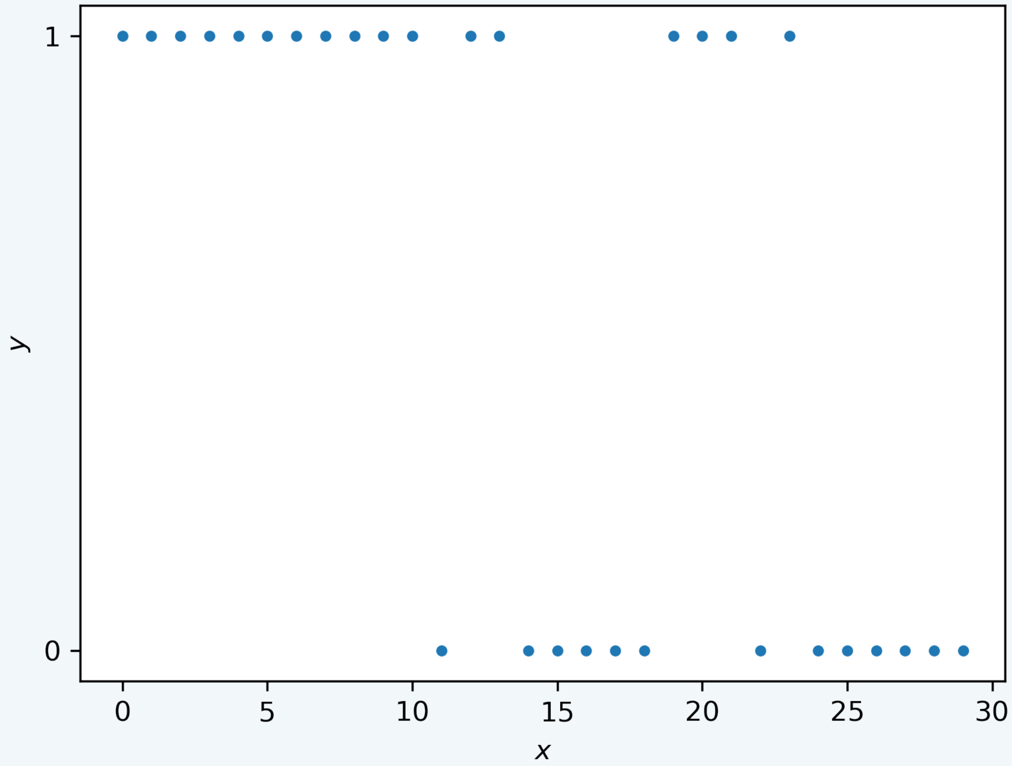
LOGISTIC REGRESSION

Objectives

- Recognize similarities between linear and logistic regression
- Understand the role of the logit and inverse logit in logistic regression

Example: Shooting Baskets



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- What effect would you expect? What do you see?

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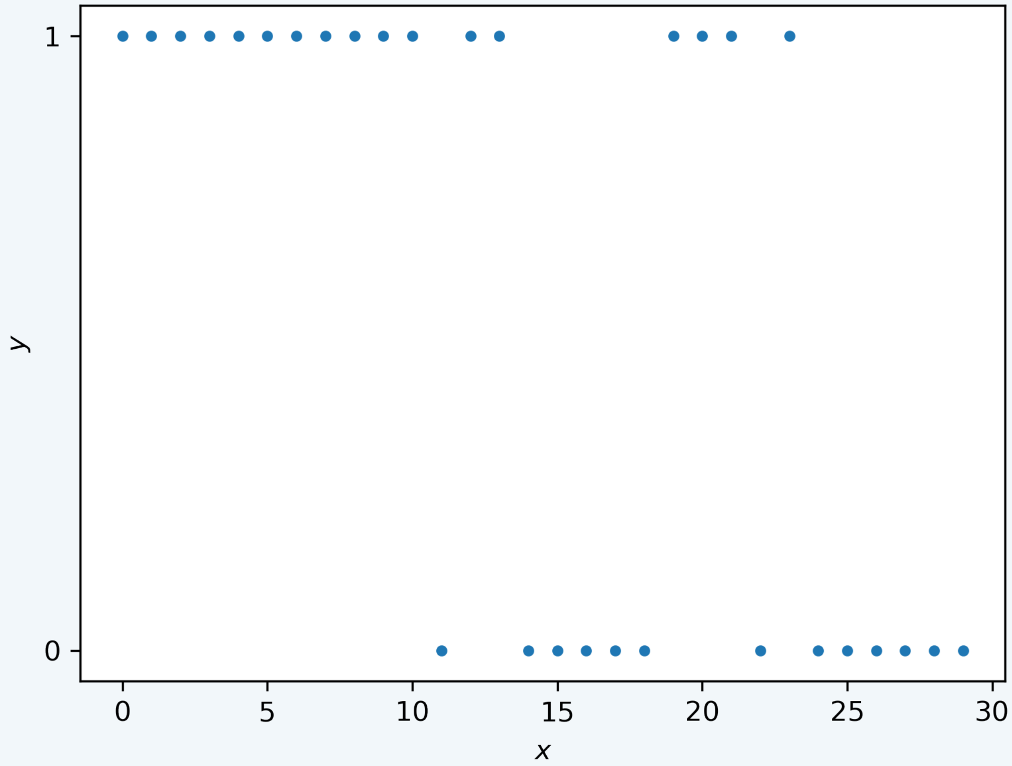

