MATH CAMP ASSIGNMENT 11

Dynamic Programming

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1 Q1

We are asked to solve the problem:

$$\max \sum_{t=0}^{2} [1 - (x_t^2 + 2u_t^2)]$$

subject to $x_{t+1} = x_t - u_t$ for $t = 0, 1$, with $x_0 = 5$ and $u_t \in \mathbb{R}$.

This is a finite-horizon deterministic dynamic programming problem. We solve it using backward induction. The value function $J_s(x)$ satisfies the Bellman equation:

$$J_s(x) = \max_{u_s} [1 - (x^2 + 2u_s^2) + J_{s+1}(x - u_s)]$$

The terminal value function is defined as $J_3(x) = 0$, as the summation ends at $t = 2$.

Step 1: Solve for $t = 2$

$$J_2(x_2) = \max_{u_2} [1 - (x_2^2 + 2u_2^2)]$$

To maximize this expression, we must minimize $2u_2^2$. The optimal control is $u_2^*(x_2) = 0$. The value function at $t = 2$ is:

$$J_2(x_2) = 1 - x_2^2$$

Step 2: Solve for $t = 1$

$$J_1(x_1) = \max_{u_1} [1 - (x_1^2 + 2u_1^2) + J_2(x_1 - u_1)]$$

Substitute $J_2(x_2)$:

$$J_1(x_1) = \max_{u_1} [1 - x_1^2 - 2u_1^2 + 1 - (x_1 - u_1)^2]$$

$$J_1(x_1) = \max_{u_1} [2 - x_1^2 - 2u_1^2 - (x_1^2 - 2x_1u_1 + u_1^2)]$$

$$J_1(x_1) = \max_{u_1} [2 - 2x_1^2 + 2x_1u_1 - 3u_1^2]$$

To find the maximum, we take the derivative with respect to u_1 and set it to zero:

$$\frac{\partial}{\partial u_1} = 2x_1 - 6u_1 = 0 \implies u_1^*(x_1) = \frac{x_1}{3}$$

Substitute u_1^* back into the expression for $J_1(x_1)$:

$$J_1(x_1) = 2 - 2x_1^2 + 2x_1\left(\frac{x_1}{3}\right) - 3\left(\frac{x_1}{3}\right)^2 = 2 - 2x_1^2 + \frac{2x_1^2}{3} - \frac{x_1^2}{3} = 2 - \frac{5}{3}x_1^2$$

Step 3: Solve for $t = 0$

$$J_0(x_0) = \max_{u_0}[1 - (x_0^2 + 2u_0^2) + J_1(x_0 - u_0)]$$

Substitute $J_1(x_1)$:

$$J_0(x_0) = \max_{u_0}[1 - x_0^2 - 2u_0^2 + 2 - \frac{5}{3}(x_0 - u_0)^2]$$

$$J_0(x_0) = \max_{u_0}[3 - x_0^2 - 2u_0^2 - \frac{5}{3}(x_0^2 - 2x_0u_0 + u_0^2)]$$

$$J_0(x_0) = \max_{u_0}[3 - \frac{8}{3}x_0^2 + \frac{10}{3}x_0u_0 - \frac{11}{3}u_0^2]$$

Take the derivative with respect to u_0 and set to zero:

$$\frac{\partial}{\partial u_0} = \frac{10}{3}x_0 - \frac{22}{3}u_0 = 0 \implies u_0^*(x_0) = \frac{5}{11}x_0$$

Substitute u_0^* back into the expression for $J_0(x_0)$:

$$J_0(x_0) = 3 - \frac{8}{3}x_0^2 + \frac{10}{3}x_0\left(\frac{5}{11}x_0\right) - \frac{11}{3}\left(\frac{5}{11}x_0\right)^2 = 3 - \frac{21}{11}x_0^2$$

Step 4: Find the optimal path and value Given $x_0 = 5$:

- $u_0^* = \frac{5}{11}x_0 = \frac{5}{11}(5) = \frac{25}{11}$
- $x_1 = x_0 - u_0^* = 5 - \frac{25}{11} = \frac{30}{11}$
- $u_1^* = \frac{x_1}{3} = \frac{1}{3}\left(\frac{30}{11}\right) = \frac{10}{11}$
- $x_2 = x_1 - u_1^* = \frac{30}{11} - \frac{10}{11} = \frac{20}{11}$
- $u_2^* = 0$

The maximum value of the objective function is $J_0(x_0)$:

$$J_0(5) = 3 - \frac{21}{11}(5^2) = 3 - \frac{525}{11} = \frac{33 - 525}{11} = -\frac{492}{11}$$

The solution is:

- Optimal controls: $u_0^* = \frac{25}{11}$, $u_1^* = \frac{10}{11}$, $u_2^* = 0$.
- Optimal states: $x_0 = 5$, $x_1 = \frac{30}{11}$, $x_2 = \frac{20}{11}$.
- Maximum value: $-\frac{492}{11}$.

