ECE-GY 5253
Applied Matrix Theory

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About This Course

1. This is <u>not</u> a Math. Class.

Links to ECE courses:

- **Systems, Control, Robotics, Signal Processing**:

  ECE-GY 6113, 6243, 6253, 7133, 7253, ROB-GYxxxx,…

- **Power Engineering**:

  ECE-GY 5613, 6603, 6623, 6633, 6653, 6663,…

- **Communications, Networking, CompE**:

ECE-GY 5363, 6023, 6033, 6313, 7353, 5483, …

- **Machine learning and optimization**

  ECE-GY6143 / CS-GY6923, ECE-GY 6233, …

About This Course

2. **Suitable for both upper-level undergraduates and graduate students from diverse fields of engineering & science:**

- **Electrical engineering (**wireless, control/robotics, communications, networking…**)**
- **Mechanical and chemical engineering**
- **Financial engineering**
- **CS, Applied mathematics, etc**

Background Knowledge

This course assumes only elementary knowledge about:

- Algebra
- Calculus

Tips for Getting "A"

- **Well prepared:** Reading the recommended textbook before each class
- **Practice:** exercise, homework
- **Problem-driven:** applications
- **Engaging in teaching**: Ask questions

For advanced topics, consult the recommended textbooks of Horn & Johnson (2013) and of Gantmacher (1960)

What is a Matrix?

A matrix is a rectangular collection of numbers (J.J. Sylvester, 1848). For example,

A $m \times n$ matrix is often written as:

$$A = \left(a_{ij}\right)_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

What is a Matrix?

So, a $m \times n$ matrix has m row vectors:

$$\left(a_{i1},\ \dots\ ,\ a_{in}\right),\ 1 \le i \le m$$

and n column vectors $(1 \le j \le n)$:

$$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \doteq \mathrm{col}\left(a_{1j},\ \dots\ ,\ a_{mj}\right)$$

$$\doteq \left(a_{1j},\ \dots\ ,\ a_{mj}\right)^T$$

What is a Matrix?

It can be considered as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ (A. Cayley, 1855):

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \mapsto y = Ax \in \mathbb{R}^m$$

with $y_i \doteq \sum_{j=1}^{n} a_{ij} x_j = a_{i1} x_1 + \cdots + a_{in} x_n$

Comment on the computation of *A*

Let $e_i^n = col(0,..,1,0,...,0)$, with 1 as the ith element,

be a vector in the coordinate basis of $\mathbb{R}^n$.

Then, for any linear mapping F (preserving the origin),

$$Fe_i^n = \sum_{j=1}^{m} a_{ji} e_j^m$$

These coefficients a_{ji} form the $m \times n$ matrix

$A = (a_{ji})_{m \times n}$, associated with F.

Comment on the computation of *A*

Proof : Let $x = \sum_{i=1}^{n} x_i e_i^n$. By linearity,

$$y = Fx = \sum_{i=1}^{n} x_i F e_i^n = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ji} e_j^m$$

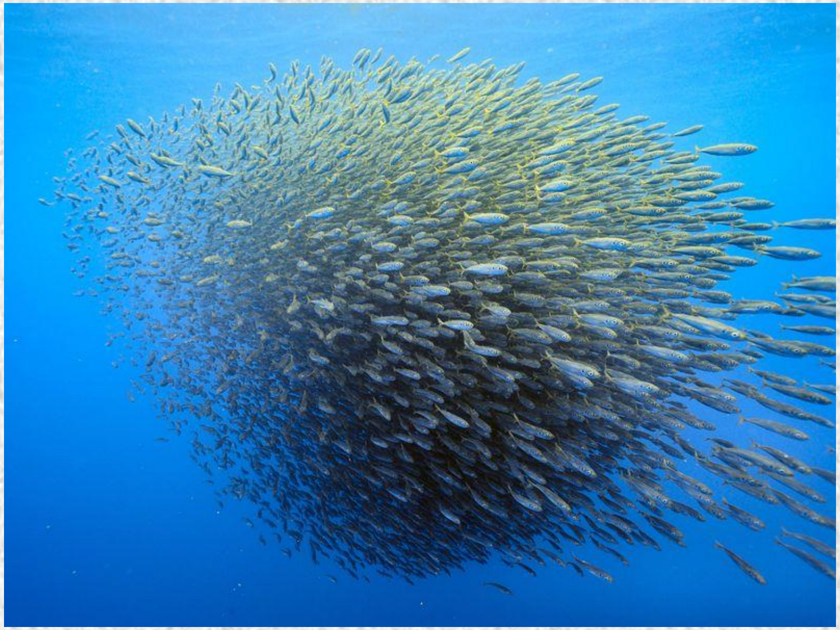
$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i \right) e_j^m \triangleq \sum_{j=1}^{m} y_j e_j^m$$

In other words,

$y = Ax.$ DONE.

