Some Pleasant Sequence-Space Arithmetic In Continuous Time*

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Abstract

This paper proposes an analytic representation of sequence-space Jacobians in heterogeneous agent models with aggregate shocks in continuous time. Our approach is based on a pen-and-paper perturbation of individual policy functions with respect to price changes, rather than numerical or automatic differentiation. We obtain linear partial differential equations that can be solved efficiently. Our continuous time algorithm speeds up computation of Jacobians and impulse responses threefold relative to discrete time. Continuous time is key to take the analytic perturbation in the presence of binding borrowing constraints. We illustrate our approach in leading heterogeneous agent models with and without nominal rigidities.

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Introduction

Heterogeneous agent macroeconomics has tremendously progressed in the last decade along multiple directions. Continuous time methods have made it possible to solve large-scale models with multiple assets and non-convexities (Kaplan et al., 2018; Achdou et al., 2021). Perturbation methods in discrete time have remarkably accelerated the computation of first-order impulse response functions. Seminal work by Auclert et al. (2021) proposes a highly efficient algorithm that relies on sequence-space Jacobians. In this paper, we ask whether it is possible to design sequence-space Jacobians in continuous time and, if so, whether there are any additional benefits to doing so.

This paper constructs sequence-space Jacobians in continuous time heterogeneous agent economies and demonstrates that doing so leads to substantial speed gains relative to discrete time approaches. Our innovation is to analytically differentiate policy functions with respect to prices rather than relying on numerical or automatic differentiation. We obtain linear backward partial differential equations that characterize how policy functions respond to prices and use their solution to construct sequence-space Jacobians. These observations deliver an algorithm that improves speed threefold relative to discrete time sequence-space Jacobian methods in standard heterogeneous agent models. A code repository accompanies this paper, in which we provide specific examples and general-purpose routines.[1]

For concreteness, consider first a version of the Aiyagari (1994) economy with aggregate shocks, as in Krusell and Smith (1998). We refer to this economy as the Heterogeneous Agent economy, or HA for short. Households face uninsurable idiosyncratic labor productivity risk, and may borrow and save in a risk-free asset. They face an occasionally binding borrowing constraint. A representative firm rents capital and hires labor. Households' consumption and savings forward-looking decisions are fully determined by the sequence of future interest and wage rates. These prices in turn depend on the underlying distribution of asset holdings and idiosyncratic productivity through the firms' decisions and market clearing. The distribution of assets and productivity evolves over time according to the optimal savings decisions of individuals.

The classic difficulty is that individual decisions are forward-looking in time, while the evolution of the infinite-dimensional distribution is backward-looking in time. Prices are the

[1] See https://github.com/ShlokG/CT-SSJ/ or https://sites.google.com/site/adrienbilal/ for routines in Matlab and Python.

fixed point of this forward-backward system that clear the capital and labor market. In this paper, we follow the logic from Auclert et al. (2021) and seek a linearized version of this fixed point problem for moderate aggregate shocks. The key objects in this approach are sequence-space Jacobians: the derivative of aggregate capital and labor with respect to past and expected future price deviations, which depend on individual policy functions through the law of motion of the distribution.

The first step to obtain analytic sequence-space Jacobians is to differentiate individual policy functions with respect to price paths. In continuous time, we show that the response of the consumption function to future interest and wage rate shocks satisfies a simple Bellman equation that involves only steady-state transition probabilities and a component that encodes the steady-state substitution effect. The solution to these partial differential equations can be calculated with a single time iteration. This result mirrors the numerical derivative in Auclert et al. (2021). Crucially however, we obtain an analytic expressions inside the model rather than relying first on discretization and second on numerical differentiation.

Continuous time facilitates the analytic perturbation. It allows seamlessly handling binding credit constraints. In continuous time, the first-order optimality condition for consumption always holds with equality, sidestepping the need to carry Lagrange multipliers in the differentiation. Continuous time also eases the differentiation with respect to the infinite-dimensional distribution, because the distribution has a well-behaved density. We make these notions precise in our heterogeneous agent economy by building on the weak derivative and base measure formalism introduced in Bilal (2023).

The second step is to differentiate the law of motion of the distribution with respect to price sequences and construct market clearing conditions. This step closely follows Auclert et al. (2021) and relies on three substeps: differentiating the law of motion of the distribution with respect to savings rates; expressing savings rates by substituting in individual policy functions; and constructing market clearing conditions. We obtain the sequence-space Jacobians that determine the equilibrium of the economy by mapping expected price paths into realized price paths.

Of course, the logic underpinning the analytic sequence-space Jacobians is more general than the HA example. For illustration, we also develop our approach for a Heterogeneous Agent New Keynesian model as in Auclert et al. (2024b), which we refer to as HANK for brevity. Finally, in the Online Appendix we show how to use analytic sequence-space Jacobians in continuous time in a generic economy that encompasses a large class of continuous-time dynamic general equilibrium frameworks.

We implement our analytic sequence-space Jacobian approach in the HA and HANK economies in continuous time. In both economies, we compare sequence-space Jacobians in the continuous and discrete time cases. Reassuringly but not surprisingly, they coincide. Therefore, the associated impulse response functions also coincide.

We illustrate the computational gains of our analytic construction of sequence-space Jacobians. We report computation times for both economies with 2,000 to 250,000 gridpoints for the individual state space. Across all cases, steady-state computation times are comparable. However, computation for Jacobians and impulse responses is three times faster in the continuous time case. We show that these speed gains are entirely driven by the computation of individual policy functions: by leveraging linear partial differential equations in continuous time, we sidestep the need for numerically or automatically differentiating a nonlinear Euler equation.

Literature. Our paper adds to the literature proposing computational methods for first-order perturbations in heterogeneous agent economies in sequence space in discrete time (Boppart et al., 2018, Auclert et al., 2021). These methods first discretize, then linearize to first order, an economy with heterogeneity. They treat the resulting finite but high-dimensional system as a standard linear system. By reversing the order—linearizing first, discretizing next—this paper proposes a foundation for this computational approach and obtains substantial speed improvements.

Our paper complements contemporaneous work by Glawion (2023) that implements Auclert et al. (2021)'s method in a continuous time setting. Glawion (2023) also discretizes the economy first and uses numerical differentiation for individual policy functions, leading to an algorithm that exactly mirrors Auclert et al. (2021). By contrast, we differentiate entirely within the continuous time model, which delivers additional speed gains.

Our paper relates to the literature that leverages continuous time techniques for heterogeneous agent economies (Kaplan et al., 2018; Achdou et al., 2021). We build on this literature for the steady-state of our models and contribute to it by proposing an analytic sequence-space Jacobian method to solve for impulse response functions to aggregate shocks.

Finally, our paper connects to the emerging literature on perturbation methods in continuous time. Bilal (2023) develops state-space perturbations to first and second order. Alvarez et al. (2016) fully characterize a version of the sequence-space Jacobian in a model that admits closed form solutions. We complement these papers by proposing a flexible approach to sequence-space

Jacobians that applies to a wide class of economies.[2]

The remainder of this paper is organized as follows. Section 1 sets up the HA framework in continuous time. Section 2 constructs sequence-space Jacobians in this economy. Section 3 constructs sequence-space Jacobians in the HANK economy. Section 4 implements our approach for both economies. The last section concludes. Proofs, our general formulation and additional details are in the Appendix and Online Appendix.

1 A workhorse heterogeneous agent model

The setup follows closely the continuous-time version of the Krusell and Smith (1998) economy in Achdou et al. (2021). We refer to this economy as HA, for Heterogeneous Agent. Time $t \geq 0$ is continuous and runs forever. We consider perfect foresight transitions as in Auclert et al. (2021). The exposition of the weak derivative and base measure formalism directly follows Bilal (2023).

1.1 Household decisions

There is a unit continuum of households. Households are endowed with idiosyncratic time-varying productivity y_t which follows a stationary stochastic process. This process is independent across individuals and is defined by its generator Λ. The generator is a functional operator that encodes conditional expectations under the income process. For instance, if productivity follows a multivariate Poisson process with income states $\{y_1, ..., y_J\}$ and transition rates $\lambda \equiv \{\lambda_{ij}\}_{i,j}$ from states y_i to y_j, then $\Lambda_i[V] = \sum_j \lambda_{ij}(V_j - V_i)$. $\Lambda_i[V]$ corresponds to the conditional expectation $\mathbb{E}_t[dV(y_t)|y_t = y_i]$ as the income process evolves. The operator Λ corresponds to the matrix with entries $\lambda_{ij} - \mathbb{1}_{i=j} \sum_k \lambda_{ik}$. For simplicity of exposition, we will focus on this multivariate Poisson process, but none of our results rely on it as we show in Appendix F.[3] When unambiguous, we use interchangeably y_t for the income state at time t, and y_i or y_j for the value of the income in state i or j.

Households have a strictly increasing and concave flow utility function u and time-separable preferences with rate of time preference ρ. They solve a standard income fluctuation problem by deciding how much to consume and save every period in a single risk-free asset. Households

[2]See also Bhandari et al. (2023) for a discrete time approach to state space analytic perturbations.

[3]If productivity follows a diffusion process $dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$, then the generator is the functional operator $\Lambda(y)[V] = \mu(y)V'(y) + \frac{\sigma(y)^2}{2}V''(y)$.

are endowed with initial asset holdings a_0, and assets $a_t \geq \underline{a}$ must remain above a borrowing constraint $\underline{a}$. The value function $V_{it}(a)$ of households with income state y_i and assets a satisfies the Hamilton-Jacobi-Bellman equation:[4]

$$\begin{aligned} \rho V_{it}(a) &= \max_{c \geq 0} u(c) + L_{it}(a,c)[V_{it}] + \frac{\partial V_{it}(a)}{\partial t} \\ L_{it}(a,c)[V_t] &\equiv (r_t a + w_t y_i - c) V'_{it}(a) + \Lambda_i[V_t(a)] \\ V'_{it}(\underline{a}) &\geq u'(r_t \underline{a} + w_t y_i) \end{aligned} \tag{1}$$

The Bellman equation (1) states that the flow value of households is the sum of three terms. The first is simply the flow utility from consumption $u(c)$.

The second term $L_{it}(a,c)[V_t]$ encodes the continuation value from endogenous changes in assets a_t that evolve according to the budget constraint $da_t = (r_t a_t + w_t y_t - c_t)dt$, and exogenous changes in productivity y_t. The operator $L_{it}(a,c)$ is the generator of the stochastic process for the pair of idiosyncratic states of households (a_t, y_t), and encodes expected changes in values. $V'_{it}(a)$ denotes the partial derivative with respect to a: $V'_{it}(a) \equiv \frac{\partial V_{it}(a)}{\partial a}$.

Together, the first two terms lead to the first-order optimality for consumption $u'(c_{it}(a)) = V'_{it}(a)$. In continuous time, it holds for all assets $a \geq \underline{a}$.

The third term $\frac{\partial V_{it}(a)}{\partial t}$ encodes the continuation value due to changes in prices r_t and w_t over time. We denote by $c_{it}(a)$ the optimal consumption decision of households.

The Bellman equation (1) also includes a boundary condition $V'_{it}(\underline{a}) \geq u'(r_t \underline{a} + w_t y_i)$ at the borrowing constraint $\underline{a}$. This condition states that savings $S_{it}(a) \equiv r_t a + w_t y_i - c_{it}(a)$ must be weakly positive at the constraint $a = \underline{a}$. Applying u' to the condition $c_{it}(\underline{a}) \leq r_t \underline{a} + w_t y_i$ and using that the first-order optimality condition always holds with equality $u'(c_{it}(a)) = V'_{it}(a)$ delivers the boundary condition.

To obtain the Euler equation, we differentiate the Bellman equation (1) with respect to assets for all $a > \underline{a}$, use the envelope condition and obtain:

$$(\rho - r_t) V'_{it}(a) = L_{it}(a, c_{it}(a))[V'_{it}] + \frac{\partial V'_{it}(a)}{\partial t}, \quad a > \underline{a}. \tag{2}$$

To see why equation (2) encodes the Euler equation, we use the first-order optimality condition for consumption $V'_{it}(a) = u'(c_{it}(a))$ and recognize that the right-hand-side of (2) represents $\mathbb{E}_t[du'(c_t)]$ by Ito's lemma. We then obtain the familiar Euler equation in continuous time which holds with equality away from the borrowing constraint: $\mathbb{E}_t\left[\frac{du'(c_t)}{dt}\right] = (\rho - r_t)u'(c_t),\ a_t > \underline{a}$.

[4]The value function is restricted to have at most linear growth as assets and income approach infinity. This restriction is the recursive analogue to the No-Ponzi condition in the sequential formulation of problem (1).

1.2 Distribution of households

The savings decisions and labor market shocks that households experience lead to the emergence of a distribution of households over asset and productivity states (a, y_i). For all $a > \underline{a}$, denote by $g_{it}(a)$ the density of households at productivity state y_i and assets a. The evolution of the density follows its law of motion, the Kolmogorov Forward equation:

$$\frac{\partial g_{it}(a)}{\partial t} = -\frac{\partial}{\partial a}\Big(S_{it}(a)g_{it}(a)\Big) + \Lambda_i^*[g_t(a)] \equiv L_{it}^*(a, c_{it}(a))[g_t], \quad a > \underline{a}, \tag{3}$$

where $S_{it}(a) = r_t a + w_t y_i - c_{it}(a)$ denotes the equilibrium savings rate. Λ^* is the transpose of the matrix Λ, so that $\Lambda_i^*[g] = \sum_j \lambda_{ji} g_j - (\sum_j \lambda_{ij}) g_i$. The operator L^* is the *adjoint* of the operator L, which generalizes the transpose property to infinite-dimensional operators. Ultimately, once we discretize the operators L, L^* into matrices for computation, their discretized counterparts are transposes of each other.

The first component in the law of motion encodes how savings move households in the state space over time. The second component encodes how labor market productivity shocks lead households to switch across labor market states.

The occasionally binding borrowing constraint $a \geq \underline{a}$ leads to potential mass points in the distribution at $a = \underline{a}$ (Achdou et al., 2021). Denote by $g_{it}(\underline{a})$ the mass of households at $\underline{a}$. Denote by $\overline{g}_{it}$ the total measure of households in income state y_i, that satisfies $\frac{\partial \overline{g}_{it}}{\partial t} = \Lambda_i^* \overline{g}_t$ since it only depends on the productivity process. Integrating the law of motion (3) over assets a, from $\underline{a} + \varepsilon$ to $+\infty$, taking $\varepsilon \downarrow 0$ and subtracting $\frac{d\overline{g}_{it}}{dt} = \Lambda_i^* \overline{g}_t$ from each side leads to the law of motion for the mass of constrained households:

$$\frac{\partial g_{it}(\underline{a})}{\partial t} = -\lim_{a \downarrow \underline{a}} S_{it}(a) g_{it}(a) + \Lambda_i^*[g_t(\underline{a})]. \tag{4}$$

The economy starts at a given distribution $g_{i0}(a)$.

To construct sequence-space Jacobians, we need to differentiate the law of motion of the distribution (3)-(4) with respect to savings rates. To make that differentiation precise, we introduce a formalism that jointly summarizes the evolution of the distribution (3)-(4) on the interior of the domain $(\underline{a}, +\infty)$ and at the mass point $\underline{a}$. We do so by following Bilal (2023). This formalism allows our sequence-space Jacobian method to seamlessly generalize to a wide class of models instead of having to work through the specificities of each state variable process.

We define the *base measure* $d\eta(a)$ such that we can define well-posed densities with respect to this base measure for the entire domain $[\underline{a}, +\infty)$. Namely, we define $\eta(A) = \mathbb{1}\{\underline{a} \in A\} + \int_A da$

for any measurable set A. Equivalently, $d\eta(a) = \delta_{\underline{a}}(a) + da$ is the sum of a Dirac mass point at $\underline{a}$ and the usual Lebesgue measure over $(\underline{a}, +\infty)$. We define the *density* of households in productivity state y_i and with assets around a as $g_{it}(a)$. The total *measure* of these households is then $g_{it}(a)d\eta(a)$.

The advantage of this formalism is that the Kolmogorov equation (3) now holds on the entire domain $[\underline{a}, +\infty)$ as long as we interpret the derivative $\frac{\partial}{\partial a}$ as a *weak derivative* (Evans, 2010). That is, we consider derivatives in a space that contains functions but also includes Dirac mass points. For instance, the weak derivative of a function with a jump at $\underline{a}$ is a Dirac mass point at $\underline{a}$ times the size of the jump. We provide details in Online Appendix D.1.

To gain intuition, consider the savings component in the law of motion (3). $s_{it}(a)g_{it}(a)$ is discontinuous as a function of a from just below $\underline{a}$ where it is normalized to zero, and just above $\underline{a}$ where it takes the value $\lim_{a\downarrow\underline{a}} s_{it}(a)g_{it}(a)$. Its weak derivative is therefore $\left(\lim_{a\downarrow\underline{a}} s_{it}(a)g_{it}(a)\right) \times \delta_{\underline{a}}(a)$, which corresponds to the savings component in the evolution of the mass point (4). Thus, we obtain that *densities with respect to the base measure* satisfy the Kolmogorov Forward equation with *weak derivatives* on the full domain $a \geq \underline{a}$ (Bogachev et al., 2015):[5]

$$\frac{\partial g_{it}(a)}{\partial t} = L^*_{it}(a, c_{it}(a))[g_t], \quad a \geq \underline{a}. \tag{5}$$

1.3 Firms and market clearing

A representative firm operates a production technology $Y_t(K, N) = e^{z_t} K^\alpha N^{1-\alpha}$ and rents capital and labor from households at the interest and wage rates r_t, w_t. Capital depreciates at rate Δ and firms cover the cost of depreciation. z_t denotes aggregate productivity and follows a perfectly foreseen path. The firm solves:

$$\max_{K_t, N_t} e^{z_t} K_t^\alpha N_t^{1-\alpha} - (r_t + \Delta)K_t - w_t N_t,$$

and the capital and labor markets clear:

$$K_t = \sum_i \int a g_{it}(a) d\eta(a) \qquad\qquad N_t = \sum_i \int y_i g_{it}(a) d\eta(a)$$

[5]Equivalently, the adjoint of L on the space of continuously differentiable functions on $[\underline{a}, +\infty)$ is L^* on the space of densities with respect to the base measure η. We provide details in Online Appendix D.2.

The optimality conditions of the firm together with market clearing imply that the real interest rate r_t and the wage rate w_t clear the capital and labor markets:

$$r_t + \Delta = \alpha e^{z_t} \left(\frac{\sum_i \int y_i g_{it}(a) d\eta(a)}{\sum_i \int a g_{it}(a) d\eta(a)} \right)^{1-\alpha} \qquad w_t = (1-\alpha) e^{z_t} \left(\frac{\sum_i \int a g_{it}(a) d\eta(a)}{\sum_i \int y_i g_{it}(a) d\eta(a)} \right)^{\alpha}. \tag{6}$$

1.4 Equilibrium definition

A steady-state equilibrium of the economy is a collection of a time-invariant value function $V_i^{ss}(a)$, a consumption function $c_i^{ss}(a)$, a distribution $g_i^{ss}(a)$ and prices r^{ss}, w^{ss} that satisfy (1,5,6) when $z_t \equiv 0$ and $g_0 = g^{ss}$. For the sequel, we define $\mathcal{L}_i(a) \equiv L_i^{ss}(a, c_i^{ss}(a))$ the steady-state generator, that encodes expectations under the steady-state evolution of assets and productivity process. We also denote by $c_i'^{ss}(a) \equiv \frac{\partial c_i^{ss}(a)}{\partial a}$ the steady-state marginal propensity to consume out of a windfall transfer.

A dynamic equilibrium of the economy is a time-varying collection of value functions $V_{it}(a)$, consumption functions $c_{it}(a)$, distributions $g_{it}(a)$ and prices r_t, w_t that satisfy (1,5,6) for any sequence z_t and initial distribution g_0.

As with all perturbation methods, we take a locally isolated steady-state as given and seek a dynamic equilibrium when $|z_t|$ and $|g_t - g^{ss}|$ are small enough that we can consider first-order perturbations. The next section shows how to efficiently construct sequence-space Jacobians to solve for dynamic equilibria to first order.

2 Sequence-space Jacobians

2.1 Policy functions

We consider small deviations of values, policy functions, distributions and prices from steady-state when $|z_t|$ and $|g_t - g^{ss}|$ are small. We denote deviations from steady-state with hats. We start with the main innovation in this paper: obtain a simple equation for the 'policy function Jacobians', that is, the response of consumption to expected price changes. A useful intermediate step is to characterize how the marginal utility of consumption responds to expected price changes. We denote by $v_{it}(a) = u'(c_{it}(a)) = V_{it}'(a)$.

Consider a small deviation in the sequence of prices $\widehat{r}_{t+s}, \widehat{w}_{t+s}$ around steady-state. We posit that, to first order, the marginal value of assets is a present discounted valuation of these price changes:

$$\widehat{v}_{it}(a) = v_{it}(a) - v_i^{ss}(a) = \int_0^\infty e^{-\rho s}\Big\{\varphi_{is}^r(a)\widehat{r}_{t+s} + \varphi_{is}^w(a)\widehat{w}_{t+s}\Big\}ds. \tag{7}$$

Our goal is to characterize the functions $\varphi_{is}^r, \varphi_{is}^w$ that translate future price changes into marginal utility today. These functions correspond to the sequence-space partial derivatives $e^{-\rho s}\varphi_{is}^r(a) = \frac{\partial u'(c_{it}(a))}{\partial r_{t+s}}$, and similarly for wages. Auclert et al. (2021) compute these derivatives φ by directly numerically differentiating the nonlinear Euler equation after discretizing it. Instead, we leverage continuous time to make this differentiation fully analytic within the model.

Our strategy is simple: substitute the definition (7) into the Euler equation (2) which we then linearize with respect to expected price sequences $\widehat{r}_{t+s}, \widehat{w}_{t+s}$. Finally, we recognize that the resulting equality must hold for all such expected price sequences, which allows us to 'identify coefficients' and obtain restrictions on the functions $\varphi_{is}^r(a), \varphi_{is}^w(a)$. That the first-order optimality condition holds for all assets $a \geq \underline{a}$ and the Euler equation holds for all assets $a > \underline{a}$ greatly facilitates this calculation. Otherwise, and as in discrete time, we would need to carry Lagrange multipliers through the derivation. These derivations prove the following result.

Proposition 1. *(Marginal utility Jacobians)*
The functions $\varphi_{is}^r(a), \varphi_{is}^w(a)$ satisfy the following partial differential equations. For all pairs (a, y_i) for which the borrowing constraint does not bind in steady-state—in particular for $a > \underline{a}$:

$$\varphi_{i0}^r(a) = \frac{\partial}{\partial a}\Big(a u'(c_i^{ss}(a))\Big) \qquad \frac{\partial \varphi_{is}^r(a)}{\partial s} = \left(r^{ss} - c_i'^{ss}(a)\right)\varphi_{is}^r(a) + \mathcal{L}_i(a)[\varphi_s^r]$$

$$\varphi_{i0}^w(a) = \frac{\partial}{\partial a}\Big(y_i u'(c_i^{ss}(a))\Big) \qquad \frac{\partial \varphi_{is}^w(a)}{\partial s} = \left(r^{ss} - c_i'^{ss}(a)\right)\varphi_{is}^w(a) + \mathcal{L}_i(a)[\varphi_s^w]$$

For all pairs $(\underline{a}, y_i)$ for which the borrowing constraint binds in steady-state:

$$\varphi_{it}^r(\underline{a}) = u''(c_i^{ss}(\underline{a}))\underline{a}\ \delta_0(t) \qquad \varphi_{it}^w(\underline{a}) = u''(c_i^{ss}(\underline{a}))y_i\ \delta_0(t),$$

where $\delta_0(t)$ denotes a Dirac mass function with respect to time.

