1. Proof of $(A+B)^T = A^T+B^T$

Let $A=(a_{ij})$, $B=(b_{ij})$ and $C = A+B = (c_{ij})$

then $c_{ij} = a_{ij}+b_{ij}$

we have

$$(A+B)^T = [(a_{ij})+(b_{ij})]^T = (c_{ij})^T = (c_{ji}) = (a_{ji})+(b_{ji}) = (a_{ij})^T+(b_{ij})^T = A^T+B^T$$

Proof of $(AB)^T = B^TA^T$

Let $(A^T)_{ij}$ denote the $(i,j)^{th}$ entry of A^T, and likewise for B and AB

then $[(AB)^T]_{ji} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n}(A^T)_{ki}(B^T)_{jk} = \sum_{k=1}^{n}(B^T)_{jk}(A^T)_{ki}$

The product is the $(j,i)^{th}$ entry of B^TA^T, while $[(AB)^T]_{ji}$ is the $(j,i)^{th}$ entry of $(AB)^T$. Therefore, $(AB)^T = B^TA^T$.

Proof of $(A_1A_2\cdots A_n)^T = A_n^T\cdots A_2^TA_1^T$.

Based on the proof above,

we know that $(AB)^T = B^TA^T$,

Thus $(A_1A_2\cdots A_n)^T = [(A_1)(A_2\cdots A_n)]^T = (A_2\cdots A_n)^TA_1^T$

$= (A_3\cdots A_n)^TA_2^TA_1^T = \cdots = A_n^T\cdots A_2^TA_1^T$

2. Proof:

if A and B are symmetric, then $A=A^T$, $B=B^T$

$(AB)^T = B^TA^T = BA$

AB is symmetric only when $AB = (AB)^T = BA$

Specifically, $A=\begin{pmatrix}1&0\\0&2\end{pmatrix}$, $B=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, then $AB=\begin{pmatrix}0&1\\2&0\end{pmatrix}$

Thus, AB is not necessarily symmetric

3. If $A+jB$ is Hermitian, A, B real

then $A+jB = (A+jB)^* = \overline{(A+jB)}^T = (A-jB)^T = A^T - jB^T$

Thus $A^T = A$, $B^T = -B$

4. If $\det(A_1)=0$, then $\det(A) = 0 = \det A_1 \cdot \det A_2$

Suppose $\det(A_1) \neq 0$, then A is invertible.

$$\begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & A_1^{-1}A_2 \\ 0 & A_2 \end{pmatrix}$$

$$\det A = \det \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} = \det \begin{pmatrix} A_1 & 0 \\ 0 & I_n \end{pmatrix} \det \begin{pmatrix} I_m & A_1^{-1}A_2 \\ 0 & A_2 \end{pmatrix} = \det A_1 \cdot \det A_2$$