MATH CAMP ASSIGNMENT

on Differential Equations

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1 Q1

Qualitative stability (phase-line). Consider the following autonomous differential equations $\dot{y} = f(y)$.

(a) $\dot{y} = y - 7$**.** The unique equilibrium is $y^{=7}$. Since $f'(y) = 1 > 0$, y is an *unstable equilibrium* (repeller). For $y_0 < 7$, solutions decrease to $-\infty$; for $y_0 > 7$, solutions diverge to $+\infty$ monotonically.

(b) $\dot{y} = 1 - 5y$**.** The equilibrium is $y^{=\frac{1}{5}}$. Since $f'(y) = -5 < 0$, y is *asymptotically stable*. Every solution converges monotonically to $\frac{1}{5}$ regardless of initial condition.

(c) $\dot{y} = (1+y)^2 - 16$**,** $y \geq 0$**.** The zeros are $y = -5$ and $y = 3$, but only $y^{=3}$ is admissible due to $y \geq 0$. Since $f'(y) = 2(1+y)$ and $f'(3) = 8 > 0$, $y^{=3}$ is *unstable*. For $0 \leq y_0 < 3$, $\dot{y} < 0$, and trajectories move left until they exit the feasible region at $y = 0$ in finite time; for $y_0 > 3$, $\dot{y} > 0$ and solutions diverge to $+\infty$.

(d) $\dot{y} = \frac{1}{2}y - y^2$**,** $y \geq 0$**.** Equilibria are $y^{1=0}$ and $y^{2=\frac{1}{2}}$. Because $f'(0) = \frac{1}{2} > 0$, y^1 is *unstable*; because $f'(\frac{1}{2}) = -\frac{1}{2} < 0$, y^2 is *asymptotically stable*. For any $y_0 > 0$, solutions converge monotonically to $\frac{1}{2}$ (logistic-type saturation).

2 Q2

Phase-line analysis. We study $\dot{y} = f(y)$ for each case, identify equilibria ($f(y^{)=0}$), and determine local stability from $f'(y^{)}$ and the sign of $f(y)$.

(a) $\dot{y} = 3 - y - \ln y$ (domain $y > 0$)

Let $g(y) = y + \ln y$. Since $g'(y) = 1 + \frac{1}{y} > 0$ for $y > 0$, g is strictly increasing with $\lim_{y\downarrow 0} g(y) = -\infty$ and $\lim_{y\uparrow\infty} g(y) = +\infty$. Hence there is a unique equilibrium $y^{>0}$ solving $y + \ln y = 3$, i.e.