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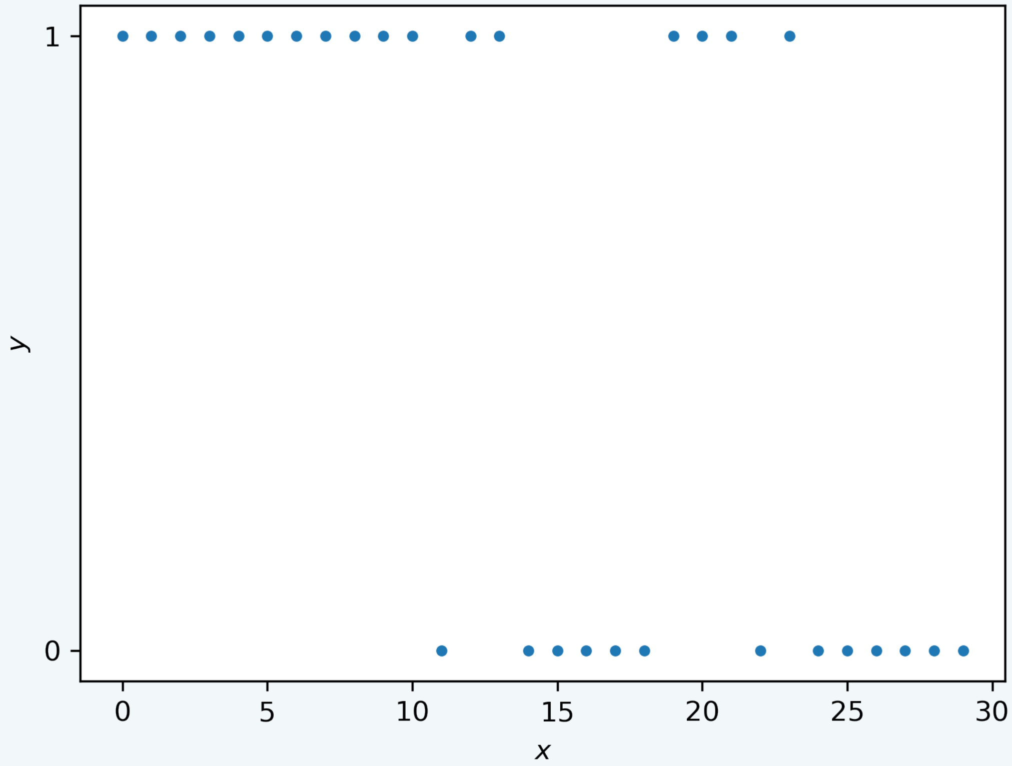

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- What can we do differently?

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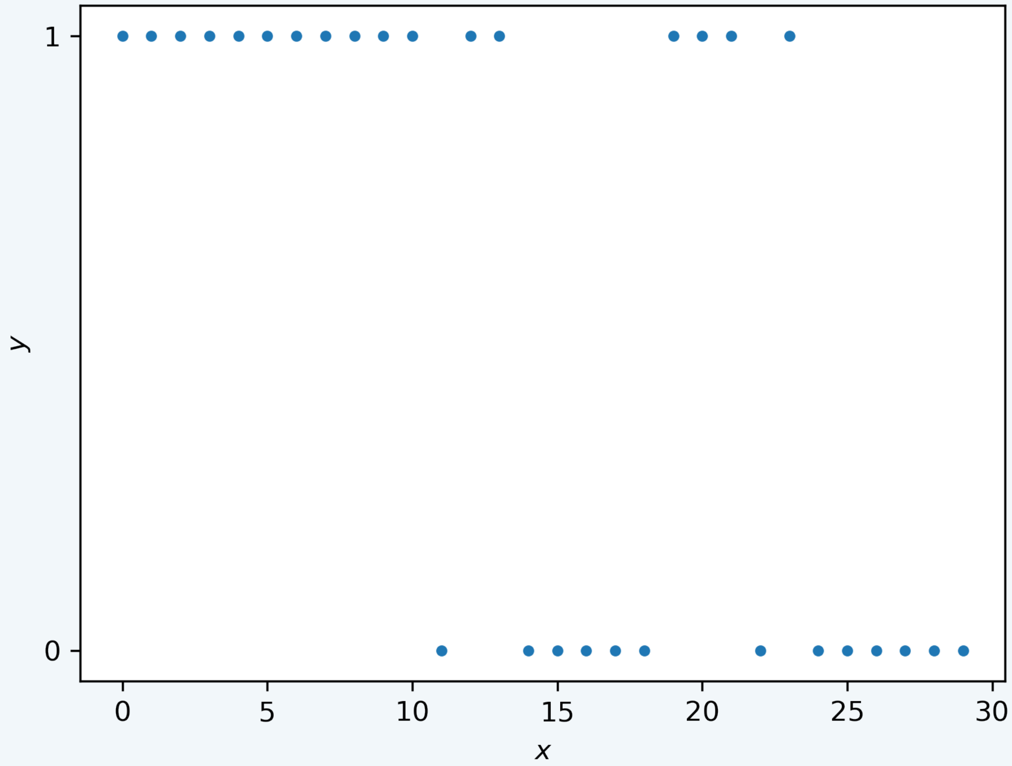

