Uncertainly Quantification 1
Exercise 1

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1 Conditioning 1

a) For arithmetic operations $f(x_1, x_2) = \frac{x_1}{x_2}$ $(x_2 \neq 0)$
computing condition number:

$$f_{x_1} = \frac{1}{x_2}, f_{x_2} = -\frac{x_1}{x_2}$$
$$K_1 = |\frac{1}{x_2} \times \frac{x_1}{x_1/x_2}| = 1$$
$$K_2 = |-\frac{x_1}{x_2^2} \times \frac{x_2}{x_1/x_2}| = 1$$

is well-conditioned.

b) For arithmetic operations $f(x_1, x_2) = x_1^{x_2}$ $(x_1 > 0)$, computing condition number:

$$K_1 = |x_2 x_1^{(x_2-1)} \times \frac{x_1}{x_1^{x^2}}| = |x_2|$$
$$K_2 = |x_1^{x_2} \ln x_1 \times \frac{x_2}{x_1^{x_2}}| = |x_2 \ln x_1|$$

since $x_1 > 0$, $\lim_{x_1 \to \infty} \to \infty$, we easily see that

$$\lim_{x_2 \to \infty} K_1 \to \infty$$
$$\lim_{x_2 \to \infty} K_2 \to \infty$$

Then it is ill-conditioned.

For the simple operations, s.t. $f(x_1, x_2) = f(1, x) = \frac{1}{x}$. Compute condition number, s.t. $K = 1$. That is well-conditional.

For the simple operations, s.t. $f(x) = \sqrt{x}(x > 0)$.Then we have $K = \frac{1}{2}$, then it is well-conditioned.

2 Conditioning 2

a) The differentiation of $f(x)$ regard to x is as follows:

$$
\begin{aligned}
f(x)' &= \frac{sin(x) * x - (1 - cos(x))}{x^2} \\
&= \frac{x * sin(x) - (1 - cos(x))}{x^2}
\end{aligned}
$$

The conditional number K is equal to:

$$
\begin{aligned}
K(x) &= |\frac{df}{f(x)}\frac{x}{dx}| \\
&= \left|\frac{\frac{x\sin x + \cos x - 1}{x^2}dx}{\frac{1-\cos x}{x}}\frac{x}{dx}\right| \\
&= \left|\frac{x\sin x + \cos x - 1}{1 - \cos x}\right| \\
&= \left|\frac{x\sin x}{1 - \cos x} - 1\right| \\
&\approx \left|\frac{x\sin x}{1 - \cos x}\right|
\end{aligned}
$$

$f(x)$ is well-conditioned when k is small enough, which means that we should decide the threshold for the condition number and find the interval during which k is smaller than the threshold.

$\left|\frac{x\sin x}{1-\cos x}\right|$ is continuous and differential at points where $x \neq 2k\pi$. When $x = 2k\pi$, $1 - \cos x = 0$, $x \sin x = 0$.

$$
\begin{aligned}
\lim_{x \to 2k\pi} \frac{x\sin x}{1 - \cos x} &= 1 + \lim_{x \to 2k\pi} \frac{x\cos x}{\sin x} \\
&= \begin{cases} 1 + \infty, & k \neq 0 \\ 1, & k = 0 \end{cases}
\end{aligned}
$$

Which means $f(x)$ will be sure ill conditioned at any points very close to $2k\pi, k \neq 0$.

Considering the interval between $(0, 2\pi]$, $\left|\frac{x\sin x}{1-\cos x}\right|$ is relative small between $(0, 2\pi - \delta]$, δ is a large enough number in $(0, 2\pi)$ which makes the conditional number $K(x)$ small. E.g. let's take $\delta = \frac{\pi}{2}, then 2\pi - \delta = \frac{3\pi}{2}$, which makes the conditional number $K(\frac{3\pi}{2}) = \left|\frac{x\sin x}{1-\cos x}\right|$ relatively small, actually $\left|\frac{x\sin x}{1-\cos x}\Big|_{x=\frac{3\pi}{2}}\right| \approx 4.7124$.

So at any point $2k\pi, k \neq 0$, it is ill conditioned. Also we have $\forall x \neq k\pi, \sin x \neq 0$, $1 - \cos x \neq= 0$, $\lim_{x\to\infty} \left|\frac{x\sin x}{1-\cos x}\right| = +\infty$.

Then for a large enough k, we will find all points in $(2k\pi, 2(k+1)\pi]$, except the point $(2k+1)\pi$, will make K(x) very big, and $f(x)$ ill conditioned. Assuming $K(\frac{3\pi}{2}) = 4.7124$ is the threshold for the condition number.

