

Robotics 811 - HW 3

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1 Q1

1(a)

Taylor series can be written as the following equation:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^n(a)}{n!} (x-a)^n$$

If we expand the taylor series for $f(x)$ around $x = 0$, we will get the following expansion

$$f(x) = \sinh(0) + \cosh(0)(x) + \frac{\sinh(0)}{2!}x^2 + \frac{\cosh(0)}{3!}x^3 + \dots$$

By solving numerically, we find that $\sinh(0) = 0$ and $\cosh(0) = 1$. By entering the values into the previous equation, we can simplify the expansion of $f(x)$ to

$$f(x) = x + \frac{x^3}{6} + \frac{x^5}{5!} + \dots$$

1(b)

Following is the graph of $f(x)$ over the interval $[-2, 2]$ in matlab

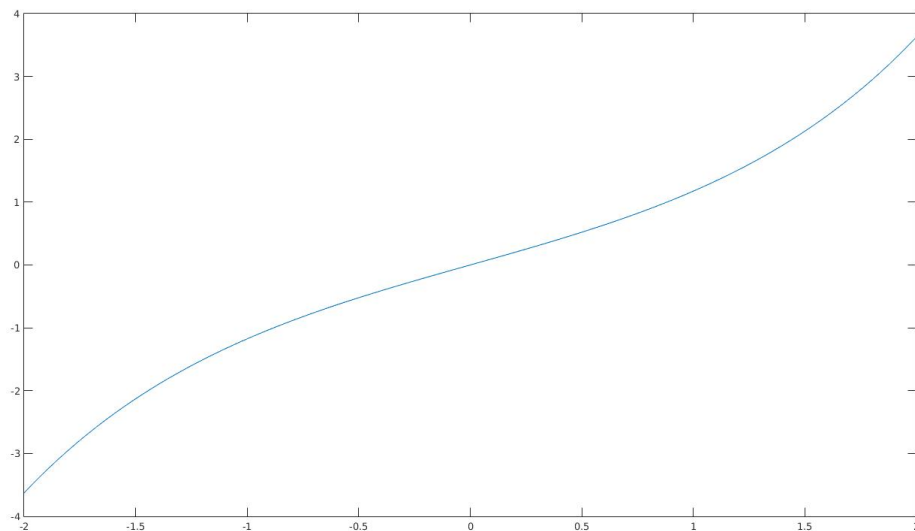


Figure 1: graph of $f(x) = \sinh(x)$ over $[-2, 2]$

1(c)

This question was done with help and discussion from Reuben Aronson

As we are trying to approximate the function with a quadratic functions($n = 2$), by the theorem given in the class and notes, we know there are $n + 2$ points that are $\varepsilon \|f - p\|_\infty$, which are the maximum error of the best quadratic approximation function. Our approximation function could be written in the following form

$$p(x) = a + bx + cx^2$$

However, a quick observation of the graphing of the function(as seen in question 1(b)), we can see that the function $f(x)$ is actually an odd function, shown by how the values of $f(x)$, where $x < 0$ is just the reflection of $f(x)$, where $x > 0$ along the $y = -x$ line. As a odd functions will be best approximate by another odd functions, that simplifies our approximation function to:

$$p(x) = bx \tag{1}$$

From the theorem, we know that the error function $E(x) = \|f - p\|_\infty$ is maximized at 4 points, that allows us to construct the following equations

$$\begin{aligned} e(x_0) &= f(x_0) - p(x_0) = \sinh(x_0) - b(x_0) \\ e(x_1) &= f(x_1) - p(x_1) = \sinh(x_1) - b(x_1) \\ e(x_2) &= f(x_2) - p(x_2) = \sinh(x_2) - b(x_2) \\ e(x_3) &= f(x_3) - p(x_3) = \sinh(x_3) - b(x_3) \end{aligned}$$

By the theorem, as $f^{n+1}(x)$ does not change sign on $[a, b]$, we know that $x_0 = -2$ and $x_1 = 2$. We also know that as a maximum error point, the first derivative of the error function at that point must be 0. The derivative of $e(x)$ is as following, and we can represent the function in terms of x

$$\begin{aligned} e'(x) &= \cosh(x) - b = 0 \\ \cosh(x) &= b \\ x &= \pm \cosh^{-1}(b) \end{aligned} \tag{2}$$

