Appendix: Combining Tail and Reaction Wheel for Underactuated Spatial Reorientation in Robot Falling with Quadratic Programming

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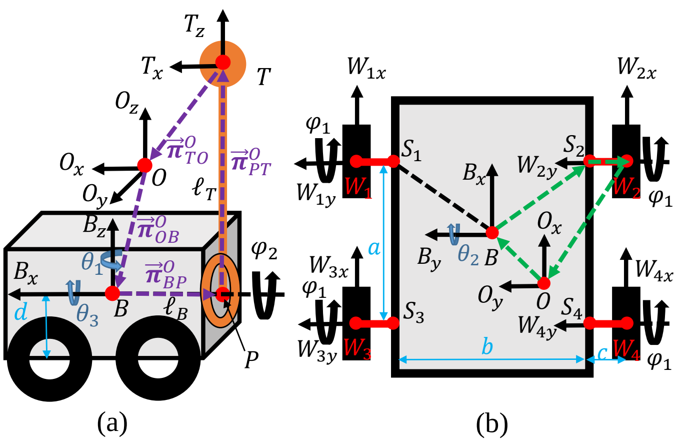

Fig. 1: Sketch of the underactuated hybrid tail-wheel robot. (a) Side view. (b) Top view.

TABLE I: Nomenclature

m_B, m_T, m_{W_i}	Mass of the body, tail and i-th wheel, respectively
$\boldsymbol{H}^O$, $\boldsymbol{H}^B \in \mathbb{R}^3$	Total angular momentum expressed in the inertial frame and the body frame, respectively.
$\boldsymbol{H}^O_* \in \mathbb{R}^3$	Angular momentum expressed in the inertial frame. $*$ = body, tail, wheel.
$\boldsymbol{I}^O_*$, $\boldsymbol{I}^B_* \in \mathbb{R}^{3\times 3}$	Moment of inertia of the body or the wheel expressed in the inertial frame and the body frame, respectively. $* = B, W_i$.
$\boldsymbol{I}^{W_i}_{W_i} \in \mathbb{R}^{3\times 3}$	Moment of inertia of the wheel expressed in the wheel frame. $\boldsymbol{I}^{W_i}_{W_i} = \boldsymbol{I}_W$ is assumed and $\boldsymbol{I}_W = \mathrm{diag}([(\boldsymbol{I}_W)_{xx}, (\boldsymbol{I}_W)_{yy}, (\boldsymbol{I}_W)_{zz}])$.
$\boldsymbol{\omega}^O_*$, $\boldsymbol{\omega}^B_* \in \mathbb{R}^3$	Angular velocity of the body frame or the wheel frame expressed in the inertial frame and the body frame, respectively. $* = B, W_i$.
$\boldsymbol{\rho}^O_*$, $\boldsymbol{\rho}^B_* \in \mathbb{R}^3$	Position of CoM expressed in the inertial frame and the body frame, respectively. $* = B, T, W_i$.
$\boldsymbol{\pi}^Z_{XY} \in \mathbb{R}^3$	Position vector from the point X to Y expressed in the frame $\{Z\}$.
$\theta_1, \theta_2, \theta_3$	Yaw, pitch, and roll, which are the rotation order.
φ_1, φ_2	Angular position of the wheel and tail, respectively
$\boldsymbol{R}^Y_X \in SO(3)$	Rotation matrix from the frame $\{X\}$ to $\{Y\}$, e.g., $\boldsymbol{\rho}^B_* = \boldsymbol{R}^B_O \boldsymbol{\rho}^O_*$ and $\boldsymbol{R}^O_B = (\boldsymbol{R}^B_O)^T$ hold.

I. Modelling

To derive the reduced model, we start from an angular momentum equation expressed in the inertial frame

$$\boldsymbol{H}^O = \boldsymbol{H}^O_{Body} + \boldsymbol{H}^O_{Tail} + \boldsymbol{H}^O_{Wheel}, \tag{1}$$

where $\boldsymbol{H}^O_{Body} = \boldsymbol{I}^O_B \boldsymbol{\omega}^O_B + m_B \boldsymbol{\rho}^O_B \times \dot{\boldsymbol{\rho}}^O_B$, $\boldsymbol{H}^O_{Tail} = m_T \boldsymbol{\rho}^O_T \times \dot{\boldsymbol{\rho}}^O_T$, and $\boldsymbol{H}^O_{Wheel} = \sum_{i=1}^4 m_{W_i} \boldsymbol{\rho}^O_{W_i} \times \dot{\boldsymbol{\rho}}^O_{W_i} + \sum_{i=1}^4 \boldsymbol{I}^O_{W_i} \boldsymbol{\omega}^O_{W_i}$.

