Key Properties of the Poisson Distribution and Applications

Xiankang Wang

November 7, 2025

Contents

1 Fundamental Theorems of the Poisson Distribution

This document outlines two fundamental properties of the Poisson distribution: the decomposition of a Poisson process (Splitting) and the sum of independent Poisson variables. We will first state and prove these theorems, then apply them to solve two related problems.

1.1 Theorem 1: The Poisson Splitting Theorem

This theorem describes how a Poisson process behaves when its events are independently classified into different categories.

Theorem 1 (Poisson Splitting). *Let $N \sim Poisson(\lambda)$ be the total number of events. If each event is independently classified as "Type 1" with probability p or "Type 2" with probability $1-p$, then the number of Type 1 events, X, and the number of Type 2 events, Y, are independent Poisson random variables with $X \sim Poisson(\lambda p)$ and $Y \sim Poisson(\lambda(1-p))$.*

Proof. Let X be the number of "Type 1" events and Y be the number of "Type 2" events. We find their joint PMF, $\mathbb{P}(X = x, Y = y)$, by conditioning on the total number of events, $N = X + Y$.

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x, Y = y|N = x + y)\mathbb{P}(N = x + y)$$

The term $\mathbb{P}(X = x, Y = y|N = x + y)$ is the probability of getting x successes in $x + y$ trials, given by the binomial distribution:

$$\mathbb{P}(X = x, Y = y|N = x + y) = \binom{x+y}{x} p^x(1-p)^y = \frac{(x+y)!}{x!y!}p^x(1-p)^y$$

The term $\mathbb{P}(N = x + y)$ is given by the Poisson PMF:

$$\mathbb{P}(N = x + y) = \frac{e^{-\lambda}\lambda^{x+y}}{(x+y)!}$$

Multiplying these together yields the joint PMF:

$$\begin{aligned}
\mathbb{P}(X = x, Y = y) &= \left(\frac{(x+y)!}{x!y!}p^x(1-p)^y\right) \times \left(\frac{e^{-\lambda}\lambda^{x+y}}{(x+y)!}\right) \\
&= \frac{e^{-\lambda}(\lambda p)^x(\lambda(1-p))^y}{x!y!} \\
&= \frac{e^{-(\lambda p+\lambda(1-p))}(\lambda p)^x(\lambda(1-p))^y}{x!y!} \\
&= \left(\frac{e^{-\lambda p}(\lambda p)^x}{x!}\right) \times \left(\frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!}\right)
\end{aligned}$$

This shows that the joint PMF is the product of two Poisson PMFs, one with mean λp and the other with mean $\lambda(1-p)$. This proves both the distributions and their independence. □

1.2 Theorem 2: The Sum of Independent Poisson Variables

This theorem describes the distribution of the sum of two or more independent Poisson random variables.

Theorem 2 (Sum of Independent Poissons). *If $X_1, X_2, \ldots, X_k$ are independent random variables where each $X_i \sim Poisson(\lambda_i)$, then their sum $S = \sum_{i=1}^{k} X_i$ is also a Poisson random variable with parameter equal to the sum of the individual parameters.*

$$S \sim Poisson\left(\sum_{i=1}^{k}\lambda_i\right)$$

Proof using MGFs. The Moment Generating Function (MGF) for a Poisson(λ) variable is $M(t) = e^{\lambda(e^t-1)}$. A key property of MGFs is that for independent variables, the MGF of their sum is the product of their individual MGFs. For $S = X_1 + X_2$, we have:

$$\begin{aligned} M_S(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \\ &= \left(e^{\lambda_1(e^t-1)}\right) \cdot \left(e^{\lambda_2(e^t-1)}\right) \\ &= e^{\lambda_1(e^t-1)+\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

This is the MGF of a Poisson random variable with parameter $\lambda_1 + \lambda_2$. By the uniqueness property of MGFs, this proves that $S \sim \text{Poisson}(\lambda_1 + \lambda_2)$. The result extends to a sum of k variables by induction. $\square$

2 Applications of Poisson Properties

In this section, we solve two problems that directly apply the theorems established above.

