Impacts of traffic congestion on resident inter-and intra-city choices of housing, workplace and commuting: A Social optimal model

Abstract

Owing to the rapid urbanization, traffic congestion deteriorates and affects the selection of employment and residence and commuting behaviors of urban residents. This study examines the impact of traffic congestion in the cities on people's choices of housing, workplace and commuting within a two-city system. Congestion coefficients the two cities were proposed. Instead of microscopic road resistance function, we proposed from the macroscopic perspective that the congestion cost of any point in the city is proportional to its distance to the downtown and the population of this city. In a two-city system, some residents living near the HSR Station of the satellite city will be attracted to work in the metropolis to increase the overall welfare of the urban system, while the congestion effect indicates that the over-congestion in the metropolis would reduce that attraction. This social welfare is dependent on salary, commuting cost and agricultural rent. According to the essence of social optimum conditions, non-convex planning was developed and solved to clarify optimized employment/residence distribution and commuting population in the two-city system. Based on the proposed model, the gap between the current status of the two-city system and its optimized status can be evaluated and the key factors limiting further improvement of social welfare can be identified. This study provides references for development and modification of relevant policies. Specifically, traffic congestion is a significant negative externality and a key factor limiting further improvement of social welfare as it accounts for 14% of the welfare loss of the two-city system.

Keywords: traffic congestion; two-city system; employment/residence distribution; social optimum; nonlinear planning

1. Introduction

Recently, urban traffic congestion has become a key factor hindering national economic development and functioning of cities. Meanwhile, it causes various issues in environment, energy and society. According to statistics by the Ministry of Transport of China, the economic loss caused by traffic congestion in China reached 250 billion yuan in 2014, which equals to 20% of disposable income of urban population or 5%~8% of annual GDP of China. The crude oil consumed by the ‘stop-and-go’ traffic reaches 20% of total crude oil consumption in China, resulting in huge energy consumption and severe exhaust emission. Additionally, ‘stop-and-go’ traffic causes severe time delay, thus increasing the traffic cost of residents, resulting in degraded traffic experience of residents. Unfortunately, this has been continuously exacerbated owing to the rapidly increasing vehicle ownership.

Owing to accelerated flow of social and economic factors, the connections of cities have been continuously strengthened and traffic demand is developing towards urban agglomeration and metropolitan area. For instance, great intercity travel has been observed in the Yangtze River Delta Area. According to 2020 Annual Report of Intercity Commuting in Yangtze River Delta issued by Tongji University, intercity commuting between Shanghai and Nantong, Suzhou, Jiaxing and Wuxi were estimated to be over 72,000 person-time in 2019, which is 26% higher than that in 2018 [2]. Residents start to thoroughly evaluate the influences of traffic congestion, housing price and living environment when making such decisions. Those living in satellite cities around megacities often stuck themselves in bumper-to-bumper traffic, despite low housing prices. There is no doubt that the drastically increasing intercity commuting is directly related to the extremely high housing price in metropolis. Owing to high housing price in metropolis, some people choose to live in adjacent cities. As a result, they are exposed to daily intercity commuting. Hence, those living in satellite cities around megacities often stuck themselves in bumper-to-bumper traffic, despite low housing prices. For instance, the average commuting time of residents in the Silicon Valley increased by 20% (43 hours per year) from 2007 to 2017. Indeed, commuting time of some intercity commuters can be up to 6 h per day [3]. In China, commuters who live in Yanjiao but work in downtown of Beijing typically spend 3-4 h per day on commuting [4]. Residents have to endure high housing price if they want to reduce commuting time. Residents start to thoroughly evaluate the influences of traffic congestion, housing price and living environment when making such decisions.

To explore employment/residence and intercity travel in metropolitan area, we investigated the effects of traffic congestion on employment/residence and intercity travel. This study aims to reveal the optimal mechanism of solving congestion based on the commuting model. In terms of residence choice, Zhou *et al.* [5] defined personal commuting cost as part of the producer's remaining cost in the social optimum model based on distance and time value. However, the impacts of external costs (e.g., traffic congestion) on social welfare was not considered in this study. Studies of economy of single city are typically based on the single-center city model [6-8] and this model has been improved and extended [9-11]. These single-center models focus on the spatial location of an isolated city and its commuting cost and housing price. Hence, these models can reflect the basic characteristics of urban structures. In some studies, single-center city model of traffic congestion was introduced. For instance, Xu *et al.* [12] (2017) focused on a bottleneck section of the highway connecting two highly concentrated areas for analysis of traffic congestion. However, practical cities cannot be simplified as two highly concentrated areas. Instead, the influences of traffic congestion at different locations of cities may vary. Additionally, traffic expenses may also be generated in the region. To quantify the cost of traffic congestion, Maddison [13] established a model to estimate time loss caused by traffic congestion. Nevertheless, this model requires homogeneity of all traffic participants, which means that all travelers and vehicles are identical and their unit travel cost, operating cost and the unit time value are consistent. TTI [14] estimated external costs of passenger vehicles and trucks using the model for estimation of external cost of traffic congestion. Herein, external cost of passenger vehicles includes time delay cost of passengers and additional fuel cost. The total cost of traffic congestion on the road is the sum of external costs of passenger vehicles and trucks. Zhu [15] established a microeconomic simulation model of traffic congestion using cellular automata and obtained the microeconomic cost. However, this study considered the congestion cost on the road section only, while a quantitative model of traffic congestion at the road network level was not established. QUINET [16] estimated various external costs, including emissions, under different scenarios using meta-analysis. However, it is a universal quantitative model instead of a specific one for congestion. Bigazzi *et al.* [17] investigated external costs of highway caused by road pollution and congestion, especially those caused by road pollution.

For studies of multiple cities, Henderson [18] introduced the single-center city model into an urban system to investigate the sizes of cities in social optimum. In previous studies, spatial models of two-city economy system are usually based on the assumption that traffic congestion is homogeneously distributed in the cities. For instance, Ren *et al.* [19] proposed a social optimum model for two-city economy system. It consists of two two-dimensional single-center cities. This model considered the difference in agricultural rents of different cities and the fact that many high-speed rail (HSR) stations are located in suburban areas. In this way, the properties of the system at the optimal state were analyzed. However, this study provides no practical solution as the cost induced by traffic congestion is assumed to be constant. Specifically, case analysis of social optimum does not reflect the actual situation. In this study, the effects of congestion on population distributions and commuting populations of the two cities were investigated for the first time based on two-city model. Herein, congestion was involved in the intercity commuting model accordingly. Travel cost per resident is denoted as a function of population and commuting population and used as an optimization variable of the model to reflect traffic congestion caused by population agglomeration. In numerical experiments, the optimal population in the commuter area of Shanghai and Jiaxing was investigated with linear function as an example.

2. Model

In this study, an economy system composed of two cities (City 1 and City 2, see Fig. 1) were established and the basic characteristics of the two cities were reflected by a single-center model [6-11]. Herein, locations of downtowns of City 1 and City 2 were fixed and urban areas are dependent on the distribution of the population. Meanwhile, the two cities are connected by HSR and both HSR stations are located in suburban areas [20]. Residents in both cities need to work and both working places are available to them. Two options are available for employment location. The first option is the downtown of the respective city and the second option is the downtown of the other city. Therefore, residents in the two cities would be exposed to commuting expenses in different forms. Additionally, time delay caused by traffic congestion is considered as a large portion of commuting expenses.

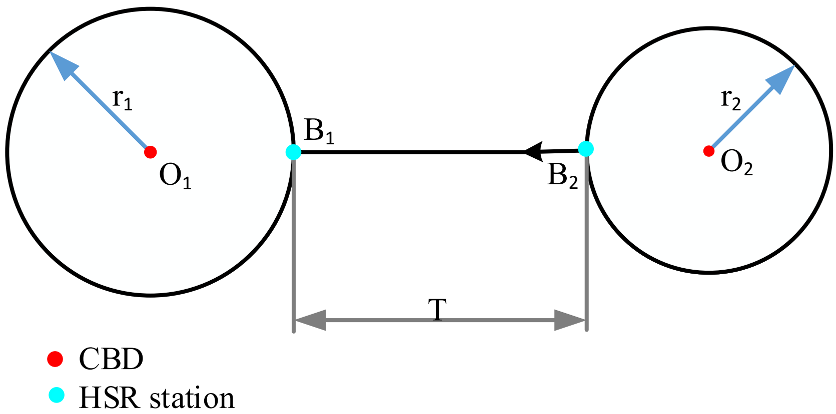


Fig. 1 Two-city system

In this study, the corresponding model was established based on six hypotheses.

Hypothesis 1: All cities in the system are circular and single-centered [21]. In this study, r_i denotes the radius of City i. Inside the city, all road networks covering the city are circular and radial [22-23].

Hypothesis 2: There are differences in resource endowment and infrastructure between the two cities [24] and these differences are reflected by the total factor productivity. A_i denotes the total factor productivity of City i. $A_1 > A_2$. City 1 and City 2 are called metropolis and satellite city, respectively [7].

Hypothesis 3: All production activities take place in a production agglomeration area that does not occupy land in downtown. All enterprises are in a completely competitive market and labor force is the sole production factor. Meanwhile, enterprises are not held by the residents in the system, meaning that the enterprise profits are not regarded as the income of the two-city system [25].

Hypothesis 4: Production output is proportional to input and the growth rate of output is greater than that of input. $P_i(W_i)$ denotes the overall output of City i and W_i denotes the number of employees in City i. It is assumed that $P_i = A_i W_i^{1+\gamma} (\gamma > 0)$. Housing consumption by all residents is set as 1. Additionally, the land rent at boundary of City i is determined by the agricultural rent R_i and $R_1 > R_2$.

Hypothesis 5: The system has a definite and constant population (N) and all people are homogeneous and evenly distributed in the city. For simplicity, the population density is assumed to be "1 /km^2" [26-27]. Residents can choose to work in their respective city or the other city. Residents who live and work in the same city and those who live and work in different cities are denoted as intracity and intercity commuters, respectively.

Hypothesis 6: People are rational, that is, people always choose the path with minimum cost. Herein, the cost includes the commuting expense of free flow and additional cost caused by congestion.

Based on hypotheses mentioned above, intracity commuters travel directly from residence to downtown; intercity commuters shall travel to the HSR station of the living city at minimum expense and then to the other city by train. Hence, traffic volumes of different roads in the city may vary. Herein, a macroscopic model of traffic congestion was established. Local residents travel to the downtown along different central angles, resulting in uniform traffic congestion in different directions of central angles in the city; intercity commuters also generate a congestion effect in the sector with the axis of downtown - HSR station. In this study, it is assumed that the congestion cost of this city can be obtained by linear superposition of the two congestion effects. In the proposed model, the congestion level decreases with the increase of the distance from the downtown. The commuting cost consists of intercity and intracity commuting costs. Since HSR pricing and speed are not likely to fluctuate drastically in practical, it is assumed that intercity commuting cost is a definite constant denoted as T. For intracity commuting costs,

h_i denotes the unit travel cost of a single person in City i $(0 \le h_i < +\infty)$. In order to investigate the effects of traffic congestion on intracity and intercity commuting, the congestion effect will be thoroughly discussed in the following section.

