MATH CAMP ASSIGNMENT 2

ANALYSIS

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Q1. The function $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$

Proposition 1. *Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |x - y|$. Then d is a metric on $\mathbb{R}$.*

Proof. We verify the three axioms.

(M1) Nonnegativity and identity of indiscernibles. By the definition of absolute value, $|x - y| \geq 0$ for all x, y. Moreover, $|x - y| = 0$ iff $x - y = 0$ (by the definition again), i.e. iff $x = y$.

(M2) Symmetry. We have $x - y = -(y - x)$, hence

$$d(x, y) = |x - y| = |-(y - x)| = |-1| \cdot |y - x| = |y - x| = d(y, x),$$

where $|ab| = |a||b|$ from Q10(a) and $|-1| = 1$ since $|1| = 1$ and $|1| = |-1| \cdot |-1|$.

(M3) Triangle inequality. For any $x, y, z \in \mathbb{R}$,

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y),$$

by the triangle inequality $|a + b| \leq |a| + |b|$ from Q10(b) with $a = x - z$ and $b = z - y$.

Thus d satisfies (M1)–(M3), so it is a metric on $\mathbb{R}$. □

Q2. If S is bounded, then $S \subset B_\rho(y)$ for every $y \in X$ and some $\rho > 0$

Proposition 2. *Let (X, d) be a metric space and let $S \subset X$ be bounded, i.e. there exist $a \in X$ and $R > 0$ such that $S \subset B_R(a) = \{x \in X : d(x, a) < R\}$. Then for every $y \in X$ there exists $\rho > 0$ with $S \subset B_\rho(y)$.*

Proof. Fix $y \in X$. Define

$$\rho := R + d(a,y) \quad (> 0).$$

Take any $x \in S$. Since $S \subset B_R(a)$ we have $d(x,a) < R$. By the triangle inequality,

$$d(x,y) \leq d(x,a) + d(a,y) < R + d(a,y) = \rho.$$

Hence $x \in B_\rho(y)$. Because $x \in S$ was arbitrary, $S \subset B_\rho(y)$. Since y was arbitrary as well, the claim holds for every $y \in X$. $\square$

Q3. $(0,1) = \bigcup_{n=1}^{\infty} [1/n,\, 1 - 1/n]$

Proposition 3. *For the standard metric on* $\mathbb{R}$,

$$(0,1) = \bigcup_{n=1}^{\infty} [1/n,\, 1 - 1/n].$$

(We adopt the convention $[a,b] = \varnothing$ *if* $a > b$*; thus the* $n = 1$ *term is empty and harmless.)*

Proof. *(⊆)* Let $x \in [1/n,\, 1 - 1/n]$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then $1/n \leq x \leq 1 - 1/n$, hence $x > 0$ and $x < 1$. Therefore $x \in (0,1)$. So $\bigcup_{n=1}^{\infty} [1/n,\, 1 - 1/n] \subseteq (0,1)$.

(⊇) Let $x \in (0,1)$. Choose $n \in \mathbb{N}$ such that

$$n > \max\left\{\frac{1}{x},\, \frac{1}{1-x}\right\}.$$

(For instance, take $n = \lceil \max\{1/x,\, 1/(1-x)\} \rceil + 1$.) Then $1/n < x$ and $1/n < 1 - x$, hence

$$1/n < x < 1 - 1/n,$$

so $x \in [1/n,\, 1 - 1/n]$. As x was arbitrary, $(0,1) \subseteq \bigcup_{n=1}^{\infty} [1/n,\, 1 - 1/n]$.

Combining the two inclusions gives the desired equality. $\square$

Remark. Each set $[1/n,\, 1 - 1/n]$ is closed in $\mathbb{R}$, yet their (countably) infinite union equals the open interval $(0,1)$. Hence an infinite union of closed sets can be open.

Q4. Show that $\operatorname{ext} A = \operatorname{int}(A^c)$

Proposition 4. *Let (X, d) be a metric space and $A \subset X$. Then*

$$\operatorname{ext} A = \operatorname{int}(A^c).$$

Proof. ($\subseteq$) If $x \in \operatorname{ext} A$, by definition there exists $r > 0$ such that $B_r(x) \subset A^c$. This exactly says that x is an interior point of A^c, i.e. $x \in \operatorname{int}(A^c)$.

($\supseteq$) Conversely, if $x \in \operatorname{int}(A^c)$, then some ball $B_r(x)$ is contained in A^c; hence x is an exterior point of A, i.e. $x \in \operatorname{ext} A$.

