MATH CAMP ASSIGNMENT 1

LOGIC AND SET THEORY

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1 Q1

According to the definitions of logical operations, the completed truth table is as follows:

p	q	$\neg q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

2 Q2

We use the truth table to verify that $p \Rightarrow q \equiv \neg(p \wedge \neg q)$.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

From the table, we observe that $\neg(p \wedge \neg q)$ and $p \Rightarrow q$ have exactly the same truth values in all cases. Therefore, we conclude that:

$$p \Rightarrow q \equiv \neg(p \wedge \neg q).$$

3 Q3

Setup. Let X and Y be sets and let $f : X \to Y$ be a function. For $S \subset X$, the *image* of S under f is

$$f(S) := \{ f(x) : x \in S \}.$$

For $T \subset Y$, the *inverse image* (preimage) of T under f is

$$f^{-1}(T) := \{ x \in X : f(x) \in T \}.$$

(a) If $S_1 \subset S_2$, then $f(S_1) \subset f(S_2)$.

Proof. Take any $y \in f(S_1)$. By the definition of image, there exists $x \in S_1$ such that $y = f(x)$. Since $S_1 \subset S_2$, we also have $x \in S_2$, hence $y = f(x) \in f(S_2)$ by the definition of image. Because every element of $f(S_1)$ is an element of $f(S_2)$, we conclude $f(S_1) \subset f(S_2)$. □

(b) If $T_1 \subset T_2$, then $f^{-1}(T_1) \subset f^{-1}(T_2)$.

Proof. Take any $x \in f^{-1}(T_1)$. By the definition of preimage, $f(x) \in T_1$. Since $T_1 \subset T_2$, it follows that $f(x) \in T_2$, and therefore $x \in f^{-1}(T_2)$ by the definition of preimage. Hence $f^{-1}(T_1) \subset f^{-1}(T_2)$. □

Remark. In (a) the inclusion need not be equality in general (it may be strict); similarly for (b).

(c) Prove that

$$A - (B - C) = (A \cap C) \cup (A \cap B^c).$$

Proof. Recall the definitions:

- $B - C := \{x : x \in B \ \wedge \ x \notin C\} = B \cap C^c$.
- $A - D := \{x : x \in A \ \wedge \ x \notin D\}$ for any set D.
- Two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$.

We will prove the equality by showing both inclusions.

(1) Show that $A - (B - C) \subseteq (A \cap C) \cup (A \cap B^c)$**.**

Let $x \in A - (B - C)$. By definition, this means:

$$x \in A \quad \text{and} \quad x \notin (B - C).$$

Since $B - C = B \cap C^c$, we have:

$$x \notin (B \cap C^c) \quad \Longrightarrow \quad x \notin B \text{ or } x \in C.$$

Thus, two cases arise:

- If $x \in C$, then $x \in A \cap C$.
- If $x \notin B$, then $x \in A \cap B^c$.

In both cases, $x \in (A \cap C) \cup (A \cap B^c)$. Therefore:

$$A - (B - C) \subseteq (A \cap C) \cup (A \cap B^c).$$

(2) Show that $(A \cap C) \cup (A \cap B^c) \subseteq A - (B - C)$**.**

Take any $x \in (A \cap C) \cup (A \cap B^c)$. Then either $x \in A \cap C$ or $x \in A \cap B^c$.

- If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, we know $x \notin B \cap C^c = B - C$. Hence $x \in A - (B - C)$.
- If $x \in A \cap B^c$, then $x \in A$ and $x \notin B$. Thus $x \notin B \cap C^c$, so $x \in A - (B - C)$.

Therefore:

$$(A \cap C) \cup (A \cap B^c) \subseteq A - (B - C).$$

(3) Conclusion.

Since we have shown both inclusions, the two sets are equal:

$$A - (B - C) = (A \cap C) \cup (A \cap B^c).$$

$\square$

Remark. We can also use

$$\begin{aligned}
A - (B - C) &= A \setminus (B \setminus C) \\
&= A \cap (B \setminus C)^c \qquad (\text{by } A \setminus D = A \cap D^c) \\
&= A \cap (B \cap C^c)^c \\
&= A \cap (B^c \cup C) \qquad (\text{De Morgan}) \\
&= (A \cap B^c) \cup (A \cap C) \qquad (\text{distributivity}) \\
&= (A \cap C) \ \cup \ (A \cap B^c).
\end{aligned}$$

4 Q4

Let $f : X \to Y$ be a function. Recall that

$$X \times Y := \{(x, y) : x \in X,\ y \in Y\},$$

and by the definition of a function, for every $x \in X$ there exists a *unique* $y \in Y$ such that $y = f(x)$.

