Derivations for Common Continuous Distributions

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Contents

1 Introduction and Setup

This document provides a detailed, step-by-step derivation of the Expectation, Variance, and Moment Generating Function (MGF) for several common continuous distributions. For a continuous random variable X with Probability Density Function (PDF) $f(x)$, the key quantities are defined as:

- **Expectation (Mean)**: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$
- **Second Moment**: $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x$
- **Variance**: $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- **Moment Generating Function (MGF)**: $M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x$

The moments can be found from the MGF by differentiation: $\mathbb{E}[X^n] = M_X^{(n)}(0)$.

2 The Uniform Distribution

Distribution 1 (Uniform)**.** *Let $X \sim$ Uniform(a, b).*

- ***Parameters:*** $a < b$
- ***Support:*** $x \in [a, b]$
- ***Probability Density Function (PDF):*** $f(x) = \frac{1}{b-a}$

2.1 Derivations from PDF

2.1.1 Expectation

$$\mathbb{E}[X] = \int_a^b x \cdot \frac{1}{b-a}\,\mathrm{d}x = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{(b-a)(b+a)}{2(b-a)}$$

$\mathbb{E}[X] = \frac{a+b}{2}$

2.1.2 Variance

First, we find the second moment $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\,\mathrm{d}x = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

Now, we use the variance formula.

$$\begin{aligned}\mathrm{Var}(X) &= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{4(a^2+ab+b^2) - 3(a^2+2ab+b^2)}{12} \\ &= \frac{a^2-2ab+b^2}{12}\end{aligned}$$

$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

2.2 Derivations from MGF

2.2.1 Moment Generating Function (MGF)

$$M_X(t) = \int_a^b e^{tx} \frac{1}{b-a} \,\mathrm{d}x = \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$ for $t \neq 0$

Note: For $t = 0$, $M_X(0) = 1$. The moments can be found by taking limits or using Taylor series. For instance, $\mathbb{E}[X] = \lim_{t \to 0} M_X'(t)$.

3 The Gamma Distribution

The Gamma distribution is a flexible two-parameter family of continuous distributions. It is the parent distribution for the Exponential and Chi-Squared distributions. (See Appendix for the Gamma function definition).

Distribution 2 (Gamma)**.** *Let* $X \sim$ *Gamma*(α, β)*.*

- ***Parameters:*** *Shape* $\alpha > 0$*, Rate* $\beta > 0$
- ***Support:*** $x > 0$
- ***PDF:*** $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

3.1 Derivations from PDF

3.1.1 Expectation

We use the identity $\Gamma(z+1) = z\Gamma(z)$.

$$\mathbb{E}[X] = \int_0^\infty x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \, \mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} \, \mathrm{d}x$$

The integral resembles a Gamma function. Let $u = \beta x$, so $\mathrm{d}x = du/\beta$.

$$\int_0^\infty x^\alpha e^{-\beta x} \, \mathrm{d}x = \int_0^\infty \left(\frac{u}{\beta}\right)^\alpha e^{-u} \frac{du}{\beta} = \frac{1}{\beta^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} du = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}$$

Substituting back:

$$\mathbb{E}[X] = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta}$$

$\mathbb{E}[X] = \frac{\alpha}{\beta}$

3.1.2 Variance

Similarly, for $\mathbb{E}[X^2]$:

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^\infty x^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \, \mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x} \, \mathrm{d}x \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta^2} = \frac{\alpha(\alpha+1)}{\beta^2} \end{aligned}$$

The variance is:

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2 + \alpha - \alpha^2}{\beta^2}$$

$\mathrm{Var}(X) = \frac{\alpha}{\beta^2}$

3.2 Derivations from MGF

3.2.1 Moment Generating Function (MGF)

$$M_X(t) = \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \, \mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} \, \mathrm{d}x$$

This integral converges for $t < \beta$. Let $\lambda = \beta - t$. The integral is $\frac{\Gamma(\alpha)}{\lambda^\alpha}$.

$$M_X(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - t)^\alpha}$$

$M_X(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$ for $t < \beta$

4 Special Cases of the Gamma Distribution

4.1 The Exponential Distribution

The Exponential distribution models the time until the first event in a Poisson process.