Proof. See Appendix A.1. □

Proposition 1 reveals that a standard, time-dependent Bellman equation determines the first-order response of the marginal utility to interest rate changes. These Bellman equations depend only on steady-state objects. The initial conditions φ_0^r, φ_0^w depend on the disposable income gain from a contemporaneous price change, converted to utils using the marginal utility of steady-state consumption. To obtain the impact on marginal utility, we apply a derivative with respect to assets. Theorem 1 in Online Appendix F.3 generalizes these results to a generic dynamic economy.

Typically, Bellman equations feature a terminal condition and time runs backwards. Here, we specified the marginal value (7) such that the resulting Bellman equations for φ^r, φ^w have an initial condition instead of a terminal condition. Time then runs forward instead of backward in the partial differential equations of Proposition 1. We have essentially reversed the direction of time to leverage a fundamental property of marginal utility Jacobians. The response of marginal utility at time t to a price change at time s only depends on the difference $s-t$, not separately on calendar times t, s. By running time from time t to time s, we can leverage this property and minimize computation similarly to the numerical differentiation in Auclert et al. (2021).

The partial differential equations of Proposition 1 also only depend on steady-state objects. The time evolution of the marginal utility Jacobians depends on two components. The first component encodes the steady-state substitution effect. To see this, consider a special case without idiosyncratic risk, without binding borrowing constraints, and with a Constant Relative Risk Aversion (CRRA) utility function with elasticity of intertemporal substitution σ. In that special case, permanent income logic implies that consumption satisfies $c_i^{ss}(a) = (r^{ss} + \sigma(\rho - r^{ss}))\left(a + \int_0^\infty e^{-\rho s} y_{t+s}\right)$ with $y_t = y_i$, and $r^{ss} - \frac{\partial c_i'^{ss}(a)}{\partial a} = \sigma(\rho - r^{ss})$: the substitution effect.

The second component in the time evolution of marginal utility Jacobians depends on expectations under steady-state decisions. Households expect to keep consuming and saving as the economy experiences price changes, and thus expect that their marginal utility may respond differently to these price changes going forward. The second component encodes this force, and Proposition 1 ensures that it suffices to evaluate it at steady-state decisions to first order.

Proposition 1 also highlights the effects of the borrowing constraint on marginal utility. The partial differential equations hold at pairs (a, y_i) where the borrowing constraint does not bind in steady-state. When the borrowing constraint binds in steady-state at some pair $(\underline{a}, y_i)$, then consumption is simply disposable income $r_t\underline{a} + w_t y_i$. The marginal utility of consumption then responds accordingly to price changes: it only depends on contemporaneous price changes, not any future price changes. This feature materializes as a boundary condition on φ^r, φ^w that follows a Dirac delta function.

Proposition 1 immediately characterizes how policy functions—consumption—respond to price changes. We leverage that the first-order optimality condition always holds with equality, so that $\widehat{c}_{it}(a) = \widehat{v}_{it}(a)/u''(c_i^{ss}(a))$.

Corollary 1. *(Policy function Jacobians)*

$$\widehat{c}_{it}(a) = \frac{1}{u''(c_i^{ss}(a))} \int_0^\infty e^{-\rho s} \Big\{ \varphi_{is}^r(a) \widehat{r}_{t+s} + \varphi_{is}^w(a) \widehat{w}_{t+s} \Big\} ds,$$

where φ^r, φ^w satisfy the conditions in Proposition 1.

Proof. See Appendix A.2 □

Corollary 1 determines how consumption responds to future price changes. Corollary 4 in Online Appendix F.3 generalizes this result to a generic dynamic economy.

Corollary 1 reveals an important feature of the consumption function in continuous time that may at first seem surprising. In continuous time, as the interest rate change becomes close to contemporaneous, the response of consumption to the interest rate change is $\lim_{s \downarrow 0} \frac{\partial c_{it}(a)}{\partial r_{t+s}} = c_i'^{ss}(a) a + \frac{u'(c_i^{ss}(a))}{u''(c_i^{ss}(a))}$, where we have used the expression for φ_0 from Proposition 1. This formula may seem surprising because an unanticipated interest rate shock $\widehat{r}_t$ at time t is isomorphic to a cash transfer of size $a\widehat{r}_t$. Under this logic, the reaction of consumption should be $c_i'^{ss}(a)a$, not $c_i'^{ss}(a)a + \frac{u'(c_i^{ss}(a))}{u''(c_i^{ss}(a))}$. The second component in this expression captures an anticipation effect: households respond to the interest rate increase by saving more to boost the resulting interest income gain.

Why is there an anticipation effect in response to a seemingly unanticipated interest rate shock? The answer is that all interest rate shocks $\widehat{r}_{t+s}$ for any $s > 0$ are, in fact, anticipated. In continuous time, households learn about the future sequence $\{\widehat{r}_{t+s}\}_{s \geq 0}$ at time t and are thus surprised only in the initial instant but not at any later instants. Thus, the *limit* $\lim_{s \downarrow 0} \frac{\partial c_{it}(a)}{\partial r_{t+s}}$ always contains the anticipation effect.

Why does the surprise response of consumption $c_i'^{ss}(a)a$ to the contemporaneous interest rate shock not appear in Corollary 1? In fact, this consumption response still occurs: $\frac{c_{it}(a)}{\partial r_t} = c_i'^{ss}(a)a$, and the policy function Jacobian $\frac{\partial c_{it}(a)}{\partial r_{t+s}}$ is discontinuous at $s = 0$. However, since the discontinuity occurs only for a vanishingly small instant $[t, t + dt)$, it does not contribute materially to household decisions as highlighted in the integral in Corollary 1.

In Online Appendix E, we formalize these arguments by deriving the discrete time counterpart of Corollary 1 for small time steps $\varepsilon > 0$. Our arguments are meant to be illustrative only because this discrete time derivation imposes that that the borrowing constraint never binds—a feature that is not needed for the proof of Corollary 1. Online Appendix E indicates that the relevant limit expression for Corollary 1 is:

$$\widehat{c}_{it}(a) = \lim_{\varepsilon \downarrow 0} \left\{ \varepsilon c_i'^{ss}(a) a \widehat{r}_t + \frac{1}{u''(c_i^{ss}(a))} \int_\varepsilon^\infty e^{-\rho s} \Big\{ \varphi_{is}^r(a) \widehat{r}_{t+s} + \varphi_{is}^w(a) \widehat{w}_{t+s} \Big\} ds \right\}.$$

This expression highlights that the surprise only affects consumption proportionally to the length of the time period ε. In continuous time, this effect vanishes.

In discrete time, the surprise effect $\frac{\partial c_{it}(a)}{\partial r_t} = c_i'^{ss}(a)a$ does not vanish because it occurs for a full period. Discrete time features *information aggregation*: households can only make a single choice per period, and thus act as if surprised for this entire period. In continuous time, households can re-optimize rapidly once information is revealed, and thus act as if most shocks are anticipated. Of course, as the period length in discrete time shrinks, we recover the continuous time formula.

Ultimately, the choice of the time period in discrete time, and time discretization in numerical implementations in continuous time, should reflect the frequency at which households are likely to adjust their consumption and savings in practice. In Section 2.4, we provide two variants of our algorithm: one with information aggregation that matches exactly discrete time; and one without information aggregation which thus slightly differs from discrete time.

Equipped with these results, we construct sequence-space Jacobians that characterize the general equilibrium of our economy.

2.2 Distribution

We now connect changes in consumption to the evolution of the distribution and, ultimately, market clearing conditions. We denote by $\widehat{S}_t(a,y) = \widehat{r}_t a + \widehat{w}_t y - \widehat{c}_t(a,y)$ the first-order change in savings rates. To first order, the law of motion of the distribution (5) becomes:

$$\frac{\partial \widehat{g}_{it}(a)}{\partial t} = \mathcal{L}_i^*(a)[\widehat{g}_t] - \frac{\partial}{\partial a}\Big(\widehat{S}_{it}(a) g_i^{ss}(a)\Big), \tag{8}$$

where $\mathcal{L}_i^*$ is the transition operator of the distribution, again evaluated at steady-state decisions.

The first order law of motion (8) is linear, and therefore admits a simple solution with a similar structure to a scalar linear ordinary differential equation. Solving forward and using functional notation $\widehat{g}_t \equiv \{\widehat{g}_{it}(a)\}_{i=1\ldots J, a\geq \underline{a}}$, we obtain:

$$\widehat{g}_t = T_t^*[\widehat{g}_0] - \int_0^t T_{t-s}^* \left[\frac{\partial}{\partial a}\Big(\widehat{S}_s g^{ss}\Big)\right] ds. \tag{9}$$

In equation (9), T_t^* is the analogue of the t-th power of the transition operator in discrete time, and the t-th power of the transition matrix if the state space was discretized. T_t^* is the continuous time steady-state transition operator that propagates the distribution forward by t

time periods. T_t^* satisfies the following evolution equation:

$$\frac{\partial T_t^*}{\partial t} = \mathcal{L}^*[T_t^*], \quad T_{0,ij}^*(a,a') = \mathbb{1}\{i=j\}\ \delta(a-a'). \tag{10}$$

Hence, the transition operator T_t^* is the *integrated* counterpart of the *local* transition operator $\mathcal{L}^*$.[6]

The solution (9) highlights that the distribution at time t is the sum of two components. The first component is the natural mean-reversion back to steady-state of any initial condition that deviated from steady-state. The steady-state transition operator T_t^* governs that mean-reversion.

The second component is the accumulated effect of past savings rates. If savings rates deviate from steady-state at time s, the distribution changes between times s and $s+ds$ by $-\frac{\partial}{\partial a}\left(\widehat{S}_s g^{ss}\right)$ as per the law of motion (8). This deviation in the distribution then mean-reverts back to steady-state once more according to the steady-state transition operator T_{t-s}^*.

Of course, we still need to relate savings rates $\widehat{S}_s$ to the consumption Jacobians that we obtained in Corollary 1. Doing so lets us express the distribution at any time as a function of the marginal utility Jacobians φ^r, φ^w.

Proposition 2. *(Distributional Jacobians)*

$$\widehat{g}_t = T_t^*[\widehat{g}_0] + \int_0^\infty \int_0^{\min\{t,\tau\}} \left\{T_{t-s}^*[\mathcal{D}_{\tau-s}^r]\widehat{r}_\tau + T_{t-s}^*[\mathcal{D}_{\tau-s}^w]\widehat{w}_\tau\right\} ds d\tau,$$

where:

$$\mathcal{D}_{is}^r(a) = -\frac{\partial}{\partial a}\left(g_i^{ss}(a)\left(a\delta_0(s) - \frac{e^{-\rho s}\varphi_{is}^r(a)}{u''(c_i^{ss}(a))}\right)\right), \quad \mathcal{D}_{is}^w(a) = -\frac{\partial}{\partial a}\left(g_i^{ss}(a)\left(y_i\delta_0(s) - \frac{e^{-\rho s}\varphi_{is}^w(a)}{u''(c_i^{ss}(a))}\right)\right).$$

Proof. See Appendix A.3. □

Proposition 2 starts from the solution (9) and expresses the deviation in savings rates $\widehat{S}_s$ as a function of consumption deviations $\widehat{c}_s$, themselves a function of the marginal utility Jacobians φ^r, φ^w. Theorem 2 in Online Appendix F.4 generalizes this result to a generic dynamic economy.

Proposition 2 indicates that savings rates influence the evolution of the distribution at time t through a specific structure. The integral over all times τ reveals that past and future price changes matter. Future price changes at $\tau > t$ influence consumption and savings decisions at all times $s < \tau$. Past price changes at $\tau < t$ have already affected savings rates at times $s < \tau$.

[6] T_t^* is also called the *semigroup* associated with $\mathcal{L}^*$ and is sometimes denoted $T_t^* = e^{t\mathcal{L}^*}$ by analogy with the finite-dimensional case.

Therefore, the integral over the time of savings rate changes s runs to the minimum of time t and time τ.

The effect of an expected price change at time τ results in a deviation in the distribution at time s given by $\mathcal{D}^r_{\tau-s}, \mathcal{D}^w_{\tau-s}$ that only depends on the relative time $\tau - s$, as with the marginal utility Jacobians. These deviations in the distribution $\mathcal{D}^r, \mathcal{D}^w$ in turn depend on the direct effect on disposable income that occurs only if the price change occurs immediately. This effect materializes in the components $a\delta_0(s)$ and $y_i\delta_0(s)$. The deviations $\mathcal{D}^r, \mathcal{D}^w$ also depend on the response of consumption to future price changes, which materializes in the components containing φ^r, φ^w following Corollary 1.

Finally, as in equation (9), the deviations in the distribution at time s propagate to time t according to the steady-state transition operator T^*_{t-s}.

2.3 Market clearing

Having related price deviations to the distribution, we turn to market clearing. To first order, the market clearing conditions (6) become:

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \sum_i \int \mathcal{E}^g_i(a)\widehat{g}_{it}(a)d\eta(a) \qquad \widehat{w}_t = w^{ss}\widehat{z}_t + \sum_i \int \mathcal{E}^w_i(a)\widehat{g}_{it}(a)d\eta(a),$$

for functions $\mathcal{E}^g_i(a) = \alpha(1-\alpha)\left(\frac{Y^{ss}}{K^{ss}}\right)^{1-\alpha}\left\{\frac{y_i}{Y^{ss}} - \frac{a}{K^{ss}}\right\}$ and $\mathcal{E}^w_i(a) = \alpha(1-\alpha)\left(\frac{K^{ss}}{Y^{ss}}\right)^{\alpha}\left\{\frac{a}{K^{ss}} - \frac{y_i}{Y^{ss}}\right\}$. We re-write these relations more compactly as:

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \mathcal{E}^{r,*}\widehat{g}_t \qquad \widehat{w}_t = w^{ss}\widehat{z}_t + \mathcal{E}^{w,*}\widehat{g}_t, \tag{11}$$

where $\mathcal{E}^* g = \sum_i \int \mathcal{E}_i(a)g_i(a)d\eta(a)$ denotes the inner product for any function $\mathcal{E}$. Combining the market clearing conditions (11) with Proposition 2, we obtain the following result.

Corollary 2. *(Sequence-space Jacobians)*

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \mathcal{E}^{r,*}_t\widehat{g}_0 + \int_0^\infty \left\{J^{r,r}_{t,\tau}\widehat{r}_\tau + J^{r,w}_{t,\tau}\widehat{w}_\tau\right\}d\tau, \quad \widehat{w}_t = w^{ss}\widehat{z}_t + \mathcal{E}^{w,*}_t\widehat{g}_0 \int_0^\infty \left\{J^{w,r}_{t,\tau}\widehat{r}_\tau + J^{w,w}_{t,\tau}\widehat{w}_\tau\right\}d\tau,$$

where the expectation functions $\mathcal{E}^r_t, \mathcal{E}^w_t$ satisfy:

$$\mathcal{E}^r_0 = \mathcal{E}^r \qquad \frac{\partial \mathcal{E}^r_t}{\partial t} = \mathcal{L}[\mathcal{E}^r_t] \qquad \mathcal{E}^w_0 = \mathcal{E}^w \qquad \frac{\partial \mathcal{E}^w_t}{\partial t} = \mathcal{L}[\mathcal{E}^w_t],$$

and for all pairs (p,q) such that $p \in \{r,w\}, q \in \{r,w\}$, the fake news operators $\mathcal{F}$ and the Jacobians J are:

$$\mathcal{F}^{p,q}_{t,\tau} = \mathcal{E}^{p,*}_t\mathcal{D}^q_\tau \qquad J^{p,q}_{t,\tau} = \int_0^{\min\{t,\tau\}} \mathcal{F}^{p,q}_{t-s,\tau-s}ds.$$

Proof. See Appendix A.4. □

The proof of Corollary 2 follows the logic in Auclert et al. (2021). By integrating the solution for the distribution in Proposition 2 against the market clearing functions in (11), we summarize the underlying market clearing prices through the Jacobians $J^{r,r}, J^{r,w}, J^{w,r}, J^{w,w}$. These Jacobians $J_{t,\tau}$ encode how a price change at time τ feeds back to the equilibrium price at time t through changes in household decisions and the subsequent law of motion of the distribution. Theorem 3 in Online Appendix F.4 generalizes this result to a generic dynamic economy.

The main content of Corollary 2 is to relate these Jacobians to the 'fake news' operators $\mathcal{F}^{r,r}, \mathcal{F}^{r,w}, \mathcal{F}^{w,r}, \mathcal{F}^{w,w}$. These scalar operators $\mathcal{F}_{t,\tau}$ encode how a price change τ periods in the future, announced at time 0 and retracted at time $0 + ds$ affects equilibrium prices at t periods in the future.

Consistently with this interpretation, the inner product of two objects parsimoniously summarizes these fake news operators. The first object is the initial deviation in the distribution in response to the price announcement, $\mathcal{D}_\tau$, from Proposition 2. The second object is the expectation function $\mathcal{E}_t$. The expectation function starts from the market clearing functions (11) and then evolves according to the steady-state policy functions summarized in the operator $\mathcal{L}$. It allows us to evaluate the expected value of the market clearing condition at all times given any starting distribution and given the subsequent evolution of the distribution under steady-state policy functions.

Corollary 2 defines a linear fixed point equation in the space of price paths. By stacking the equations and the operators, we define:

$$\overline{J} = \begin{pmatrix} J^{r,r} & J^{r,w} \\ J^{w,r} & J^{w,w} \end{pmatrix} \qquad \mathcal{J} = \mathrm{Id} - \overline{J} \qquad \widehat{p} = \begin{pmatrix} \widehat{r} \\ \widehat{w} \end{pmatrix} \qquad \widehat{Z}_t = \begin{pmatrix} r^{ss}\widehat{z}_t + \mathcal{E}_t^{r,*}\widehat{g}_0 \\ w^{ss}\widehat{z}_t + \mathcal{E}_t^{w,*}\widehat{g}_0 \end{pmatrix},$$

where Id denotes the identity operator. With this notation, we obtain:

$$\mathcal{J}\widehat{p} = \widehat{Z}. \tag{12}$$

When the combined Jacobians define an invertible operator $\mathcal{J}$, there exists a unique dynamic equilibrium and the sequence of equilibrium prices satisfies $\widehat{p} = \mathcal{J}^{-1}\widehat{Z}$. Armed with these results, we describe how to construct impulse response functions in practice.

2.4 Computation of impulse response functions

The structure of the Jacobians in Corollary 2 delivers a constructive recursive algorithm.

Corollary 3. *(Computation of sequence-space Jacobians)*
For all pairs (p, q) such that $p \in \{r, w\}, q \in \{r, w\}$, the Jacobians satisfy the ordinary differential equation:

$$\frac{dJ^{p,q}_{t+s,\tau+s}}{ds} = \mathcal{F}^{p,q}_{t+s,\tau+s} \qquad J^{p,q}_{t,0} = \mathcal{E}^{p,*}_t \overline{\mathcal{D}}^q \qquad J^{p,q}_{0,\tau} = 0,$$

where $\overline{\mathcal{D}}^r_i(a) = -\frac{\partial}{\partial a}\big(g^{ss}_i(a)a\big)$ and $\overline{\mathcal{D}}^w_i(a) = -\frac{\partial}{\partial a}\big(g^{ss}_i(a)y_i\big)$.

Proof. See Appendix A.5. □

Corollary 3 indicates that the 'first column' of the Jacobians $J_{t,0}$ is the change in the distribution that follows from the contemporaneous disposable income effect of price changes, propagated t time periods forward using the expectation function. At horizon $\tau = 0$, only this disposable income effect affects the distribution because it leads to an immediate jump in individual wealth. Of course, at any subsequent horizon $\tau > 0$, the response of consumption to this disposable income change affects equilibrium prices through the ordinary differential equation $\frac{dJ_{t+s,\tau+s}}{ds} = \mathcal{F}_{t+s,\tau+s}$. Similarly, the 'first row' of the Jacobians $J_{0,\tau}$ is zero because there is no immediate shift in the distribution at time 0 in response to a price change at time τ. From a practical perspective, Corollary 3 implies that we can obtain Jacobians by specifying initial conditions for $(t, 0)$ and $(0, \tau)$ and then solve an ordinary differential equation diagonally, by changing both times t and τ by the same amount s. Theorem 3 in Online Appendix F.4 generalizes this result to a generic dynamic economy.

We can now leverage Corollaries 1, 2 and 3 to obtain an algorithm that computes counterfactuals in our economy.

1. *Time step.* Choose a numerical time step dt

2. *Policy functions.* Compute φ^r_t, φ^w_t in a single time iteration using the partial differential equations in Proposition 1. Compute $\frac{\partial c_t}{\partial r_{t+s}}, \frac{\partial c_t}{\partial w_{t+s}}$ using Corollary 1.

   Under information aggregation, additionally set $\frac{\partial c_{it}(a)}{\partial r_t} = ac'^{ss}(a)$ ex-post.

3. *Distribution deviations.* Compute $\mathcal{D}^r_t, \mathcal{D}^w_t$ in Proposition 2.

4. *Expectation functions.* Compute $\mathcal{E}_t^r, \mathcal{E}_t^w$ in a single time iteration using the partial differential equations in Corollary 2.

5. *Fake news operators.* Compute $\mathcal{F}_{t,\tau}^{r,r}, \mathcal{F}_{t,\tau}^{r,w}, \mathcal{F}_{t,\tau}^{w,r}, \mathcal{F}_{t,\tau}^{w,w}$ through the inner products in Corollary 2.

6. *Jacobians.* Compute $J_{t,\tau}^{r,r}, J_{t,\tau}^{r,w}, J_{t,\tau}^{w,r}, \mathcal{F}_{t,\tau}^{w,w}$ using the ordinary differential equations in Corollary 3.

7. *Impulse response functions.* Solve the scalar linear system (12).

The discretization imposes constant choices during time steps $[t, t+dt)$. Thus, when imposing the information aggregation adjustment $\frac{\partial c_{it}(a)}{\partial r_t} = ac'^{ss}(a)$ ex-post in step 2, the algorithm imposes that households act as if constantly surprised throughout the first numerical period $[t, t+dt)$. This adjustment allows the algorithm to exactly reproduce the information structure inherent to discrete time. Of course, it is also possible to omit this adjustment and calculate Jacobians with rapid diffusion of information in continuous time. We compare both cases in Section 4.

Step 2 of our algorithm solves a Bellman equation. As in Achdou et al. (2021), we can choose whether to use an implicit or an explicit method to discretize the associated linear partial differential equation. The explicit method is faster but, if applied naively, can be unstable. Appendix C.2 describes how to amend the explicit method—by adjusting the discretization of the transition operator $\mathcal{L}$—to obtain an explicit method that is always stable and thus maximize speed relative to an implicit method.

Our algorithm delivers the only possible impulse response functions if the equilibrium exists and is unique. This is the case if and only if the Jacobian $\mathcal{J}$ is invertible when seen as an operator that maps price paths to market clearing conditions. Because our Jacobians share the same structure as the discrete-time ones, the determinacy criterion in Auclert et al. (2024a) applies after discretization.