It can be also expressed in the body frame

$$\begin{aligned}\boldsymbol{H}^B &= \boldsymbol{I}^B_B \boldsymbol{\omega}^B_B + \sum_{i=1}^4 \boldsymbol{I}^B_{W_i} \boldsymbol{\omega}^B_{W_i} + m_B \boldsymbol{R}^B_O \boldsymbol{\rho}^O_B \times \dot{\boldsymbol{\rho}}^O_B \\ &+ m_T \boldsymbol{R}^B_O \boldsymbol{\rho}^O_T \times \dot{\boldsymbol{\rho}}^O_T + \sum_{i=1}^4 m_{W_i} \boldsymbol{R}^B_O \boldsymbol{\rho}^O_{W_i} \times \dot{\boldsymbol{\rho}}^O_{W_i}.\end{aligned} \tag{2}$$

The different momentum components in (2) are computed. Let us start from $m_B \boldsymbol{R}^B_O \boldsymbol{\rho}^O_B \times \dot{\boldsymbol{\rho}}^O_B$; with $\boldsymbol{\rho}^O_B = \boldsymbol{R}^O_B \boldsymbol{\rho}^B_B$, $\dot{\boldsymbol{\rho}}^O_B = \dot{\boldsymbol{R}}^O_B \boldsymbol{\rho}^B_B + \boldsymbol{R}^O_B \dot{\boldsymbol{\rho}}^B_B$ holds. Thus, we have

$$\begin{aligned}& m_B \boldsymbol{R}^B_O \boldsymbol{\rho}^O_B \times \dot{\boldsymbol{\rho}}^O_B \\ &= m_B (\boldsymbol{R}^O_B)^T (\boldsymbol{R}^O_B \boldsymbol{\rho}^B_B) \times (\dot{\boldsymbol{R}}^O_B \boldsymbol{\rho}^B_B + \boldsymbol{R}^O_B \dot{\boldsymbol{\rho}}^B_B) \\ &= m_B (\boldsymbol{R}^O_B)^T (\boldsymbol{R}^O_B \boldsymbol{\rho}^B_B) \times (\dot{\boldsymbol{R}}^O_B \boldsymbol{\rho}^B_B) \\ &\quad + m_B (\boldsymbol{R}^O_B)^T (\boldsymbol{R}^O_B \boldsymbol{\rho}^B_B) \times \boldsymbol{R}^O_B \dot{\boldsymbol{\rho}}^B_B) \\ &= m_B ((\boldsymbol{R}^O_B)^T \boldsymbol{R}^O_B \boldsymbol{\rho}^B_B) \times ((\boldsymbol{R}^O_B)^T \dot{\boldsymbol{R}}^O_B \boldsymbol{\rho}^B_B) \\ &+ m_B ((\boldsymbol{R}^O_B)^T \boldsymbol{R}^O_B \boldsymbol{\rho}^B_B) \times ((\boldsymbol{R}^O_B)^T \boldsymbol{R}^O_B \dot{\boldsymbol{\rho}}^B_B) \\ &= m_B \boldsymbol{\rho}^B_B \times ((\boldsymbol{R}^O_B)^T \dot{\boldsymbol{R}}^O_B \boldsymbol{\rho}^B_B) + m_B \boldsymbol{\rho}^B_B \times \dot{\boldsymbol{\rho}}^B_B \\ &= -m_B [\boldsymbol{\rho}^B_B \times][\boldsymbol{\rho}^B_B \times] \boldsymbol{\omega}^B_B + m_B \boldsymbol{\rho}^B_B \times \dot{\boldsymbol{\rho}}^B_B,\end{aligned} \tag{3}$$

