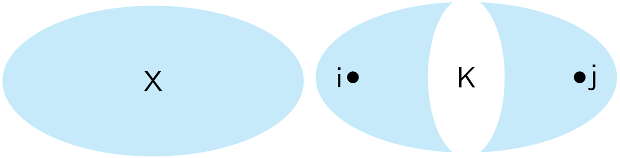

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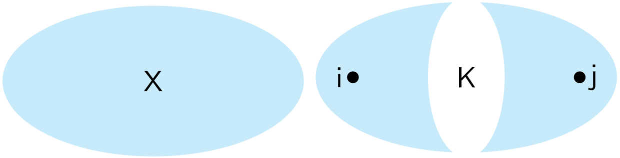

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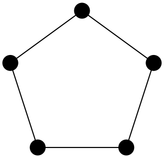

The cycle graph C_5, which is not topologizable.

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$X_{[n]}$ satisfies the Markov property associated to the graph $G = ([n], E)$ if

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Remark: A regular Gaussian random vector $X_{[n]}$ satisfies the Markov property iff its covariance matrix Σ satisfies

$$(\Sigma^{-1})_{ij} = 0 \qquad \text{for all } i \neq j \text{ and } ij \notin E.$$

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Lemma ([Mat02])

$$\det(\Sigma_{kL}) \det(\Sigma_{ij|L}) = \det(\Sigma_L) \det(\Sigma_{ij|kL}) + \det(\Sigma_{ik|L}) \det(\Sigma_{jk|L}).$$

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Therefore $\mathcal{G}$ satisfies

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(Composition) $\{(ij|L), (ik|L)\} \subseteq \mathcal{G} \Rightarrow \{(ij|kL), (ik|jL)\} \subseteq \mathcal{G}$,

(Weak transition) $\{(ij|L), (ij|kL)\} \subseteq \mathcal{G} \Rightarrow (ik|L)$ or $(jk|L) \in \mathcal{G}$.

Semigraphoids, graphoids, gaussoids

Let $\mathcal{A}_n$ be the set $\{(ij|K) : K \subseteq [n], i \neq j \in [n]\backslash K\}$ of conditional independence statements.
A conditional independence structure is a subset of $\mathcal{A}_n$.

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A CI-structure $\mathcal{G} \subseteq \mathcal{A}_n$ is

- a semigraphoid if it satisfies (Semigraphoid),
- and is a graphoid if it satisfies additionally (Intersection),
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$[[X]] := \{(ij|K) : i \perp\!\!\!\perp j | K\}$ is a semigraphoid for any random vector X.
$[[X]]$ is a graphoid for any random vector X with positive density function.
$[[X]]$ is a gaussoid for any regular Gaussian random vector X.

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The principal minors of a positive definite matrix Σ satisfy the Hadamard-Fischer inequalities

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Therefore,

$$\begin{aligned} [[\Sigma]] &= \{(ij|K) \in \mathcal{A}_n : X_i \perp\!\!\!\perp X_j | X_K\} = \{(ij|K) \in \mathcal{A}_n : \det(\Sigma_{ij|K}) = 0\} \\ &= \{(ij|K) \in \mathcal{A}_n : \log\det(\Sigma_{iK}) + \log\det(\Sigma_{jK}) = \log\det(\Sigma_{ijK}) + \log\det(\Sigma_K)\}. \end{aligned}$$

Semimatroid

A set function $\omega : 2^{[n]} \to \mathbb{R}$ is called submodular if for all $A, B \subseteq [n]$,

$$\omega(A) + \omega(B) \geq \omega(A \cap B) + \omega(A \cup B).$$

Definition

A semigraphoid $\mathcal{G} \subseteq \mathcal{A}_n$ is a semimatroid if there is a submodular function $\omega : 2^{[n]} \to \mathbb{R}$ with $\omega(\emptyset) = 0$ such that

$$\mathcal{G} = [[\omega]] := \{(ij|K) \in \mathcal{A}_n : \omega(Ki) + \omega(Kj) = \omega(Kij) + \omega(K)\}.$$