Distribution 3 (Exponential)**.** *It is a special case of the Gamma distribution with shape* $\alpha = 1$ *and rate* $\beta = \lambda$*.* $X \sim \text{Exponential}(\lambda) \equiv \text{Gamma}(1, \lambda)$*.*

- ***Parameter****:* *Rate* $\lambda > 0$
- ***Support****:* $x > 0$
- ***PDF****:* $f(x) = \lambda e^{-\lambda x}$

By substituting $\alpha = 1, \beta = \lambda$ into the Gamma results:

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

$$M_X(t) = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda$$

4.2 The Chi-Squared (χ^2) Distribution

The Chi-Squared distribution is widely used in hypothesis testing. It is the distribution of a sum of the squares of k independent standard normal random variables.

Distribution 4 (Chi-Squared)**.** *It is a special case of the Gamma distribution with shape* $\alpha = k/2$ *and rate* $\beta = 1/2$*.* $X \sim \chi^2(k) \equiv \text{Gamma}(k/2, 1/2)$*.*

- ***Parameter****:* *Degrees of freedom* $k > 0$ *(an integer)*
- ***Support****:* $x > 0$
- ***PDF****:* $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$

By substituting $\alpha = k/2, \beta = 1/2$ into the Gamma results:

$$\mathbb{E}[X] = \frac{k/2}{1/2} = k$$

$$\text{Var}(X) = \frac{k/2}{(1/2)^2} = 2k$$

$$M_X(t) = \left(\frac{1/2}{1/2 - t}\right)^{k/2} = (1 - 2t)^{-k/2} \text{ for } t < 1/2$$

5 The Beta Distribution

Distribution 5 (Beta)**.** *Let* $X \sim$ *Beta*(α, β)*. (See Appendix for the Beta function).*

- ***Parameters****:* *Shape* $\alpha > 0$*, Shape* $\beta > 0$
- ***Support****:* $x \in (0, 1)$
- ***PDF****:* $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$

The MGF for the Beta distribution does not have a simple closed form. We derive moments directly from the PDF.

5.1 Derivations from PDF

5.1.1 Expectation

$$\mathbb{E}[X] = \int_0^1 x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \, \mathrm{d}x = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1} \, \mathrm{d}x$$

The integral is the definition of $B(\alpha + 1, \beta)$.

$$\mathbb{E}[X] = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}$$

$\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$

5.1.2 Variance

We find $\mathbb{E}[X^2]$ by evaluating $\int_0^1 x^2 f(x) \, \mathrm{d}x$, which involves $B(\alpha + 2, \beta)$.

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}\end{aligned}$$

Then the variance is:

$$\begin{aligned}\mathrm{Var}(X) &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\ &= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\end{aligned}$$

$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

6 The Normal Distribution

Distribution 6 (Normal)**.** *Let* $X \sim \mathcal{N}(\mu, \sigma^2)$*.*

- ***Parameters****:* *Mean* $\mu \in \mathbb{R}$*,* *Variance* $\sigma^2 > 0$
- ***Support****:* $x \in (-\infty, \infty)$
- ***PDF****:* $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

6.1 Derivations from MGF

It is easiest to first find the MGF of a standard normal $Z \sim \mathcal{N}(0, 1)$, where $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

$$M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, \mathrm{d}z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2tz}{2}\right) \mathrm{d}z$$

Complete the square: $z^2 - 2tz = (z-t)^2 - t^2$.

$$\begin{aligned} M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2 - t^2}{2}\right) \mathrm{d}z \\ &= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-t)^2}{2}\right) \mathrm{d}z \end{aligned}$$

