2015 & 2016 Adv-Econometrics Midterms: Detailed Solutions

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Contents

Todo list

Source and Scope

This document provides full, step-by-step solutions to the 2015 and 2016 mid-term problems. Minor typographical glitches in the scans are resolved by adopting the standard probabilistic parameterizations noted in each solution.

1 2015 Mid-Term: Detailed Solutions

Problem 1 (30 pts)

Given. The joint pdf is $f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$ for $x \geq 0,\ y \geq 0, \lambda > 0$.

(a) Independence of X and Y

Compute marginals:

$$f_X(x) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)}\, dy = \lambda e^{-\lambda x}, \qquad f_Y(y) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)}\, dx = \lambda e^{-\lambda y}.$$

Hence $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, so $X \perp\!\!\!\perp Y$.

(b) Pdf of $U = X + Y$

Convolution for independent exponentials (same rate):

$$f_U(u) = \int_0^u f_X(x) f_Y(u-x)\, dx = \int_0^u \lambda e^{-\lambda x}\, \lambda e^{-\lambda(u-x)}\, dx = \lambda^2 u\, e^{-\lambda u}, \quad u \geq 0,$$

i.e. $U \sim \text{Gamma}(k = 2,\ \text{rate}\ \lambda)$.

(c) $\text{Var}(X+Y)$

Since $X, Y \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$ with $\text{Var}(X) = \text{Var}(Y) = 1/\lambda^2$ and independence,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) = \boxed{\tfrac{2}{\lambda^2}}.$$

Problem 2 (20 pts) Negative Binomial

Parameterization. $X \sim \text{NB}(r, p)$ with pmf

$$\Pr(X = x) = \binom{r+x-1}{x}(1-p)^x p^r, \quad x = 0, 1, 2, \ldots, \quad r \in \mathbb{N},\ 0 < p < 1.$$

(a) MGF of X

Use the negative binomial series $\sum_{x=0}^\infty \binom{r+x-1}{x} z^x = (1-z)^{-r}$ for $|z| < 1$. Then,

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^\infty \binom{r+x-1}{x}(1-p)^x p^r e^{tx} = p^r \sum_{x=0}^\infty \binom{r+x-1}{x}\left((1-p)e^t\right)^x = \left(1 - (1-p)e^t\right)^{-r} p^r.$$

Equivalently,

$$\boxed{M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r}, \qquad t < -\ln(1-p).$$

(b) Mean and variance

A quick route is via the p.g.f. $G_X(s) = \left(\dfrac{p}{1-(1-p)s}\right)^r$:

$$\mathbb{E}[X] = G_X'(1) = \frac{r(1-p)}{p}, \qquad \mathrm{Var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2 = \frac{r(1-p)}{p^2}.$$

Hence $\boxed{\mathbb{E}[X] = \frac{r(1-p)}{p}}$, $\boxed{\mathrm{Var}(X) = \frac{r(1-p)}{p^2}}$.

Problem 3 (20 pts)

Given. $f_{X,Y,Z}(x,y,z) = (x+y)e^{-z}$ on $0 < x < 1,\ 0 < y < 1,\ z > 0$.

(a) Marginal $f_{Y,Z}$

$$f_{Y,Z}(y,z) = \int_0^1 (x+y)e^{-z}\,dx = \left(\tfrac{1}{2}+y\right)e^{-z}, \quad 0 < y < 1,\ z > 0.$$

(b) Conditional $f_{Y|Z}$

First find $f_Z(z)$:

$$f_Z(z) = \int_0^1\int_0^1 (x+y)e^{-z}\,dx\,dy = \left(\int_0^1\int_0^1 (x+y)\,dx\,dy\right)e^{-z} = 1\cdot e^{-z}.$$

Thus

$$f_{Y|Z}(y|z) = \frac{f_{Y,Z}(y,z)}{f_Z(z)} = \tfrac{1}{2}+y, \qquad 0 < y < 1,$$

which is independent of z (so $Y \perp\!\!\!\perp Z$; similarly Z is independent of (X,Y)).

Problem 4 (20 pts) Double Exponential (Laplace)

Convention. Take the standard Laplace$(0,\lambda)$ pdf $f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|}$, $\lambda > 0$.

(a) $Y = |X|$: pdf and MGF

By symmetry, for $y \geq 0$,

$$f_Y(y) = f_X(y) + f_X(-y) = \tfrac{\lambda}{2}e^{-\lambda y} + \tfrac{\lambda}{2}e^{-\lambda y} = \boxed{\lambda e^{-\lambda y}}, \quad (Y \sim \mathrm{Exp}(\lambda)).$$

Thus $M_Y(t) = \mathbb{E}[e^{tY}] = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$.

