MAT3004 Lecture Notes: Alternating, Dihedral, and Product Groups

1 Alternating Group (A_n)

1.1 Even and Odd Permutations

Recall: For all $\sigma \in S_n$: If $\sigma = \tau_1 \cdots \tau_m$ (where τ_i are transpositions), then for m odd (or even), <u>all</u> other expressions of $\sigma = \tau_1' \cdots \tau_\ell'$ also have ℓ odd (or even).

Definition 1. $\sigma \in S_n$ is an <u>odd</u> (or <u>even</u>) permutation if there is one expression $\sigma = \tau_1 \cdots \tau_m$ such that m is odd (or even).

Remark 1.

- If σ is odd, τ is even, then $\sigma \cdot \tau$ is odd.
- If σ, τ are even $\implies \sigma\tau$ is even.

Definition 2. The Alternating Group A_n is defined as:

$$A_n := \{\sigma \in S_n \mid \sigma \text{ is even}\}$$

1.2 Properties of A_n

Proposition 1. *$A_n \leq S_n$ is a subgroup.*

Proof. Let $\sigma, \sigma' \in A_n$. We show that:

1. $\sigma \cdot \sigma' \in A_n$ (Closure)
2. $\sigma^{-1} \in A_n$ (Inverse)

(Standard subgroup proof follows from the parity properties of permutations). □

Proposition 2. $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$ *(for $n \geq 2$).*

Proof. Note that $S_n = A_n \sqcup \{\sigma \in S_n \mid \sigma \text{ is odd}\}$. i.e., $A_n \cap \{\text{odd perms}\} = \emptyset$.

Claim 1. $|A_n| = |\{\sigma \text{ odd}\}|$.

To see so, consider a map $f : A_n \to \{\sigma \text{ odd}\}$ defined by:

$$\sigma \mapsto f(\sigma) := \sigma\tau$$

where τ is a fixed transposition in S_n (e.g., (12)).

Note that f is bijective. The inverse is given by $f^{-1} = f$ (conceptually similar, mapping odd back to even).

$$(f \circ f)(\sigma) = (\sigma\tau)\tau = \sigma\tau^2 = \sigma e = \sigma$$

$$f^{-1} : \{\sigma \text{ odd}\} \to A_n$$
$$f^{-1}(\gamma) := \gamma\tau$$

Then $f^{-1} \circ f(\sigma) = \sigma\tau\tau = \sigma$. Since these are identity maps, the sets have the same cardinality. Therefore, $|A_n| = |\{\sigma \text{ odd}\}|$ and combining this with the disjoint union gives the desired result. $\square$

Remark 2. A_3 describes the rotational symmetries of an equilateral triangle.

$$S_3 \text{ vs } A_3 : \quad (123) \leftrightarrow \text{rotation by } 120^\circ$$

2 A_4 and Tetrahedrons

A_4: Rotational symmetries of a tetrahedron.

[Diagram: Tetrahedron with vertices 1, 2, 3, 4]

Rotations correspond to permutations:

- Rotation about a vertex axis: $\leftrightarrow (123)$
- Rotation about edge midpoints: $\leftrightarrow (12)(34)$

3 Dihedral Group D_n

Informally, D_n describes all symmetries (motion & reflection) of a regular n-gon (in $\mathbb{R}^2$).

3.1 Examples

- $n = 3$: Triangle. $120^\circ, 240^\circ, 0^\circ$. 3 rotations + 3 reflections.
- $n = 4$: Square. $90^\circ, 180^\circ, 270^\circ, 0^\circ$. 4 rotations + 4 reflections.
- $n = 5$: Pentagon. 5 rotations + 5 reflections.

To describe them mathematically, for a regular n-gon, there should be $n+n = 2n$ symmetries.

$$|D_n| = 2n$$

3.2 Generators: Rotation and Reflection

1. Rotation (r) Given by the cycle ($1\ 2\ \cdots\ n-1\ n$). This represents a rotation of $(\frac{360}{n})^\circ$.

$$r \leftrightarrow (1\ 2\ \cdots\ n)$$

Then all rotations are r^i for $0 \leq i < n$ (with $r^n = e$).

2. Reflection (s) Consider a reflection along an axis passing through vertex 1 (for odd n) or appropriate edges.

$$s \leftrightarrow (2\ n)(3\ n-1)\cdots$$

All reflections can be expressed as $r^i s$ (for $0 \leq i < n$) with $s^2 = e$.

Example 1 (D_4)**.** Apply $r^3 s$ to a square.

- Start with square labeled 1, 2, 3, 4.
- s reflects it.
- r^3 rotates it 270°.
- Result is the same as reflection along a different axis.

