Robotics 811 - Homework 5

Xiang Zhi Tan

November 12, 2015

I discussed with Rick Goldstein and Reuben Aronson on parts of this problem sets.

1 Q1

We are trying to minimze the following equation

$$minA = 2\pi \int_{x_0}^{x_1} y\sqrt{1+y'^2} \tag{1}$$

$$F = y\sqrt{1+y'^2} \tag{2}$$

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}} \tag{3}$$

Because F does not rely on x, the following is true

$$y'F_{y'} - F = C \tag{4}$$

We then insert 2 and 3 into 4.

$$\begin{aligned}
y'\frac{yy'}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} &= C \\
\frac{y(y')^2}{\sqrt{1+y'^2}} - \frac{y(1+y'^2)}{\sqrt{1+y'^2}} &= C \\
\frac{y(y')^2 - y - yy'^2)}{\sqrt{1+y'^2}} &= C \\
\frac{-y}{\sqrt{1+y'^2}} &= C \\
-y &= C(\sqrt{1+y'^2}) \\
y^2 &= C(1+y'^2) \\
\frac{y^2}{C} &= 1+y'^2 \\
\frac{y^2}{C} - 1 &= y'^2 \\
\sqrt{\frac{y^2}{C} - 1} &= y' \\
\frac{dy}{dx} &= \sqrt{\frac{y^2}{C} - 1} \\
\frac{dx}{dy} &= \frac{1}{\sqrt{\frac{y^2}{C} - 1}}
\end{aligned} \tag{5}$$

Now, we can use solve the differential equation by seperation.

$$\begin{aligned}
dx &= \frac{1}{\sqrt{\frac{y^2}{C}-1}}dy \\
\int dx &= \int \frac{1}{\sqrt{\frac{y^2}{C}-1}}dy \\
\int dx &= \int \frac{1}{\sqrt{\frac{y^2}{C}-1}}dy \\
x+k_1 = cosh^{-1}1(\frac{y}{C}) + k_2 cosh(x+k) \quad &= \frac{y}{C}
\end{aligned} \tag{6}$$

The constant terms, C and k will depend on the constraints of $y_0 = y(x_0)$ and $y_1 = y(x_1)$. This shows that from the Euler-Lagarange equation, the curve that will minimize the surface of revolution is a *cosh* curve. However, this might not be the true for all y. Another possible solution that is not C^2 is a curve where it has 2 disk on the starting and the ending point that is connected with a line in between. Whether that solution or the cosh solution is the minimum surface area will depend on the initialization.

2 Q2

$$\sqrt{dx^2+dy^2+dz^2} \tag{7}$$

$$\begin{aligned}
x &= R\sin v\cos u \\
dx &= R\cos v\cos u du - R\sin v\sin u du \\
y &= R\sin v\sin u \\
dy &= R\cos v\sin u dv + R\sin v\cos u du \\
z &= R\cos v \\
dz &= -R\sin v dv
\end{aligned} \tag{8}$$

Insert 8 into 7

$$\sqrt{(R\cos v\cos u du - R\sin v\sin u du)^2 + (R\cos v\sin u dv + R\sin v\cos u du)^2 + (-R\sin v dv)^2} \tag{9}$$

First, let simplify the term inside the square root.

$$\begin{aligned}
R\cos v\cos u du - R\sin v\sin u du)^2 + (R\cos v\sin u dv + R\sin v\cos u du)^2 + (-R\sin v dv)^2 \\
= R^2\cos^2 v\cos^2 u dv^2 - 2R\cos v\cos u\sin v\sin u dv du + R^2\sin^2 v\sin^2 u du^2 \\
+R^2\cos^2 v\sin^2 u dv^2 + 2R\cos v\cos u\sin v\sin u dv du + R^2\sin^2 v\cos^2 u du^2 \\
+R^2\sin^2 v dv^2 \\
= R^2\cos^2 v\cos^2 u dv^2 + R^2\sin^2 v\sin^2 u du^2 + R^2\cos^2 v\sin^2 u dv^2 + R^2\sin^2 v\cos^2 u du^2 + R^2\sin^2 v dv^2 \\
= R^2(\sin^2 v du^2(\sin^2 u + \cos^2 u) + \cos^2 v dv^2(\cos^2 u + \sin^2 u) + \sin^2 v dv^2) \\
= R^2(\sin^2 v du^2 + \cos^2 v dv^2 + \sin^2 v dv^2) \\
= R^2(\sin^2 v du^2 + dv^2(\cos^2 v + \sin^2 v)) \\
= R^2(\sin^2 v du^2 + dv^2
\end{aligned} \tag{10}$$