Then for all x in the interval of $[2k\pi + \delta_k, 2(k+1)\pi - \delta_k], \delta_k <= \pi/2, k = \pm 1, \pm 2, ...$, where δ_k satisfies $K((2k+1) * \pi - \delta_k) = K\frac{3\pi}{2}$ for each k and all the points of $(2k+1)\pi$, $f(x)$ is well conditioned. Obviously, $(2k+1)\pi$ is in between $[2k\pi + \delta_k, 2(k+1)\pi - \delta_k]$

In summary: for any point in:

$$[2k\pi+\delta_k, 2(k+1)\pi-\delta_k], \delta_k <= \pi/2, k = \pm 1, \pm 2, ..., K((2k+1)\pi-\delta_k) = K(\frac{3\pi}{2})$$

$f(x)$ is well conditioned. Otherwise, $f(x)$ is ill conditioned.

b)

$$\begin{aligned}
\frac{1-\cos(x)}{x} &= \frac{1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}{x} \\
&= \frac{1 - 1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 ...}{x} \\
&= \frac{1}{2!}x - \frac{1}{4!}x^3 ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2(n+1))!} x^{2n+1}
\end{aligned}$$

compare with $\sin(x)$ in Taylor sums: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. It is easy to see, that $f(x) = \frac{1-\cos(x)}{x} = C(n)\sin(x)$, where $C(n)$ is depend on n. When $|x| \ll 1$ Taylor_sin is stable. Therefore this algorithm is also stable.

3 Machine precision and stable algorithm

a) Machine epsilon: 16 (test code please see Figure 1)

b) Plot relative error please see Figure 2

For x is negative, when n goes larger, the terms of Exp Taylor sums gets closer, s.t. result into disaster.
New method please see Algorithm 1. And plot relative error please see Figure 3

Data: x, n, function: Taylor_exp(x,n)
if $x >= 0$ **then**
  return Taylor_exp(x, n);
else
  return 1/Taylor_exp(-x, n);
end

Algorithm 1: mean idea: $e^{-x} = 1/e^{x}$ for $x > 0$ and $e^{x}(x > 0)$ is well-conditioned

```
import math
import numpy as np
a = 2.0 ;
b = 1.0;
x1 = np.zeros(100);

for i in range(0, 50):
   a = a / 10.0
   b = b / 10.0
   x1[i] = math.ceil((1+ a)/(1+b))
print (x1)
```

Figure 1: Test code for Machine Precision

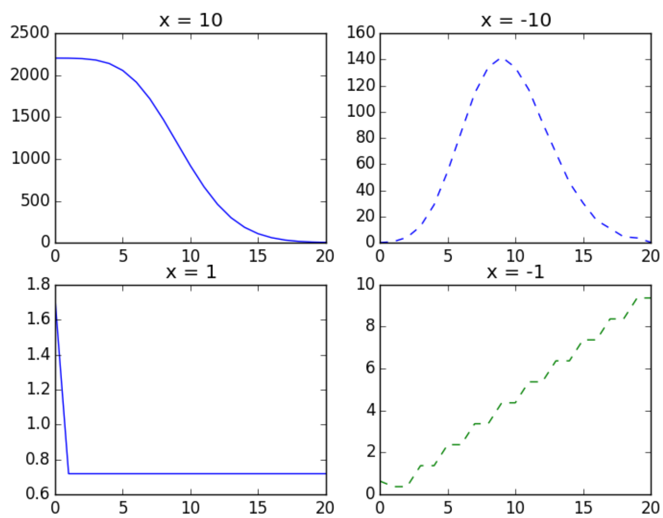

Figure 2: Relative error for the arguments $x \in \{10, 1, 1, 10\}$

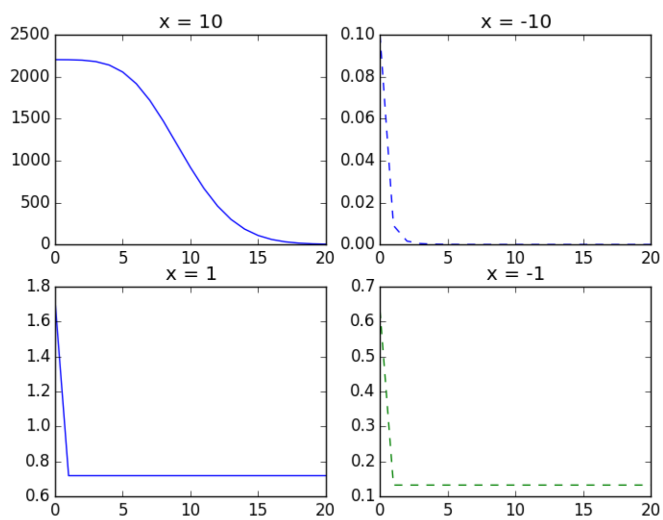

Figure 3: Relative error for new method