Solutions to Probability Problems

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November 7, 2025

Contents

1 Problem 1: Event Probabilities

1.1 Problem Statement

Let A, B denote two events. $\bar{A}$ is the complement set of A, and $\bar{B}$ is the complement set of B. Given that $P(A) = x$, $P(B) = y$, $P(A \cap B) = z$, please find the probability of the following using x, y, z:

(a) $P(\bar{A} \cap B)$.

(b) $P(\bar{A} \cap \bar{B})$.

(c) $P(\bar{A} \cup B)$.

(d) $P(\bar{A} \cup \bar{B})$.

1.2 Solution

(a) $P(\bar{A} \cap B)$**.** This is the probability that event B occurs but event A does not. We can find this by subtracting the probability of both events occurring from the probability of event B.

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = \boxed{y - z}.$$

(b) $P(\bar{A} \cap \bar{B})$**.** This is the probability that neither A nor B occurs. By De Morgan's laws, $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B).$$

Using the principle of inclusion-exclusion for $P(A \cup B)$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - z.$$

Therefore,

$$P(\bar{A} \cap \bar{B}) = 1 - (x + y - z) = \boxed{1 - x - y + z}.$$

(c) $P(\bar{A} \cup B)$**.** Using the probability addition rule:

$$P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B).$$

We know $P(\bar{A}) = 1 - P(A) = 1 - x$, and from part (a), $P(\bar{A} \cap B) = y - z$. Substituting these in:

$$P(\bar{A} \cup B) = (1 - x) + y - (y - z) = \boxed{1 - x + z}.$$

(d) $P(\bar{A} \cup \bar{B})$**.** By De Morgan's laws, $\bar{A} \cup \bar{B} = \overline{A \cap B}$.

$$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = \boxed{1 - z}.$$

2 Problem 2: Exponential Distribution Transformation

2.1 Problem Statement

Let random variable X have an exponential distribution with $EX = 1$. Let $Y = X^2 - 2$.

(a) Find the moment generating function of X.

(b) Find the probability density function of Y.

(c) Find the mean and the variance of Y.

2.2 Solution

For an exponential distribution with parameter λ, the PDF is $f(x) = \lambda e^{-\lambda x}$ and the expectation is $EX = 1/\lambda$. Given $EX = 1$, we have $1/\lambda = 1$, so $\lambda = 1$. Thus, X follows a standard exponential distribution with PDF $f_X(x) = e^{-x}$ for $x > 0$.

(a) Moment generating function (MGF) of X. The MGF is defined as $M_X(t) = E[e^{tX}]$.

$$M_X(t) = \int_0^\infty e^{tx} f_X(x)dx = \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{(t-1)x} dx.$$

For the integral to converge, we require $t - 1 < 0$, or $t < 1$.

$$M_X(t) = \left[\frac{1}{t-1} e^{(t-1)x}\right]_0^\infty = 0 - \frac{1}{t-1} = \boxed{\frac{1}{1-t}}, \quad \text{for } t < 1.$$

(b) Probability density function (PDF) of Y. We first find the CDF of Y, $F_Y(y) = P(Y \leq y)$, for y in the support of Y. Since $X > 0$, $X^2 > 0$, so $Y = X^2 - 2 > -2$. For $y > -2$:

$$F_Y(y) = P(X^2 - 2 \leq y) = P(X^2 \leq y + 2).$$

Since X is always positive, this is equivalent to:

$$F_Y(y) = P(0 < X \leq \sqrt{y+2}) = F_X(\sqrt{y+2}).$$

The CDF of $X \sim \text{Exp}(1)$ is $F_X(x) = 1 - e^{-x}$ for $x > 0$. So,

$$F_Y(y) = 1 - e^{-\sqrt{y+2}}.$$

To find the PDF, we differentiate the CDF with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(1 - e^{-\sqrt{y+2}}) = -e^{-\sqrt{y+2}} \cdot \left(-\frac{1}{2\sqrt{y+2}}\right).$$

$$f_Y(y) = \boxed{\frac{e^{-\sqrt{y+2}}}{2\sqrt{y+2}}}, \quad \text{for } y > -2.$$

(c) Mean and variance of Y. We use the properties of expectation and variance. For $X \sim \text{Exp}(1)$, the k-th moment is $E[X^k] = k!$. The mean of Y:

$$E[Y] = E[X^2 - 2] = E[X^2] - 2.$$

$E[X^2] = 2! = 2$.

$$E[Y] = 2 - 2 = \boxed{0}.$$

The variance of Y:

$$\text{Var}(Y) = \text{Var}(X^2 - 2) = \text{Var}(X^2) = E[(X^2)^2] - (E[X^2])^2 = E[X^4] - (E[X^2])^2.$$

$E[X^4] = 4! = 24$.

$$\text{Var}(Y) = 24 - (2)^2 = 24 - 4 = \boxed{20}.$$

3 Problem 3: Joint, Marginal, and Conditional Densities

3.1 Problem Statement

The joint probability density function of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} c(x^2 - y^2)e^{-x}, & 0 < x < \infty, -x < y < x \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of c?

(b) Find the marginal probability density function of X.