3. Congestion effect

The intracity commuting cost (h_i) can be divided into two parts $(h_i = t_i + f_i)$. Herein, t_i refers to unit distance cost per capita in City i under free flow; f_i refers to agglomeration-induced unit congestion cost per capita in City i. The units of both parameters are 10,000 yuan/(capita×meter).

$$f_i = f_{ia} + f_{ib}$$

where f_{ia} refers to the congestion cost induced by intracity commuters. As mentioned above, f_{ia} increases as the distance from downtown decreases; f_{ib} refers to the congestion cost induced by intercity commuters and f_{ib} induces additional congestion cost in the zone affected by the HSR station only. As shown in Fig. 2, $\Phi \ge 0$ reflects the range of additional congestion cost induced by intercity commuters. Specifically, intercity commuters travel to the downtown via a radial path within Sector $O_iS_iT_i$ upon arrival to the HSR station. Especially, $\Phi = 0$ suggests that all intercity commuters select roads that are directly connected to downtown and HSR station. Hence, the value of Φ is related to the traffic network near the HSR station of City i and generally accepted detour distances.

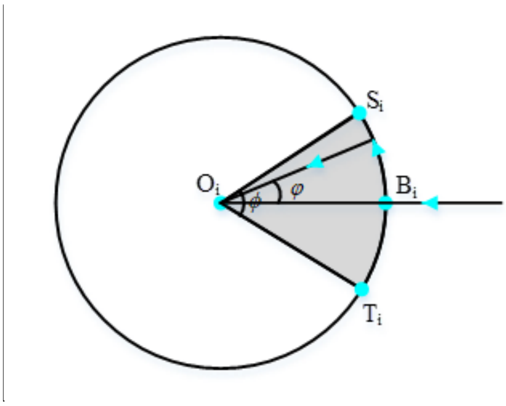


Fig. 2 Scope of influence of intercity commuters on commuting near the HSR station

With downtown as the pole and the HSR station direction as the positive direction, a polar coordinate system is established. For any point (r, θ) in City i, f_{ia} is a function of r and N_i (or r_i, as indicated in Hypothesis 5), namely $f_{ia} = f_{ia}(r_i, r)$. Without losing generality, it can be considered that $f_{ia}(r_i, r_i) = 0$; N_{ji} refers to the population who live in City j but work in City i; within Φ, f_{ib} is a function of N_{ji}, namely $f_{ib} = f_{ib}(N_{ji})$. Herein, φ indicates the angle at which the commuters enter the downtown.

f_{ib} is within the range of Φ and can be expressed as

$$f_{ib} = \begin{cases} f_{ib}(N_{ji}) & -\frac{\Phi}{2} \leq \varphi \leq \frac{\Phi}{2} \\ 0 & Other \end{cases} \tag{1}$$

4. Establishment of social optimum model

Social optimum refers to the case where total welfare of the system is maximized. In social optimum, the central government has the absolute power to determine the residential populations of all cities in the urban agglomeration and the intercity commuting populations, thus maximizing social welfare. Hence, social welfare = aggregate output - (aggregate commuting cost + aggregate agricultural rent). Consider the population endogenesis case, that is, the residential populations of the two cities are employed as the optimization variables in the model.

Theorem 1: In case of population endogenesis, residents in City 1 do not commute to City 2 in social optimum.

Proof. see Appendix A.

Theorem 1 demonstrated that residents in City 1 will not commute to City 2, suggesting that $N_{12} = 0$ and $N_{21} \geq 0$. Residents in City 1 always prefer radial paths as the way to downtown. Therefore, social optimum can be expressed as:

$$\max_{N_1, N_2, N_{21}} SW = P_1 + P_2 - (ACC_1 + ACC_2 + N_{21}T) - (R_1N_1 + R_2N_2) \tag{2}$$

$s.t.$

$$N_1 + N_2 = N \tag{2.1}$$

$$N_{21} \leq N_2 \tag{2.2}$$

$$N_1 \geq 0, N_2 \geq 0, N_{21} \geq 0 \tag{2.3}$$

where SW refers to social welfare, ACC_1 and ACC_2 refer to commuting costs in the city induced by intracity and intercity commuters, respectively. Herein, objective function (2) includes aggregate output, aggregate commuting cost and aggregate agricultural rent. The decision variables include population of City 1 (N_1), population of City 2 (N_2) and quantity of intercity commuting (N_{21}). Constraint (2.1) indicates that total population of the system is determined and remains constant, Constraint (2.2) indicates that the quantity of intercity commuting does not exceed the population of City 2 and Constraint (2.3) is a non-negative constraint.

In order to solve the planning problem mentioned above, the three parts of objective function shall be expressed as functions of decision variables.

For the aggregate output,

$$P_1 = A_1(N_1 + N_{21})^{1+\gamma}$$
$$P_2 = A_2(N_1 - N_{21})^{1+\gamma} \tag{3}$$

The house price (including rent) is transferred between subjects in the two-city system and aggregate agricultural rent reflects the cost of land reclamation by the central government. Since housing consumptions by all residents are 1, the aggregate agricultural rent is $R_1N_1 + R_2N_2$ (sum of the products of population and agricultural rent of the two cities).

For the aggregate commuting cost, ACC_1 and ACC_2 shall be denoted as functions of decision variables. Herein, ACC_1 is generated by both residents in City 1 and intercity commuters and ACC_2 includes the commuting costs to the HSR station and to the downtown of City 2. In order to reflect ACC_1 and ACC_2, identity and travel path of residents in City 2 must be solved. Identity refers to the role of intracity commuter or intercity commuter, while travel path refers to the selection of travel path ("circumferential + radial" or "radial + radial") to the HSR station by intercity commuters. As shown in Fig. 3, residents at Points M and Q travel to the HSR station via "circumferential + radial" and "radial + radial" paths, respectively. In both cases, congestion has key influences on identity and travel path. If City 1 is over-crowded, it is possible that negligible or no intercity commuters from City 2 are observed; if City 2, especially its downtown, is over-crowded, it is possible that negligible or no commuters select the "radial +radial" path.

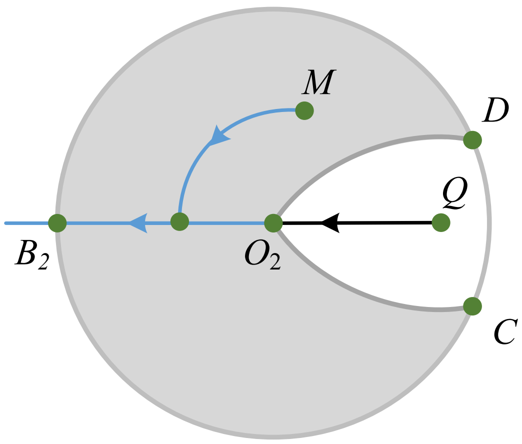


Fig. 3 Different paths to HSR Station

First, the traveling paths of intercity commuters in City 2 are determined. Indeed, selection of "circumferential + radial" or "radial + radial" paths traveling to HSR station is dependent on the starting point of a commuter. If the commuting cost of "circumferential + radial" is lower than that of "radial + radial", the area consisting of the starting points of corresponding commuters is denoted as Area L.

Otherwise, the area is denoted as Area S. Fig. 4 shows these two areas. Herein, B_2 refers to the HSR station in City 2, Area L refers to the gray area corresponding to major Arc C_2D_2. The boundaries of these two areas are Curves C_2O_2 and D_2O_2. Notably, Areas L and S are only present in cities with intercity commuters, meaning that Areas L and S are not present in City 1 under population endogenesis. Then, the specific forms of Curves C_2O_2 and D_2O_2 in City 2 are deduced. As shown in Fig. 4, only points on Curve D_2O_2 in City 2 shall be considered owing to the symmetry. Its polar coordinates are set as $(r, \bar{\theta})$ and the costs traveling to the HSR station via "circumferential + radial" and "radial + radial" paths are equivalent. Hence,

$$h_2\bar{\theta}r + \int_r^{r_2} h_2 dr = \int_0^r h_2 dr + \int_0^{r_2} h_2 dr \tag{4}$$

where $h_2\bar{\theta}r$ refers to the commuting cost traveling to Point Y via circumferential paths; $\int_r^{r_2} h_2 dr$ refers to the commuting cost traveling from Point Y to Point B_2; $\int_0^r h_2 dr$ refers to the commuting cost traveling to Point O_2 via radial paths; $\int_0^{r_2} h_2 dr$ refers to the commuting cost traveling from Point O_2 to Point B_2.

$$\bar{\theta}_2(r) = \frac{2\int_0^r h_2 dr}{h_2 r} \tag{5}$$

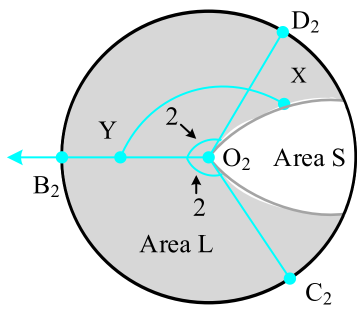


Fig. 4 Geographic division of City 2

The conclusions about equations of boundary curves of Areas L and S are as follows:

Conclusion 1: if $r = 0$, $\bar{\theta}_2$ is minimized and $\bar{\theta}_2^{min} = 2$.

Proof. See Appendix A.

Inference 1: both Areas L and S shall be present in City 2 regardless of the congestion situation.

Proof. See Appendix A.

Social optima can be divided as Type-I and Type-II social optimum according to Inference 1. In Type-I social optimum, all intercity commuters are from Area L in City 2; in Type-II social optimum, intercity commuters are from Areas L and S in City 2.

Then, identities of residents (i.e., spatial distribution of intercity commuters) in City 2 are determined based on Theorem 2 to determine expressions of ACC_1 and ACC_2. The distribution areas of intercity commuters should be continuous and symmetrical about B_2O_2, while residents outside this area work locally.

Theorem 2: In social optimum, the differences of practical commuting cost and opportunity commuting cost of all points on the boundary of intercity commuting area AC_1 are consistent.

Proof. See Appendix A.

The following inference can be obtained based on Theorem 2:

Inference 2: If two residents with different identities have equal differences between practical costs and opportunity commuting costs, switching of their identities do not affect the overall commuting cost.

Theorem 2 and Inference 2 may be employed to deduce the boundary of intercity commuters. Owing to symmetry, only a random point E (see Fig. 5), whose polar coordinates are $(r, \tilde{\theta}_2)$, on Boundary AC_1, shall be considered. In this way, equations for intercity commuting boundaries under population endogenesis in Type-I and Type-II social optima can be deduced.

4.1 Type-I social optimum

In order to deduce the equation of boundary of intercity commuting area in Type-I social optimum, the intersection point of boundary curve and B_2O_2 in intercity commuting area (i.e., Point A in Fig. 5) is described using η. Herein, $\eta \in [0,1], |O_2A| = \eta r_2$, η reflects the relative position of Point A to the downtown. Nevertheless, η would be employed as optimization variable in model optimization in the following section and η reflects the attractiveness of City 1 to City 2. Specifically, the attractiveness of City 1 to residents in City 2 increases as η decreases.