2 Q2

We are asked to solve the problem:

$$\max_{u_t \in [0,1]} \left[\sum_{t=0}^{T-1} \left(-\frac{2}{3} u_t \right) + \ln x_T \right]$$

subject to $x_{t+1} = x_t(1 + u_t)$ for $t = 0, \dots, T-1$, with $x_0 > 0$.

We solve this using backward induction. The Bellman equation is:

$$J_t(x_t) = \max_{u_t \in [0,1]} \left[-\frac{2}{3} u_t + J_{t+1}(x_{t+1}) \right]$$

The terminal value function is given by the terminal reward:

$$J_T(x_T) = \ln x_T$$

Step 1: Solve for $t = T-1$

$$J_{T-1}(x_{T-1}) = \max_{u_{T-1} \in [0,1]} \left[-\frac{2}{3} u_{T-1} + J_T(x_{T-1}(1 + u_{T-1})) \right]$$

$$J_{T-1}(x_{T-1}) = \max_{u_{T-1} \in [0,1]} \left[-\frac{2}{3} u_{T-1} + \ln(x_{T-1}(1 + u_{T-1})) \right]$$

Using the property of logarithms, $\ln(ab) = \ln a + \ln b$:

$$J_{T-1}(x_{T-1}) = \ln x_{T-1} + \max_{u_{T-1} \in [0,1]} \left[-\frac{2}{3} u_{T-1} + \ln(1 + u_{T-1}) \right]$$

Let $h(u) = -\frac{2}{3} u + \ln(1 + u)$. To find the maximum, we take the derivative:

$$h'(u) = -\frac{2}{3} + \frac{1}{1+u}$$

Setting $h'(u) = 0$ gives $1 + u = 3/2 \implies u = 1/2$. This value is in the domain $[0, 1]$. The second derivative is $h''(u) = -1/(1+u)^2 < 0$, so this is a maximum. The optimal control is $u^*_{T-1} = 1/2$. Substitute this back to find $J_{T-1}(x_{T-1})$:

$$J_{T-1}(x_{T-1}) = \ln x_{T-1} - \frac{2}{3}\left(\frac{1}{2}\right) + \ln\left(1 + \frac{1}{2}\right) = \ln x_{T-1} - \frac{1}{3} + \ln\left(\frac{3}{2}\right)$$

Step 2: Generalize for any time t Let's assume a form for $J_{t+1}(x)$ and solve for $J_t(x)$. Assume $J_{t+1}(x) = \ln x + C_{t+1}$ for some constant C_{t+1}.

$$J_t(x_t) = \max_{u_t \in [0,1]} \left[-\frac{2}{3}u_t + J_{t+1}(x_t(1+u_t)) \right]$$

$$J_t(x_t) = \max_{u_t \in [0,1]} \left[-\frac{2}{3}u_t + \ln(x_t(1+u_t)) + C_{t+1} \right]$$

$$J_t(x_t) = \ln x_t + C_{t+1} + \max_{u_t \in [0,1]} \left[-\frac{2}{3}u_t + \ln(1+u_t) \right]$$

The maximization problem is identical to the one in the previous step. The optimal control is always $u^*_t = 1/2$, and the maximized value of the bracketed term is $-\frac{1}{3} + \ln(\frac{3}{2})$. Thus, we have a recursive relationship for the constant part: $C_t = C_{t+1} - \frac{1}{3} + \ln(\frac{3}{2})$. Since $J_T(x_T) = \ln x_T$, we have $C_T = 0$. So, $C_{T-1} = 0 - \frac{1}{3} + \ln(\frac{3}{2})$. $C_{T-2} = C_{T-1} - \frac{1}{3} + \ln(\frac{3}{2}) = 2(-\frac{1}{3} + \ln(\frac{3}{2}))$. By induction, $C_t = (T-t)(-\frac{1}{3} + \ln(\frac{3}{2}))$.

Solution The optimal control policy is constant for all time periods:

$$u^*_t = \frac{1}{2} \quad \text{for } t = 0, \ldots, T-1$$

The maximum value of the problem is given by $J_0(x_0)$:

$$J_0(x_0) = \ln x_0 + C_0 = \ln x_0 + (T-0)\left(-\frac{1}{3} + \ln\left(\frac{3}{2}\right)\right)$$

$$J_0(x_0) = \ln x_0 + T\left(\ln\left(\frac{3}{2}\right) - \frac{1}{3}\right)$$

3 Q3

The problem is:

$$\max_{u_t \in \mathbb{R}} \left[\sum_{t=0}^{T-1} (-e^{-\gamma u_t}) - \alpha e^{-\gamma x_T} \right]$$

subject to $x_{t+1} = 2x_t - u_t$, with $x_0 > 0$ and $\alpha, \gamma > 0$.

