2014 Adv Econometrics Midterm: Detailed Solutions

Xiankang Wang

November 7, 2025

Contents

Todo list

1 Problem 1: Mixed germination and Bayes posterior

1.1 Problem Statement

Rutabaga seeds from growers A, B, C have germination rates $0.90, 0.80, 0.85$ respectively. A packet is a mixture drawn with proportions 0.25 (A), 0.35 (B), 0.40 (C). (i) Find the probability a random seed germinates. (ii) Given a seed did not germinate, find the posterior probability it came from grower B.

1.2 Solution

Let G be the event of germination.

(i) Mixture success probability.

$$\Pr(G) = 0.25 \cdot 0.90 + 0.35 \cdot 0.80 + 0.40 \cdot 0.85 = \boxed{0.845}.$$

(ii) Posterior for source B given failure. Let $\bar{G}$ denote failure. Then $\Pr(\bar{G}) = 0.25 \cdot 0.10 + 0.35 \cdot 0.20 + 0.40 \cdot 0.15 = \boxed{0.155}$. Bayes:

$$\Pr(B \mid \bar{G}) = \frac{\Pr(\bar{G} \mid B)\Pr(B)}{\Pr(\bar{G})} = \frac{0.20 \cdot 0.35}{0.155} = \boxed{\tfrac{7}{15.5} \approx 0.4516}.$$

2 Problem 2: Functions of i.i.d. standard normals

2.1 Problem Statement

Suppose $X_i \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, $i = 1, \ldots, k$. (i) Find the distribution of $Y = \sum_{i=1}^k X_i^2$. (ii) For constants $a_1, \ldots, a_k \in \mathbb{R}$, find the distribution of $Z = \sum_{i=1}^k a_i X_i$ (in particular, specialize to $\sum a_i^2 = 1$).

2.2 Solution

(i) χ^2 law. By definition $Y \sim \chi^2_k$; its mgf is $M_Y(t) = (1-2t)^{-k/2}$ for $t < 1/2$.

(ii) Linear form. Vectorize $X \sim \mathcal{N}(0, I_k)$. Then $Z = a'X \sim \mathcal{N}\big(0,\, a'a\big)$. In particular, if $a'a = 1$, then $Z \sim \mathcal{N}(0,1)$.

3 Problem 3: Normalizing constant, marginals, and a conditional probability

3.1 Problem Statement

Let X and Y be the fractions of time during a working day when the dining room and drive-thru are busy. The joint density is

$$f_{X,Y}(x,y) = k\left(x^2+y^2\right), \qquad 0<x<1,\ 0<y<1,\ 0 \text{ otherwise}.$$

(i) Find k. (ii) Find the marginal densities f_X, f_Y. (iii) Compute $\Pr\left(Y>0.75 \mid X<0.10\right)$.

3.2 Solution

(i) Normalizing constant.

$$1=\int_0^1\int_0^1 k(x^2+y^2)\,dy\,dx = k\left(\tfrac{1}{3}+\tfrac{1}{3}\right)=\tfrac{2k}{3}\ \Rightarrow\ \boxed{k=\tfrac{3}{2}}.$$

(ii) Marginals.

$$f_X(x)=\int_0^1 \tfrac{3}{2}(x^2+y^2)\,dy=\tfrac{3}{2}\left(x^2+\tfrac{1}{3}\right),\quad 0<x<1;\qquad f_Y(y)=\tfrac{3}{2}\left(\tfrac{1}{3}+y^2\right),\ 0<y<1.$$

(iii) Conditional probability.

$$\Pr(Y>0.75\mid X<0.10)=\frac{\int_0^{0.10}\int_{0.75}^1\frac{3}{2}(x^2+y^2)\,dy\,dx}{\int_0^{0.10}\int_0^1\frac{3}{2}(x^2+y^2)\,dy\,dx}=\frac{\frac{3}{2}\left[\frac{(0.1)^3}{3}\cdot\frac{1}{4}+0.1\cdot\frac{1-0.75^3}{3}\right]}{\frac{3}{2}\left[\frac{(0.1)^3}{3}+0.1\cdot\frac{1}{3}\right]}=\boxed{\tfrac{929}{1616}\approx 0.5749}.$$

4 Problem 4: Ratio $W=X/(X+Y)$ for independent Gammas

4.1 Problem Statement

Let X and Y be independent with densities proportional to $x^{\alpha-1}e^{-x}$ and $y^{\beta-1}e^{-y}$ on $(0,\infty)$, with $\alpha,\beta>0$ (normalizing constants not required). Find the density of $W=\dfrac{X}{X+Y}$.