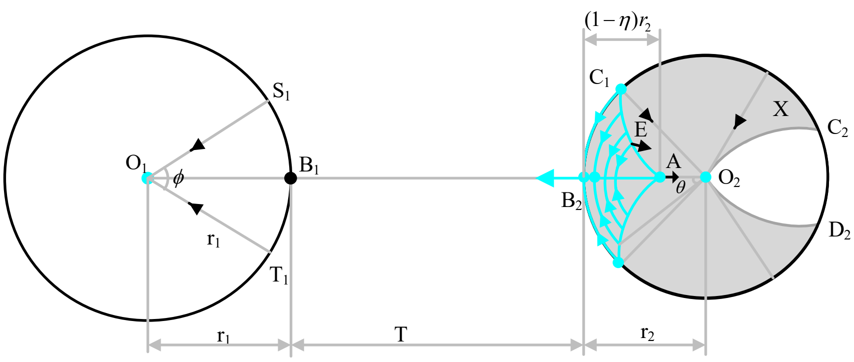


Fig. 5 Intercity commuting area in Type-I social optimum

According to Theorem 2, differences of commuting costs of Points E and A are equivalent to each other. At Type-I social optimum, all intercity commuters are from Area L:

$$h_2\tilde{\theta}r + \int_r^{r_2} h_2 dr + T + \int_0^{r_1} h_1 dr - \int_0^{r} h_2 dr = \int_{\eta r_2}^{r_2} h_2 dr + T + \int_0^{r_1} h_1 dr - \int_0^{\eta r_2} h_2 dr \tag{6}$$

Terms on the left side of the equation reflect the difference between practical and opportunity commuting costs of Point E, while terms on the right side of the equation reflect the difference between practical and opportunity commuting costs of Point A.

Further simplification leads to:

$$\tilde{\theta}_2(r) = \frac{2\int_{\eta r_2}^{r} h_2\, dr}{h_2 r} \tag{7}$$

If $\eta > 0$, $\tilde{\theta}_2$ is minimized at $r = \eta r_2$ and $\tilde{\theta}_2^{min} = 0$.

In this way, expression of boundary curves of intercity commuters in the distribution area of City 2 is obtained. N_{21} can be easily obtained based on boundary equation and N_{21} will be uniquely identified by η in this case. Hence, N_{21} can be substituted by η as decision variable. The area of intercity commuters in Area L increases as η decreases. Hence, η denotes the attractiveness of City 1 on City 2.

The boundary curve equation of intercity commuters has a determining effect on the identity (intercity or intracity commuters) of residents in City 2, while the boundary curve equation of Areas L and S ($\bar{\theta}_2$) has a determining effect on intercity commuters' selection of the path to HSR station. The correlation of $\bar{\theta}_2$ and $\tilde{\theta}_2$ reflects identity and path selection of residents in City 2.

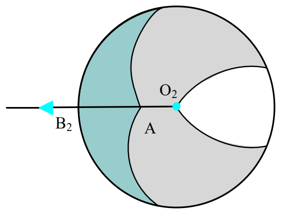
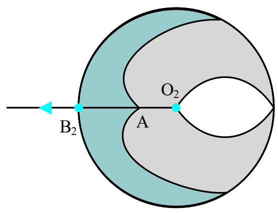
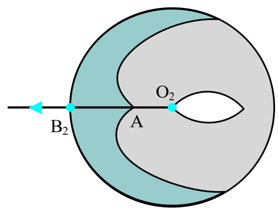
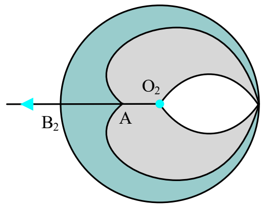
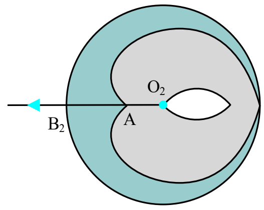
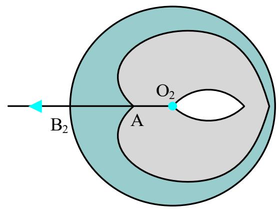
Conclusion 2: At Type-I social optimum, regional boundary curves include boundary curves of commuters, namely $\tilde{\theta}_2 \leq \bar{\theta}_2$.

Conclusion 3: If Points A and O_2 coincide (see Fig. 5), regional boundary curves coincide with boundary curves of commuters. In other words, $\tilde{\theta}_2 \leq \bar{\theta}_2$ if $\eta = 0$.

Since maximum angles of the two boundary curves ($\bar{\theta}_2^{max}$ and $\tilde{\theta}_2^{max}$) cannot be determined, Type-I social optimum may be present in various patterns (see Table 1) and this is directly related to the traffic congestion level. As shown in Table 1, blue area denotes intercity commuters, while grey and white areas denote intracity commuters. If forced to commute to City 1, residents in grey and white areas would travel to HSR station via the "circumferential + radial" path and the "radial + radial" path, respectively.

Table 1. Possible cases of boundary curves

$\bar{\theta}_2^{max}$ \ $\tilde{\theta}_2^{max}$	$\bar{\theta}_2^{max} < \pi$	$\bar{\theta}_2^{max} = \pi$	$\bar{\theta}_2^{max} > \pi$

$\tilde{\theta}_2^{max} < \pi$			
$\tilde{\theta}_2^{max} = \pi$			
$\tilde{\theta}_2^{max} > \pi$			

According to Inference 2 and Table 1, $\tilde{\theta}_2^{max} > \pi$ is possible if $\bar{\theta}_2^{max} > \pi$, which interferes determination of N_{21} and ACC_2. For consistent expression, it is assumed that $\tilde{\theta}_2 = \tilde{\theta}_2^{max}$ when $r = \widetilde{r_2}$. Based on that, $\widetilde{r_2}$ and $\tilde{\theta}_2^{max}$ are re-defined: $\widetilde{r_2} = min\{r_2, \widetilde{r_2}\}, \tilde{\theta}_2^{max} = min\{\tilde{\theta}_2^{max}, \pi\}$. In this case, all inferences mentioned above are still valid. Additionally, the expressions of N_{21} and ACC_2 using N_1, N_2 and η were deduced. N_{21}, which equals to the area of intercity commuting zone, can be obtained by double integral:

$$N_{21} = 2\int_{\eta r_2}^{\widetilde{r_2}} \int_0^{\tilde{\theta}_2(r)} r dr d\theta + \pi(r_2^2 - \widetilde{r_2}^2) \tag{8}$$

ACC_2 consists of the costs of intercity and intracity commuters:

$$\begin{aligned} ACC_2 = 2\Bigg[\int_{\eta r_2}^{\widetilde{r_2}} r dr \int_0^{\tilde{\theta}(r)} \left(h_2\theta r + \int_r^{r_2} h_2 dr\right) d\theta + \int_{\widetilde{r_2}}^{r_2} r dr \int_0^{\pi} \left(h_2\theta r + \int_r^{r_2} h_2\right) d\theta \\ + \int_0^{\widetilde{r_2}} \left(\pi r \int_0^r h_2\, dr\right) dr - \int_{\eta r_2}^{\widetilde{r_2}} \left(\tilde{\theta} r \int_0^r h_2\, dr\right) dr\Bigg] \end{aligned} \tag{9}$$

Herein, the first part refers to the cost of intercity commuters traveling to the HSR station via "circumferential + radial" paths, while the second part refers to intracity commuters traveling to the downtown via "radial + radial" paths.

Conclusion 4: $\tilde{\theta}_2^{max} > \pi$ is not possible when $\eta \to 1$ is presented.

Proof. See Appendix A.

4.2 Type-II social optimum

Before deduction of equation of boundary curve of intercity commuting area in Type-II social optimum, the correlation of the two social optima is clarified based on Theorem 3.

Theorem 3: In social optimum, residents in Area S shall not commute to City 1 for work unless all residents in Area L are required to commute to City 1 for work.

Proof. See Appendix A.

Theorem 3 indicates that residents in Area S shall commute to City 1 for work only if $\eta = 0$.

In Type-II social optimum, intercity commuters are from Areas L and S, while all residents in Area L are intercity commuters. Similar to the case of Type-I social optimum, the distribution area of intercity computers in Area S is determined first. Indeed, the difference of practical and opportunity commuting costs of a random Point (r, θ) in Area S ($\int_0^r h_2\,dr + \int_0^{r_2} h_2\,dr + T + \int_0^{r_1} h_1\,dr - \int_0^r h_2\,dr = \int_0^{r_2} h_2\,dr + T + \int_0^{r_1} h_1\,dr$) is a constant. According to Inference 2, the aggregate commuting cost is independent from the residence of intercity commuters, suggesting that the residence of intercity commuters can be arranged randomly if the overall quantity of intercity commuters in Area S remains constant.

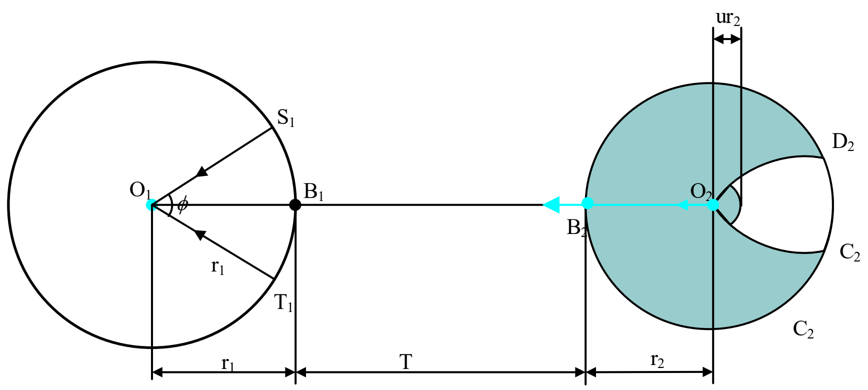


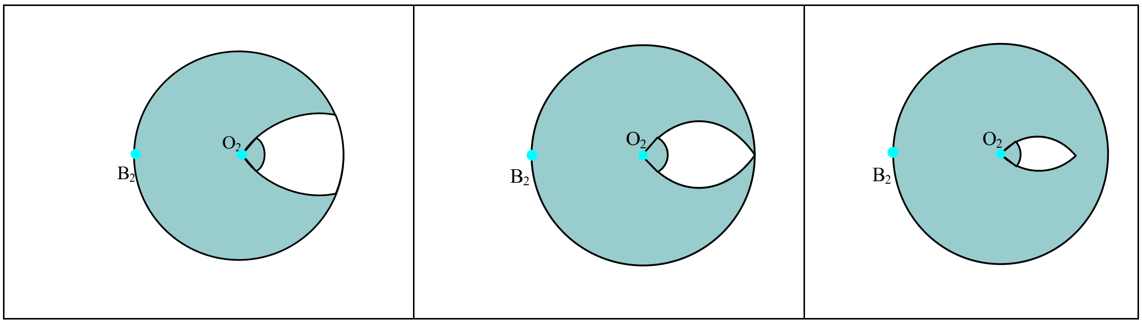
Fig. 6 Intercity commuting area in Type-II social optimum

As shown in Fig. 6, the intercity commuting area of Area S is reflected by a sector with radius of $\mu r_2 (0 \leq \mu \leq 1)$. Similar to η in Type-I social optimum, μ would be employed as optimization variable in model optimization, μ reflects the attractiveness of City 1 to City 2 in Type-II social optimum. Specifically, the attractiveness of City 1 to residents in City 2 increases as μ increases.

