Assignment 5 Solutions

Robotics 811, Fall 2015

1. Consider a plane curve $y(x)$ over the interval $[x_0, x_1]$, with specified endpoints $y_0 = y(x_0)$ and $y_1 = y(x_1)$. Assume that $y_0 > 0$ and $y_1 > 0$ and that $y(x) \geq 0$ for $x_0 \leq x \leq x_1$. Now imagine rotating the curve about the x-axis to obtain a surface of revolution. Find the C^2 curve $y(x)$ with specified endpoints that minimizes the surface area of this surface of revolution. [Hint: This problem explores further some of the limitations of the Calculus of Variations. Depending on the endpoint conditions there may or may not be a C^2 solution. How do the endpoint conditions matter, and what happens when there is no solution?]

The surface area of a surface of revolution is given by the integral

$$\begin{aligned} SA &= \int_{x_0}^{x_1} 2\pi y ds \\ &= 2\pi \int_{x_0}^{x_1} y\sqrt{1+y'^2}dx \\ F(x,y,y') &= y\sqrt{1+y'^2} \end{aligned}$$

where ds is the infinitesimal arc length of $y = y(x)$ at x, and $ds = \sqrt{1+y'^2}dx$.

Since F does not depend on x ($F_x = 0$), we can use the *Beltrami identity* instead of solving the Euler-Lagrange equation directly. The Beltrami identity states that

$$F - y'F_{y'} = c,$$

where c is a constant. Since

$$F = y\sqrt{1+y'^2}$$

we take the partial with respect to y' to find

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}}.$$

Plugging this back into the Beltrami identity, we have

$$y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = c.$$

Simplifying, we get

$$\begin{aligned} y(1+y'^2) - yy'^2 &= c\sqrt{1+y'^2} \\ y &= c\sqrt{1+y'^2} \\ y' &= \sqrt{\frac{y^2}{c^2} - 1}, \end{aligned}$$

and since $y' = \frac{dy}{dx}$, we can write

$$\begin{aligned} \frac{dx}{dy} &= \frac{c}{\sqrt{y^2-c^2}} \\ dx &= c\,\frac{dy}{\sqrt{y^2-c^2}} \end{aligned}$$

and after integrating both sides, we have

$$x = c\ \cosh^{-1}\frac{y}{c} + k,$$

where k is a constant of integration. Solving for y, we find

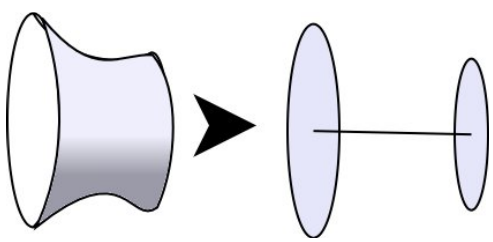
$$y = c\ \cosh\frac{x-k}{c}.$$

This curve is called a *catenary*, the curve a hanging rope assumes when supported at its ends and acted upon by a uniform gravitational force. The surface of revolution obtained from a catenary is called a *catenoid.*

Using the boundary conditions, we can solve for the constants c and k:

$$\begin{aligned} y(x_0) &= y_0 \\ y(x_1) &= y_1 \\ y_0 &= c\ \cosh\frac{x_0-k}{c} \\ y_1 &= c\ \cosh\frac{x_1-k}{c} \end{aligned}$$

Note that these equations do not have analytic solutions and must be solved numerically. Also, note that in certain situations, there will not be real valued solutions for the constants (see `http://mathworld.wolfram.com/SurfaceofRevolution.html` for more details).



Intuitively, you can think of the the catenoid as a soap bubble being stretched between two rings of wire. As the rings move apart and the bubble elongates, the minimum surface dips closer and closer to the x axis. At a certain point, the bubble breaks and the minimum surface is actually the two discs themselves, joined by a line. At this point the curve is discontinuous – so the Euler-Lagrange solution is invalid.

2. Show that the shortest curve between two points on a sphere is an arc of a great circle.
[Hints: Use spherical (u, v) coordinates, where $x = R\sin v\cos u$, $y = R\sin v\sin u$, $z = R\cos v$, with R the radius of the sphere. Cast 3D arclength $\sqrt{dx^2+dy^2+dz^2}$ into (u, v) space, and parameterize the curve in terms of the coordinate u. Observe that the integrand in the expression for arclength does not depend on u. You may find the following identity useful:

$$\int \frac{a\ dw}{\sqrt{\sin^4 w - a^2\sin^2 w}} = -\sin^{-1}\left(\frac{\cot w}{\sqrt{\frac{1}{a^2}-1}}\right) + k,$$

where a and k are appropriate constants.]

