

Problem 1. *For the matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identify the space $p_g(\lambda)$ and the principal verctors of grade 2.

Solution. We have a given matrix $A \in \mathbb{R}^{3\times 3}$. So,

$$det(\lambda I - A) = det \begin{bmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = 0$$

which is

$$\lambda_1 = 1 \qquad \lambda_2 = 2$$

We know,

$$(\lambda_i I - A)^g p = 0$$

Now, for $\lambda = 1$,

$$(I - A)^2 = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} p = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So,

$$P_2(\lambda_1 = 1) = \{p^1 | p^1 = col(\alpha, 0, \gamma)\}$$

Now, for λ_2,

$$(2I - A)^2 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So,

$$P_1(\lambda_2 = 2) = \{p^2 | p^2 = col(0, \beta, 0)\}$$

Hence,

$$p^1 = \begin{bmatrix} \alpha \\ 0 \\ \gamma \end{bmatrix} \qquad p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

Problem 2. *Express the following vectors as unique representations of principle vectors found in Problem 1:*

$$x = \begin{bmatrix} \sqrt{2} \\ -9 \\ 84 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 9.3 \\ 0 \end{bmatrix}$$

Solution.
i) Given:

$$x = \begin{bmatrix} \sqrt{2} \\ -9 \\ 84 \end{bmatrix}$$

Express X as unique representations of principle vectors found in problem 1,
from problem 1,

$$p^1 \begin{bmatrix} \alpha \\ 0 \\ \gamma \end{bmatrix}, \quad p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

Then,

$$x = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 9 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 84 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ii) Given:

$$x = \begin{bmatrix} 0 \\ 9.3 \\ 0 \end{bmatrix}$$

Use same method,

$$x = 9.3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where} \quad p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \quad \text{and} \quad \beta = 1$$

Problem 3. *Can you transform the following matrix into a Jordan form:*

$$A = \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{bmatrix}, \quad \lambda \neq 0$$

Solution.

$$A = \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \lambda I - A = \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$null(P_1) = < 1, 0, 0 >$$

$$(\lambda I - A)^2 = \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$null(P_2) =< 1, 0, 0 >; < 0, 1, 0 >$$

$$(\lambda I - A)^3 = \begin{bmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$null(P_2) =< 1, 0, 0 >; < 0, 1, 0 >; < 0, 0, 1 >$$

Hence,

$$v^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v^2 = (A - \lambda I)v^1 = \begin{bmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix}$$

$$v^3 = (A - \lambda I)v^2 = \begin{bmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 \\ 0 \\ 0 \end{bmatrix}$$

So,

$$P = \begin{bmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & -\frac{1}{\lambda^2} & 0 \end{bmatrix}$$

Therefore,

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & -\frac{1}{\lambda^2} & 0 \end{bmatrix} \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$