

Robotics 811 - HW 2

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1 Q1

$$\begin{aligned}P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\&= f[x_0] + f[x_0, x_1]x - f[x_0, x_1]x_0 + f[x_0, x_1, x_2](x^2 - xx_0 - xx_1 + x_0x_1) \\P'_2(x) &= f[x_0, x_1] + f[x_0, x_1, x_2](2x - x_0 - x_1)\end{aligned}$$

where:

$$\begin{aligned}f[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\f[x_0, x_1, x_2] &= \frac{\frac{f(x_0) - f(x_1)}{x_0 - x_1} - \frac{f(x_1) - f(x_2)}{x_1 - x_2}}{x_0 - x_2}\end{aligned}$$

Now, we insert the point at $i = 1$ into $P'_2(x)$

$$\begin{aligned}P'_2\left(\frac{x_0 + x_1}{2}\right) &= f[x_0, x_1] + f[x_0, x_1, x_2]\left(2 \times \frac{x_0 + x_1}{2} - x_0 - x_1\right) \\&= f[x_0, x_1] + f[x_0, x_1, x_2] \times 0 \\&= f[x_0, x_1]\end{aligned}$$

This proves $P'_2(x)$ at $i = 1$ is equal to $f[x_0, x_1]$.

Now, we insert the point at $i = 2$ into $P'_2(x)$

$$\begin{aligned}P'_2\left(\frac{x_1 + x_2}{2}\right) &= f[x_0, x_1] + f[x_0, x_1, x_2]\left(2 \times \frac{x_1 + x_2}{2} - x_0 - x_1\right) \\&= \frac{f(x_0) - f(x_1)}{x_0 - x_1} + \frac{\frac{f(x_0) - f(x_1)}{x_0 - x_1} - \frac{f(x_1) - f(x_2)}{x_1 - x_2}}{x_0 - x_2}(x_0 - x_2) \times -1 \\&= \frac{f(x_0) - f(x_1)}{x_0 - x_1} - \frac{f(x_0) - f(x_1)}{x_0 - x_1} + \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\&= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\&= f[x_1, x_2]\end{aligned}$$

This proves $P'_2(x)$ at $i = 2$ is equal to $f[x_1, x_2]$.

2 Q2

a

Firstly, we calculate the values of $f(x_0), \dots, f(x_i)$ which gives us the vector 0, 0.0434, 0.1269, 0.2217, 0.3187, 0.4145, 0.5075. By plugging it in into the method with the following variables. *dividedDifferences*(6, 2.25, *xs*, *fs*). The method will return the result: 0.1737 which is equal to the answer given by matlab which is 0.1737.

b

- Using the procedure at $x = 0.05$ with $n = 2$, gave the estimate of 4.9878.
- Using the procedure at $x = 0.05$ with $n = 4$, gave the estimate of 4.9442
- Using the procedure at $x = 0.05$ with $n = 40$, gave the estimate of 4.5872

The actual value of $f(0.05)$ is 4.5872 which is same as the estimate of 4.5872 by $n = 40$

2(c)

The Error estimate is as following:

n	E_n
2	3.4911
4	2.3584
6	3.5367
8	6.4873
10	12.9702
12	27.1445
14	58.4298
16	128.1824
18	285.0742
20	640.9705
40	2830400

The Error make sense, because we are trying to fit a polynomial on a non-polynomial function. As the interpolating polynomial will never become the function that we are trying to interpolating and it will only continue to over fitting the function causing massive error at the end of the functions

3 Q3

I received help from Reuben Aronson for this question

Firstly, we assume the table contains the values $f(X_i), i = 0, \dots, n$ with $n = \frac{1}{h}$ and $x_i = -\frac{\pi}{2} + i \times h$. If $\bar{x} \in [x_{i-1}, x_i]$ then we are approximating the function

with a polynomial of degree 2, $p_1(x)$. From the notes, we know the error must be:

$$e_1(\bar{x}) = \frac{f'(xi)}{2!}(\bar{x} - x_{i-1})(\bar{x} - x_i)$$

The error of the first term $f'''(x)$ must be 1 as the function is a simple cos function. The error of the second term is estimated as following:

$$|(\bar{x} - x_{i-1})(\bar{x} - x_i)| \leq \max_{y \in [-h, 0]} |(y - h)y|$$

Let us consider the function $g(y) = (y - h)y = y^2 - yh$. To find the maximum on the interval, we find the derivative of the function, $g'(y) = 2y - h$ and set it to zero $g'(y) = 0$. This means $g'(y) = 0$ if and only if $y = \frac{h}{2}$. By inserting the value into the $g(y)$, we find the maximum of $g(y)$. Which is:

$$\begin{aligned} g\left(\frac{h}{2}\right) &= \frac{h^2}{2} - h\left(\frac{h}{2}\right) \\ &= -\frac{h^2}{4} \end{aligned}$$