In Type-II social optimum, all residents in Area L are intercity commuters and the two boundaries coincide. Table 2 lists all possible cases of boundary curves in Type-II social optimum.

Table 2. Possible cases of boundary curves in Type-II social optimum

$\bar{\theta}_2^{max} < \pi$	$\bar{\theta}_2^{max} = \pi$	$\bar{\theta}_2^{max} > \pi$



Similar to the case of Type-I social optimum, N_{21} and ACC_2 in Type-II social optimum can be expressed as:

$$N_{21} = 2\left(\int_0^{\widetilde{r_2}}\int_0^{\tilde{\theta}(r)} r dr d\theta + \int_0^{\mu\widetilde{r_2}}(\pi - \tilde{\theta}) r dr\right) + \pi(r_2^2 - \widetilde{r_2}^2) \tag{10}$$

$$\begin{aligned} ACC_2 = 2(&\int_0^{\widetilde{r_2}} r dr \int_0^{\tilde{\theta}(r)}\left(h_2\theta r + \int_r^{r_2} h_2 dr\right) d\theta \\ &+ \int_0^{\mu\widetilde{r_2}}\left((\pi - \tilde{\theta}) r (\int_0^r h_2\, dr + \int_0^{r_2} h_2\, dr)\right) dr \\ &+ \int_{\mu\widetilde{r_2}}^{\widetilde{r_2}}\left((\pi - \tilde{\theta}) r \int_0^r h_2\, dr\right) dr + \int_{\widetilde{r_2}}^{r_2} r dr \int_0^{\pi}\left(h_2\theta r + \int_r^{r_2} h_2\right) d\theta) \end{aligned} \tag{11}$$

4.3 Social optimum model

In this section, ACC_1 is deduced. Residents in City 1 always travel to the downtown via radial paths, while intercity commuters tend to travel along circumferential paths of the boundary of City 1 before moving to the downtown via radial paths. We simply think that intercity commuters are evenly distributed on the boundary, so the average distance they detour is $\frac{\phi r_1}{4}$. With h_1^* denoting the commuting cost per unit distance of intercity commuters on the boundary of City 1, ACC_1 can be expressed as:

$$ACC_1 = \int_0^{r_1}(2\pi r \int_0^r h_1 dr)\, dr + N_{21}(\frac{\phi r_1}{4} h_1^* + \int_0^{r_1} h_1 dr) \tag{12}$$

where the first term denotes the commuting cost of residents in City 1 traveling to the downtown via radial paths and the second term denotes the commuting cost of intercity commuters traveling to the downtown via "circumferential + radial" paths.

N_{21} and ACC_2 can be expressed as:

$$N_{21} = 2\left(\int_{\eta r_2}^{\widetilde{r_2}} r dr \int_0^{\tilde{\theta}(r)} d\theta + \int_0^{\mu\widetilde{r_2}}(\pi - \tilde{\theta}) r dr\right) + \pi(r_2^2 - \widetilde{r_2}^2) \tag{13}$$

$$ACC_2 = 2(\int_{\eta r_2}^{\widetilde{r_2}} rdr \int_0^{\widetilde{\theta}(r)} \left(h_2\theta r + \int_r^{r_2} h_2 dr\right) d\theta + \int_0^{\eta r_2} \left(\pi r \int_0^r h_2\, dr\right) dr$$

$$+ \int_0^{\mu\widetilde{r_2}} \left((\pi - \widetilde{\theta}) r \left(\int_0^r h_2\, dr + \int_0^{r_2} h_2\, dr\right)\right) dr \tag{14}$$

$$+ \int_{\eta r_2 + \mu\widetilde{r_2}}^{\widetilde{r_2}} \left((\pi - \widetilde{\theta}) r \int_0^r h_2\, dr\right) dr + \int_{\widetilde{r_2}}^{r_2} rdr \int_0^{\pi} \left(h_2\theta r + \int_r^{r_2} h_2\right) d\theta)$$

According to Theorem 3, $\mu > 0$ is possible only if $\eta = 0$, otherwise $\mu = 0$. Hence, with 0-1 variable δ introduced, social optimum (4) can be equivalent to:

$$\max_{N_1,N_2,\eta,\mu,\delta} SW = P_1 + P_2 - (ACC_1 + ACC_2 + N_{21}T) - (R_1N_1 + R_2N_2) \tag{15}$$

$s.t.$

$$N_1 + N_2 = N \tag{15.1}$$

$$\eta\delta = 0 \tag{15.2}$$

$$N_1 \geq 0 \tag{15.3}$$

$$N_2 \geq 0 \tag{15.4}$$

$$0 \leq \mu \leq \delta \tag{15.5}$$

$$0 \leq \eta \leq 1 \tag{15.6}$$

$$\delta \in \{0, 1\} \tag{15.7}$$

As the proposed social optimum model is essentially nonlinear planning and the subject of this study is relatively small, three solving algorithms in MATLAB were employed and tested. Additionally, parameters in these algorithms were determined.

The three solving algorithms exhibit the following features:

1. $fmincon$ and default interior point algorithm in Matlab are used as the model solver and the initial values before iterations are selected by random simulation.

2. The genetic algorithm (GA) kit in Matlab is used as the model solver.

3. The advantage of $fmincon$ in local optimization and the advantage of GA in global searching are combined. In other words, iteration results by GA are used as input of $fmincon$ for local optimization and the results are used as input of next iteration cycle. This is denoted as $GA + f$.

4.4 Population exogenesis

Section 4.3 introduces the social optimum model under population endogenesis. Under population exogenesis induced by household registration (i.e., both N_1 and N_2 are constants), residents in City 1 may commute to City 2 in social optimum due to the congestion effect. This may be attributed to the fact that social welfare can be enhanced by residents in City 1 commuting to City 2 for work as congestion-induced additional costs are high if City 1 is over-crowded. As a result, Theorem 1 is not valid and the problem would evolve. Therefore, the following theorem shall be proved first:

Theorem 4: Simultaneous commuting between metropolis and satellite city is not possible.

Proof. See Appendix A.

Theorem 4 suggests that intercity commuting is either from City 2 to City 1 or from City 1 to City 2. Therefore, population exogenesis can be divided as two population endogenesis by the following method to determine the social optimum.

City A and City B constitute a two-city system and the total factor productivity of City A is higher than that of City B. Under population endogenesis, City A is denoted as City 1 and City B is denoted as City 2. Apparently, residents commute from City 2 to City 1. Under population exogenesis, City A and B are defined as City 1 and 2, respectively; N_1 and N_2 denoted populations of City A and B, respectively. the optimized solution (SW^1) can be obtained; by defining City B as City 1 and City A as City 2. The optimized solution (SW^2) can be obtained by solving the social optimum model mentioned in Section 4.3. Additionally, the larger of them is taken as the optimized solution under population exogenesis, namely $SW = \max\{SW^1, SW^2\}$.

In this way, a social optimum model of two-city system can be established and the corresponding solution method is given.

Additionally, the possibility of intercity commuting and that residents in City 2 commute to City 1 for work when N_1 and N_2 remain constant is investigated based on the proposed social optimum model.

The presence of intercity commuting guarantees that:

$$\lim_{N_{21}\to 0} \frac{\partial SW}{\partial N_{21}} > 0$$

Since $N_{21} \to 0$ is equivalent to $\eta \to 1$, it has been demonstrated that $\tilde{\theta}_2^{max} > \pi$ is not possible in this case. Hence,

$$\begin{aligned}\frac{\partial SW}{\partial N_{21}} &= (1+\gamma)(A_1(N_1+N_{21})^\gamma - A_2(N_2-N_{21})^\gamma) - \frac{\partial ACC_1}{\partial N_{21}} - \frac{\partial ACC_2}{\partial N_{21}} - T \\ &= (1+\gamma)(A_1(N_1+N_{21})^\gamma - A_2(N_2-N_{21})^\gamma) \\ &\quad - \int_0^{r_1}\left(\phi r \int_0^r \frac{\partial h_1}{\partial N_{21}} dr\right) dr - \frac{\phi r_1}{4} h_1^* - \int_0^{r_1} h_1 dr - N_{21}\int_0^{r_1} \frac{\partial h_1}{\partial N_{21}} dr \\ &\quad -2\left(\int_{\eta r_2}^{r_2} h_2\, dr - \frac{\int_{\eta r_2}^{r_2} \frac{\int_0^r h_2 dr}{h_2} dr}{\int_{\eta r_2}^{r_2} \frac{1}{h_2} dr}\right) - T\end{aligned} \tag{16}$$

It can be obtained from Eq (16):

$$\lim_{N_{21}\to 0}\frac{\partial SW}{\partial N_{21}} = (1+\gamma)(A_1{N_1}^{\gamma} - A_2{N_2}^{\gamma}) + 2\int_0^{r_2} h_2 - \int_0^{r_1}\left(\phi r\int_0^{r}\frac{\partial h_1}{\partial N_{21}}dr\right)dr - \int_0^{r_1}(t_1 + f_{1a})dr - \frac{t_1\phi r_1}{4} - T \tag{17}$$

As ϕ is typically small, In this case, the presence of intercity commuting leads to:

$$\lim_{N_{21}\to 0}\frac{\partial SW}{\partial N_{21}} = (1+\gamma)(A_1{N_1}^{\gamma} - A_2{N_2}^{\gamma}) + 2\int_0^{r_2} h_2 - T - \int_0^{r_1}(t_1 + f_{1a})dr > 0$$

Namely

$$T < (1+\gamma)(A_1{N_1}^{\gamma} - A_2{N_2}^{\gamma}) + 2\int_0^{r_2} h_2 - \int_0^{r_1}(t_1 + f_{1a})dr \tag{18}$$

When the population difference between City 1 and City 2 is huge, Eq (13) tend to be valid. Generally, intercity commuting will always be observed if the overall population of the system exceeds a critical level.

The case where all residents in City 2 commute to City 1 for work is equivalent to the case where $\eta = 0$ and $\mu = 1$. This situation is indeed Type-II social optimum. The population of Area L of City 2 is denoted as N_2^L, which can be determined by Eq (15). In this case, $N_{21} \in [N_2^L, N_2]$ and $\eta = 0, \mu = 1$ is equivalent to:

$$(\frac{\partial SW}{\partial N_{21}})_{min} \geq 0$$

Additionally, $\frac{\partial SW}{\partial N_{21}} \geq 0$ is always valid if $N_{21} = N_2^L$ as the situation is Type-II social optimum.

According to Eq (18):

$$\begin{aligned}\frac{\partial SW}{\partial N_{21}} &= (1+\gamma)(A_1(N_1+N_{21})^{\gamma} - A_2(N_2-N_{21})^{\gamma}) - \frac{\partial ACC_1}{\partial N_{21}} - \frac{\partial ACC_2}{\partial N_{21}} - T \\ &= (1+\gamma)(A_1(N_1+N_{21})^{\gamma} - A_2(N_2-N_{21})^{\gamma}) - \int_0^{r_1}\left(\phi r\int_0^{r}\frac{\partial h_1}{\partial N_{21}}dr\right)dr - \frac{\phi r_1}{4}h_1^* \\ &\quad - \int_0^{r_1} h_1 dr - N_{21}\int_0^{r_1}\frac{\partial h_1}{\partial N_{21}}dr - \int_0^{r_2} h_2\, dr - T\end{aligned} \tag{19}$$

If $-1 < \gamma < 0$, $\lim_{N_{21}\to N_2}\frac{\partial SW}{\partial N_{21}} = -\infty$ as suggested by Eq (19), demonstrate that $(\frac{\partial SW}{\partial N_{21}})_{min} < 0$. Therefore, not all residents in City 2 would choose to commute to City 1 for work.