3.1 (a) Compute $J_T(x)$, $J_{T-1}(x)$, and $J_{T-2}(x)$

We use backward induction. The Bellman equation is $J_t(x) = \max_{u_t}[-e^{-\gamma u_t} + J_{t+1}(2x - u_t)]$.

For $t = T$: The value function is the terminal reward.

$$J_T(x) = -\alpha e^{-\gamma x}$$

For $t = T-1$:

$$J_{T-1}(x) = \max_{u_{T-1}} \left[-e^{-\gamma u_{T-1}} + J_T(2x - u_{T-1})\right]$$

$$J_{T-1}(x) = \max_{u_{T-1}} \left[-e^{-\gamma u_{T-1}} - \alpha e^{-\gamma(2x - u_{T-1})}\right]$$

$$J_{T-1}(x) = \max_{u_{T-1}} \left[-e^{-\gamma u_{T-1}} - \alpha e^{-2\gamma x} e^{\gamma u_{T-1}}\right]$$

To find the maximum, take the derivative with respect to u_{T-1} and set to zero:

$$\gamma e^{-\gamma u_{T-1}} - \gamma \alpha e^{-2\gamma x} e^{\gamma u_{T-1}} = 0 \implies e^{-2\gamma u_{T-1}} = \alpha e^{-2\gamma x}$$

This gives $e^{-\gamma u_{T-1}} = \sqrt{\alpha} e^{-\gamma x}$ and $e^{\gamma u_{T-1}} = \frac{1}{\sqrt{\alpha}} e^{\gamma x}$. Substituting back into the expression for $J_{T-1}(x)$:

$$J_{T-1}(x) = -(\sqrt{\alpha} e^{-\gamma x}) - \alpha e^{-2\gamma x} \left(\frac{1}{\sqrt{\alpha}} e^{\gamma x}\right) = -\sqrt{\alpha} e^{-\gamma x} - \sqrt{\alpha} e^{-\gamma x}$$

$$J_{T-1}(x) = -2\sqrt{\alpha} e^{-\gamma x}$$

For $t = T-2$:

$$J_{T-2}(x) = \max_{u_{T-2}} \left[-e^{-\gamma u_{T-2}} + J_{T-1}(2x - u_{T-2})\right]$$

$$J_{T-2}(x) = \max_{u_{T-2}} \left[-e^{-\gamma u_{T-2}} - 2\sqrt{\alpha} e^{-\gamma(2x - u_{T-2})}\right]$$

$$J_{T-2}(x) = \max_{u_{T-2}} \left[-e^{-\gamma u_{T-2}} - (2\sqrt{\alpha} e^{-2\gamma x}) e^{\gamma u_{T-2}}\right]$$

This maximization problem has the same form as before, with α replaced by $2\sqrt{\alpha}$. The maximum value is $-2\sqrt{2\sqrt{\alpha} e^{-2\gamma x}}$.

$$J_{T-2}(x) = -2\sqrt{2\sqrt{\alpha}}\sqrt{e^{-2\gamma x}} = -2(2\sqrt{\alpha})^{1/2} e^{-\gamma x} = -2^{3/2} \alpha^{1/4} e^{-\gamma x}$$

3.2 (b) Prove form of $J_t(x)$ and find difference equation

We prove by induction (backwards in time) that $J_t(x)$ can be written in the form $J_t(x) = -\alpha_t e^{-\gamma x}$.

Base Case: For $t = T$, $J_T(x) = -\alpha e^{-\gamma x}$. This holds with $\alpha_T = \alpha$.

Inductive Step: Assume for some $t+1 \leq T$, the hypothesis holds, i.e., $J_{t+1}(x) = -\alpha_{t+1} e^{-\gamma x}$ for some constant $\alpha_{t+1} > 0$. Now consider $J_t(x)$:

$$J_t(x) = \max_{u_t} \left[-e^{-\gamma u_t} + J_{t+1}(2x - u_t) \right]$$

$$J_t(x) = \max_{u_t} \left[-e^{-\gamma u_t} - \alpha_{t+1} e^{-\gamma(2x - u_t)} \right]$$

$$J_t(x) = \max_{u_t} \left[-e^{-\gamma u_t} - (\alpha_{t+1} e^{-2\gamma x}) e^{\gamma u_t} \right]$$

As shown in part (a), the maximum value of an expression of the form $\max_z[-e^{-z} - Ce^z]$ is $-2\sqrt{C}$. Here, $z = \gamma u_t$ and $C = \alpha_{t+1} e^{-2\gamma x}$. So, the maximum value is:

$$J_t(x) = -2\sqrt{\alpha_{t+1} e^{-2\gamma x}} = -2\sqrt{\alpha_{t+1}} \cdot e^{-\gamma x}$$

This expression is of the form $-\alpha_t e^{-\gamma x}$, where we can identify:

$$\alpha_t = 2\sqrt{\alpha_{t+1}}$$

This is the difference equation for α_t. The induction is complete.