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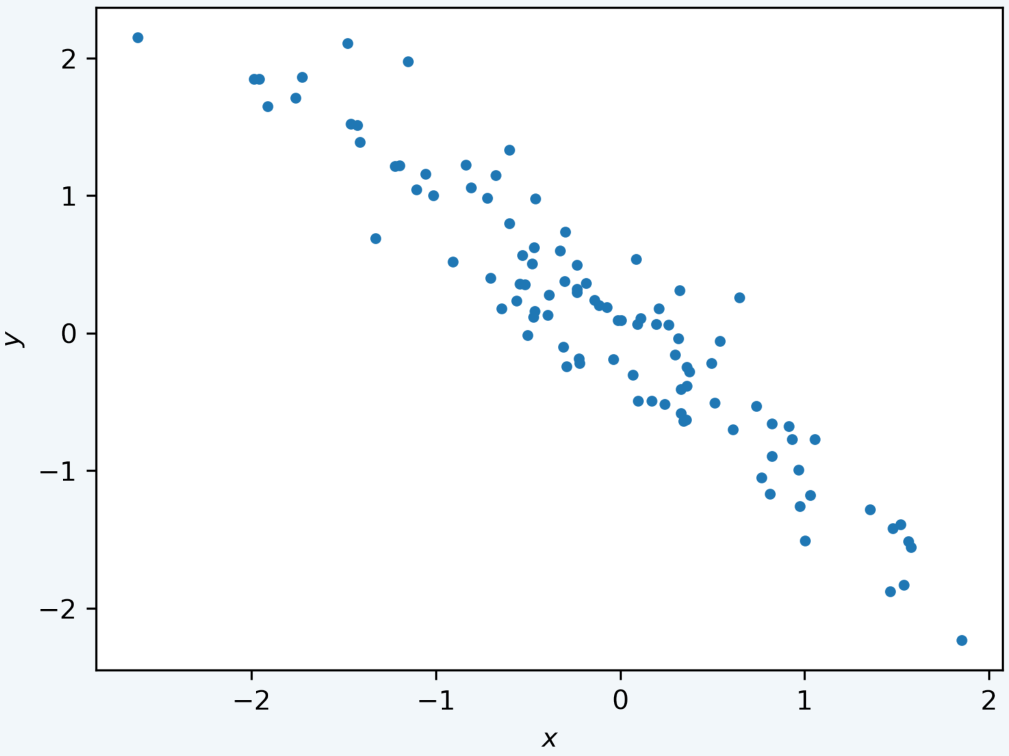
- Let’s think some more about regression and come back to this

More Regression



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- Do you see a conditional expectation here?

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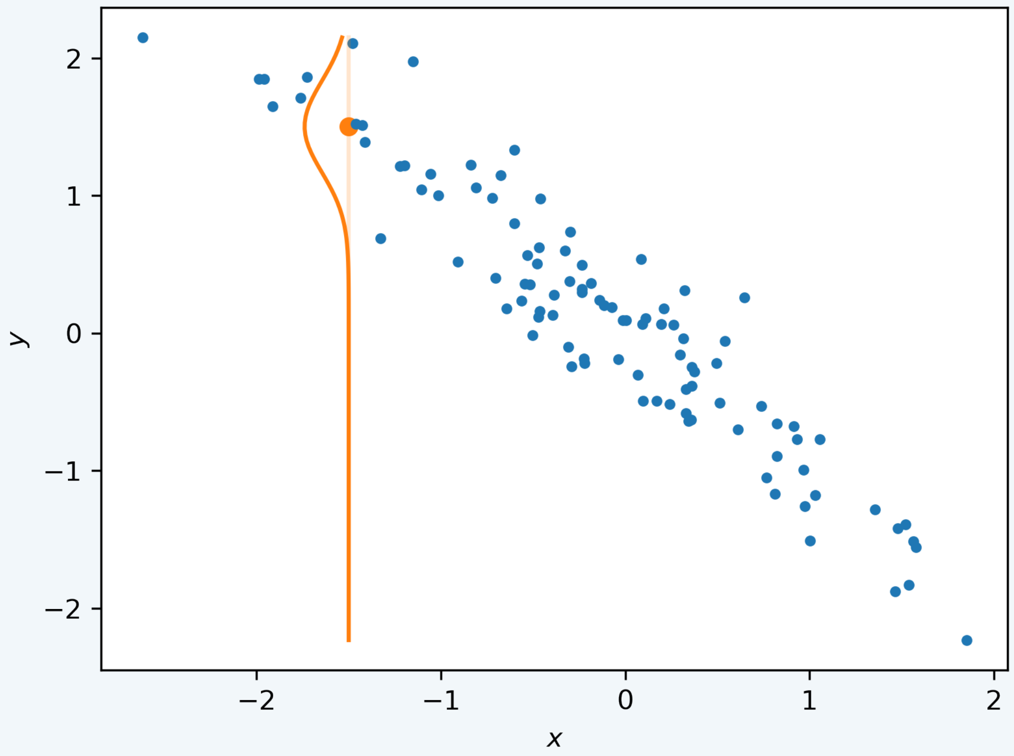

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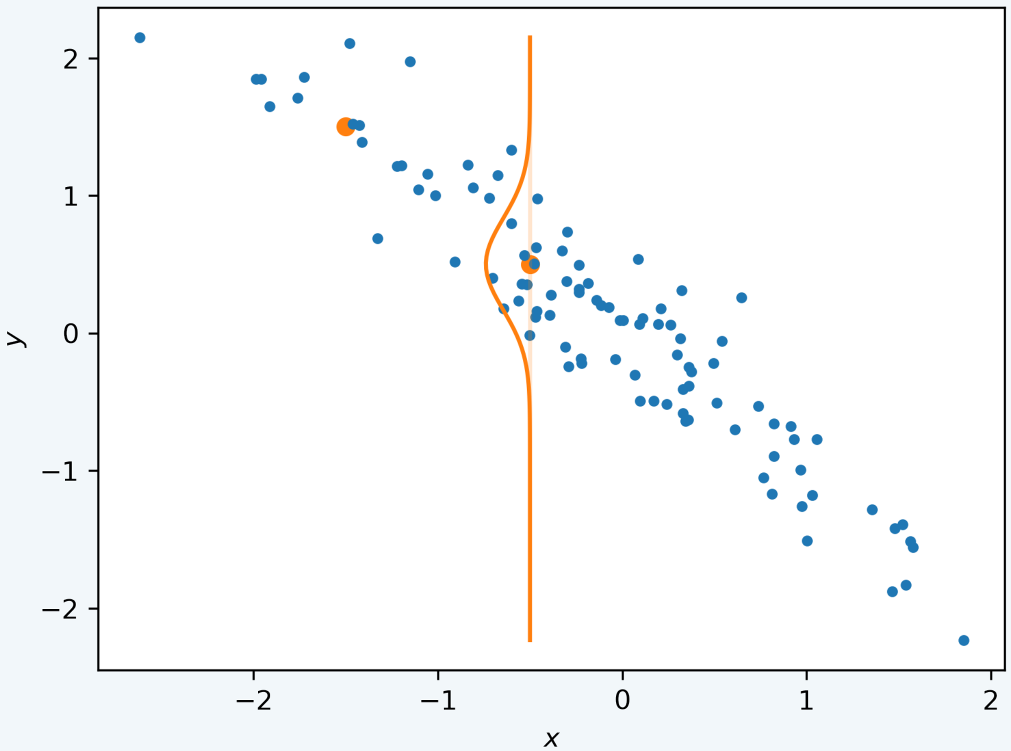

