Problem 1. *Using the Gram-Schmidt process find a set of mutually orthonormal vectors u^1, u^2, u^3, based on*

$$x^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, x^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, x^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

Solution. By definition of Gram-Schmidt process, we have to use

$$y^i = x^i + \sum_{k=1}^{i-1} a_{(i-1)k} x^k$$

where

$$< y^i, y^j >= 0 \qquad \forall i \neq j$$

Hence,

$$y^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + 0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$y^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 1 \\ 0 \\ -1 - a_{11} \end{bmatrix}$$

$$y^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_{21} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_{22} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{22} \\ 1 \\ -1 - a_{21} - a_{22} \end{bmatrix}$$

So,

$$< y^1, y^2 > = 0 \quad \rightarrow \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} a_{11} & 1 & 0 & -1 - a_{11} \end{bmatrix} = 0$$

$$< y^2, y^3 > = 0 \quad \rightarrow \quad \begin{bmatrix} a_{11} \\ 1 \\ 0 \\ -1 - a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & 1 & 0 & -1 - a_{11} \end{bmatrix} = 0$$

$$< y^3, y^1 > = 0 \quad \rightarrow \quad \begin{bmatrix} a_{21} \\ a_2 2 \\ 1 \\ -1 - a_{21} - a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = 0$$

Then,

$$a_{11} = -\frac{1}{2} \qquad a_{21} = -\frac{1}{3} \qquad a_{22} = -\frac{1}{3}$$

Hence,

$$y^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad y^2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \qquad y^3 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix}$$

In order to find orthonoraml vectors,

$$u^i \triangleq \frac{y^i}{||y^i||}, \qquad i = 1, 2, 3, \ldots N$$

Therefore,

$$u^1 = \frac{y^1}{||y^1||} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} / \sqrt{1^2 + 0^2 + 0^2 + (-1)^2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$u^2 = \frac{y^2}{||y^2||} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} / \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 0^2 + \left(-\frac{1}{2}\right)^2} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ 0 \\ -\frac{\sqrt{6}}{6} \end{bmatrix}$$

$$u^3 = \frac{y^3}{||y^3||} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix} / \sqrt{\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + 1^2 + \left(-\frac{1}{3}\right)^2} = \begin{bmatrix} -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} \end{bmatrix}$$