$$y^{=W(e^3)} \quad \text{(Lambert-}W\text{)}, \qquad y^{\approx 2.207.}$$

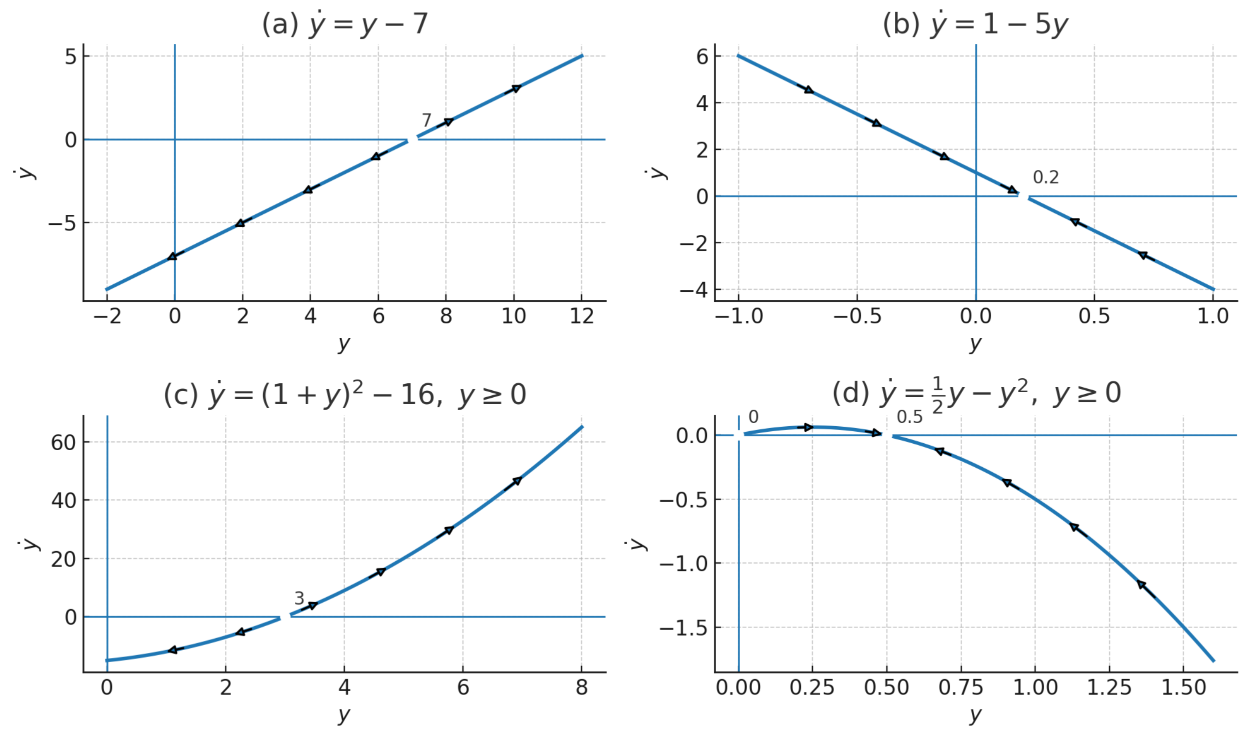


Figure 1: Q1

Moreover $f'(y) = -1 - \frac{1}{y} < 0$ for all $y > 0$, so the phase line points upward when $y < y$ and downward when $y > y$. Therefore y is *globally asymptotically stable* on $(0, \infty)$: every solution with $y_0 > 0$ moves monotonically toward y (no overshooting in 1D).

(b) $\dot{y} = e^y - (y + 2)$ (domain $y \in \mathbb{R}$)

Let $h(y) = e^y - y - 2$. Then $h'(y) = e^y - 1$, $h''(y) = e^y > 0$, so h is convex with a unique minimum at $y = 0$, where $h(0) = -1 < 0$. Since $\lim_{y \to \pm\infty} h(y) = +\infty$, there are exactly two equilibria $y_1 < 0 < y_2$ solving $e^y = y + 2$. They can be written in closed form as

$$y_k = -2 - W_k\left(-e^{-2}\right), \qquad k \in \{0, -1\},$$

with numerical values $y_1 \approx -1.84$ (principal branch W_0) and $y_2 \approx 1.15$ (lower branch W_{-1}). Stability follows from $f'(y) = e^y - 1$: since $e^{y_1} < 1$ and $e^{y_2} > 1$,

y_1 is *asymptotically stable*, $\qquad$ y_2 is *unstable*.

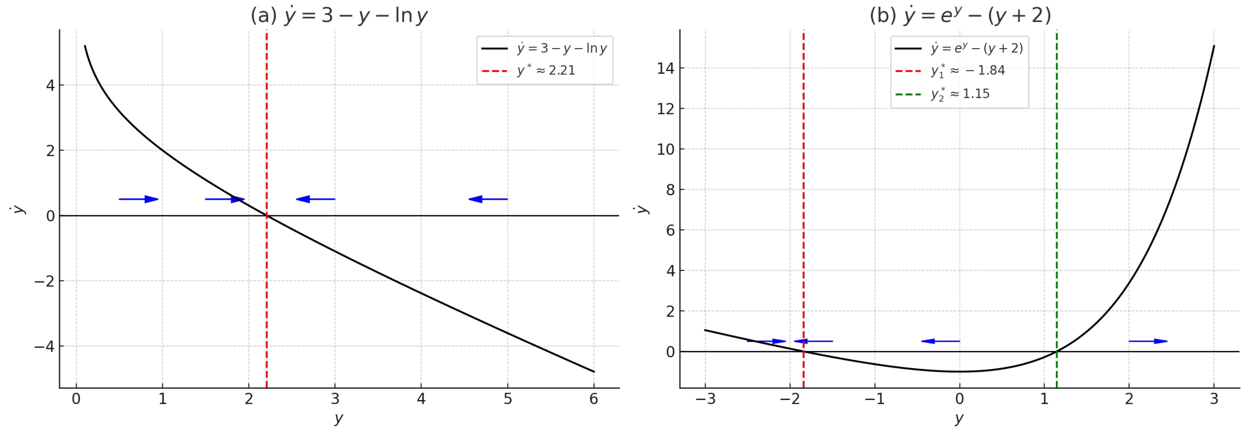


Figure 2: Q2

Phase-line directions: $f(y) > 0$ for $y < y_1$, $f(y) < 0$ on (y_1, y_2), and $f(y) > 0$ for $y > y_2$. Hence all trajectories with $y_0 < y_2$ converge monotonically to y_1, while those with $y_0 > y_2$ diverge to $+\infty$.