2.1 Problem 1: Decomposing a Poisson Process

Problem 1. Suppose that the number of events that occur is a Poisson random variable with mean λ and that each event is independently counted with probability p. Show that the number of counted events and the number of uncounted events are independent Poisson random variables with respective means λp and $\lambda(1-p)$.

2.1.1 Solution

This problem is a direct application of the **Poisson Splitting Theorem** (Theorem 1).

Let's define our variables according to the theorem:

- The total number of events is $N \sim \text{Poisson}(\lambda)$.
- We classify events into two types: "counted" (Type 1) and "uncounted" (Type 2).
- The probability of an event being "counted" is p.
- The probability of an event being "uncounted" is $1-p$.
- Let X be the number of counted events and Y be the number of uncounted events.

According to Theorem 1, if a Poisson(λ) process is split into two streams with probabilities p and $1-p$, the resulting streams are independent Poisson processes.

Therefore, we can directly apply the conclusion of the theorem:

1. The number of counted events, X, follows a Poisson distribution with mean $\lambda \times p$.
2. The number of uncounted events, Y, follows a Poisson distribution with mean $\lambda \times (1-p)$.
3. The random variables X and Y are independent.

The full mathematical derivation for this result is provided in the proof of Theorem 1 in the previous section.

2.2 Problem 2: Grocery Store Customers

Problem 2. Assume that customers visit a grocery store independently. The number of customers visiting the store each day is a Poisson random variable with parameter λ_1 if it is a weekday or λ_2 if it is the weekend. Suppose the ratio of males to female in the population is p/q.

(a) If Y is the total number of female customers visiting the store in one week, what is the distribution of Y?

(b) Find the mean and variance of Y.

2.2.1 Solution

This solution uses both the Poisson Splitting Theorem and the Sum of Independent Poissons Theorem.

Part (a): Finding the Distribution of Y

1. **Probability of a Female Customer:** The ratio p/q means the probability that a random customer is female is $P(\text{female}) = \frac{q}{p+q}$.

2. **Daily Female Customer Distributions (Theorem 1):** By the Poisson Splitting Theorem, the number of female customers each day is also a Poisson variable.

   - For a weekday: $Y_{\text{weekday}} \sim \text{Poisson}\left(\lambda_1 \frac{q}{p+q}\right)$.
   - For a weekend day: $Y_{\text{weekend}} \sim \text{Poisson}\left(\lambda_2 \frac{q}{p+q}\right)$.

3. **Weekly Total (Theorem 2):** The total for the week is the sum of 5 weekday counts and 2 weekend counts. Since these are all independent Poisson variables, their sum is also a Poisson variable by Theorem 2. The parameter Λ_Y is the sum of the individual parameters.

$$\begin{aligned}\Lambda_Y &= 5 \times \left(\lambda_1 \frac{q}{p+q}\right) + 2 \times \left(\lambda_2 \frac{q}{p+q}\right) \\ &= (5\lambda_1 + 2\lambda_2) \frac{q}{p+q}\end{aligned}$$

Distribution of Y

The total number of female customers in one week, Y, follows a Poisson distribution:

$$Y \sim \text{Poisson}\left((5\lambda_1 + 2\lambda_2) \frac{q}{p+q}\right)$$

Part (b): Finding the Mean and Variance of Y For any random variable $K \sim \text{Poisson}(\Lambda)$, its expectation and variance both equal Λ.

Mean and Variance of Y

The mean and variance of Y are equal:

$$\begin{aligned}\mathbb{E}[Y] &= (5\lambda_1 + 2\lambda_2) \frac{q}{p+q} \\ \text{Var}(Y) &= (5\lambda_1 + 2\lambda_2) \frac{q}{p+q}\end{aligned}$$