Assignment 3 Solutions
Robotics 811, Fall 2015

1. Consider the function $f(x) = \sinh x$ over the interval $[-2, 2]$.

(a) What is the Taylor series expansion for $f(x)$ around $x = 0$?

In the Taylor series expansion for $f(x)$ around $x = 0$, the derivative alternates between 0 and 1 ($\cosh(0)$ and $\sinh(0)$). The result is that the even Taylor terms drop out leaving

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

(b) Graph $f(x)$ over the interval $[-2, 2]$.

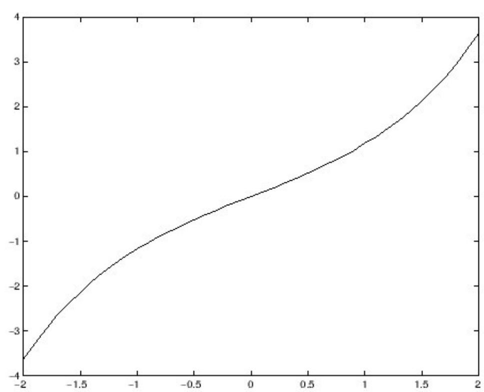


Figure 1: $\sinh(x)$ on the range $[-2, 2]$

(c) Determine the best uniform approximation by a quadratic to $f(x)$ over the interval $[-2, 2]$.
What are the L_∞ and L_2 errors for this approximation?

We know that the best quadratic uniform approximation (BQUA) $p(x) = ax^2 + bx + c \approx \sinh(x)$ will have the following characteristics:

- There will be four points $(n + 2)$ at which the error is maximal.
- Since the third derivative ($f'''(x) = \cosh(x)$) does not change sign on the interval $[-2, 2]$, two of those points (x_0, x_3) will be the end points.
- Since the function is symmetric over this interval, $x_1 = -x_2$.
- Since the function is odd over this interval, the even coefficients will be zero (fairly straightforward to prove).

Given these characteristics, the BQUA will be of the form $p(x) = bx$, and the maximum error will be at four points ($x_0 = -2, -x_2, x_2, x_3 = 2$). To solve for the two unknowns (x_2 and b), we note that the derivative of the error reaches 0 at x_2

$$e'(x_2) = \cosh(x_2) - b = 0 \text{ and } e(x_1) = -e(x_2)$$

and combined together, we have

$$\sinh(2) + 2\cosh(x_2) - \sinh(x_2) + x_1 \cosh(x_2) = 0$$

Using numerical methods to find the roots of this equation, we find that $x_2 = 1.0472$ and $b = 1.6002$.

Finding the L_∞ norm is easy, since we know already know the maximum error points:

$$||e(x)||_\infty = |\sinh(x_0) - b(x_0)| = 0.4264$$

For the L_2 norm, we can just solve numerically to find

$$||e(x)||_2 = \sqrt{\int_{-2}^{2} (\sinh(x) - b(x))^2 dx} = 0.5953$$

(d)

Determine the best least-squares approximation by a quadratic to $f(x)$ over the interval $[-2, 2]$. What are the L_∞ and L_2 errors for this approximation?

Several of you discretized the interval (usually with a stepsize of 0.1) and ran SVD on some system of equations $Ac = f$, which didn't give a very good answer. If you discretize the interval finely (say 0.001), your answer would be close to the answer given by polynomial projection:

$$p(x) = \sum_{i=1}^{2} \frac{< \sinh(x), p_i >}{< p_i, p_i >} p_i(x)$$

Where the basis functions, the Legendre polynomials, are defined as in the notes, and are

$$\begin{aligned} p_0 &= 1 \qquad < p_0, p_0 >= 4 \\ p_1 &= x \qquad < p_1, p_1 >= \frac{16}{3} \\ p_2 &= x^2 - \frac{4}{3} \end{aligned}$$

