MATH CAMP ASSIGNMENT 6

Optimal Control Theory

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1 Q1

Problem. Maximize

$$\int_0^1 \left(y - u^2\right)\, dt \quad \text{s.t. } \dot{y} = u,\ y(0) = 2,\ y(1) = a.$$

Hamiltonian

$$H(y, u, \lambda) = y - u^2 + \lambda u.$$

PMP conditions

$$\begin{aligned}
\dot{y} &= \frac{\partial H}{\partial \lambda} = u, \\
\dot{\lambda} &= -\frac{\partial H}{\partial y} = -1, \\
0 &= \frac{\partial H}{\partial u} = -2u + \lambda \ \Rightarrow\ u = \frac{\lambda}{2}.
\end{aligned}$$

Both endpoints of y are fixed, so $\lambda(1)$ is free.

Solution From $\dot{\lambda} = -1$, $\lambda(t) = -t + c$. Hence

$$u(t) = \frac{c-t}{2}, \qquad y(t) = 2 + \int_0^t \frac{c-s}{2}\, ds = 2 - \frac{t^2}{4} + \frac{c}{2}t.$$

Impose $y(1) = a$: $a = 2 - \frac{1}{4} + \frac{c}{2} \Rightarrow c = 2a - \frac{7}{2}$.

Optimal paths

$$\boxed{\lambda^*(t) = 2a - \frac{7}{2} - t, \quad u^*(t) = a - \frac{7}{4} - \frac{t}{2}, \quad y^*(t) = 2 - \frac{t^2}{4} + \left(a - \frac{7}{4}\right)t}.$$

Because H is strictly concave in u and the dynamics are linear, this candidate is globally optimal.

2 Q2

Problem. Maximize

$$\int_0^2 (2y - 3u - au^2)\,dt, \qquad \dot{y} = u + y, \quad y(0) = 5,\ y(2) \text{ free},\ a > 0.$$

Hamiltonian

$$H = 2y - 3u - au^2 + \lambda(u + y).$$

PMP conditions

$$\begin{aligned}
\dot{y} &= u + y, \\
\dot{\lambda} &= -\frac{\partial H}{\partial y} = -(2 + \lambda), \\
0 &= \frac{\partial H}{\partial u} = -3 - 2au + \lambda \ \Rightarrow\ u = \frac{\lambda - 3}{2a}.
\end{aligned}$$

Since $y(2)$ is free, the transversality condition is $\lambda(2) = 0$.

Costate & control Solve $\dot{\lambda} + \lambda = -2$:

$$\lambda(t) = Ce^{-t} - 2, \quad 0 = \lambda(2) = Ce^{-2} - 2 \Rightarrow C = 2e^2.$$

Thus

$$\boxed{\lambda^*(t) = 2\left(e^{2-t} - 1\right)}, \qquad \boxed{u^*(t) = \frac{2e^{2-t} - 5}{2a}}.$$

State Solve the linear ODE $\dot{y} - y = u^*(t)$:

$$(e^{-t}y)' = e^{-t}u^*(t), \qquad y(0) = 5.$$

Integrating gives

$$\boxed{y^*(t) = e^t\left[5 + \frac{1}{2a}\left(e^2\left(1 - e^{-2t}\right) - 5\left(1 - e^{-t}\right)\right)\right]}.$$

Strict concavity of H in u ensures global optimality.

3 Q3

Problem. Maximize

$$\int_0^T \left(K - aK^2 - I^2\right)\,dt, \qquad \dot{K} = I - \delta K, \quad K(0) = K_0,\ K(T) \text{ free},\ a > 0,\ \delta > 0.$$

Hamiltonian

$$H = K - aK^2 - I^2 + \lambda(I - \delta K).$$

PMP conditions

$$\begin{aligned}
\dot{K} &= I - \delta K,\\
\dot{\lambda} &= -\frac{\partial H}{\partial K} = -1 + 2aK + \delta\lambda,\\
0 &= \frac{\partial H}{\partial I} = -2I + \lambda \ \Rightarrow\ \boxed{I^* = \frac{\lambda}{2}}.
\end{aligned}$$

Free terminal state $\Rightarrow \lambda(T) = 0$.

Reduction to a second-order ODE From $\dot{K} = I - \delta K$ and $I = \lambda/2$,

$$\lambda = 2(\dot{K} + \delta K).$$

Differentiate and substitute the costate equation:

$$2(\ddot{K} + \delta\dot{K}) = -1 + 2aK + \delta \cdot 2(\dot{K} + \delta K) \ \Rightarrow\ \boxed{\ddot{K} - (a+\delta^2)K = -\tfrac{1}{2}}.$$

Let $\gamma := \sqrt{a+\delta^2} > 0$ and $K_p = \dfrac{1}{2(a+\delta^2)}$. The general solution is

$$K(t) = K_p + \alpha\cosh(\gamma t) + \beta\sinh(\gamma t).$$

Use $K(0) = K_0 \Rightarrow \alpha = K_0 - K_p$. The terminal condition $\lambda(T) = 0$ is equivalent to

$$\dot{K}(T) + \delta K(T) = 0,$$

which yields

$$\boxed{\beta = -\frac{\alpha\left[\gamma\sinh(\gamma T) + \delta\cosh(\gamma T)\right] + \delta K_p}{\gamma\cosh(\gamma T) + \delta\sinh(\gamma T)}}.$$

Optimal paths

$$\boxed{\begin{aligned} &K^*(t) = K_p + \alpha \cosh(\gamma t) + \beta \sinh(\gamma t), \\ &\lambda^*(t) = 2\big(\dot{K}^*(t) + \delta K^*(t)\big), \\ &I^*(t) = \frac{\lambda^*(t)}{2} = \dot{K}^*(t) + \delta K^*(t). \end{aligned}}$$

Since H is strictly concave in (K, I) and dynamics are linear, the solution is globally optimal.

