MATH CAMP ASSIGNMENT 4

DIFFERENTIATION & CONVEXITY

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1 Q1

Claim. If $f'(x) > 0$ on (a, b), then f is strictly increasing on (a, b) and, assuming f extends continuously to $[a, b]$, f is a bijection from (a, b) onto $(f(a), f(b))$.

Proof. For $a < x < y < b$, the Mean Value Theorem (MVT) yields $c \in (x, y)$ with

$$f(y) - f(x) = f'(c)(y - x) > 0.$$

Hence f is strictly increasing and therefore injective. For $t \in (f(a), f(b))$, set $h(x) = f(x) - t$. Then $h(a) < 0 < h(b)$, so by the Intermediate Value Theorem there is $x \in (a, b)$ with $h(x) = 0$, i.e. $f(x) = t$. Thus f is surjective onto $(f(a), f(b))$. □

2 Q2

Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable with $|g'(x)| \leq M$ for all x and some $M > 0$. Fix $\varepsilon > 0$ and define $f(x) = x + \varepsilon g(x)$.

Claim. If $\varepsilon M < 1$, then f is a bijection $\mathbb{R} \to \mathbb{R}$ (in fact strictly increasing).

Proof. $f'(x) = 1 + \varepsilon g'(x) \geq 1 - \varepsilon M =: c > 0$, hence f is strictly increasing and injective. By the MVT, for $x < y$,

$$f(y) - f(x) = f'(\xi)(y - x) \geq c(y - x),$$

so $f(x) \to \pm\infty$ as $x \to \pm\infty$, implying surjectivity onto $\mathbb{R}$. □

3 Q3

Suppose f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Define $g(x) = f(x+1) - f(x)$.

Claim. $g(x) \to 0$ as $x \to \infty$.

Proof. By the MVT applied on $[x, x+1]$, there exists $\xi_x \in (x, x+1)$ with

$$g(x) = f(x+1) - f(x) = f'(\xi_x) \to 0 \quad (x \to \infty).$$

$\square$

4 Q4

Let f be continuous, differentiable for $x \neq 0$, and $f'(x) \to 3$ as $x \to 0$.

Claim. $f'(0)$ exists and equals 3.

Proof. For $h > 0$, by the MVT on $[0, h]$ there is $\xi_h \in (0, h)$ with $\dfrac{f(h) - f(0)}{h} = f'(\xi_h) \to 3$ as $h \downarrow 0$. Thus the right derivative at 0 equals 3. Similarly on $[-h, 0]$ we obtain the left derivative equals 3. Hence $f'(0) = 3$. $\square$

5 Q5

Assume f is twice continuously differentiable and $f^{(3)}$ exists in a neighborhood of $\hat{x}$. By Taylor's theorem with Lagrange remainder, for x near $\hat{x}$ there is ξ between x and $\hat{x}$ such that

$$f(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x}) + \frac{1}{2} f''(\hat{x})(x - \hat{x})^2 + \frac{f^{(3)}(\xi)}{6}(x - \hat{x})^3.$$

If $|f^{(3)}| \leq K$ near $\hat{x}$, the remainder is bounded by $\frac{K}{6}|x - \hat{x}|^3$.

6 Q6

Let $A \in L(X, Y)$ be linear.

(i) If $Ax = 0$ only when $x = 0$, then A is injective: indeed, if $Ax_1 = Ax_2$ then $A(x_1 - x_2) = 0$, so $x_1 = x_2$.

(ii) Conversely, if A is injective and $Ax = 0$, then $Ax = A0$, hence $x = 0$.

7 Q7

Let X be an n-dimensional vector space and $A \in L(X)$. Fix a basis $\{v_1, \ldots, v_n\}$.

(a)

Claim. $\text{Range}(A) = \text{span}\{Av_1, \ldots, Av_n\}$.

Proof. For $x = \sum_i \alpha_i v_i$, linearity gives $Ax = \sum_i \alpha_i Av_i$, so $\text{Range}(A) \subset \text{span}\{Av_i\}$. The reverse inclusion is obvious since each Av_i is in the range. □

(b)

Claim. A is surjective $\iff$ $\{Av_1, \ldots, Av_n\}$ is linearly independent.

Proof. By (a), A is surjective iff its range equals X, i.e. iff $\text{span}\{Av_i\} = X$. In an n-dimensional space, n vectors span X iff they are linearly independent. □

(c)

Claim. If A is injective, then $\{Av_1, \ldots, Av_n\}$ is linearly independent.

Proof. If $\sum_i \alpha_i Av_i = 0$, then $A(\sum_i \alpha_i v_i) = 0$. Injectivity implies $\sum_i \alpha_i v_i = 0$, hence all $\alpha_i = 0$. □

(d)

Claim. If $\{Av_1, \ldots, Av_n\}$ is linearly independent, then A is injective.

Proof. If $Ax = 0$ with $x = \sum_i \alpha_i v_i$, then $0 = Ax = \sum_i \alpha_i Av_i$. Linear independence yields all $\alpha_i = 0$, hence $x = 0$. □

Therefore, on X finite-dimensional, A is injective iff A is surjective.

8 Q8 (Inverse Function Theorem: 1D case)

Let $f : (a,b) \to \mathbb{R}$ be C^1 and let $x_0 \in (a,b)$ with $f'(x_0) \neq 0$. Then there exist neighborhoods $U \ni x_0$ and $V \ni f(x_0)$ such that $f : U \to V$ is a bijection with C^1 inverse $g = f^{-1}$ and

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (x \in U).$$

Equivalently, $(f^{-1})'(y) = 1/f'(x)$ with $y = f(x)$.