Because all the errors are the same values with different signs, we now that $e(x_2) = -e(x_3)$. By substituting equation 2 into that equation, we will get following equation

$$\sinh(\cosh^{-1}(b)) - b(\cosh^{-1}(b)) = -\sinh(2) + 2b$$

By solving the previous equation numerically, we get

$$b \approx 1.600233$$

This will give us our approximation function by substituting b into equation 1.

$$p(x) = 1.600233x$$

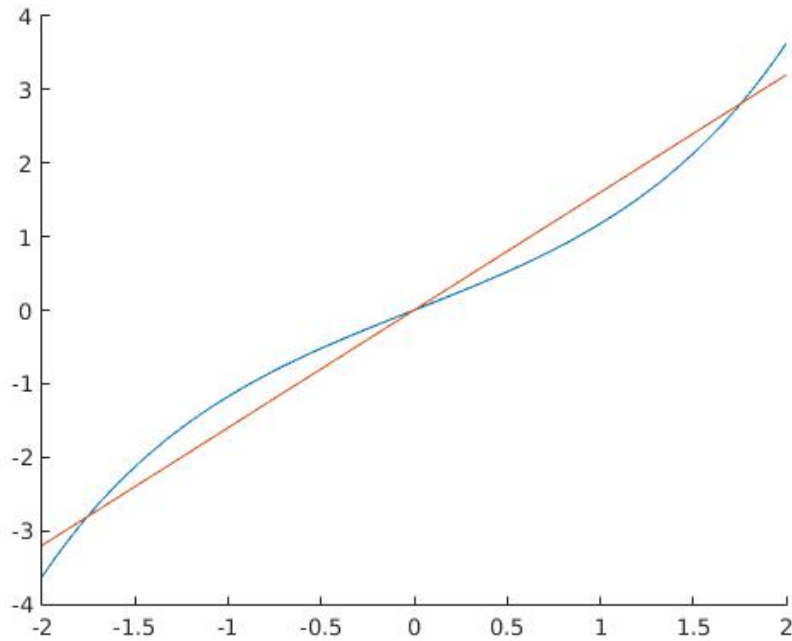


Figure 2: graph of $f(x) = \sinh(x)$ over $[-2, 2]$ with the approximation function, $p(x) = 1.600233x$

Now we will determine the L_∞ and L_2 error of the approximation function. L_∞ error would be the just the error function used in the approximation.

$$\begin{aligned} |e(x_0)| &= \sinh(-2) - 1.600233(-2) \\ &\approx 0.4264 \end{aligned}$$

L_2 error can be found through integration of the error function

$$\begin{aligned} \int_{-2}^2 (f(x) - p(x))^2 &= \int_{-2}^2 (\sinh(x) - bx)^2 \\ &\approx 0.354438 \end{aligned}$$

1(d)

Now we will be approximating the best least squares approximation by a quadratic function. First we will construction a orthogonal sequence of polynomials with the given interval using the recurrence function given in page 33 of the notes on approximation. We will define $p_0(x) \equiv 1$. Following is the construction of $p_1(x)$

and $p_2(x)$.

$$\begin{aligned}
\langle p_0, p_0 \rangle &= \int_{-2}^2 1 dx = 4 \\
\langle xp_0, p_0 \rangle &= \int_{-2}^2 x dx = 0 \\
p_1(x) &= [x - 0]p_0(x) = x \\
\langle p_1, p_1 \rangle &= \int_{-2}^2 x \cdot x dx = \frac{16}{3} \\
\langle xp_1, p_1 \rangle &= \int_{-2}^2 x^2 \cdot x dx = 0 \\
p_2(x) &= [x - 0]p_1(x) - \frac{\frac{16}{3}}{4} = x^2 - \frac{4}{3} \\
\langle p_2, p_2 \rangle &= \int_{-2}^2 (x^2 - \frac{4}{3}) \cdot (x^2 - \frac{4}{3}) dx = \frac{256}{45}
\end{aligned} \tag{3}$$