4 Q4

Problem. Maximize

$$\int_0^1 (2x - x^2)\, dt, \qquad \dot{x} = u,\;\; x(0) = 0,\; x(1) = 0,\; u \in [-1, 1].$$

Note $2x - x^2 = 1 - (x-1)^2$ is concave in x.

Hamiltonian

$$H = 2x - x^2 + \lambda u.$$

Since H is linear in u and u is box-constrained, the optimal control is bang–bang.

PMP conditions

$$\begin{aligned} \dot{x} &= u, \\ \dot{\lambda} &= -\frac{\partial H}{\partial x} = -(2 - 2x) = 2(x - 1), \\ u^*(t) &= \arg\max_{u \in [-1,1]} \{\lambda(t)\, u\} = \begin{cases} +1, & \lambda(t) > 0, \\ -1, & \lambda(t) < 0. \end{cases} \end{aligned}$$

No singular arc exists: $\lambda \equiv 0$ would imply $x \equiv 1$, incompatible with the rate bound and $x(0) = x(1) = 0$ on a horizon of length 1.

Structure and switching time With a single switch at $t = \tau$,

$$u^*(t) = \begin{cases} +1, & 0 \le t < \tau, \\ -1, & \tau < t \le 1, \end{cases} \qquad x^*(t) = \begin{cases} t, & 0 \le t \le \tau, \\ 2\tau - t, & \tau \le t \le 1. \end{cases}$$

Imposing $x(1) = 0$ gives $2\tau - 1 = 0 \Rightarrow \boxed{\tau = \frac{1}{2}}$.

Costate (consistency check) For $t \le \frac{1}{2}$, $\dot{\lambda} = 2(t-1)$, hence

$$\lambda(t) = (t-1)^2 - \tfrac{1}{4} > 0 \;\Rightarrow\; u = +1.$$

For $t \ge \frac{1}{2}$, $\dot{\lambda} = -2t$, hence

$$\lambda(t) = \tfrac{1}{4} - t^2 < 0 \;\Rightarrow\; u = -1.$$

Optimal paths

$$\boxed{u^*(t) = \begin{cases} +1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1, \end{cases} \quad x^*(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2}, \\ 1 - t, & \frac{1}{2} \le t \le 1, \end{cases} \quad \lambda^*(t) = \begin{cases} (t-1)^2 - \frac{1}{4}, & 0 \le t \le \frac{1}{2}, \\ \frac{1}{4} - t^2, & \frac{1}{2} \le t \le 1. \end{cases}}$$

Concavity of the integrand in x ensures global optimality.

5 Q5

Problem. Maximize

$$\int_0^4 3y\,dt, \qquad \dot{y} = y + u, \quad y(0) = 5, \quad y(4) \ge 300, \quad u \in [0, 2].$$

Hamiltonian and PMP

$$H = 3y + \lambda(y + u), \qquad \dot{\lambda} = -(3 + \lambda).$$

The maximizing control is

$$u^*(t) = \begin{cases} 2, & \lambda(t) > 0, \\ 0, & \lambda(t) < 0. \end{cases}$$

Terminal inequality $y(4) \geq 300$ gives the transversality condition

$$\lambda(4) = \mu, \qquad \mu \geq 0, \qquad \mu\,[y(4) - 300] = 0.$$

Unconstrained terminal multiplier Assume the constraint is slack so that $\lambda(4) = 0$. Solve $\dot{\lambda} + \lambda = -3$:

$$\lambda(t) = Ce^{-t} - 3, \quad 0 = \lambda(4) = Ce^{-4} - 3 \Rightarrow C = 3e^4.$$

Hence $\lambda(t) = 3(e^{4-t} - 1) > 0$ on $[0, 4)$, so

$$\boxed{u^*(t) = 2 \text{ for all } t \in [0, 4].}$$

State and feasibility Solve $\dot{y} - y = 2$ with $y(0) = 5$:

$$y^*(t) = 7e^t - 2, \qquad y^*(4) = 7e^4 - 2 \approx 380.19 > 300.$$

Thus the terminal constraint is non-binding and the multiplier is $\mu = 0$; the above solution is self-consistent.

Optimal paths

$$\boxed{u^*(t) = 2, \qquad y^*(t) = 7e^t - 2, \qquad \lambda^*(t) = 3\big(e^{4-t} - 1\big).}$$

Sufficiency Remarks

In Q1–Q3 the Hamiltonians are strictly concave in the control (and state, where relevant), and the dynamics are linear; therefore any trajectory satisfying PMP is globally optimal. In Q4–Q5 the Hamiltonian is affine in the control with box constraints, yielding bang–bang (or boundary) controls; the integrands are concave in the state, which, together with feasibility, ensures global optimality of the PMP solution.