Robotics 811 - Homework 5

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I discussed with Rick Goldstein and Reuben Aronson on parts of this problem sets.

1 Q1

We are trying to minimize the following equation

$$minA = 2\pi \int_{y(x_0)}^{y(x_1)} y\sqrt{1+y'^2} \tag{1}$$

This means that

$$F = y\sqrt{1+y'^2} \tag{2}$$

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}} \tag{3}$$

Because F does not rely on x, the following is true

$$y'F_{y'} - F = C \tag{4}$$

We then insert 2 and 3 into 4.

$$\begin{aligned}
y'\frac{yy'}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} &= C \\
\frac{y(y')^2}{\sqrt{1+y'^2}} - \frac{y(1+y'^2)}{\sqrt{1+y'^2}} &= C \\
\frac{y(y')^2 - y - yy'^2)}{\sqrt{1+y'^2}} &= C \\
\frac{-y}{\sqrt{1+y'^2}} &= C \\
-y &= C(\sqrt{1+y'^2}) \\
y^2 &= C(1+y'^2) \\
\frac{y^2}{C} &= 1+y'^2 \\
\frac{y^2}{C} - 1 &= y'^2 \\
\sqrt{\frac{y^2}{C} - 1} &= y' \\
\frac{dy}{dx} &= \sqrt{\frac{y^2}{C} - 1} \\
\frac{dx}{dy} &= \frac{1}{\sqrt{\frac{y^2}{C} - 1}}
\end{aligned} \tag{5}$$

Now, we can use solve the differential equation by separation.

$$\begin{aligned}
dx &= \frac{1}{\sqrt{\frac{y^2}{C} - 1}}dy \\
\int dx &= \int \frac{1}{\sqrt{\frac{y^2}{C} - 1}}dy \\
\int dx &= \int \frac{1}{\sqrt{\frac{y^2}{C} - 1}}dy \\
x + k_1 &= cosh^-1(\frac{y}{C}) + k_2 \\
cosh(x + k) &= \frac{y}{C}
\end{aligned} \tag{6}$$

The constant terms, C and k will depend on the constraints of $y_0 = y(x_0)$ and $y_1 = y(x_1)$. This shows that from the Euler-Lagrange equation, the C^2 curve that will minimize the surface of revolution is a *cosh* curve. However, this might not be the true for all y. Another 'optimal curve' that is not C^2 is a 'curve' where it has 2 disk on the starting and the ending point that is connected with a line in between. Whether that solution or the cosh solution is solution with the minimum surface area will depend on the constraints of $y_0 = y(x_0)$ and $y_1 = y(x_1)$.

2 Q2

We want to find the shortest curve between two points on a sphere. The function that we are trying to minimize is

$$\int \sqrt{dx^2 + dy^2 + dz^2} \tag{7}$$

First, we need to transform the following arc length equation into (u, v) space.

$$\sqrt{dx^2 + dy^2 + dz^2} \tag{8}$$

$$\begin{aligned}
x &= R \sin v \cos u \\
dx &= R \cos v \cos u du - R \sin v \sin u du \\
y &= R \sin v \sin u \\
dy &= R \cos v \sin u dv + R \sin v \cos u du \\
z &= R \cos v \\
dz &= -R \sin v dv
\end{aligned} \tag{9}$$

Insert 9 into 8

$$\sqrt{(R \cos v \cos u du - R \sin v \sin u du)^2 + (R \cos v \sin u dv + R \sin v \cos u du)^2 + (-R \sin v dv)^2} \tag{10}$$

First, let simplify the term inside the square root.

$$
\begin{aligned}
R\cos v\cos udu - R\sin v\sin udu)^2 + (R\cos v\sin udv + R\sin v\cos udu)^2 + (-R\sin vdv)^2 \\
= R^2\cos^2 v\cos^2 udv^2 - 2R\cos v\cos u\sin v\sin udvdu + R^2\sin^2 v\sin^2 udu^2 \\
+R^2\cos^2 v\sin^2 udv^2 + 2R\cos v\cos u\sin v\sin udvdu + R^2\sin^2 v\cos^2 udu^2 \\
+R^2\sin^2 vdv^2 \\
= R^2\cos^2 v\cos^2 udv^2 + R^2\sin^2 v\sin^2 udu^2 + R^2\cos^2 v\sin^2 udv^2 + R^2\sin^2 v\cos^2 udu^2 + R^2\sin^2 vdv^2 \\
= R^2(\sin^2 vdu^2(\sin^2 u + \cos^2 u) + \cos^2 vdv^2(\cos^2 u + \sin^2 u) + \sin^2 vdv^2) \\
= R^2(\sin^2 vdu^2 + \cos^2 vdv^2 + \sin^2 vdv^2) \\
= R^2(\sin^2 vdu^2 + dv^2(\cos^2 v + \sin^2 v)) \\
= R^2(\sin^2 vdu^2 + dv^2)
\end{aligned}
\tag{11}
$$