3 Q3

Consider the planar autonomous system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y),$$

and let $(\bar{x}, \bar{y})$ be an equilibrium $f(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y}) = 0$. The Jacobian at the equilibrium is

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \Big|_{(\bar{x}, \bar{y})}.$$

The local phase portrait is governed by the linearization $\dot{\mathbf{z}} = J\mathbf{z}$ with $\mathbf{z} = (x - \bar{x},\ y - \bar{y})^\top$.

(a) $f_x = 0,\ f_y > 0,\ g_x > 0,\ g_y = 0$

Here

$$J = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad a = f_y > 0,\ \ b = g_x > 0.$$

Eigenvalues solve $\lambda^2 - ab = 0$, hence

$$\lambda_{1,2} = \pm\sqrt{ab} \in \mathbb{R} \setminus \{0\}.$$

Because the eigenvalues are real with opposite signs, the equilibrium is a *saddle*. The corresponding eigenvectors have slopes

$$m_\pm = \frac{dy}{dx} = \frac{\lambda_\pm}{a} = \pm\sqrt{\frac{b}{a}},$$

so the stable manifold is the line with slope $m_- < 0$ and the unstable manifold is the line with slope $m_+ > 0$ through $(\bar{x}, \bar{y})$.

Nullclines and streamlines. The nullclines are the axes of the linearization: $\dot{x} = 0$ on $y = \bar{y}$ and $\dot{y} = 0$ on $x = \bar{x}$. Moreover the quadratic invariant

$$\frac{d}{dt}\big(b(x-\bar{x})^2 - a(y-\bar{y})^2\big) = 2b(x-\bar{x})\dot{x} - 2a(y-\bar{y})\dot{y} = 0$$

implies that the trajectories are the hyperbolas

$$b(x-\bar{x})^2 - a(y-\bar{y})^2 = C,$$

with asymptotes given by the eigendirections $m_\pm$; orbits approach the stable manifold and are repelled along the unstable manifold.

(b) $f_x < 0,\ f_y > 0,\ g_x < 0,\ g_y < 0$

Write $J = -\alpha\beta$
$-\gamma - \delta$ with $\alpha, \beta, \gamma, \delta > 0$. Then

$$\text{trace } \tau = \operatorname{tr} J = -(\alpha+\delta) < 0, \qquad \text{determinant } \Delta = \det J = \alpha\delta + \beta\gamma > 0.$$

Hence every eigenvalue has negative real part and the equilibrium is *asymptotically stable*. The precise nature is determined by the discriminant:

$$D = \tau^2 - 4\Delta = (\alpha-\delta)^2 - 4\beta\gamma.$$

- If $D > 0$ (i.e. $\beta\gamma < \frac{(\alpha-\delta)^2}{4}$), the eigenvalues are real and negative: a *stable node*. Trajectories are tangent to the two eigendirections and approach monotonically.
- If $D = 0$, a *degenerate/stellar node* (repeated negative eigenvalue).
- If $D < 0$ (i.e. $\beta\gamma > \frac{(\alpha-\delta)^2}{4}$), the eigenvalues are complex with negative real part: a *stable focus* (spiral sink).