If $\gamma > 0$, only cases where congestion function is constant (i.e., h_1, h_2 are constants) are investigated. According to Eq (19):

$$\frac{\partial SW}{\partial N_{21}} = (1+\gamma)(A_1(N_1+N_{21})^{\gamma} - A_2(N_2-N_{21})^{\gamma}) - h_1 r_1 - \frac{h_1\phi r_1}{4} - h_2 r_2 - T \tag{20}$$

Further evolution leads to:

$$\frac{\partial^2 SW}{\partial {N_{21}}^2} = \gamma(1+\gamma)(A_1(N_1+N_{21})^{\gamma-1} + A_2(N_2-N_{21})^{\gamma-1}) > 0 \tag{21}$$

According to Eq (21), $\frac{\partial SW}{\partial N_{21}}$ increases monotonically with N_{21}. Hence, $(\frac{\partial SW}{\partial N_{21}})_{min} \geq 0$, demonstrating that all residents in City 2 would commute to City 1 for work.

5. Empirical study

5.1 Target area

Shanghai and Jiaxing, both of which are in the Yangtze River Delta region, were selected as the object of empirical study. Shanghai is the core metropolitan of the Yangtze River Delta region. It is also a financial center, transportation center and technological innovation center globally. Jiaxing is a second-tier city adjacent to Shanghai. Before the operation of HSR, a train trip from Jiaxing to Shanghai took 1.5 h. After the operation of HSR, such trip takes 0.5 h. Therefore, various residents in Jiaxing commute to Shanghai for work, resulting in huge intercity commuting. Despite their large areas, considerable parts of both Shanghai and Jiaxing are still under-developed or undeveloped. In 2018, the GDP of central urban area of Shanghai is approximately 2.13 trillion, which is 2/3 of the total GDP of Shanghai, indicating that economic activities in Shanghai are concentrated in this region. Hence, central urban area of Shanghai and Jiaxing (red line in Fig. 7) were defined as urban boundaries of Shanghai and Jiaxing and all parameters were extracted from central urban area. Additionally, both Shanghai Hongqiao Station and Jiaxing South Station are located at the urban boundaries and multiple HSR shifts are available to satisfy commuting needs. The practical situation is highly consistent with the model.

Fig. 7 Downtowns of Shanghai and Jiaxing

5.2 Parameter calibration

Previous studies demonstrated that γ typically equals to 0.05 [18] and $\gamma = 0.05$ in this study. Other parameters were calibrated based on practical data. Table 1 lists data required for parameter calibration (mainly extracted from statistics issued by the government in 2018). Herein, million (¥1,000,000) is employed as the unit of currency-related parameters. Notably, the statistical data of 2018 are used to calibrate relevant parameters, while social welfare and resident utility are calculated in years. Hence, the resident income refers to annual income and the commuting cost refers to annual commuting cost.

Table 3 Definitions of input symbols

Input	Description	Unit	Value	Data source
$\bar{N}_i$	Population of City i		Shanghai: 12,436,800 Jiaxing: 1,288,700	Shanghai Bureau of Statistics; Jiaxing Bureau of Statistics
$\bar{N}_{21}$	Current intercity commuting population		3100	*Annual Report on Intercity Commuting in the Yangtze River Delta in 2019*[26]

s_i	Housing space per capita in City i	m^2	Shanghai: 37 Jiaxing: 41.26	Interface news; *2018 Statistical Bulletin of National Economic and Social Development of Jiaxing City*
v_i	Average vehicle travel speed in City i	km/h	Shanghai: 60 Jiaxing: 40	
λ_i	Money cost of commuting per unit km in City i	¥10,000	Shanghai: 0.00015 Jiaxing: 0.0001	Estimated according to taxi pricing.
tc	Time value of intercity commuting	Hour	0.65	Official website of China Railway 12306
mc	Money cost of intercity commuting	¥ 10,000	0.00385	Official website of China Railway 12306
GDP_i	GDP of City i	¥ 10,000	Shanghai: 213,388,800 Jiaxing: 12,464,200	Shanghai Bureau of Statistics; Jiaxing Bureau of Statistics
tw	Total annual working days	Day	232	Notice of the General Office of the State Council on the Arrangement of Some Holidays in 2018

The total output of City i can be measured by its GDP (GDP_i) and population $\bar{N}_i$:

$$A_i = \frac{GDP_i}{\bar{N}_i^{1+\gamma}}.$$

In this study, A_1 and A_2 were 7.58 and 4.79, respectively.

In 2018, average annual incomes of residents in Shanghai and Jiaxing were 140270 and 89311 yuan, respectively. With 10 working hours per day, the average hourly wage of residents in Shanghai was about 74 yuan. Based on that, the time value of residents involved is set to be 74 yuan/h (denoted as vot, vot = 74 yuan/h). The intercity commuting cost per year can be determined using working days per year (tw), time value (vot) and monetary cost of intercity commuting per trip (mc):

$$T = tw \times (2 \times mc + 2 \times vot \times tc) = 40182 \text{ yuan}$$

The annual commuting cost under intracity free flow can be determined using tw, vot, monetary cost per km of City i (λ_i), housing space per capita (s_i) and vehicle speed under free flow (v_i):

$$t_i = tw \times (2 \times \lambda_i \times \frac{\sqrt{s_i}}{1000} + 2 \times vot \times \frac{\sqrt{s_i}}{1000 v_i})$$

$$t_1 = 7.7\ yuan$$

$$t_2 = 8.5\ yuan$$

With agricultural rent R_i substituted by the average housing price of suburban area of City i, R_1 and R_2 are 75200 and 23100 yuan, respectively [19].

5.3 Determination of congestion cost

As discussed above, congestion costs of City 1 and City 2 can be expressed as:

$$h_1 = t_1 + f_{1a} + f_{1b}, h_2 = t_2 + f_{2a}$$

Then, f_{1a}, f_{2a} and f_{1b} are determined.

5.3.1 Determination of $\boldsymbol{f_{1a}}$ *and* $\boldsymbol{f_{2a}}$

It is assumed that f_{1a} and f_{2a} are linear:

$$f_{1a} = k_{1a}(r_1 - r) \tag{22}$$

$$f_{2a} = k_{2a}(r_2 - r) \tag{23}$$

where k_{1a} and k_{2a} are two undetermined constants that can be determined based on current data:

$$\int_0^{\bar{r}_i} f_{ia}\, dr = m_{ia} t_i \bar{r}_i \quad i = 1,2 \tag{24}$$

$$k_{ia} = \frac{2 m_{ia} t_i}{\bar{r}_i}$$

where $\bar{r}_i = \sqrt{\frac{\bar{N}_i}{\pi}}$.

In City i, congestion cost caused by population agglomeration is m_{ia} times of that under free flow and m_{ia} can be calibrated by the congestion delay coefficient of City i.

In 2018, the congestion delay coefficients of Shanghai and Jiaxing are 1.82 and 1.25, respectively. With travel cost considered, $m_{1a} = 0.82$ and $m_{2a} = 0.25$. The calibration results of t_i and N_i mentioned in Section 5.2 were substituted into Eq (24):

$$k_{1a} = 6.3588 \times 10^{-7}$$

$$k_{2a} = 6.3029 \times 10^{-7}$$

5.3.2 Determination of f_{1b}

According to the analysis in Section 2, the congestion cost caused by intercity commuters is expressed as a linear function of intercity commuters:

$$f_{1b} = k_{1b} N_{21} \tag{25}$$

Similar to determination of k_{1a},based on current data, k_{1b} can be determined by:

$$k_{1b}N_{21}\bar{r}_1 = m_{1b}t_1\bar{r}_1 \tag{26}$$

where $\bar{r}_1 = \sqrt{\frac{\bar{N}_1}{\pi}}$. In other words, intercity commuters induced congestion cost near the HSR station of City 1 is m_{1b} times of that under free flow.

Considering that the congestion near the high-speed railway station of city 1 is caused by the population of city 1 and the commuter population, in this paper, we use the following formula to determine m_{1b}

$$m_{1b} = \frac{\bar{N}_{21}}{\bar{N}_{21} + \frac{1}{2}\Phi r^2} m_{1a}$$

And we obtain

$$k_{1b} = 1.8150 \times 10^{-9}$$

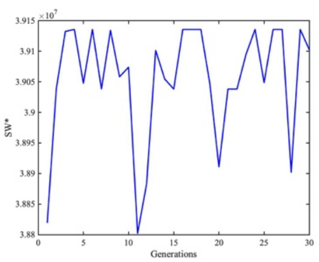
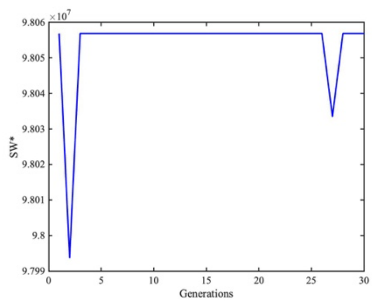
5.4 Comparison of the three algorithms

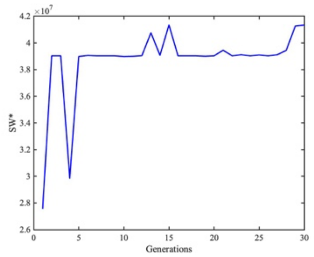
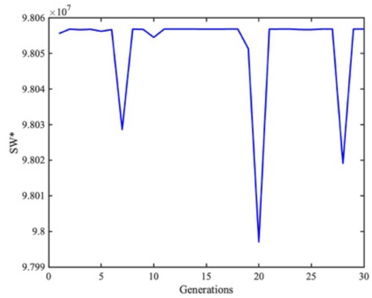
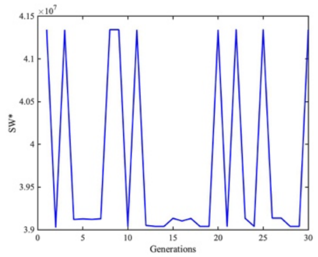
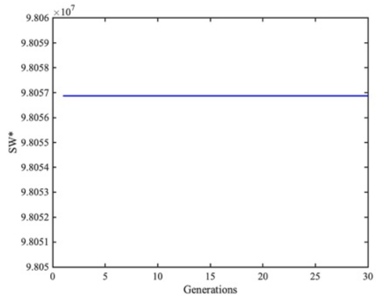
The three solving algorithms involved in this study are compared with each other.

In terms of parameter determination, the iteration numbers were determined while other parameters were set default in MATLAB. The iteration numbers were confined in the range of 0-30 as large iteration numbers lead to significantly reduced solving efficiency.

Table 4 illustrate the results obtained by the three solving algorithms with total population of 5 and 13 million, respectively.