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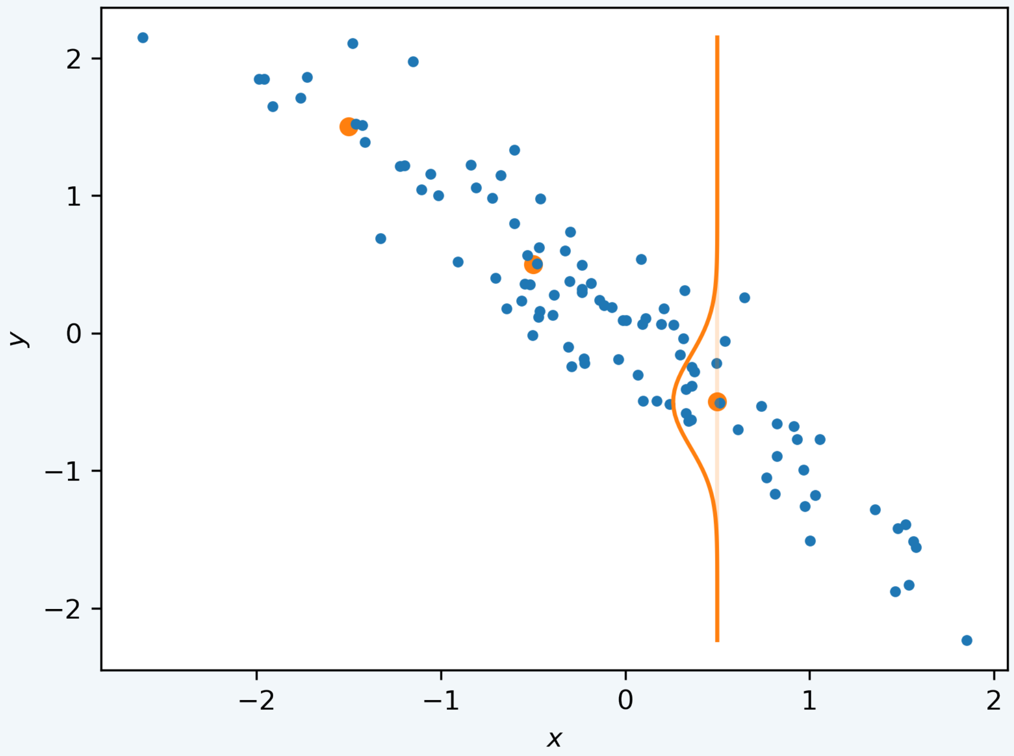

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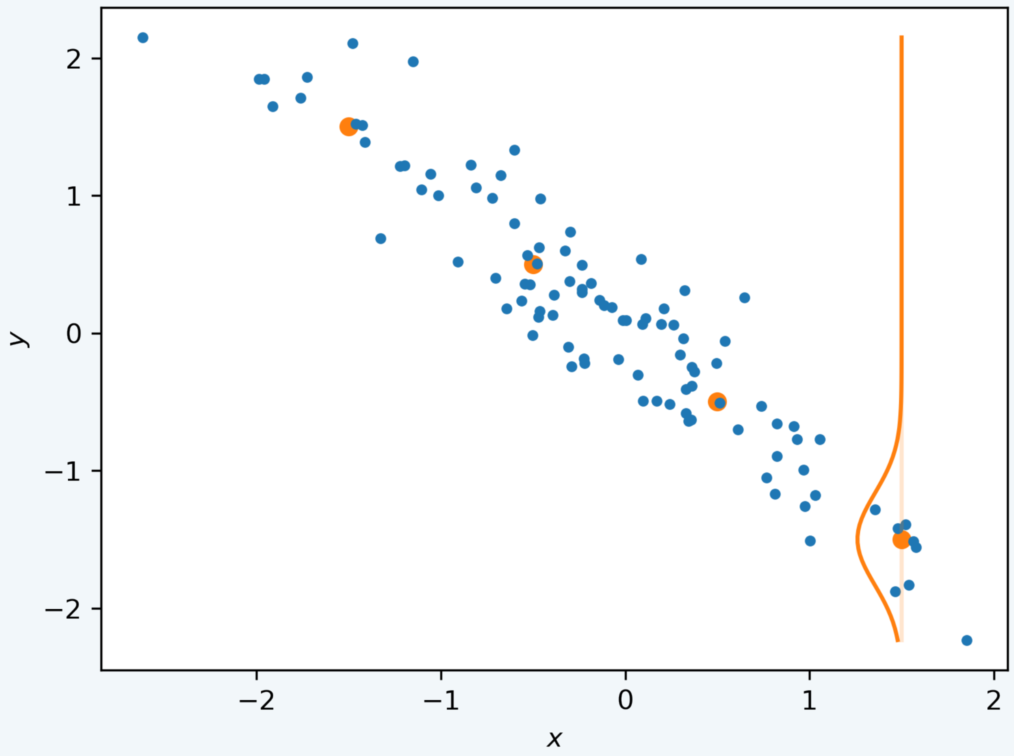