9 Q9 (Implicit Function Theorem: 1D–1D case)

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^1, write variables as (x,u), and suppose $F(x_0,u_0) = 0$ with $F_x(x_0,u_0) \neq 0$. Then there exists a neighborhood $U \ni u_0$ and a unique C^1 function $x = g(u)$ such that $F(g(u),u) = 0$ for $u \in U$. Moreover,

$$g'(u) = -\frac{F_u(g(u),u)}{F_x(g(u),u)}.$$

10 Q10 (Perturbation Methods)

We wish to solve $f(x,\varepsilon) = 0$ for $x = x(\varepsilon)$ near $\varepsilon = 0$. Assume there is x_0 with $f(x_0,0) = 0$.

(a) First derivative via the Implicit Function Theorem

Assume $f \in C^1$ near $(x_0,0)$ and $f_x(x_0,0) \neq 0$. Then by Q9 there is a unique C^1 function $x(\varepsilon)$ with $f(x(\varepsilon),\varepsilon) = 0$ for $|\varepsilon|$ small and

$$x'(\varepsilon) = -\frac{f_\varepsilon(x(\varepsilon),\varepsilon)}{f_x(x(\varepsilon),\varepsilon)}, \qquad x'(0) = -\frac{f_\varepsilon(x_0,0)}{f_x(x_0,0)}.$$

(b) First-order approximation (Taylor)

Since x is C^1 at 0,

$$x(\varepsilon) = x_0 + x'(0)\,\varepsilon + o(\varepsilon) = x_0 - \frac{f_\varepsilon(x_0,0)}{f_x(x_0,0)}\,\varepsilon + o(\varepsilon).$$

(c) Second derivative and second-order approximation

Assume further $f \in C^2$ near $(x_0, 0)$ and f_x stays nonzero along the solution curve. Differentiate $f_x\, x' + f_\varepsilon = 0$ once more in ε to obtain

$$f_x\, x'' + f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon} = 0,$$

all functions evaluated at $(x(\varepsilon), \varepsilon)$. Hence

$$x''(\varepsilon) = -\,\frac{f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon}}{f_x}.$$

At $\varepsilon = 0$, let

$$A = f_x(x_0, 0), \quad B = f_\varepsilon(x_0, 0), \quad C = f_{xx}(x_0, 0), \quad D = f_{x\varepsilon}(x_0, 0), \quad E = f_{\varepsilon\varepsilon}(x_0, 0).$$

Then $x'(0) = -B/A$ and

$$x''(0) = -\,\frac{CB^2 - 2DAB + EA^2}{A^3}.$$

Therefore the second-order Taylor approximation around $\varepsilon = 0$ is

$$x(\varepsilon) = x_0 - \frac{B}{A}\,\varepsilon - \frac{CB^2 - 2DAB + EA^2}{2A^3}\,\varepsilon^2 + o(\varepsilon^2).$$

11 Q11 (Convex hull of finitely many points)

Let V be a vector space and $S = \{x_1, \ldots, x_n\} \subset V$. Denote

$$A_n := \Big\{ \sum_{i=1}^{n} \lambda_i x_i :\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \Big\}.$$

Claim. $\mathrm{Co}(x_1, \ldots, x_n) = A_n$.

Proof. First, A_n contains S (take one coefficient 1, others 0) and is convex: if $y = \sum \alpha_i x_i$, $z = \sum \beta_i x_i$ are in A_n and $\lambda \in [0, 1]$, then

$$\lambda y + (1-\lambda)z = \sum_{i=1}^{n} \big(\lambda\alpha_i + (1-\lambda)\beta_i\big)x_i \in A_n.$$

By minimality of the convex hull, $\mathrm{Co}(S) \subset A_n$.

Conversely, let C be any convex set containing S. We show $A_n \subset C$ by induction in n. The cases $n = 1, 2$ are trivial by convexity. Suppose true for $n-1$. For $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$, set $s = \sum_{i=1}^{n-1} \lambda_i$. If $s = 0$ then $y = x_n \in C$. Otherwise put $\mu_i = \lambda_i / s$, then $z = \sum_{i=1}^{n-1} \mu_i x_i \in C$ by the induction hypothesis and $y = sz + (1-s)x_n \in C$ by convexity. Hence $A_n \subset C$. Taking the intersection of all such C yields $A_n \subset \mathrm{Co}(S)$. $\square$

12 Q12 (Epigraph characterization)

Let $S \subset \mathbb{R}^n$ be convex and $f : S \to \mathbb{R}$. The epigraph of f is

$$\operatorname{epi} f = \{(x,t) \in S \times \mathbb{R} : \; t \geq f(x)\}.$$

Claim. f is convex $\iff$ $\operatorname{epi} f$ is convex.

Proof. ($\Rightarrow$) Take $(x,t),(y,s) \in \operatorname{epi} f$ and $\lambda \in [0,1]$. Then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda t + (1-\lambda)s,$$

so $(\lambda x + (1-\lambda)y, \lambda t + (1-\lambda)s) \in \operatorname{epi} f$.

($\Leftarrow$) If $\operatorname{epi} f$ is convex, apply it to $(x, f(x))$ and $(y, f(y))$ to get $(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \operatorname{epi} f$, i.e. $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. $\square$

13 Q13 (Convex $\Rightarrow$ Quasi-convex)

A function $f : S \to \mathbb{R}$ is quasi-convex iff for all x, y and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

If f is convex, then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \max\{f(x), f(y)\}.$$

Equivalently, all lower level sets $L_a = \{x : f(x) \leq a\}$ are convex because for $x, y \in L_a$, $f(\lambda x + (1-\lambda)y) \leq \lambda a + (1-\lambda)a = a$. Hence convexity implies quasi-convexity. $\square$