Projecting our function $\sinh(x)$ onto these basis functions, we have

$$\begin{aligned} < \sinh(x), p_0 > &= \int_{-2}^{2} \sinh(x) dx = 0 \\ < \sinh(x), p_1 > &= \int_{-2}^{2} x \sinh(x) dx = 7.7951 \\ < \sinh(x), p_2 > &= \int_{-2}^{2} (x^2 - \frac{4}{3}) \sinh(x) dx = 0 \end{aligned}$$

So we can write our interpolating polynomial as

$$p(x) = \frac{< \sinh(x), p_1 >}{< p_1, p_1 >} p_1(x) = 1.4616x$$

and our L_2 norm can be computed numerically:

$$||e(x)||_2 = 0.5019.$$

The L_∞ norm is

$$||e(x)||_\infty = \max_{-2 \le x \le 2} |\sinh(x) - b(x)| = 0.7037,$$

which can be found by checking at the endpoints and points where $e'(x) = 0$.

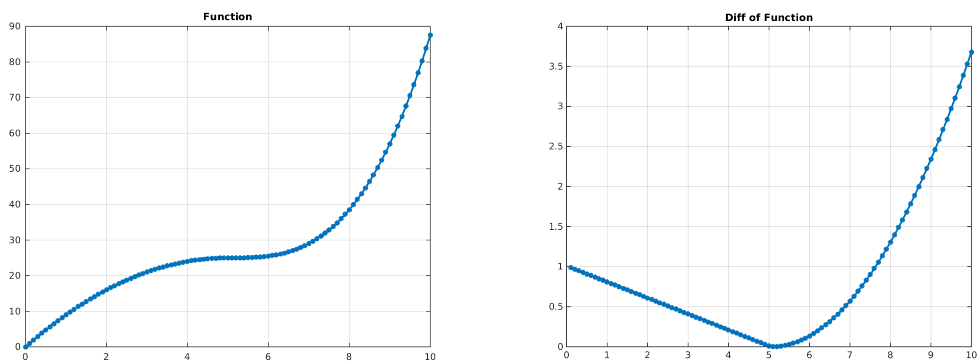
2.

> Suppose very accurate values of some function $f(x)$ are given at the points $0 = x_0, x_1, \ldots, x_{100} = 10$, with $x_i = i/10$, $i = 0, \ldots, 100$.
> What is the function $f(x)$? [Provide a succinct description using analytic expressions.]

The general approach here is to use SVD to find a set of basis functions that fit the data over this range. SVD is used to find the weighting or mixing coefficient of each basis function, solving for x in the equation $Ax = d$, where A is a matrix of basis function values over the range, x are the mixing coefficients, and d is the data given in the data file.

A number of solutions achieved very low error by just throwing together a large number of polynomial or trigonometric functions, or even both.

The key to solving the problem is to look at the difference data, from which its clear that function is piecewise $[0, 5], (5, 10]$.



(a) Function Plot (b) Function Difference Plot

Figure 2: Plotted using plot_problem2.m

SVD can then be used to find the 2 polynomials.

$$f(x) = \begin{cases} 10x - x^2 & : 0 \leq x \leq 5 \\ -37.5 + 37.5x - 7.5x^2 + 0.5x^3 & : 5 < x \leq 10 \end{cases}$$

3.

> The Chebyshev polynomials of the first kind are defined according to: $T_n(\cos\theta) = cos(n\theta)$ for $n \geq 0$.

(a)

> Derive T_6 and T_7 and show that they are orthogonal polynomials relative to the inner product
> $$< g, h > = \int_{-1}^{1} (1 - x^2)^{-1/2} g(x) h(x) dx.$$

$$\begin{aligned} T_6(\cos\theta) &= \cos(6\theta) \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\ T_7(\cos\theta) &= \cos(7\theta) \\ T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \end{aligned}$$

To show that T_6 and T_7 are orthogonal, we must show that $< T_6, T_7 >= 0$ relative to the inner product given above. Plugging in, we see that

$$< T_6, T_7 >= \int_{-1}^{1} (1-x^2)^{-1/2} T_6(x) T_7(x) dx$$

must be an odd function of x, so it will integrate to zero on the interval $[-1, 1]$.