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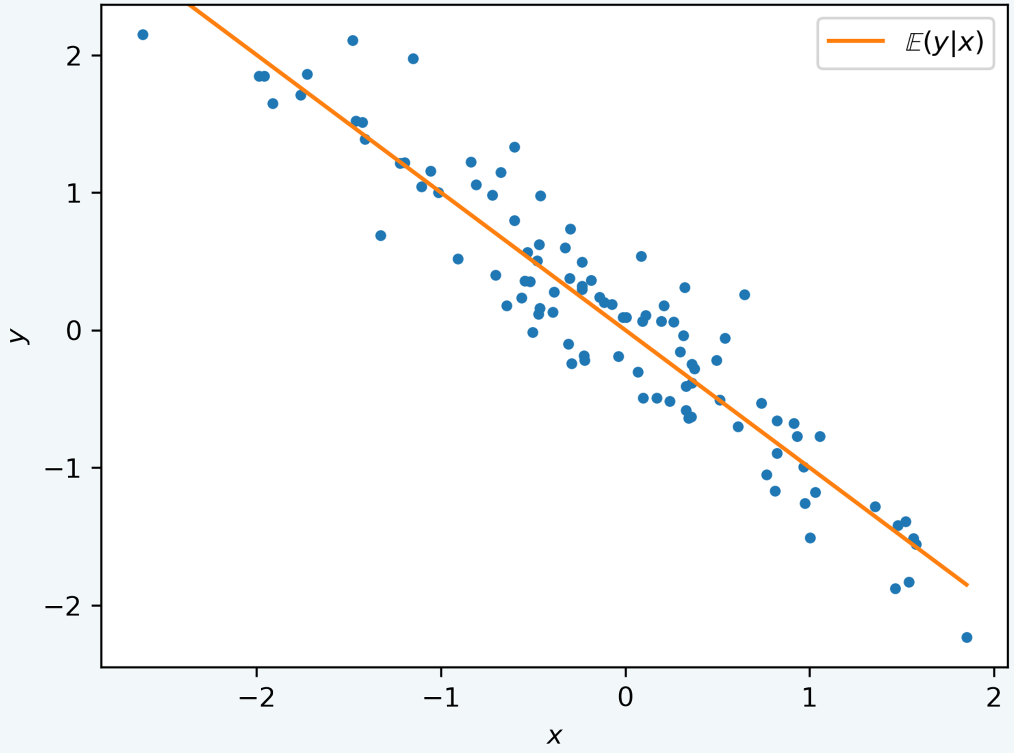

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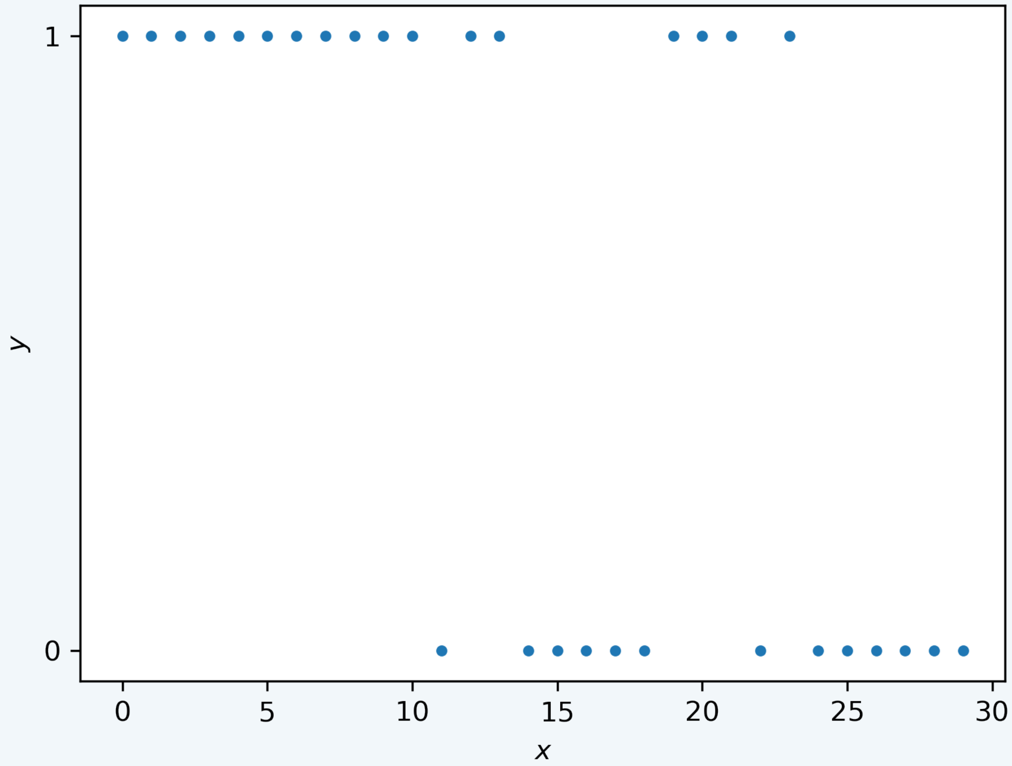
- For linear regression, this "conditional expectation" curve is linear (in the parameters)