The permutohedral fan

The permutohedral fan $\Sigma_{A_{n-1}}$ is the normal fan of the permutohedron

$$\Pi_{n-1} = \operatorname{conv}\left\{(\delta^{-1}(n), \ldots, \delta^{-1}(1))^\top : \delta \in \mathfrak{S}_n\right\} = \frac{n}{2}\mathbf{1} + \sum_{1 \le i < j \le n} \left[-\frac{\mathbf{e}_j - \mathbf{e}_i}{2}, \frac{\mathbf{e}_j - \mathbf{e}_i}{2}\right]$$

It can also be defined by the hyperplane arrangement $\{\{x_i = x_j\} : 1 \le i < j \le n\}$ in $\mathbb{R}^n$.

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We associate this wall to a CI statement $\delta(i) \perp\!\!\!\perp \delta(i+1)|\delta(1)\cdots\delta(i-1)$ in $\mathcal{A}_n$.

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Every CI statement $(ij|K) \in \mathcal{A}_n$ corresponds to $|K|!(n-|K|-2)!$ walls of $\Sigma_{A_{n-1}}$.

The permutohedral fan

- The ridges of $\Sigma_{A_{n-1}}$ are either of the form

$$\{\mathbf{x} : x_{\delta(1)} \geq \cdots \geq x_{\delta(i)} = x_{\delta(i+1)} = x_{\delta(i+2)} \geq \cdots \geq x_{\delta(n)}\}$$

for some $\delta \in \mathfrak{S}_n$ and $1 \leq i \leq n-2$, or

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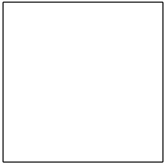
for some $\delta \in \mathfrak{S}_n$ and $1 \leq i \leq j-2 \leq n-3$.
Let s_i be the reflection across the hyperplane $\{x_i = x_{i+1}\}$, $i = 1, \ldots, n-1$.
Then $\langle s_1, \ldots, s_{n-1} \rangle = \mathfrak{S}_n$, and a 2-face of Π_{n-1} is a coset $\delta \cdot \langle s_i, s_j \rangle$, which is a square if $j > i+1$ and a hexagon if $j = i+1$.


$(L_1|j|i|L_2|i'|j'|L_3)$
$i \perp\!\!\!\perp j | L_1$
$(L_1|i|j|L_2|i'|j'|L_3)$
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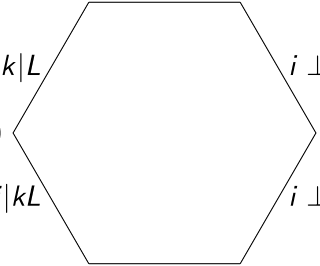
$(L|i|k|j|[n]\setminus ijkL)$
$j \perp\!\!\!\perp k | iL$
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$j \perp\!\!\!\perp k | L$
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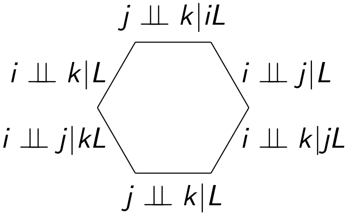
Semigraphoids



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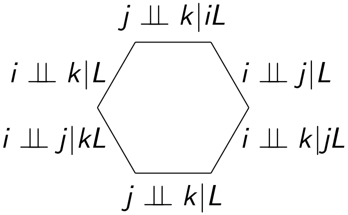

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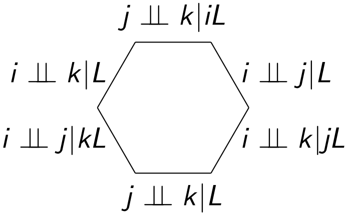

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A set M of edges of the permutohedron Π_{n-1} is a semigraphoid iff it satisfies

(Square) if an edge of a square is in M, then the opposite edge is also in M.

(Hexagon) if two adjacent edges of a hexagon are in M, then the two opposite edges are also in M.