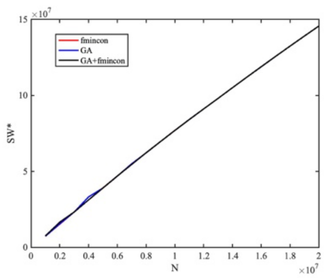
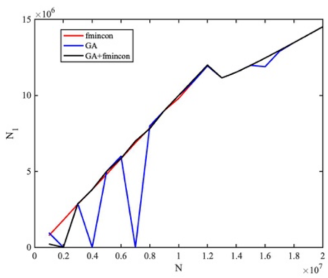
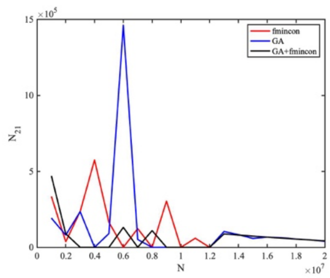
Table 4. Determination of parameters in the three solving algorithms

population / algorithm	$N = 5\,million$	$N = 13\,million$
$fmincon$		

GA		
$GA+f$		

In case of small total population, no algorithms delivered stabilized solutions, while Algorithm 1 was superior to the other two algorithms. In case of large total population, $fmincon$ could only deliver stabilized solutions with over 15 random simulations of initial values; GA required an iteration number larger than 10 and the results were not stabilized; $GA+f$ exhibited highest converging rate and most stabilized results. Therefore, the iteration numbers of $fmincon$, GA and $GA+f$ were 20, 15 and 5, respectively. Table 5 shows the results obtained by the three solving algorithms in case of different total populations. Herein, the initial and final total populations were set as 1 and 20 million, respectively, with a step of 1 million.

Table 5. Results obtained by the three solving algorithms in case of different total populations

SW^*	N_1	N_{21}
		

Maximum social welfares obtained by the three solving algorithms were highly consistent, while the corresponding populations were significantly different, especially intercity commuting populations. Hence, it is essential to identify the optimized solutions. As discussed above, intercity commuting will always be observed in case of large total population of the system, while Algorithm 1 may deliver results where no intercity commuting is observed. Therefore, Algorithm 3 exhibits superior stability and accuracy in case of large populations, compared with Algorithms 1 and 2.

Considering solving efficiency and stability, Algorithms 1 and 3 were used in case of small and large total populations, respectively.

5.5 Analysis of social optimum

The commuting in Shanghai and Jiaxing under both population exogenesis and population endogenesis are analyzed. Analysis of population exogenesis revealed the difference between current and optimized commuting status of Jiaxing and Shanghai, thus providing references to short-term and medium-term commuting policies by the government. Likewise, analysis of population endogenesis also provides references to development strategy and long-term planning by the government. Current social welfare can be obtained by substituting current residential and commuting populations of Jiaxing and Shanghai into the model. Indeed, social welfares with and without congestion taken into consideration were 1.0302×10^{12} and 1.1621×10^{12} yuan, respectively.

5.5.1 Population exogenesis

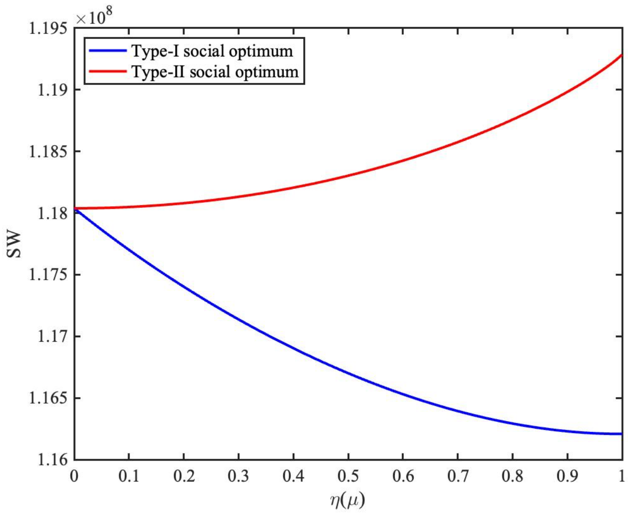


Fig. 8 Social optimum without congestion taken into consideration

Fig. 8 shows the trend of social welfare as a function of η (μ) in Type-I (II) social optimum without congestion taken into consideration. As observed, social optimum can only be reached if all residents in Jiaxing commute to Shanghai for work. In this case, the social welfare is maximized (1.1928×10^{12} yuan). According to statistics, the population commuting from Jiaxing to Shanghai is currently 3,062. In summary, results obtained without congestion taken into consideration reveal that this system is far away from social optimum, while it is apparently unrealistic for all residents in Jiaxing to commute to Shanghai for work.

With congestion taken into consideration, trigonometric function was employed to verify the rationality of linear f_{1a} and f_{2a}. The trigonometric function can be expressed as:

$$f_{1a} = k_{1a}\frac{r_1}{2}\left(1 + cos\frac{\pi r}{r_1}\right) \tag{27}$$

$$f_{2a} = k_{2a}\frac{r_2}{2}\left(1 + cos\frac{\pi r}{r_2}\right) \tag{28}$$

The results are shown in Fig. 9.

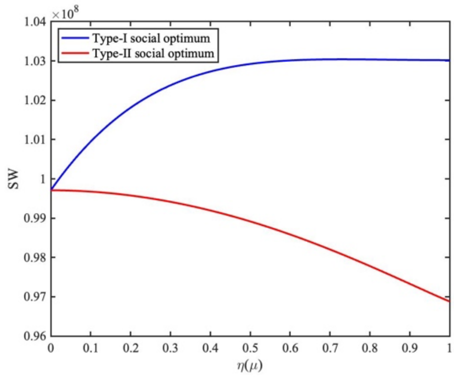


Fig. 9-1 linear function

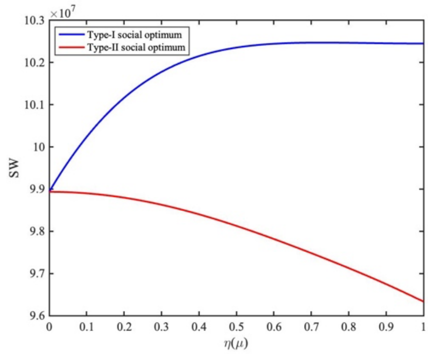


Fig. 9-2 trigonometric function

As observed, the results of trigonometric function and linear function are consistent, demonstrating that the linear congestion function is rational. As shown in Fig. 9-1, social optimum was achieved at $\eta = 0.73$，$\mu = 0$, demonstrating that if the congestion level of Shanghai remains unchanged, social optimum (1.0304×10^{12} yuan) needs to prohibit Jiaxing people from commuting to Shanghai. Therefore, under the condition of exogenous population, the systems of Jiaxing and Shanghai still has much room to improve.

5.5.2 Population endogenesis

Without congestion taken into consideration, maximum population distribution and social welfare of Shanghai and Jiaxing obtained are:

$$N_1 = 11.52\ million$$

$$N_2 = 2.19\ million$$

$$N_{21} = 2.19 million$$

$$SW^* = 1.1962 \times 10^{12}\ yuan$$

In this case, social optimum can only be achieved if everyone is a resident in Shanghai, which is similar to the population exogenesis case.

With congestion taken into consideration, maximum population distribution and social welfare of

Shanghai and Jiaxing obtained are:

$$N_1 = 11.41 \text{ million}$$

$$N_2 = 2.30 \text{ million}$$

$$N_{21} = 82{,}800$$

$$SW^* = 1.1962 \times 10^{12}\ yuan$$

In social optimum, the populations of Shanghai and Jiaxing shall be 11.41 and 2.30 million, respectively, and the intercity commuting population between Shanghai and Jiaxing shall be approximately 82.800. In practical, the populations of Shanghai and Jiaxing are 12.43 and 1.29 million, respectively, and the intercity commuting population between Shanghai and Jiaxing is 3,100.

Optimized population distribution and social welfare under different scenarios were investigated.

Table 6 shows optimized population distributions in different situations.

Table 6. Optimized population distribution of the Shanghai-Jiaxing two-city system

Scenario		N_1	N_2	N_{21}
Current population		12.43 million	1.28 million	3,100
Without congestion	Exogenous population	12.43 million	1.28 million	1.28 million
	Endogenous population	11.52 million	2.19 million	2.19 million
With congestion	Exogenous population	12.43 million	1.28 million	62,400
	Endogenous population	11.41 million	2.30 million	82,800

Investigation of optimized population distribution and social welfare under different scenarios can provide references to policies by government. As observed, the results obtained with congestion taken into consideration are more consistent with the practical situation compared to those obtained without congestion taken into consideration. Specifically, exogenous population requires 20 times increase of the intercity commuting population compared with the case of current population, indicating that the positive externalities brought by population agglomeration are still large for Shanghai if the residential population remains unchanged. Meanwhile, population agglomeration induces large congestion cost. As a result, social welfare under population endogenesis can only be enhanced by reducing the population of the metropolis and increasing the population of the satellite city and the intercity commuting population. Additionally, population distributions in social optimum under current and exogenesis populations are significantly different from that in ideal cases.

Table 7 summarizes optimized social welfare under different scenarios compared with the current situation. This reflects the effects of policies implemented.

Table 7 Optimized social welfare of the Shanghai-Jiaxing two-city system

<table>
<tr><th colspan="2">Scenario</th><th>Social optimum</th><th>Increase (compared with current situation)</th><th>Social welfare loss induced by congestion</th></tr>
<tr><td rowspan="3">Without congestion</td><td>Current Population</td><td>1.1621×10^{12}</td><td></td><td></td></tr>
<tr><td>Exogenous Population</td><td>1.1928×10^{12}</td><td>2.64%</td><td></td></tr>
<tr><td>Endogenous Population</td><td>1.1962×10^{12}</td><td>2.93%</td><td></td></tr>
<tr><td rowspan="3">With congestion</td><td>Current Population</td><td>1.0302×10^{12}</td><td></td><td>11.35%</td></tr>
<tr><td>Exogenous Population</td><td>1.0304×10^{12}</td><td>0.02%</td><td>13.61%</td></tr>
<tr><td>Endogenous Population</td><td>1.0316×10^{12}</td><td>0.14%</td><td>13.76%</td></tr>
</table>

The results reveal that improvements of social welfare with congestion taken into consideration are not significant (up to 0.14%) although population distribution and intercity commuting population in this case are still significantly different from those in ideal cases, suggesting that the Shanghai-Jiaxing two-city system is currently in a relatively good state. Additionally, congestion causes severe loss of social welfare (about 14%) practically, indicating that the commuting model in urban areas shall consider the influences by congestion.

