MATH CAMP ASSIGNMENT

on Differential Equations

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1 Q1 Lotka–Volterra phase analysis

Consider the predator–prey system ($x > 0,\ y > 0$, parameters $a, b, h, k > 0$)

$$\dot{x} = x\,(k - ay), \qquad \dot{y} = y\,(-h + bx).$$

Nullclines and equilibria. The x–nullclines are $x = 0$ and $y = \frac{k}{a}$; the y–nullclines are $y = 0$ and $x = \frac{h}{b}$. Hence the equilibria are

$$(0, 0) \quad \text{and} \quad (x^*, y^*) = \left(\tfrac{h}{b}, \tfrac{k}{a}\right) \text{ in } \mathbb{R}^2_{>0}.$$

The axes are invariant sets: on $y = 0$ we have $\dot{x} = kx$ (exponential growth); on $x = 0$ we have $\dot{y} = -hy$ (exponential decay).

First integral (closed orbits). Define

$$H(x, y) = bx - h \ln x + ay - k \ln y.$$

Along trajectories,

$$\dot{H} = b\dot{x} - h\frac{\dot{x}}{x} + a\dot{y} - k\frac{\dot{y}}{y} = bx(k - ay) - h(k - ay) + ay(-h + bx) - k(-h + bx) = 0.$$

Therefore $H(x(t), y(t)) \equiv C$ and the level sets $H(x, y) = C$ are the *streamlines*: in the positive quadrant they are closed simple loops surrounding (x^*, y^*). Crossing the nullclines determines the orientation: on $y = \frac{k}{a}$, $\dot{y} = y(-h + bx)$ changes sign at $x = \frac{h}{b}$, and on $x = \frac{h}{b}$, $\dot{x} = x(k - ay)$ changes sign at $y = \frac{k}{a}$. As a result the motion around (x^*, y^*) is *counterclockwise*: (right $\to$ up $\to$ left $\to$ down).

Linearization and local stability. The Jacobian

$$J(x,y) = \begin{pmatrix} k - ay & -ax \\ by & -h + bx \end{pmatrix}.$$

At $(0,0)$,

$$J(0,0) = \begin{pmatrix} k & 0 \\ 0 & -h \end{pmatrix}, \qquad \det J(0,0) = -kh < 0,$$

so $(0,0)$ is a *hyperbolic saddle* (repelling along the x–axis and attracting along the y–axis).

At the positive equilibrium $(x^*, y^*) = (h/b, k/a)$,

$$J(x^*, y^*) = \begin{pmatrix} 0 & -\frac{ah}{b} \\ \frac{bk}{a} & 0 \end{pmatrix}, \quad \operatorname{tr} J = 0, \quad \det J = hk > 0.$$

Thus the linearization has eigenvalues $\lambda_{1,2} = \pm i\sqrt{hk}$: a *linear center*. Since the nonlinear system admits the first integral H, the equilibrium is in fact a *nonlinearly (Lyapunov) stable center*: solutions neither converge nor diverge but lie on closed orbits determined by $H(x,y) = C$. Hence the coexistence point is *neutrally stable* (not asymptotically stable).

Phase diagram summary. - Nullclines: $x = 0$, $y = 0$, $x = h/b$, $y = k/a$. - $(0,0)$: saddle. - $(h/b, k/a)$: center; nearby trajectories are counterclockwise closed loops given by $H(x,y) = C$.