4.2 Solution

Take the one-to-one transform $(U,W)=(X+Y,\ X/(X+Y))$ so that $(X,Y)=(UW,\ U(1-W))$ with Jacobian $\left|\partial(x,y)/\partial(u,w)\right|=u$. The joint density is, up to a constant,

$$f_{U,W}(u,w)\propto (uw)^{\alpha-1}e^{-uw}\left(u(1-w)\right)^{\beta-1}e^{-u(1-w)}\,u=u^{\alpha+\beta-1}\,e^{-u}\,w^{\alpha-1}(1-w)^{\beta-1}.$$

Integrating out $u > 0$ yields a finite constant (the Gamma function) times $w^{\alpha-1}(1-w)^{\beta-1}$ on $w \in (0,1)$. Hence

$$\boxed{f_W(w) \propto w^{\alpha-1}(1-w)^{\beta-1}, \quad 0 < w < 1} \quad \text{(i.e., } W \sim \text{Beta}(\alpha, \beta)\text{)}.$$

5 Problem 5: Triangular support $0 < y < x < 1$

5.1 Problem Statement

Let (X, Y) have joint density $f_{X,Y}(x,y) = c$ on $\{(x,y) : 0 < y < x < 1\}$ and 0 otherwise. Compute: (i) the conditional pdf $f_{Y|X}(y \mid x)$ for $0 < x < 1$; (ii) $\mathbb{E}[Y \mid X = x]$; (iii) $\text{Cov}(X, Y)$.

5.2 Solution

The normalizing constant is $c = 2$ (area of the triangle is $1/2$).

(i) Conditional pdf. For fixed $x \in (0,1)$,

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{\int_0^x 2\,dy} = \boxed{\frac{1}{x},\ 0 < y < x}.$$

(ii) Conditional mean. Since $Y \mid X = x \sim \text{Unif}(0, x)$, $\mathbb{E}[Y \mid X = x] = \boxed{x/2}$.

(iii) Covariance. With $f_X(x) = \int_0^x 2\,dy = 2x$, $\mathbb{E}[X] = \int_0^1 x\,2x\,dx = 2/3$, and $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}(Y \mid X)] = \mathbb{E}[X/2] = 1/3$. Moreover,

$$\mathbb{E}[XY] = \mathbb{E}\left[X\,\mathbb{E}(Y \mid X)\right] = \mathbb{E}\left[X \cdot X/2\right] = \tfrac{1}{2}\,\mathbb{E}[X^2] = \tfrac{1}{2}\int_0^1 x^2\,2x\,dx = \tfrac{1}{4}.$$

Thus $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \boxed{\tfrac{1}{36}}$.

6 Problem 6: Thinning a Poisson process

6.1 Problem Statement

Suppose the number of events $N \sim \text{Pois}(\lambda)$. Each event is independently counted with probability p. Show that the counted and uncounted numbers are independent Poisson with means λp and $\lambda(1-p)$.

6.2 Solution

Conditional on $N = n$, the counted number $C \mid N = n \sim \mathrm{Binom}(n, p)$ and the uncounted $U = n - C$. For $c, u \in \mathbb{N}$,

$$\begin{aligned}
\Pr(C = c, U = u) &= \sum_{n=c+u}^{\infty} \Pr(N = n) \Pr(C = c \mid N = n) \\
&= \sum_{n=c+u}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{n}{c} p^c (1-p)^{n-c} \\
&= e^{-\lambda} \frac{(\lambda p)^c}{c!} \frac{(\lambda(1-p))^u}{u!} \sum_{n=c+u}^{\infty} \frac{\lambda^{n-c-u}}{(n-c-u)!} \\
&= e^{-\lambda} \frac{(\lambda p)^c}{c!} \frac{(\lambda(1-p))^u}{u!} e^{\lambda} \\
&= \underbrace{e^{-\lambda p} \frac{(\lambda p)^c}{c!}}_{\mathrm{Pois}(\lambda p)} \underbrace{e^{-\lambda(1-p)} \frac{(\lambda(1-p))^u}{u!}}_{\mathrm{Pois}(\lambda(1-p))}.
\end{aligned}$$

Thus $C \sim \mathrm{Pois}(\lambda p)$, $U \sim \mathrm{Pois}(\lambda(1-p))$, and the factorization shows $C \perp U$.