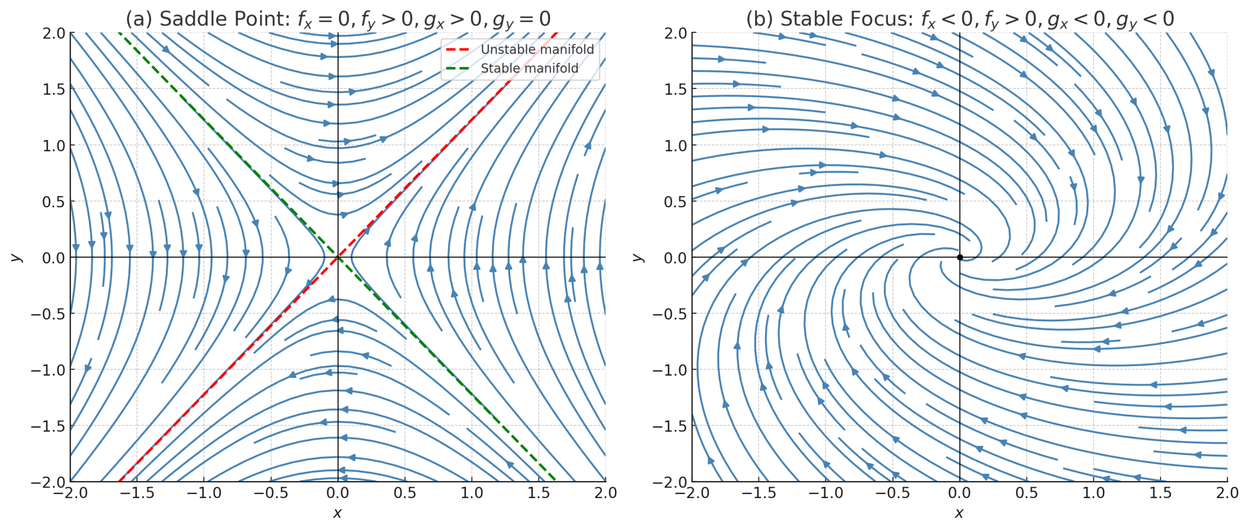


Figure 3: Q3

Orientation (clockwise vs. counterclockwise). Evaluate the vector field on the axes: at $(x > 0, y = 0)$, $\dot{x} = -\alpha x < 0$, $\dot{y} = -\gamma x < 0$ (moves left and down); at $(x = 0, y > 0)$, $\dot{x} = \beta y > 0$, $\dot{y} = -\delta y < 0$ (moves right and down). Thus the rotation around the equilibrium is *clockwise*. Streamlines either spiral clockwise into the equilibrium (focus) or approach it along two inward-pointing directions (node), depending on D as above.

4 Q4 Local stability

For each system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, we find equilibria and classify them via the Jacobian $J = f_x f_y$ $g_x g_y$.

(a) $\dot{x} = e^x - 1,\ \dot{y} = ye^x$

Equilibria solve $e^x - 1 = 0$ and $ye^x = 0$, hence only $(0, 0)$.

$$J(x, y) = \begin{pmatrix} e^x & 0 \\ ye^x & e^x \end{pmatrix}, \qquad J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Eigenvalues $(1, 1)$: a *repelling (unstable) node*. Therefore $(0, 0)$ is a *locally unstable equilibrium*.

(b) $\dot{x} = x + 2y,\ \dot{y} = x^2 + y$

Equilibria from $x + 2y = 0$ and $x^2 + y = 0$ give

$$(x, y) = (0, 0), \qquad (x, y) = \left(\tfrac{1}{2}, -\tfrac{1}{4}\right).$$

Jacobian $J(x, y) = 12$
$2x1$.

$$J(0,0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = (1, 1) \text{ (defective)} \Rightarrow \textit{unstable improper node (source)}.$$

$$J\left(\tfrac{1}{2}, -\tfrac{1}{4}\right) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \lambda = 1 \pm \sqrt{2} \Rightarrow \textit{saddle}.$$

(c) $\dot{x} = 1 - e^y,\ \dot{y} = 5x - y$

Equilibrium $(0, 0)$. The Jacobian

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 5 & -1 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 + \lambda + 5.$$