A Modern Engineering Example

We have observed many interesting biological group behaviors: Bird flocking, fish schooling, etc



- Why?
- How?

A Modern Engineering Example

Hot research topic:
Consider a group of "connected" autonomous agents (say, birds, robots or humans).

- What leads to the desired group behavior?
- Why do local interactions lead to emerging group behavior?
- How to take advantage of it?

Also, see the computer demonstration of *Reynolds's BOIDS model* of bird flocking at: http://www.red3d.com/cwr/boids/

Motivation (cont'd)

<u>Assume</u> that each agent updates her/his/its heading using the "average" of her/his/its nearby "neighbors".

Nearest neighbor rule:

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in N_i}\theta_j(t)\right)$$

$$\theta := \mathrm{col}\left(\theta_1, \theta_2, \ldots, \theta_n\right)$$

Motivation (cont'd)

Nearest neighbor rule:

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in N_i} \theta_j(t)\right)$$

$$\theta := \mathrm{col}\left(\theta_1, \theta_2, \ldots, \theta_n\right)$$

Equivalently, in compact matrix notation

$$\theta\left(t+1\right) = F_{\sigma(t)}\theta\left(t\right),\ t = 0,1,2,\ldots$$

$$F_{\sigma(t)} = \left(I + D_{\sigma(t)}\right)^{-1}\left(I + A_{\sigma(t)}\right)$$

Motivation (cont'd)

It is shown in (Jadbabaie, Lin and Morse, IEEE Transactions on Automat. Control, 2003); also see (Bertsekas-Tsitsiklis, 1986) that, under mild assumptions on graph connectivity,

$$\theta(t) = \mathrm{col}\left(\theta_1, \theta_2, \ldots, \theta_n\right) \rightarrow \theta_{ss} col\left(1, 1, \ldots., 1\right)$$

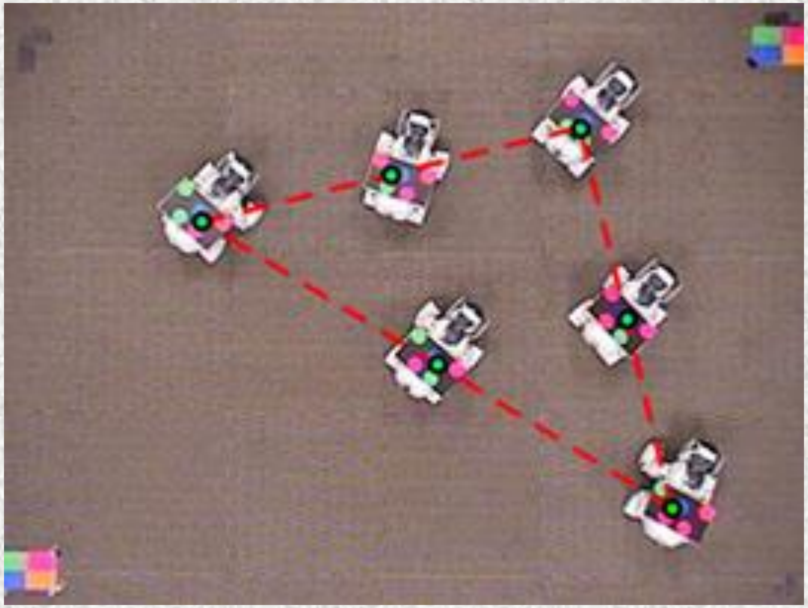
i.e.,

$$\theta_i\left(t\right) \rightarrow \theta_{ss}$$

for all i and all initial conditions $\theta_i\left(0\right)$!

Engineering Applications

Coordinated Control of Groups of Unmanned Vehicles:



Special Types of Matrices

$A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$ is a

- Square matrix, if $n = m$.
- Symmetric matrix, if $n = m$ and $a_{ij} = a_{ji}$, or $A = A^T$.
- Hermitian matrix, if $A = A^* (\doteq \bar{A}^T)$.
- Non-square matrix, if $n \neq m$.
- Any scalar number is a 1×1 matrix.
- A column vector is a $m\times 1$ matrix.
- A row vector is a $1\times n$ matrix.