Definition 3. The Dihedral Group D_n is:

$$D_n = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2 s, \ldots, r^{n-1} s\}$$

3.3 Generators and Relations

The group satisfies the following properties:

1. $r^n = e$
2. $s^2 = e$
3. Interaction between r and s:

$$(r^i s)^2 = e \implies r^i s r^i s = e \implies r^i s r^i = s \implies s r^i s^{-1} = r^{-i}$$

Often written as $srs = r^{-1}$ (or $rs = sr^{-1} = sr^{n-1}$).

Alternatively, D_n can be expressed using "generators and relations":

$$D_n = \langle r, s \mid r^n = e, s^2 = e, srs = r^{-1} \rangle$$

4 Dihedral Group Calculation (D_7 Example)

Example 2 (Setup)**.** Simplify $r^{19}s^5r^{-4}s^{10}r$ in D_7.

- In D_7, $r^7 = e$ and $s^2 = e$.
- $r^{19} = r^{14} \cdot r^5 = r^5$.
- $s^5 = s^4 \cdot s = s$.
- $s^{10} = (s^2)^5 = e$.

Expression becomes: $r^5sr^{-4}r = r^5sr^{-3}$.

Example 3 (Detailed Step-by-Step)**.** We simplify expressions using the relations $r^7 = e$, $s^2 = e$, and $sr^i = r^{-i}s$.

Consider the expression: $r^{19}s^5r^{-4}s^{10}r$.

$$
\begin{aligned}
&= r^{14} \cdot r^5 \cdot s^4 \cdot s \cdot r^{-4} \cdot (s^2)^5 \cdot r \\
&= e \cdot r^5 \cdot e \cdot s \cdot r^{-4} \cdot e \cdot r \\
&= r^5sr^{-4}r \\
&= r^5sr^{-3} \\
&= r^5(r^3s) \quad \text{(using } sr^{-3} = r^3s \text{ because } sr^k = r^{-k}s\text{)} \\
&= r^8s \\
&= rs \quad \text{(since } r^7 = e\text{)}
\end{aligned}
$$

5 Product Group

Definition 4. Let $G_1, G_2, \ldots, G_n$ be groups. The **product group** is defined as:

$$G_1 \times G_2 \times \cdots \times G_n := \{(g_1, g_2, \ldots, g_n) \mid g_i \in G_i\}$$

with the operation defined component-wise:

$$(g_1, \ldots, g_n) \cdot (h_1, \ldots, h_n) = (g_1h_1, \ldots, g_nh_n)$$

The identity element is $e = (e_{G_1}, \ldots, e_{G_n})$.
The inverse is $(g_1, \ldots, g_n)^{-1} = (g_1^{-1}, \ldots, g_n^{-1})$.

Remark 3.

1. Order: $|G_1 \times \cdots \times G_n| = |G_1| \times \cdots \times |G_n|$.
2. If $G_1, \ldots, G_n$ are abelian, then $G_1 \times \cdots \times G_n$ is abelian.
3. If $G_1, \ldots, G_n$ are cyclic, it does **NOT** imply $G_1 \times \cdots \times G_n$ is cyclic. (Recall this is not true in general).

Example 4. Consider $\mathbb{Z}_2 \times \mathbb{Z}_5 = \{([a], [b]) \mid 0 \leq a \leq 1, 0 \leq b \leq 4\}$. Calculation example:

$$([1], [3]) + ([0], [4]) = ([1 + 0], [3 + 4]) = ([1], [2])$$

Exercise 1. Show that $\mathbb{Z}_2 \times \mathbb{Z}_5$ is cyclic.

6 Group Homomorphisms

6.1 Idea / Motivation

Recall in D_4: Consider the symmetries of a square labeled 1, 2, 3, 4.

- Rotation r corresponds to the permutation $(1\ 2\ 3\ 4) \in S_4$.
- Reflection s (across the diagonal 1-3) corresponds to $(2\ 4) \in S_4$.

We can form a "dictionary" (mapping) $\phi : D_4 \to S_4$:

$$\phi(r) = (1\ 2\ 3\ 4)$$

$$\phi(s) = (2\ 4)$$

Let's check if this mapping respects the operations. Consider the element $r^3 s$ in D_4. In D_4, we know relations like $srs = r^{-1}$ etc. Let's see the image in S_4:

$$\begin{aligned}\phi(r^3 s) &= \phi(r)^3 \cdot \phi(s) \\ &= (1\ 2\ 3\ 4)^3 \cdot (2\ 4) \\ &= (1\ 4\ 3\ 2) \cdot (2\ 4) \\ &= (1\ 4)(2\ 3)\end{aligned}$$

This result matches the geometric transformation corresponding to $r^3 s$. The multiplication structure of D_4 "determines" the multiplication structure of the image in S_4.

6.2 Definition

Definition 5. Let $(G, \cdot)$ and $(H, *)$ be groups. A function

$$\phi : G \to H$$

is a **homomorphism** if

$$\phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$$

for all $g_1, g_2 \in G$.

Note: The operation on the LHS is in G, and the operation on the RHS is in H.