Therefore the error for a linear interpolation is:

$$\begin{aligned} |e_1(\bar{x})| &\leq \frac{1}{2!} \times 1 \times \frac{h^2}{4} \\ &= \frac{h^2}{8} \end{aligned}$$

To get the 6 place accuracy, h need to be choose as following:

$$\begin{aligned} \frac{h^2}{8} &\leq 5 \times 10^{-6} \\ h &= 0.002 \end{aligned}$$

Therefore, the number of table entries for a linear interpolation is as following:

$$\frac{\frac{pi}{2} + \frac{3\pi}{2}}{0.002} + 1 \approx 3143$$

If we are using a quadratic interpolation, the interval for \bar{x} will be $[x_{i-1}, x_{i+1}]$ instead. The maximum of the error of $(\bar{x} - x_{i-1})(\bar{x} - x_i)(\bar{x} - x_{i+1})$ can be find in the notes, which is $\frac{2}{3} \frac{1}{\sqrt{3}} h^3$. The Error term for the quadratic equation is as following:

$$\begin{aligned} |e_2(\bar{x})| &\leq \frac{1}{3!} \times 1 \times \frac{2}{3} \frac{h^3}{\sqrt{3}} \\ &= \frac{h^3}{9\sqrt{3}} \end{aligned}$$

To get the 6 place accuracy, h need to be choose as following:

$$\frac{h^3}{9\sqrt{3}} \leq 5 \times 10^{-6}$$

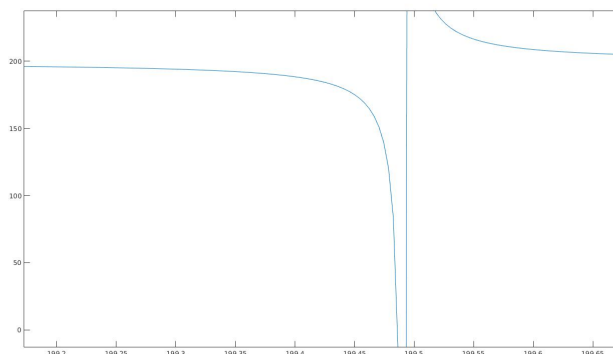
$$h \approx 0.019827$$

Therefore, the number of table entries for a linear interpolation is as following:

$$\frac{\frac{pi}{2} + \frac{3\pi}{2}}{0.019827} + 1 \approx 318$$

4 Q4

First we observe the graph of the function $x - \tan(x) = 0$ around the 200 point to find good values to be tested.



By visual inspection, we found that the root must be between 199.49 and 199. We tested this by inserting both values into the function and it gave us both -682.7305 and 197.1303 . Since 199.49 won't be able to converge using Newton Method(implemented in the file titled *newtonMethod.m*), we first use those two values in the bisection method to give an estimate root. The code for the function is attached to this paper under the title *bisectionMethod.m*. The bisectionMethod returns 199.4861 as the root. As we plugged in the value into newton method, it gave us the same value of 199.4861. Therefore, the closet root to 200 is 199.4861

5 Q5

I received help from Reuben Aronson for this question
First, we look at the Newton Method:

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$$

Minus ξ from both sides will give us

$$\xi - X_{n+1} = \xi - \left(X_n - \frac{f(x_n)}{f'(x_n)}\right)$$

Because $E_n = \xi - x_n$

$$E_{n+1} = E_n + \frac{f(x_n)}{f'(x_n)}$$

Let us set $\frac{f(x_n)}{f'(x_n)}$ to be $h(x)$

$$E_{n+1} = E_n + h(x)$$

Inserting $E_n = \xi - x_n$ into $h(x)$, we get

$$E_{n+1} = E_n + h(\xi - E_n)$$

$h(\xi - E_n)$ can be expressed as a Taylor series

$$h(\xi - E_n) = h(\xi) - E_n h'(\xi) + \frac{E_n^2}{2} h''(\xi) + \dots$$

First we solve $h(\xi)$.

$$h(\xi) = \frac{f(\xi)}{f'(\xi)}$$

Because $f'(\xi) = 0$, the equation is indeterminate. Therefore, to solve this, we could apply the L'Hôpital's rule.

$$h(\xi) = \lim_{x \rightarrow \xi} \frac{f'(\xi)}{f''(\xi)} = 0$$

Now we solve $h'(\xi)$. By the notes, we know the form would as following

$$h'(\xi) = 1 - \frac{f(x)f''(x)}{[f'(x)]^2}$$

Again, the part of $\frac{f(x)f''(x)}{[f'(x)]^2}$ is indeterminate, we could apply the L'Hôpital's rule to it.