where $\boldsymbol{R}(\boldsymbol{x} \times \boldsymbol{y}) = (\boldsymbol{R}\boldsymbol{x}) \times (\boldsymbol{R}\boldsymbol{y})$ and $[\boldsymbol{\omega}^B_B \times] = \boldsymbol{R}^B_O \dot{\boldsymbol{R}}^O_B$ are used. $[(\cdot)\times]$ denotes a skew-symmetric matrix operator. Similarly, we calculate the last two terms in the right side of (2). Then we can update (2) to

$$\begin{aligned}\boldsymbol{H}^B &= \boldsymbol{I}^B_B \boldsymbol{\omega}^B_B + \sum_{i=1}^4 \boldsymbol{I}^B_{W_i} \boldsymbol{\omega}^B_{W_i} + m_B [\boldsymbol{\rho}^B_B \times] \dot{\boldsymbol{\rho}}^B_B + m_T [\boldsymbol{\rho}^B_T \times] \dot{\boldsymbol{\rho}}^B_T \\ &+ \sum_{i=1}^4 m_{W_i} [\boldsymbol{\rho}^B_{W_i} \times] \dot{\boldsymbol{\rho}}^B_{W_i} - m_B [\boldsymbol{\rho}^B_B \times][\boldsymbol{\rho}^B_B \times] \boldsymbol{\omega}^B_B \\ &- m_T [\boldsymbol{\rho}^B_T \times][\boldsymbol{\rho}^B_T \times] \boldsymbol{\omega}^B_B - \sum_{i=1}^4 m_{W_i} [\boldsymbol{\rho}^B_{W_i} \times][\boldsymbol{\rho}^B_{W_i} \times] \boldsymbol{\omega}^B_B.\end{aligned} \tag{4}$$

The task becomes the determination of each term in (4).

Since we have assumed that the CoM of the whole robot locates at the origin of the inertial frame, the CoM of the whole robot can be expressed as

$$m_B \boldsymbol{\rho}^O_B + m_T \boldsymbol{\rho}^O_T + \sum_{i=1}^4 m_{W_i} \boldsymbol{\rho}^O_{OW_i} = \boldsymbol{0}. \tag{5}$$

To find the relation between $\boldsymbol{\rho}^O_T$ and $\boldsymbol{\rho}^O_B$, a close path involving the tail (see purple dashed arrows in Fig. 1 (a)) gives

$$\boldsymbol{\pi}^O_{OB} - \boldsymbol{\pi}^O_{PB} + \boldsymbol{\pi}^O_{PT} - \boldsymbol{\pi}^O_{OT} = \boldsymbol{0}, \tag{6}$$

where

$$\boldsymbol{\pi}_{OB}^{O} = \boldsymbol{\rho}_{B}^{O},\ \boldsymbol{\pi}_{OT}^{O} = \boldsymbol{\rho}_{T}^{O}$$

$$\boldsymbol{\pi}_{PB}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{\pi}_{PB}^{B} = \boldsymbol{R}_{B}^{O}[\ell_B, 0, 0]^T,$$

$$\boldsymbol{\pi}_{PT}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{T}^{B}\boldsymbol{\pi}_{PT}^{T} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{T}^{B}[0, 0, \ell_T]^T.$$

Pre-multiplying both sides of (6) by m_T yields

$$m_T\boldsymbol{\rho}_{OT}^{O} = m_T\boldsymbol{\rho}_{OB}^{O} - m_T(\boldsymbol{\pi}_{PB}^{O} - \boldsymbol{\pi}_{PT}^{O}). \tag{7}$$

Similarly, close paths involving wheels (see an example of green dashed arrows in Fig. 1 (b)) give

$$\begin{aligned}
\boldsymbol{\pi}_{OB}^{O} - \boldsymbol{\pi}_{S_1B}^{O} + \boldsymbol{\pi}_{S_1W_1}^{O} - \boldsymbol{\pi}_{OW_1}^{O} &= \boldsymbol{0},\\
\boldsymbol{\pi}_{OB}^{O} - \boldsymbol{\pi}_{S_2B}^{O} + \boldsymbol{\pi}_{S_2W_2}^{O} - \boldsymbol{\pi}_{OW_2}^{O} &= \boldsymbol{0},\\
\boldsymbol{\pi}_{OB}^{O} - \boldsymbol{\pi}_{S_3B}^{O} + \boldsymbol{\pi}_{S_3W_3}^{O} - \boldsymbol{\pi}_{OW_3}^{O} &= \boldsymbol{0},\\
\boldsymbol{\pi}_{OB}^{O} - \boldsymbol{\pi}_{S_4B}^{O} + \boldsymbol{\pi}_{S_4W_4}^{O} - \boldsymbol{\pi}_{OW_4}^{O} &= \boldsymbol{0},
\end{aligned} \tag{8}$$