Plug the equation back, we will get

$$\begin{aligned}
\sqrt{R^2(\sin^2 v du^2 + dv^2} &= \sqrt{R^2}\sqrt{\sin^2 v du^2 + dv^2} \\
&= R\sqrt{\sin^2 v du^2 + dv^2} \\
&= R\sqrt{\sin^2 v du^2 + dv^2\frac{du^2}{du^2}}
\end{aligned} \tag{11}$$

Let's define $v' = \frac{dv}{du}$

$$
\begin{aligned}
R\sqrt{\sin^2 v du^2 + dv^2\frac{du^2}{du^2}} &= R\sqrt{\sin^2 v du^2 + dv^2\frac{du^2}{du^2}} \\
&= R\sqrt{\sin^2 v du^2 + v'^2 du^2} \\
&= R\sqrt{du^2(\sin^2 v + v'^2)} \\
&= R\sqrt{\sin^2 v + v'^2} du
\end{aligned} \tag{12}
$$

Our arc length function will be now

$$
L = R\int \sqrt{\sin^2 v + v'^2} du \tag{13}
$$

Our F function will be

$$
\sqrt{\sin^2 v + v'^2} \tag{14}
$$

It's partial derivate to relative to v' is

$$
\begin{aligned}
F_{v'} &= \frac{\partial}{\partial v'}\sqrt{\sin^2 v + v'^2} \\
&= \frac{1}{2}(\sin^2 v + v'^2)^{-\frac{1}{2}} 2v' \\
&= \frac{v'}{\sin^2 v + v'^2}
\end{aligned} \tag{15}
$$

Because F does, not relies on u, the following is true:

$$
v'F_{v'} - F = C \tag{16}
$$

3 Q3

$$
F(x, y, y') = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1 + y'^2}}{\sqrt{y_0 - y_1}} \tag{17}
$$

$$
F_{y'} = \frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0 - y)(1 + y'^2)}} \tag{18}
$$

$$
F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0 \tag{19}
$$

Insert 17 and 18 into 19.

$$
\begin{aligned}
\frac{1}{\sqrt{-2g}}\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} - \frac{g_y\sqrt{1+y'^2}}{(\sqrt{-2g}\sqrt{y_0-y_1})(g_x+g_yy')} &= 0 \\
\frac{1}{\sqrt{-2g}}\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} &= \frac{g_y\sqrt{1+y'^2}}{(\sqrt{-2g}\sqrt{y_0-y_1})(g_x+g_yy')} \\
\frac{1}{\sqrt{-2g}}\frac{1}{\sqrt{y_0-y_1}}\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{1}{\sqrt{-2g}}\frac{1}{\sqrt{y_0-y_1}}\frac{g_y\sqrt{1+y'^2}}{(g_x+g_yy')} \\
\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{g_y\sqrt{1+y'^2}}{(g_x+g_yy')} \\
\frac{y'}{\sqrt{(1+y'^2)}} &= \frac{g_y(1+y'^2)}{\sqrt{(1+y'^2)}(g_x+g_yy')} \\
y' &= \frac{g_y(1+y'^2)}{(g_x+g_yy')} \\
g_xy'+g_yy'^2 &= g_y+g_yy'^2 \\
g_xy' &= g_y \\
y' &= \frac{g_x}{g_y}
\end{aligned} \tag{20}
$$

You can observe that $\frac{g_x}{g_y}$ gives us the slope for the gradient of the iso-contour. As we know, the gradient of an iso-contour are outwards from the center and perpendicular to the tangient of the iso-contuor. There, showing that the direction of the function, y' is the same as the slope, meants the direction of the curve must be parallel to the gradient which is perpendicular to the iso-contour.

4 Q4

As learned in class, the Lagrangian is defined as $L = T - V$. Because we are not calculating gravity, the potential energy, V is zero and the T is the sum of all kinectic energy. Therefore,

$$
L = T - V = T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_2v_2^2 \tag{21}
$$

Now we will try to derive all the variables in the equation above. Because there are two end points, the first point will have the subscript of 2 and the second point will have the subscript of 3. This will change the previous equation to $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$. First we will get those variables that could immediately obtain from the figure.

$$
\begin{aligned}
&v_1 = l_1\dot{\theta}_1 \\
&x_1 = l_1cos_1 + l_2cos_{12} \\
&x_2 = l_1cos_1 - l_2cos_{12} \\
&y_1 = l_1sin_1 + l_2sin_{12} \\
&y_2 = l_1sin_1 - l_2sin_{12}
\end{aligned} \tag{22}
$$