Back to Baskets



- For each x, y follows a Bernoulli (“biased coin flip”) distribution
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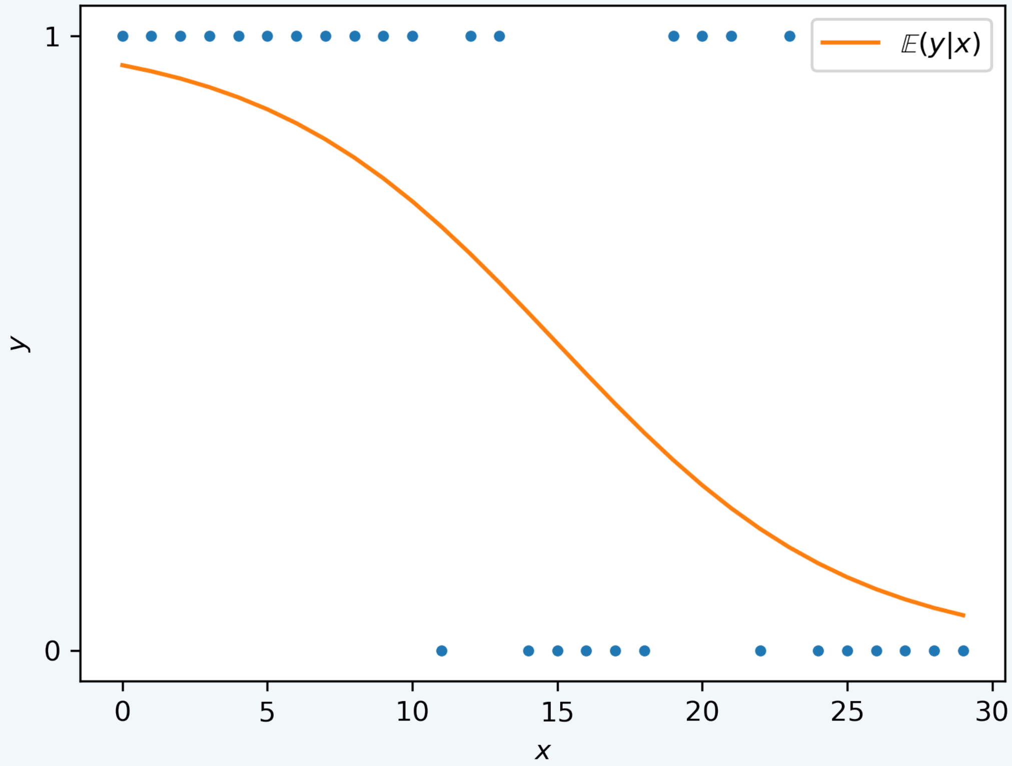

- For each x, y follows a Bernoulli ("biased coin flip") distribution
- What might conditional expectation look like here?
- In this case "conditional expectation" is also a conditional probability,

$$\mathbb{E}(y \,|\, x) = P(y = 1 \,|\, x)$$

- Hey, that's what we need for classification!
- But… how do we get there from x?

Finding the Link

- For linear modeling, our model was

$$y = X\beta + \varepsilon$$
$$\varepsilon \sim \text{Normal}(0, \sigma)$$

- We used the data to find the best $\hat{\beta}$
- For a new "row vector" x, our prediction was $\mathbb{E}(y \mid x) = x\hat{\beta}$
- In that case the mean and linear predictor were the same
- For the baskets example, we'll need some transformation to "link" them

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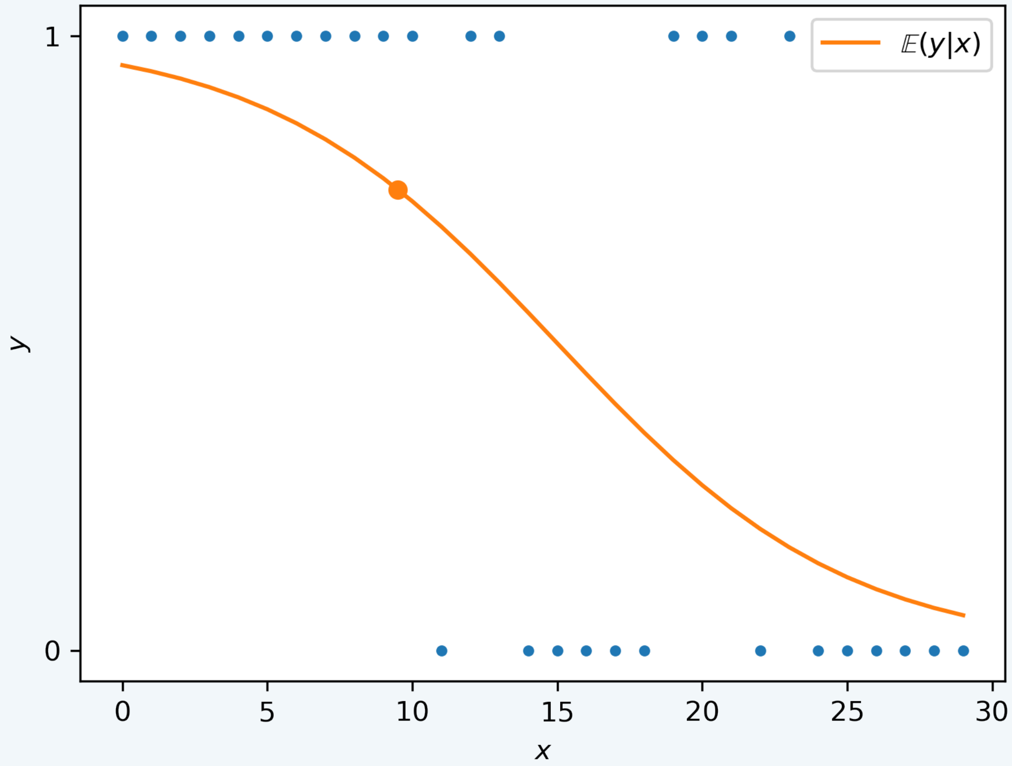

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- Probabilities are always in (0,1)