Hermitian adjoint

Matrix Addition and Multiplication

1) Addition of two matrices with the same dimensions:

$$A+B=\left(a_{ij}\right)_{m\times n}+\left(b_{ij}\right)_{m\times n}=\left(a_{ij}+b_{ij}\right)_{m\times n}.$$

2) Multiplication of two matrices A, B with matched dimensions:

$$AB=\left(a_{ij}\right)_{m\times n}\left(b_{ij}\right)_{n\times p}=\left(c_{ij}\right)_{m\times p}$$

with $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$.

Extension (Semi-tensor product): D. Cheng, 2005

Matrix Addition and Multiplication

3) *Associative* Property:

$$(AB)C = A(BC).$$

4) *Distributive* Property:

$$(A+B)C = AC + BC,$$

$$A(B+C) = AB + AC.$$

Notions about Vectors

- **Linear dependence**

A set of vectors $\{x_1, x_2, \cdots, x_k\}$, of the same size, is said to be linearly dependent, if $\exists$ constants $\{\alpha_j\}_{j=1}^{k}$, not all zero, s.t.

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0.$$

Or, equivalently, $\exists j \in \{1, \cdots, k\}$ such that

$$x_j = \sum_{l \neq j} c_l x_l$$

$$:= c_1 x_1 + \cdots + c_{j-1} x_{j-1} + c_{j+1} x_{j+1} + \cdots + c_k x_k.$$

Notions about Vectors

- **Linear independence**

A set of vectors $\{x_1, x_2, \cdots, x_k\}$, of the same size, is said to be linearly independent, if they are not linearly dependent.

Or, equivalently,

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0, \ \forall \alpha_i$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_k = 0.$$

Remark

1) Ax is a linear combination of the columns of A, with the coordinates of x as the coefficients.

2) $y^T A$ is a linear combination of the rows of A, with the coordinates of y as the coefficients.

Examples

Are the following sets of vectors linearly dependent or independent?

1) $\left\{(1,0,0)^T,\ (1,1,0)^T,\ (1,1,1)^T\right\}$

2) $\{0\}$

3) $\{1\}$

4) $\left\{(1,1,1)^T,\ (1,2,3)^T,\ (2,0,-2)^T\right\}$

Notions about Vectors

- **Basis**

  Consider a subspace V of $\mathbb{R}^n$ or $\mathbb{C}^n$ (being itself a vector space).

  A set of vectors $\{x_1, \cdots, x_k\}$ is said to be a basis for V, *if* :

  1) They are linearly independent;

  2) They span V, *i.e.*

  $$span\{x_1, \cdots, x_k\} = \{a_1 x_1 + \cdots + a_k x_k : \forall a_j\} = V.$$

Standard Basis

The basis $\{e_1, e_2, \cdots, e_n\}$ is called the standard basis of $\mathbb{R}^n$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \cdots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Comments on Basis

1) In this case, the dimension of V is k.

2) $k \leq n$.

3) A subspace has an infinite number of bases. Nonetheless, all bases must be composed of the same number, k, of vectors.

Remark

It is worth noting that *some* vector spaces may be infinite-dimensional, i.e. there does not exist any basis consisting of a finite number of elements. Such an example is

$$V = \{\text{all continuous functions } f : [0,1] \to \mathbb{R}\}.$$

Note: such a space occurs in several optimization problems!

Illustrative Example

Consider the following subspace of $\mathbb{R}^3$:

$$V = \left\{ \left(x_1, x_2, 0 \right)^T : \ \forall x_1, x_2 \in \mathbb{R} \right\}.$$

Examples of a basis for V include:

1) $\left\{ (1,0,0)^T, \ (0,1,0)^T \right\}$;

2) $\left\{ (1,1,0)^T, \ (0,1,0)^T \right\}$;

3) $\left\{ (1,2,0)^T, \ (3,1,0)^T \right\}$.

1^{st} Application: Solving Linear Equations

Solving n equations for n unknowns x_i, $i = 1, 2, \ldots, n$:

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad Ax = b.$$

When does a solution exist? When unique?

Special Case 1: ***n*=1**

In this case, the equation becomes

$$a_{11}x_1 = b_1$$

Clearly,

$$a_{11} \neq 0 \Rightarrow x_1 = \frac{b_1}{a_{11}}, \quad \text{unique}.$$

However, when $a_{11} = b_1 = 0$,

an infinite number of solutions exist!