Thus $\operatorname{ext} A = \operatorname{int}(A^c)$. □

Q5. Show that $\operatorname{int} A$ and $\operatorname{ext} A$ are open

Proposition 5. *For any $A \subset X$ in a metric space (X, d), both* $\operatorname{int} A$ *and* $\operatorname{ext} A$ *are open subsets of X.*

Proof. **(a)** $\operatorname{int} A$ **is open.** Take $x \in \operatorname{int} A$. Then there exists $R > 0$ with $B_R(x) \subset A$. Let $y \in B_R(x)$ and set $\rho := R - d(x, y) > 0$. If $z \in B_\rho(y)$, then by the triangle inequality

$$d(z, x) \leq d(z, y) + d(y, x) < \rho + d(x, y) = R,$$

so $z \in B_R(x) \subset A$. Hence $B_\rho(y) \subset A$, which means $y \in \operatorname{int} A$. Therefore $B_R(x) \subset \operatorname{int} A$, showing that every $x \in \operatorname{int} A$ has an open ball contained in $\operatorname{int} A$; thus $\operatorname{int} A$ is open.

(b) $\operatorname{ext} A$ **is open.** Let $x \in \operatorname{ext} A$. Then some $R > 0$ satisfies $B_R(x) \subset A^c$. For any $y \in B_R(x)$ define $\rho := R - d(x, y) > 0$. If $z \in B_\rho(y)$, then

$$d(z, x) \leq d(z, y) + d(y, x) < \rho + d(x, y) = R,$$

hence $z \in B_R(x) \subset A^c$. Therefore $B_\rho(y) \subset A^c$ and $y \in \operatorname{ext} A$. Thus $B_R(x) \subset \operatorname{ext} A$, proving $\operatorname{ext} A$ is open. □

Q6. $(1/2, 1]$ is open in the subspace $X = [0, 1]$

Proposition 6. *Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ restricted to X. Then the subset $U = (1/2, 1]$ is open in X.*

Proof. Recall that an open ball in the subspace X is

$$B_r^X(x) = \{y \in X : \ d(x, y) < r\} = (x - r, x + r) \cap [0, 1].$$

We must show that for every $x \in U$ there exists $r > 0$ with $B_r^X(x) \subset U$.

Case 1: $x = 1$. Take $r = \frac{1}{2}$. Then

$$B_{1/2}^X(1) = (1 - \tfrac{1}{2}, 1 + \tfrac{1}{2}) \cap [0, 1] = (1/2, 1] \subset U.$$

Case 2: $x \in (1/2, 1)$. Set

$$r := \min\{x - \tfrac{1}{2}, \ 1 - x\} > 0.$$

If $y \in B_r^X(x)$ then $|y - x| < r \leq x - \frac{1}{2}$ and $|y - x| < r \leq 1 - x$. Hence $y > x - (x - \frac{1}{2}) = \frac{1}{2}$ and $y < x + (1 - x) = 1$, i.e. $y \in (1/2, 1)$. Therefore $B_r^X(x) \subset (1/2, 1) \subset U$.

In both cases there exists $r > 0$ with $B_r^X(x) \subset U$. Thus U is open in X. □

Q7. (x_n) with $x_n = \frac{1}{n^2}$ converges to 0

Proposition 7. *In the metric space $(\mathbb{R}, |\cdot|)$, the sequence $x_n = 1/n^2$ satisfies $x_n \to 0$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\sqrt{\varepsilon}}$ (equivalently $1/N^2 < \varepsilon$). Then for every $n \geq N$,

$$d(x_n, 0) = \left|x_n - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

By the definition of convergence in a metric space, $x_n \to 0$. □

Q8. If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$

Proposition 8. *Let (X, d) be a metric space. If $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.*

Proof. By the triangle inequality,

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n), \qquad d(x, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y).$$

Rearranging gives, for all n,

$$\big|\, d(x_n, y_n) - d(x, y) \,\big| \le d(x_n, x) + d(y_n, y). \tag{$*$}$$

Since $x_n \to x$ and $y_n \to y$, we have $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$. By Proposition 18(1) (squeeze: if $0 \le u_n \le v_n$ and $v_n \to 0$ then $u_n \to 0$), from $(*)$ it follows that $\left|d(x_n, y_n) - d(x, y)\right| \to 0$. Therefore $d(x_n, y_n) \to d(x, y)$. ☐

Q9. Completeness of $\mathbb{R}^n$ via Prop. 21, Thm. 29 and BW (Thm. 30)

Proposition 9 (Proposition 23: $\mathbb{R}^n$ is complete)**.** *Let (x_n) be a sequence in $\mathbb{R}^n$ with the Euclidean metric d_E. Then (x_n) converges if and only if it is Cauchy. In particular, every Cauchy sequence in $\mathbb{R}^n$ converges.*

Proof. ($\Rightarrow$) If (x_n) converges, then it is Cauchy by Proposition 21(2).

