MATH CAMP ASSIGNMENT

on Difference Equations

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1 Q1

Problem. A multiplier–accelerator model is given by

$$S_t = \alpha Y_t, \qquad I_{t+1} = \beta\,(Y_{t+1} - Y_t), \qquad S_t = I_t,$$

with constants $\beta > \alpha > 0$. Deduce a difference equation for $\{Y_t\}$, given Y_0, and solve it.

Solution. Market clearing holds each period, hence also at $t+1$:

$$S_{t+1} = I_{t+1} \quad \Rightarrow \quad \alpha Y_{t+1} = \beta\,(Y_{t+1} - Y_t).$$

Rearranging,

$$(\beta - \alpha) Y_{t+1} - \beta Y_t = 0 \quad \Longleftrightarrow \quad Y_{t+1} = \lambda Y_t, \qquad \lambda := \frac{\beta}{\beta - \alpha} > 1.$$

Therefore $\{Y_t\}$ satisfies a first–order homogeneous linear difference equation with constant coefficient λ. With Y_0 given,

$$\boxed{Y_t = Y_0\,\lambda^{\,t} = Y_0 \left(\frac{\beta}{\beta - \alpha}\right)^t, \qquad t = 0, 1, 2, \ldots}$$

Since $\beta > \alpha > 0$, we have $\lambda > 1$, so the path is explosive (monotone growth).

2 Q2

(a) Solve $x_{t+1} = 2x_t + 4$ with $x_0 = 1$.

Method. Write $x_{t+1} - 2x_t = 4$. A constant particular solution satisfies $x = 2x + 4$, so $x^* = -4$. Let $y_t := x_t - x^* = x_t + 4$. Then

$$y_{t+1} = x_{t+1} + 4 = 2x_t + 4 + 4 = 2(x_t + 4) = 2y_t,$$

hence $y_t = y_0\, 2^t = (x_0 + 4)2^t = 5 \cdot 2^t$. Therefore

$$\boxed{x_t = -4 + 5 \cdot 2^t\,, \qquad t \geq 0.}$$

(b) Solve $x_{t+1} - x_t + 3 = 0$ with $x_0 = 3$.

Method. The recursion is $x_{t+1} = x_t - 3$. Iterating,

$$x_t = x_0 - 3t = 3 - 3t, \qquad t \geq 0,$$

so

$$\boxed{x_t = 3 - 3t\,, \qquad t \geq 0.}$$

3 Q3

Problem. Investigate the stability of

$$x_{t+2} - \tfrac{1}{6} x_{t+1} - \tfrac{1}{6} x_t = c_t.$$

Solution. First, consider the homogeneous equation

$$x_{t+2} - \tfrac{1}{6} x_{t+1} - \tfrac{1}{6} x_t = 0.$$

The characteristic equation is

$$r^2 - \tfrac{1}{6} r - \tfrac{1}{6} = 0 \quad \Longleftrightarrow \quad 6r^2 - r - 1 = 0,$$

whose roots are

$$r_1 = \tfrac{1}{2}, \qquad r_2 = -\tfrac{1}{3}.$$

Therefore, the homogeneous solution is

$$x_t^{(h)} = A \left(\tfrac{1}{2}\right)^t + B \left(-\tfrac{1}{3}\right)^t \xrightarrow[t \to \infty]{} 0.$$

Both roots lie inside the unit circle, so the difference operator is *asymptotically stable*. Thus:

- When $c_t \equiv 0$, any solution converges to 0.

- When $\{c_t\}$ is bounded, since the system is BIBO (Bounded-Input, Bounded-Output) stable, there exists a unique bounded solution, and all solutions are given by the sum of a bounded particular solution and the decaying homogeneous solution.
- If $c_t \equiv c$ is a constant, let the steady state x^* satisfy $x^* - \frac{1}{6}x^* - \frac{1}{6}x^* = c$, then $x^* = \frac{3}{2}c$, and

$$x_t = x^* + A\left(\frac{1}{2}\right)^t + B\left(-\frac{1}{3}\right)^t \xrightarrow[t\to\infty]{} x^*.$$

In summary, the equation is stable; with a constant exogenous term, it converges exponentially to the steady state $\frac{3}{2}c$.