Special Case 2: ***n*=2**

In this case, the equation becomes

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Solving for x_1 and x_2 gives

$$\left(a_{11}a_{22} - a_{12}a_{21}\right)x_1 = b_1a_{22} - b_2a_{12}$$
$$\left(a_{11}a_{22} - a_{12}a_{21}\right)x_2 = a_{11}b_2 - a_{21}b_1$$

Special Case 2: ***n*=2**

Denote $a_{11}a_{22} - a_{12}a_{21} \doteq \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$

When $\det A \neq 0,$ the equation has the unique solution:

$$x_1 = \det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} / \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$x_2 = \det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix} / \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Cramer's Rule, 1750

Exercise

Solve

$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

Determinant

Given any square matrix $A = \left(a_{ij}\right)_{n\times n}$,

its determinant is defined by

$$\det A \triangleq \sum_{(j_1,\ldots,j_n)} s\left(j_1,\ldots,j_n\right) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where $\left(j_1,\ldots,j_n\right)$ is one of the $n!$ permutations of $1, \ldots, n,$ whose sign is given as

$$s\left(j_1,\ldots,j_n\right) = sign \prod_{1\le p<q\le n} \left(j_q - j_p\right).$$

Facts about Permutations

Fact 1: If two numbers in a permutation are inter-changed, the sign of the permutation is reversed.

For example, $s(3,1,4,2) = -s(4,1,3,2)$.

Examples

$$\mathit{Case}\ 1: n = 2$$

$$s(1,2) = 1, \quad s(2,1) = -1.$$

$$\mathit{Case}\ 2: n = 3$$

$$s(1,2,3) = s(3,1,2) = s(2,3,1) = 1,$$

$$s(1,3,2) = s(2,1,3) = s(3,2,1) = -1.$$

Computing a Determinant

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \sum_{(j)=(j_1\ j_2)} s(j) a_{1j_1} a_{2j_2}$$

$$= s(1\ \ 2) a_{11} a_{22} + s(2\ \ 1) a_{12} a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

Facts about Permutations

Fact 2: Let the permutation $j_1, \ldots, j_n$ be formed from 1, 2, …, n by k successive inter-changes of pairs of numbers.

Then, $s(j_1, \ldots, j_n) = (-1)^k$.

The permutation j is even, if k is even. Otherwise, it is called an odd permutation.

A Puzzle?

Is it possible to rearrange the letters of the alphabet a, b, …, z in reverse order

z, y, …, a by exactly 100 successive interchanges of pairs of letters?

Facts about Determinants

Fact 1: If A is a square matrix, $\det A = \det A^T$.

Proof. It follows directly from the definition of determinant. $\square$

An Application of Fact 1

For any given $m \times n$ matrix A, the "row rank of A" is equal to its "column rank".

Facts about Determinants

Fact 2: If two rows (or columns) of a square matrix A are interchanged, the sign of the determinant is reversed.

Examples

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\det\begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\det\begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$$

Exercise

What is the determinant of

$$\begin{pmatrix} 3 & 2 & 1 \\ 7 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

Proof of Fact 2

Let B be generated by inter-changing the rows r and s of A. That is, $b_{rj} = a_{sj}$, $b_{sj} = a_{rj}$, $b_{ij} = a_{ij}$ if $i \neq r, s$.

By definition,

$$\det B = \sum_{(j)} s(j) b_{1j_1} \dots b_{rj_r} \dots b_{sj_s} \dots b_{nj_n}$$

Let k be the permutation produced from j by interchanging j_r and j_s. Then, $s(k) = -s(j)$.

Thus, $\det B = \sum_{(j)} -s(k) b_{1j_1} \dots b_{sj_s} \dots b_{rj_r} \dots b_{nj_n}$

$$= -\sum_{(k)} s(k) a_{1j_1} \dots a_{rj_s} \dots a_{sj_r} \dots a_{nj_n} = -\det A.$$

An Implication of Fact 2

If a square matrix has two identical rows (or columns), then the determinant must be zero.

Facts about Determinants

Fact 3: If a row (or column) of a square matrix is multiplied by a constant c, the determinant is also multiplied by c.

Fact 4: If a multiple of one row (or column) is subtracted from another row (or column) of a square matrix, the determinant is unchanged.

Exercise

Show that

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0.$$

Proof of Fact 4

Because of Fact 1, we only need to prove the row part.