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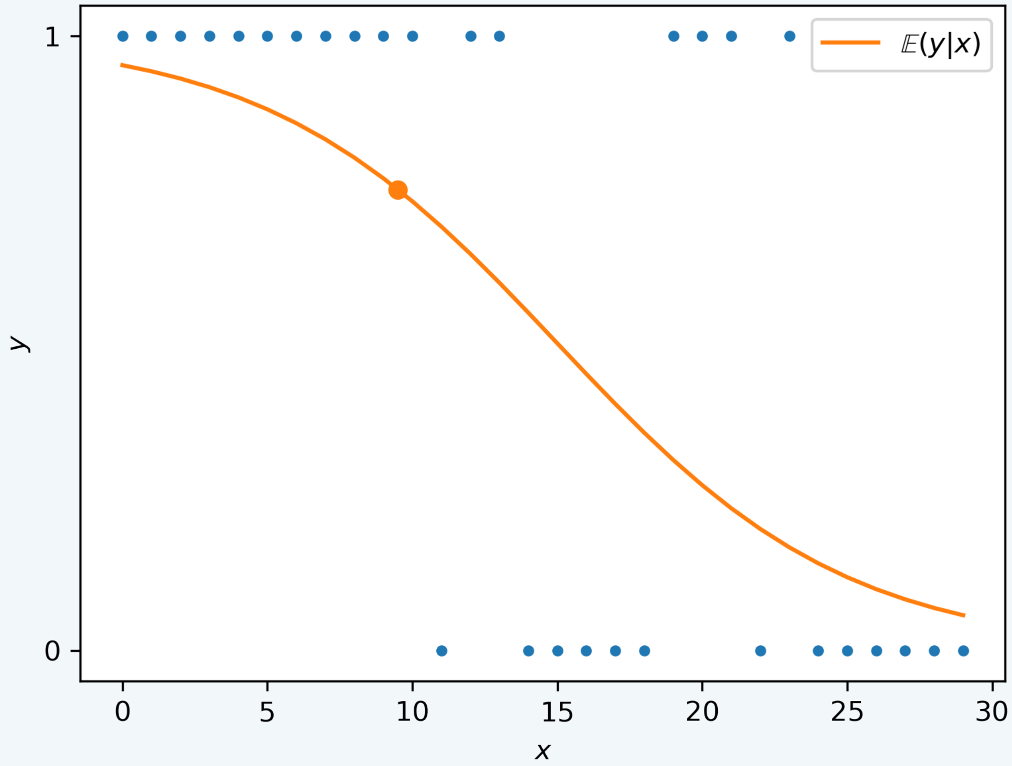

- Here's an x with $\mathbb{E}(y \mid x) = 3/4$
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- We can also work in terms of *odds*, which are 3:1 or 3/1
- If the mean is p, odds are $\dfrac{p}{1-p}$
- What values are possible for odds?

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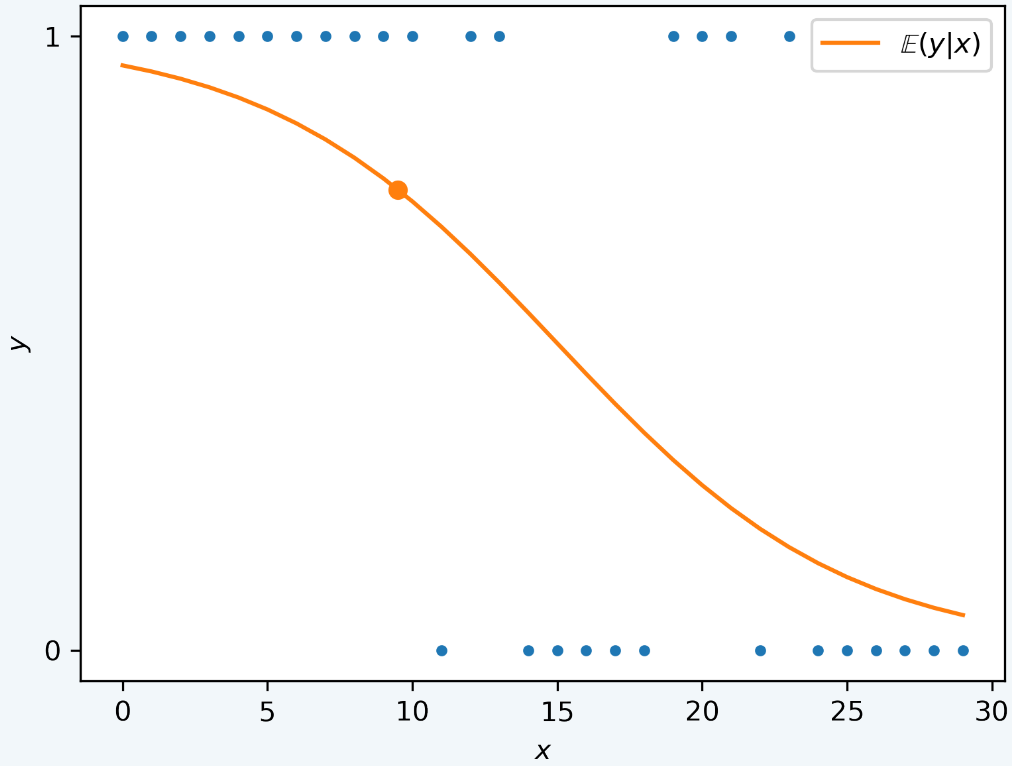

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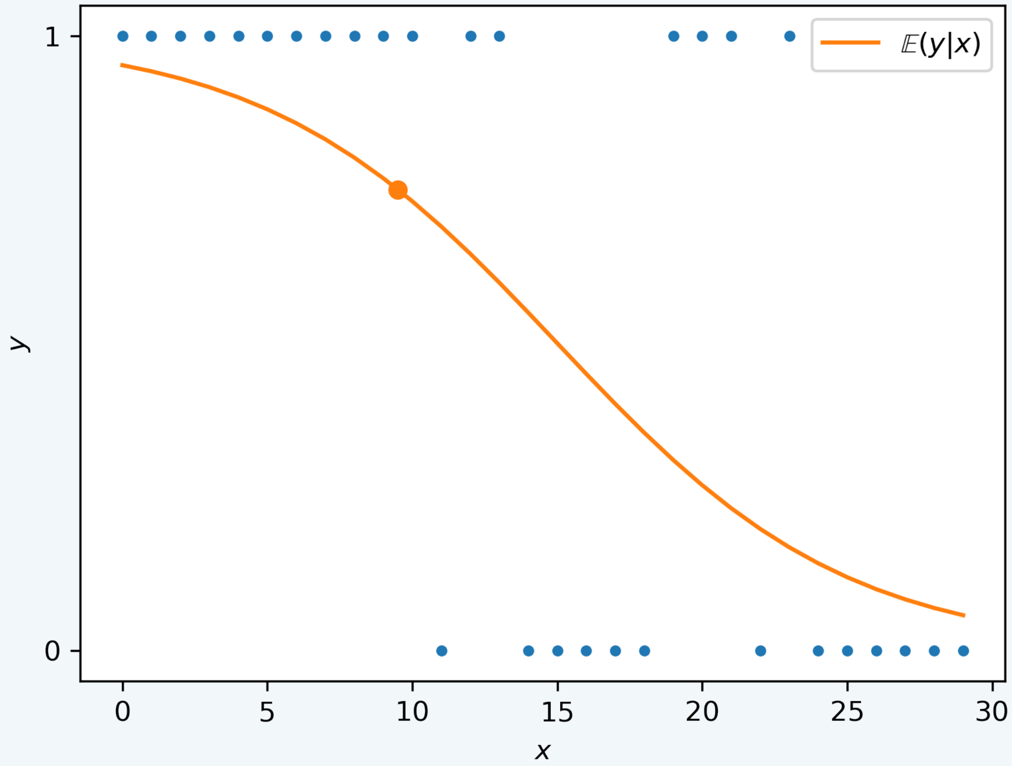

