2013 Adv-Econometrics Midterm: Detailed Solutions

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Contents

Todo list

1 Problem 1: Normal MGF and moments

1.1 Problem Statement

(a) Find the MGF of the standard normal distribution.

(b) Let $Z \sim N(0, 1)$ and $X \sim N(\mu, \sigma^2)$. Express X in terms of Z, and use (a) to find the MGF of X.

(c) Using (b), compute $\mathbb{E}\left[(X-\mu_X)^2\right]$ and $\mathbb{E}[X^3]$, where μ_X is the mean of X.

1.2 Solution

(a) $Z \sim N(0,1)$: $M_Z(t)$. Completing the square,

$$M_Z(t) = \mathbb{E}[e^{tZ}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big(tz - \tfrac{z^2}{2}\Big)\, dz = \exp\Big(\tfrac{t^2}{2}\Big).$$

(b) Affine mapping and MGF of X. Write $X = \mu + \sigma Z$. Then

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{\mu t}\ \mathbb{E}\left[e^{t\sigma Z}\right] = e^{\mu t} M_Z(\sigma t) = \boxed{\exp\big(\mu t + \tfrac{1}{2}\sigma^2 t^2\big)}.$$

(c) Variance and third raw moment. From (b),

$$M'_X(t) = (\mu + \sigma^2 t) M_X(t), \quad M''_X(t) = \big(\sigma^2 + (\mu+\sigma^2 t)^2\big) M_X(t),$$

$$M'''_X(t) = \Big(3\sigma^2(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^3\Big) M_X(t).$$

Evaluate at $t = 0$ to get

$$\mathbb{E}[X] = M'_X(0) = \mu, \qquad \mathbb{E}[X^2] = M''_X(0) = \sigma^2 + \mu^2, \qquad \boxed{\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2}.$$

Hence $\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right] = \mathbb{E}[X^2] - \mu^2 = \boxed{\sigma^2}$.

2 Problem 2: Joint density $f(x,y) = e^{-y}$ on $0 \le x \le y < \infty$

2.1 Problem Statement

Given $f_{X,Y}(x,y) = e^{-y}$ for $0 \le x \le y < \infty$ and 0 otherwise, compute:

(a) The marginal densities f_X and f_Y.

(b) The conditional densities $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

(c) The means and variances of X and Y.

(d) $\mathrm{Cov}(X,Y)$ and the correlation coefficient ρ.

2.2 Solution

(a) Marginals. For $y \ge 0$,

$$f_Y(y) = \int_{x=0}^{y} e^{-y}\, dx = y\, e^{-y} \quad \Rightarrow \quad Y \sim \mathrm{Gamma}(2,1).$$

For $x \ge 0$,

$$f_X(x) = \int_{y=x}^{\infty} e^{-y}\,dy = e^{-x} \quad \Rightarrow \quad X \sim \mathrm{Exp}(1).$$

(b) Conditionals.

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)}, \quad y \ge x \text{ (and 0 otherwise)},$$

so $Y \mid X = x \overset{d}{=} x + E$ with $E \sim \mathrm{Exp}(1)$. Likewise,

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad 0 \le x \le y,$$

so $X \mid Y = y \sim \mathrm{Unif}(0, y)$.

(c) Means and variances.

$$\mathbb{E}[X] = 1, \quad \mathrm{Var}(X) = 1 \quad (\mathrm{Exp}(1)); \qquad \mathbb{E}[Y] = 2, \quad \mathrm{Var}(Y) = 2 \quad (\mathrm{Gamma}(2,1)).$$

(d) Covariance and correlation. Using $\mathbb{E}[Y \mid X = x] = x + 1$,

$$\mathbb{E}[XY] = \mathbb{E}\left[X\,\mathbb{E}(Y \mid X)\right] = \mathbb{E}[X(X+1)] = \mathbb{E}[X^2] + \mathbb{E}[X] = 2 + 1 = 3.$$

Thus

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 3 - (1)(2) = \boxed{1}, \qquad \rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \boxed{1/\sqrt{2}}.$$

3 Problem 3: Zero covariance $\Leftrightarrow$ independence (two cases)

3.1 Problem Statement

(a) If (X, Y) are jointly normal, show $\mathrm{Cov}(X,Y) = 0$ iff X and Y are independent.

(b) If X and Y are discrete and each takes only two values, show $\mathrm{Cov}(X,Y) = 0$ iff X and Y are independent.

3.2 Solution

(a) Bivariate normal. The joint pdf is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left[\left(\tfrac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\tfrac{x-\mu_X}{\sigma_X}\right)\left(\tfrac{y-\mu_Y}{\sigma_Y}\right) + \left(\tfrac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}.$$

If $\mathrm{Cov}(X,Y) = \rho\sigma_X\sigma_Y = 0$, then $\rho = 0$ and the exponent separates, yielding

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\Big(-\tfrac{1}{2}\big(\tfrac{x-\mu_X}{\sigma_X}\big)^2\Big)\exp\Big(-\tfrac{1}{2}\big(\tfrac{y-\mu_Y}{\sigma_Y}\big)^2\Big) = f_X(x)f_Y(y),$$

so $X \perp Y$. The converse ($X \perp Y \Rightarrow \mathrm{Cov} = 0$) holds for any variables with finite second moments.