Let $\tilde{A}$ be the new matrix produced from subtracting row s by λ times row r. So,

$$\det \tilde{A} = \sum_{(j)} s(j) a_{1j_1} \ldots a_{rj_r} \ldots \left(a_{sj_s} - \lambda a_{rj_r} \right) \ldots a_{nj_n}$$

$$= \sum_{(j)} s(j) a_{1j_1} \ldots a_{rj_r} \ldots a_{sj_s} \ldots a_{nj_n}$$

$$-\lambda \sum_{(j)} s(j) a_{1j_1} \ldots a_{rj_r} \ldots a_{rj_r} \ldots a_{nj_n}$$

$$= \det A - 0 := \det A.$$

Facts about Determinants

Fact 5: For any "upper-triangular" square matrix

$$A = \left(a_{ij}\right)_{n\times n}, \text{ i.e., } a_{ij} = 0 \text{ for } i > j,$$

$$\det A = a_{11}a_{22}\cdots a_{nn}.$$

Fact 6: For any "lower-triangular" square matrix

$$A = \left(a_{ij}\right)_{n\times n}, \text{ i.e., } a_{ij} = 0 \text{ for } i < j,$$

$$\det A = a_{11}a_{22}\cdots a_{nn}.$$

Facts about Determinants

Fact 7 (Row expansion): Consider any row i of a matrix $A = \left(a_{ij}\right)_{n\times n}$. Then,

$$\det A = c_{i1}a_{i1} + c_{i2}a_{i2} + \cdots + c_{in}a_{in}$$

where $c_{ik} = \left(-1\right)^{i+k} \det A_{ik}$,

$A_{ik} = (n-1)\times(n-1)$ matrix formed by deleting row i and column k from A.

Remark: Because of Fact 1, the same can be stated for "column expansion".

An Exercise

Give a simple expression for *Vandermonde's Determinant*:

$$V_n\left(x_1,\ldots,x_n\right)=\det\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Solution

$$V_n(x_1,\ldots,x_n)=\prod_{i>j}(x_i-x_j).$$

Idea of Proof :

If $x_i = x_j$ for some $i \neq j$, then the above is obvious.

Otherwise, x_i, $i=1,\ldots,n$, are distinct.

Then, V_n is a polynomial of degree $n-1$ in $x=x_n$,

with distinct roots x_1, x_2, $\ldots$, x_{n-1}. That is,

$$V_n=\alpha(x-x_1)\cdots(x-x_{n-1}),$$

$$\text{with } \alpha \doteq \begin{cases}\text{the coefficient of } x^{n-1}=x_n^{n-1}, \text{ i.e.}\\ V_{n-1}(x_1,\ldots,x_{n-1}).\end{cases}$$

Then, the identity follows by induction.

RREF: Row-reduced Echelon Form

This is a canonical form, useful for solving the system of linear equations $Ax=b$.

Indeed, it suffices to bring the augmented matrix $[A, b]$ down to a RREF.

What is a RREF?

Any RREF must meet the following requirements:

1) Each nonzero row has 1 as its first nonzero entry;

2) All other entries in the column of such a leading 1 are equal to 0;

3) Any rows consisting entirely of zeroes occur at the bottom of the matrix;

4) The leading 1's occur in a "stairstep" pattern, left to right.

An Example of RREF

$$\begin{pmatrix} \color{red}{1} & -1 & 0 & 0 & 3 \\ 0 & 0 & \color{red}{1} & 0 & -1 \\ 0 & 0 & 0 & \color{red}{1} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An Example of RREF

$$\left(\begin{array}{cccc|c} \underbrace{\begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}}_{A} & & & & \begin{matrix} 3 \\ -1 \\ 5 \\ 0 \end{matrix} \\ & & & & b \end{array}\right)$$

So, the solutions are:

$$\begin{cases} x_1 - x_2 = 3, \\ x_3 = -1, \\ x_4 = 5. \end{cases}$$

Conversion to RREF

Any matrix can be converted into a (unique) RREF, via the following elementary (row!) operations:

Type 1) Interchange of two rows;

Type 2) Multiplication of a row by a nonzero scalar;

Type 3) Addition of a scalar multiple of one row to another row.

An Exercise

Give the RREFs of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Circulant matrix

Or, Toeplitz matrix

Homework #1

1. Prove the following identities:

$$(A+B)^T = A^T + B^T,$$

$$(AB)^T = B^T A^T,$$

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T.$$

2. Show that AB is not necessarily symmetric if A and B are symmetric.

Homework #1

3. If $A + jB$ is Hermitian, $A,\ B$ real, then

$$A^T = A\ , \quad B^T = -B.$$

4. For any square matrix $A = \begin{pmatrix} A_1 & * \\ O & A_2 \end{pmatrix}$

with A_1, A_2 two square submatrices,

show that $\det A = \det A_1 \bullet \det A_2$.

Homework #1

5. (*Optional*) Give a simple expression for

$$\det \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$