Now we want to project the function $f(x)$ on the subspace spanned by the polynomials(basis) that we generated. Which can be done with the following formula:

$$p(x) = \sum_{i=0}^2 \frac{\langle f(x), p_i \rangle}{\langle p_i, p_i \rangle} p_i(x)$$

Where p_i are the polynomial we generated in equation 3. Here we will calculate $\langle f(x), p_i \rangle$ for all i .

$$\begin{aligned}
\langle f(x), p_0 \rangle &= \int_{-2}^2 \sinh(2) \cdot 1 dx = \cosh(x)|_{-2}^2 \\
&= 0 \\
\langle f(x), p_1 \rangle &= \int_{-2}^2 \sinh(2) \cdot x dx = x \cosh(x)|_{-2}^2 - \sinh(x)|_{-2}^2 \\
&= 7.7951 \\
\langle f(x), p_2 \rangle &= \int_{-2}^2 \sinh(2) \cdot (x^2 - \frac{4}{3}) dx = (x^2 - \frac{4}{3}) \cosh(x)|_{-2}^2 - 2(x \sinh(x))|_{-2}^2 - \cosh(x)|_{-2}^2 \\
&= 0
\end{aligned}$$

Let d_i be $\frac{\langle f(x), p_i \rangle}{\langle p_i, p_i \rangle}$. Then,

$$\begin{aligned}
d_0 &= \frac{0}{4} = 0 \\
d_1 &= \frac{7.7951}{\frac{163}{3}} \approx 1.4616 \\
d_2 &= \frac{0}{\frac{256}{45}} = 0
\end{aligned}$$

These will give us our least square approximation of

$$p(x) = 1.46146p_1(x) = 1.4616x \tag{4}$$

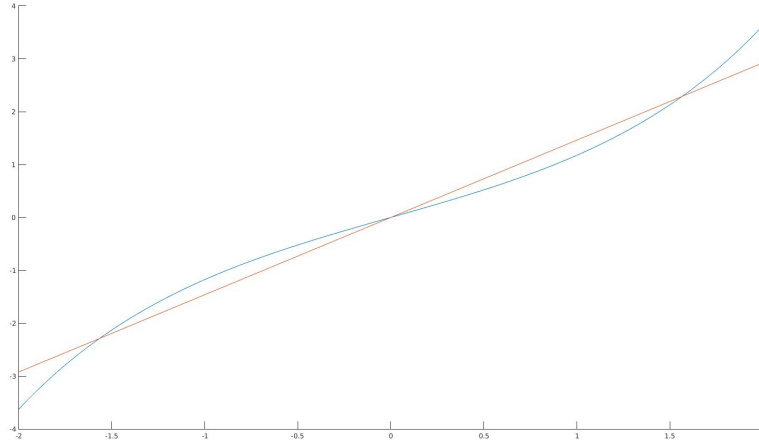


Figure 3: graph of $f(x) = \sinh(x)$ over $[-2, 2]$ with the approximation function $p(x) = 1.4616x$

Now we will determine the L_∞ and L_2 error of the approximation function. The x value that will give the L_∞ error would either be at the boundaries or when the derivative of the error function is 0.

$$\begin{aligned}
 e(x) &= (f(x) - p(x))^2 = (\sinh(x) - 1.4616x)^2 \\
 e'(x) &= 2(\sinh(x) - 1.4616x)(\cosh(x) - 1.4616) = 0 \\
 x &\approx \pm 0.927255 \\
 x &= 0 \\
 x &\approx \pm 1.56559
 \end{aligned}$$

This means x for the maximum error would be either 2, -2 or one of the x in the previous equation. After substituting all x in equation 4. We found the maximum error given from all x is 0.7037 at the points $x = 2$ and $x = -2$.

