2012 Prob Midterm

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Contents

Todo list

1 Problem 1: Poisson failures across weeks

1.1 Problem Statement

Let the number of failures in a week be $N \sim \text{Pois}(\lambda)$.

(i) Compute the probability of no failures in a given week.

(ii) For the next 10 weeks, compute

(a) the probability that there are at least two weeks with no failures; and

(b) the probability that the first week with at least one failure is week 10.

1.2 Solution

(a) One week, no failures.

$$\Pr(N = 0) = e^{-\lambda}.$$

(b.i) At least two "zero-failure" weeks in 10 weeks. Let $p_0 \equiv \Pr(N = 0) = e^{-\lambda}$. If K denotes the number of zero-failure weeks out of 10, then $K \sim \text{Binom}(10, p_0)$. Hence

$$\Pr(K \geq 2) = 1 - \Pr(K = 0) - \Pr(K = 1) = 1 - (1 - p_0)^{10} - 10\, p_0(1 - p_0)^9.$$

(b.ii) Week 10 is the first week with any failure. This means the first 9 weeks have $N = 0$ and week 10 has $N \geq 1$:

$$\Pr = (e^{-\lambda})^9 \left(1 - e^{-\lambda}\right) = e^{-9\lambda} \left(1 - e^{-\lambda}\right).$$

2 Problem 2: Joint density $f(x, y) = 6x$ on $0 < x < y < 1$

2.1 Problem Statement

The joint density of (X, Y) is $f(x, y) = 6x$ for $0 < x < y < 1$, and 0 otherwise. Compute:

(i) the marginal densities f_X and f_Y;

(ii) $\mathrm{Cov}(X, Y)$ and the correlation ρ;

(iii) the conditional expectation $\mathbb{E}[X \mid Y = y]$ or $\mathbb{E}[Y \mid X = x]$ (specify your choice and compute it).

2.2 Solution

(a) Marginals.

$$f_X(x) = \int_{y=x}^{1} 6x\, dy = 6x(1 - x), \quad 0 < x < 1; \qquad f_Y(y) = \int_{x=0}^{y} 6x\, dx = 3y^2, \quad 0 < y < 1.$$

(b) Moments, covariance, and correlation.

$$\mathbb{E}[X] = \int_0^1 x\, f_X(x)\, dx = \tfrac{1}{2}, \qquad \mathbb{E}[Y] = \int_0^1 y\, f_Y(y)\, dy = \tfrac{3}{4}.$$

$$\mathbb{E}[XY] = \int_0^1 \int_0^y xy \cdot 6x\, dx\, dy = \int_0^1 6y \left[\tfrac{y^3}{3}\right] dy = \int_0^1 2y^4\, dy = \tfrac{2}{5}.$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 6x(1-x)\,dx = \tfrac{3}{10} \Rightarrow \mathrm{Var}(X) = \tfrac{3}{10} - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{20}.$$

$$\mathbb{E}[Y^2] = \int_0^1 y^2 \cdot 3y^2\,dy = \tfrac{3}{5} \Rightarrow \mathrm{Var}(Y) = \tfrac{3}{5} - \left(\tfrac{3}{4}\right)^2 = \tfrac{3}{80}.$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \tfrac{2}{5} - \tfrac{1}{2} \cdot \tfrac{3}{4} = \tfrac{1}{40}.$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\frac{1}{40}}{\sqrt{\frac{1}{20} \cdot \frac{3}{80}}} = \frac{1}{\sqrt{3}}.$$

(c) Conditional expectation. Using $\mathbb{E}[Y \mid X = x]$:

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{6x}{6x(1-x)} = \frac{1}{1-x}, \quad y \in (x, 1),$$

so $Y \mid X = x \sim \mathrm{Unif}(x, 1)$ and

$$\mathbb{E}(Y \mid X = x) = \frac{x+1}{2}.$$

3 Problem 3: Maximum of three i.i.d. variables

3.1 Problem Statement

Let X_1, X_2, X_3 be i.i.d. with common CDF F (and density f if continuous). Define $X_{\max} = \max\{X_1, X_2, X_3\}$. Find its distribution.

3.2 Solution

(a) CDF. By independence,

$$F_{X_{\max}}(t) = \Pr(X_1 \le t, X_2 \le t, X_3 \le t) = F(t)^3.$$

(b) PDF/PMF. If continuous,

$$f_{X_{\max}}(t) = \frac{d}{dt} F(t)^3 = 3F(t)^2 f(t).$$

If discrete,

$$\Pr(X_{\max} = x) = F(x)^3 - \big(F(x^-)\big)^3 \quad \text{(e.g., for integer support, } F(x)^3 - F(x-1)^3\text{)}.$$

4 Problem 4: Normal mixture with heteroskedastic mean

4.1 Problem Statement

Let $X \sim \mathrm{Unif}(0,1)$ and, conditional on $X = x$, $Y \mid X = x \sim N(x, x^2)$.

(i) Compute $\mathbb{E}[Y]$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X,Y)$.

(ii) Show that Y/X is independent of X.

4.2 Solution

(a) Moments. Tower property and law of total variance give

$$\mathbb{E}[Y] = \mathbb{E}\left[\,\mathbb{E}(Y \mid X)\right] = \mathbb{E}[X] = \tfrac{1}{2}.$$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\,\mathbb{E}(Y \mid X)\right) + \mathbb{E}\left[\,\mathrm{Var}(Y \mid X)\right] = \mathrm{Var}(X) + \mathbb{E}[X^2] = \tfrac{1}{12} + \tfrac{1}{3} = \tfrac{5}{12}.$$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}\left(X, \mathbb{E}(Y \mid X)\right) = \mathrm{Cov}(X,X) = \mathrm{Var}(X) = \tfrac{1}{12}.$$

(b) Independence of Y/X and X. Let $Z \equiv Y/X$. For any fixed $x > 0$,

$$Z \mid X = x = \frac{Y}{x} \sim N\Big(\frac{x}{x}, \frac{x^2}{x^2}\Big) = N(1,1),$$

which does not depend on x. A change-of-variables calculation confirms factorization:

$$f_{Z,X}(z,x) = f_{Y|X}(xz \mid x)\, f_X(x) \left|\frac{\partial(xz)}{\partial z}\right| = \phi(z;1,1)\, f_X(x),$$

so $Z \perp X$.

5 Problem 5: Normal MGF and the third raw moment

5.1 Problem Statement

Let $X \sim N(\mu, \sigma^2)$.

(i) Derive the moment generating function (MGF) $M_X(t) = \mathbb{E}[e^{tX}]$.

(ii) Use (a) to compute $\mathbb{E}[X^3]$.

5.2 Solution

(a) MGF. Completing the square in the Gaussian integral yields

$$M_X(t) = \exp\big(\mu t + \tfrac{1}{2}\sigma^2 t^2\big), \quad t \in \mathbb{R}.$$

(b) Third raw moment from MGF. Differentiate three times and evaluate at $t = 0$:

$$M_X'(t) = (\mu + \sigma^2 t) M_X(t), \quad M_X''(t) = \big(\sigma^2 + (\mu + \sigma^2 t)^2\big) M_X(t),$$

$$M_X'''(t) = \Big(3\sigma^2(\mu + \sigma^2 t) + (\mu + \sigma^2 t)^3\Big) M_X(t) \Rightarrow \boxed{\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2}.$$