The integral is the total probability of a $\mathcal{N}(t, 1)$ distribution, which is 1. Thus, $M_Z(t) = e^{t^2/2}$. For $X = \mu + \sigma Z$, we have $M_X(t) = \mathbb{E}[e^{t(\mu+\sigma Z)}] = e^{\mu t}\mathbb{E}[e^{(t\sigma)Z}] = e^{\mu t} M_Z(t\sigma)$.

$$\boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

6.1.1 Expectation and Variance

$$\begin{aligned} M_X'(t) &= (\mu + \sigma^2 t) M_X(t) \implies \mathbb{E}[X] = M_X'(0) = \mu \\ M_X''(t) &= \sigma^2 M_X(t) + (\mu + \sigma^2 t)^2 M_X(t) \implies \mathbb{E}[X^2] = M_X''(0) = \sigma^2 + \mu^2 \end{aligned}$$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2$.

$$\boxed{\mathbb{E}[X] = \mu \quad \text{and} \quad \mathrm{Var}(X) = \sigma^2}$$

6.2 Stein's Lemma

A powerful property of the normal distribution.

Theorem (Stein's Lemma) If $X \sim \mathcal{N}(\mu, \sigma^2)$ and g is a differentiable function such that the expectations $\mathbb{E}[g(X)(X-\mu)]$ and $\mathbb{E}[g'(X)]$ exist, then:

$$\boxed{\mathbb{E}[g(X)(X-\mu)] = \sigma^2 \mathbb{E}[g'(X)]}$$

Proof We start with the definition of $\mathbb{E}[g'(X)]$:

$$\begin{aligned}
\sigma^2\mathbb{E}[g'(X)] &= \sigma^2 \int_{-\infty}^{\infty} g'(x)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}/\sigma}\int_{-\infty}^{\infty} g'(x)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,\mathrm{d}x
\end{aligned}$$

We use integration by parts, with $u = \exp(-\frac{(x-\mu)^2}{2\sigma^2})$ and $dv = g'(x)dx$. Then $du = -\frac{x-\mu}{\sigma^2}\exp(-\dots)dx$ and $v = g(x)$.

$$\begin{aligned}
&\int u\,dv = [uv]_{-\infty}^{\infty} - \int v\,du \\
\implies &\int g'(x)\exp(\dots) = [g(x)\exp(\dots)]_{-\infty}^{\infty} - \int g(x)\left(-\frac{x-\mu}{\sigma^2}\right)\exp(\dots)\,\mathrm{d}x
\end{aligned}$$

The first term $[g(x)\exp(\dots)]$ goes to 0 at both $\pm\infty$ because the exponential term decays much faster than any polynomial growth in $g(x)$ (assuming it's well-behaved).

$$\begin{aligned}
\sigma^2\mathbb{E}[g'(X)] &= \frac{\sigma^2}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\left(\frac{x-\mu}{\sigma^2}\right)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} g(x)(x-\mu)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,\mathrm{d}x \\
&= \mathbb{E}[g(X)(X-\mu)]
\end{aligned}$$

7 The Log-Normal Distribution

Distribution 7 (Log-Normal)**.** *A random variable X is log-normally distributed if its logarithm, $Y = \ln(X)$, is normally distributed. Let $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then $X = e^Y \sim$ LogNormal(μ, σ^2).*

- ***Parameters:*** $\mu \in \mathbb{R}$, $\sigma^2 > 0$ *(from the underlying normal)*
- ***Support:*** $x > 0$
- ***PDF:*** $f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$

The MGF of the log-normal distribution does not have a simple closed form. The moments are easily found using the MGF of the underlying normal distribution.