L_2 error can be found through integration of the error function

$$\begin{aligned}
 \int_{-2}^2 (f(x) - p(x))^2 &= \int_{-2}^2 (\sinh(x) - 1.4616x)^2 \\
 &= 0.251898
 \end{aligned}$$

2 Q2

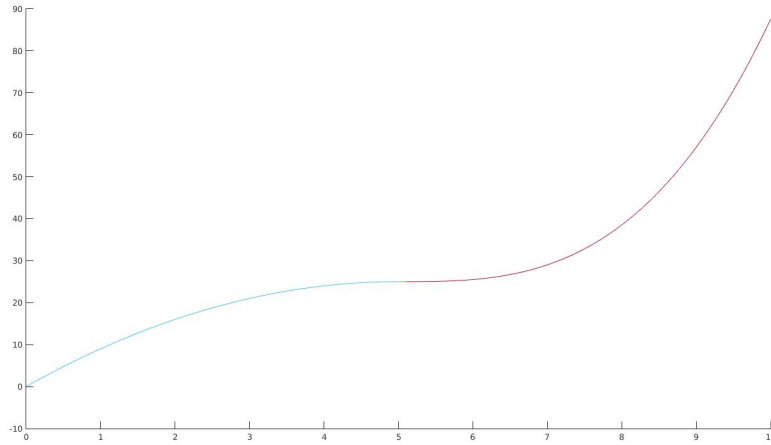


Figure 4: The graph plotted with the given points overlaid with the piecewise function. Note how by observation, we see the transition between functions at point $x = 5$

This question was done through discussion with Devin Schwab

First we observed the graph, we noticed that the graph's changes are different for $x > 5$ and $x < 5$. This might have been because of a piecewise function. Using the data fitting method learned in class, we were able to fit two different functions (that have a rational/whole coefficient term) for the given points at the range $0 \rightarrow 5$ and $5 \rightarrow 10$.

The data fitting process is as follows. First pick possible basis, in our implementation, we picked the standard polynomials $(1, x, x^2, x^3 \dots)$ and create a matrix, A with each term as its own column with each row as the $x \times \text{term}$ value. We then try to solve the equation $Av = b$, where b is a $n \times 1$ vector of each point's y value. The equation $Av = b$ is solved through SVD. The values of v would be the coefficients for each of the possible basis. The implementation and script to create a graph of the functions can be found in **q2.m**

Using that method we found function $f(x)$ to be:

$$f(x) = \begin{cases} 10x - x^2 & x \leq 5 \\ -37.5 + 37.5x - 7.5x^2 & x > 5 \end{cases}$$

3 Q3

3(a)

We can derive T_6 and T_7 using the recurrence function on page 11 of the notes on approximation with the initial value of $T_0 = 1$ and $T_1 = x$.

$$\begin{aligned}
T_{i+1}(x) &= 2x(T_i(x)) - T_{i-1}(x) \\
T_2(x) &= 2x(x) - 1 \\
&= 2x^2 - 1 \\
T_3(x) &= 2x(2x^2 - 1) - x \\
&= 4x^3 - 3x \\
T_4(x) &= 2x(4x^3 - 3x) - (2x^2 - 1) \\
&= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) \\
&= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) \\
&= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) \\
&= 64x^7 - 112x^5 + 56x^3 - 7x
\end{aligned}$$

To prove that T_6 and T_7 are orthogonal polynomials, we need to show that $\langle T_6, T_7 \rangle = 0$. First we will do a value substitution of $x = \cos\theta$

$$\begin{aligned}
x &= \cos\theta \\
dx &= -\sin(\theta)d\theta \\
x = 1 &\rightsquigarrow \cos^{-1}(1) = 0 \\
x = -1 &\rightsquigarrow \cos^{-1}(-1) = \pi
\end{aligned}$$