where

$$\boldsymbol{\pi}_{OW_i}^{O} = \boldsymbol{\rho}_{W_i}^{O}, i = 1, \cdots, 4,$$

$$\boldsymbol{\pi}_{S_1B}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{\pi}_{S_1B}^{B} = \boldsymbol{R}_{B}^{O}[-a/2, -b/2, d]^T,$$

$$\boldsymbol{\pi}_{S_1W_1}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_1}^{B}\boldsymbol{\pi}_{S_1W_1}^{W_1} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_1}^{B}[0, c, 0]^T,$$

$$\boldsymbol{\pi}_{S_2B}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{\pi}_{S_2B}^{B} = \boldsymbol{R}_{B}^{O}[-a/2, b/2, d]^T,$$

$$\boldsymbol{\pi}_{S_2W_2}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_2}^{B}\boldsymbol{\pi}_{S_2W_2}^{W_2} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_2}^{B}[0, -c, 0]^T,$$

$$\boldsymbol{\pi}_{S_3B}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{\pi}_{S_3B}^{B} = \boldsymbol{R}_{B}^{O}[a/2, -b/2, d]^T,$$

$$\boldsymbol{\pi}_{S_3W_3}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_3}^{B}\boldsymbol{\pi}_{S_3W_3}^{W_3} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_3}^{B}[0, c, 0]^T,$$

$$\boldsymbol{\pi}_{S_4B}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{\pi}_{S_4B}^{B} = \boldsymbol{R}_{B}^{O}[a/2, b/2, d]^T,$$

$$\boldsymbol{\pi}_{S_4W_4}^{O} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_4}^{B}\boldsymbol{\pi}_{S_4W_4}^{W_4} = \boldsymbol{R}_{B}^{O}\boldsymbol{R}_{W_4}^{B}[0, -c, 0]^T.$$

Rearranging (8) yields

$$\sum_{i=1}^{4} m_{W_i}\boldsymbol{\rho}_{W_i}^{O} = \sum_{i=1}^{4} m_{W_i}\boldsymbol{\rho}_{B}^{O} - \sum_{i=1}^{4} m_{W_i}(\boldsymbol{\pi}_{S_iB}^{O} - \boldsymbol{\pi}_{S_iW_i}^{O}) \tag{9}$$

Substituting (7) and (9) into (5) gives

$$\begin{aligned}
(m_B + m_T + \sum_{i=1}^{4} m_{W_i})\boldsymbol{\rho}_{B}^{O} = m_T(&\boldsymbol{\pi}_{PB}^{O} - \boldsymbol{\pi}_{PT}^{O})\\
&+ \sum_{i=1}^{4} m_{W_i}(\boldsymbol{\pi}_{S_iB}^{O} - \boldsymbol{\pi}_{S_iW_i}^{O}),
\end{aligned} \tag{10}$$

thus,

$$\boldsymbol{\rho}_{B}^{O} = \frac{m_T(\boldsymbol{\pi}_{PB}^{O} - \boldsymbol{\pi}_{PT}^{O}) + \sum_{i=1}^{4} m_{W_i}(\boldsymbol{\pi}_{S_iB}^{O} - \boldsymbol{\pi}_{S_iW_i}^{O})}{m_B + m_T + \sum_{i=1}^{4} m_{W_i}},$$

and

$$\boldsymbol{\rho}_{B}^{B} = \frac{m_T(\boldsymbol{\pi}_{PB}^{B} - \boldsymbol{\pi}_{PT}^{B}) + \sum_{i=1}^{4} m_{W_i}(\boldsymbol{\pi}_{S_iB}^{B} - \boldsymbol{\pi}_{S_iW_i}^{B})}{m_B + m_T + \sum_{i=1}^{4} m_{W_i}}.$$