This section developed sequence-space Jacobians in a heterogeneous agent economy without nominal rigidities. The next section shows how to construct them in a HANK economy. Online Appendix F generalizes the approach to a wide class of heterogeneous agent economies.

3 Sequence-space Jacobians in a HANK economy

We modify the economy from Section 1 to illustrate our sequence-space Jacobian approach in the presence of nominal rigidities. Unions set wages subject to nominal rigidity. A central bank sets monetary policy, and the government sets fiscal policies. There is no capital, only bonds in zero net supply. These modifications let us develop a continuous time equivalent of the framework in Auclert et al. (2024b), which we refer to as HANK.

3.1 Setup

Household decisions. Households now solve:

$$
\begin{aligned}
\rho V_{it}(a) &= \max_c u(c) - v(n_{it}(a)) + L_{it}(a,c)[V_t] + \frac{\partial V_{it}(a)}{\partial t} \qquad (13)\\
L_{it}(a,c)[V] &\equiv \left(r_t a + \tau_t (w_t y_i n_{it}(a))^{1-\theta} - c\right) \frac{\partial V}{\partial a}(x) + \Lambda_i[V]\\
V'_{it}(\underline{a}) &\geq u'\left(r_t \underline{a} + \tau_t (w_t y_i n_{it}(a))^{1-\theta}\right).
\end{aligned}
$$

Flow utility $u(c) - v(n_{it})$ now includes disutility $v(n_{it}(a))$ from labor $n_{it}(a)$. Households supply the amount of labor $n_{it}(a)$ that is demanded from them. r_t is the real interest rate. Real post-tax labor income $\tau_t(w_t y_i n)^{1-\theta}$ depends on the overall level of taxes τ_t set by the government and on the progressivity of the tax schedule encoded in θ.[7] To construct sequence-space Jacobians, it is useful to relate individual post-tax labor income to aggregate post-tax income $Y_t - T_t$ through the identity: $\tau_t(w_t y_i n_{it}(a))^{1-\theta} = (Y_t - T_t)\frac{y_{it}^{1-\theta}}{\int y_{jt}^{1-\theta} dj}$. This identity guarantees that individual decisions only depend on the path of aggregate post-tax income $X_t \equiv Y_t - T_t$ rather than on the sequence of wages w_t and tax rates τ_t separately.

Distribution. The distribution $g_{it}(a)$ evolves as in (5).

Final goods firms. There is a perfectly competitive representative final goods firm with production function $Y_t = e^{z_t} N_t$, where N_t is a an aggregator of intermediate labor varieties that sells at price W_t. Goods prices P_t are flexible, so that $W_t/P_t = e^{z_t}$.

Unions. There is a unit continuum of unions indexed by k that hire a representative sample of workers. Each union produces a variety k of labor services N_{kt}. A competitive la-

[7] We use the Bénabou-Hornstein-Storesletten-Violante tax schedule $\overline{\tau}_t(I) = 1 - \tau_t I^{1-\theta}$.

bor aggregator packages the union-produced labor with a constant elasticity of substitution: $N_t = \left(\int N_{kt}^{\frac{\varepsilon-1}{\varepsilon}} dk\right)^{\frac{\varepsilon}{\varepsilon-1}}$. Unions pay all workers the same wage per efficiency unit and require them to work the same number of hours $n_{it}(a) \equiv n_t$.

In equilibrium, all unions choose the same wage and hours: $W_{kt} = W_t$ and $N_{kt} = N_t$, implying that real wages are equal to productivity $w_t = W_t/P_t = e^{z_t}$. Thus, nominal rigidity affects post-tax income only through the equilibrium number of hours chosen by the union—itself a function of nominal wages. In Appendix B, we show that the unions' decisions lead to a wage Phillips curve:

$$\rho\pi_t = \kappa_1 u'(C_t)X_t + \kappa_2 v'(N_t)N_t + \frac{\partial \pi_t}{\partial t}, \tag{14}$$

where $\pi_t = \frac{\partial W_t}{\partial t}/W_t$ denotes nominal wage inflation, $\kappa_1 = (1-\varepsilon)(1-\theta)/\psi$ and $\kappa_2 = \varepsilon/\psi$.

Monetary policy. The central bank sets the nominal interest rate i_t according to the Taylor rule $i_t = r^{ss} + \phi\pi_t + \varepsilon_t$, where ε_t is a monetary policy shock. The Fisher equation is $r_t = i_t - \pi_t$, so that the real interest rate satisfies $r_t = r^{ss} + (\phi - 1)\pi_t + \varepsilon_t$.

Fiscal policy. The government exogenously sets spending G_t. We consider a monetary dominant regime in which taxes T_t always adjust to match the present discounted value of spending along any transition. The government finances deficits by issuing bonds to satisfy its budget constraint: $\partial B_t/\partial t = r_t B_t + G_t - T_t$.[8] The initial level of real government debt B_0 is equal to household assets. We linearize the budget constraint as $\partial \widehat{B}_t/\partial t = r^{ss}\widehat{B}_t + B^{ss}\widehat{r}_t + \widehat{G}_t - \widehat{T}_t$. We solve the linearized equation forward: $\widehat{B}_t = e^{r^{ss}t}\widehat{B}_0 + \int_0^t e^{r^{ss}(t-s)}(B^{ss}\widehat{r}_s + \widehat{G}_s - \widehat{T}_s)ds$.

3.2 Sequence-space Jacobians

Household policy functions. We construct policy functions similarly to the real economy from Section 1. Inspecting the household decision problem (13) reveals that only two aggregates matter for decisions: the real interest rate r_t and post-tax aggregate income X_t—provided we start from the steady-state distribution of productivity— similarly to the wage w_t in Section 1. Thus, we write $\widehat{c}_t = \int_0^\infty e^{-\rho s}(\varphi_s^r \widehat{r}_{t+s} + \varphi_s^X \widehat{X}_{t+s})ds$, where the sequence-space derivatives φ^r, φ^X satisfy nearly identical partial differential equations to those in Corollary 1.

[8] These choices are not required by our approach but, as in Auclert et al. (2024b), keep the exposition simple by abstracting from considerations related to the Fiscal Theory of the Price Level. We normalize the initial nominal wage level so that initial real government debt B_0 is equal to aggregate household assets A^{ss}.

Union, central bank and government policy functions. Given consumption policy functions, we obtain the response of nominal wages, interest rates and government bonds as in a standard representative agent economy. For instance, the Phillips curve (14) implies that the sequence-space partial derivative of inflation with respect to post-tax income is $\frac{\partial \pi_t}{\partial X_{t+s}} = \kappa_1 e^{-\rho s} u'(C^{ss})$, holding the path of consumption and hours constant. To solve for counterfactuals in general equilibrium, we also derive the sequence-space partial derivatives with respect to the path of consumption and hours. Appendix B provides more details.

Similarly, the sequence-space partial derivative of the real interest rate to inflation is $\frac{\partial r_t}{\partial \pi_{t+s}} = (\phi - 1)\delta_0(s)$. The sequence-space partial derivative of government debt to government spending is $\frac{\partial B_t}{\partial G_s} = e^{r^{ss}(t-s)} \mathbb{1}\{s \leq t\}$. We combine these sequence-space derivatives to solve for general equilibrium counterfactuals in Appendix B.

Distribution. The evolution of the distribution satisfies a law of motion virtually identical to (5). Therefore, we obtain a similar result to Proposition 2 after replacing $\widehat{w}_\tau$ with $\widehat{X}_\tau$.

Market clearing. We use the aggregate resource constraint $Y_t = C_t + G_t$ to close our economy:

$$Z_t N_t - G_t = \sum_i \int c_{it}(a) g_{it}(a) d\eta(a).$$

To first order around steady-state:

$$N^{ss}\widehat{z}_t + \widehat{N}_t - \widehat{G}_t = c^{ss,*}\widehat{g}_t + \widehat{c}_t^* g^{ss}. \tag{15}$$

We recognize a market clearing equation of the form of (11). The left-hand-side of (15) only features aggregate objects. The first component on the right-hand-side $c^{ss,*}\widehat{g}_t$ is the inner product between the steady-state consumption function and the deviation in the distribution. Its structure is similar to the right-hand-side of the market clearing condition (11). Hence, we construct this component as in Corollary 2.

The second component $\widehat{c}_t^* g^{ss}$ on the right-hand-side of the market clearing condition (15) is the inner product between the deviation in the consumption function $\widehat{c}_t$ and the steady-state distribution g^{ss}. We directly construct this component by leveraging our characterization of policy functions, so that $\widehat{c}_t^* g^{ss} = \int_0^\infty e^{-\rho s}\big[(\varphi_s^{r,*} g^{ss})\widehat{r}_{t+s} + (\varphi_s^{X,*} g^{ss})\widehat{X}_{t+s}\big] ds$.

4 Implementation

We now illustrate our continuous time sequence-space Jacobian approach in the context of the HA economy of Section 1 and the HANK economy of Section 3. We compare our approach with the discrete time sequence-space Jacobian approach in Auclert et al. (2021). Appendix C.1 describes our choice of parameters and the value of moments in our continuous time and discrete time economies.

For comparability of results and runtimes to discrete time, we discretize the continuous time economy with the same time step. We compute continuous-time Jacobians following Propositions 1 and 3. We compute discrete time Jacobians using the method in Auclert et al. (2021).

4.1 The HA economy

We display the general equilibrium Jacobians of aggregate capital with respect to interest rates and wages in Figure 1. These Jacobians correspond to the component of the expectation function in Corollary 2 that summarizes the evolution of the capital stock. We plot three 'columns' of the capital Jacobian with respect to an interest rate and wage shock at specific times. That is, we plot $\left\{\frac{\partial K_t}{\partial r_s}, \frac{\partial K_t}{\partial w_s}\right\}_{t\geq 0}$ for fixed $s = 0, 100, 200$. Of course, once we have discretized time for computation, these Jacobians become matrices and we display the corresponding actual column of that matrix.

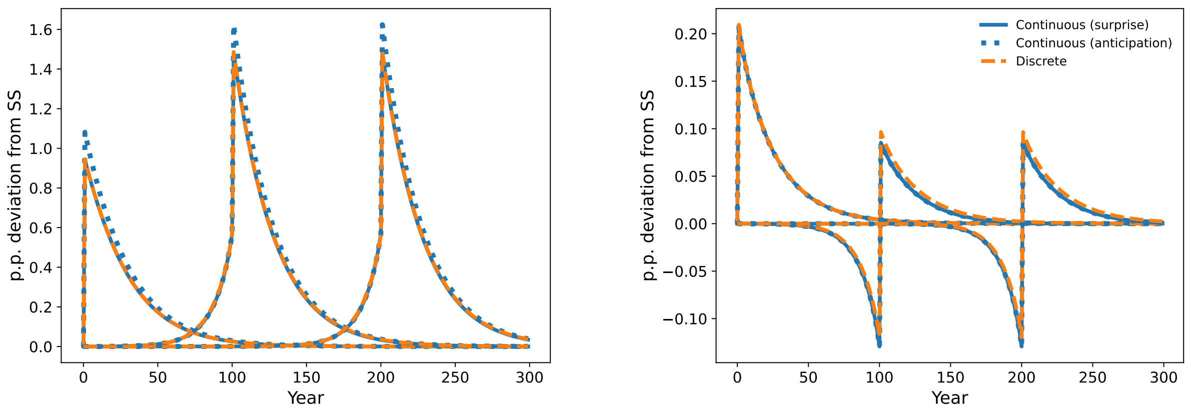
When the interest rate and wages increase at time 0, households are wealthier and immediately save a fraction of this increase. They then gradually consume this saved capital over time. When we discretize the continuous-time economy to feature information aggregation as in discrete time, the capital Jacobians virtually overlap. When we turn off information aggregation, households in the continuous time economy save more: they have nearly a full period in which they anticipate future elevated interest rates that they can take advantage of. In that case, the capital Jacobian in continuous time displays larger responses, although the difference is small.

When the interest rate increases in the future, households preemptively save to benefit from the interest rate hike when it takes place. Thus, the capital stock peaks at a higher value than when the shock is entirely anticipated at time 0. After the interest rate rise, households draw down these additional savings as in the time 0 case. The role of information aggregation is similar for future interest rate changes than it is for contemporaneous interest rate changes.

The response to an anticipated future wage increase is different. There is no reason to

Figure 1: Capital Jacobians in the HA economy

(a) Capital Jacobian w.r.t. interest rates (b) Capital Jacobian w.r.t. wages



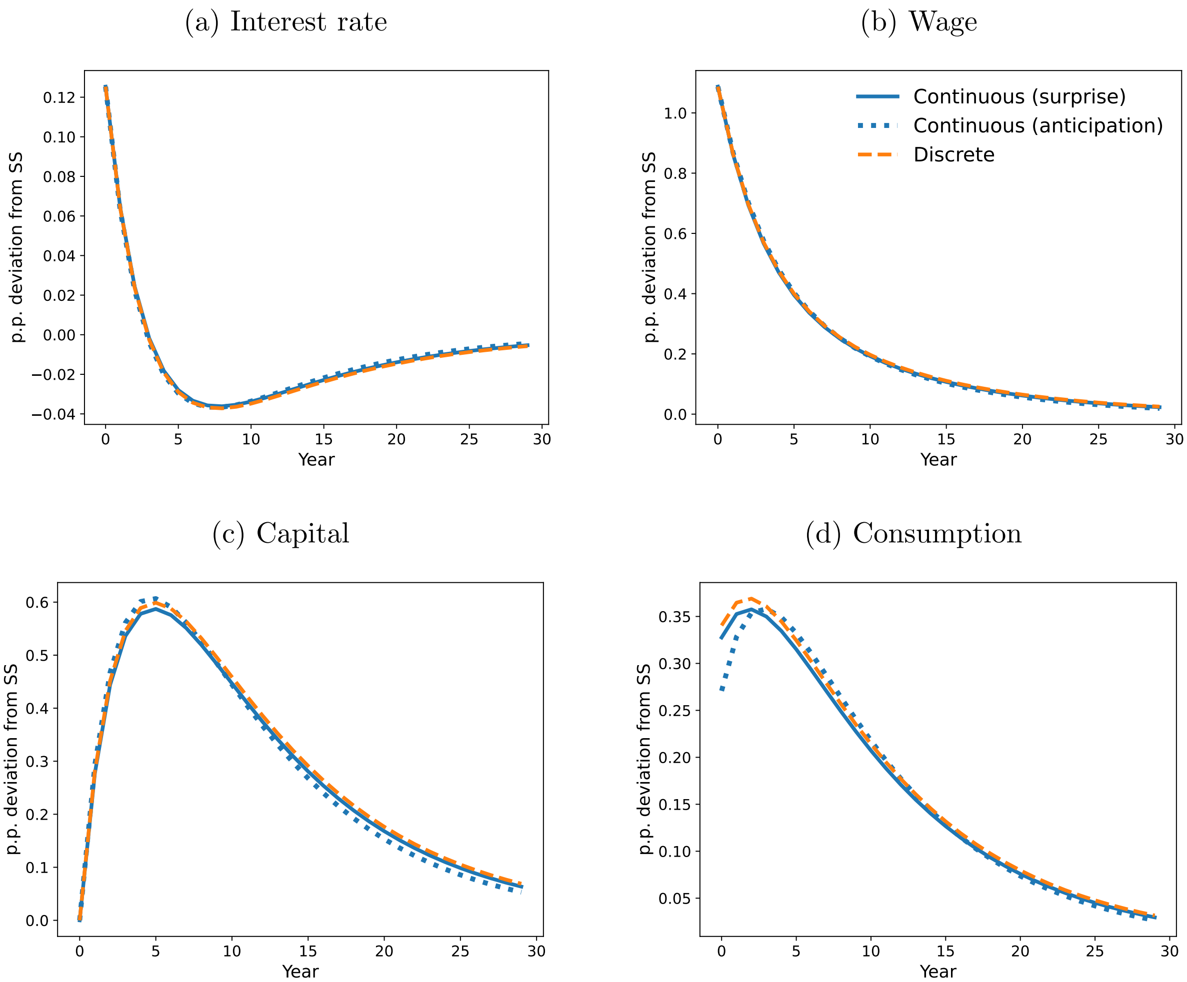
Note: Columns of the capital Jacobian with respect to interest rates $\{\partial K_t/\partial r_s\}_{t\geq 0}$ (left) and wages $\{\partial K_t/\partial w_s\}_{t\geq 0}$ (right), for values of $s = 0, 100, 200$, in the HA economy. Solid blue: continuous time with a discretization that imposes information aggregation as in discrete time. Dotted blue: continuous time with a discretization that does not impose information aggregation. Dashed orange: discrete time. 50 income states and 5,000 asset gridpoints.

preemptively save because wage increases do not raise the value of assets when they take place. On the contrary, households preemptively dissave and consume more, anticipating that this decline in their assets will be compensated by higher wages in the future. After the wage increase has taken place, this behavior reverses and households save and gradually consume part of the excess income. There is no role for information aggregation in response to a wage change because there is no additional incentive to save to take advantage of a higher wage.

With information aggregation, our continuous time capital Jacobians are almost indistinguishable from their discrete time counterparts. Of course, we do not expect them to exactly coincide: even when discretized with the same time period, a continuous time economy is slightly different from its discrete time counterpart with the same parameters. That the continuous time and discrete time Jacobians are extremely similar indicates that the continuous time calculation captures the same economic forces as in discrete time. Even without information aggregation, the difference is small.

Figure 2 depicts the general equilibrium impulse responses of interest rates, wages, the capital stock and aggregate consumption to a 1% aggregate productivity shock on impact, that subsequently follows an AR(1) path with annual autocorrelation 0.8. As for the capital Jacobians, the continuous and discrete time impulse response functions closely track each other, with a more noticeable discrepancy without information aggregation in continuous time for the initial response of consumption, as expected.

Figure 2: Impulse response functions to a productivity shock in the HA economy



Note: Impulse response functions to an aggregate productivity shock $\widehat{z}_t$, for values of $s = 0, 100, 200$, in the HA economy. Productivity rises by 1% on impact and follows an AR(1) path with annual autocorrelation 0.8. Panel (a): interest rate $\widehat{r}_t$. Panel (b): wage $\widehat{w}_t$. Panel (c): capital $\widehat{K}_t$. Panel (d): consumption $\widehat{C}_t$. Solid blue: continuous time with a discretization that imposes information aggregation as in discrete time. Dotted blue: continuous time with a discretization that does not impose information aggregation. Dashed orange: discrete time. 50 income states and 5,000 asset gridpoints.

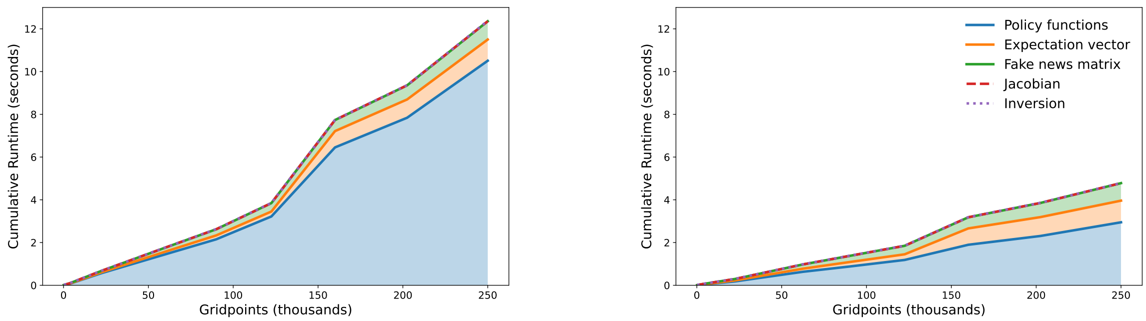
A positive aggregate productivity shock initially increases the marginal product of capital and labor, thus increasing both interest rates and capital. As households save in response to higher interest rates, capital accumulates and the marginal product of capital drops, putting downward pressure on the interest rate. Labor supply is set to unity so that wages largely track the path of the productivity shock.

As higher productivity raises output, consumption jumps on impact. Capital accumulates, raising output and consumption. After a few years however, productivity has largely mean-reverted and hence capital and consumption gradually return to their steady-state values.

Jacobians and impulse response functions in continuous and discrete time are virtually iden-

Figure 3: Computation times in the HA economy

(a) Discrete time (b) Continuous time



Note: Runtimes for calculation of Jacobians and impulse response functions in discrete time (panel (a)) and in continuous time (panel (b)) in the HA economy. 'Policy functions': computation of response of marginal utility and consumption to price changes. 'Expectation vector': computation of $\mathcal{E}_t$. 'Fake news matrix': computation of the $\mathcal{F}_{t,s}$. 'Jacobian': computation of $\mathcal{J}$. 'Inversion' inversion of $\mathcal{J}$ and computation of impulse responses. Steady-state computation time in Figure 6, Appendix C.3. Exact computation times in Tables 5 and 6, Appendix C.3.

tical, but does continuous time confer additional benefits? Figure 3 reveals that the calculation of Jacobians and of impulse response functions is three times faster with our continuous time algorithm, relative to the discrete time algorithm. To understand this improvement, we break down the calculation into five steps.

The first step is to calculate policy functions: the response of marginal utility and consumption to price changes, encoded in φ^r, φ^w. In continuous time, we do so using Proposition 1 and Corollary 1. In discrete time, we apply the algorithm in Auclert et al. (2021) that solves the nonlinear Euler equation backwards for a small price perturbation, and uses numerical or automatic differentiation. This is the only step that markedly differs between continuous and discrete time.

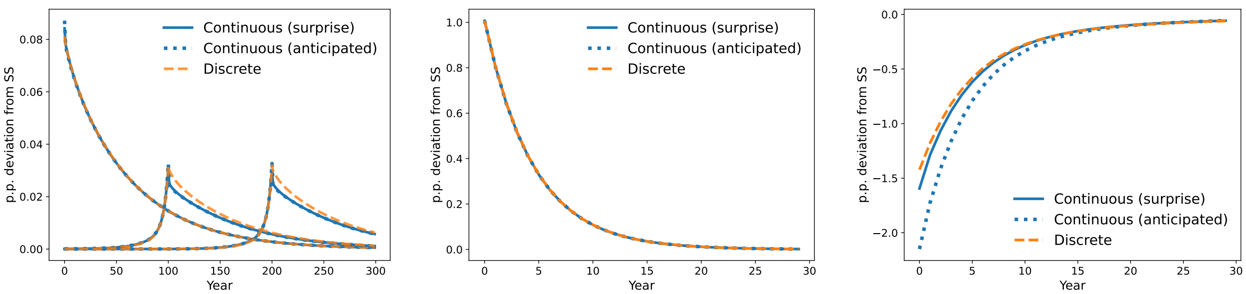
The second step computes the expectation vector $\mathcal{E}_t$. The third step constructs the fake news matrices $\mathcal{F}_{t,s}$. The fourth step constructs the Jacobians $\mathcal{J}$. The fifth step inverts the Jacobian to obtain impulse response functions. These last four steps are similar in continuous and discrete time.

Panel 3(a) indicates that the policy function step is by far the most computationally expensive in discrete time. Panel 3(b) reveals that our continuous algorithm cuts this computation time threefold by leveraging the closed-form partial differential equations from Proposition 1. Virtually all the speed gains occur in this step. All the subsequent steps mirror Auclert et al. (2021) and there is thus no reason to expect an improvement there.