$$\lim_{x \rightarrow \xi} \frac{f(x)f''(x)}{[f'(x)]^2} = \lim_{x \rightarrow \xi} \frac{f'(x)f''(x) + f(x)f'''(x)}{2f'(x)f''(x)}$$

Because the solution is again indeterminate, we apply L'Hôpital's rule again.

$$\begin{aligned} \lim_{x \rightarrow \xi} \frac{f'(x)f''(x) + f(x)f'''(x)}{2f'(x)f''(x)} &= \lim_{x \rightarrow \xi} \frac{f'(x)f'''(x) + f''(x)f''(x) + f'(x)f'''(x) + f(x)f''''(x)}{2f''(x)f''(x) + 2f'(x)f'''(x)} \\ &= \frac{f''(x)f''(x)}{2f''(x)f''(x)} \\ &= \frac{1}{2} \end{aligned}$$

Plugging $\frac{1}{2}$ back into the original equation, we get

$$h'(\xi) = 1 - \frac{1}{2} = \frac{1}{2}$$

Now we plugging the value of $h'(x)$ and $h(x)$ back into $E_{n+1} = E_n + h(\xi - E_n)$

$$\begin{aligned} E_{n+1} &= E_n + h(\xi - E_n) \\ &= E_n + h(\xi) - E_n h'(\xi) + \dots = E_n + 0 - \frac{E_n}{2} + O(E_n^2) = \frac{1}{2}E_n + O(E_n^2) \end{aligned}$$

This establishes the linear convergence of Newton's Method when $f(\xi) = 0, f'(\xi) = 0$.

$$|E_{n+1}| \sim C|E_n|^1, \text{ where } C = \frac{1}{2} + O(E_n^2)$$

This proves that the method now converges linearly.

Now we change the newton's method function to $x_{n+1} = x_{n-1} - 2\frac{f(x_n)}{f'(x_n)}$. The only change would be that now $E_{n+1} = E_n + 2h(\xi - E_n)$. By expanding it:

$$\begin{aligned} E_{n+1} &= E_n + 2h(\xi - E_n) \\ &= E_n + 2(h(\xi) - E_n h'(\xi) + \frac{E_n^2}{2} h''(\xi) + O(E_n^3)) \\ &= E_n + 2(0) - 2(\frac{E_n}{2}) + E_n^2 h''(\xi) + 2(O(E_n^3)) \\ &= E_n^2 h''(\xi) + 2(O(E_n^3)) \end{aligned}$$

Now we only need to prove that $h''(\xi) \neq 0$. This could be done through calculation on matlab or wolframalpha. If done, it will prove that $h''(\xi) \neq 0$. Therefore,

$$|E_{n+1}| \sim C|E_n|^2, \text{ where } C = O(E_n^3)$$

This proves that the method now converges quadratically.

6 Q6

6.1 6(a)

The Muller Method was implemented based on the notes and algorithm given on the website: <http://www.physics.arizona.edu/~restrepo/475A/Notes/sourcea/node25.html>. The implementation of the algorithm can be found in the file titled *mullerMethod.m* attached to the homework.

6.2 6(b)

first route: $x_0 = 4; x_1 = 4.35; x_2 = 4.5$; Root at: 4.4934 second root:

6.3 6(c)

We wrote a script called *p6c.m* that runs the algorithm to find the correct k and real roots. First, we visually inspected the graph of the Bessel function at $K = 100$. This allow us to determine the ranges for the possible roots. Then, we set the initial k to be 3 and loop through the Muller function and slowly increasing k until the root become stable(they did not change since the last iteration). We found the roots to be stable at $k = 31$. The following is the roots found by the algorithm

x_0	x_1	x_2	roots
3	4	5	3.3817
5	6	7	7.0156
9	10	11	10.1735
12	13	14	13.3237

7 Q7

7(a)

From the two polynomials given, we can construct a Sylvester's matrix:

$$M = \begin{pmatrix} 1 & -12 & 41 & -42 & 0 \\ 0 & 1 & -23 & 41 & -42 \\ 1 & -2 & -35 & 0 & 0 \\ 0 & 0 & 1 & -2 & -35 \end{pmatrix}$$

Calculating the determinant will give us

$$\det(M) = -7.1623e - 12 \approx 0$$

Because the determinant is approximately 0, this means that $p(x)$ and $q(x)$ shares a common root.