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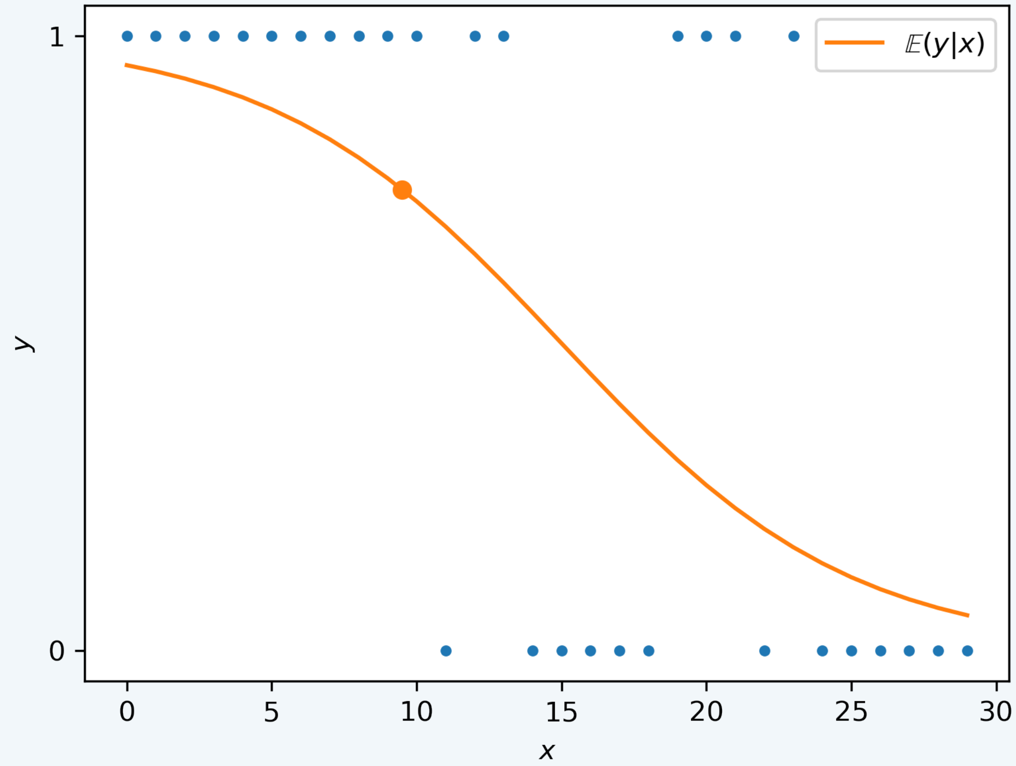

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- We can also work in terms of *odds*, which are 3:1 or 3/1
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- What values are possible for odds?
- Odds can be any positive number!
- But our linear predictor can be any real number
- How can we get from "positives" to "reals"?
- Use the log-odds!

$$p \mapsto \log \frac{p}{1-p}$$

The Logit

- The *logit* ("low jit") maps a probability to its log-odds

$$\operatorname{logit} p = \log \frac{p}{1-p}$$

- For a normal linear model, the linear predictor $x\hat{\beta}$ gets us directly to $\mathbb{E}(y \mid x)$
- For logistic regression, it gives us a transformed version, $x\hat{\beta} = \operatorname{logit} \mathbb{E}(y \mid x)$
- The logit "links" the linear predictor to the conditional expectation
- **For logistic regression, the logit is the *link function***
- (We'll see other link functions another time)

A Link and its Inverse

These mean the same thing:

$$\text{logit}\left(\mathbb{E}(y\,|\,x)\right) = x\beta$$

$$\mathbb{E}(y\,|\,x) = \text{logit}^{-1}(x\beta)$$

A Tale of Two Models

Linear Regression

$$x\beta \in \mathbb{R}$$

Identity

Identity

$$\mathbb{E}(y\,|\,x) \in \mathbb{R}$$

Logistic Regression

$$x\beta \in \mathbb{R}$$

Logit

Inverse Logit

$$\mathbb{E}(y\,|\,x) \in (0,1)$$

Sorry for the Synonyms

These are all the same thing:

- Inverse logit
- logistic function
- expit (in scipy)
- sigmoid (this means “S-shaped”, but people usually mean this specifically)

$$S(x) = \frac{1}{1 + e^{-x}} = \frac{e^x}{e^x + 1}$$

Sigmoid Function

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

x can range from **-infinity** to **infinity**

As **x** approaches -**infinity**

S(x) approaches **0**

As **x** approaches **infinity**

S(x) approaches **1**

We get **probabilities** which is what we want for supervised classification

Final Thoughts

- Loosely speaking, logistic regression is "linear regression for probabilities"
- Two differences:
  - Distribution is binomial instead of normal
  - Link function is logit instead of identity
- Many ideas from linear regression apply:
  - Feature engineering
  - Regularization (L_1, L_2, Elastic net)
  - Diagnostics (*p*-values, log-likelihood, etc)