MATH CAMP ASSIGNMENT 5

Comparative Static Analysis

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1 Q1: Utility Maximization and Consumer Demand

Problem. Maximize $U(x,y) = (x+2)(y+1)$ subject to a binding budget constraint $P_x x + P_y y = m$, with $P_x, P_y > 0$.

(i) Lagrangian and FOCs

$$\mathcal{L}(x,y,\lambda) = (x+2)(y+1) + \lambda\,(m - P_x x - P_y y).$$

FOCs:

$$\begin{aligned}
\partial_x \mathcal{L} &: \ (y+1) - \lambda P_x = 0, \\
\partial_y \mathcal{L} &: \ (x+2) - \lambda P_y = 0, \\
\partial_\lambda \mathcal{L} &: \ P_x x + P_y y = m.
\end{aligned}$$

Equating the first two gives $P_y(y+1) = P_x(x+2)$.

(ii) Optimal choices and shadow value

Solve the linear system

$$\begin{cases} P_x x + P_y y = m, \\ -P_x x + P_y y = 2P_x - P_y, \end{cases} \quad \Rightarrow \quad \boxed{x^* = \frac{m - 2P_x + P_y}{2P_x} = \frac{m}{2P_x} + \frac{P_y}{2P_x} - 1\,,}$$

$$\boxed{y^* = \frac{m + 2P_x - P_y}{2P_y} = \frac{m}{2P_y} + \frac{P_x}{P_y} - \frac{1}{2}\,,}$$

and from $\lambda = (y+1)/P_x = (x+2)/P_y$,

$$\boxed{\lambda^* = \frac{m + 2P_x + P_y}{2P_x P_y} > 0\,,}$$

which is the marginal utility of income.

(iii) Comparative statics (Marshallian)

$$\frac{\partial x^*}{\partial m} = \frac{1}{2P_x} > 0, \quad \frac{\partial x^*}{\partial P_x} = -\frac{m + P_y}{2P_x^2} < 0, \quad \frac{\partial x^*}{\partial P_y} = \frac{1}{2P_x} > 0;$$

$$\frac{\partial y^*}{\partial m} = \frac{1}{2P_y} > 0, \quad \frac{\partial y^*}{\partial P_y} = -\frac{m + 2P_x}{2P_y^2} < 0, \quad \frac{\partial y^*}{\partial P_x} = \frac{1}{P_y} > 0.$$

Interpretation. Both goods are normal; own-price effects are negative; cross-price effects are positive, so the goods are (net) substitutes.

2 Q2: Income Determination with Taxes

Problem.

$$Y = C + I_0 + G_0, \quad C = \alpha + \beta(Y - T), \quad T = \gamma + \delta Y,$$

with $\alpha > 0$, $0 < \beta < 1$, $\gamma > 0$, $0 < \delta < 1$.

(i) Equilibrium income

Substitute T into C:

$$C = \alpha + \beta\big(Y - (\gamma + \delta Y)\big) = \alpha - \beta\gamma + \beta(1-\delta)Y.$$

Hence

$$Y = \alpha - \beta\gamma + \beta(1-\delta)Y + I_0 + G_0 \;\Rightarrow\; \boxed{Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta(1-\delta)}.}$$

The denominator is positive since $0 < \beta < 1$ and $0 < \delta < 1$.

(ii) Comparative statics

Let $D := 1 - \beta(1-\delta) > 0$.

$$\boxed{\frac{\partial Y^*}{\partial G_0} = \frac{1}{D} > 0,} \qquad \boxed{\frac{\partial Y^*}{\partial \gamma} = -\frac{\beta}{D} < 0,} \qquad \boxed{\frac{\partial Y^*}{\partial \delta} = -\frac{\beta Y^*}{D} < 0.}$$

Interpretation. Government purchases raise Y with the (tax-damped) Keynesian multiplier $1/D$. Higher lump-sum taxes γ and a higher marginal tax rate δ both reduce equilibrium income; δ also shrinks the multiplier.

3 Q3: Open-Economy Fiscal Expansion (IS–LM–BP)

Equilibrium is characterized by

$$Y = C(Y-T) + I(r) + G_0 + X(E) - M(Y,E), \tag{1}$$

$$L(Y,r) = M_0^s, \tag{2}$$

$$X(E) - M(Y,E) + K(r, r_w) = 0. \tag{3}$$

Assumptions on derivatives: $c := C'(Y-T) \in (0,1)$, $I_r := I'(r) < 0$, $X_E > 0$, $M_Y > 0$, $M_E < 0$, $L_Y > 0$, $L_r < 0$, $K_r > 0$.

(i) Linearization and Jacobian

Write each condition as $F_i(Y,r,E) = 0$ and totally differentiate with respect to (Y,r,E) and G_0:

$$J \begin{bmatrix} dY \\ dr \\ dE \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dG_0, \qquad J = \begin{bmatrix} A & \alpha & \beta \\ \phi & \psi & 0 \\ -\mu & \kappa & \chi \end{bmatrix},$$

where

$$A := 1 - c + M_Y > 0, \quad \alpha := -I_r > 0, \quad \beta := M_E - X_E < 0, \quad \phi := L_Y > 0,\ \psi := L_r < 0,\ \mu := M_Y > 0,\ \kappa := K_r > 0,\ \chi := X$$

(ii) Cramer's rule

Let $\Delta = \det J$. Replacing column i by $(1,0,0)'$ yields

$$\Delta_Y = \psi\chi < 0, \qquad \Delta_r = -\phi\chi < 0, \qquad \Delta_E = \phi\kappa + \psi\mu \ (=: S).$$

The determinant can be written

$$\Delta = \chi(A\psi - \alpha\phi) + \beta(\phi\kappa + \psi\mu).$$

Since $A > 0$, $\psi < 0$, $\alpha\phi > 0$, the first term is negative; with the empirically standard $S > 0$ (capital flows sufficiently interest-sensitive), the second term is also negative (because $\beta < 0$). Hence $\Delta < 0$.