(c) Find the EY.

(d) Find the conditional density function of Y, given $X = x$.

3.2 Solution

(a) Value of c. The total probability must be 1.

$$\int_0^\infty \int_{-x}^{x} c(x^2 - y^2)e^{-x}\, dy\, dx = 1.$$

First, we solve the inner integral with respect to y:

$$\int_{-x}^{x} (x^2 - y^2)\, dy = \left[x^2 y - \frac{y^3}{3}\right]_{-x}^{x} = \left(x^3 - \frac{x^3}{3}\right) - \left(-x^3 + \frac{x^3}{3}\right) = \frac{2x^3}{3} - \left(-\frac{2x^3}{3}\right) = \frac{4x^3}{3}.$$

Now, we solve the outer integral:

$$c\int_0^\infty \frac{4x^3}{3} e^{-x}\, dx = 1 \implies \frac{4c}{3}\int_0^\infty x^3 e^{-x}\, dx = 1.$$

The integral is the Gamma function $\Gamma(4) = 3! = 6$.

$$\frac{4c}{3} \cdot 6 = 1 \implies 8c = 1 \implies \boxed{c = \frac{1}{8}}.$$

(b) Marginal PDF of X. We integrate the joint PDF over all possible values of y.

$$f_X(x) = \int_{-x}^{x} f_{XY}(x,y)\, dy = \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}\, dy.$$

This is $\frac{1}{8}e^{-x}$ times the inner integral from part (a):

$$f_X(x) = \frac{1}{8}e^{-x}\left(\frac{4x^3}{3}\right) = \boxed{\frac{1}{6}x^3 e^{-x}}, \quad \text{for } x > 0.$$

(c) Expectation of Y.

$$E[Y] = \int_0^\infty \int_{-x}^{x} y f_{XY}(x,y)\, dy\, dx = \int_0^\infty \int_{-x}^{x} y\frac{1}{8}(x^2 - y^2)e^{-x}\, dy\, dx.$$

Consider the inner integral:

$$\int_{-x}^{x} y(x^2 - y^2)\, dy = \int_{-x}^{x} (x^2 y - y^3)\, dy.$$

The integrand $g(y) = x^2 y - y^3$ is an odd function of y, and it is integrated over a symmetric interval $[-x, x]$. Therefore, the integral is 0.

$$E[Y] = \int_0^\infty \frac{1}{8}e^{-x}(0)\, dx = \boxed{0}.$$

(d) Conditional PDF of Y given $X = x$. The conditional density is given by $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$ for $f_X(x) > 0$.

$$f_{Y|X}(y|x) = \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{1}{6}x^3 e^{-x}}.$$

The term e^{-x} cancels out. The domain for y is $-x < y < x$.

$$f_{Y|X}(y|x) = \frac{1/8}{1/6} \cdot \frac{x^2 - y^2}{x^3} = \frac{6}{8}\frac{x^2 - y^2}{x^3} = \boxed{\frac{3}{4x^3}(x^2 - y^2)}, \quad \text{for } -x < y < x.$$

4 Problem 4: Functions of Exponential Variables

4.1 Problem Statement

Let Z be an exponentially distributed random variable with mean β.

(a) Find the probability density function of $X = -\ln \frac{Z}{\beta}$.

(b) Suppose X_1 and X_2 are two independent random variables following the same distribution as X. Find the probability density function of $X_1 - X_2$.

(c) Suppose $X_1, X_2, \ldots, X_n$ are independent random variables following the same distribution as X. Show that $\max(X_2, X_3, \ldots, X_n)$ and $X_1 + \ln(n-1)$ are identically distributed.

4.2 Solution

The PDF of $Z \sim \text{Exp}(\text{mean} = \beta)$ is $f_Z(z) = \frac{1}{\beta}e^{-z/\beta}$ for $z > 0$. The CDF is $F_Z(z) = 1 - e^{-z/\beta}$.

(a) PDF of $X = -\ln(Z/\beta)$. We find the CDF of X first.

$$F_X(x) = P(X \leq x) = P(-\ln(Z/\beta) \leq x) = P(\ln(Z/\beta) \geq -x)$$
$$= P(Z/\beta \geq e^{-x}) = P(Z \geq \beta e^{-x}).$$

This is $1 - F_Z(\beta e^{-x}) = 1 - (1 - e^{-(\beta e^{-x})/\beta}) = e^{-e^{-x}}$. So, the CDF is $F_X(x) = e^{-e^{-x}}$. The PDF is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx}\left(e^{-e^{-x}}\right) = e^{-e^{-x}} \cdot (-e^{-x}) \cdot (-1) = \boxed{e^{-x}e^{-e^{-x}}}.$$

This is the standard Gumbel distribution.