4 Q4

Problem. Model

$$C_t = cY_{t-1}, \qquad K_t = \sigma Y_{t-1}, \qquad Y_t = C_t + K_t - K_{t-1},$$

where $c, \sigma > 0$. Derive the second-order difference equation for Y_t, and give the necessary and sufficient condition for the solution to exhibit *explosive oscillations*.

Solution. Substituting gives

$$Y_t = (c+\sigma)Y_{t-1} - \sigma Y_{t-2} \quad \Longleftrightarrow \quad Y_t - (c+\sigma)Y_{t-1} + \sigma Y_{t-2} = 0.$$

The characteristic equation is

$$r^2 - (c+\sigma)r + \sigma = 0.$$

If the roots are a complex conjugate pair $r = \rho e^{\pm i\theta}$, then

$$\rho^2 = r_1 r_2 = \sigma, \qquad 2\rho\cos\theta = r_1 + r_2 = c + \sigma.$$

Therefore

$$(\text{complex roots}) \iff (c+\sigma)^2 < 4\sigma, \qquad \rho = \sqrt{\sigma}.$$

The general solution can be written as

$$Y_t = \rho^t\left[A\cos(t\theta) + B\sin(t\theta)\right], \qquad \rho = \sqrt{\sigma},\ \ \cos\theta = \frac{c+\sigma}{2\sqrt{\sigma}}.$$

So-called *explosive oscillations* mean that the solution oscillates and the amplitude diverges, which is equivalent to having complex roots with modulus $\rho > 1$. Thus

$$\boxed{\text{Explosive oscillations} \iff \begin{cases} (c+\sigma)^2 < 4\sigma, \\ \sigma > 1. \end{cases}}$$

When $\sigma < 1$, the oscillations are damped. When $(c+\sigma)^2 \geq 4\sigma$, the roots are real, and the trajectory is non-oscillatory (it may converge/diverge monotonically or alternate in sign, but not in a sinusoidal manner).

5 Q5

Problem. Given the system

$$y_t = 0.49\, y_{t-1} + 0.68\, i_{t-1}, \qquad i_t = 0.032\, y_{t-1} + 0.43\, i_{t-1},$$

where y_t is output and i_t is investment.

(a) Derive the second-order difference equation for y_t and its characteristic equation

From the first equation, we get $i_{t-1} = (y_t - 0.49y_{t-1})/0.68$. Substituting into the second equation gives

$$i_t = 0.032\, y_{t-1} + 0.43\, \frac{y_t - 0.49y_{t-1}}{0.68}.$$

Substituting this back into $y_{t+1} = 0.49y_t + 0.68i_t$ and simplifying, we get

$$y_{t+1} = 0.92\, y_t - 0.18894\, y_{t-1}.$$

Therefore

$$\boxed{y_{t+1} - 0.92\, y_t + 0.18894\, y_{t-1} = 0}$$

Its characteristic equation is

$$\boxed{r^2 - 0.92\, r + 0.18894 = 0}.$$

(b) Approximate characteristic roots and the general solution

Solving the above quadratic equation yields

$$r_{1,2} = \frac{0.92 \pm \sqrt{0.92^2 - 4 \times 0.18894}}{2} \approx 0.61053,\ 0.30947.$$

Both roots are inside the unit circle, so the system is stable. The general solution for y_t is

$$\boxed{y_t = A\, r_1^t + B\, r_2^t, \qquad r_1 \approx 0.61053,\ r_2 \approx 0.30947,}$$

where A and B are determined by initial values. From $y_{t+1} = 0.49 y_t + 0.68 i_t$, we can express i_t as

$$i_t = \frac{y_{t+1} - 0.49 y_t}{0.68} = A\, \frac{r_1 - 0.49}{0.68}\, r_1^t + B\, \frac{r_2 - 0.49}{0.68}\, r_2^t \approx A\,(0.1773)\, r_1^t + B\,(-0.2655)\, r_2^t.$$

This gives the general solution representation for the two-dimensional system.