2 Q2 Local stability in Kaldor's model

Consider

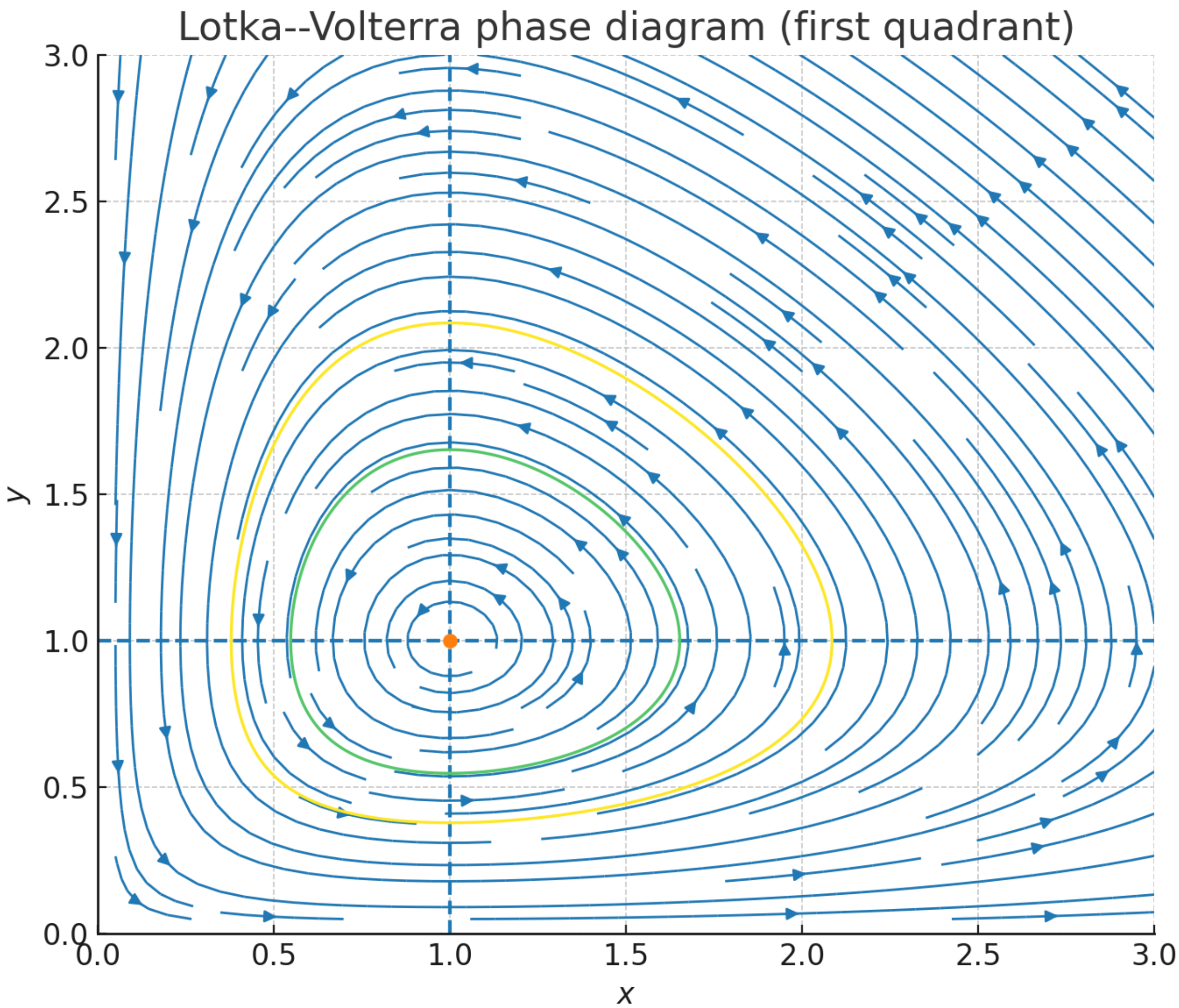
$$\dot{Y} = \alpha\,[I(Y,K) - S(Y,K)], \qquad \dot{K} = I(Y,K), \qquad \alpha > 0,$$

with partial derivatives (evaluated at the steady state)

$$I_Y > 0,\ I_K < 0, \qquad S_Y > 0,\ S_K < 0, \qquad I_K - S_K < 0.$$

Steady state. A stationary point (Y^*, K^*) must satisfy

$$\dot{Y} = \dot{K} = 0 \implies I(Y^*, K^*) = S(Y^*, K^*) = 0.$$



Lotka--Volterra phase diagram (first quadrant)
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x

Figure 1: Q1

Linearization. Let $F(Y,K) = (\dot{Y}, \dot{K})^{\top}$. The Jacobian at (Y^*, K^*) is

$$J = \begin{pmatrix} \alpha(I_Y - S_Y) & \alpha(I_K - S_K) \\ I_Y & I_K \end{pmatrix}.$$

Hence

$$\operatorname{tr} J = \alpha(I_Y - S_Y) + I_K, \qquad \det J = \alpha\,(S_K I_Y - I_K S_Y).$$

Local (asymptotic) stability. For a 2×2 continuous-time system, the steady state is locally asymptotically stable iff

$$\det J > 0 \quad \text{and} \quad \operatorname{tr} J < 0.$$

With $\alpha > 0$ this becomes

$$\boxed{S_K I_Y - I_K S_Y \;>\; 0} \qquad \text{and} \qquad \boxed{\alpha(I_Y - S_Y) + I_K \;<\; 0}.$$

Sign implications / economics. Since $I_Y, S_Y > 0$ and $I_K, S_K < 0$,

$$\det J > 0 \iff |I_K|\,S_Y \;>\; |S_K|\,I_Y \iff \frac{S_Y}{I_Y} > \frac{-S_K}{-I_K}.$$

Thus the (income) sensitivity of saving must dominate the corresponding weighted sensitivity of investment. The trace condition can be read as:

$$\alpha(I_Y - S_Y) < -I_K,$$

so, e.g., a sufficient (not necessary) condition is $S_Y > I_Y$ together with $I_K < 0$. Under these inequalities the steady state is a *locally asymptotically stable* node/focus (no center), whereas violation of either $\det J > 0$ or $\operatorname{tr} J < 0$ yields instability.