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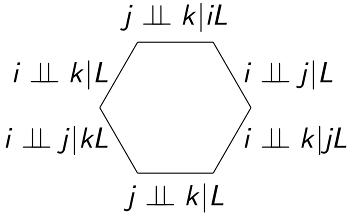

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Theorem ([Mor+09])

A set of walls of the permutohedral fan $\Sigma_{A_{n-1}}$ is a semigraphoid iff removing them from $\Sigma_{A_{n-1}}$ results a fan.

Semimatroids

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A polytope $P \subseteq \mathbb{R}^n$ is a generalized permutohedron if its normal fan coarsens $\Sigma_{A_{n-1}}$. Equivalently, there exists a submodular function $\omega : 2^{[n]} \to \mathbb{R}$ with $\omega(\emptyset) = 0$ such that

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i \in I} x_i \leq \omega(I) \ \forall \emptyset \neq I \subseteq [n], \sum_{i \in [n]} x_i = \omega([n]) \right\}. \tag{1}$$

Theorem ([Mor+09])

A semigraphoid is a semimatroid iff the corresponding coarsening of the permutohedral fan is polytopal. In particular, it is the normal fan of the generalized permutohedron (1) defined by the submodular function.

Root systems

Root systems

A root system $\Phi \subset V$ is a finite set of vectors, called roots, which satisfies

(R0) $\mathrm{span}(\Phi) = V$,

(R1) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for any $\alpha \in \Phi$,

(R2) $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.

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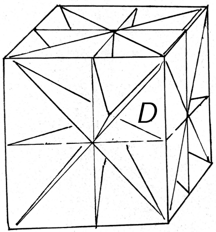
Theorem

The irreducible root systems can be completely classified into four infinite families A_d, B_d, C_d, D_d, the exceptional types $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ in the dimensions indicated by their subscripts, and $I_2(m)$ for $m \geq 3$.

Root systems

More definitions

$\Phi = C_3 = \{\pm\mathbf{e}_1 \pm \mathbf{e}_2, \pm\mathbf{e}_1 \pm \mathbf{e}_3, \pm\mathbf{e}_2 \pm \mathbf{e}_3, \pm 2\mathbf{e}_1, \pm 2\mathbf{e}_2, \pm 2\mathbf{e}_3\}$



Let Φ be a root system.
The Coxeter complex Σ_Φ is the simplicial fan defined by the hyperplane arrangements with the roots as normal vectors.
Fix a chamber D of Σ_Φ called the fundamental domain.

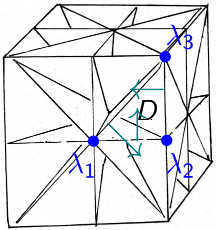
Root systems

More definitions

$\Phi = C_3 = \{\pm\mathbf{e}_1 \pm \mathbf{e}_2, \pm\mathbf{e}_1 \pm \mathbf{e}_3, \pm\mathbf{e}_2 \pm \mathbf{e}_3, \pm 2\mathbf{e}_1, \pm 2\mathbf{e}_2, \pm 2\mathbf{e}_3\}$

$\Delta_{C_3} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, 2\mathbf{e}_3\}$

$\lambda_1 = \mathbf{e}_1, \lambda_2 = \mathbf{e}_1 + \mathbf{e}_2,$
$\lambda_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$



The simple roots $\Delta = \{\alpha_1, \dots, \alpha_d\} \subseteq \Phi$ are the roots in Φ which are the inner normals of the walls in D.
The fundamental weights $(\lambda_1, \dots, \lambda_d)$ is the basis of V dual to the simple coroots $(\alpha_1^\vee, \dots, \alpha_d^\vee)$, that is, $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, where $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$.

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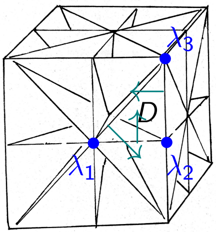

$W_{C_3} \cong \mathbb{Z}_2 \rtimes \mathfrak{S}_3$
s_1 = exchange the 1st and 2nd coord.
s_2 = exchange the 2nd and 3rd coord.
s_3 = change the sign of the 3rd coord.