5.5.3 Regional congestions

The influences of congestion on residents were investigated in this section. Geographic division and intercity commuting area division of City 2 can be obtained by determining the specific function patterns of $\bar{\theta}_2(r)$ and $\tilde{\theta}_2(r)$, as shown in Fig. 10.

$$\bar{\theta}_2(r) = \frac{2t_2 + k_2(2r_2 - r)}{t_2 + k_2(r_2 - r)} \tag{29}$$

$$\tilde{\theta}_2(r) = \frac{2t_2(r - \eta r_2) + k_2(2r_2 r - r^2 - 2\eta r_2^2 + \eta^2 r_2^2)}{\big(t_2 + k_2(r_2 - r)\big)r} \tag{30}$$

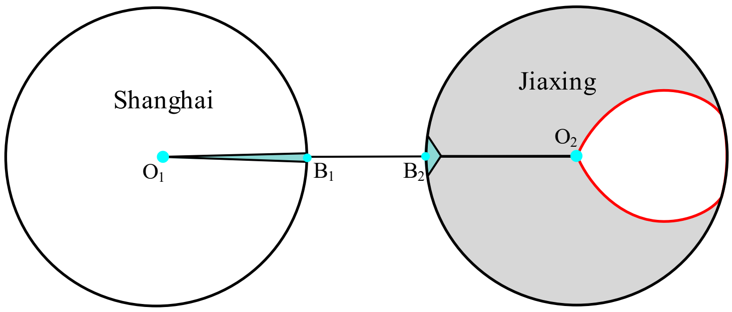


Fig. 10 Intercity commuting area from Jiaxing to Shanghai

In Fig. 10, the blue area refers to the intercity commuting area of Jiaxing, the grey and white areas refer to Area L and S, respectively. As observed, $\tilde{\theta}_2 < \bar{\theta}_2$, $\tilde{\theta}_2^{max} < \bar{\theta}_2^{max} < \pi$, which belongs to the first case in Table 1.

In order to investigate the loss of residents' travel caused by congestion in social optimum, the congestion coefficients of different commuters C_{ij} are defined:

$$C_{ij} = \frac{ACC_{ij}}{FACC_{ij}}, i, j = 1, 2 \tag{31}$$

where ACC_{ij} and $FACC_{ij}$ refer to the traveling cost of residents of City *i* commuting to City *j* for work under congestion and free flow, respectively.

The commuting costs of different commuters can be calculated by introducing social optimum under population endogenesis into the model. Hence,

$$C_{11} = 1.98$$

$$C_{22} = 1.41$$

$$C_{21} = 1.39$$

$$C_{21}^* = 2.01$$

where C_{21}^* refers to the congestion coefficient of intercity commuters from City 2 in City 1.

In social optimum, the congestion coefficient is approximately 2 in City 1, indicating that the commuting cost in presence of congestion is two times of that under free flow. Additionally, the congestion coefficient of intracity commuters between Shanghai and Jiaxing is relatively high, suggesting that Jiaxing is also exposed to severe congestion during peak hours in social optimum.

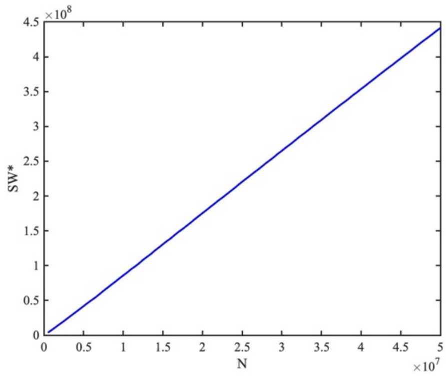
5.6 Sensitivity

The effects of parameters such as congestion on social welfare per capita （SW per capita） and SW^* （optimized social welfare） are discussed in this section. Sensitivity analysis aims to explore the influences of parameters such as congestion on social optimum were investigated. Eight parameters and their corresponding ranges (see Table 8) were selected and the control variable method was employed for analysis.

Table 8 Parameters selected

Parameter	Definition	Amplitude
N	Population scale of the system	$[0, 50\text{ million}]$
γ	Production function parameters	$[\frac{\gamma}{2}, 2\gamma]$
tc	Time value of intercity commuting	$[\frac{tc}{2}, 2tc]$
ϕ	Scope of influence of intercity commuters	$[\frac{\phi}{2}, 2\phi]$
R_i	agricultural rent of City i	$[\frac{R_i}{2}, 2R_i]$
m_{1i}	Congestion-induced delay index of City i	$[\frac{m_{1i}}{2}, 2m_{1i}]$

5.6.1 Effects of population scale on social optimum



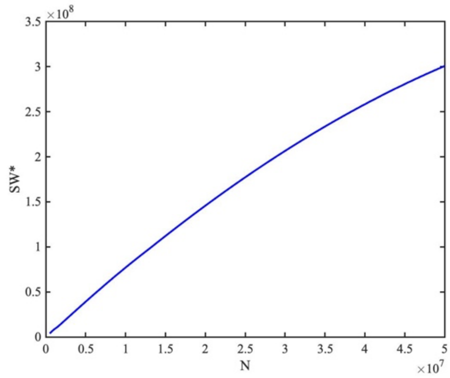
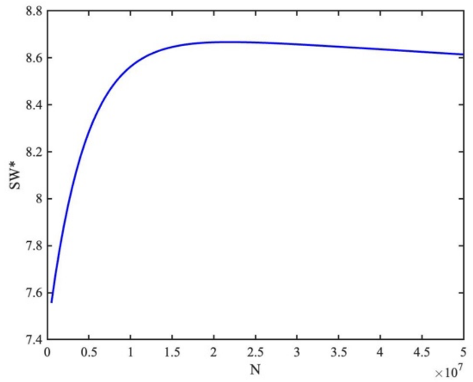


Fig. 11-1 Results obtained without congestion taken into consideration

Fig. 11-2 Results obtained with congestion taken into consideration

The effects of the population scale of the two-city system on social optimum (under population endogenesis) are discussed in this section. As shown in Figs. 11-1 and 11-2, the optimal solution of social welfare increases with the total population, suggesting that increased investment of labor force leads to enhanced production capacity and increased social fortune. Nevertheless, social welfare increases linearly with the population scale without congestion taken into consideration, with social welfare increases gradually with the population scale with congestion taken into consideration. Additionally, the effects of the population scale of the system on SW per capita were investigated. Figs. 12-1 and 12-2 shows the results without and with congestion taken into consideration.



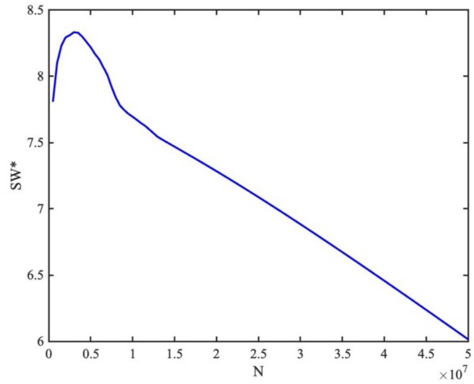


Fig. 12-1 SW per capita vs. population scale without congestion taken into consideration

Fig. 12-2 SW per capita vs. population scale with congestion taken into consideration

Without congestion taken into consideration, SW^* per capita and N^* are:

$$SW^* = 86700 \text{ yuan}$$

$$N^* = 22\, million$$

With congestion taken into consideration, SW^* per capita and N^* are:

$$SW^* = 83300\, yuan$$

$$N^* = 3\, million$$

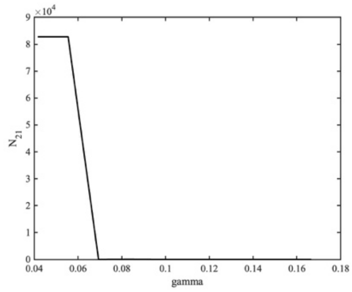
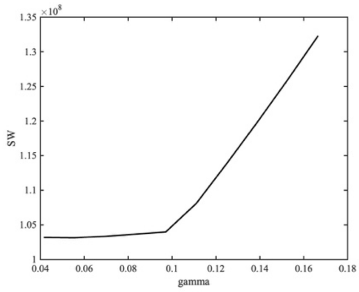
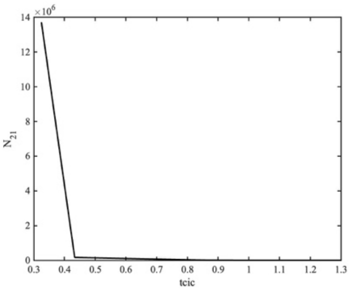
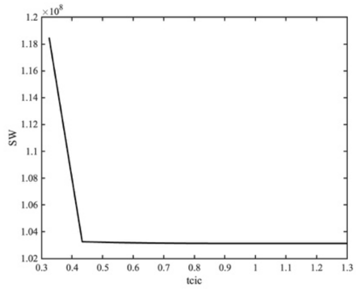
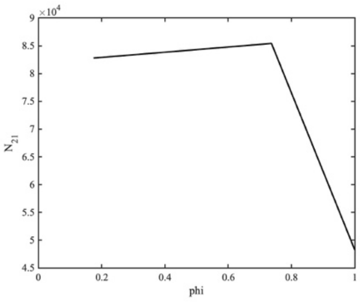
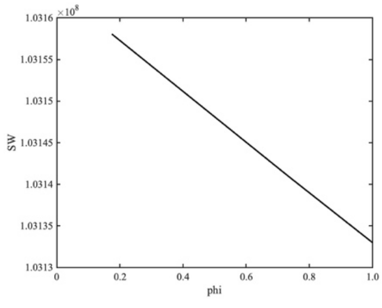
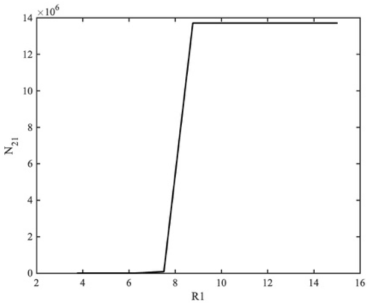
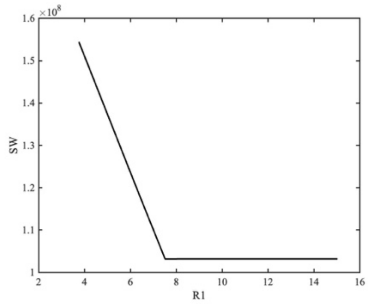
As observed, SW^* per capitaand corresponding population scale without congestion taken into consideration are 86700 yuan and 22 million, respectively. As the population increases, SW^*remains stabilized despite a slight decrease. Optimized $SW^*$and corresponding population scale with congestion taken into consideration are 83300 yuan and 3 million, respectively. As the population increases,SW^* decreases.