(b) PDF of $Y = X_1 - X_2$. Let $U_i = Z_i/\beta \sim \text{Exp}(1)$. Then $X_i = -\ln U_i$. So, $Y = X_1 - X_2 = (-\ln U_1) - (-\ln U_2) = \ln(U_2/U_1)$. We find the CDF of Y: $F_Y(y) = P(Y \leq y) = P(\ln(U_2/U_1) \leq y) = P(U_2 \leq U_1 e^y)$. We integrate over the distribution of U_1:

$$F_Y(y) = \int_0^\infty P(U_2 \leq u_1 e^y) f_{U_1}(u_1) du_1 = \int_0^\infty (1 - e^{-u_1 e^y})e^{-u_1} du_1$$
$$= \int_0^\infty (e^{-u_1} - e^{-(1+e^y)u_1}) du_1 = \left[-e^{-u_1}\right]_0^\infty - \left[-\frac{e^{-(1+e^y)u_1}}{1+e^y}\right]_0^\infty$$
$$= (1) - (\frac{1}{1+e^y}) = \frac{e^y}{1+e^y} = \frac{1}{1+e^{-y}}.$$

This is the CDF of the standard logistic distribution. Differentiating to find the PDF:

$$f_Y(y) = \frac{d}{dy}\left(\frac{e^y}{1+e^y}\right) = \frac{e^y(1+e^y) - e^y(e^y)}{(1+e^y)^2} = \boxed{\frac{e^y}{(1+e^y)^2}}.$$

(c) Show identical distribution. Let $V = X_1 + \ln(n-1)$ and $M = \max(X_2, \dots, X_n)$. We show they have the same CDF. First, the CDF of V:

$$F_V(v) = P(V \le v) = P(X_1 + \ln(n-1) \le v) = P(X_1 \le v - \ln(n-1)).$$

$$= F_X(v - \ln(n-1)) = \exp(-\exp(-(v - \ln(n-1)))) = \exp(-\exp(-v + \ln(n-1)))$$
$$= \exp(-e^{-v} \cdot e^{\ln(n-1)}) = \exp(-(n-1)e^{-v}).$$

Next, the CDF of M:

$$F_M(m) = P(M \le m) = P(\max(X_2, \dots, X_n) \le m).$$

This is equivalent to $P(X_2 \le m, X_3 \le m, \dots, X_n \le m)$. Due to independence and identical distribution:

$$F_M(m) = P(X_2 \le m) \cdots P(X_n \le m) = [F_X(m)]^{n-1}.$$
$$= (e^{-e^{-m}})^{n-1} = \exp(-(n-1)e^{-m}).$$

Since $F_V(x) = F_M(x)$ for all x, the two random variables are identically distributed.

5 Problem 5: Derived Discrete Variables

5.1 Problem Statement

Suppose the probability density function of random variable Y is $f_Y(y) = e^{-y}$ for $y > 0$. Define the random variables X_i as follows:

$$X_i = \begin{cases} 1, & Y > i \\ 0, & Y \le i \end{cases} \quad \text{for } i = 1, 2.$$

(a) Let $Z = e^{X_1}$. Find the probability mass function of Z.

(b) Find the joint probability mass function of (X_1, X_2).

5.2 Solution

The given distribution for Y is standard exponential, $Y \sim \text{Exp}(1)$. We have $P(Y > y) = e^{-y}$ and $P(Y \le y) = 1 - e^{-y}$.

(a) PMF of $Z = e^{X_1}$. First, we find the PMF of X_1. X_1 can take values 0 or 1.

$$P(X_1 = 1) = P(Y > 1) = e^{-1}.$$

$$P(X_1 = 0) = P(Y \le 1) = 1 - e^{-1}.$$

Now, we find the possible values of Z and their probabilities.

- If $X_1 = 0$, then $Z = e^0 = 1$. This occurs with probability $P(X_1 = 0) = 1 - e^{-1}$.
- If $X_1 = 1$, then $Z = e^1 = e$. This occurs with probability $P(X_1 = 1) = e^{-1}$.

So, the PMF of Z, denoted $p_Z(z)$, is:

$$\boxed{p_Z(z) = \begin{cases} 1 - e^{-1}, & z = 1 \\ e^{-1}, & z = e \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Joint PMF of (X_1, X_2). The vector (X_1, X_2) can take four possible values: $(0,0), (0,1), (1,0), (1,1)$. We compute the probability for each case.

- $P(X_1 = 0, X_2 = 0) = P(Y \le 1 \text{ and } Y \le 2) = P(Y \le 1) = \boxed{1 - e^{-1}}$.
- $P(X_1 = 0, X_2 = 1) = P(Y \le 1 \text{ and } Y > 2)$. These are disjoint events, so the probability is $\boxed{0}$.
- $P(X_1 = 1, X_2 = 0) = P(Y > 1 \text{ and } Y \le 2) = P(1 < Y \le 2) = P(Y \le 2) - P(Y \le 1) = (1 - e^{-2}) - (1 - e^{-1}) = \boxed{e^{-1} - e^{-2}}$.
- $P(X_1 = 1, X_2 = 1) = P(Y > 1 \text{ and } Y > 2) = P(Y > 2) = \boxed{e^{-2}}$.

The joint PMF $p(x_1, x_2)$ can be summarized in a table:

$p(x_1, x_2)$	$X_2 = 0$	$X_2 = 1$
$X_1 = 0$	$1 - e^{-1}$	0
$X_1 = 1$	$e^{-1} - e^{-2}$	e^{-2}