Why does our continuous time approach speed up the calculation of policy functions? The

Figure 4: Consumption Jacobian and impulse response functions in the HANK economy

(a) Cons. Jacobian to income (b) Output to gov. spending (c) Output to monetary shock



Note: Panel 4(a): partial-equilibrium aggregate consumption Jacobian to aggregate income $\{\partial C_t/\partial Y_{t+s}\}_{t\geq 0}$ for $s = 0, 100, 200$, in the HANK economy. Panel 4(b): impulse response of output to a government spending shock of 1% of output that fades away at rate $\partial \widehat{G}_t/\partial t = -\chi \widehat{G}_t$ where $e^{-\chi} = 0.8$. Government debt follows: $\partial \widehat{B}_t/\partial t = -\beta \widehat{B}_t + \widehat{G}_t$ with $e^{-\beta} = 0.5$. Taxes adjust to satisfy the government budget constraint: $\widehat{T}_t = \widehat{G}_t + r^{ss}\widehat{B}_t + \widehat{r}_t B^{ss} - \partial \widehat{B}_t/\partial t$. Panel 4(c): impulse response of output to a 100 basis point monetary shock, that follows an AR(1) with autocorrelation 0.8. Solid blue: continuous time with a discretization that imposes information aggregation as in discrete time. Dotted blue: continuous time with a discretization that does not impose information aggregation. Dashed orange: discrete time. 50 income states and 5,000 asset gridpoints.

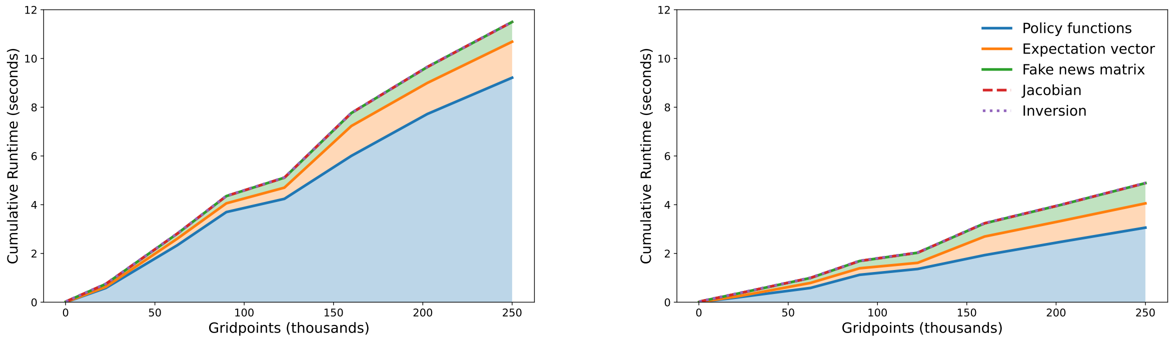
discrete time algorithm relies on solving the Euler equation nonlinearly with the endogenous gridpoint method, therefore requiring to calculate expectations and interpolate at each time step. Our continuous time algorithm requires only one sparse matrix multiplication per time step, bypassing the need to evaluate nonlinear functions and interpolate.

4.2 The HANK economy

We now turn to our HANK economy. We start by depicting the relevant Jacobians that determine counterfactuals. As there is no capital in the HANK framework, we plot instead the columns of the aggregate consumption Jacobian with respect to aggregate income changes in partial equilibrium, $\frac{\partial C_t}{\partial Y_s}$. Figure 4(a) shows that the continuous and discrete time Jacobians are close. In both cases, consumption progressively rises in anticipation of an income increase, and gradually declines after it.

Figure 4(b) shows the impulse response of output to a government spending shock of 1% of output that fades away at rate: $\partial \widehat{G}_t/\partial t = -\chi \widehat{G}_t$ where $e^{-\chi} = 0.8$. Government debt follows: $\partial \widehat{B}_t/\partial t = -\beta \widehat{B}_t + \widehat{G}_t$ with $e^{-\beta} = 0.5$. Taxes adjust to satisfy the government budget constraint: $\widehat{T}_t = \widehat{G}_t + r^{ss}\widehat{B}_t + \widehat{r}_t B^{ss} - \partial \widehat{B}_t/\partial t$. In our calibration, there are not many constrained households, and thus the multiplier is just slightly above 1. Figure 4(c) displays the response of output to a 100 basis point monetary shock. As expected, output contracts. In both cases, the continuous and discrete time impulse response functions are virtually identical with information aggrega-

Figure 5: Computation times in the HANK economy



(a) Discrete time (b) Continuous time

Note: Runtimes for calculation of Jacobians and impulse response functions in discrete time (panel (a)) and in continuous time (panel (b)) in the HANK economy. 'Policy functions': computation of response of marginal utility and consumption to price changes. 'Expectation vector': computation of $\mathcal{E}_t$. 'Fake news matrix': computation of the $\mathcal{F}_{t,s}$. 'Jacobian': computation of $\mathcal{J}$. 'Inversion' inversion of $\mathcal{J}$ and computation of impulse responses. Steady-state computation time in Figure 6, Appendix C.3. Exact computation times in Tables 5 and 6, Appendix C.3.

tion. Without information aggregation, output responds slightly more strongly to a monetary shock because interest rate changes have an additional anticipation effect on consumption.

The computational gains associated with continuous time also materialize in the HANK economy. Figure 5 compares runtimes in discrete and continuous time. Comparing panels (a) and (b) reveals that the continuous time implementation of policy functions again leads to a threefold speed gain.

Conclusion

This paper proposes an analytic derivation of sequence-space Jacobians in heterogeneous agent models with aggregate shocks in continuous time. This analytic derivation leads to efficient and easy-to-implement algorithms that do not rely on numerical differentiation. These algorithms confer substantial speed gains relative to their discrete time counterparts.

References

Achdou, Yves, Han, Jiequn, Lasry, Jean-Michel, Lions, Pierre-Louis, and Moll, Benjamin (Apr. 2021). "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach". *The Review of Economic Studies* 89.1, pp. 45–86.

Aiyagari, S. Rao (1994). “Uninsured Idiosyncratic Risk and Aggregate Saving”. *The Quarterly Journal of Economics* 109.3, pp. 659–684.

Alvarez, Fernando, Le Bihan, Hervé, and Lippi, Francesco (2016). “The Real Effects of Monetary Shocks in Sticky Price Models: A Sufficient Statistic Approach”. *American Economic Review* 106.10, 2817–51.

Auclert, Adrien, Bardóczy, Bence, Rognlie, Matthew, and Straub, Ludwig (2021). “Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models”. *Econometrica* 89.5, pp. 2375–2408.

Auclert, Adrien, Rognlie, Matthew, and Straub, Ludwig (2024a). “Determinacy and Existence in the Sequence Space”. *Working Paper.*

— (2024b). “The Intertemporal Keynesian Cross”. *Journal of Political Economy.*

Bhandari, Anmol, Bourany, Thomas, Evans, David, and Golosov, Mikhail (2023). “A Perturbational Approach for Approximating Heterogeneous-Agent Models”. *National Bureau of Economic Research Working Paper Series 31744.*

Bilal, Adrien (2023). “Solving Heterogeneous Agent Models with the Master Equation”. *National Bureau of Economic Research Working Paper Series 31103.*

Bogachev, Vladimir I., Krylov, Nicolai V., Rockner, Michael, and Shaposhnikov, Stanislav V. (2015). *Fokker–Planck–Kolmogorov Equations.* American Mathematical Society.

Boppart, Timo, Krusell, Per, and Mitman, Kurt (2018). “Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative”. *Journal of Economic Dynamics and Control* 89. Fed St. Louis-JEDC-SCG-SNB-UniBern Conference, titled: “Fiscal and Monetary Policies”., pp. 68–92.

Cardaliaguet, Pierre, Delarue, Francois, Lasry, Jean-Michel, and Lions, Pierre-Louis (2019). *The Master Equation and the Convergence Problem in Mean Field Games.* ISBN: 9780691190709.

Carmona, René and Delarue, Francois (2018). *Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control and Games.* Springer.

Evans, Lawrence C. (2010). *Partial Differential Equations.* American Mathematical Society.

Fleming, Wendell H. and Soner, Halil Mete (2006). *Controlled Markov Processes and Viscosity Solutions.* Springer.

Glawion, René (2023). “Sequence-Space Jacobians in Continuous Time”. *Social Science Research Network Working Paper Series 4504829*.

Kaplan, Greg, Moll, Benjamin, and Violante, Giovanni L. (2018). “Monetary Policy According to HANK”. *American Economic Review* 108.3, 697–743.

Krusell, Per and Smith Jr., Anthony A. (1998). “Income and Wealth Heterogeneity in the Macroeconomy”. *Journal of Political Economy* 106.5, pp. 867–896.

Appendix

A Proofs for Section 1

A.1 Proof of Proposition 1

We start by evaluating the Euler equation (2) for a small marginal value deviation $\widehat{v}_{it}(a)$. We obtain:

$$(\rho - r^{ss})\widehat{v}_{it}(a) = \widehat{r}_t v_i^{ss}(a) + \mathcal{L}_i(a)[\widehat{v}_t] + \Big(\widehat{r}_t a + \widehat{w}_t y_i - \widehat{c}_{it}(a)\Big) v_i'^{ss}(a) + \frac{\partial \widehat{v}_{it}(a)}{\partial t}.$$

Using the first-order optimality for consumption, we express:

$$u''(c_i^{ss}(a))\widehat{c}_{it}(a) = \widehat{v}_{it}(a) \qquad v_i'^{ss}(a) = u''(c_i^{ss}(a))c_i'^{ss}(a).$$

Substituting these conditions back into the linearized Euler equation:

$$\Big(\rho - r^{ss} + c_i'^{ss}(a)\Big)\widehat{v}_{it}(a) = \widehat{r}_t\Big(v_i^{ss}(a) + a v_i'^{ss}(a)\Big) + \widehat{w}_t y_i v_i'^{ss}(a) + \mathcal{L}_i(a)[\widehat{v}_t] + \frac{\partial \widehat{v}_{it}(a)}{\partial t}.$$

We can re-write the linearized Euler equation more compactly by recognizing derivatives with respect to assets:

$$\Big(\rho - r^{ss} + c_i'^{ss}(a)\Big)\widehat{v}_{it}(a) = \widehat{r}_t \frac{\partial\big(a v_i^{ss}(a)\big)}{\partial a} + \widehat{w}_t \frac{\partial\big(y_i v_i^{ss}(a)\big)}{\partial a} + \mathcal{L}_i(a)[\widehat{v}_t] + \frac{\partial \widehat{v}_{it}(a)}{\partial t}.$$

We are now almost ready to substitute (7) into the linearized Euler equation. In order to simplify the calculation, we use the equivalent formula after changing variables in the time integration:

$$\widehat{v}_{it}(a) = \int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s + \varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds.$$

Substituting into the linearized Euler equation wherever the constraint does not bind, we obtain:

$$\begin{aligned}
&\Big(\rho - r^{ss} + c_i'^{ss}(a)\Big)\int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s + \varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds \qquad (16)\\
&= \widehat{r}_t \frac{\partial\big(a v_i^{ss}(a)\big)}{\partial a} + \widehat{w}_t \frac{\partial\big(y_i v_i^{ss}(a)\big)}{\partial a}\\
&+ \int_t^\infty e^{-\rho(s-t)}\Big\{\mathcal{L}_i(a)[\varphi^r_{s-t}]\widehat{r}_s + \mathcal{L}_i(a)[\varphi^w_{s-t}]\widehat{w}_s\Big\}ds\\
&+ \frac{\partial}{\partial t}\int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s + \varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds
\end{aligned}$$

where in the third line, we pull the operator $\mathcal{L}_i(a)$ inside the integral. The same component on the third line further writes as:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s+\varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds\\
&=\int_t^\infty e^{-\rho(s-t)}\left\{\left(\rho\varphi^r_{i,s-t}(a)-\frac{\partial\varphi^r_{i,s-t}(a)}{\partial s}\right)\widehat{r}_s+\left(\rho\varphi^w_{i,s-t}(a)-\frac{\partial\varphi^w_{i,s-t}(a)}{\partial s}\right)\widehat{w}_s\right\}ds\\
&-\varphi^r_{i,0}(a)\widehat{r}_t-\varphi^w_{i,0}(a)\widehat{w}_t
\end{aligned}
$$

Substituting back into (16) and re-arranging, we obtain:

$$
\begin{aligned}
0=&\left\{\frac{\partial\big(av^{ss}_i(a)\big)}{\partial a}-\varphi^r_{i,0}(a)\right\}\widehat{r}_t+\left\{\frac{\partial\big(y_iv^{ss}_i(a)\big)}{\partial a}-\varphi^w_{i,0}(a)\right\}\widehat{w}_t\\
&+\int_t^\infty e^{-\rho(s-t)}\left\{\big(r^{ss}-c'^{ss}_i(a)\big)\varphi^r_{i,s-t}(a)+\mathcal{L}_i(a)[\varphi^r_{s-t}]-\frac{\partial\varphi^r_{i,s-t}(a)}{\partial s}\right\}\widehat{r}_sds\\
&+\int_t^\infty e^{-\rho(s-t)}\left\{\big(r^{ss}-c'^{ss}_i(a)\big)\varphi^w_{i,s-t}(a)+\mathcal{L}_i(a)[\varphi^w_{s-t}]-\frac{\partial\varphi^w_{i,s-t}(a)}{\partial s}\right\}\widehat{w}_sds
\end{aligned}
\tag{17}
$$

Equation (17) must hold for all price paths $\{\widehat{r}_s,\widehat{w}_s\}_{s\geq t}$. Therefore, we may 'identify coefficients' and we obtain:

$$
\begin{aligned}
\varphi^r_{i,0}(a)&=\frac{\partial\big(av^{ss}_i(a)\big)}{\partial a} & \frac{\partial\varphi^r_{i,s}(a)}{\partial s}&=\big(r^{ss}-c'^{ss}_i(a)\big)\varphi^r_{i,s}(a)+\mathcal{L}_i(a)[\varphi^r_s]\\
\varphi^w_{i,0}(a)&=\frac{\partial\big(y_iv^{ss}_i(a)\big)}{\partial a} & \frac{\partial\varphi^w_{i,s}(a)}{\partial s}&=\big(r^{ss}-c'^{ss}_i(a)\big)\varphi^w_{i,s}(a)+\mathcal{L}_i(a)[\varphi^w_s]
\end{aligned}
$$

at all points where the credit constraint does not bind. These points include pairs (a,y_i) for which $a>\underline{a}$.

We now turn to the borrowing constraint. The borrowing constraint is encoded in the state constraint inequality: $v_{it}(a)\geq u'(r_t\underline{a}+w_ty_i)$. Taking a first-order perturbation and substituting in (7), we obtain to first order:

$$
v^{ss}_i(a)+\int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s+\varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds\geq u'(r^{ss}\underline{a}+w^{ss}y_i)+u''(r^{ss}\underline{a}+w^{ss}y_i)\Big(\underline{a}\widehat{r}_t+y_i\widehat{w}_t\Big),
\tag{18}
$$

with equality if and only if the borrowing constraint binds.

For all (a,y_i) where the constraint does not bind in steady-state, the inequality $v^{ss}_i(a)>u'(r^{ss}\underline{a}+w^{ss}y_i)$ is strict. The borrowing constraint can only bind when $a=\underline{a}$ even outside of steady-state, in which case $s_{it}(a)<0$ as a approaches $\underline{a}$.

In steady-state, the borrowing constraint binds at $(\underline{a},y_i)$ if and only if $s^{ss}_i(a)<0$ as a approaches $\underline{a}$. Hence, given our discrete income process, the set of income states y_i for which

$s_{it}(a) < 0$ as a approaches $\underline{a}$ does not generically change for small perturbations of the consumption function. We consider continuous income processes in Online Appendix F.

Our claim is 'generic' in the sense that it holds for all parameters up to a set of Lebesgue measure zero. It could be that for a knife-edge combination of parameters, in steady-state we have $s_i^{ss}(a) = 0$ for a in a neighborhood of $\underline{a}$ and a particular income state y_i. In that case, small perturbations in prices may change whether households are constrained at $(\underline{a}, y_i)$ outside of steady-state. We focus on parameter combinations in which this situation does not arise, which are almost all parameter combinations. Accordingly, in simulations this case never arises.

These arguments imply that, in our first-order perturbation, the borrowing constraint binds outside of steady-state at $(\underline{a}, y_i)$ if and only if it binds in steady-state at $(\underline{a}, y_i)$. In that case, (18) holds with equality at $(\underline{a}, y_i)$, and we also have $v_i^{ss}(\underline{a}) = u'(r^{ss}\underline{a} + w^{ss}y_i)$. Combining both, we obtain:

$$\int_t^\infty e^{-\rho(s-t)}\Big\{\varphi^r_{i,s-t}(a)\widehat{r}_s + \varphi^w_{i,s-t}(a)\widehat{w}_s\Big\}ds = u''(r^{ss}\underline{a} + w^{ss}y_i)\Big(\underline{a}\widehat{r}_t + y_i\widehat{w}_t\Big)$$

for all price paths. Therefore, we identify coefficients once more and obtain that:

$$\varphi^r_{it}(\underline{a}) = u''(c_i^{ss}(\underline{a}))\underline{a}\ \delta_0(t) \qquad\qquad \varphi^w_{it}(\underline{a}) = u''(c_i^{ss}(\underline{a}))y_i\ \delta_0(t),$$

where $\delta_0(t)$ denotes a Dirac mass function with respect to time.

A.2 Proof of Corollary 1

We start by stating the full first-order optimality condition in continuous time:

$$u'(c_{it}(a)) = \lim_{\varepsilon\downarrow 0} e^{-r_t\varepsilon}v_{it}(a) = v_{it}(a).$$

To see how to obtain the expression with the limit $\varepsilon \downarrow 0$, consider the associated discrete time problem with a small time step ε, omitting binding constraints in discrete time (see Online Appendix E for details):

$$V_{it}(a) = \max_c \varepsilon u(c) + e^{-\rho\varepsilon}\mathbb{E}_t V_{j,t+\varepsilon}\big(e^{r_t\varepsilon}a + (y_i - c)\varepsilon\big),$$

where the j index denotes next period's income state, and we used the budget constraint $a_{t+\varepsilon} = e^{r_t\varepsilon}a + (y_t - c_t)\varepsilon$. The first-order optimality condition and the envelope condition lead to:

$$u'(c_{it}) = e^{-\rho\varepsilon}\mathbb{E}_t V'_{j,t+\varepsilon}(a_{t+\varepsilon}) \qquad\qquad V'_{it}(a) = e^{(r_t-\rho)\varepsilon}\mathbb{E}_t V'_{j,t+\varepsilon}(a_{t+\varepsilon}),$$

implying that $u'(c_{it}(a)) = e^{-r_t\varepsilon}V'_{it}(a)$.

In most cases, the limit $\varepsilon \downarrow 0$ is innocuous. However, when we consider shocks to the time-t interest rate, this limit is not innocuous because it does not commute with the limit associated with the perturbation with respect to the interest rate sequence.

For all $s > 0$, the expression in Corollary 1 immediately follows from the standard first-order optimality condition $u'(c_{it}(a)) = v_{it}(a)$, because the limit $\varepsilon \downarrow 0$ only involves r_t, not r_{t+s}, $s > 0$.

For $s = 0$, an additional term appears due to the two limits. We must take limits jointly in the size of the time step and in the change in the interest rate. This occurs because the partial derivative with respect to r_t involves a limit over an interest rate change specified over the same small time step ε. As the time step ε vanishes, we must also rescale the derivative by ε. Formally:

$$\begin{aligned}\frac{\partial[u'(c_{it}(a))]}{\partial r_t} &\equiv \lim_{\varepsilon\downarrow 0}\lim_{\widehat{r}_t\downarrow 0}\frac{\left[e^{-(r_t+\widehat{r}_t)\varepsilon}\big(v_{it}(a)+\widehat{v}_{it}(a)\big)-e^{-r_t\varepsilon}v_{it}(a)\right]}{\varepsilon\widehat{r}_t}\\ &= \frac{\partial v_{it}(a)}{\partial r_t} - v_{it}(a)\end{aligned}$$

This expression is analogous to the one we obtain in the discrete time formula: $u''(c_{it}(a))\widehat{c}_{it}(a) = e^{-r_t\varepsilon}\widehat{V}'_{it}(a) - u'(c_{it}(a))e^{-r_t\varepsilon}\widehat{r}_t\varepsilon$.

Hence, we obtain in continuous time around steady-state:

$$u''(c_i^{ss}(a))\frac{\partial c_{it}(a)}{\partial r_t}\bigg|^{ss} = \frac{\partial v_{it}(a)}{\partial r_t}\bigg|^{ss} - v_i^{ss}(a).$$

From Proposition 1, $\frac{\partial v_{it}(a)}{\partial r_t}\big|^{ss} = \varphi_{i0}(a) = \frac{\partial(av_i^{ss}(a))}{\partial a}$. Therefore:

$$u''(c_i^{ss}(a))\frac{\partial c_{it}(a)}{\partial r_t}\bigg|^{ss} = av_i'^{ss}(a) + v_i^{ss} - v_i^{ss}(a) = av_i'^{ss}(a) = au''(c_i^{ss}(a))c_i'^{ss}(a),$$

leading to:

$$\frac{\partial c_{it}(a)}{\partial r_t}\bigg|^{ss} = ac_i'^{ss}(a).$$

An unanticipated change in the interest rate $\widehat{r}_t$ at time t only is equivalent to a pure transfer or size $a\widehat{r}_t$. Thereafter, the interest rate reverts to steady-state and so decisions also revert back to steady-state policy functions. Thus, the consumption response is given by the steady-state marginal propensity to consume out of wealth, $c_i'^{ss}(a)$, times the size of the transfer.

Without the adjustment associated with the limit $\epsilon \downarrow 0$, we would have erroneously obtained $\frac{\partial c_{it}(a)}{\partial r_t}\big|^{ss} = ac_i'^{ss}(a) + u'(c_i^{ss}(a))$, which is at odds with the argument of an unanticipated transfer.