Plug the equation back, we will get

$$
\begin{aligned}
\sqrt{R^2(\sin^2 vdu^2 + dv^2} &= \sqrt{R^2}\sqrt{\sin^2 vdu^2 + dv^2} \\
&= R\sqrt{\sin^2 vdu^2 + dv^2} \\
&= R\sqrt{\sin^2 vdu^2 + dv^2\frac{du^2}{du^2}}
\end{aligned}
\tag{12}
$$

Let's define $v' = \frac{dv}{du}$

$$
\begin{aligned}
R\sqrt{\sin^2 vdu^2 + dv^2\frac{du^2}{du^2}} &= R\sqrt{\sin^2 vdu^2 + dv^2\frac{du^2}{du^2}} \\
&= R\sqrt{\sin^2 vdu^2 + v'^2du^2} \\
&= R\sqrt{du^2(\sin^2 v + v'^2)} \\
&= R\sqrt{\sin^2 v + v'^2}du
\end{aligned}
\tag{13}
$$

Our arc length function will be now

$$
L = R\int\sqrt{\sin^2 v + v'^2}du \tag{14}
$$

Our F function will be

$$
\sqrt{\sin^2 v + v'^2} \tag{15}
$$

It's partial derivate to relative to v' is

$$
\begin{aligned}
F_{v'} &= \frac{\partial}{\partial v'}\sqrt{\sin^2 v + v'^2} \\
&= \frac{1}{2}(\sin^2 v + v'^2)^{-\frac{1}{2}}2v' \\
&= \frac{v'}{\sqrt{\sin^2 v + v'^2}}
\end{aligned}
\tag{16}
$$

Because F does, not relies on u, the following is true:

$$
v'F_{v'} - F = C \tag{17}
$$

We then insert the 15 and 16 into the previous equation.

$$
\begin{aligned}
v'(\frac{v'}{\sqrt{\sin^2 v + v'^2}}) - \sqrt{\sin^2 v + v'^2} &= C \\
\frac{(v')^2}{\sqrt{\sin^2 v + v'^2}}) - \sqrt{\sin^2 v + v'^2} &= C \\
\frac{(v')^2}{\sqrt{\sin^2 v + v'^2}}) - \frac{\sin^2 v + v'^2}{\sqrt{\sin^2 v + v'^2}} &= C \\
\frac{(v')^2}{\sqrt{\sin^2 v + v'^2}}) - \frac{\sin^2 v + v'^2}{\sqrt{\sin^2 v + v'^2}} &= C \\
\frac{(v')^2 - \sin^2 v + v'^2}{\sqrt{\sin^2 v + v'^2}} &= C \\
\frac{\sin^2 v}{\sqrt{\sin^2 v + v'^2}} &= C \\
\sin^2 v &= C\sqrt{\sin^2 v + v'^2} \\
\sin^4 v &= C^2(\sin^2 v + v'^2) \\
\sin^4 v &= C^2 \sin^2 v + C^2 v'^2 \\
\sin^4 v - C^2 \sin^2 v &= C^2 v'^2 \\
\sin^4 v - C^2 \sin^2 v &= C^2 v'^2 \\
\frac{\sin^4 v - C^2 \sin^2 v}{C^2} &= v'^2 \\
\frac{\sqrt{\sin^4 v - C^2 \sin^2 v}}{\sqrt{C^2}} &= v' \\
\frac{\sqrt{\sin^4 v - C^2 \sin^2 v}}{C} &= \frac{dv}{du} \\
\frac{C}{\sqrt{\sin^4 v - C^2 \sin^2 v}} &= \frac{du}{dv} \\
\frac{C dv}{\sqrt{\sin^4 v - C^2 \sin^2 v}} &= du \\
\int \frac{C dv}{\sqrt{\sin^4 v - C^2 \sin^2 v}} &= \int du
\end{aligned}
\tag{18}
$$

