MATH CAMP ASSIGNMENT 3

LINEAR ALGEBRA

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August 21, 2025

1 Q1

Claim. $\mathbb{R}^n$ with componentwise addition

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

and scalar multiplication

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n),$$

and with zero vector $0 = (0, \ldots, 0)$, is a vector space.

Proof. Closure holds because sums and scalar multiples of real numbers are real. Each of the eight axioms follows componentwise from the corresponding axiom in $\mathbb{R}$: commutativity and associativity of $+$, existence of 0 and $-x$, distributivity $\lambda(x + y) = \lambda x + \lambda y$, $(\lambda + \mu)x = \lambda x + \mu x$, associativity $(\lambda\mu)x = \lambda(\mu x)$, and $1 \cdot x = x$. Hence $(\mathbb{R}^n, +, \cdot)$ is a vector space. $\square$

2 Q2

Let $\mathcal{F}$ be the set of all real functions on $\mathbb{R}$. Define

$$(f + g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda f(x),$$

and $0(x) \equiv 0$. For every $x \in \mathbb{R}$ the vector space axioms hold pointwise because they hold in $\mathbb{R}$: $(f + g) = (g + f)$, $(f + (g + h)) = ((f + g) + h)$, $f + 0 = f$, $f + (-f) = 0$, $(\lambda\mu)f = \lambda(\mu f)$, $1 \cdot f = f$, $\lambda(f + g) = \lambda f + \lambda g$, $(\lambda + \mu)f = \lambda f + \mu f$. Therefore $\mathcal{F}$ is a vector space with zero vector the zero function. $\square$

3 Q3

(a)

Statement. If $v_1, \ldots, v_m$ are linearly independent, then $2v_1 - 3v_2$, $v_2, \ldots, v_m$ are linearly independent.

Proof. Suppose

$$a(2v_1 - 3v_2) + bv_2 + \sum_{i=3}^{m} c_i v_i = 0.$$

Then $(2a)v_1 + (-3a + b)v_2 + \sum_{i=3}^{m} c_i v_i = 0$. By independence of $v_1, \ldots, v_m$, we get $2a = 0$ (so $a = 0$), $-3a + b = 0$ (so $b = 0$), and $c_i = 0$ for $i \geq 3$. Hence the new list is independent. $\square$

(b)

Statement. If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are two linearly independent lists, then $v_1 + w_1, \ldots, v_m + w_m$ are linearly independent.

Verdict. *False.* Counterexample with $m = 1$ in $\mathbb{R}$: take $v_1 = 1$, $w_1 = -1$. Both lists are linearly independent, but $v_1 + w_1 = 0$, so the singleton $\{0\}$ is linearly dependent. $\square$

4 Q4

Equip $\mathbb{R}^n$ with the Euclidean inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Then (i) symmetry $\langle x, y \rangle = \langle y, x \rangle$ holds, (ii) linearity in the first variable $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ holds by distributivity of real multiplication, and (iii) positive definiteness $\langle x, x \rangle = \sum_{i=1}^{n} x_i^2 \geq 0$ with equality iff $x = 0$. Hence $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is an inner product space. $\square$

5 Q5

(**Parallelogram identity**) For any u, v in an inner product space,

$$\|u + v\|^2 = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle, \qquad \|u - v\|^2 = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle.$$

Adding gives

$$\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2),$$

which generalizes the Euclidean parallelogram law. $\square$

6 Q6

(**Rule of Sarrus**) For

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

Leibniz's formula with the six permutations of $\{1,2,3\}$ yields

$$\det(A)=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32},$$

which is exactly Sarrus' rule (sum of the three "downward" products minus the three "upward" ones). $\square$

7 Q7

Let $A=(a_{ij})\in\mathbb{R}^{m\times n}$. Define

$$T_A:\mathbb{R}^n\to\mathbb{R}^m,\qquad T_A(x)=Ax,$$

so that the i-th component is $(Ax)_i=\sum_{j=1}^n a_{ij}x_j\in\mathbb{R}$. For each input x the output Ax is uniquely determined, hence T_A is a well-defined function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (indeed a linear map). $\square$

8 Q8

Let $L(\mathbb{R}^n,\mathbb{R}^m)=\{T_A:x\mapsto Ax\mid A\in\mathbb{R}^{m\times n}\}$. Define

$$T_A+T_B:=T_{A+B},\qquad \lambda T_A:=T_{\lambda A}.$$

Since $(A+B)x=Ax+Bx$ and $(\lambda A)x=\lambda(Ax)$, these coincide with pointwise operations. Vector space axioms reduce to the corresponding matrix axioms. The zero vector is T_0 (the map induced by the zero matrix), because $T_0(x)=0$ for all x. Therefore $L(\mathbb{R}^n,\mathbb{R}^m)$ is a vector space. $\square$

9 Q9

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T(x_1, x_2, x_3) = (2x_2, 0, 5x_3)$. With respect to the standard basis,

$$[T] = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

The characteristic polynomial is $\chi(\lambda) = \det([T] - \lambda I) = \lambda^2(5 - \lambda)$, hence eigenvalues are $\lambda = 0$ (algebraic multiplicity 2) and $\lambda = 5$. Eigenvectors: for $\lambda = 5$, solve $([T] - 5I)x = 0$, giving $x_2 = 0$, $x_1 = 0$, x_3 free, so $E_5 = \operatorname{span}\{(0, 0, 1)\}$; for $\lambda = 0$, solve $[T]x = 0$, giving $x_2 = x_3 = 0$, so $E_0 = \operatorname{span}\{(1, 0, 0)\}$. (Thus T is not diagonalizable: geometric multiplicities sum to $2 < 3$.) □

10 Q10

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be

$$T(x_1, \ldots, x_n) = (x_1, 2x_2, \ldots, nx_n).$$

Then $[T] = \operatorname{diag}(1, 2, \ldots, n)$. The eigenvalues are $1, 2, \ldots, n$ (all distinct), and the corresponding eigenspaces are $E_i = \operatorname{span}\{e_i\}$ for $i = 1, \ldots, n$. Therefore T is diagonalizable by the standard basis. □

11 Q11

(**Product of orthogonal matrices**) If $Q_1^\top Q_1 = I$ and $Q_2^\top Q_2 = I$, then

$$(Q_1 Q_2)^\top (Q_1 Q_2) = Q_2^\top Q_1^\top Q_1 Q_2 = Q_2^\top I Q_2 = I,$$

hence $Q_1 Q_2$ is orthogonal. Equivalently, orthogonal matrices satisfy $Q^{-1} = Q^\top$ and are closed under multiplication. □

12 Q12

Suppose

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$$

is orthogonal. Let the columns of the unscaled matrix be

$$c_1 = \begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix}, \quad c_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 2 \\ -2 \\ y \end{pmatrix}.$$

Since $A^\top A = I$, we have $M^\top M = 9I$ for $M = 3A$, so the columns of M are pairwise orthogonal with squared norms 9. From $\langle c_1, c_2 \rangle = 1 \cdot 2 + 2 \cdot 1 + x \cdot 2 = 4 + 2x = 0$ we get $x = -2$. From $\langle c_2, c_3 \rangle = 2 \cdot 2 + 1 \cdot (-2) + 2 \cdot y = 2 + 2y = 0$ we get $y = -1$. Finally $\|c_1\|^2 = 1 + 4 + (-2)^2 = 9$, $\|c_3\|^2 = 4 + 4 + (-1)^2 = 9$, and $\langle c_1, c_3 \rangle = 2 - 4 + 2 = 0$, confirming orthogonality. Thus

$$\boxed{x = -2, \qquad y = -1.}$$

□