(iii) Effects of $G_0 \uparrow$

By Cramer,

$$\boxed{\frac{dY^*}{dG_0} = \frac{\Delta_Y}{\Delta} = \frac{\psi\chi}{\Delta} > 0}, \qquad \boxed{\frac{dr^*}{dG_0} = \frac{\Delta_r}{\Delta} = \frac{-\phi\chi}{\Delta} > 0}, \qquad \boxed{\frac{dE^*}{dG_0} = \frac{\Delta_E}{\Delta} = \frac{S}{\Delta} < 0}.$$

Interpretation. Fiscal expansion shifts IS to the right: output and the domestic interest rate rise. Higher r induces capital inflow; to restore external balance, the domestic currency appreciates (E falls), which crowds out net exports and dampens the output effect. With perfect capital mobility ($\kappa \to \infty$) under floating E, $dY^*/dG_0 \to 0^+$ and the adjustment occurs mainly via appreciation.

4 Q4: KKT Problems

We use the KKT system for a maximization problem with inequality constraints written as $a_i(x) \geq 0$. The Lagrangian is

$$\mathcal{L}(x,\mu) = f(x) + \sum_i \mu_i a_i(x), \quad \mu_i \geq 0.$$

KKT conditions: primal feasibility $a_i(x^*) \geq 0$; dual feasibility $\mu_i \geq 0$; complementary slackness $\mu_i a_i(x^*) = 0$; stationarity $\nabla f(x^*) + \sum_i \mu_i \nabla a_i(x^*) = 0$.

Q4.1 $\max\,(x_1-4)^2+(x_2-4)^2$

Constraints:

$$a_1 = 2x_1+3x_2-6 \geq 0, \quad a_2 = -3x_1-2x_2+12 \geq 0, \quad a_3 = x_1 \geq 0,\ a_4 = x_2 \geq 0.$$

The feasible set is a convex quadrilateral in the first quadrant. Since the objective is convex in (x_1, x_2), its maximum over a convex polytope is attained at an extreme point. The corner points are $(3,0)$, $(4,0)$, $(0,2)$, $(0,6)$ with objective values $17, 16, 20, 20$ respectively. Thus

$$\boxed{\text{Maximum value } = 20 \text{ at } (x_1,x_2) = (0,2) \text{ and } (0,6).}$$

One set of KKT multipliers at $(0,2)$ is $\mu_1 = \frac{4}{3}$, $\mu_3 = \frac{16}{3}$, others zero (active: $a_1 = a_3 = 0$). At $(0,6)$, take $\mu_2 = 2$, $\mu_3 = 14$, others zero (active: $a_2 = a_3 = 0$). Both satisfy stationarity, feasibility, and complementary slackness.

Q4.2 $\max\ \frac{1}{2}x - y$ **s.t.** $x + e^{-x} + z^2 \leq y$, $x \geq 0$

Write $a_1 = y - x - e^{-x} - z^2 \geq 0$, $a_2 = x \geq 0$. Lagrangian

$$\mathcal{L} = \tfrac{1}{2}x - y + \mu_1(y - x - e^{-x} - z^2) + \mu_2 x.$$

Stationarity gives $-1+\mu_1 = 0 \Rightarrow \mu_1 = 1$; $-2\mu_1 z = 0 \Rightarrow z^* = 0$; and

$$\tfrac{1}{2} + \mu_1(-1+e^{-x}) + \mu_2 = 0 \ \Rightarrow\ \mu_2 = \tfrac{1}{2} - e^{-x} \geq 0 \ \Rightarrow\ x \geq \ln 2.$$

If $x > 0$, complementary slackness for a_2 implies $\mu_2 = 0$, hence $e^{-x} = 1/2$ and

$$\boxed{x^* = \ln 2, \quad z^* = 0, \quad y^* = x^* + e^{-x^*} + z^{*2} = \ln 2 + \tfrac{1}{2}.}$$

The optimal value is $f^* = \frac{1}{2}\ln 2 - (\ln 2 + \frac{1}{2}) = -\frac{1}{2}(\ln 2 + 1)$. Slater holds, so KKT is necessary and sufficient.

Q4.3 max $x^2 + 2y$ **s.t.** $x^2 + y^2 \leq 5$**,** $y \geq 0$

Let $a_1 = 5 - x^2 - y^2 \geq 0$, $a_2 = y \geq 0$ and

$$\mathcal{L} = x^2 + 2y + \mu_1(5 - x^2 - y^2) + \mu_2 y.$$

Stationarity:

$$2x(1 - \mu_1) = 0, \qquad 2 - 2\mu_1 y + \mu_2 = 0.$$

No interior optimum exists. On the circle ($a_1 = 0$) with $y > 0$ ($\mu_2 = 0$), either $x = 0$ (gives $y = \sqrt{5}$, value $2\sqrt{5}$) or $\mu_1 = 1$ (yields $y = 1$ and $x = \pm 2$, value 6). Points with $y = 0$ violate dual feasibility. Hence

$$\boxed{(x^*, y^*) = (2, 1) \text{ or } (-2, 1), \quad f^* = 6.}$$