A.3 Proof of Proposition 2

We start by expressing savings rate compactly:

$$\begin{aligned}\widehat{S}_{is}(a) &= \widehat{r}_s a + \widehat{w}_s y_i - \frac{1}{u''(c_i^{ss}(a))}\int_0^\infty e^{-\rho q}\Big\{\varphi_{iq}^r(a)\widehat{r}_{s+q} + \varphi_{iq}^w(a)\widehat{w}_{s+q}\Big\}dq \\ &\equiv \int_0^\infty \Big\{\psi_{iq}^r(a)\widehat{r}_{s+q} + \psi_{iq}^w(a)\widehat{w}_{s+q}\Big\}dq,\end{aligned}$$

where:

$$\psi_{iq}^r(a) = a\delta_0(q) - \frac{e^{-\rho q}\varphi_{iq}^r(a)}{u''(c_i^{ss}(a))} \qquad\qquad \psi_{iq}^w(a) = y_i\delta_0(q) - \frac{e^{-\rho q}\varphi_{iq}^w(a)}{u''(c_i^{ss}(a))}.$$

Then substitute this expression for the savings rate into (9) and obtain:

$$\widehat{g}_t - T_t^*[\widehat{g}_0] = -\int_0^t T_{t-s}^*\left[\frac{\partial}{\partial a}\left(g_i^{ss}(a)\int_0^\infty \Big\{\psi_{iq}^r(a)\widehat{r}_{s+q} + \psi_{iq}^w(a)\widehat{w}_{s+q}\Big\}dq\right)\right]ds.$$

Change variables to $\tau = s+q$:

$$\widehat{g}_t - T_t^*[\widehat{g}_0] = -\int_0^t T_{t-s}^*\left[\frac{\partial}{\partial a}\left(g_i^{ss}(a)\int_s^\infty \Big\{\psi_{i,\tau-s}^r(a)\widehat{r}_{\tau} + \psi_{i,\tau-s}^w(a)\widehat{w}_{\tau}\Big\}\right)\right]d\tau ds.$$

Change the order of integration:

$$\begin{aligned}\widehat{g}_t - T_t^*[\widehat{g}_0] &= -\int_0^\infty\int_0^t T_{t-s}^*\left[\frac{\partial}{\partial a}\left(g_i^{ss}(a)\mathbb{1}\{s\le\tau\}\Big[\psi_{i,\tau-s}^r(a)\widehat{r}_{\tau} + \psi_{i,\tau-s}^w(a)\widehat{w}_{\tau}\Big]\right)\right]dsd\tau \\ &= -\int_0^\infty\int_0^{\min\{t,\tau\}} T_{t-s}^*\left[\frac{\partial}{\partial a}\left(g_i^{ss}(a)\Big[\psi_{i,\tau-s}^r(a)\widehat{r}_{\tau} + \psi_{i,\tau-s}^w(a)\widehat{w}_{\tau}\Big]\right)\right]dsd\tau,\end{aligned}$$

where in the first line we integrate τ from 0 to $+\infty$ provided we include the indicator $\mathbb{1}\{s\le\tau\}$, and in the second line we subsume this indicator in the upper bound of the domain of integration for s, from 0 to $\min\{t,\tau\}$ instead of t. This last equation concludes the proof.

A.4 Proof of Corollary 2

We treat only the case for the interest rate, as the one for wages is entirely symmetric. Integrating the expression in Proposition 2 against $\mathcal{E}^r$, we obtain:

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \mathcal{E}^{r,*}T_t^*[\widehat{g}_0] + \int_0^\infty\left\{\int_0^{\min\{t,\tau\}}\Big(\mathcal{E}^{r,*}T_{t-s}^*[\mathcal{D}_{\tau-s}^r]\widehat{r}_\tau + \mathcal{E}^{r,*}T_{t-s}^*[\mathcal{D}_{\tau-s}^w]\widehat{w}_\tau\Big)ds\right\}d\tau. \quad (19)$$

Because $\mathcal{L}$ and $\mathcal{L}^*$ are adjoints (the generalization of the matrix transpose) for our inner product, T_t and T_t^* are also adjoints for our inner product, where T_t satisfies:

$$T_{0,ij}(a,a') = \mathbb{1}\{i=j\}\ \delta(a-a') \qquad\qquad \frac{\partial T_t}{\partial t} = \mathcal{L}[T_t].$$

This observation is the exact analogue of a similar result in finite dimensions. Using this observation and the definition of the inner product, we obtain that:

$$\mathcal{E}^{r,*}\big(T_t^*[\widehat{g}_0]\big) = \big(T_t[\mathcal{E}^r]\big)^*\widehat{g}_0.$$

We define the expectation function $\mathcal{E}_t^r = T_t[\mathcal{E}^r]$. It satisfies the partial differential equation:

$$\frac{\partial \mathcal{E}_t^r}{\partial t} = \frac{\partial T_t}{\partial t}[\mathcal{E}^r] = \mathcal{L}[T_t[\mathcal{E}^r]] = \mathcal{L}[\mathcal{E}_t^r],$$

with initial condition $\mathcal{E}_0^r = \mathcal{E}^r$.

Next, we define the fake news operators $\mathcal{F}_{t,s}^{r,r} = \mathcal{E}_t^{r,*}\mathcal{D}_s^r$ and $\mathcal{F}_{t,s}^{r,w} = \mathcal{E}_t^{r,*}\mathcal{D}_s^w$. Substituting these results into (19), we obtain:

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \mathcal{E}_t^{r,*}\widehat{g}_0 + \int_0^\infty \left\{ \int_0^{\min\{t,\tau\}} \left(\mathcal{F}_{t-s,\tau-s}^{r,w}\widehat{r}_\tau + \mathcal{F}_{t-s,\tau-s}^{r,w}\widehat{w}_\tau \right) ds \right\} d\tau. \tag{20}$$

Finally, we define the Jacobians $J_{t,\tau}^{r,r} = \int_0^{\min\{t,\tau\}} \mathcal{F}_{t-s,\tau-s}^{r,r} ds$ and similarly for $J_{t,\tau}^{r,w}$. Substituting these definitions into (20), we obtain:

$$\widehat{r}_t = r^{ss}\widehat{z}_t + \mathcal{E}_t^{r,*}\widehat{g}_0 + \int_0^\infty \left\{ J_{t,\tau}^{r,w}\widehat{r}_\tau + J_{t,\tau}^{r,w}\widehat{w}_\tau \right\} d\tau.$$

A.5 Proof of Corollary 3

For brevity, we omit the superscripts $(r,r),(r,w)$, etc. of the Jacobians and fake news operators. Using the definition of any of the Jacobians $J_{t,\tau}$, we obtain for any q such that $0 < q < \min\{t,\tau\}$:

$$J_{t+q,\tau+q} = \int_0^{\min\{t+q,\tau+q\}} \mathcal{F}_{t+q-s,\tau+q-s} ds = \int_0^q \mathcal{F}_{t+q-s,\tau+q-s} ds + \int_q^{\min\{t+q,\tau+q\}} \mathcal{F}_{t+q-s,\tau+q-s} ds.$$

Changing variables $s' = s - q$ in the second integral, we obtain:

$$J_{t+q,\tau+q} = \int_0^q \mathcal{F}_{t+q-s,\tau+q-s} ds + \int_0^{\min\{t,\tau\}} \mathcal{F}_{t-s',\tau-s'} ds',$$

and so:

$$J_{t+q,\tau+q} - J_{t,\tau} = \int_0^q \mathcal{F}_{t+q-s,\tau+q-s} ds.$$

Taking $q \downarrow 0$, we obtain:

$$\lim_{q\downarrow 0} \frac{J_{t+q,\tau+q} - J_{t,\tau}}{q} = \mathcal{F}_{t+q,\tau+q}.$$

To determine the initial conditions $J_{t,0}$ and $J_{0,\tau}$, we must take into account that $\mathcal{F}$ contains

Dirac mass functions. We obtain:

$$J_{t,0} \equiv \lim_{\tau \downarrow 0} J_{t,\tau} = \lim_{\tau \downarrow 0} \int_0^\tau \mathcal{F}_{t-s,\tau-s} ds = \lim_{\tau \downarrow 0} \int_0^\tau \mathcal{F}_{t,s} ds = \mathcal{E}_t^* \overline{\mathcal{D}},$$

where for the interest rate $\overline{\mathcal{D}}_i^r(a) = -\frac{\partial}{\partial a}\big(g_i^{ss}(a)a\big)$ and for wages $\overline{\mathcal{D}}_i^w(a) = -\frac{\partial}{\partial a}\big(g_i^{ss}(a)y_i\big)$.

For $J_{0,\tau}$, the Dirac mass point does not affect the calculation, and we obtain:

$$J_{0,\tau} \equiv \lim_{t \downarrow 0} J_{t,\tau} = \lim_{t \downarrow 0} \int_0^t \mathcal{F}_{t-s,\tau-s} ds = \lim_{t \downarrow 0} \int_0^t \mathcal{F}_{s,\tau} ds = 0.$$

B Details for the HANK economy

This provides additional details for our HANK economy in continuous time.

B.1 Wage Phillips Curve

Given constant elasticity of substitution demand for labor, union k faces labor demand:

$$N_{kt} = \left(\frac{W_{kt}}{W_t}\right)^{-\varepsilon} N_t$$

Workers allocate uniformly to unions, and unions set wages to maximize the utility of their members. They solve:

$$\begin{aligned} J_t(W_k) = \max_{\{\pi_{ks}\}_s} \mathbb{E}_t \int_0^\infty e^{-\rho s} \left\{ \int_0^1 \Big[u(c_{j,t+s}) - v(n_{j,t+s})\Big] dj - \frac{\psi}{2}\pi_{k,t+s}^2 \right\} ds \\ \text{s.t. } \frac{\partial W_{ks}}{\partial s} = \pi_{ks} W_{ks},\ W_{kt} = W_k,\ \text{and household and labor aggregator behavior,} \end{aligned}$$

where j indexes households (not income states) employed by the union. t is the initial time and W_{kt} is the initial condition for the wage. In principle, the distribution of assets of these households employed by the union should be a state variable of the union problem, but we subsume it in the time subscript t.

We can rewrite this problem as:

$$\begin{aligned} J_t(W_k) = \max_{\{\pi_{ks}\}_s} \int V_{i_j t t}(a_{jt}) dj - \frac{\psi}{2} \int_0^\infty e^{-\rho s} \pi_{k,t+s}^2 ds \\ \text{s.t. } \frac{\partial W_{ks}}{\partial s} = \pi_{ks} W_{ks},\ W_{kt} = W_k,\ \text{labor aggregator behavior,} \end{aligned}$$

We will need a simple lemma that builds on the envelope theorem. We define net income as $x_{it} = \tau_t (w_t y_i n_{it})^{1-\theta}$.

Lemma 1. *(Envelope theorem)*

For any small change in wages $\widehat{w}_{t+q}$ holding employment n_{it} fixed:

$$\widehat{V}_{it}(a) = \mathbb{E}_t \int_0^\infty e^{-\rho s} u'(c_{t+s}) \widehat{x}_{t+s} ds.$$

Proof. We start from the Bellman equation:

$$\rho V_{it}(a) = \max_c u(c) - v(n_{it}) + (r_t a + x_{it} - c) V'_{it}(a) + L_{it}(a,c)[\omega_{it}] + \frac{\partial V_{it}(a)}{\partial t},$$

where, importantly, there is no optimization over n because the households do not choose hours: the union does. The first-order optimality condition for consumption is $u'(c_{it}(a)) = V'_{it}(a)$. Now differentiate the value function to obtain:

$$\rho \widehat{V}_{it}(a) = V'_{it}(a)\widehat{x}_{it} + L_{it}(a)[\widehat{V}_{it}] + \frac{\partial \widehat{V}_{it}(a)}{\partial t} = u'(c_{it}(a))\widehat{x}_{it} + L_{it}(a)[\widehat{V}_{it}] + \frac{\partial \widehat{V}_{it}(a)}{\partial t}.$$

where we have used the envelope theorem with respect to c. Going back to the sequential formulation concludes the proof. □

We form the Lagrangian associated with the union's sequential problem, use $t = 0$ without loss of generality for expositional simplicity, and omit time t indices when unambiguous:

$$\begin{aligned}
\mathbb{L}_0 &= \int V_{i_j 0}(a_j) dj - \frac{\psi}{2} \int_0^\infty e^{-\rho s} \pi_s^2 ds + \int_0^\infty e^{-\rho s} \lambda_s \left(w_s \pi_s - \frac{\partial w_s}{\partial s} \right) ds \\
&= \int V_{i_j 0}(a_j) dj + \int_0^\infty e^{-\rho s} \left\{ w_s \pi_s \lambda_s - \frac{\psi}{2} \pi_s^2 - \left(\rho \lambda_s - \frac{\partial \lambda_s}{\partial s} \right) w_s \right\} ds + \lambda_0 w_0.
\end{aligned}$$

by integration by parts. The first-order condition with respect to π_s leads to:

$$\psi \pi_s = w_s \lambda_s.$$

The first-order condition with respect to w_s is:

$$0 = \int \frac{\partial V_{i_j 0}(a_j)}{\partial w_s} dj + e^{-\rho s} \left\{ \pi_s \lambda_s - \rho \lambda_s + \frac{\partial \lambda_s}{\partial s} \right\}.$$

Applying Lemma 1, we have:

$$\frac{\partial V_{i_j 0}(a_j)}{\partial w_s} = e^{-\rho s} u'(c_{js}) \frac{\partial x_{j,s}}{\partial w_s} - e^{-\rho s} v'(n_{js}) \frac{\partial n_{js}}{\partial w_s},$$

because now the union chooses n simultaneously and internalizes the demand curve. Substituting into the previous equation we obtain:

$$(\rho - \pi_s)\lambda_s - \frac{\partial \lambda_s}{\partial s} = \int \left\{ u'(c_{js}) \frac{\partial x_{js}}{\partial w_s} - v'(n_{js}) \frac{\partial n_{js}}{\partial w_s} \right\} dj.$$

Substituting in the first-order condition $\psi\pi_s = w_s\lambda_s$, we obtain:

$$\begin{aligned}(\rho-\pi_s)\lambda_s - \frac{\partial\lambda_s}{\partial s} &= (\rho-\pi_s)\frac{\psi\pi_s}{w_s} - \lambda_s\left(\frac{1}{\pi_s}\frac{\partial\pi_s}{\partial s} - \frac{1}{w_s}\frac{\partial w_s}{\partial s}\right)\\ &= (\rho-\pi_s)\frac{\psi\pi_s}{w_s} - \lambda_s\left(\frac{1}{\pi_s}\frac{\partial\pi_s}{\partial s} - \pi_s\right)\\ &= \rho\frac{\psi\pi_s}{w_s} - \frac{\psi}{w_s}\frac{\partial\pi_s}{\partial s}\\ &= \frac{\psi}{w_s}\left(\rho\pi_s - \frac{\partial\pi_s}{\partial s}\right).\end{aligned}$$

Hence, we obtain:

$$\rho\pi_t = \frac{w_t}{\psi}\int\left\{u'(c_{jt})\frac{\partial x_{jt}}{\partial w_t} - v'(n_{jt})\frac{\partial n_{jt}}{\partial w_t}\right\}dj + \frac{\partial\pi_t}{\partial t}.$$

Using the expression for the demand curve and net income in a symmetric equilibrium, we obtain:

$$\frac{\partial n_t}{\partial w_t} = -\varepsilon\frac{n_t}{w_t} \qquad\qquad \frac{\partial x_{jt}}{\partial w_t} = (1-\theta)x_{jt}\left(\frac{1}{w_t} + \frac{1}{n_t}\frac{\partial n_t}{\partial w_t}\right) = (1-\theta)(1-\varepsilon)\frac{x_{it}}{w_t}.$$

Substituting in for a symmetric equilibrium $n_{jt} \equiv N_t$, we obtain:

$$\rho\pi_t = \frac{\varepsilon}{\psi}\left(\frac{1-\varepsilon}{\varepsilon}(1-\theta)\int u'(c_{jt})x_{jt}dj + v'(N_t)N_t\right) + \frac{\partial\pi_t}{\partial t}.$$

Define C_t^* such that $u'(C_t^*)(Y_t - T_t) = \int u'(c_{jt})x_{jt}dj$. Then we obtain the Phillips curve:

$$\rho\pi_t = \frac{\varepsilon}{\psi}\left(\frac{1-\varepsilon}{\varepsilon}(1-\theta)u'(C_t^*)(Y_t - T_t) + v'(N_t)N_t\right) + \frac{\partial\pi_t}{\partial t}.$$

Note that this Phillips curve does not involve directly the marginal utility of aggregate consumption, but instead a weighted average of individual marginal utilities of consumption.

Auclert et al. (2024b) impose $C_t = C_t^*$ ex post in the union's problem. Instead, we microfound this assumption by setting set up a modified union problem to obtain $u'(C_t)$ in the wage Phillips curve. Define the weights $w_{jt} = \frac{u''(C_t)}{u''(c_{jt})}\frac{\int y_{jt}^{1-\theta}di}{y_{jt}^{1-\theta}}$. Then if the union places weight w_{jt} on household j and takes this weight as given, then repeating the steps above delivers the wage Phillips curve:

$$\rho\pi_t = \frac{\varepsilon}{\psi}\left(\frac{1-\varepsilon}{\varepsilon}(1-\theta)u'(C_t)(Y_t - T_t) + v'(N_t)N_t\right) + \frac{\partial\pi_t}{\partial t}.$$

B.2 Linearized Wage Phillips Curve

We specify functional forms for $u(c)$ and $v(n)$. Let $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ and $v(n) = \frac{n^{1+\xi}}{1+\xi}$. Let $X_t = Y_t - T_t$ denote aggregate post-tax income. We obtain:

$$\rho\pi_t = \frac{\varepsilon}{\psi}\Big[\frac{1-\varepsilon}{\varepsilon}(1-\theta)C_t^{-\sigma}X_t + N_t^{1+\xi}\Big] + \frac{\partial\pi_t}{\partial t}$$

We then linearize around the zero inflation steady state. To ease notation, letters without time subscripts denote steady-state quantities. We obtain:

$$\rho\widehat{\pi}_t = \frac{(1+\xi)\varepsilon}{\psi}N^{1+\xi}\frac{\widehat{N}_t}{N} + \frac{(1-\varepsilon)(1-\theta)}{\psi}C^{-\sigma}X\left(-\sigma\frac{\widehat{C}_t}{C} + \frac{\widehat{X}_t}{X}\right) + \frac{\partial\widehat{\pi}_t}{\partial t}.$$

Using that at steady-state, $0 = \frac{\varepsilon}{\psi}\Big[\frac{1-\varepsilon}{\varepsilon}(1-\theta)C^{-\sigma}X + N^{1+\xi}\Big]$, we obtain:

$$\rho\widehat{\pi}_t = \frac{\varepsilon}{\psi}N^{1+\xi}\left[(1+\xi)\frac{\widehat{N}_t}{N} + \sigma\frac{\widehat{C}_t}{C} - \frac{\widehat{X}_t}{X}\right] + \frac{\partial\widehat{\pi}_t}{\partial t}.$$

B.3 General Equilibrium Jacobians to Aggregate Shocks

In what follows, we always denote by Id the identity operator.

We obtain the Jacobians of output, Y, and the real interest rate, r, to all aggregate shocks using the market clearing condition, $Y_t = C_t + G_t$, and the Taylor rule, which combined with the Fisher equation is $r_t = (\phi-1)\pi_t + \epsilon_t$.

As interest rates change, the government balances its budget by adjusting taxes. We denote by T^e this endogenous component of taxes. We thus need, in sequence notation: $\widehat{T}^e = \mathcal{J}^{Tr}\widehat{r}$ where we define the Jacobian $\mathcal{J}^{Tr}$ such that $q^*\mathcal{J}^{Tr} = Bq^*$ as in Auclert et al. (2024b), where $q_t = e^{-rt}$ and B is the government debt level. There are thus many possible choices, and in the implementation, we will set $\mathcal{J}^{Tr} = B\mathrm{Id}$ so that taxes rise contemporaneously to counteract the increase in the debt value.

B.3.1 G shock

Differentiating the Taylor rule and market-clearing equations with respect to G_s yields:

$$\begin{aligned}\mathcal{J}^{rG} &= \Big(\phi-1\Big)\Big((\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\ \mathcal{J}^{Tr})\mathcal{J}^{rG} + \mathcal{J}^{\pi Y}\mathcal{J}^{YG} + \mathcal{J}^{\pi G}\Big)\\ &= \underbrace{\Big(\mathrm{Id} - (\phi-1)(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\ \mathcal{J}^{Tr})\Big)^{-1}\Big(\phi-1\Big)}_{\equiv J^{rY,G}}\Big[\mathcal{J}^{\pi Y}\mathcal{J}^{YG} + \mathcal{J}^{\pi G}\Big]\end{aligned}$$

$$\mathcal{J}^{YG} = \mathcal{J}^{CY}\mathcal{J}^{YG} + \mathcal{J}^{Cr}\mathcal{J}^{rG} + \mathcal{J}^{CT}\mathcal{J}^{Tr}\mathcal{J}^{rG} + \mathrm{Id}$$
$$= \Big(\mathrm{Id} - \mathcal{J}^{CY} - \mathcal{J}^{Cr}\underbrace{J^{rY,G}\mathcal{J}^{\pi Y}}_{\equiv \mathcal{J}^{rY}} + \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rY}\Big)^{-1}\Big[\mathrm{Id} + \mathcal{J}^{Cr}J^{rY,G}\mathcal{J}^{\pi G} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,G}\mathcal{J}^{\pi G}\Big]$$

Above, we plugged in $\mathcal{J}^{CT} = -\mathcal{J}^{CY}$ since the household only cares about $Y - T$ so the response to T is just the negative of the response to Y. The expression above is the same as $\mathcal{J}^{YG} = (\mathrm{Id} - \mathcal{J}^{CY})^{-1}$ in Auclert et al. (2024b) but accounting for the case when monetary policy is active ($\phi > 1$). Importantly, $\mathcal{J}^{Cr}$ and $\mathcal{J}^{CY}$ are calculated using:

$$\widehat{C}_t = \sum_i \int \widehat{c}_{it}(a) g_i^{ss}(a) d\eta(a) + \sum_i \int c_i^{ss}(a)\widehat{g}_{it}(a) d\eta(a).$$

In continuous time, the equation is $r_t = i_t - \pi_t$, but corresponds to $r_t = i_t - \lim_{\varepsilon \downarrow 0} \pi_{t+\varepsilon}$. In discrete time, this distinction is directly visible because the Fisher equation is $r_t = i_t - \pi_{t+1}$. Once we discretize the continuous-time economy, as for information aggregation, we can decide which timing to use. To align the results between the two methods, we use a version of discretization that mirrors the the discrete-time Fisher equation. We make this distinction explicit by introducing a notation for a lead operator $\mathbf{F}$ (with the understanding that this lead operator collapses to the identity operator in the truly continuous time case). The expression for $\mathcal{J}^{rG}$ becomes:

$$\mathcal{J}^{rG} = (\phi\mathrm{Id} - \mathbf{F})((\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})\mathcal{J}^{rG} + \mathcal{J}^{\pi Y}\mathcal{J}^{YG} + \mathcal{J}^{\pi G})$$
$$= \underbrace{(\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi\mathrm{Id} - \mathbf{F})}_{\equiv J^{rY,G}}(\mathcal{J}^{\pi Y}\mathcal{J}^{YG} + \mathcal{J}^{\pi G})$$

Above, $\mathbf{F}$ is the lead operator. Now, $\mathcal{J}^{rY} = (\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi\mathrm{Id} - \mathbf{F})\mathcal{J}^{\pi Y}$. Note from the wage Phillips curve that $\mathcal{J}^{\pi r} = 0$.

As in Auclert et al. (2024b), the operator $\mathcal{J}^{YG}$ is not invertible. Thus, instead of defining $\mathcal{J}^{YG}$ through its first component $(\mathrm{Id} - \mathcal{J}^{CY} - \mathcal{J}^{Cr}\mathcal{J}^{rY} + \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rY})^{-1}$, we use $\mathcal{J}^{YG} = \mathbf{A}^{-1}\mathbf{K}$ where $\mathbf{A} = \mathbf{K}(\mathrm{Id} - \mathcal{J}^{CY} - \mathcal{J}^{Cr}\mathcal{J}^{rY} + \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rY})$ and $\mathbf{K} = -\int_0^\infty e^{-rt}\mathbf{F}_t$, and $\mathbf{F}_t$ is the t^{th} iterate of the forward operator.