4 Q4

The problem is to solve the infinite-horizon discounted problem:

$$\max_{u_t \in (0,\infty)} \left[\sum_{t=0}^{\infty} \beta^t \left(-e^{-u_t} - \frac{1}{2} e^{-x_t} \right) \right]$$

subject to $x_{t+1} = 2x_t - u_t$, with $\beta \in (0, 1)$. We suppose the value function has the form $J(x) = -\alpha e^{-x}$ for some $\alpha > 0$ and we need to determine α.

The value function for a stationary infinite-horizon problem satisfies the Bellman equation:

$$J(x) = \max_{u \in (0,\infty)} \left[-e^{-u} - \frac{1}{2} e^{-x} + \beta J(2x - u) \right]$$

Substitute the proposed form of $J(x)$:

$$-\alpha e^{-x} = \max_{u\in(0,\infty)} \left[-e^{-u} - \frac{1}{2}e^{-x} + \beta(-\alpha e^{-(2x-u)})\right]$$

Isolate the terms not involving u from the maximization:

$$-\alpha e^{-x} = -\frac{1}{2}e^{-x} + \max_{u\in(0,\infty)} \left[-e^{-u} - \beta\alpha e^{-2x}e^{u}\right]$$

Let's solve the maximization problem. Let $h(u) = -e^{-u} - (\beta\alpha e^{-2x})e^{u}$. Take the derivative with respect to u and set to zero:

$$h'(u) = e^{-u} - \beta\alpha e^{-2x}e^{u} = 0 \implies e^{-2u} = \beta\alpha e^{-2x}$$

The second derivative is $h''(u) = -e^{-u} - \beta\alpha e^{-2x}e^{u} < 0$, confirming a maximum. The maximum value is found by substituting the condition back into $h(u)$. From $e^{-2u} = \beta\alpha e^{-2x}$, we have $e^{-u} = \sqrt{\beta\alpha}e^{-x}$ and $e^{u} = \frac{1}{\sqrt{\beta\alpha}}e^{x}$.

$$\max_u h(u) = -(\sqrt{\beta\alpha}e^{-x}) - \beta\alpha e^{-2x}\left(\frac{1}{\sqrt{\beta\alpha}}e^{x}\right) = -2\sqrt{\beta\alpha}e^{-x}$$

Now substitute this result back into the Bellman equation:

$$-\alpha e^{-x} = -\frac{1}{2}e^{-x} - 2\sqrt{\beta\alpha}e^{-x}$$

Since this equation must hold for all x, we can divide by $-e^{-x}$:

$$\alpha = \frac{1}{2} + 2\sqrt{\beta\alpha}$$

Let $y = \sqrt{\alpha}$ (note $y > 0$ since $\alpha > 0$). The equation becomes:

$$y^2 = \frac{1}{2} + 2\sqrt{\beta}y$$

Rearranging gives a quadratic equation in y:

$$y^2 - 2\sqrt{\beta}y - \frac{1}{2} = 0$$

Using the quadratic formula to solve for y:

$$y = \frac{-(-2\sqrt{\beta}) \pm \sqrt{(-2\sqrt{\beta})^2 - 4(1)(-\frac{1}{2})}}{2(1)} = \frac{2\sqrt{\beta} \pm \sqrt{4\beta+2}}{2} = \sqrt{\beta} \pm \frac{\sqrt{4\beta+2}}{2}$$

Since $y = \sqrt{\alpha}$ must be positive, and $\sqrt{4\beta+2} > \sqrt{4\beta} = 2\sqrt{\beta}$, the negative root is not valid. Thus, we take the positive root:

$$y = \sqrt{\alpha} = \sqrt{\beta} + \frac{\sqrt{4\beta + 2}}{2}$$

Finally, we find α by squaring y:

$$\alpha = \left(\sqrt{\beta} + \frac{\sqrt{4\beta + 2}}{2} \right)^2$$

$$\alpha = (\sqrt{\beta})^2 + 2(\sqrt{\beta}) \left(\frac{\sqrt{4\beta + 2}}{2} \right) + \left(\frac{\sqrt{4\beta + 2}}{2} \right)^2$$

$$\alpha = \beta + \sqrt{\beta(4\beta + 2)} + \frac{4\beta + 2}{4}$$

$$\alpha = \beta + \sqrt{4\beta^2 + 2\beta} + \beta + \frac{1}{2}$$

$$\alpha = 2\beta + \frac{1}{2} + \sqrt{4\beta^2 + 2\beta}$$