Claim. The function f can be identified with the subset

$$\Gamma_f \ := \ \{\, (x, y) \in X \times Y : y = f(x) \,\} \subseteq X \times Y,$$

called the *graph* of f.

Proof. First, for any $x \in X$ we have $f(x) \in Y$ (codomain of f), hence $(x, f(x)) \in X \times Y$. Therefore every ordered pair in the set above lies in $X \times Y$, so indeed $\Gamma_f \subseteq X \times Y$.

Second, Γ_f encodes exactly the action of f: by the definition of a function, for each $x \in X$ there exists a unique $y \in Y$ with $y = f(x)$. Equivalently,

$$(\forall x \in X)\ (\exists!\, y \in Y)\ \big((x, y) \in \Gamma_f\big).$$

Thus from Γ_f one recovers f via

$$f(x) \text{ is the unique } y \in Y \text{ such that } (x, y) \in \Gamma_f.$$

Conversely, any subset $R \subseteq X \times Y$ satisfying $(\forall x \in X)(\exists!\, y \in Y)\,((x, y) \in R)$ is the graph of a unique function $X \to Y$, namely $x \mapsto y$ where $(x, y) \in R$. Hence identifying f with its graph Γ_f is legitimate. $\square$

Therefore, as a subset of $X \times Y$,

$$\boxed{f \;\equiv\; \Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y}$$

and the correspondence $f \leftrightarrow \Gamma_f$ is a bijection between functions $X \to Y$ and those subsets of $X \times Y$ with the "exactly one Y-partner per $x \in X$" property.

5 Q5

Let $Id_X : X \to X$ be the identity map defined by $Id_X(x) = x$ for all $x \in X$.

Injective. Assume $Id_X(x_1) = Id_X(x_2)$. By the definition of Id_X, this means $x_1 = x_2$. Hence for every $y \in X$ there is at most one $x \in X$ such that $y = Id_X(x)$, so Id_X is injective.

Surjective. Let $y \in X$ be arbitrary. Taking $x = y$ gives $Id_X(x) = Id_X(y) = y$. Therefore for every $y \in X$ there exists $x \in X$ with $y = Id_X(x)$, so Id_X is surjective.

Conclusion. Since Id_X is both injective and surjective, it is bijective. (Indeed, $Id_X^{-1} = Id_X$.)

6 Q6

(a) Is $(f(A))^c = f(A^c)$ correct?

Answer: No in general. Consider $X = Y = \{0, 1\}$ and the constant map $f : X \to Y$ given by $f(0) = f(1) = 0$. Let $A = \{0\}$. Then

$$f(A) = \{0\}, \qquad (f(A))^c = Y \setminus f(A) = \{1\},$$

while

$$A^c = X \setminus A = \{1\}, \qquad f(A^c) = f(\{1\}) = \{0\}.$$

Hence $(f(A))^c = \{1\} \neq \{0\} = f(A^c)$.

Remark. In general only the preimage distributes over complement: $(f^{-1}(T))^c = f^{-1}(T^c)$. For the image, equality $(f(A^c) = (f(A))^c)$ holds if f is bijective (indeed, injective gives $f(X \setminus A) = f(X) \setminus f(A)$, and surjectivity upgrades $f(X)$ to Y).

(b) If $A \subset B$, $B \subset C$, and $C \subset A$, then $A = B = C$.

Proof. From $A \subset B$ and $B \subset C$ we have $A \subset C$. Together with $C \subset A$ it follows that $A = C$. Then $A \subset B \subset C = A$ implies $A \subset B \subset A$, hence $A = B$. Therefore $A = B = C$. □

7 Q7

(a)

For $k \in \mathbb{N}$, let

$$A_k = [0,1] \cup (1,k].$$

Then $\bigcup_k A_k = [0,\infty)$ and $\bigcap_k A_k = [0,1]$, and the A_k are pairwise distinct.

Another solution is

$$A_k = [0,k]$$

(b)

Let

$$A_1 = (0,\infty), \qquad A_k = \left(3 - \frac{1}{k-1}, 3 + \frac{1}{k-1}\right) \ (k \geq 2).$$

Then $\bigcup_k A_k = (0,\infty)$ and $\bigcap_k A_k = \{3\}$, with all A_k distinct.

(c)

For $k \in \mathbb{N}$, let

$$A_k = \mathbb{N} \cup (-k,k).$$

Then $\bigcup_k A_k = \mathbb{R}$ and $\bigcap_k A_k = \mathbb{N}$, and the A_k are pairwise distinct.

8 Q8

(1) If $f : \mathbb{R} \to \mathbb{R}$ is surjective, then f is unbounded. Suppose, to the contrary, that f is bounded above by some $M \in \mathbb{R}$: $f(\mathbb{R}) \subset (-\infty, M]$. Since f is surjective, there exists $x_0 \in \mathbb{R}$ with $f(x_0) = M + 1$, which contradicts $f(x_0) \le M$. Hence f is not bounded above. The same argument with $-f$ shows that f is not bounded below. Therefore f is unbounded.