Now we could used the identity given in the homework. This will give us

$$\begin{aligned}
u + k_1 &= -\sin^- 1(\frac{\cot v}{\sqrt{\frac{1}{C^2} - 1}}) + k_2 \\
u + k_3 &= -\sin^- 1(\frac{\cot v}{\sqrt{\frac{1}{C^2} - 1}}) \\
\sin(u + k_3) &= -(\frac{\cot v}{\sqrt{\frac{1}{C^2} - 1}}) \\
\sin(u + k_3) &= -(\frac{\cos v}{\sin v\sqrt{\frac{1}{C^2} - 1}})) \\
\sin u \cos k_3 + \cos u \sin k_3 &= -(\frac{\cos v}{\sin v\sqrt{\frac{1}{C^2} - 1}}) \\
(\sin u \cos k_3 + \cos u \sin k_3)(\sin v) &= -(\frac{\cos v}{\sqrt{\frac{1}{C^2} - 1}}) \\
\sin u \cos k_3 \sin v + \cos u \sin k_3 \sin v &= -(\frac{\cos v}{\sqrt{\frac{1}{C^2} - 1}}) \\
\sin u \cos k_3 \sin v + \cos u \sin k_3 \sin v + (\frac{\cos v}{\sqrt{\frac{1}{C^2} - 1}}) &= 0
\end{aligned} \tag{19}$$

Now we substitute 9 which where $\frac{x}{R} = \sin v \cos u$, $\frac{y}{R} = \sin v \sin u$ and $\frac{z}{R} = \cos v$ into the previous equation, we will get

$$\cos k_3 \frac{y}{R} + \sin k_3 \frac{x}{R} + \frac{z}{R}\frac{1}{\sqrt{\frac{1}{C^2} - 1}} = 0. \tag{20}$$

Equation 20 is an infinite plane equation with the origin at $(0, 0, 0)$. This means that the plane will intercept the sphere at the center of the sphere. The interception of the plane and sphere will create a greater circle. Therefore, the shortest curve between the two points on the sphere will be the arch of this greater circle.

3 Q3

As, this is an open boundary problem similar to the question discussed in class. We know that F is the following

$$F(x, y, y') = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1 + y'^2}}{\sqrt{y_0 - y_1}} \tag{21}$$

We then take the partial of F.

$$F_{y'} = \frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0 - y)(1 + y'^2)}} \tag{22}$$

As discussed in class, the relationship is described as the following.

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0 \tag{23}$$

Insert 21 and 22 into 23.

$$
\begin{aligned}
\frac{1}{\sqrt{-2g}}\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} - \frac{g_y\sqrt{1+y'^2}}{(\sqrt{-2g}\sqrt{y_0-y_1})(g_x+g_yy')} &= 0 \\
\frac{1}{\sqrt{-2g}}\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} &= \frac{g_y\sqrt{1+y'^2}}{(\sqrt{-2g}\sqrt{y_0-y_1})(g_x+g_yy')} \\
\frac{1}{\sqrt{-2g}}\frac{1}{\sqrt{y_0-y_1}}\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{1}{\sqrt{-2g}}\frac{1}{\sqrt{y_0-y_1}}\frac{g_y\sqrt{1+y'^2}}{(g_x+g_yy')} \\
\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{g_y\sqrt{1+y'^2}}{(g_x+g_yy')} \\
\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{g_y(1+y'^2)}{\sqrt{(1+y'^2)}(g_x+g_yy')} \\
y' &= \frac{g_y(1+y'^2)}{(g_x+g_yy')} \\
g_xy'+g_yy'^2 &= g_y+g_yy'^2 \\
g_xy' &= g_y \\
y' &= \frac{g_x}{g_y}
\end{aligned} \tag{24}
$$

You can observe that $\frac{g_x}{g_y}$ gives us the slope for the gradient of the iso-contour. As we know, the gradient of an iso-contour are outwards from the center and perpendicular to the tangent of the iso-contour. Therefore, showing that the direction of the function, y' is the same as the slope, means the direction of the curve must be parallel to the gradient which is perpendicular to the iso-contour.