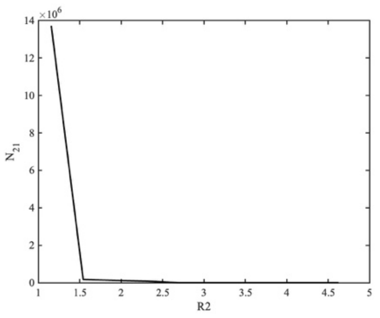
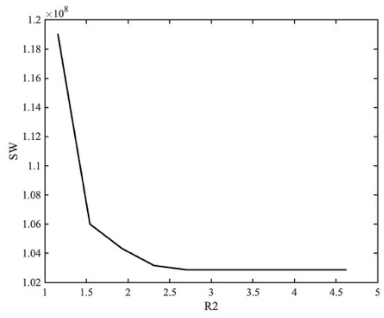
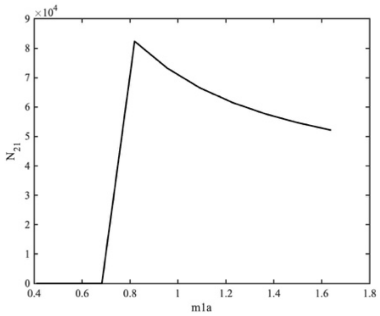
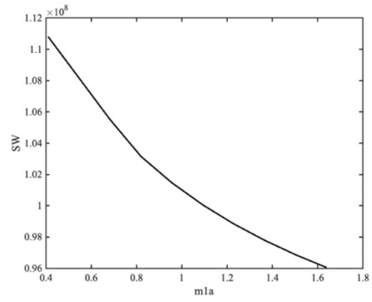
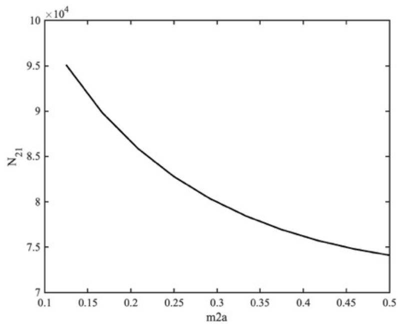
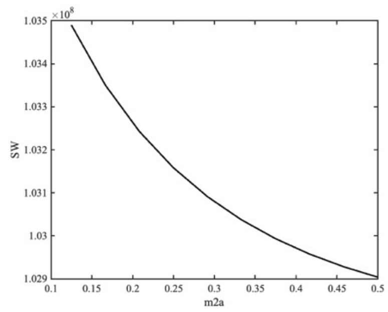
Macroscopically, despite the increases of overall social welfare induced by increasing population, traffic congestion has a negative influence on the improvement of social welfare, resulting in limited improvement. Microscopically, personal welfare is sensitive to population scale; increasing population leads to rapidly increasing SW^* when the population scale is small. With traffic congestion taken into consideration, SW^* starts to decrease in a short time as the total population increases further and the correlation is linearly negative. The results demonstrate that over-large population scale may lead to severe loss of personal welfare although the overall social welfare is enhanced. Additionally, increasing total population has negligible effects on social welfare under congestion if the population scale is large. In order to significantly enhance both social and personal welfare, government shall pay great attention to congestion management.

5.6.2 Effects of other parameters on social optimum results

Table 9 shows the effects of other parameters on commuting population and social welfare results in social optimum with congestion taken into consideration. The sensitivity analysis revealed that γ, tc, ϕ and R_i significantly affected N_{21}^{*}; R_i and m_{1a} affected SW^{*}. Herein, the influence level follows the sequence of $R_1 > m_{1a} > tc$, demonstrating that congestion is the main factor hindering improvement of social welfare.

Table 9 Sensitivity analysis

<table>
<tr><th>Variable
Parameter</th><th>N_{21}^{*}</th><th>SW^{*}</th></tr>
<tr><td>γ</td><td>
</td><td>
</td></tr>
<tr><td>tc</td><td>
</td><td>
</td></tr>
<tr><td>ϕ</td><td>
</td><td>
</td></tr>
<tr><td>R_1</td><td>
</td><td>
</td></tr>
</table>

R_2		
m_{1a}		
m_{2a}		

Since agricultural rent R_i does not fluctuate in short term, government shall pay great attention to congestion management, especially in metropolis. Meanwhile, increasing social welfare basically corresponds to increasing intercity commuting population (see Fig. 9-1), suggestion that the infrastructure of intercity traffic shall also be enhanced to provide sufficient supply of intercity traffic.

6. Conclusions and outlook

6.1 Main conclusions

The congestion effect is introduced to accurately describe the social optimum of two-city system. An optimization model was established to clarify most rational residential and commuting populations of the two-city system under both location conditions and population migration. Additionally, an empirical study of the Shanghai-Jiaxing two-city system was conducted. Social optimum results obtained under population exogenesis and endogenesis were investigated, the Shanghai-Jiaxing two-city system was evaluated, measures by government to enhance social welfare of two-city system were discussed. This study provides references and suggestions to social development and population issues based on two-city model considering congestion. The main conclusions are as follows:

1. Excessive population agglomeration leads to high congestion cost in the city, thereby severe loss of social welfare. Specifically, traffic congestion is a significant negative externality.

2. Intercity commuting can enhance social welfare via re-allocation of labor force. Nevertheless, cost induced by traffic congestion is the main factor hindering further improvement of social welfare if the population scale is large. Additionally, severe time delay of individuals may be caused.

3. The population scale shall be controlled according to the objectives of traffic management. Nevertheless, attractiveness of metropolis to satellite cities would be enhanced as a result of congestion management. Therefore, the infrastructure of intercity traffic shall be further improved and large volume transportation shall be encouraged.

6.2 Outlook

First, it is assumed in this study that all residents in the system are homogeneous (e.g., consistent housing consumption and time value). Future studies may assume that residents in different cities have different time values. Then, it is assumed in this study that the urban system has one single industry only and residents have one single consumption only (housing consumption). Future studies may involve multiple industries in urban system and residents have multiple consumptions to establish an improved social optimum model. Additionally, social optimum of two-city system was investigated in this study. Future studies may extend social optimum to urban agglomeration based on two-city systems.

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Appedix A.

Proof of Theorem 1.

Contradiction method. If some residents commute from City 1 to City 2 in social optimum, define the optimized population distribution as (N_1^*, N_2^*, N_{12}^*), N_{12}^* refers to the total quantity of commuting from City 1 to City 2. Based on that, a new population distribution (N_1', N_2') is established under the premise of population endogenesis. Meanwhile, $N_1' = N_1^* - N_{12}^*, N_2' = N_2^* + N_{12}^*$ and no intercity commuting is observed. Apparently, these two population distributions have no influences on the overall output and commuting cost of the two-city system, while they lead to reduced aggregate agricultural rent $(N_{12}^*(R_1 - R_2))$ as $R_1 > R_2$, demonstrating that new population distributions lead to enhanced social welfare. In other words, the original population distribution is not optimal. This conflicts with the hypothesis.

Proof of Conclusion 1.

According to the mean value theorem of integrals, $0 < \xi < r$ so that the following equation holds:

$$\bar{\theta}_2(r) = \frac{2\int_0^r h_2 dr}{h_2 r} = \frac{2h_2(\xi)r}{h_2 r}$$

Since h_2 decreases monotonically with the distance (i.e., $\frac{\partial h_2}{\partial r} < 0$), then

$$\bar{\theta}_2(r) > \frac{2h_2(r)r}{h_2 r} = 2$$

Since $\bar{\theta}_2$ has no definition at $r = 0$, the ultimate value of $\bar{\theta}_2$ at $r = 0$ $(\lim_{r\to 0}\frac{\partial h_2}{\partial r} > -\infty)$ is investigated:

$$\lim_{r\to 0}\bar{\theta}_2(r) = \lim_{r\to 0}\frac{2\int_0^r h_2 dr}{h_2 r} = 2\lim_{r\to 0}\frac{h_2}{\frac{\partial h_2}{\partial r}r + h_2} = 2$$

According to Eq above, it is defined that $\bar{\theta}_2(0) = 2$ so that $\bar{\theta}_2^{min} = 2$.

Proof of Inference 1.

According to Conclusion 1, the minimum angle of boundary curves of Areas L and S satisfies

$\bar{\theta}_2^{min} = 2 < \pi$. Hence, Area S is definitely present in the city.

Proof of Theorem 2.

Contradiction method. As shown in Fig. 5, Point E is defined as any point different from Point A on Curve AC_1, a_1 and e_1 denote the practical commuting costs of A and E, respectively, while a_2 and e_2 denote the opportunity commuting costs of A and E, respectively. ACC_2^* denotes aggregate commuting cost in City 2 in social optimum. Apparently, ACC_2^* is the minimized in social optimum. Assuming that $a_1 - a_2 \neq e_1 - e_2$ if the commuting cost of City 2 reaches ACC_2^*, it can be deduced that $a_1 - a_2 > e_1 - e_2$. Then, Points A' and E' are identified near Points A and E so that practical and opportunity commuting costs of Point A' are $a_1 - \Delta$ and $a_2 + \Delta$, respectively, while practical and opportunity commuting costs of Point E' are $e_1 + \Delta$ and $e_2 - \Delta$, respectively. Herein, $\Delta > 0$ refers to minimal increment. Owing to the continuity of the Area, residents of Point A' and E' are intercity and intracity commuters, respectively. If identities of residents of Point A' and E' are switched (the overall quantity of intercity commuters remains constant, while the aggregate commuting cost of City 2 increases by $e_1 - e_2 - (a_1 - a_2) + 4\Delta$), $\Delta^* > 0$ must be possible so that $e_1 - e_2 - (a_1 - a_2) + 4\Delta^* < 0$ as $\lim_{\Delta \to 0}(e_1 - e_2 - (a_1 - a_2) + 4\Delta) = e_1 - e_2 - (a_1 - a_2) < 0$. Therefore, ACC_2^* is not the optimal solution, which is conflict with the hypothesis.

Proof of Conclusion 4.

If $\tilde{\theta}_2^{max} > \pi$, $\widetilde{r_2}$ that is strictly less than r_2 can be found so that $\tilde{\theta}_2$ is maximized at $\widetilde{r_2}$. Hence,

$$2\left(\frac{1}{r} - \frac{\left(h_2 + \frac{\partial h_2}{\partial r} r\right)\int_{\eta r_2}^{r_2} h_2\, dr}{h_2^2 r^2}\right) < \frac{\partial \tilde{\theta}_2}{\partial r} = 2\left(\frac{1}{r} - \frac{\left(h_2 + \frac{\partial h_2}{\partial r} r\right)\int_{\eta r_2}^{r} h_2\, dr}{h_2^2 r^2}\right) < \frac{2}{r}$$

Therefore,

$$\lim_{\eta \to 1} \frac{\partial \tilde{\theta}_2}{\partial r} = \frac{2}{r} > 0$$

When $\eta \to 1$, $\tilde{\theta}_2$ increases monotonically with r. In this case, $\widetilde{r_2}$ that is strictly less than r_2 cannot be found so that $\tilde{\theta}_2$ is maximized at $\widetilde{r_2}$.

Proof of Theorem 3.

the commuting costs of working in this city and the other city are defined as opportunity and practical commuting costs, respectively. Assuming that, some residents in Area S commute to City 1 for work in social optimum when $\eta > 0$, residents at Point Q (in Area S), whose distance from the downtown is r, are intercity commuters and $r < \eta r_2$. As a result, there must be Point Q', whose distance from the downtown is r, on the connection of HSR station and downtown in Area L. Residents at Point Q' are intracity commuters as it is on Segment OA. Apparently, opportunity commuting costs of Points Q and Q' are equivalent, while the practical commuting cost of Point Q' is lower than that of Point Q. Hence, social welfare can be enhanced by switching identities of residents at Points Q and Q', which conflicts to the hypothesis.

Proof of Theorem 4.

Assuming that simultaneous commuting between metropolis and satellite city is present in social optimum, there must be Point A near the HSR station of metropolis so that residents at this point commute from metropolis to satellite city; likewise, there must be Point E near the HSR station of satellite city so that residents at this point commute from satellite city to metropolis. a_1 and e_1 refer to practical commuting costs of Points A and E, respectively; a_2 and e_2 refer to opportunity commuting costs of Points A and E; $a_1 < a_2, e_1 < e_2$. In this case, $a_1 > e_2, a_2 < e_1$. Since it is assumed that $a_1 < a_2$, then $e_1 > e_2$. This is contradictory to the hypothesis that $e_1 < e_2$.