Using the spherical coordinates given in the hint, we derive 3D arclength as follows (abbreviating $\cos v$ as c_v, $\sin v$ as s_v, and so forth):

$$x = Rs_vc_u$$

$$\begin{aligned}
dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\
&= R(-s_v s_u du + c_v c_u dv) \\
dx^2 &= R^2(s_v^2 s_u^2 du^2 - 2 s_v s_u c_v c_u du\ dv + c_v^2 c_u^2 dv^2)
\end{aligned}$$

$$\begin{aligned}
y &= R s_v s_u \\
dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\
&= R(s_v s_u du + c_v s_u dv) \\
dy^2 &= R^2(s_v^2 c_u^2 du^2 + 2 s_v s_u c_v c_u du\ dv + c_v^2 s_u^2 dv^2)
\end{aligned}$$

$$\begin{aligned}
z &= R c_v \\
dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\
&= -R s_v dv \\
dz^2 &= R^2 s_v^2 dv^2
\end{aligned}$$

So now we can write

$$\begin{aligned}
dx^2 + dy^2 + dz^2 &= R^2(s_v^2 s_u^2 du^2 - 2 s_v s_u c_v c_u du\ dv + c_v^2 c_u^2 dv^2 + s_v^2 c_u^2 du^2 + \\
&\quad 2 s_v s_u c_v c_u du\ dv + c_v^2 s_u^2 dv^2 + s_v^2 dv^2) \\
&= R^2(s_v^2 s_u^2 du^2 + c_v^2 c_u^2 dv^2 + s_v^2 c_u^2 du^2 + c_v^2 s_u^2 du^2 + s_v^2 dv^2) \\
&= R^2(s_v^2 du^2 + c_v^2 dv^2 + s_v^2 dv^2) \\
&= R^2\left(s_v^2 + \left(\frac{dv}{du}\right)^2\right) du^2
\end{aligned}$$

The length of the arc from a to b can now be written as

$$\begin{aligned}
L &= \int_a^b \sqrt{R^2\left(s_v^2 + \left(\frac{dv}{du}\right)^2\right) du^2} \\
&= R\int_a^b \sqrt{s_v^2 + \left(\frac{dv}{du}\right)^2} du \\
&= R\int_a^b \sqrt{s_v^2 + v'^2} du
\end{aligned}$$

where $v' = dv/du$.

We want to find the arc $v = v(u)$ that minimizes L, so we now have an expression for F:

$$F(u, v, v') = \sqrt{s_v^2 + v'^2}$$

By observation, F does not depend on u, so we can again use the Beltrami identity and write

$$F - v' F_{v'} = a$$

We already have an expression for F, and for $F_{v'}$ we have

$$F_{v'} = \frac{v'}{\sqrt{s_v^2 + v'^2}}$$

Substituting these into Beltrami, we get:

$$\sqrt{s_v^2 + v'^2} - \frac{v'^2}{\sqrt{s_v^2 + v'^2}} = a$$

Simplifying and solving for v', we get

$$v' = \frac{dv}{du} = \sqrt{\frac{s_v^4}{a^2} - s_v^2}$$

or

$$\frac{a\ dv}{\sqrt{s_v^4 - a^2 s_v^2}} = du.$$

We can then integrate this using the hint given above, to find

$$u = -\sin^{-1}\left(\frac{\cot(v)}{\sqrt{\frac{1}{a^2} - 1}}\right) + k$$

where k is yet another constant, which (dropping the c_v shorthand for $\cos v$) simplifies to

$$\sin(k-u) = \frac{\cot(v)}{\sqrt{\frac{1}{a^2} - 1}}$$

We can convert this back to x, y, z coordinates as follows

$$\sin(k)\cos(u) - \cos(k)\sin(u) - \frac{\cot(v)}{\sqrt{\frac{1}{a^2} - 1}} = 0$$

$$\sin(k)\ (R\sin(v)\cos(u)) - \cos(k)\ (R\sin(v)\sin(u)) - \frac{R\cos(v)}{\sqrt{\frac{1}{a^2} - 1}} = 0$$

$$\sin(k)x - \cos(k)y - \frac{z}{\sqrt{\frac{1}{a^2} - 1}} = 0$$

This is the equation of a plane which passes through the origin of the sphere, so the minimizing curve which lies on that sphere is obtained by the intersection of the sphere and the plane through the origin, which is the definition of a great circle.

3. In the brachistochrone problem, suppose the right endpoint is constrained only to touch some curve given implicitly by an equation of the form $g(x,y) = 0$. Show that the optimizing curve $y(x)$ must touch $g(x,y)$ orthogonally.

Solution: In this case, the endpoint is determined by the condition

$$F'_y \frac{g_y F}{g_x + g_y y'} = 0$$

at (x_1, y_1) (we derived this in class). From the brachistochrone problem we solved in class, recall that

$$F = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}}$$

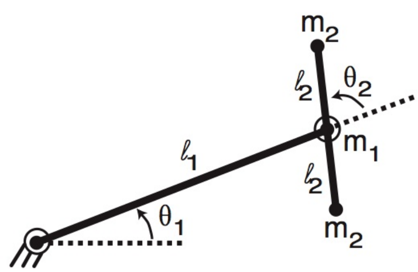
$$F'_y = \frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0 - y)(1+y'^2)}}$$

So the condition at (x_1, y_1) becomes

$$\begin{aligned}\frac{1}{\sqrt{-2g}}\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} - \frac{g_y}{g_x+g_y y'}\frac{1}{\sqrt{-2g}}\frac{\sqrt{1+y'^2}}{\sqrt{y_0-y}} \\ \frac{y'}{\sqrt{1+y'^2}} - \frac{g_y}{g_x+g_y y'}\sqrt{1+y'^2} &= 0 \\ y' - \frac{g_y}{g_x+g_y y'}(1+y'^2) &= 0 \\ g_x y' + g_y y'^2 - g_y - g_y y'^2 &= 0 \\ y' &= \frac{g_y}{g_x}\end{aligned}$$

So at (x_1, y_1) , the slope of y is in the same direction as ∇g, i.e. $y(x)$ touches $g(x, y)$ orthogonally.