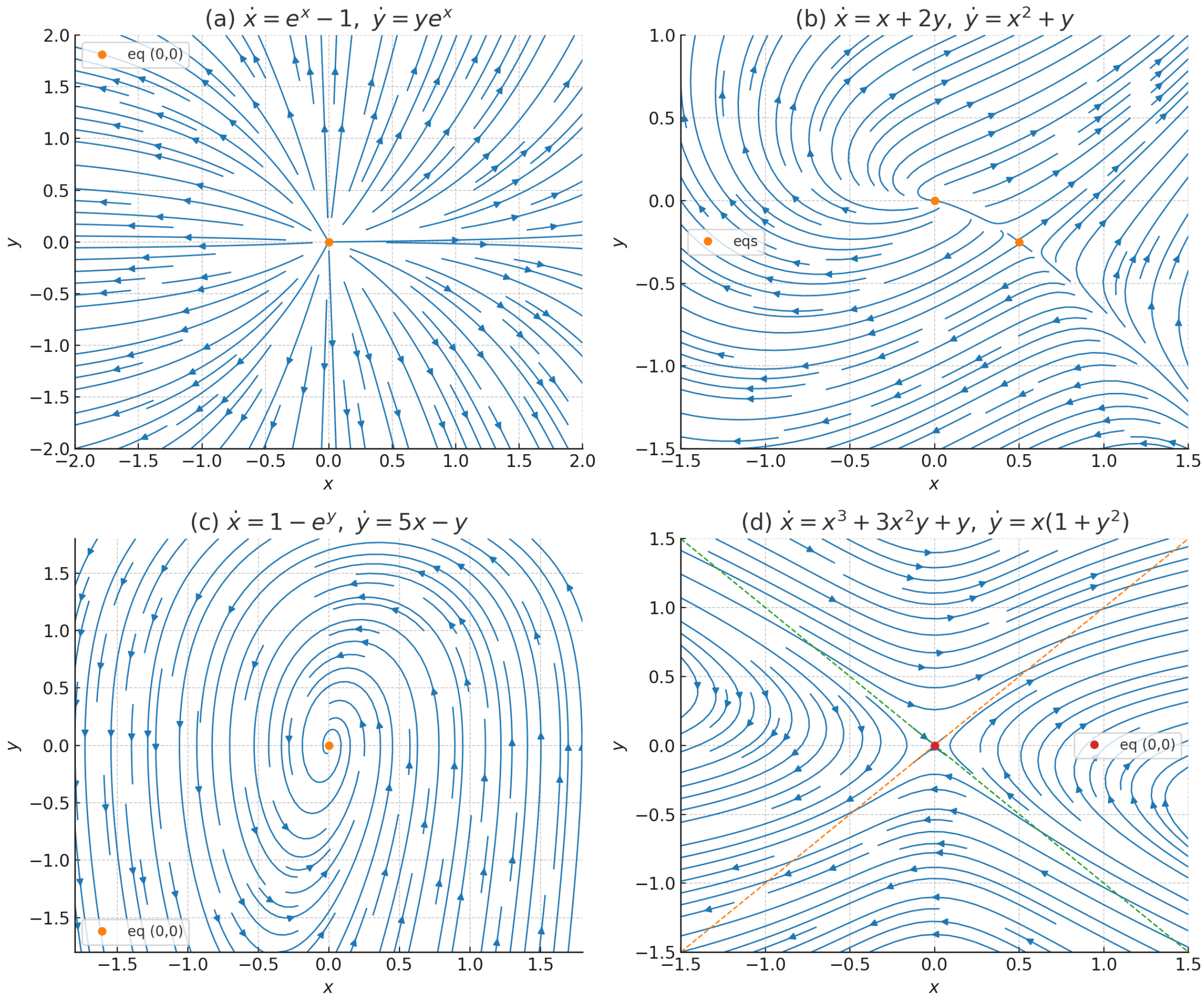
Eigenvalues $\lambda = \frac{-1 \pm i\sqrt{19}}{2}$ have negative real part; thus $(0, 0)$ is a *locally asymptotically stable focus (spiral sink)*. From $\dot{x}(0, y > 0) < 0$ and $\dot{y}(x > 0, 0) > 0$, the rotation is *clockwise*.

(d) $\dot{x} = x^3 + 3x^2y + y,\ \dot{y} = x(1 + y^2)$

Equilibria: $x(1 + y^2) = 0 \Rightarrow x = 0$, then $\dot{x} = y \Rightarrow y = 0$; only $(0, 0)$. The Jacobian

$$J(x, y) = \begin{pmatrix} 3x^2 + 6xy & 3x^2 + 1 \\ 1 + y^2 & 2xy \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues $\lambda = \pm 1$ of opposite sign $\Rightarrow$ *saddle*. Stable/unstable eigendirections are $y = \mp x$.


(a) $\dot{x} = e^x - 1,\ \dot{y} = ye^x$
eq (0,0)
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eqs
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x
y

Figure 4: Q4