$\langle s_1, s_2 \rangle \cong \mathfrak{S}_3 \cong D_6$
$\langle s_1, s_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_4$
$\langle s_2, s_3 \rangle \cong \mathbb{Z}_2 \rtimes \mathfrak{S}_2 \cong D_8$

The Weyl group of Φ is $W_\Phi := \langle s_\alpha : \alpha \in \Phi \rangle = \langle s_\alpha : \alpha \in \Delta \rangle \subseteq \mathrm{GL}(V)$.
The parabolic subgroups of W_Φ are the subgroups

$$(W_\Phi)_I := \langle s_\alpha : \alpha \in I \rangle \subseteq W_\Phi \text{ for } I \subseteq \Delta.$$

Φ-permutohedra

The Coxeter complex Σ_Φ is the normal fan of the Φ-permutohedron

$$\Pi_\Phi := \sum_{\alpha \in \Phi_+} [-\alpha/2, \alpha/2] = \mathrm{conv}\{w \cdot \rho : w \in W\},$$

where $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi_+} \alpha) = \lambda_1 + \cdots + \lambda_d$ is the sum of fundamental weights.

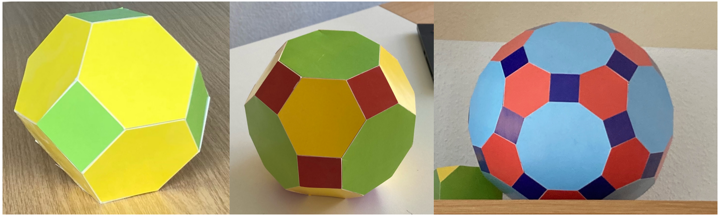

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

Φ-permutohedra

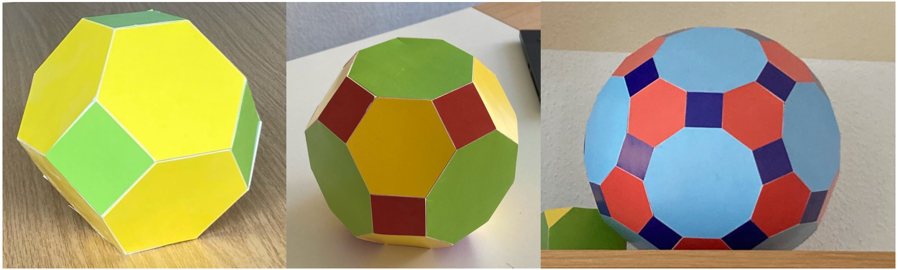

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

polytope	fan	toric variety
truncation	stellar subdivision	blow-up
omnitruncation	Coxeter complex	wonderful compactification

Φ-permutohedra

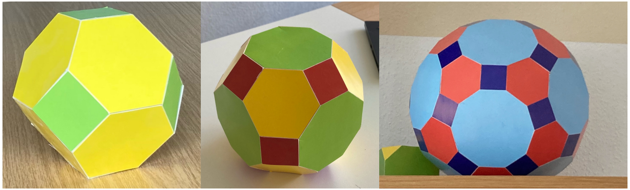

Figure: The A_3, B_3 (C_3) and H_3 permutohedra

Coxeter complex Σ_Φ	Φ-permutohedron Π_Φ	parabolic cosets
chambers	vertices	$w \cdot D : w \in W$
walls	edges	$\{w, ws_\alpha\} = w\langle s_\alpha \rangle$ for $w \in W$ and $\alpha \in \Delta$
ridges	2-faces	$w\langle s_{\alpha_1}, s_{\alpha_2} \rangle$ for $w \in W$, $\alpha_1 \neq \alpha_2 \in \Delta$
rays	facets	$W\{\lambda_1, \ldots, \lambda_d\} =: \mathcal{R}_\Phi$

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Definition

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Theorem ([Rea12])

Let Z be a zonotope and let Σ_Z be the normal fan of Z. Then a set E of edges of Z is corresponding to the set of walls of Σ_Z whose removal results a coarser fan if and only if for every $2k$-gonal 2-face F of Z, whenever F contains any $k-1$ consecutive edges of F, then E also contains the opposite $k-1$ consecutive edges of F.