To see why, start from the aggregate household budget constraint discretized with small time intervals $\varepsilon > 0$:

$$B_t = (1 + r_t\varepsilon)B_{t-\varepsilon} + \underbrace{Y_t - T_t}_{X_t} - C_t$$
$$\implies \frac{\partial B_t}{\partial X_s} = (1 + r^{ss}\varepsilon)\frac{\partial B_{t-\varepsilon}}{\partial X_s} + \frac{\partial r_t}{\partial X_s}B^{ss}\varepsilon + \mathbb{1}_{t=s} - \frac{\partial C_t}{\partial X_s} - \sum_{s'}\frac{\partial C_t}{\partial r_{s'}}\frac{\partial r_{s'}}{\partial X_s}$$

$$
\begin{aligned}
\mathcal{J}^{BY} &= (1+r^{ss}\varepsilon)\mathbf{L}\mathcal{J}^{BY} + \mathcal{J}^{rY}B^{ss}\varepsilon + \mathrm{Id} - \mathcal{J}^{CY} - \mathcal{J}^{Cr}\mathcal{J}^{rY} \\
(\mathrm{Id} - (1+r^{ss}\varepsilon)\mathbf{L})\mathcal{J}^{BY} &= [\mathrm{Id} - \mathcal{J}^{Cr}\mathcal{J}^{rY} - \mathcal{J}^{CY} + \mathcal{J}^{rY}B^{ss}\varepsilon] \\
\underbrace{\mathbf{K}(\mathrm{Id} - (1+r^{ss}\varepsilon)\mathbf{L})}_{=\mathrm{Id}}\mathcal{J}^{BY} &= \mathbf{K}[\mathrm{Id} - \mathcal{J}^{Cr}\mathcal{J}^{rY} - \mathcal{J}^{CY} + \mathcal{J}^{rY}B^{ss}\varepsilon] \\
\mathcal{J}^{BY} &= \mathbf{K}[\mathrm{Id} - \mathcal{J}^{Cr}\mathcal{J}^{rY} - \mathcal{J}^{CY} + \mathcal{J}^{rY}B^{ss}\varepsilon]
\end{aligned}
$$

Above, $\mathbf{L}$ is the lag operator such that $\mathbf{FL} = \mathrm{Id}$. Lemma 3 in the Online Appendix of Auclert et al. (2024b) proves that $\mathbf{K}(\mathrm{Id} - (1+r^{ss}\varepsilon)\mathbf{L}) = \mathrm{Id}$. Note that $\mathcal{J}^{rY}B^{ss}\varepsilon$ captures the effect of the change in interest rates on debt. Since we assume that the government adjusts taxes to ensure that there is no effect on debt, this term can be replaced by the effect on consumption of the change in taxes induced by the change in interest rates: $\mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rY}$ yielding

$$
\begin{aligned}
&\mathcal{J}^{BY} = \mathbf{K}[\mathrm{Id} - \mathcal{J}^{Cr}\mathcal{J}^{rY} - \mathcal{J}^{CY} + \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rY}] \\
\implies &\mathcal{J}^{YG} = (\mathcal{J}^{BY})^{-1}\mathbf{K}(\mathrm{Id} + \mathcal{J}^{Cr}J^{rY,G}J^{\pi G} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,G}\mathcal{J}^{\pi G})
\end{aligned}
$$

Let $\mathbf{A}^{-1}\mathbf{K} \equiv (\mathcal{J}^{BY})^{-1}\mathbf{K}$ to be consistent with the notation in Auclert et al. (2024b), which we will use later.

B.3.2 T Shock

The setup is similar for the T shock. We again account for the feedback effect of r on T through debt:

$$
\begin{aligned}
\mathcal{J}^{rT} &= (\phi-1)((\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})\mathcal{J}^{rT} + \mathcal{J}^{\pi Y}\mathcal{J}^{YT} + \mathcal{J}^{\pi T}) \\
&= \underbrace{(\mathrm{Id} - (\phi-1)(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi-1)}_{\equiv J^{rY,T}}[\mathcal{J}^{\pi Y}\mathcal{J}^{YT} + \mathcal{J}^{\pi T}] \\
\mathcal{J}^{YT} &= \mathcal{J}^{Cr}\mathcal{J}^{rT} + \mathcal{J}^{C,Y-T}\mathcal{J}^{Y-T,T} - \mathcal{J}^{C,Y-T}\mathcal{J}^{Tr}\mathcal{J}^{rT} \\
&= \mathcal{J}^{Cr}\mathcal{J}^{rT} + \mathcal{J}^{CY}(\mathcal{J}^{YT} - \mathrm{Id} - \mathcal{J}^{Tr}\mathcal{J}^{rT}) \\
&= (\mathrm{Id} - \mathcal{J}^{Cr}J^{rY,T}\mathcal{J}^{\pi Y} - \mathcal{J}^{CY} + \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,T}\mathcal{J}^{\pi Y})^{-1}[\mathcal{J}^{Cr}J^{rY,T}\mathcal{J}^{\pi T} - \mathcal{J}^{CY} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,T}\mathcal{J}^{\pi T}] \\
&= \mathbf{A}^{-1}\mathbf{K}[\mathcal{J}^{Cr}J^{rY,T}\mathcal{J}^{\pi T} - \mathcal{J}^{CY} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,T}\mathcal{J}^{\pi T}]
\end{aligned}
$$

The first line follows from the goods market clearing condition, $Y_t = C_t + G_t$ and noting that T does not affect G, but affects C by changing r and X. The effect on X is both direct (by changing Y and T) and indirect (T responds to balance the government budget as r changes). The term being inverted in the second-to-last line is identical to what we had for $\mathcal{J}^{YG}$ which yields the last line.

If we use the version of the Fisher equation that is discretized to mirror the discrete time case, $\mathcal{J}^{rT}$ becomes:

$$\begin{aligned}
\mathcal{J}^{rT} &= (\phi\mathrm{Id} - \mathbf{F})((\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})\mathcal{J}^{rT} + \mathcal{J}^{\pi Y}\mathcal{J}^{YT} + \mathcal{J}^{\pi T}) \\
&= [\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})]^{-1}(\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi Y}\mathcal{J}^{YT} + \mathcal{J}^{\pi T}) \\
\text{and } J^{rY,T} &= (\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi\mathrm{Id} - \mathbf{F})
\end{aligned}$$

B.3.3 Z Shock

Next, we consider an aggregate productivity shock:

$$\begin{aligned}
\mathcal{J}^{rZ} &= (\phi - 1)(\mathcal{J}^{\pi r}\mathcal{J}^{rZ} + \mathcal{J}^{\pi Y}\mathcal{J}^{YZ} + \mathcal{J}^{\pi Z} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}\mathcal{J}^{rZ}) \\
&= \underbrace{(\mathrm{Id} - (\phi - 1)(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi - 1)}_{\equiv J^{rY,Z}}[\mathcal{J}^{\pi Y}\mathcal{J}^{YZ} + \mathcal{J}^{\pi Z}] \\
\mathcal{J}^{YZ} &= \mathcal{J}^{CZ} + \mathcal{J}^{GZ} \\
&= \mathcal{J}^{CY}\mathcal{J}^{YZ} + \mathcal{J}^{Cr}\mathcal{J}^{rZ} - \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{rZ} \\
&= (\mathrm{Id} - \mathcal{J}^{CY} - \mathcal{J}^{Cr}J^{rY,Z}\mathcal{J}^{\pi Y} + \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,Z}\mathcal{J}^{\pi Y})^{-1}(\mathcal{J}^{Cr}J^{rY,Z}\mathcal{J}^{\pi Z} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,Z}\mathcal{J}^{\pi Z}) \\
&= \mathbf{A}^{-1}\mathbf{K}\Big[(\mathcal{J}^{Cr} - \mathcal{J}^{CY}\mathcal{J}^{Tr})J^{rY,Z}\mathcal{J}^{\pi Z}\Big]
\end{aligned}$$

If we use the discrete-time version of the Fisher equation, $\mathcal{J}^{rZ}$ becomes:

$$\begin{aligned}
\mathcal{J}^{rZ} &= (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r}\mathcal{J}^{rZ} + \mathcal{J}^{\pi Y}\mathcal{J}^{YZ} + \mathcal{J}^{\pi Z} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}\mathcal{J}^{rZ}) \\
&= [\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})]^{-1}(\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi Y}\mathcal{J}^{YZ} + \mathcal{J}^{\pi Z}) \\
\text{and } J^{rY,Z} &= (\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}(\phi\mathrm{Id} - \mathbf{F})
\end{aligned}$$

B.3.4 ϵ Shock

Finally, we consider a real interest rate shock, ϵ:

$$\begin{aligned}
\mathcal{J}^{r\epsilon} &= (\phi - 1)(\mathcal{J}^{\pi r}\mathcal{J}^{r\epsilon} + \mathcal{J}^{\pi Y}\mathcal{J}^{Y\epsilon} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}\mathcal{J}^{r\epsilon}) + \mathrm{Id} \\
&= \underbrace{\Big[\mathrm{Id} - (\phi - 1)(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})\Big]^{-1}}_{\equiv J^{rY,\epsilon}}\Big[(\phi - 1)\mathcal{J}^{\pi Y}\mathcal{J}^{Y\epsilon} + \mathrm{Id}\Big] \\
\mathcal{J}^{Y\epsilon} &= \mathcal{J}^{Cr}\mathcal{J}^{r\epsilon} + \mathcal{J}^{CY}\mathcal{J}^{Y\epsilon} - \mathcal{J}^{CY}\mathcal{J}^{Tr}\mathcal{J}^{r\epsilon} \\
&= \underbrace{\Big(\mathrm{Id} + (\mathcal{J}^{CY}\mathcal{J}^{Tr} - \mathcal{J}^{Cr})J^{rY,\epsilon}(\phi - 1)\mathcal{J}^{\pi Y} - \mathcal{J}^{CY}\Big)^{-1}}_{=\mathcal{J}^{YG}}\Big[\mathcal{J}^{Cr}J^{rY,\epsilon} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,\epsilon}\Big]
\end{aligned}$$

Table 1: HA Parameters

Parameter	Description	Value
γ	γ in $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$	2
ρ	Continuous Time Discount Rate	0.05
β	Discrete Time Discount Rate	0.95
δ	Depreciation Rate	0.1
α	Capital Share	0.33
$\overline{Z}$	Aggregate Productivity	1
μ_e	Idiosyncratic Productivity Persistence	0.91
σ_e	Idiosyncratic Productivity Cross-Sectional Standard Deviation	0.5
$\underline{a}$	Min. Asset	0

$$= \mathbf{A}^{-1}\mathbf{K}\Big[\mathcal{J}^{Cr}J^{rY,\epsilon} - \mathcal{J}^{CY}\mathcal{J}^{Tr}J^{rY,\epsilon}\Big]$$

If we use the discrete-time version of the Fisher equation, $\mathcal{J}^{r\epsilon}$ becomes:

$$\begin{aligned}
\mathcal{J}^{r\epsilon} &= (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r}\mathcal{J}^{r\epsilon} + \mathcal{J}^{\pi Y}\mathcal{J}^{Y\epsilon} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}\mathcal{J}^{r\epsilon}) \\
&= [\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr})]^{-1}(\phi\mathrm{Id} - \mathbf{F})\mathcal{J}^{\pi Y}\mathcal{J}^{Y\epsilon} \\
\text{and } J^{rY,\epsilon} &= (\mathrm{Id} - (\phi\mathrm{Id} - \mathbf{F})(\mathcal{J}^{\pi r} + \mathcal{J}^{\pi T}\mathcal{J}^{Tr}))^{-1}
\end{aligned}$$

C Numerical appendix

C.1 Parameters and steady-state

This section reports our choice of parameters and compares the steady-state values between the discrete-time and continuous-time economies. The parameters and numerical approximations are chosen to match exactly, with $\beta = \exp(-\rho)$. The resulting steady-state values are quite close.

C.2 Implementation details for Proposition 1

Proposition 1 implies that, for a small time interval τ:

$$\varphi_{t+\tau} = \tau(r^{ss} - c'^{ss})\varphi_t + (\mathrm{Id} + \tau\mathcal{L})\varphi_t. \tag{21}$$

When we set $\tau = 1$ to align with discrete time, iterating numerically on this equation can lead to explosive dynamics. The reason is that households might save or dissave enough to move

Table 2: HA Steady State

SS Object	Description	Cont Time Value	Discrete Time Value
r	Real Interest Rate	0.025	0.025
w	Wage	1.087	1.087
K	Aggregate Capital	4.337	4.337
Y	Aggregate Output	1.631	1.631
C	Aggregate Consumption	1.197	1.197
Share Constrained	Share Constrained	0.046	0.046

Table 3: HANK Parameters

Parameter	Description	Value
γ	γ in $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$	2
ρ	Continuous Time Discount Rate	0.063
β	Discrete-Time Discount Rate	0.939
$\overline{Z}$	Aggregate Productivity	1
μ_w	Wage Markup	1.1
ϕ	Taylor Rule Coefficient on π	1
κ_w	Wage Flexibility	0.03
θ	Tax Parameter	0.181
ξ	Labor Supply Elasticity	1
N^{ss}	Steady State Labor	1
Y^{ss}	Steady State Output	1
$\frac{G}{Y}$	Government Share	0.2
$\frac{T}{Y}$	Transfer Share	0.529
ρ_e	Idiosyncratic Productivity Persistence	0.91
σ_e	Idiosyncratic Productivity Cross-Sectional Standard Deviation	0.753
$\underline{a}$	Minimum Asset Constraint	0

Table 4: HANK Steady State

SS Object	Description	Cont Time Value	Discrete Time Value
r	Real Interest Rate	0.048	0.048
K	Aggregate Capital	6.855	6.855
C	Aggregate Consumption	0.800	0.800
Share Constrained	Share Constrained	0.005	0.005

more than one gridpoint in the asset space in a time interval $[t, t+\tau)$.

Standard continuous time discretization formulas assume that τ is small enough that households only move by at most one gridpoint. When savings rates imply that households would move by multiple gridpoints, standard continuous time discretization formulas compensate by imposing that a larger mass of households moves by one gridpoint. For example, if the household at idiosyncratic state (a, y_i) saves $2d_a$ where d_a is the distance between asset gridpoints and the mass of households at (a, y_i) is $g_i(a)d_a$, then standard discretization formulas impose that $2g_i(a)d_a$ agents move to $a + d_a$ instead of $g_i(a)d_a$ agents move to $a + 2d_a$.

Since there are only $g_i(a)d_a$ agents at (a, y_i), this calculation leaves $-g_i(a)d_a$ agents at (a, y_i). The fact that negative mass of households appears leads to instability and results in an explosive solution over multiple iterations. This explosivity at larger τ is well-understood in finite difference methods, and arises because the time iteration scheme we proposed is an *explicit* scheme (Achdou et al., 2021).

There are three possible solutions to avoid instability. The first solution is to use a smaller time step τ in (21), that implies smaller savings from each time period to the next. However, this choice results in a larger Jacobian to ultimately invert and slows down the calculation of impulse response functions.

The second solution is to use an *implicit* scheme when iterating on (21), according to:

$$(\mathrm{Id} - \tau\mathcal{L})\varphi_{t+\tau} = \tau(r^{ss} - c'^{ss})\varphi_t + \varphi_t. \tag{22}$$

Iterating on (22) requires solving a linear system involving $(\mathrm{Id} - \tau\mathcal{L})$ each period. While this implicit scheme is relatively fast because $\mathcal{L}$ is sparse, it is still slower than using the explicit scheme (21).

The third solution is to mimic the transition matrix in discrete time. We construct a version of the transition matrix $\mathcal{L}$ in which we allow agents to move more than by one gridpoint. Returning to our example, we impose that d_a agents move to $a+2d_a$ after applying the transition operator. This approach is our preferred one and results in fast and stable dynamics when iterating on (21).

C.3 Computation times

We use three different methods to solve for the steady-state. "Discrete" uses a discrete time formulation and the endogenous gridpoints method. "Continuous Time, EGM" and "Continuous

Time, Implicit" use the continuous-time method. "Continuous Time EGM" uses the endogenous gridpoints method in continuous time, similarly to the discrete time case. "Continuous Time, Implicit" uses the implicit method in Achdou et al. (2021) for the steady state.

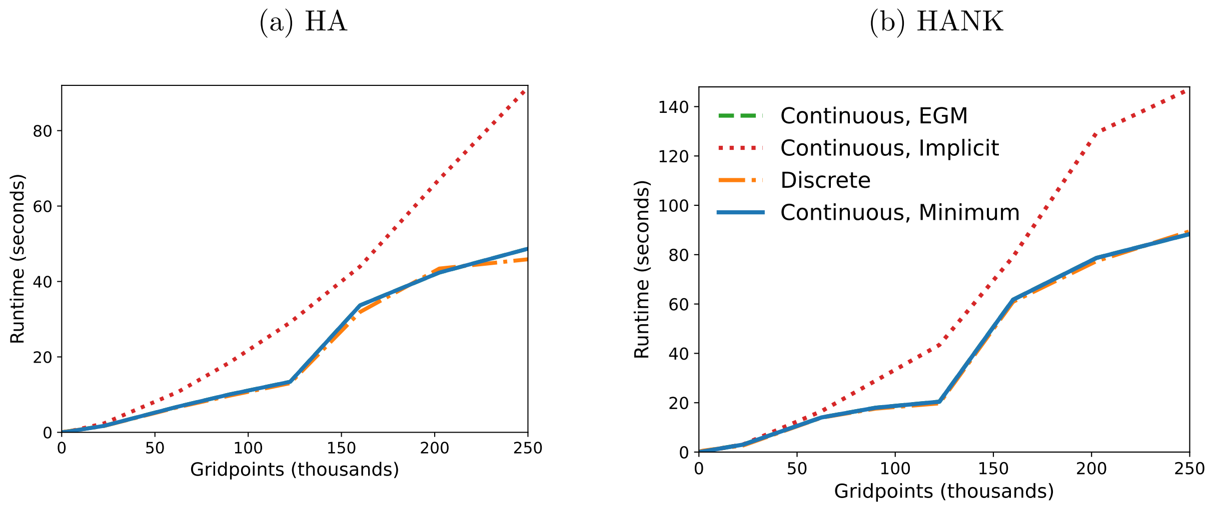
When we use the endogenous gridpoints method to calculate the steady-state, we compute $\frac{\partial V_i^{ss}(a)}{\partial a} = u'(c_i^{ss}(a))$ with a backward iteration. In this calculation, we leverage a (well-known) trick to reduce the dimensionality of the state space: we note that the idiosyncratic productivity process is independent of the household's asset level. This allows us to use the endogenous gridpoints method for the savings decision separately from evaluating the expectation of future idiosyncratic productivity.

We calculate the steady-state distribution with a forward iteration given policy functions. This iteration has comparable computational performance to inverting the linear system associated with the steady-state equation $\mathcal{L}g^{ss} = 0$. A root-finding algorithm on r then returns the steady state.

We find that the endogenous gridpoints method generally slightly outperforms the implicit method for a large number of gridpoints. For smaller numbers of gridpoints, the implicit method dominates. In fact, in other models or when the dimensionality of the productivity process is smaller, the implicit method outlined in Achdou et al. (2021) can be substantially superior. When the transition matrix is general—for instance, if the idiosyncratic productivity process was not independent from current assets—the discrete time calculation is significantly slower because we cannot leverage this separability anymore in the endogenous gridpoints method. In that case, the implicit method, which involves inverting a sparse $N_g \times N_g$ matrix (where N_g is the number of gridpoints) tends to be faster than the backward iteration.

Figure 6 displays steady-state calculation times for continuous and discrete time for multiple computation methods. Tables 5 and 6 report exact steady-state and transitional dynamics runtimes. The steady-state computation takes approximately the same amount of time under discrete time and continuous time with the endogenous gridpoint method. The implicit method is faster when the number of idiosyncratic productivity gridpoints is low but slower when it is high. The Jacobian calculation is consistently around three times faster in continuous time because of the faster calculation of policy functions.

Figure 6: Steady-state computation times



Note: Runtimes for steady-state calculation. Discrete time: with Endogenous Gridpoint Method (EGM, dash-dot orange). Continuous time: with Implicit scheme as in Achdou et al., 2021 (dotted red); with Endogenous Gridpoint Method (EGM, dashed green); and minimum of both methods (solid blue). Exact computation times in Tables 5 and 6.