7.1 7(b)

Using the ratio method discussed in class we constructed the following equation:

$$\begin{aligned}
 x &= \frac{x^4}{x^3} \\
 &= (-1)^{1+2} \frac{\det(M_1)}{\det(M_2)} \\
 &= (-1) \frac{\begin{vmatrix} -12 & 41 & -42 & 0 \\ 1 & -12 & 41 & -42 \\ -2 & -35 & 0 & 0 \\ 1 & -2 & -35 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 41 & -42 & 0 \\ 0 & -12 & 41 & -42 \\ 1 & -35 & 0 & 0 \\ 0 & -2 & -35 & 0 \end{vmatrix}} \\
 &= (-1) \frac{806736}{-115248} \\
 &= 7
 \end{aligned}$$

8 Q8

8.1 8(a)

The following is the graphing of the following two functions using matlab

$$p(x, y) = 2x^2 + 2y^2 - 1$$

$$q(x, y) = x^2 + y^2 + 2xy + x - y$$

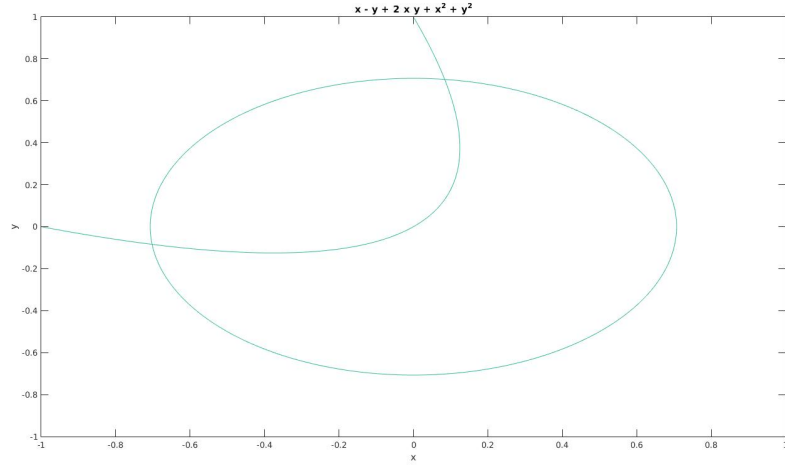


Figure 1: graphing of the two functions

8.2 8(b)

To find the roots of the equations, we first construct the resultant by eliminating y . The functions are re-written with x as constants.

$$p(y) = 2y^2 + (2x^2 - 1)$$

$$q(y) = y^2 + y(2x - 1) + (x^2 + x)$$

This then could be use to construct the Q matrix

$$Q = \begin{pmatrix} 2 & 0 & 2x^2 - 1 & 0 \\ 0 & 2 & 0 & 2x^2 - 1 \\ 1 & 2x - 1 & x^2 + x & 0 \\ 0 & 1 & 2x - 1 & x^2 + x \end{pmatrix}$$

Using the symbolic toolkit inside matlab, we could determine the determinant of the matrix, $\det(Q) = 16x^4 - 16x^3 + 12x - 1$. Using the build in solver in matlab, we were able to solve the 4th degree polynomial, and it gave us 4 different roots (2 real roots and 2 imaginary roots).

$$0.8090 + 0.6360i$$

$$0.8090 - 0.6360i$$

$$-0.7021$$

$$0.0841$$

We plugged in the two real root into the original algorithm and it gave us the following values

$$x : -0.7021 \quad y : 0.0840$$

$$x : 0.0840 \quad y : 0.7021$$

8.3 8(c)

The result of the roots marked on the graph

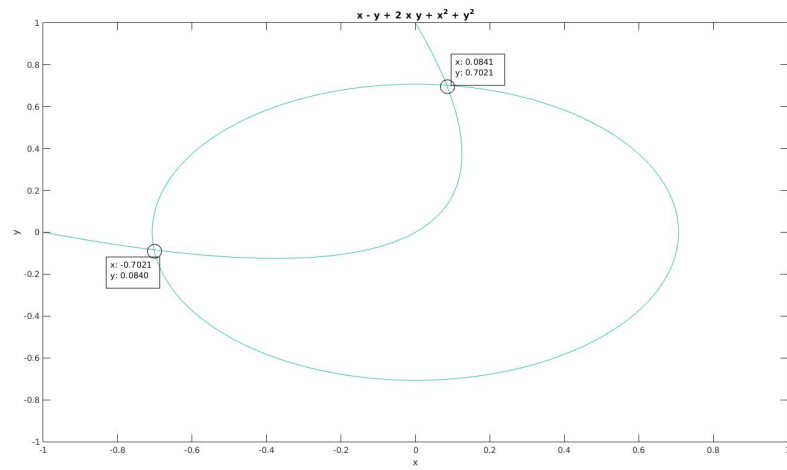


Figure 2: graphing of the two functions

9 Q9