If we assume that all wheels are the same ($\boldsymbol{I}_{W_i}^{W_i} = \boldsymbol{I}_W = \mathrm{diag}([(\boldsymbol{I}_W)_{xx}, (\boldsymbol{I}_W)_{yy}, (\boldsymbol{I}_W)_{zz}])$, $m_{W_i} = m_W$) and rotate with the same velocity, the rotation matrix related to the wheels is

$$\boldsymbol{R}_{W_i}^{B} = \mathrm{Rot}_y(\varphi_1) = \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix}.$$

Then,

$$\begin{aligned}
\boldsymbol{\pi}_{S_1B}^{B} - \boldsymbol{\pi}_{S_1W_1}^{B} &= \begin{bmatrix} -a/2 \\ -b/2 \\ d \end{bmatrix} - \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}\\
&= \begin{bmatrix} -a/2 \\ -b/2 - c \\ d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\pi}_{S_2B}^{B} - \boldsymbol{\pi}_{S_2W_2}^{B} &= \begin{bmatrix} -a/2 \\ b/2 \\ d \end{bmatrix} - \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 \\ -c \\ 0 \end{bmatrix}\\
&= \begin{bmatrix} -a/2 \\ b/2 + c \\ d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\pi}_{S_3B}^{B} - \boldsymbol{\pi}_{S_3W_3}^{B} &= \begin{bmatrix} a/2 \\ -b/2 \\ d \end{bmatrix} - \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}\\
&= \begin{bmatrix} a/2 \\ -b/2 - c \\ d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\pi}_{S_4B}^{B} - \boldsymbol{\pi}_{S_4W_4}^{B} &= \begin{bmatrix} a/2 \\ b/2 \\ d \end{bmatrix} - \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 \\ -c \\ 0 \end{bmatrix}\\
&= \begin{bmatrix} a/2 \\ b/2 + c \\ d \end{bmatrix}.
\end{aligned}$$

The rotation matrix related to the tail is given by

$$\boldsymbol{R}_{T}^{B} = \mathrm{Rot}_x(\varphi_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \cos\varphi_2 & -\sin\varphi_2 \\ 0 & \sin\varphi_2 & \cos\varphi_2 \end{bmatrix}.$$

Then,

$$\begin{aligned}
\boldsymbol{\pi}_{PB}^{B} - \boldsymbol{\pi}_{PT}^{B} &= \begin{bmatrix} \ell_B \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & \cos\varphi_2 & -\sin\varphi_2 \\ 0 & \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ell_T \end{bmatrix}\\
&= \begin{bmatrix} \ell_B \\ \ell_T\sin\varphi_2 \\ -\ell_T\cos\varphi_2 \end{bmatrix}.
\end{aligned}$$

So far, we are able to calculate $\boldsymbol{\rho}_{B}^{B}$ as

$$\boldsymbol{\rho}_{B}^{B} = \frac{m_T\begin{bmatrix} \ell_B \\ \ell_T\sin\varphi_2 \\ -\ell_T\cos\varphi_2 \end{bmatrix} + m_W\begin{bmatrix} 0 \\ 0 \\ 4d \end{bmatrix}}{m_B + m_T + 4m_W}.$$

Once we obtain $\boldsymbol{\rho}_B^B$, it is easy to calculate $\boldsymbol{\rho}_T^B$ and $\boldsymbol{\rho}_{W_i}^B$. Firstly, $\boldsymbol{\rho}_T^O$ is calculated:

$$\begin{aligned}\boldsymbol{\rho}_T^O &= \boldsymbol{\rho}_B^O - (\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O)\\ &= \frac{m_T(\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O) + \sum_{i=1}^4 m_W(\boldsymbol{\pi}_{S_iB}^O - \boldsymbol{\pi}_{S_iW_i}^O)}{m_B + m_T + 4m_W} - (\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O)\\ &= \frac{\sum_{i=1}^4 m_W(\boldsymbol{\pi}_{S_iB}^O - \boldsymbol{\pi}_{S_iW_i}^O) - (m_B + 4m_W)(\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O)}{m_B + m_T + 4m_W}.\end{aligned}$$

Expressing $\boldsymbol{\rho}_T^O$ in the body frame gives

$$\boldsymbol{\rho}_T^B = \frac{m_W\begin{bmatrix}0\\0\\4d\end{bmatrix} - (m_B + 4m_W)\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix}}{m_B + m_T + 4m_W}.$$

Similarly, $\boldsymbol{\rho}_{W_1}^O$ is calculated first and $\boldsymbol{\rho}_{W_1}^B$ can be obtained.