(b)

Relative to this inner product, all the T_n, with $n > 0$, have the same length. Compute that length.

Using the definition of the Chebyshev polynomials, we can show that for $k > 0$, The length of T_n is the same for all n.

$$T_n(\cos\theta) = \cos(n\theta) \quad \text{where} \quad x = \cos(\theta), dx = -\sin(\theta)d\theta$$

And computing the inner product using this definition gives

$$\begin{aligned}
< T_n, T_n > &= \int_{-1}^{1} \frac{T_n^2}{\sqrt{1-x^2}} dx && (1)\\
&= -\int_0^{\pi} \frac{(\cos(n\theta))^2}{\sqrt{1-\cos^2(\theta)}} \sin(\theta) d\theta && (2)\\
&= \frac{1}{2}\int_0^{\pi} (\cos(2n\theta)+1) d\theta && (3)\\
&= \frac{1}{2}(\theta + \frac{1}{2n}\sin(2n\theta))\Big|_0^{\pi} && (4)\\
&= \frac{\pi}{2} && (5)
\end{aligned}$$

So the length of all Chebyshev polynomials T_k, with $k > 0$, is

$$\sqrt{< T_n, T_n >} = \sqrt{\frac{\pi}{2}}$$

4.

After weeks of work you have finally completed construction of a gecko robot. It is a quadruped robot with suctioning feet that allow it to walk on walls. It is also equipped with a Kinect-like sensor, providing a 3D point cloud observation of the world. You want to use these point clouds to reason about the environment and aid in navigation.

(a)

You boot up the robot and place it on a table, taking an initial observation. The observation is saved in the provided `clear_table.txt`, and lists (x,y,z) locations in the following format:

$$\begin{matrix} x_1 & y_1 & z_1 \\ & \vdots & \\ x_n & y_n & z_n \end{matrix}$$

Points are in units of meters and the positive x-direction is right, positive y-direction is down, and positive z-direction is forward. Find the least-squares approximation plane that fits the data. Visualize your fitted plane along with the data. What is the average distance of a point in our data set to the fitted plane? (i.e., how accurate is our sensor?)

Here we find the simple least-squares estimate of the plane intersecting all of the sample points. Using SVD we find the least-squares solution with the average distance of points to the fitted plane being 0.002737.

(b)

Interested in your gecko robot, your cat jumps up on the table. You take a second observation, saved as the provided `cluttered_table.txt`. Using the same method as above, find the least-squares fit to the new data. How does it look? Why?

In this situation the least-squares solution from SVD is heavily impacted by the presence of the cat.

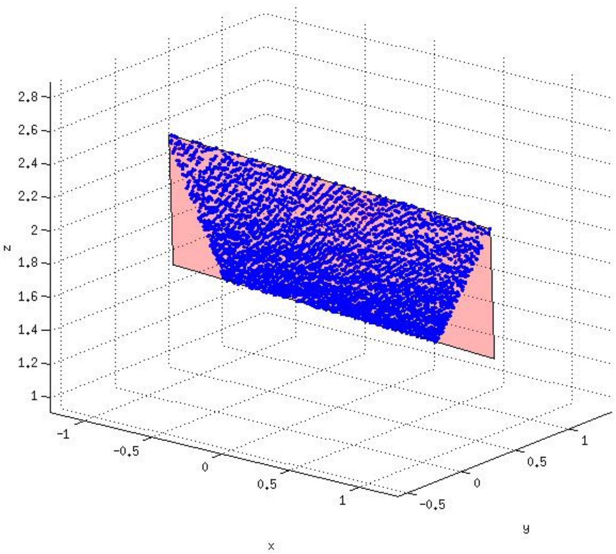


Figure 3: SVD solution to a planar scan.

(c) Can you suggest a way to still find a fit to the plane on the table regardless of clutter? Verify your idea by writing a program that can successfully find the dominant plane in a list of points regardless of outliers. [Hint: You may assume that the number of points on the plane is much larger than the number of points not on the plane.] Visualize `cluttered_table.txt` with your new plane.