(b) Transformation $Y = (Z-\mu)^3$

Since $Y \geq 0$, necessarily $Z \geq \mu$ and $y = (z-\mu)^3$. Jacobian $dy/dz = 3(z-\mu)^2$. Hence

$$\boxed{f_Z(z) = f_Y\big((z-\mu)^3\big) \cdot 3(z-\mu)^2 = 3\lambda(z-\mu)^2 e^{-\lambda(z-\mu)^3}, \quad z \geq \mu}.$$

(c) Hazard function of Z

$F_Z(z) = \Pr\big(Y \leq (z-\mu)^3\big) = 1 - e^{-\lambda(z-\mu)^3}$ $(z \geq \mu)$, so

$$S_Z(z) = 1 - F_Z(z) = e^{-\lambda(z-\mu)^3}, \qquad \lambda_Z(z) = \frac{f_Z(z)}{S_Z(z)} = \boxed{3\lambda(z-\mu)^2}, \; z \geq \mu.$$

Problem 5 (10 pts) Waiting Time for the r-th Poisson Event

Events arrive as a Poisson process with rate λ; let Y be the waiting time to the r-th event. Then

$$\Pr(Y \leq y) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda y)^k}{k!} e^{-\lambda y}, \quad f_Y(y) = \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y}, \quad y > 0,$$

i.e. $\boxed{Y \sim \text{Gamma}(r, \text{ rate } \lambda)}$.

2 2016 Mid-Term: Detailed Solutions

Problem 1 (16 pts) Poisson with Thinning by Gender

On weekdays, daily customers $\sim$ Poisson(λ_1); on weekends, $\sim$ Poisson(λ_2). Let population male:female ratio be $p : q$; the female probability is $\pi_f = \dfrac{q}{p+q}$. By thinning, the number of female customers each day is Poisson with rate $\pi_f \lambda_d$. Summing over a week (5 weekdays, 2 weekend days) yields

$$\boxed{Y \sim \text{Poisson}\big(\pi_f(5\lambda_1 + 2\lambda_2)\big)}.$$

Hence

$$\boxed{\mathbb{E}[Y] = \text{Var}(Y) = \pi_f(5\lambda_1 + 2\lambda_2)}.$$

Problem 2 (17 pts) Urn: Multivariate Hypergeometric

Urn: 3 red, 4 white, 5 blue, draw $n = 3$ without replacement. Let $X = \#\text{red}$, $Y = \#\text{white}$.

(a) Joint pmf

For $x, y \geq 0$ integers with $x \leq 3$, $y \leq 4$, $x + y \leq 3$,

$$\boxed{\Pr(X = x, Y = y) = \frac{\binom{3}{x}\binom{4}{y}\binom{5}{3-x-y}}{\binom{12}{3}}}.$$

(b) Marginals

Sum over y (resp. x):

$$\boxed{\Pr(X = x) = \frac{\binom{3}{x}\binom{9}{3-x}}{\binom{12}{3}}, \quad \Pr(Y = y) = \frac{\binom{4}{y}\binom{8}{3-y}}{\binom{12}{3}}}.$$

(c) $\mathbb{E}[X \mid Y = 2]$

Given $Y = 2$, the remaining one ball is drawn from 8 non-white (3 red, 5 blue). Thus $\Pr(X = 1 \mid Y = 2) = \frac{3}{8}$, $\Pr(X = 0 \mid Y = 2) = \frac{5}{8}$, so

$$\boxed{\mathbb{E}[X \mid Y = 2] = \tfrac{3}{8}}.$$

Problem 3 (16 pts) Cauchy

(a) If $X \sim \text{Cauchy}(0,1)$, find pdf of $1/X$

Let $W = 1/X$. Then

$$f_W(w) = f_X(1/w)\left|\frac{d}{dw}\frac{1}{w}\right| = \frac{1}{\pi\big(1+(1/w)^2\big)}\cdot\frac{1}{w^2} = \frac{1}{\pi(1+w^2)},$$

so $W \sim \text{Cauchy}(0,1)$.

(b) If $Y, Z \overset{\text{iid}}{\sim} N(0,1)$, show $Y/Z \sim \text{Cauchy}(0,1)$

Use the transformation $(T,S) = (Y/Z,\, Z)$ with inverse $(Y,Z) = (TS, S)$ and Jacobian $|J| = |S|$. The joint density of (Y,Z) is $\phi(y)\phi(z)$; integrate out S:

$$f_T(t) = \int_{-\infty}^{\infty}\phi(ts)\phi(s)\,|s|\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2+1)s^2}\,|s|\,ds = \frac{1}{\pi(1+t^2)}.$$

Thus $T = Y/Z \sim \text{Cauchy}(0,1)$.

Problem 4 (15 pts) Change of Variables

Given. Joint pdf of (X_1, X_2) is $f_{X_1,X_2}(x_1,x_2) = C\,x_1x_2$ on $0 < x_1 < x_2 < 1$, 0 otherwise.

