Interdisciplinary Centre
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Exercise Sheet 2

November 2nd, 2017 – November 9th, 2017

Uncertainty Quantification 1 (Winter Term 2017)

Practical Exercise 1. Numerical Differentiation (2+3 = 5 Points)

Consider the function

$$f(x) = e^{\sin(x)}.$$

We intend to compute the first derivative $f'(x)$ at the point $x = 0$ with the aid of the first order approximation (*finite difference*)

$$f'(0) \approx g(h) := \frac{f(h) - f(0)}{h}.$$

The *absolute* error $\varepsilon(h)$ is defined as

$$\varepsilon(h) := \left| f'(0) - g(h) \right|.$$

a. Write a program which computes $\varepsilon(h)$ for $h \in \{10^0, 10^{-1}, \dots, 10^{-14}\}$.

b. Visualize the error as a function of h via a double-logarithmic plot. Explain the plot! What is happening here?

Practical Exercise 2. Stability of numerical evaluation (3+3 = 6 Points)

Consider the expressions

a. $\frac{1}{1+2x} - \frac{1-x}{1+x}$ für $\|x\| \ll 1$

b. $\sqrt{x + \frac{1}{x}} - \sqrt{x - \frac{1}{x}}$ für $x \gg 1$.

Transform each of these expression such that their numerical evaluation becomes more stable. Reason why the new expressions are more stable! Write a program which computes and compares the original and the stable expression.

Exercise 3. Probability Space (3+2+2 = 7 Points)

A (fair) dice is thrown three times.

a. Determine the sample space Ω. What is the appropriate σ-algebra $\mathcal{F}$ of events in this case?

b. Which probability distribution in meaningful in this case due to the fairness of the dice? Name the distribution and state the probability measure P in form of a density $\rho(\omega)$, $\omega \in \Omega$.

c. Determine for the event "*A: At least one time 'six'*" the set $A \subset \Omega$ as well as the probability $P(A)$.

Hint: If we deal with an event containing *at least*, then it might be helpful to consider the complementary event.

Reminder:[1] A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ consisting of:

- The *sample space* Ω – an arbitrary non-empty set,
- the σ-*algebra* $\mathcal{F} \subseteq \mathcal{P}(\Omega) = 2^\Omega$ (also called σ-*field*) - a set of subsets of Ω, called events, such that:
  - $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$,
  - $\mathcal{F}$ is closed under complements: if $A \in \mathcal{F}$, then also $(\Omega \setminus A) \in \mathcal{F}$,
  - $\mathcal{F}$ is closed under countable unions: if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$, then also $(\bigcup_{i=1}^\infty A_i) \in \mathcal{F}$
- The *probability measure* $P : \mathcal{F} \to [0, 1]$ – a function on $\mathcal{F}$ such that:
  - P is countably additive (also called σ-additive): if $\{A_i\}_{i=1}^\infty \subseteq \mathcal{F}$ is a countable collection of pairwise disjoint sets, then $P(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty P(A_i)$,
  - The measure of entire sample space is equal to one: $P(\Omega) = 1$.

Submission: until November 9th, 2017, 11 a.m. (EMCL, INF 205, 1st floor, room 1/214 *or* room 1/232)

[1] https://en.wikipedia.org/wiki/Probability_space