$$\begin{aligned}\boldsymbol{\rho}_{W_1}^O &= \boldsymbol{\rho}_B^O - (\boldsymbol{\pi}_{S_1B}^O - \boldsymbol{\pi}_{S_1W_1}^O)\\ &= \frac{m_T(\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O) + \sum_{i=1}^4 m_W(\boldsymbol{\pi}_{S_iB}^O - \boldsymbol{\pi}_{S_iW_i}^O)}{m_B + m_T + 4m_W} - (\boldsymbol{\pi}_{S_1B}^O - \boldsymbol{\pi}_{S_1W_1}^O)\\ &= \frac{m_T(\boldsymbol{\pi}_{PB}^O - \boldsymbol{\pi}_{PT}^O) + \sum_{i=2}^4 m_W(\boldsymbol{\pi}_{S_iB}^O - \boldsymbol{\pi}_{S_iW_i}^O) - (m_B + m_T + 3m_W)(\boldsymbol{\pi}_{S_1B}^O - \boldsymbol{\pi}_{S_1W_1}^O)}{m_B + m_T + 4m_W}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\rho}_{W_1}^B &= \frac{m_T\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix} + m_W\begin{bmatrix}a/2\\ b/2+c\\ 3d\end{bmatrix} - (m_B + m_T + 3m_W)\begin{bmatrix}-a/2\\ -b/2-c\\ d\end{bmatrix}}{m_B + m_T + 4m_W}\\ &= \frac{m_T\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix} + \begin{bmatrix}(m_B + m_T + 4m_W)a/2\\ (m_B + m_T + 4m_W)(b/2+c)\\ -(m_B + m_T)d\end{bmatrix}}{m_B + m_T + 4m_W}\end{aligned}$$

To not repeat, $\boldsymbol{\rho}_{W_2}^B$, $\boldsymbol{\rho}_{W_3}^B$, and $\boldsymbol{\rho}_{W_4}^B$ are given by

$$\boldsymbol{\rho}_{W_2}^B = \frac{m_T\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix} + \begin{bmatrix}(m_B + m_T + 4m_W)a/2\\ -(m_B + m_T + 4m_W)(b/2+c)\\ -(m_B + m_T)d\end{bmatrix}}{m_B + m_T + 4m_W},$$

$$\boldsymbol{\rho}_{W_3}^B = \frac{m_T\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix} + \begin{bmatrix}-(m_B + m_T + 4m_W)a/2\\ (m_B + m_T + 4m_W)(b/2+c)\\ -(m_B + m_T)d\end{bmatrix}}{m_B + m_T + 4m_W},$$

$$\boldsymbol{\rho}_{W_4}^B = \frac{m_T\begin{bmatrix}\ell_B\\ \ell_T\sin\varphi_2\\ -\ell_T\cos\varphi_2\end{bmatrix} + \begin{bmatrix}-(m_B + m_T + 4m_W)a/2\\ -(m_B + m_T + 4m_W)(b/2+c)\\ -(m_B + m_T)d\end{bmatrix}}{m_B + m_T + 4m_W}.$$

Let $m_r = m_B + m_T + 4m_W$, we have

$$\dot{\boldsymbol{\rho}}_B^B = \frac{m_T}{m_r}\begin{bmatrix}0\\ \ell_T\cos\varphi_2\dot{\varphi}_2\\ \ell_T\sin\varphi_2\dot{\varphi}_2\end{bmatrix},$$

$$\dot{\boldsymbol{\rho}}_T^B = \frac{-(m_B + 4m_W)}{m_r}\begin{bmatrix}0\\ \ell_T\cos\varphi_2\dot{\varphi}_2\\ \ell_T\sin\varphi_2\dot{\varphi}_2\end{bmatrix},$$

$$\dot{\boldsymbol{\rho}}_{W_i}^B = \frac{m_T}{m_r}\begin{bmatrix}0\\ \ell_T\cos\varphi_2\dot{\varphi}_2\\ \ell_T\sin\varphi_2\dot{\varphi}_2\end{bmatrix}.$$