($\Leftarrow$) Suppose (x_n) is Cauchy. By Proposition 21(1), every Cauchy sequence is bounded; hence (x_n) is bounded. By the Bolzano–Weierstrass Theorem (Theorem 30), a bounded sequence in $\mathbb{R}^n$ has a convergent subsequence: there exist indices n_k and some $x \in \mathbb{R}^n$ with $x_{n_k} \to x$. Now apply Theorem 29: a Cauchy sequence that has a subsequence converging to x must itself converge to x. Hence $x_n \to x$.

Combining the two implications gives the equivalence, and in particular the completeness of $\mathbb{R}^n$. ☐

Q10. ε–δ proof that $\sqrt{x}$ is continuous at $x = 1$

Proposition 10. *The function $f(x) = \sqrt{x}$ is continuous at $x = 1$.*

Proof. Let $\varepsilon > 0$ be given. Choose

$$\delta := \min\{1, \varepsilon\}.$$

Assume $|x-1| < \delta$. Then $\delta \le 1$ implies $x \in (0,2)$, hence $\sqrt{x} > 0$ and

$$|\sqrt{x} - 1| = \frac{|x-1|}{\sqrt{x}+1} \le \frac{|x-1|}{1} = |x-1| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-1| < \delta \Rightarrow |\sqrt{x}-1| < \varepsilon$. By the ε–δ definition of continuity, f is continuous at $x = 1$. ☐

Q11. Continuity of $f(x) = \sqrt{x}$ on $[0, \infty)$ via preimages of open sets

Proposition 11. *Let $f : [0,\infty) \to [0,\infty)$ be given by $f(x) = \sqrt{x}$. Then f is continuous.*

Proof. We use the topological characterization of continuity: f is continuous iff for every open set $U \subset [0,\infty)$ (with the subspace topology) the preimage $f^{-1}(U)$ is open in $[0,\infty)$.

Let $U \subset [0,\infty)$ be open. Then there exists an open set $V \subset \mathbb{R}$ such that $U = V \cap [0,\infty)$. By the stated theorem, we can write

$$V = \bigcup_{i \ge 1} I_i$$

as a countable union of pairwise disjoint open intervals $I_i = (a_i, b_i)$, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ and $a_i < b_i$. Hence

$$U = V \cap [0,\infty) = \bigcup_{i \ge 1} \big(I_i \cap [0,\infty)\big), \qquad f^{-1}(U) = \bigcup_{i \ge 1} f^{-1}\big(I_i \cap [0,\infty)\big).$$

It therefore suffices to show that $f^{-1}(J)$ is open in $[0,\infty)$ for each interval J of the form $I_i \cap [0,\infty)$.

There are four possibilities (empty intersections are irrelevant):

(1) $J = (a,b)$ with $0 \le a < b < \infty$. Then

$$f^{-1}(J) = \{x \ge 0 : \ a < \sqrt{x} < b\} = (a^2, b^2),$$

which is open in $[0,\infty)$.

(2) $J = [0,b)$ with $0 < b < \infty$ (this occurs when $a_i < 0 < b_i$). Then

$$f^{-1}(J) = \{x \ge 0 : \ 0 \le \sqrt{x} < b\} = [0, b^2),$$

which is open in the subspace $[0,\infty)$ since $[0,b^2) = (-1,b^2) \cap [0,\infty)$.

(3) $J = (a,\infty)$ with $a \geq 0$ (this occurs when $b_i = +\infty$ and $a_i \geq 0$). Then

$$f^{-1}(J) = \{x \geq 0 : \sqrt{x} > a\} = (a^2,\infty),$$

open in $[0,\infty)$.

(4) $J = [0,\infty)$ (this occurs when $b_i = +\infty$ and $a_i < 0$). Then

$$f^{-1}(J) = [0,\infty),$$

which is open in $[0,\infty)$.

Thus in every case $f^{-1}(J)$ is open in $[0,\infty)$. Since a (countable) union of open sets is open, $f^{-1}(U)$ is open in $[0,\infty)$ for every open $U \subset [0,\infty)$. Hence f is continuous on $[0,\infty)$. $\square$