Now, we will calculate the first derivative of the end points

$$
\begin{aligned}
\dot{x}_1 &= l_1(-sin_1)\dot{\theta}_1 + l_2(-sin_{12})(\dot{\theta}_1+\dot{\theta}_2) \\
&= -l_1sin_1\dot{\theta}_1 - l_2sin_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{x}_2 &= l_1(-sin_1)\dot{\theta}_1 - l_2(-sin_{12})(\dot{\theta}_1+\dot{\theta}_2) \\
&= -l_1sin_1\dot{\theta}_1 + l_2sin_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{y}_1 &= l_1cos_1\dot{\theta}_1 + l_2cos_{12}(\dot{\theta}_1+\dot{\theta}_2) \\
\dot{y}_2 &= l_1cos_1\dot{\theta}_1 - l_2cos_{12}(\dot{\theta}_1+\dot{\theta}_2)
\end{aligned} \tag{23}
$$

Now we will be using the previous equation to derive v_3 and v_2 using the fact that $v^2 = \dot{x}^2 + \dot{y}^2$. For the first point

$$\begin{aligned} v_2^2 &= l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \\ v_3^2 &= l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \end{aligned} \tag{24}$$

From all the equations above, we get our final Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2(l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) + \frac{1}{2}m_2(l_1^2\theta_1^2 + l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - 2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1) + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 - m_2l_1l_2cos_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2l_1^2\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot{\theta}_1 - \dot{\theta}_2)^2 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + m_2l_1^2\theta_1^2 + m_2l_2^2\dot{\theta}_1^2 + 2m_2l_2^2\dot{\theta}_1\dot{\theta}_2 + m_2l_2^2\dot{\theta}_2^2 \end{aligned} \tag{25}$$

Now we can apply Lagrangian dynamics. First we find the partial derivative of the system.

$$\begin{aligned} \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{\partial L}{\partial \theta_2} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}_1} &= m_1l_1^2\dot{\theta}_1 + 2m_2l_1^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_2 \\ \frac{\partial L}{\partial \dot{\theta}_1} &= 2m_2l_2^2\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_2 \end{aligned} \tag{26}$$

Now we derive some of the partial derivation respective to time.

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= m_1l_1^2\ddot{\theta}_1 + 2m_2l_1^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \end{aligned} \tag{27}$$

As we learn in class, the torque, τ is equal to $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$. Following is the two torques for the system.

$$\begin{aligned} \tau_1 &= m_1l_1^2\ddot{\theta}_1 + 2m_2l_1^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ &= \ddot{\theta}_1(m_1l_2^2 + 2m_2l_1^2 + 2m_2l_2^2) + \ddot{\theta}_2(2m_2l_2^2) \\ \tau_2 &= 2m_2l_2^2\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2 \\ &= \ddot{\theta}_1(2m_2l_2^2) + \ddot{\theta}_2(2m_2l_2^2) \end{aligned} \tag{28}$$

4(b)

When the $\ddot{\theta}_2 = 0$, tau_1 would be

$$\begin{aligned} \tau_1 &= \ddot{\theta}_1(m_1l_1^2 + 2m_2l_1^2 + 2m_2l_2^2) + 0(2m_2l_2^2) \\ &= \ddot{\theta}_1(m_1l_1^2 + 2m_2l_1^2 + 2m_2l_2^2) \end{aligned} \tag{29}$$

In angular rotation, the term relating to $\ddot{\theta}_1$ is known as the **Moment of Inertia**. According to wikipedia, the moment of inertia determines how much force is needed for a desired angular acceleration,$\ddot{\theta}_1$. The moment of inertia for a point object is $I = mr^2$ where r is the length to the point. This simple equation explained the first term of the equation which is $m_1 l_1^2$ that is the moment of inertia for the point at the end of l_1. The moment of inertia for the two endpoints relative to the end of l_1 would be $m_2 l_2^2$ for each. To transform the moment of inertia back to the origin point, we could use the **Parallel Axis Theorem** which states that to transform the rotation axis to a new axis, the Inertia is related by $I = I_{cm} + md^2$ where I_{cm} is the inertia relative to the current point, m is the mass and d being the distance between the old point and new point. We can them sum up the inertia of all three masses.

$$\begin{aligned}
I &= m_1 l_1^2 + m_2 l_2^2 + m_2 l_1 + m_2 l_2^2 + m_2 l_1 \\
I &= m_1 l_1^2 + 2m_2 l_2^2 + 2m_2 l_1
\end{aligned} \tag{30}$$

We have successfully derived the moment of Inertia from the information in the figure which is the same term related to $\ddot{\theta}_1$ in 29.