(2) A function $g : \mathbb{R} \to \mathbb{R}$ that is unbounded but not surjective. Define

$$g(x) := \begin{cases} x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then for $|x| \to \infty$ we have $g(x) = x \to \pm\infty$, so g is unbounded (indeed unbounded above and below). However, $0 \notin g(\mathbb{R})$, because $g(x) = 0$ would force $x = 0$ (by $x \neq 0 \Rightarrow g(x) = x$), but $g(0) = 1 \neq 0$. Hence $g(\mathbb{R}) = \mathbb{R} \setminus \{0\}$, so g is not surjective.

9 Q9

Let $7\mathbb{N} := \{7n : n \in \mathbb{N}\}$ be the set of multiples of 7. Define $f : \mathbb{N} \to 7\mathbb{N}$ by $f(n) = 7n$. If $f(n_1) = f(n_2)$ then $7n_1 = 7n_2$ hence $n_1 = n_2$; thus f is injective. For any $m \in 7\mathbb{N}$ there exists $k \in \mathbb{N}$ with $m = 7k$, and then $f(k) = m$; thus f is surjective. Therefore f is a bijection and $|\mathbb{N}| = |7\mathbb{N}|$.

Q10. Properties of Absolute Value

$$|a| = \begin{cases} a, & a \ge 0, \\ -a, & a < 0. \end{cases}$$

Lemma 1. *For any $x \in \mathbb{R}$, we have $-|x| \le x \le |x|$.*

Proof. If $x \ge 0$, then $|x| = x$ and $-|x| = -x \le 0 \le x = |x|$. If $x < 0$, then $|x| = -x$, hence $-|x| = x \le -x = |x|$. Thus $-|x| \le x \le |x|$ in both cases. □

Lemma 2. *If $M \ge 0$ and $-M \le x \le M$, then $|x| \le M$.*

Proof. If $x \geq 0$ then $|x| = x \leq M$. If $x < 0$ then $-M \leq x$ implies $-x \leq M$, but $|x| = -x$, so $|x| \leq M$. $\square$

Proposition 1 (Absolute value identities and inequalities)**.** *For all real numbers a, b,*

(a) $|ab| = |a|\,|b|$;

(b) $|a+b| \leq |a| + |b|$;

(c) $\big||a| - |b|\big| \leq |a-b|$.

Proof. **(a)** By the definition of absolute value, consider four sign cases for a and b. In each case one checks directly that $|ab|$ equals $|a|\,|b|$; e.g., if $a \geq 0, b < 0$ then $|ab| = a(-b) = |a|\,|b|$, etc.

(b) By Lemma 1, we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding inequalities yields

$$-(|a| + |b|) \;\leq\; a + b \;\leq\; |a| + |b|.$$

Since $|a| + |b| \geq 0$, Lemma 2 gives $|a+b| \leq |a| + |b|$.

(c) Apply part (b) to $(a-b) + b = a$:

$$|a| = |(a-b) + b| \leq |a-b| + |b| \quad \Rightarrow \quad |a| - |b| \leq |a-b|.$$

Swapping a and b gives $|b| - |a| \leq |a-b|$. Hence

$$-|a-b| \;\leq\; |a| - |b| \;\leq\; |a-b|.$$

By Lemma 2 (with $x = |a| - |b|$ and $M = |a-b|$), we obtain $\big||a| - |b|\big| \leq |a-b|$. $\square$

Q11. Infinitely many reals between two distinct reals

Proposition 2. *For any real numbers $a < b$, there are infinitely many real numbers in the open interval (a, b).*

Proof. Fix $a < b$. For each integer $n \in \mathbb{N}$ with $n \geq 2$, define

$$x_n := a + \frac{b-a}{n}.$$

Since $b - a > 0$ and $n \geq 2$, we have $\frac{b-a}{n} > 0$, hence

$$a \; < \; a + \frac{b-a}{n} \; = \; x_n.$$

Also $\frac{b-a}{n} < b - a$, so adding a to both sides yields

$$x_n \; = \; a + \frac{b-a}{n} \; < \; a + (b-a) \; = \; b.$$

Thus $x_n \in (a, b)$ for every $n \geq 2$.

Finally, if $m \neq n$ with $m, n \geq 2$, then

$$x_m - x_n = \frac{b-a}{m} - \frac{b-a}{n} = (b-a)\left(\frac{1}{m} - \frac{1}{n}\right) \neq 0,$$

so the x_n are pairwise distinct. Therefore there are infinitely many distinct real numbers in (a, b). $\square$