7.1 Moments

The k-th moment of X is given by:

$$\mathbb{E}[X^k] = \mathbb{E}[(e^Y)^k] = \mathbb{E}[e^{kY}]$$

This is exactly the MGF of $Y \sim \mathcal{N}(\mu, \sigma^2)$, evaluated at $t = k$.

$$\mathbb{E}[X^k] = M_Y(k) = e^{\mu k + \frac{1}{2}\sigma^2 k^2}$$

7.1.1 Expectation

Set $k = 1$:

$$\boxed{\mathbb{E}[X] = e^{\mu + \sigma^2/2}}$$

7.1.2 Variance

Set $k = 2$ to find $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = e^{2\mu + \frac{1}{2}\sigma^2(2^2)} = e^{2\mu + 2\sigma^2}$$

The variance is:

$$\begin{aligned}\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 &= e^{2\mu+2\sigma^2} - (e^{\mu+\sigma^2/2})^2 \\ &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)\end{aligned}$$

$$\boxed{\mathrm{Var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)}$$

8 The Cauchy Distribution

Distribution 8 (Cauchy)**.** *Let* $X \sim \textit{Cauchy}(x_0, \gamma)$*.*

- ***Parameters****: Location* $x_0 \in \mathbb{R}$*, Scale* $\gamma > 0$
- ***Support****:* $x \in (-\infty, \infty)$
- ***PDF****:* $f(x) = \frac{1}{\pi\gamma\left(1+\left(\frac{x-x_0}{\gamma}\right)^2\right)}$

8.1 Non-existence of Moments

The Cauchy distribution is famous for having undefined moments. Let's check the expectation for a standard Cauchy ($x_0 = 0, \gamma = 1$).

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \frac{1}{\pi(1+x^2)}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\mathrm{d}x$$

This is an improper integral. We evaluate it as a limit:

$$\int_0^R \frac{x}{1+x^2}\,\mathrm{d}x = \left[\frac{1}{2}\ln(1+x^2)\right]_0^R = \frac{1}{2}\ln(1+R^2)$$

As $R \to \infty$, this integral diverges to ∞. Since the integral does not converge absolutely, the expectation $\mathbb{E}[X]$ is undefined.

The Mean, Variance, and all higher moments of the Cauchy distribution are undefined. Its MGF does not exist.

8.2 Characteristic Function

Since the MGF does not exist, we use the characteristic function $\phi_X(t) = \mathbb{E}[e^{itX}]$, which always exists. For the Cauchy distribution, it has a simple form (derived using complex analysis).

$\phi_X(t) = e^{ix_0 t - \gamma|t|}$

The characteristic function makes the Cauchy distribution "stable" and shows that the average of i.i.d. Cauchy variables follows the same Cauchy distribution, a rather counter-intuitive result.

9 Summary Table

Table 1: Comparison of Properties for Common Continuous Distributions

Distribution	**PDF** $f(x)$	**Support** $\mathcal{X}$	**Expectation** $\mathbb{E}[X]$	**Variance** $\mathrm{Var}(X)$	**MGF** $M_X(t)$
Uniform	$\frac{1}{b-a}$	$[a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{R}$	μ	σ^2	$e^{\mu t+\sigma^2 t^2/2}$
Log-Normal	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$	$\mathbb{R}^+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$	Does not exist
Exponential	$\lambda e^{-\lambda x}$	$\mathbb{R}^+$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$	$\mathbb{R}^+$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$(\frac{\beta}{\beta-t})^\alpha$
Chi-Squared	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{\frac{k}{2}-1}e^{-x/2}$	$\mathbb{R}^+$	k	$2k$	$(1-2t)^{-k/2}$
Beta	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$	$(0, 1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	No simple form
Cauchy	$\frac{1}{\pi\gamma(1+(\frac{x-x_0}{\gamma})^2)}$	$\mathbb{R}$	Undefined	Undefined	Does not exist

A The Gamma and Beta Functions

A.1 The Gamma Function

The Gamma function, $\Gamma(z)$, is an extension of the factorial function to complex numbers. For a positive real number z, it is defined by the integral:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, \mathrm{d}x$$

A key property, derived via integration by parts, is:

$$\Gamma(z+1) = z\Gamma(z)$$

For an integer n, this means $\Gamma(n) = (n-1)!$. Other special values include $\Gamma(1/2) = \sqrt{\pi}$.

A.2 The Beta Function

The Beta function, $B(x, y)$, is defined by the integral:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$$

It is related to the Gamma function by the identity:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$