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Corollary

A set of edges $\mathcal{G}$ of Π_Φ is a Φ-semigraphoid iff it satisfies

> *For every $2k$-gonal 2-face F of Π_Φ, whenever $\mathcal{G}$ contains any $k-1$ consecutive edges of F, then $\mathcal{G}$ also contains the opposite $k-1$ consecutive edges of F.*

Φ-semimatroid

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A Φ-semigraphoid, regarded as a fan, is a Φ-semimatroid if it is a polytopal fan. That is, a fan coarsening Σ_Φ which is the normal fan of a polytope Q. Such a polytope Q is called a generalized Φ-permutohedron.

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A function $h : \mathcal{R}_\Phi \to \mathbb{R}$ is Φ-submodular if it is convex when regarded as a piecewise linear function $|\Sigma_\Phi| \to \mathbb{R}$. Equivalently, it is the support function of a generalized Φ-permutohedron.

Φ-semimatroid

Theorem ([Ard+20])

A function $h : \mathcal{R}_\Phi \to \mathbb{R}$ is Φ-submodular iff for every $w \in W_\Phi$ and every simple reflection s_i and corresponding fundamental weight λ_i, the following local Φ-submodularity inequalities hold:

$$h(w\lambda_i) + h(ws_i\lambda_i) \geq \sum_{j \in N(i)} -2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} h(w\lambda_j),$$

where $N(i)$ is the set of neighbors of i in the Dynkin diagram and α_i the simple root corresponding to s_i.

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where $N(i)$ is the set of neighbors of i in the Dynkin diagram and α_i the simple root corresponding to s_i.

A Φ-semigraphoid is a Φ-semimatroid iff there is a Φ-submodular function $h : \mathcal{R}_\Phi \to \mathbb{R}$ such that the equality is attended in the local Φ-submodularity inequalities exactly at its elements.

Φ-CI-statements

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The cone SF_Φ of Φ-submodular functions: parameter space of generalized Φ-permutohedra

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Theorem ([Ard+20])

Each local Φ-submodularity inequality associated with a pair (w, s_i), for $w \in W$ and $i \in [d]$, gives a facet of the Φ-submodular cone SF_Φ. Two pairs (w, s_i) and $(w', s_{i'})$ define the same facet iff $i = i'$ and $w^{-1}w' \in W_{[d] \setminus N(i)}$.

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The Φ-CI statements are exactly the orbits

$$\mathcal{A}_\Phi = \{W_{[d]\setminus N(i)} \cdot wW_{\{i\}} : w \in W, i \in [d]\}.$$

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The Φ-CI statements are exactly the orbits

$$\mathcal{A}_\Phi = \{W_{[d]\setminus N(i)} \cdot wW_{\{i\}} : w \in W, i \in [d]\}.$$

A subset $\mathcal{G}$ of $\mathcal{A}_\Phi$ is called a semigraphoid resp. semimatroid if $\bigcup \mathcal{G}$ is a semigraphoid resp. semimatroid as a set of edges of Π_Φ.

Type B and C

Write $[\pm n] := \{1, \dots, n, -1, \dots, -n\}$, and $S \sqsubseteq [\pm n]$ if $\emptyset \neq S \subseteq [\pm n]$ and $j \in S \Rightarrow -j \notin S$. We define the set of C-conditional independence (CI) statements to be

$$\begin{aligned}\mathcal{C}_n :=&\{(ij|K) : K \sqsubseteq [\pm n], \{i,j\} \sqsubseteq [\pm n] \backslash (K \cup \bar{K}), i \neq j\} \cup \\ &\cup \{(i\bar{i}|K) : K \sqsubseteq [\pm n], |K| = n-1, i \in [\pm n] \backslash (K \cup \bar{K})\}.\end{aligned}$$

A C-semigraphoid on $[n]$ is a subset $\mathcal{G} \subseteq \mathcal{C}_n$ which satisfies (Semigraphoid), and for every $L \sqsubseteq [\pm n]$, $|L| = n-2$, $\{i,j\} \sqsubseteq [\pm n] \backslash (L \cup \bar{L})$, $i \neq j$:

(CG1) $\{(ij|L), (j\bar{j}|iL), (i\bar{j}|L)\} \subseteq \mathcal{G} \Rightarrow \{(\bar{i}\bar{j}|L), (j\bar{j}|\bar{i}L), (\bar{i}j|L)\} \subseteq \mathcal{G}$,

(CG2) $\{(i\bar{i}|jL), (\bar{i}j|L), (j\bar{j}|\bar{i}L)\} \subseteq \mathcal{G} \Rightarrow \{(i\bar{i}|\bar{j}L), (i\bar{j}|L), (j\bar{j}|iL)\} \subseteq \mathcal{G}$.