Table 5: Runtime Data for HA (in seconds)

Type	Gridpoints	Steady State	Policy functions	Expectation vector	Fake news matrix	Jacobian	Inversion
Discrete	200	0.046	0.012	0.001	0.002	0.001	0.006
Continuous, EGM	200	0.046	0.005	0.001	0.004	0.001	0.006
Continuous, Implicit	200	0.035	0.005	0.001	0.002	0.001	0.006
Discrete	500	0.068	0.017	0.001	0.005	0.001	0.006
Continuous, EGM	500	0.065	0.007	0.002	0.007	0.001	0.006
Continuous, Implicit	500	0.049	0.007	0.002	0.003	0.001	0.006
Discrete	3,500	0.276	0.066	0.007	0.017	0.001	0.006
Continuous, EGM	3,500	0.275	0.033	0.007	0.018	0.001	0.005
Continuous, Implicit	3,500	0.205	0.033	0.007	0.010	0.001	0.006
Discrete	22,500	1.663	0.571	0.041	0.066	0.001	0.006
Continuous, EGM	22,500	1.696	0.182	0.045	0.068	0.001	0.005
Continuous, Implicit	22,500	2.337	0.182	0.045	0.068	0.001	0.006
Discrete	62,500	6.686	1.508	0.130	0.197	0.001	0.006

Continued on next page

Table 5: Runtime Data for HA (in seconds)

Type	Gridpoints	Steady State	Policy functions	Expectation vector	Fake news matrix	Jacobian	Inversion
Continuous, EGM	62,500	6.833	0.631	0.146	0.198	0.001	0.006
Continuous, Implicit	62,500	10.738	0.626	0.146	0.197	0.001	0.006
Discrete	90,000	9.721	2.152	0.182	0.288	0.001	0.006
Continuous, EGM	90,000	10.050	0.879	0.208	0.288	0.001	0.006
Continuous, Implicit	90,000	18.504	0.876	0.208	0.289	0.001	0.006
Discrete	122,500	13.035	3.221	0.231	0.394	0.001	0.006
Continuous, EGM	122,500	13.404	1.186	0.264	0.399	0.001	0.006
Continuous, Implicit	122,500	29.120	1.187	0.263	0.395	0.001	0.006
Discrete	160,000	31.919	6.452	0.762	0.518	0.001	0.005
Continuous, EGM	160,000	33.680	1.896	0.765	0.520	0.001	0.006
Continuous, Implicit	160,000	43.989	1.824	0.790	0.520	0.001	0.006
Discrete	202,500	43.427	7.840	0.856	0.661	0.001	0.006
Continuous, EGM	202,500	42.364	2.307	0.883	0.662	0.001	0.006
Continuous, Implicit	202,500	67.128	2.301	0.838	0.660	0.001	0.008
Discrete	250,000	45.881	10.507	0.992	0.854	0.001	0.019
Continuous, EGM	250,000	48.684	2.948	1.013	0.820	0.001	0.006
Continuous, Implicit	250,000	91.452	2.990	0.969	0.834	0.001	0.009

Table 6: Runtime Data for HANK (in seconds)

Type	Gridpoints	Steady State	Policy functions	Expectation vector	Fake news matrix	Jacobian	Inversion
Discrete	200	0.159	0.015	0.003	0.004	0.001	0.008
Continuous, EGM	200	0.113	0.005	0.001	0.004	0.001	0.013
Continuous, Implicit	200	0.061	0.005	0.001	0.002	0.001	0.009
Discrete	500	0.222	0.020	0.004	0.010	0.001	0.005
Continuous, EGM	500	0.180	0.007	0.002	0.010	0.001	0.008

Continued on next page

Table 6: Runtime Data for HANK (in seconds)

Type	Gridpoints	Steady State	Policy functions	Expectation vector	Fake news matrix	Jacobian	Inversion
Continuous, Implicit	500	0.086	0.007	0.002	0.003	0.001	0.010
Discrete	2,200	0.545	0.054	0.007	0.028	0.001	0.007
Continuous, EGM	2,200	0.372	0.019	0.005	0.029	0.001	0.009
Continuous, Implicit	2,200	0.264	0.019	0.005	0.009	0.001	0.025
Discrete	22,500	2.706	0.583	0.047	0.116	0.001	0.007
Continuous, EGM	22,500	3.026	0.204	0.045	0.116	0.001	0.010
Continuous, Implicit	22,500	3.073	0.200	0.045	0.075	0.001	0.010
Discrete	62,500	13.986	2.335	0.269	0.207	0.002	0.007
Continuous, EGM	62,500	14.039	0.585	0.206	0.205	0.001	0.009
Continuous, Implicit	62,500	16.588	0.782	0.206	0.205	0.001	0.010
Discrete	90,000	17.635	3.698	0.358	0.297	0.002	0.007
Continuous, EGM	90,000	17.960	1.124	0.268	0.298	0.001	0.009
Continuous, Implicit	90,000	28.929	1.131	0.269	0.298	0.002	0.011
Discrete	122,500	19.772	4.241	0.459	0.409	0.001	0.008
Continuous, EGM	122,500	20.409	1.363	0.252	0.415	0.001	0.007
Continuous, Implicit	122,500	43.429	1.204	0.256	0.418	0.001	0.009
Discrete	160,000	60.915	6.004	1.226	0.532	0.001	0.005
Continuous, EGM	160,000	61.736	1.932	0.763	0.542	0.001	0.009
Continuous, Implicit	160,000	79.113	1.940	0.752	0.535	0.001	0.009
Discrete	202,500	77.391	7.721	1.274	0.655	0.001	0.005
Continuous, EGM	202,500	78.673	2.474	0.856	0.660	0.001	0.007
Continuous, Implicit	202,500	129.377	2.461	0.882	0.658	0.001	0.007
Discrete	250,000	89.462	9.212	1.476	0.811	0.001	0.005
Continuous, EGM	250,000	88.350	3.059	0.996	0.831	0.001	0.007
Continuous, Implicit	250,000	147.122	2.970	0.993	0.842	0.001	0.008

Online Appendix

D Weak derivatives and duality

D.1 Weak derivatives

Let f be a η-measurable function on some η-measurable space X that is a subset of a finite-dimensional Euclidian space. Its weak derivative $\frac{\partial f}{\partial x_i}$ is, when it exists, defined by duality. Suppose that there exists a η-measurable function w_i such that

$$\int f(x)\frac{\partial \varphi}{\partial x_i}(x)d\eta(x) = -\int w_i(x)\varphi(x)d\eta(x)$$

for all continuously differentiable functions φ that vanish on the boundary ∂X. In that case, define its weak derivative $\frac{\partial f}{\partial x_i}$ to be w_i.

D.2 Borrowing constraints and adjoint property

Note: This section is a modified version of the corresponding section in Bilal (2023) and shows that the duality property holds more generally when using weak derivatives.

For concreteness, consider the Aiyagari (1994) economy with a borrowing constraint $a \geq \underline{a}$. The asset domain is $[\underline{a}, \overline{a}]$. For simplicity, focus only on the asset dimension. Consider the operators associated with the asset drift $L \equiv S(a)\partial_a$ and $L^* \equiv -\partial_a S(a)$, where S denotes the savings rate. Consider the base measure $d\eta(a) = \delta_{\underline{a}}(a) + da$. A distribution is given by a density $\{g(\underline{a}); g(a), a > \underline{a}\}$, so that the probability measure that represents the distribution of individuals is $g(\underline{a})\delta_{\underline{a}}(a) + g(a)da$. For any function f, we denote by $f(\underline{a}^+) = \lim_{a \downarrow \underline{a}} f(a)$.

A mass point arises at $\underline{a}$ if $s(a) \leq 0$ in a neighborhood of $\underline{a}$. The borrowing constraint imposes $s(\underline{a}) = 0$. The presence of a mass point requires that $g(\underline{a}^+) = +\infty$ so that the (asset-induced) inflow into the mass point, $(sg)(\underline{a}^+)$, is finite and non-zero. Integration by parts implies that the weak derivative of sg at $\underline{a}$ (with respect to the base measure η) is:

$$\partial_a(s(a)g(a)) = \begin{cases} (sg)(\underline{a}^+) & \text{if} \quad a = \underline{a} \\ \text{the classical derivative } \partial_a(s(a)g(a)) & \text{if} \quad a > \underline{a} \end{cases}$$

Thus, for any smooth test function φ that vanishes at $a = \overline{a}$ and any density g with respect to the base measure:

$$
\begin{aligned}
\int_{\underline{a}}^{\overline{a}} \varphi(a) L^*(a)[g] d\eta(a) &= -\varphi(\underline{a})(sg)(\underline{a}^+) - \int_{\underline{a}^+}^{\overline{a}} \varphi(a)\partial_a(s(a)g(a))da \\
&= -\varphi(\underline{a})(sg)(\underline{a}^+) - \Big[sg\varphi\Big]_{\underline{a}^+}^{\overline{a}} + \int_{\underline{a}^+}^{\overline{a}} s(a)g(a)\partial_a\varphi(a)da \\
&= \int_{\underline{a}^+}^{\overline{a}} L(a)[\varphi]g(a)da \\
&= \int_{\underline{a}}^{\overline{a}} L(a)[\varphi]g(a)d\eta(a) - \underbrace{s(\underline{a})g(\underline{a})}_{=0}\varphi'(\underline{a}) \\
&= \int_{\underline{a}}^{\overline{a}} L(a)[\varphi]g(a)d\eta(a),
\end{aligned}
$$

where the first line uses the definition of $d\eta(a)$ and the expression for the weak derivative of sg. The second line integrates the second term by parts. The third line cancels out terms and uses that $g(\overline{a}) = 0$. The fourth lines substracts the $\underline{a}$ terms to make the integral over the closed interval appear. The fifth line recognizes that the extra term is equal to zero.

This derivation proves that L and L^* are adjoints on the domain $[\underline{a}, \overline{a}]$, when L is defined on the space of smooth functions that vanish at $\overline{a}$, and L^* is defined on the space of densities with respect to the base measure η. This conclusion extends immediately to including an income process in the operators L, L^*.

E Information aggregation

We provide context for information aggregation by repeating our derivations in discrete time, but assuming that the borrowing constraint never binds. This is not an innocuous restriction, because of the well-known result that an occasionally binding constraint is necessary for a non-degenerate steady-state to exist. Thus, our discrete time derivations are illustrative only.

E.1 Setup

We start with the discrete time problem with a time step ε. We sometimes omit the dependence on income and write $X_t(a)$ instead of $X_t(a, y)$. When we write $V_t'(a, y)$ we always mean the derivative with respect to assets, and we sometimes write $V_t'(a)$.

The Bellman equation is:

$$
V_t(a, y) = \max_c \varepsilon u(c) + e^{-\rho\varepsilon}\mathbb{E}_t\Big[V_{t+\varepsilon}\Big(e^{r_t\varepsilon}a + (y - c)\varepsilon, y_{t+\varepsilon}\Big)\Big| a_t = a, y_t = y\Big],
$$

where we used the budget constraint $a_{t+\varepsilon} = e^{r_t\varepsilon}a_t + (y_t - c_t)\varepsilon$ and there is a borrowing constraint $a_{t+\varepsilon} \geq \underline{a}$. The first-order and envelope conditions are:

$$\varepsilon u'(c_t) = \varepsilon e^{-\rho\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(a_{t+\varepsilon}, y_{t+\varepsilon}\Big) \quad \Longrightarrow \quad u'(c_t) = e^{-\rho\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(a_{t+\varepsilon}, y_{t+\varepsilon}\Big)$$
$$V'_t(a,y) = e^{(r_t-\rho)\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(a_{t+\varepsilon}, y_{t+\varepsilon}\Big).$$

Combining both, we obtain:

$$u'(c_t) = e^{-r_t\varepsilon}V'_t(a,y) \to_{\varepsilon\downarrow 0} V'_t(a,y) \qquad\qquad u'(c_t) = e^{-\rho\varepsilon}\mathbb{E}_t[e^{r_{t+\varepsilon}\varepsilon}u'(c_{t+\varepsilon})].$$

E.2 A first derivation of the contemporaneous consumption response

Consider a perturbation of the interest rate sequence such that r_t only changes in period t by a small $\widehat{r}_t$, and then goes back to baseline. Denote with tildes values and assets under the alternative path, and with hats the difference between original and new variables. We consider what happens from t onward.

Consumption $c_t(a)$ is the implicit solution to:

$$u'(c_t(a)) = e^{-\rho\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(e^{r_t\varepsilon}a + y_t\varepsilon - c_t(a)\varepsilon, y_{t+\varepsilon}\Big).$$

For a change in the interest rate by $\widehat{r}_t$, we obtain:

$$u'(\widetilde{c}_t(a)) = e^{-\rho\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(e^{r_t\varepsilon}a + e^{r_t\varepsilon}\widehat{r}_t\varepsilon a + y_t\varepsilon - \widetilde{c}_t(a)\varepsilon\Big),$$

where importantly the value $V_{t+\varepsilon}$ is the same because from $t+\varepsilon$ onward we revert back to the original interest rate sequence. We re-arrange to express the second condition in terms of a different level of initial assets $\overline{a}$:

$$u'(\widetilde{c}_t(a)) = e^{-\rho\varepsilon}\mathbb{E}_t V'_{t+\varepsilon}\Big(e^{r_t\varepsilon}\overline{a} + y_t\varepsilon - \widetilde{c}_t(a)\varepsilon\Big), \quad \overline{a} \equiv a\left(1 + \widehat{r}_t\varepsilon\right).$$

Thus, we obtain $\widetilde{c}_t(a) = c_t(\overline{a})$, and so to first order in $\widehat{r}_t$:

$$\widehat{c}_t(a) = c'_t(a)(\overline{a} - a) = c'_t(a)\widehat{r}_t a\varepsilon.$$

E.3 Perturbation of values and consumption

We now characterize the perturbation of marginal values in discrete time for a general change in the interest rate sequence, abstracting from binding constraints. We start from:

$$\widehat{V}_t(a,y) = e^{-\rho\varepsilon}\mathbb{E}_t\widehat{V}_{t+\varepsilon}(a_{t+\varepsilon}, y_{t+\varepsilon}) + u'(c_t)\varepsilon e^{r^{ss}\varepsilon}\widehat{r}_t a.$$

Differentiating w.r.t. a and evaluating around steady-state, we obtain:

$$\widehat{V}_t'(a,y) = e^{-\rho\varepsilon}\big(e^{r^{ss}\varepsilon} - \varepsilon c'^{ss}(a,y)\big)\mathbb{E}_t\widehat{V}_{t+\varepsilon}'(a_{t+\varepsilon}, y_{t+\varepsilon}) + \varepsilon e^{r^{ss}\varepsilon}\widehat{r}_t\frac{\partial[au'(c^{ss})]}{\partial a}.$$

We now posit:

$$\widehat{V}_t'(a,y) = \sum_{s=0}^{\infty} e^{-s\rho\varepsilon}\varepsilon\phi_{s\varepsilon}(a,y)e^{r^{ss}\varepsilon}\widehat{r}_{t+s\varepsilon},$$

We calculate:

$$e^{-\rho\varepsilon}\widehat{V}_{t+\varepsilon}'(a,y) = \sum_{s=0}^{\infty} e^{-\rho\varepsilon(s+1)}\varepsilon\phi_{s\varepsilon}(a,y)e^{r_{t+(s+1)\varepsilon}}\widehat{r}_{t+(s+1)\varepsilon} = \sum_{s=1}^{\infty} e^{-\rho\varepsilon s}\varepsilon\phi_{(s-1)\varepsilon}(a,y)e^{r_{t+s\varepsilon}}\widehat{r}_{t+s\varepsilon}.$$

Substituting into the Euler equation, we obtain:

$$\sum_{s=0}^{\infty} e^{-s\rho\varepsilon}\varepsilon\phi_{s\varepsilon}(a,y)e^{r^{ss}\varepsilon}\widehat{r}_{t+s\varepsilon} = \big(e^{r_t\varepsilon} - \varepsilon c'^{ss}(a,y)\big)\sum_{s=1}^{\infty} e^{-\rho\varepsilon s}\varepsilon\mathbb{E}_t[\phi_{(s-1)\varepsilon}(a_{t+\varepsilon}, y_{t+\varepsilon})]e^{r^{ss}\varepsilon}\widehat{r}_{t+s\varepsilon} + \varepsilon e^{r^{ss}\varepsilon}\widehat{r}_t\frac{\partial[au'(c^{ss})]}{\partial a}.$$

Dividing through by ε and identifying coefficients:

$$\begin{aligned}
\phi_0(a,y) &= \frac{\partial[au'(c^{ss})]}{\partial a} \\
\phi_{s\varepsilon}(a,y) &= \big(e^{r^{ss}\varepsilon} - \varepsilon c'^{ss}(a,y)\big)\mathbb{E}\Big[\phi_{(s-1)\varepsilon}(a_{t+\varepsilon}, y_{t+\varepsilon}) \;\Big|\; a_t = a, y_t = y\Big].
\end{aligned}$$

These equations mirror those in Proposition 1 when the borrowing constraint never binds in discrete time.

E.4 Consumption

We now characterize consumption. Recall that from the first-order condition: $u'(c_t(a,y)) = e^{-r_t\varepsilon}V_t'(a,y)$. The adjustment by $e^{-r_t\varepsilon}$ is key. To first order:

$$u''(c^{ss}(a))\widehat{c}_t(a) = e^{-r^{ss}\varepsilon}\Big\{\widehat{V}_t'(a) - V'^{ss}(a)\widehat{r}_t\varepsilon\Big\}$$

For an interest rate shock only at time t, we have:

$$\begin{aligned}
u''(c^{ss}(a))\widehat{c}_t(a) &= e^{-r^{ss}\varepsilon}\Big\{\partial_a(\varepsilon e^{r^{ss}\varepsilon}\widehat{r}_t a u'(c^{ss}(a))) - e^{r^{ss}\varepsilon}u'(c^{ss}(a))\widehat{r}_t\varepsilon\Big\} \\
&= \varepsilon\widehat{r}_t\Big\{u'(c^{ss}(a)) + au''(c^{ss}(a))c'^{ss}(a) - u'(c^{ss}(a))\Big\} \\
&= \varepsilon\widehat{r}_t \times au''(c^{ss}(a))c'^{ss}(a),
\end{aligned}$$

where in the second equality we used $V'^{ss}(a) = e^{r^{ss}\varepsilon}u'(c^{ss}(a))$.

The discontinuity in the Jacobian $\frac{\partial c_t}{\partial r_{t+s\varepsilon}}$ at $s = 0$ also arises in discrete time because the interest rate r_t enters the first-order condition $u'(c_t(a,y)) = e^{-r_t\varepsilon}V_t'(a)$ separately, while all the other interest rates $r_{t+s\varepsilon}, s \geq 1$ only enter through the marginal value $V_t'(a)$.

F General framework

F.1 Setup

We now describe a general economy that encompasses many applications of interest. The setup follows closely Bilal (2023). Time $t \geq 0$ is continuous and runs forever. The economy is populated by a unit measure of agents. Agents are characterized by their individual state vector $x \in \overline{X}_0 \subset \mathbb{R}^{D_X}$, where $X_0 = (\underline{x}_1, \overline{x}_1) \times ... \times (\underline{x}_{D_X}, \overline{x}_{D_X})$ is a D_X-dimensional hypercube.[9] Let $\overline{X}_0$ denote its closure in the Euclidean norm. $\overline{X}_0$ is endowed with the Borel $\sigma-$algebra, and a base probability measure η. Individuals may choose a control variable $c \in \overline{\Gamma} \subset \mathbb{R}^{D_C}$, where Γ is open.[10]

The base probability measure η plays a key role in the sequel. It encodes a priori information about where the distribution is absolutely continuous, and where it may develop mass points. If the example of Section 1 was enriched with an occasionally binding borrowing constraint $a \geq \underline{a}$, one would define $d\eta(a,y) = (da + \delta_{\{\underline{a}\}}(da)) \otimes dy$, where $\delta_{\{\underline{a}\}}$ denotes the Dirac measure at $\underline{a}$, and $\otimes$ denotes the tensor product of measures. This definition then allows for the possibility of a mass point at the borrowing constraint $\underline{a}$.

The base measure allows to handle only densities with respect to that base measure, and thus treats mass points and smooth densities symmetrically.[11] In the sequel, we always impose that the base measure is a product measure of marginal measures. The marginal measure along dimension i in turn consists of the Lebesgue measure together with a countable set of possible mass points. These possible mass points are located on a $D_X - 1$-dimensional manifold B that may include parts of the boundary of X_0—for instance when there are credit constraints—and may also include points in the interior of X_0—for instance when there are kinks in the interest

[9]Working with a hypercube is not strictly necessary for most of the results below. However, it makes the notation much lighter—in a relative sense—to handle mass points. Without mass points, virtually all the results below go through for a general open domain X without additional notation.

[10]To keep the exposition as concise as possible, discussion of filtrations and adaptedness are omitted. See Carmona and Delarue (2018) for an in-depth exposition.

[11]In principle, it is possible to work without a base measure. In that case, the law of motion of the distribution is set in the space of measures. Consequently, one needs to develop the formalism of derivatives with respect to general measures in the Wasserstein space. See Cardaliaguet et al. (2019) for details. Introducing this formalism is beyond the scope of this paper, and is also irrelevant for many economic applications of interest in which a priori knowledge of where mass points may develop is often available.

rate. We assume that this manifold intersects any direction i at a countable number of points $\{x_{in}\}$ only. We denote by $X = X_0 \cup B$ the domain of the state x as the union of the open domain X together with the set of possible mass points B.

Assumption 1. *(Base measure)*
$d\eta(x) = d\eta_1(x_1) \otimes ... \otimes d\eta_{D_X}(x_{D_X})$, where, for all $i = 1...D_X$ and all Borel-measurable subset $Z \subset [\underline{x}_i, \overline{x}_i]$, $\eta_i(Z) = \ell(Z_i) + \sum_{n \in \mathcal{N}_i} \mathbb{1}\{x_{in} \in Z\}$, where $\ell(Z)$ denotes the Lebesgue measure of Z.

The relevant state variables of the economy are a vector of aggregate shocks $z_t \in \mathbb{R}^S$, and the current distribution of agents over the state space X, denoted by $g_t(x)d\eta(x)$. Crucially, we consider only densities with respect to the base measure η.

Agents' state x_t evolves over time according to a controlled jump-diffusion process

$$dx_t = b(x_t, c_t, p_t)dt + \sigma(x_t, c_t, p_t) \cdot dW_t + \int (y - x_t) f(x_t, c_t, p_t, y) d\eta(y) N(dt). \tag{23}$$

b is the $\mathbb{R}^{D_X}$-valued drift of the process. It may depend directly on the current state x_t, the control c_t and prices p_t.[12] Similarly, σ is the $\mathbb{R}^{D_X \times D_W}$-valued function volatility of the process. W is independent across agents. The Poisson jump measure N encodes the frequency of jump increments. The density f captures the density of increments and their frequency over the base measures η, N. Jumps are also independent across agents.

The state x_t may include discrete indicators for different types of agents, for instance employed workers and unemployed workers, workers and firms, regions or countries. The process x_t is assumed to remain within $\overline{X}$, either through reflection at the boundary of X, or through an appropriate combination of drift and volatility at the boundary. Without further restrictions, equation (23) encompasses a large class of stochastic processes in continuous time, and nests the vast majority of those commonly used in macroeconomic applications.

We use the notion of weak derivatives to handle mass points in the distribution.[13] When a η-measurable function f is continuously differentiable, the weak derivative coincides with the classical derivative. When f has a jump in direction i at some $x_0 \in X$, then the weak derivative in the sense of generalized functions is a measure: it is a Dirac mass point multiplied by the size of the jump: $\frac{\partial f}{\partial x_i}(x_0)dx \equiv \Big(\lim_{\varepsilon \downarrow 0} f(x_0 + \varepsilon \tau_i) - \lim_{\varepsilon \downarrow 0} f(x_0 - \varepsilon \tau_i)\Big)\delta_{x_0}(dx)$, where τ_i is the unit vector pointing in direction i. In that case, f admits a weak derivative only if $x_0 \in M_i$, the set of mass points. Then, the weak derivative as a η-measurable function simply the size of

[12] That it does not depend on calendar time t is without loss of generality: one can always redefine one entry of z_t to be time t, and one entry of p_t time.

[13] See Online Appendix D.1 for details

the jump at x_0: $\lim_{\varepsilon\downarrow 0} f(x_0+\varepsilon\tau_i) - \lim_{\varepsilon\downarrow 0} f(x_0-\varepsilon\tau_i)$. Thus, we can work only with densities with respect to the base measure.

Two functional spaces are useful in the sequel. The first is the space of square-integrable functions with respect to the base measure η. The second is the second Sobolev space:

$$L^2 \equiv \left\{ f : X \to \mathbb{R} \text{ is } \eta\text{-measurable} \,\Big|\, \int f(x)^2 d\eta(x) < \infty \right\}, \qquad H^2 \equiv \left\{ f \in L^2 \,\Big|\, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2} \in L^2 \right\}. \tag{24}$$

The notation L^2 should not be confused with the notation for the generator of the process, L. The second Sobolev space H^2 consists of all square-integrable functions that have square-integrable first and second weak derivatives.

We now define three functional operators related to the state process x_t. The first operator L is the generator of the state process and encodes conditional expectations of functions of x_t. For any $V \in H^2$, define:[14]

$$\begin{aligned} L(x,c,p)[V] &= \sum_{i=1}^{D_X} b_i(x,c,p)\frac{\partial V}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{D_X} \Sigma(x,c,p)\frac{\partial^2 V}{\partial x_i \partial x_j}(x) \\ &+ \int f(x,c,p,y)\Big(V(y)-V(x)\Big)d\eta(y), \end{aligned} \tag{25}$$

where $\Sigma = \sigma\sigma^T$, and recall that the ^{T} superscript denotes the matrix transpose.

The second operator is the formal adjoint operator L^*.[15] It encodes how the cross-sectional probability distribution of the process x_t evolves over time. For any $g \in H^2$, define:

$$\begin{aligned} L^*(x,c,p)[g] &= -\sum_{i=1}^{D_X} \frac{\partial}{\partial x_i}\Big(g(x)b_i(x,c,p)\Big) + \frac{1}{2}\sum_{i,j=1}^{D_X} \frac{\partial^2}{\partial x_i \partial x_j}\Big(\Sigma(x,c,p)g(x)\Big) \\ &+ \int f(y,c,p,x,\nu)g(y)d\eta(y) - g(x)\int f(x,c,y,\nu)d\eta(y). \end{aligned} \tag{26}$$

Aggregate shocks are denoted z_t. They follow a stationary Feller process with generator $\mathcal{A}(z)$ that admits a unique invariant distribution.