4. (a) Using Lagrangian Dynamics, derive the relationship between joint torques and the angular state (angles, velocities, and accelerations) of the following balanced manipulator:



There is no gravity (in practice, gravity often acts perpendicular to the sheet of the paper).
Legend: All of link #1's mass, m_1 , is concentrated at distance ℓ_1 from its rotational joint (which is attached to the ground). In turn, link #2 rotates around this distal point, with two masses, m_2, located symmetrically, each at distance ℓ_2 from the joint. In practice, these two masses might constitute one counter-balanced end-effector or two different but equally weighted end-effectors. This is a variation of a basic Scara-type robot arm, often used in industrial assembly, for instance by SONY.

Using Lagrangian dynamics, we know that

$$L = T - V.$$

Since there is no gravity we know that the potential energy, V, is zero, so

$$L = T.$$

The kinetic energy, T, can be written as the sum of the kinetic energies of the three masses:

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_{21}^2 + \frac{1}{2}m_2 v_{22}^2.$$

A very common error was to assume that $v_{21}^2 = v_{22}^2$, which is not the case except when $\theta_2 = \frac{\pi}{2}$, due to the effect of θ_1. Now, we must calculate the three velocities v_1, v_{21}, and v_{22}. The coordinates of $m1$ are given by (x_1, x_2), the coordinates of the upper m_2, as drawn in the figure, now called m_{21} are (x_{21}, y_{21}), and likewise the coordinates of the lower m_2, now called m_{22} are (x_{22}, y_{22}). In the following, we use the standard abbreviations $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij} = \sin(\theta_i + \theta_j)$, etc.

Using this notation, we can write the positions of the point masses as follows :

$$
\begin{aligned}
x_1 &= \ell_1 c_{\theta_1} \\
y_1 &= \ell_1 s_{\theta_1} \\
x_{21} &= \ell_1 c_{\theta_1} + \ell_2 c_{\theta_1\theta_2} \\
y_{21} &= \ell_1 s_{\theta_1} + \ell_2 s_{\theta_1\theta_2} \\
x_{22} &= \ell_1 c_{\theta_1} - \ell_2 c_{\theta_1\theta_2} \\
y_{22} &= \ell_1 s_{\theta_1} - \ell_2 s_{\theta_1\theta_2}
\end{aligned}
$$

By differentiating the postitions, we can derive the velocities:

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = \ell_1^2\dot{\theta}_1^2$$

$$v_{21}^2 = \dot{x}_{21}^2 + \dot{y}_{21}^2 = \ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\ell_1\ell_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 c_2$$

$$v_{22}^2 = \dot{x}_{22}^2 + \dot{y}_{22}^2 = \ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2\ell_1\ell_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 c_2$$

Now we can write down the Lagrangian:

$$
\begin{aligned}
L &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_{21}^2 + \frac{1}{2}m_2 v_{22}^2 \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[\ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\ell_1\ell_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 c_2] \\
&\quad + \frac{1}{2}m_2[\ell_1^2\dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2\ell_1\ell_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 c_2] \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + m_2\ell_1^2\dot{\theta}_1^2 + m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2
\end{aligned}
$$

We now take the various derivatives of the Lagrangian:

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= 0 \\
\frac{\partial L}{\partial \theta_2} &= 0 \\
\frac{\partial L}{\partial \dot{\theta}_1} &= (m_1 + 2m_2)\ell_1^2\dot{\theta}_1 + 2m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{\partial L}{\partial \dot{\theta}_2} &= 2m_2\ell_2^2(\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= (m_1 + 2m_2)\ell_1^2\ddot{\theta}_1 + 2m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= 2m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)
\end{aligned}
$$

We are now in a position to write down the equations for the joint torques:

$$
\begin{aligned}
\tau_1 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \\
&= (m_1 + 2m_2)\ell_1^2\ddot{\theta}_1 + 2m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \\
\tau_2 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \\
&= 2m_2\ell_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)
\end{aligned}
$$

(b) $\boxed{\text{When } \ddot{\theta}_2 = 0, \text{ explain the terms relating } \ddot{\theta}_1 \text{ to } \tau_1\ .}$

When $\ddot{\theta}_2 = 0$, we τ_1 simplifies to

$$\begin{aligned}\tau_1 &= ((m_1 + 2m_2)\ell_1^2 + 2m_2\ell_2^2)\ddot{\theta}_1 \\ &= I\ddot{\theta}_1\end{aligned}$$

In this case, the first term is just the sum of the point mass inertias around the first joint.