The common solution to a problem such as this is RANSAC, or RANdom SAmple Consensus. RANSAC works by taking a small handful of randomly-selected points (3 in this case as we are trying to find the equation of a plane), find the equation of the plane given by these points, find the number of points that lie within ϵ of the plane (inlier points), and iterate, keeping track of the best planar fit found by any random set. At the end of n iterations, return the best solution. The parameters of the process are n, the number of iterations of the algorithm, and ϵ, the distance at which points are considered inliers.

(d) Encouraged by your results when testing on a table, you move your geckobot into the hallway and take an observation saved as the provided `clean_hallway.txt`. Describe an extension to your solution to part 4c that finds the four dominant planes shown in the scene, then implement it and visualize the data and the four planes. You may assume that there are roughly the same amount of points in each plane.

This solution also uses RANSAC to find planes within the scene. Since you know there are four planes of an approximately equal number of points, you can, run RANSAC until a plane with $\approx 1/4$ points is found, remove those points from the dataset, then iterate 3 more times.

(e) You decide it's time to test your gecko robot's suction feet and move it to a different hallway. The feet are strong enough to ignore the force of gravity, allowing the robot to walk on the floor, walls, or ceiling. However, the locomotion of the legs works best on smooth surfaces with few obstacles. Using your solution from part d), describe how you can mathematically characterize the smoothness of each surface. Load the provided scan `cluttered_hallway.txt`, find and plot the four wall planes, describe which surface is safest for your robot to traverse, and provide the smoothness scores from your mathematical characterization. Note that you can no longer assume that there are roughly the same amount of points in each plane.

Here RANSAC can be extended to work with noisier data. There were two main approaches:

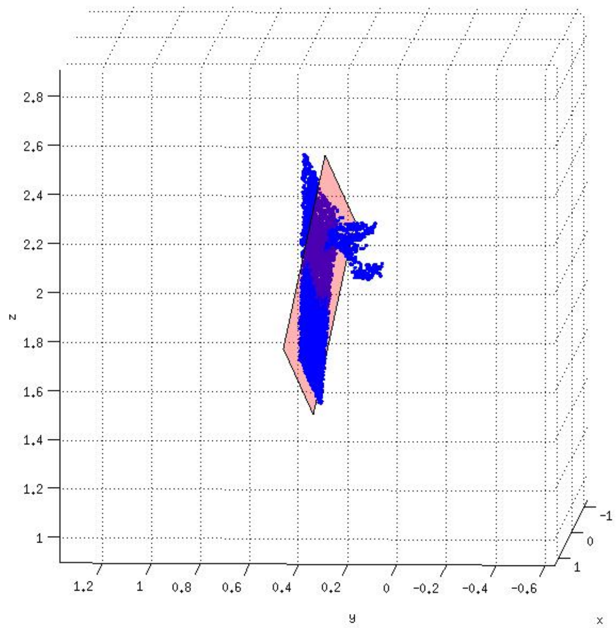


Figure 4: SVD solution to the cat scan.

i. Run RANSAC like normal, beginning with strict values for the distance required for a point to be an inlier. If a plane is found that contains a large enough number of points, remove those points. If no plane is found, relax the inlier constraint until one is.

ii. Reduce the number of inlier points required to find a plane. Find all planes in the scene, as long as they are above the minimum number of inlier points required. Once no more planes are found, take the four largest or four with the farthest centroids from each other, and iteratively join these planes with nearby unmatched planes, eventually clustering them into four main sets.

Similarly, there were two main ways that were used to calculate the smoothness of the walls:

i. Find the least-squares distance of all points in a plane to the equation of their respective planes.

ii. Find the minimum eigenvalue of the covariance matrix of the points in each plane.

The least-squares solution is compuationally more efficient and is often used in situations such as these. The eigenvector solution is more computationally intensive, but is often a good indicator of error in real-world problems. In ICP (Iterative Closest Point), these eigenvalues are an indicator of the uncertainty of your point cloud matchin a given direction. It SLAM (Simultaneous Location and Mapping) you will often see error ellipses around landmarks. These are directly related to the eigenvectors of that landmark's covariance matrix.