Finally, we posit the existence of a price functional $p(z,g)$ that maps aggregate shocks z_t and the distribution g_t into the vector of prices p_t. This price functional typically arises from market clearing conditions. However, the vector p_t can also represent any alterative moment of the distribution that is relevant for individual decisions.

[14]For points on the boundary of $\overline{X}$, $x \in \partial X$, it is understood that derivatives are taken with respect to the interior directions to X, and all functions are extended by 0 outside of $\widehat{X}$.

[15]See Appendix D.2 for the case with mass points in the distribution.

F.2 Individual decision problem and distribution

Armed with the notation above, we define agents' decision problem. They solve the following time-dependent Bellman equation with possible constraints on the state variable:

$$\rho V_t(x) = \max_{c \in \overline{C}} u(x, c, z_t, p_t, V_t) + L(x, c, p)[V_t] + \mathbb{E}_t \left[\frac{\partial V_t}{\partial t}(x) \right] \quad \text{s.t.} \quad C\left(x, p, V_t(x), \frac{\partial V_t}{\partial x}(x)\right) \geq 0, \;\; x \in B \tag{27}$$

In equation (27), the flow payoff u may depend on the current state, but also the value function V directly. This formulation embeds recursive preferences such as Epstein-Zin. The flow payoff also depends on aggregate shocks z directly, to allow for aggregate demand shocks.[16] We assume that the problem is concave: $u_{cc} + L_{cc}[V] < 0$ for all x, c, V.

The function C captures constraints on the state x_t.[17] For instance, if there was a credit constraint $a_t \geq \underline{a}$ in the economy of Section 1, consumption c must be such that $r_t\underline{a} + w - c \geq 0$, i.e. $c \leq r_t\underline{a} + w$. This constraint on the control may equivalently be re-stated on the value function by $\frac{\partial V_t}{\partial a} \geq u'(r_t\underline{a} + wy)$. Therefore, $C\left(a, y, r, V_t, \frac{\partial V_t}{\partial a}\right) = \frac{\partial V_t}{\partial a} - u'(r_t a + wy)$ for $a = \underline{a}$. The state constraint inequality holds for all x in a set B that we assume without loss of generality to be included in the set of possible mass points as well as the boundary of the domain: $B \subset M \cap \partial X$.

It is useful to define the evolution of the distribution in terms of the density $g_t(x)$ with respect to the base measure η in order to accommodate the presence of possible mass points. g_t satisfies the law of motion:

$$\frac{\partial g_t}{\partial t}(x) = L^*(x, \overline{c}_t(x), p_t)[g_t], \tag{28}$$

where $\overline{c}_t(x)$ denotes the optimal control. The notion of weak derivative with respect to the base measure η is key to systematically handle mass points in the evolution of the distribution. As long as mass points develop only where η allows them to, the weak derivative is well-defined in the space H^2: for all $g \in H^2$, $L^*(x, \overline{c}_t(x), p_t)[g] \in L^2$.

When the economy is stationary, the Bellman equation (27) becomes

$$\rho V(x) \;=\; \max_{c \in \overline{C}} u(x, c, z, p, V) + L(x, c, p)[V] \quad \text{s.t.} \quad \mathcal{C}\left(x, p, V(x), \frac{\partial V}{\partial x}(x)\right) \geq 0, \;\; x \in B \tag{29}$$

where $p = p(0, g)$, and the law of motion of the distribution (28) becomes

$$0 = L^*(x, \widehat{c}(x), p)[g] \tag{30}$$

[16] It is also possible to include idiosyncratic demand shocks with additional notation.

[17] See Fleming and Soner (2006) for more details.

F.3 Policy functions

We can now generalize the results from Section 1. We consider again only perfect foresight transitional dynamics. We start with the sequence-space Jacobian of the value function. Denote by $\mathcal{B}$ the set of points in which the state constraint binds in the steady-state.

Theorem 1. *(Value function sequence-space Jacobians)*
To first order in $\widehat{p}, \widehat{z}$, $V_t(x) = V^{SS}(x) + \widehat{V}_t(x)$, where:

$$\widehat{V}_t(x) = \int_t^\infty e^{-\rho(s-t)} \Big\{ \widehat{p}_s^* \varphi_{s-t}(x) + \widehat{z}_s^* \phi_{s-t}(x) \Big\} ds,$$

and where $\varphi_t(x), \phi_t(x) \in \mathbb{R}^P$ satisfy, for all $x \in X \backslash \mathcal{B}$:

$$\begin{aligned} \frac{\partial \varphi_t}{\partial t} &= \mathcal{L}\varphi_t \quad , \quad \varphi_0 = u_p + \mathcal{L}_p V^{SS} \\ \frac{\partial \phi_t}{\partial t} &= \mathcal{L}\phi_t \quad , \quad \phi_0 = u_z, \end{aligned}$$

and for all $x \in \mathcal{B}$:

$$\begin{aligned} \mathcal{C}_V(x)^* \varphi_0(x) + \mathcal{C}_{\partial V}(x)^* \partial_x \varphi_0(x) &= -\mathcal{C}_p(x) \\ \mathcal{C}_V(x)^* \varphi_t(x) + \mathcal{C}_{\partial V}(x)^* \partial_x \varphi_t(x) &= 0 \qquad \textit{for } t > 0 \\ \mathcal{C}_V(x)^* \phi_t(x) + \mathcal{C}_{\partial V}(x)^* \partial_x \phi_t(x) &= 0 \qquad \textit{for all } t \geq 0. \end{aligned}$$

Proof. To first order, let:

$$V_t(x) = V^{SS}(x) + \widehat{V}_t(x). \tag{31}$$

Substitute (31) into the HJB (27) and using the envelope condition, we obtain to first order:

$$\rho \widehat{V}_t(x) \quad = \quad \big\{ u_p(x) + \mathcal{L}_p(x)[V^{SS}] \big\} \cdot \widehat{p}_t + u_z(x) \cdot \widehat{z}_t + \mathcal{L}(x)[\widehat{V}_t] + \frac{\partial \widehat{V}_t(x)}{\partial t} \tag{32}$$

Now posit:

$$\widehat{V}_t(x) = \int_t^\infty e^{-\rho(s-t)} \Big\{ \varphi_{s-t}(x) \cdot \widehat{p}_s + \phi_{s-t}(x) \cdot \widehat{z}_s \Big\} ds.$$

Substituting into (32) and rearranging, we obtain:

$$\begin{aligned} 0 = &\int_t^\infty e^{-\rho(s-t)} \Bigg\{ \left(\mathcal{L}(x)[\varphi_{s-t}] - \frac{\partial \varphi_{s-t}}{\partial s} \right) \cdot \widehat{p}_s + \left(\mathcal{L}(x)[\phi_{s-t}] - \frac{\partial \phi_{s-t}}{\partial s} \right) \cdot \widehat{z}_s \Bigg\} ds \\ &+ \Big\{ u_p(x) + \mathcal{L}_p(x)[V^{SS}] - \varphi_0(x) \Big\} \cdot \widehat{p}_t + \Big\{ u_z(x) - \phi_0(x) \Big\} \cdot \widehat{z}_t. \end{aligned} \tag{33}$$

Equation (33) must hold for all sequences $\widehat{p}_t, \widehat{z}_t$. Identifying coefficients, we obtain the restrictions:

$$\begin{aligned}
\varphi_0(x) &= u_p(x) + \mathcal{L}_p(x)[V^{SS}] \quad , \quad \frac{\partial \varphi_t(x)}{\partial t} = \mathcal{L}(x)[\varphi_t] \\
\phi_0(x) &= u_z(x) \quad , \quad \frac{\partial \phi_t(x)}{\partial t} = \mathcal{L}(x)[\phi_t].
\end{aligned}$$

Finally, consider the state constraint boundary condition. As shown in Bilal (2023), in the first-order perturbation it is sufficient to consider that it binds at the same points as in the steady-state. Linearizing the state constraint condition, for all $x \in \mathcal{B}$:

$$0 = \mathcal{C}_p \cdot \widehat{p}_t + \mathcal{C}_V \cdot \widehat{V}_t + \mathcal{C}_{\partial V} \cdot \partial_x \widehat{V}_t.$$

Substituting in the expression for $\widehat{V}_t$, we obtain

$$0 = \mathcal{C}_p \cdot \widehat{p}_t + \int_t^\infty e^{-\rho(s-t)} \Big[\mathcal{C}_V \cdot \Big(\varphi_{s-t} \cdot \widehat{p}_s + \phi_{s-t} \cdot \widehat{z}_s \Big) + \mathcal{C}_{\partial V} \cdot \Big(\partial_x \varphi_{s-t} \cdot \widehat{p}_s + \partial_x \phi_{s-t} \cdot \widehat{z}_s \Big) \Big] ds.$$

Assuming that the Jacobian of the state constraint function is invertible, and identifying coefficients, we obtain for $x \in \mathcal{B}$:

$$\begin{aligned}
0 &= \mathcal{C}_V(x) \cdot \varphi_{s-t}(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \varphi_{s-t}(x) && \text{if} \quad s > t \\
0 &= \mathcal{C}_p(x) + \mathcal{C}_V(x) \cdot \varphi_{s-t}(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \varphi_{s-t}(x) && \text{if} \quad s = t \\
0 &= \mathcal{C}_V(x) \cdot \phi_{s-t}(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \phi_{s-t}(x) && \text{for all} \quad s \geq t.
\end{aligned}$$

Re-indexing, we obtain:

$$\begin{aligned}
0 &= \mathcal{C}_V(x) \cdot \varphi_t(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \varphi_t(x) && \text{if} \quad t > 0 \\
0 &= \mathcal{C}_p(x) + \mathcal{C}_V(x) \cdot \varphi_0(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \varphi_0(x) && \\
0 &= \mathcal{C}_V(x) \cdot \phi_t(x) + \mathcal{C}_{\partial V}(x) \cdot \partial_x \phi_t(x) && \text{for} \quad t \geq 0.
\end{aligned}$$

Q.E.D. □

An immediate implication of Theorem 1 is a similar linearization of individual policy functions. To keep the exposition simple, we further assume that $L_c = -\sum_i \partial_{x_i}$. While this assumption is purely expositional and is not necessary for the results to go through, it simplifies the formulas.

Corollary 4. *(Policy functions sequence-space Jacobians)*
Define $\mathcal{U} = u_{cc} + \mathcal{L}_{cc}[V^{SS}]$. To first order in $\widehat{p}, \widehat{z}, \widehat{g}_0$:

$$\widehat{c}_t(x) = -\mathcal{U}(x)^{-1} \Big\{ u_{cp}(x) \widehat{p}_t + \mathcal{L}_c(x)[\psi_t] + u_{cz}(x)^* \widehat{z}_t \Big\}.$$

Proof. Start from the first-order optimality condition:

$$0 = u_c(x, \bar{c}_t, z_t, p_t) + L_c(x, \bar{c}_t, p_t)[V_t].$$

Totally differentiating:

$$0 = u_{cc} \cdot \widehat{c}_t + u_{cp} \cdot \widehat{p}_t + u_{cz} \cdot \widehat{z}_t + \left(\mathcal{L}_{cc}[V^{SS}]\right) \cdot \widehat{c}_t(x) + \left(\mathcal{L}_{cp}[V^{SS}]\right) \cdot \widehat{p}_t + \mathcal{L}_c(x)[\psi_t].$$

Let $\mathcal{U}(x) = u_{cc}(x) + \mathcal{L}_{cc}(x)[V^{SS}]$. Re-arranging:

$$\widehat{c}_t(x) = -\mathcal{U}(x)^{-1} \left\{ u_{cp} \cdot \widehat{p}_t + u_{cz} \cdot \widehat{z}_t + \mathcal{L}_{cp}(x)[V^{SS}] \cdot \widehat{p}_t + \mathcal{L}_c(x)[\psi_t] \right\}.$$

Imposing $L_c = -\partial_x$ and so $\mathcal{L}_{cp} = 0$ concludes the proof. *Q.E.D.* □

Corollary 4 leads to useful definitions for the sequel. Similarly to Section 1, we define the change in the distribution that results from a price change t periods in the future as

$$\mathcal{D}_t^p = e^{-\rho t} \mathcal{M} \varphi_t - \boldsymbol{\delta}(t) \mathcal{P},$$

where $\mathcal{M} = -\mathcal{L}_c^*(g^{SS} \mathcal{U}^{-1} \mathcal{L}_c)$ translates changes in values to changes in decisions. $\mathcal{P} = \mathcal{L}_p^* g^{SS} - \mathcal{L}_c^* \mathcal{U}^{-1} u_{cp}$ represents the direct impact of price changes on decisions. $\boldsymbol{\delta}(t)$ denotes a Dirac delta mass function at time 0. Similarly, we define the change in the distribution that results from an aggregate shock t periods in the future as

$$\mathcal{D}_t^z = e^{-\rho t} \mathcal{M} \phi_t - \boldsymbol{\delta}(t) \mathcal{Z},$$

where $\mathcal{Z} = -\mathcal{L}_c^*(x) \mathcal{U}^{-1} u_{cz}$. The specific structure of how controls respond to price shocks delivers crucial structure that we summarize in the following result.

Corollary 5. *(Control operator)*
The operator $\mathcal{M}$ is self-adjoint and non-positive definite.

Proof. Direct application of the definition of $\mathcal{M}$ and concavity of the maximization problem. □

F.4 Distribution and prices

With the linearized policy functions at hand, we provide an analytic linearization of the law of motion of the distribution $g_t = g^{SS} + \widehat{g}_t$.

Theorem 2. *(Distribution sequence-space Jacobians)*

To first order in $\widehat{p}, \widehat{z}, \widehat{g}_0$:

$$\widehat{g}_t = T_t^* \widehat{g}_0 + \int_0^\infty \left(J_{t,s}^p \widehat{p}_s + J_{t,s}^z \widehat{z}_s\right) ds,$$

where the Jacobians $J_{t,s}^p, J_{t,s}^z$ only depend on φ, ϕ and steady-state objects, and are given in equations (35)-(36) in the proof.

Proof. Start from the law of motion:

$$\frac{dg_t}{dt} = L^*(x, \overline{c}_t(x), p_t)[g_t].$$

Totally differentiating it and using Corollary 4:

$$\frac{d\widehat{g}_t}{dt}(x) = \mathcal{L}^*(x)[\widehat{g}_t] + \mathcal{L}_c^*(x, \widehat{c}_t(x))[g^{SS}] + \mathcal{L}_p^*(x)[g^{SS}] \cdot \widehat{p}_t.$$

Using $\mathcal{L}_c = -\sum_i \partial_{x_i}$, we obtain:

$$\mathcal{L}_c^*(x, \widehat{c}_t(x))[g^{SS}] = \sum_i \partial_{x_i}\Big(g^{SS}(x)\widehat{c}_{it}(x)\Big) \equiv \mathcal{L}_c^*(x)[\widehat{c}_t(x)],$$

where with a slight abuse of notation we also denote by $\mathcal{L}_c^*(x)$ the operator that acts on the consumption deviation holding the distribution fixed at steady-state. Then, using Corollary 4:

$$\begin{aligned}
\frac{d\widehat{g}_t}{dt} &= \mathcal{L}^*(x)[\widehat{g}_t] + \mathcal{L}_p^*(x)[g^{SS}] \cdot \widehat{p}_t + \mathcal{L}_c^*(x)[\widehat{c}_t(x)] \\
&= \mathcal{L}^*(x)[\widehat{g}_t] + \underbrace{\Big\{\mathcal{L}_p^*(x)[g^{SS}] - \mathcal{L}_c^*(x)[\mathcal{U}(x)^{-1}u_{cp}(x)]\Big\}}_{\equiv \mathcal{P}(x)} \cdot \widehat{p}_t \\
&\quad \underbrace{- \mathcal{L}_c^*(x)[\mathcal{U}(x)^{-1}\mathcal{L}_c(x)[\widehat{V}_t]]}_{\equiv \mathcal{M}(x)[\widehat{V}_t]} \ \underbrace{- \mathcal{L}_c^*(x)[\mathcal{U}(x)^{-1}u_{cz}]}_{\equiv \mathcal{Z}(x)} \widehat{z}_t \\
&\equiv \mathcal{L}^*(x)[\widehat{g}_t] + \mathcal{P}(x) \cdot \widehat{p}_t + \mathcal{Z}(x) \cdot \widehat{z}_t + \mathcal{M}(x)[\psi_t]
\end{aligned} \tag{34}$$

Let T_t^* be the semigroup associated with the generator $\mathcal{L}^*(x)$, so that

$$\frac{\partial T_t^*(x,y)}{\partial t} = \mathcal{L}^*(x)[T_t^*(\cdot, y)] \quad , \quad T_0^*(x,y) = \delta(x-y).$$

For all h, we denote by $T_t^*(x)[h] = \int T_t^*(x,y)h(y)\nu(dy) \equiv T_t^* h$. The solution to equation (34) is then given by:

$$\widehat{g}_t = T_t^* \widehat{g}_0 + \int_0^t T_{t-s}^* \big(\mathcal{P}\widehat{p}_s + \mathcal{Z}\widehat{z}_s + \mathcal{M}\widehat{V}_s\big) ds.$$

Substituting in the formula for $\widehat{V}_s$ from Theorem 1:

$$\int_0^t T_{t-s}^* \mathcal{M}\widehat{V}_s ds = \iint_0^\infty e^{-\rho(\tau - s)} T_{t-s}^* \mathcal{M}\big(\varphi_{\tau-s}\widehat{p}_\tau + \phi_{\tau-s}\widehat{z}_\tau\big) \mathbb{1}\{s \le t, s \le \tau\} d\tau ds$$

$$
\begin{aligned}
&= \int_0^\infty \widehat{p}_\tau \cdot \left(\int_0^{\min\{t,\tau\}} e^{-\rho(\tau-s)} T^*_{t-s} \mathcal{M} \varphi_{\tau-s} ds \right) d\tau \\
&+ \int_0^\infty \widehat{z}_\tau \cdot \left(\int_0^{\min\{t,\tau\}} e^{-\rho(\tau-s)} T^*_{t-s} \mathcal{M} \phi_{\tau-s} ds \right) d\tau
\end{aligned}
$$

Define:

$$
\mathcal{D}^p_{\tau-s} = e^{-\rho(\tau-s)} \mathcal{M} \varphi_{\tau-s} + \boldsymbol{\delta}(\tau-s) \mathcal{P} \quad , \quad J^p_{t,\tau} = \int_0^{\min\{t,\tau\}} T^*_{t-s} \mathcal{D}^p_{\tau-s} ds, \tag{35}
$$

and:

$$
\mathcal{D}^z_{\tau-s} = e^{-\rho(\tau-s)} \mathcal{M} \phi_{\tau-s} + \boldsymbol{\delta}(\tau-s) \mathcal{Z} \quad , \quad J^z_{t,\tau} = \int_0^{\min\{t,\tau\}} T^*_{t-s} \mathcal{D}^z_{\tau-s} ds. \tag{36}
$$

Armed with this notation, we obtain:

$$
\widehat{g}_t = T^*_t h_0 + \int_0^\infty \left(J^p_{t,s} \widehat{p}_s + J^z_{t,s} \widehat{z}_s \right) ds.
$$

Q.E.D. □

Having linearized the law of motion, the linearization of the price function delivers the equilibrium fixed point. To that end, denote $\mathcal{E}(x) = \frac{\partial p}{\partial g}(0, g^{SS}, x)$ the Fréchet derivative of the price functional with respect to the distribution, and denote $\zeta = \frac{\partial p}{\partial z}(0, g^{SS})$ the gradient of the price functional with respect to aggregate shocks.

Theorem 3. *(Price sequence-space Jacobians)*
To first order in $\widehat{z}, \widehat{g}_0$:

$$
\widehat{p}_t - \int_0^\infty \mathcal{J}^p_{t,s} \widehat{p}_s = \mathcal{E}^*_t h_0 + \int_0^\infty \mathcal{J}^z_{t,s} \widehat{z}_s ds,
$$

where the the expectation vector $\mathcal{E}_t$ satisfies

$$
\frac{\partial \mathcal{E}_t}{\partial t} = \mathcal{L} \mathcal{E}_t, \quad \mathcal{E}_0 = \mathcal{E},
$$

and the Jacobians $\mathcal{J}^n_{t,s}$, for $n \in \{p, z\}$, satisfy

$$
\lim_{\tau \to 0} \frac{\mathcal{J}^n_{t+\tau, s+\tau}}{\tau} = \mathcal{E}^*_t \mathcal{D}^n_s \equiv \mathcal{F}^n_{t,s},
$$

and boundary conditions for $t \geq 0, s > 0$: $\mathcal{J}^p_{t,0} = \mathcal{E}^*_t \mathcal{P}$, $\mathcal{J}^p_{0,s} = 0$ and $\mathcal{J}^z_{t,0} = \mathcal{E}^*_t \mathcal{Z}$, $\mathcal{J}^z_{0,s} = 0$.

Proof. Linearizing the price functional and using Theorem 2:

$$
\begin{aligned}
\widehat{p}_t &= \zeta \widehat{z}_t + \mathcal{E}^* \widehat{g}_t \\
&= \zeta \widehat{z}_t + \mathcal{E}^* T^*_t h_0 + \int_0^\infty \left(\mathcal{E}^* J^p_{t,s} \widehat{p}_s + \mathcal{E}^* J^z_{t,s} \widehat{z}_s \right) ds
\end{aligned}
$$

$$= \quad \zeta\widehat{z}_t + \mathcal{E}_t^* h_0 + \int_0^\infty \left(\mathcal{J}_{t,s}^p \widehat{p}_s + \mathcal{J}_{t,s}^z \widehat{z}_s\right) ds,$$

where:

$$\mathcal{J}_{t,s}^p \quad = \quad \mathcal{E}^* J_{t,s}^p = \int_0^{\min\{t,s\}} \mathcal{E}^* T_{t-\tau}^* \mathcal{D}_{s-\tau}^p d\tau = \int_0^{\min\{t,s\}} \mathcal{E}_{t-\tau}^* \mathcal{D}_{s-\tau}^p d\tau.$$

Similarly:

$$\mathcal{J}_{t,s}^z \quad = \quad \int_0^{\min\{t,s\}} \mathcal{E}_{t-\tau}^* \mathcal{D}_{s-\tau}^z d\tau.$$

It is straightforward to check that:

$$\begin{aligned} \frac{d\mathcal{J}_{t+s,\tau+s}^p}{ds} &= \mathcal{E}_{t+s}^* \mathcal{D}_{\tau+s}^p \equiv \mathcal{F}_{t+s,\tau+s}^p \\ \frac{d\mathcal{J}_{t+s,\tau+s}^z}{ds} &= \mathcal{E}_{t+s}^* \mathcal{D}_{\tau+s}^z \equiv \mathcal{F}_{t+s,\tau+s}^z. \end{aligned}$$

In addition, for $t, s > 0$

$$\mathcal{J}_{t,0}^p = \mathcal{E}_t^* \mathcal{P} \qquad \mathcal{J}_{0,s}^p = 0 \qquad \mathcal{J}_{0,0}^p = 0.$$

Q.E.D. □