The new equation will have the following form

$$\begin{aligned}
\langle T_6, T_7 \rangle &= \int_0^\pi (1 - \cos^2(\theta))^{-\frac{1}{2}} \cos(6\theta) \cos(7\theta) - \sin(\theta) d\theta \\
&= \int_0^\pi (\sin^2(\theta))^{-\frac{1}{2}} \cos(6\theta) \cos(7\theta) - \sin(\theta) d\theta \\
&= -1 \int_0^\pi \frac{\sin(\theta)}{\sqrt{\sin^2(\theta)}} \cos(6\theta) \cos(7\theta) d\theta \\
&= -1 \int_0^\pi \cos(6\theta) \cos(7\theta) d\theta \\
&= -1 \int_0^\pi \frac{1}{2} (\cos(\theta) + \cos(13\theta)) d\theta \\
&= -\frac{1}{2} \int_0^\pi \cos(\theta) + \cos(13\theta) d\theta \\
&= -\frac{1}{2} \left(\int_0^\pi \cos(13\theta) d\theta + \int_0^\pi \cos(\theta) d\theta \right) \\
&= -\frac{1}{2} \left(\int_0^{13\pi} \cos(u) \frac{1}{13} du + \int_0^\pi \cos(\theta) d\theta \right) \\
&= -\frac{1}{2} (\sin(\theta)|_0^\pi + \frac{1}{13} (\sin(u)|_0^{13\pi})) \\
&= 0
\end{aligned}$$

This proves they are orthogonal polynomials.

3(b)

Let us calculate the length of T_1 . This can be done with the inner product of itself.

$$\langle T_1, T_1 \rangle = \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} x^2 dx$$

By solving the equation numerically, we get

$$\langle T_1, T_1 \rangle \approx 1.5708$$

4 Q4

The solution of this question was based on a similar problem on a computer vision homework that was done this semester.

4(a)

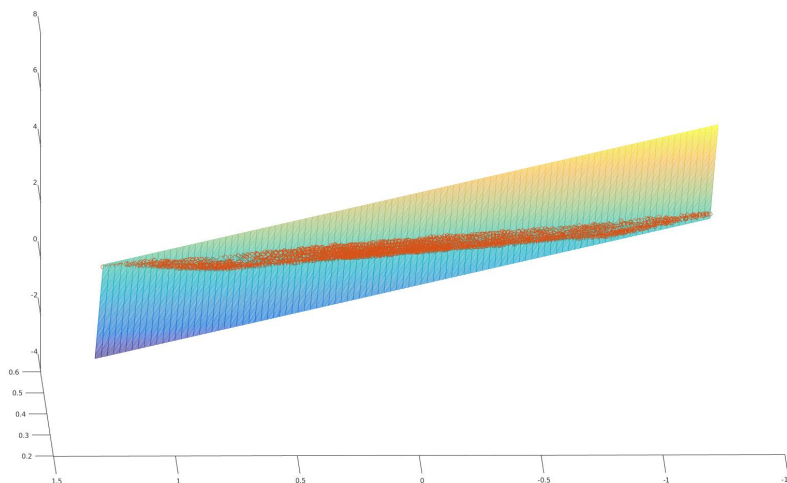
We are trying to solve the plane equation $ax + by + cz = d$, where $\langle a, b, c \rangle$ will be the normal vector of the plane. Because we are using the least square methods, that means we are trying find best parameters, v which are $\langle a, b, c, d \rangle$ such that $Av = 0$, where A are the input variables. This is equal to try minimize $|Av|^2$ subjecting to the fact that $|v| = 1$ such that it is a non trivial solution.

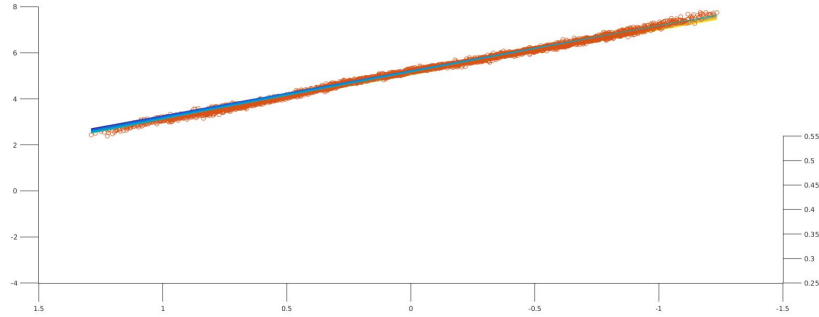
We do the minimization of $|Av|^2$ through SVD as the minimum of the equation is equal to the eigenvector of A corresponding to the smallest eigenvalue. We decompose A into $[U\Sigma V]$ where Σ is the eigenvalues of A . We pick the smallest eigenvalue and the corresponding eigenvector in V . The eigenvector will be least square solution to our plane equation.