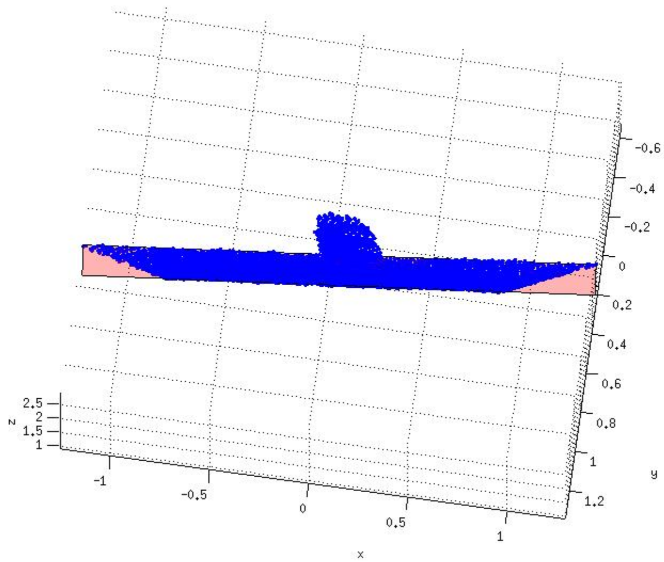


Figure 5: Improved solution to the cat scan.

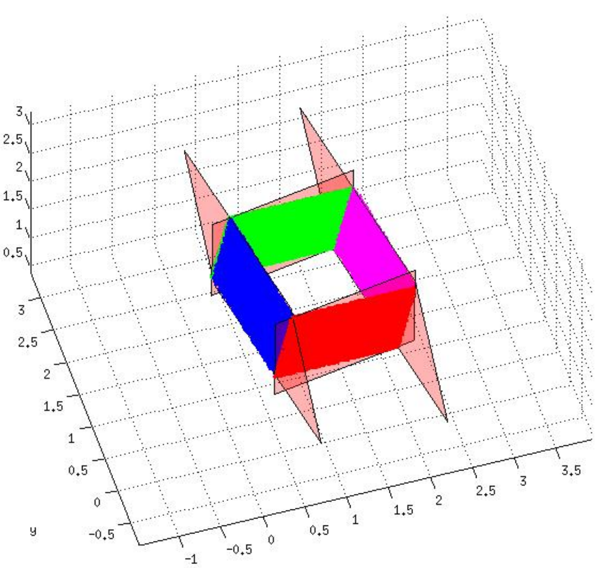


Figure 6: Solution to multiple planes in a clean hallway.

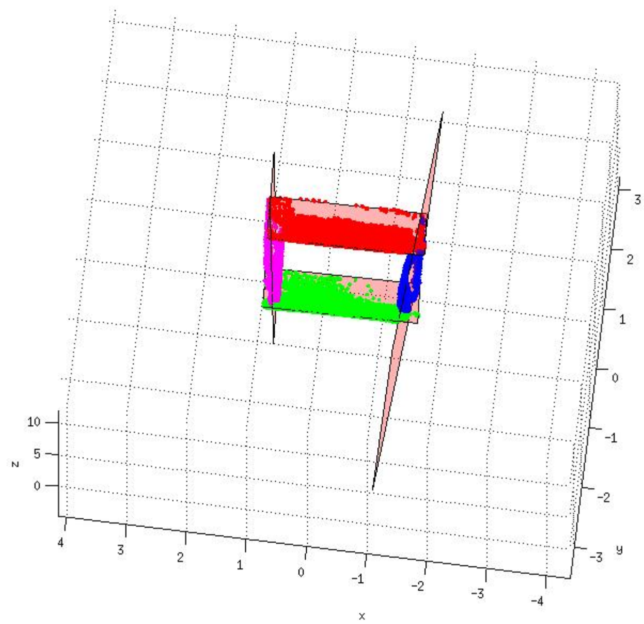


Figure 7: Solution to multiple planes in a cluttered hallway.