4 Q4

As learned in class, the Lagrangian is defined as $L = T - V$. Because we are not calculating gravity, the potential energy, V is zero and the T is the sum of all kinectic energy. Therefore,

$$L = T - V = T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_2v_2^2 \tag{25}$$

Now we will try to derive all the variables in the equation above. Because there are two end points, the first point will have the subscript of 2 and the second point will have the subscript of 3. This will change the previous equation to $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$. First we will get those variables that could be immediately obtain from the figure.

$$
\begin{aligned}
&v_1 = l_1\dot{\theta}_1 \\
&x_1 = l_1cos_1 + l_2cos_{12} \\
&x_2 = l_1cos_1 - l_2cos_{12} \\
&y_1 = l_1sin_1 + l_2sin_{12} \\
&y_2 = l_1sin_1 - l_2sin_{12}
\end{aligned} \tag{26}
$$

Now, we will calculate the first derivative of the end points

$$
\begin{aligned}
\dot{x}_1 &= l_1(-sin_1)\dot{\theta}_1 + l_2(-sin_{12})(\dot{\theta}_1+\dot{\theta}_2) \\
&= -l_1sin_1\dot{\theta}_1 - l_2sin_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{x}_2 &= l_1(-sin_1)\dot{\theta}_1 - l_2(-sin_{12})(\dot{\theta}_1+\dot{\theta}_2) \\
&= -l_1sin_1\dot{\theta}_1 + l_2sin_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{y}_1 &= l_1cos_1\dot{\theta}_1 + l_2cos_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{y}_2 &= l_1cos_1\dot{\theta}_1 - l_2cos_{12}(\dot{\theta}_1+\dot{\theta}_2)
\end{aligned} \tag{27}
$$

Now we will be using the previous equation to derive v_3 and v_2 using the fact that $v^2 = \dot{x}^2 + \dot{y}^2$. For the first point

$$\begin{aligned} v_2^2 &= l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \\ v_3^2 &= l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \end{aligned} \tag{28}$$

From all the equations above, we get our final Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2(l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) + \frac{1}{2}m_2(l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - m_2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2\dot{\theta}_1^2 + 2m_2l_2^2\dot{\theta}_1\dot{\theta}_2 + m_2l_2^2\dot{\theta}_2^2 \end{aligned} \tag{29}$$

Now we can apply Lagrangian dynamics. First we find the partial derivative of the system.

$$\begin{aligned} \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{\partial L}{\partial \theta_2} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}_1} &= m_1l_1^2\dot{\theta}_1 + 2m_2l_1^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_2 \\ \frac{\partial L}{\partial \dot{\theta}_1} &= 2m_2l_2^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_2 \end{aligned} \tag{30}$$

Now we derive some of the partial derivation respective to time.

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= m_1l_1^2\ddot{\theta}_1 + 2m_2l_1^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \end{aligned} \tag{31}$$

As we learn in class, the torque, τ is equal to $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$. Following is the two torques for the system.

$$\begin{aligned} \tau_1 &= m_1l_1^2\ddot{\theta}_1 + 2m_2l_1^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ &= \ddot{\theta}_1(m_1l_2^2 + 2m_2l_1^2 + 2m_2l_2^2) + \ddot{\theta}_2(2m_2l_2^2) \\ \tau_2 &= 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ &= \ddot{\theta}_1(2m_2l_2^2) + \ddot{\theta}_2(2m_2l_2^2) \end{aligned} \tag{32}$$

4(b)

When the $\ddot{\theta}_2 = 0$, τ_1 would be

$$\begin{aligned} \tau_1 &= \ddot{\theta}_1(m_1l_1^2 + 2m_2l_1^2 + 2m_2l_2^2) + 0(2m_2l_2^2) \\ &= \ddot{\theta}_1(m_1l_1^2 + 2m_2l_1^2 + 2m_2l_2^2) \end{aligned} \tag{33}$$

In angular rotation, the term relating to $\ddot{\theta}_1$ is known as the **Moment of Inertia**. According to wikipedia, the moment of inertia determines how much force is needed for a desired angular acceleration,$\ddot{\theta}_1$. The moment of inertia for a point object is $I = mr^2$ where r is the length to the point. This simple equation explained the first term of the equation which is $m_1 l_1^2$ that is the moment of inertia for the point at the end of l_1. The moment of inertia for the two endpoints relative to the end of l_1 would be $m_2 l_2^2$ for each. To transform the moment of inertia back to the origin point, we could use the **Parallel Axis Theorem** which states that to transform the rotation axis to a new axis, the Inertia is related by $I = I_{cm} + md^2$ where I_{cm} is the inertia relative to the current point, m is the mass and d being the distance between the old point and new point. We can them sum up the inertia of all three masses.

$$\begin{aligned}
I &= m_1 l_1^2 + m_2 l_2^2 + m_2 l_1 + m_2 l_2^2 + m_2 l_1 \\
I &= m_1 l_1^2 + 2m_2 l_2^2 + 2m_2 l_1
\end{aligned} \tag{34}$$

We have successfully derived the moment of Inertia from the information in the figure which is the same term related to $\ddot{\theta}_1$ in 33.