The implementation of this algorithm is done in the file **planeSolver.m** where the function will return the least square $[a, b, c, d]$ solution given a set of input points. The least square approximation of the plane equation is:

$$-0.0853x - 0.8920y - 0.0436z = 0.4418$$

Following is the figure of the approximated plane from 2 different angles. The code to generate the graph can be found in **q4a.m**





The average distance from the plane to each point is 0.0027, which is pretty good.

4(b)

Using the same method, we see that the plane is no longer aligned to the points and have an average distance to each point of 0.0248 which is 10 times the error of a.

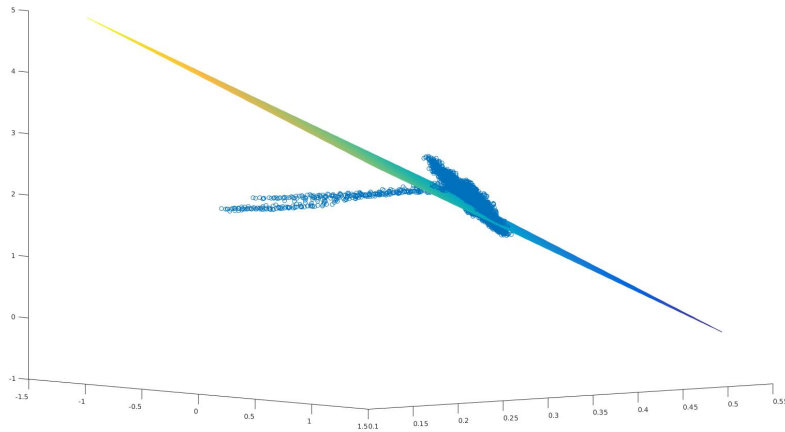


Figure 5: note how the plane does not follow the plane points

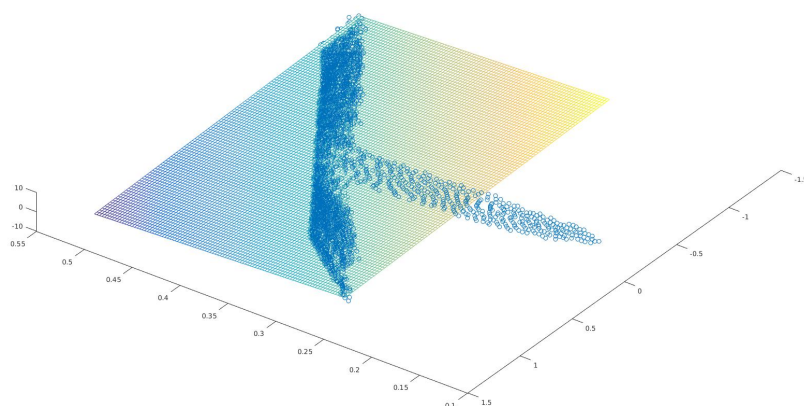
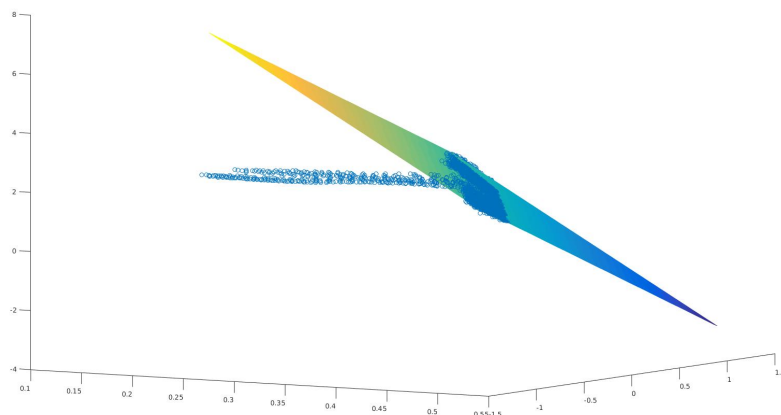
4(c)

To solve this issue, we choose to use the RANSAC algorithm to filter out outlier points. Our implementation of RANSAC does the following it.