Normalization. $\int_0^1\int_0^{x_2} Cx_1x_2\,dx_1dx_2 = \frac{C}{8} = 1 \Rightarrow C = 8$. Define $Y_1 = X_1/X_2$, $Y_2 = X_2$. Then $x_1 = y_1y_2$, $x_2 = y_2$, Jacobian $|J| = y_2$.

(a) Joint pdf of (Y_1, Y_2)

On $0 < y_1 < 1, 0 < y_2 < 1$,

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2, y_2)\,|J| = 8\,(y_1y_2)\,(y_2)\,y_2 = \boxed{8\,y_1\,y_2^3}.$$

(b) Independence?

Marginals:

$$f_{Y_1}(y_1) = \int_0^1 8y_1y_2^3\,dy_2 = 2y_1, \quad f_{Y_2}(y_2) = \int_0^1 8y_1y_2^3\,dy_1 = 4y_2^3.$$

Product $f_{Y_1}(y_1)f_{Y_2}(y_2) = 8y_1y_2^3 = f_{Y_1,Y_2}(y_1,y_2)$. Hence Y_1 *and* Y_2 *are independent*.

Problem 5 (18 pts) Joint pdf and related probabilities

Given.

$$f_{X,Y}(x,y) = \begin{cases} c\Big(x^2 + \dfrac{xy}{2}\Big), & 0 < x < 1,\ 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Normalizing constant c

$$1 = \int_0^1 \int_0^2 c\Big(x^2 + \frac{xy}{2}\Big)\ dy\,dx = \int_0^1 c\big(2x^2 + x\big)\ dx = c\Big(\frac{2}{3} + \frac{1}{2}\Big) = c \cdot \frac{7}{6},$$

hence $\boxed{c = \frac{6}{7}}$.

(b) Pdf of X

For $0 < x < 1$,

$$f_X(x) = \int_0^2 c\Big(x^2 + \frac{xy}{2}\Big)\ dy = c\ \big(2x^2 + x\big) = \boxed{\frac{6}{7}\big(2x^2 + x\big)}, \qquad f_X(x) = 0 \text{ otherwise.}$$

(c) $\Pr(X > Y)$ **and** $\Pr(X = 2Y)$

$$\Pr(X > Y) = \int_{x=0}^1 \int_{y=0}^x c\Big(x^2 + \frac{xy}{2}\Big)\ dy\,dx = \int_0^1 c\Big(x^3 + \frac{x^3}{4}\Big)\ dx = c \cdot \frac{5}{4} \cdot \frac{1}{4} = \boxed{\frac{15}{56}}.$$

Since the joint law is absolutely continuous,

$$\boxed{\Pr(X = 2Y) = 0}.$$

(d) $\Pr\big(Y \le \frac{1}{2} \mid X < \frac{1}{2}\big)$

$$\Pr(X < \tfrac{1}{2}) = \int_0^{1/2} c\big(2x^2 + x\big)\,dx = c\Big(\frac{1}{12} + \frac{1}{8}\Big) = \frac{6}{7} \cdot \frac{5}{24} = \frac{5}{28}.$$

$$\Pr\Big(Y \le \tfrac{1}{2},\ X < \tfrac{1}{2}\Big) = \int_0^{1/2} \int_0^{1/2} c\Big(x^2 + \frac{xy}{2}\Big)\ dy\,dx = \int_0^{1/2} c\Big(\frac{x^2}{2} + \frac{x}{16}\Big)\ dx = \frac{6}{7}\Big(\frac{1}{48} + \frac{1}{128}\Big) = \frac{11}{448}.$$

Therefore,

$$\boxed{\Pr\Big(Y \le \tfrac{1}{2}\ \Big|\ X < \tfrac{1}{2}\Big) = \frac{\frac{11}{448}}{\frac{5}{28}} = \frac{11}{80}}.$$

Problem 6 (18 pts) Independence, MGFs, and Poisson Additivity

Let X and Y be independent with mgfs existing in a neighborhood of 0.

(a) Factorization of expectations

For measurable h, g for which the expectations exist,

$$\mathbb{E}\left[h(X)g(Y)\right] = \int h(x)g(y)\, f_X(x) f_Y(y)\, dxdy = \left(\int h(x) f_X(x)\, dx\right)\left(\int g(y) f_Y(y)\, dy\right) = \mathbb{E}[h(X)]\ \mathbb{E}[g(Y)].$$

(b) MGF of $Z = X + Y$

$$M_Z(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX}e^{tY}] = \mathbb{E}[e^{tX}]\ \mathbb{E}[e^{tY}] = \boxed{M_X(t)M_Y(t)}.$$

(c) Poisson additivity

If $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$ independent, then

$$M_Z(t) = \exp\big(\lambda_1(e^t - 1)\big)\ \exp\big(\lambda_2(e^t - 1)\big) = \exp\big((\lambda_1 + \lambda_2)(e^t - 1)\big),$$

which is the mgf of $\text{Poisson}(\lambda_1 + \lambda_2)$. Hence $Z = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.