One of the terms associated with the wheel is calculated as

$$\begin{aligned}
&\sum_{i=1}^{4} \boldsymbol{I}_{W_i}^{B} \boldsymbol{\omega}_{W_i}^{B} = \sum_{i=1}^{4} \boldsymbol{R}_{W_i}^{B} \boldsymbol{I}_{W_i}^{W_i} \boldsymbol{\omega}_{W_i}^{W_i} = \sum_{i=1}^{4} \boldsymbol{R}_{W_i}^{B} \boldsymbol{I}_{W_i}^{W_i} \begin{bmatrix} 0 \\ \dot{\varphi}_1 \\ 0 \end{bmatrix} \\
&= 4 \begin{bmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 1 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{bmatrix} \boldsymbol{I}_W \begin{bmatrix} 0 \\ \dot{\varphi}_1 \\ 0 \end{bmatrix} \\
&= 4 \begin{bmatrix} 0 \\ (\boldsymbol{I}_W)_{yy} \dot{\varphi}_1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Up to now, we are able to calculate each term in (4). Recalling (4) gives

$$\boldsymbol{H}^{B} = \boldsymbol{A}\boldsymbol{\omega}_{B}^{B} + \boldsymbol{B}\dot{\boldsymbol{\varphi}}, \quad \boldsymbol{A} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{B} \in \mathbb{R}^{3\times 2}, \tag{11}$$

where

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{I}_{B}^{B} - m_B[\boldsymbol{\rho}_{B}^{B}\times][\boldsymbol{\rho}_{B}^{B}\times] - m_T[\boldsymbol{\rho}_{T}^{B}\times][\boldsymbol{\rho}_{T}^{B}\times] \\
&- \sum_{i=1}^{4} m_{W_i}[\boldsymbol{\rho}_{W_i}^{B}\times][\boldsymbol{\rho}_{W_i}^{B}\times], \\
\boldsymbol{B}\dot{\boldsymbol{\varphi}} &= m_B[\boldsymbol{\rho}_{B}^{B}\times]\dot{\boldsymbol{\rho}}_{B}^{B} + m_T[\boldsymbol{\rho}_{T}^{B}\times]\dot{\boldsymbol{\rho}}_{T}^{B} + \sum_{i=1}^{4} m_{W_i}[\boldsymbol{\rho}_{W_i}^{B}\times]\dot{\boldsymbol{\rho}}_{W_i}^{B} \\
&+ \sum_{i=1}^{4} \boldsymbol{I}_{W_i}^{B} \boldsymbol{\omega}_{W_i}^{B} \\
&= \begin{bmatrix} 0 & \frac{\ell_T m_T(\ell_T m_B + 4\ell_T m_W - 4 d m_W \cos\varphi_2)}{m_r} \\ 4(\boldsymbol{I}_W)_{yy} & -\frac{(\ell_B \ell_T \sin\varphi_2 (m_B + 4m_W)}{m_r} \\ 0 & \frac{(\ell_B \ell_T \cos\varphi_2 (m_B + 4m_W)}{m_r} \end{bmatrix} \dot{\boldsymbol{\varphi}}.
\end{aligned}$$

Assuming that the robot starts from rest, the relation between the body angular velocity and the appendage velocity can be derived as

$$\boldsymbol{\omega}_{B}^{B} = -\boldsymbol{A}^{-1}\boldsymbol{B}\dot{\boldsymbol{\varphi}}. \tag{12}$$

Therefore, with the relation $\boldsymbol{\omega}_{B}^{B} = \boldsymbol{Q}\dot{\boldsymbol{\theta}}$, the reduced model is given by

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}\dot{\boldsymbol{\varphi}}, \quad \boldsymbol{J} \in \mathbb{R}^{3\times 2}, \tag{13}$$

where

$$\boldsymbol{J} = -(\boldsymbol{A}\boldsymbol{Q})^{-1}\boldsymbol{B}, \quad \boldsymbol{Q} = \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2 \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2\cos\theta_3 & -\sin\theta_3 & 0 \end{bmatrix},$$

$\boldsymbol{J}$ acts like a Jacobian matrix, and $\boldsymbol{Q}$ is invertible if $\theta_2 \neq \pm(2k-1)\frac{\pi}{2}, k \in \mathbb{N}_0$. For simplicity, we assume $\boldsymbol{Q}$ is invertible in this paper. Note that the Jacobian-like matrix $\boldsymbol{J}$ is different from the Jacobian matrix in redundant robotic systems in terms of dimension.