(b) Two-point discrete variables. Let $X \in \{x_1, x_2\}$, $Y \in \{y_1, y_2\}$. Denote $a \equiv \Pr(X = x_2)$, $b \equiv \Pr(Y = y_2)$ and $p \equiv \Pr(X = x_2, Y = y_2)$. Then

$$\mathbb{E}[X] = x_1(1-a) + x_2 a, \quad \mathbb{E}[Y] = y_1(1-b) + y_2 b, \quad \mathbb{E}[XY] = x_2y_2\,p + x_2y_1\,(a-p) + x_1y_2\,(b-p) + x_1y_1\,(1-a-b$$

A short algebra shows

$$\mathrm{Cov}(X,Y) = (x_2 - x_1)(y_2 - y_1)\,(p - ab).$$

Hence $\mathrm{Cov}(X,Y) = 0$ iff $p = ab$, which implies the full 2×2 table factorizes:

$$\Pr(x_2,y_2) = ab,\ \Pr(x_2,y_1) = a(1-b),\ \Pr(x_1,y_2) = (1-a)b,\ \Pr(x_1,y_1) = (1-a)(1-b),$$

so X and Y are independent.

4 Problem 4: Diagnostic testing (Bayes)

4.1 Problem Statement

Prevalence: $\Pr(D) = 0.02$. Sensitivity: $\Pr(T+ \mid D) = 0.9$. Specificity given: $\Pr(T- \mid D^c) = 0.95$.

(a) Report sensitivity and specificity.

(b) Compute $\Pr(D \mid T+)$. (Positive predictive value.)

4.2 Solution

(a) Definitions. Sensitivity $= \Pr(T+ \mid D) = \boxed{0.9}$. Specificity $= \Pr(T- \mid D^c) = \boxed{0.95}$.

(b) Positive predictive value. Using Bayes' rule with $\Pr(T+ \mid D^c) = 1 - \text{specificity} = 0.05$ and $\Pr(D^c) = 0.98$,

$$\Pr(D \mid T+) = \frac{\Pr(T+ \mid D)\Pr(D)}{\Pr(T+ \mid D)\Pr(D) + \Pr(T+ \mid D^c)\Pr(D^c)} = \frac{0.9 \times 0.02}{0.9 \times 0.02 + 0.05 \times 0.98} = \boxed{\frac{18}{67}} \approx \boxed{0.2687}.$$

5 Problem 5: Transformations of Gaussian variables

5.1 Problem Statement

Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$.

(a) With $Z = X_2^2$, find the pdf of Z.

(b) Define $Y_1 = X_1^2 + X_2^2$ and $Y_2 = -X_1/\sqrt{Y_1}$. Find the joint distribution of (Y_1, Y_2).

(c) Are Y_1 and Y_2 independent? Explain and give a geometric interpretation.

5.2 Solution

(a) Pdf of $Z = X_2^2$. Since $(X_2/\sigma)^2 \sim \chi_1^2$, by scaling

$$f_Z(z) = \frac{1}{\sigma^2} f_{\chi_1^2}(z/\sigma^2) = \boxed{\frac{1}{\sqrt{2\pi\,\sigma^2\,z}}\,\exp\Big(-\frac{z}{2\sigma^2}\Big), \quad z \geq 0}.$$

(b) Joint law of (Y_1, Y_2). Let $U_i \equiv X_i/\sigma \sim N(0,1)$. Write polar coordinates $U_1 = R\cos\Theta$, $U_2 = R\sin\Theta$ with $R \geq 0$, $\Theta \in (-\pi, \pi]$. Then $W \equiv R^2 \sim \chi_2^2$ (i.e., $W \sim \mathrm{Exp}(\text{rate} = 1/2)$) and $\Theta \sim \mathrm{Unif}(-\pi, \pi]$, independent. Observing

$$Y_1 = X_1^2 + X_2^2 = \sigma^2 W, \qquad Y_2 = -\frac{X_1}{\sqrt{X_1^2 + X_2^2}} = -\cos\Theta,$$

we obtain the marginals

$$f_{Y_1}(y_1) = \frac{1}{2\sigma^2}\,\exp\Big(-\frac{y_1}{2\sigma^2}\Big), \quad y_1 \geq 0 \quad (\text{Exponential with mean } 2\sigma^2),$$

$$f_{Y_2}(y_2) = \frac{1}{\pi\sqrt{1 - y_2^2}}, \quad -1 < y_2 < 1 \quad (\text{arcsine law}).$$

Since $W \perp \Theta$, it follows that $Y_1 \perp Y_2$ and

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)\, f_{Y_2}(y_2), \qquad y_1 \geq 0,\ -1 < y_2 < 1.$$

(c) Independence and geometry. Yes, independent. Geometrically, (X_1, X_2) has a rotationally symmetric Gaussian density centered at the origin. The "radius" $\sqrt{Y_1}$ (energy) and the "angle" $\arccos(-Y_2)$ are independent coordinates; Y_1 encodes radial magnitude while Y_2 depends only on orientation.