Type B and C

For $f : \mathcal{R}_{C_n} = \{e_S : S \sqsubseteq [\pm n]\} \to \mathbb{R}$, write $f(S) = f(\mathbf{e}_S)$ for any $S \sqsubseteq [\pm n]$.
The function f is bisubmodular if it satisfies the local C_n-submodularity inequalities

$$\begin{cases} f(Sa) + f(Sb) \geq f(S) + f(Sab) & S \sqsubseteq [\pm n], |S| \leq n-2, ab \sqsubseteq [\pm n] \setminus (S\bar{S}), \\ f(Sa) + f(S\bar{a}) \geq 2f(S) & S \sqsubseteq [\pm n], |S| = n-1, a \in [\pm n] \setminus (S\bar{S}). \end{cases}$$

A C-semigraphoid $\mathcal{G}$ on $[n]$ is a C-semimatroid if there is a bisubmodular function $f : \{S \sqsubseteq [\pm n]\} \to \mathbb{R}$ such that the equality is attended in the local C_n-bisubmodularity inequalities exactly at the triples $(ij|K) \in \mathcal{G}$.

Type D

Let

$$\tilde{\mathcal{D}}_n := \{(ij|K) : K \sqsubseteq [\pm n], |K| \leq n-2, \{i,j\} \sqsubseteq [\pm n] \backslash K\bar{K}, i \neq j\} \subseteq \mathcal{C}_n.$$

The set of D-CI-statements is

$$\mathcal{D}_n := \tilde{\mathcal{D}}_n / \sim$$

where $\sim$ is the equivalence relation in $\mathcal{D}_n$ defined by

$$\sim := \{((ij|K), (\overline{ij}|K)) \in \tilde{\mathcal{D}}_n \times \tilde{\mathcal{D}}_n : |K| = n-2\}.$$

By abusing of notations, we write an element of $\tilde{\mathcal{D}}_n$ for its class in $\mathcal{D}_n$. In other words, we identify $(ij|K)$ with $(\overline{ij}|K)$ for $|K| = n-2$.

Type D

A D-semigraphoid on $[n]$ is a subset $\mathcal{G} \subseteq \mathcal{D}_n$ satisfying

(DG1) $\{(ij|L),(ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L),(ij|kL)\} \subseteq \mathcal{G}$.

A function $f : \mathcal{R}_{D_n} \to \mathbb{R}$ is disubmodular if

$$\begin{cases} f(Sa)+f(Sb) \geq f(S)+f(Sab) & S \sqsubseteq [\pm n], |S| \leq n-4, ab \sqsubseteq [\pm n] \setminus S\bar{S}, \\ f(Sa)+f(Sb) \geq f(S)+g(Sabc)+g(Sab\bar{c}) & S \sqsubseteq [\pm n], |S| = n-3, abc \sqsubseteq [\pm n] \setminus S\bar{S}, \\ g(Sab)+g(S\bar{a}\bar{b}) \geq f(S) & S \sqsubseteq [\pm n], |S| = n-2, ab \sqsubseteq [\pm n] \setminus S\bar{S}, \end{cases}$$

where $f(S) := f(\mathbf{e}_S)$ and $g(S) := f(\frac{1}{2}\mathbf{e}_S)$.
A D-semigraphoid $\mathcal{G}$ is a D-semimatroid if there is a disubmodular function $f : \mathcal{R}_{D_n} \to \mathbb{R}$ such that the equality in the local D_n-submodularity inequalities is attained exactly at the elements of $\mathcal{G}$.

Thank you!

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