First, it picks 4 random points. Using, these 4 points, we create a plane equation. we then measure the average distance of each point to the plane. If the point has a distance less than the **tolerance**, the points are consider as **inliers**. we loop through this process for **n iterations**, and save all the **inliers** for the iteration with the highest **inliers** count. After n iterations, we generate a new plane equation with the **inliers**. The implementation of RANSAC can be found in **RANSAC.m**.

Using this method, we found 3015 inliers from the 3537 given points. The average distance from the inliers

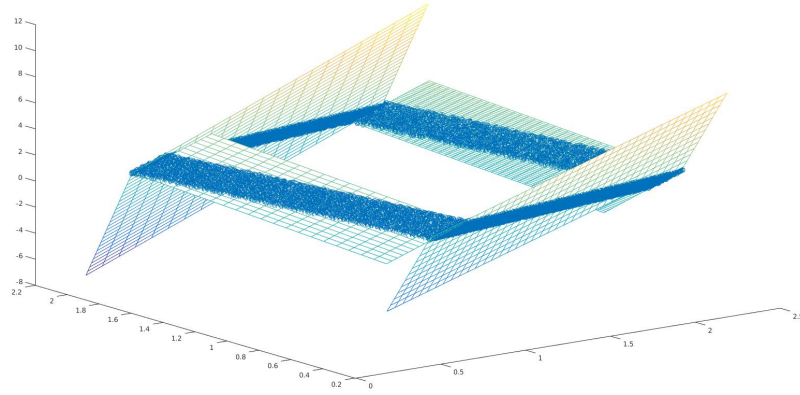
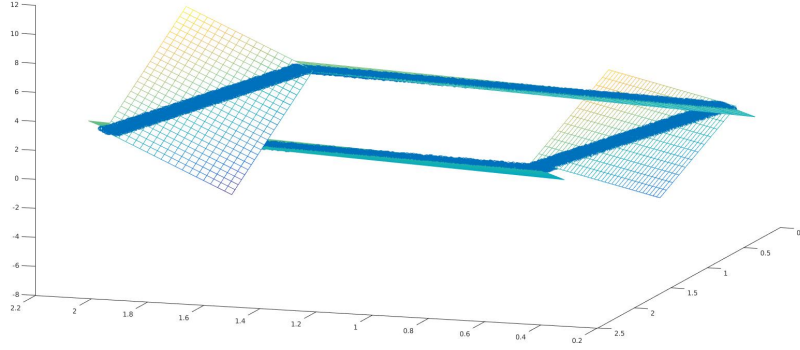
to the plane was 0.002 Following is the figures of the approximated plane from 2 different angles. The code to generate the graph can be found in **q4c.m**



4(d)

The improved RANSAC will now ran the RANSAC algorithm 4 times(not iterations, but the whole algorithm). First, It will find a plane equations that have the maximum possible number of points, then It will remove those points from the list and run RANSAC algorithm on the remain points. After 4 iterations, by theory, the algorithm should end up with 4 different plane equations that have the maximum number of inliers. The code for this improved RANSAC can be found in **RANSAC_improved.m**.

Using the improved algorithm, we found 4 planes with 3090, 2720, 2060 and 1291 points respectively. Following is the visualization of the data and 4 planes from 2 different angles. The code to generate the graph can be found in **q4d.m**



4(e)

When trying 4(e), our previous algorithm did not work because each plane no longer has the same amount of points. We improved our RANSAC to address this issue. The improved-improved RANSAC now first randomly pick one point from all possible points and instead of randomly picking other points, the algorithm will only pick points that are within a certain maximum range from the first point. The remaining algorithm is the same as **RANSAC_improved**. The implementation of the code can be found in **RANSAC_3.m**.

Once we have created the need 4 planes, we created the following smoothness score. We notice that in the graph that an unsmooth surface while generates a plane, the distance between points and the plane are larger. Our smoothness score take advantage of this and calculate the average distance to the plane for each inlier points. The plane with the lowest score would be the best plane to travel. The implementation of the score function can be found in **smoothness.m**.

Following is the visualization of the planes, the red plane is the selected plane and the points that constructed the plan is also included for reference. The code to generate the graph can be found in **q4e.m**

