1.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6(t+1)^{-2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{-t} \\ \sqrt{t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Phi_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \Phi_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Phi_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \Phi_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

① $\dot{x}_1 = x_2$

$\ddot{x}_1 = \dot{x}_2 = 6(t+1)^{-2} x_1$

initial guess to $x_1 = (t+1)^3$ and $(t+1)^{-2}$

$x_1 = c_1(t+1)^3 + c_2(t+1)^{-2}$

$x_1(0) = 1 = c_1 + c_2$

$x_2(0) = \dot{x}_1(0) = 0 = \left[3c_1(t+1)^2 - 2c_2(t+1)^{-3}\right]\Big|_{t=0} = 3c_1 - 2c_2$

$$\left.\begin{matrix} c_1 + c_2 = 1 \\ 3c_1 - 2c_2 = 0 \end{matrix}\right\} \Rightarrow c_1 = \frac{2}{5},\ c_2 = \frac{3}{5} \Rightarrow x_1(t) = \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2}$$

② $x_1 = c_1(t+1)^3 + c_2(t+1)^{-2}$

$x_1(0) = 0 = c_1 + c_2$

$x_2(0) = \dot{x}_1(0) = 1 = \left[3c_1(t+1)^2 - 2c_2(t+1)^{-3}\right]\Big|_{t=0} = 3c_1 - 2c_2$

$$\left.\begin{matrix} c_1 + c_2 = 0 \\ 3c_1 - 2c_2 = 1 \end{matrix}\right\} \Rightarrow c_1 = \frac{1}{5},\ c_2 = -\frac{1}{5} \Rightarrow x_1(t) = \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2}$$

Therefore,

$$\Phi_1(t) = \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^2 - \frac{6}{5}(t+1)^{-3} \end{bmatrix}$$

$$\Phi_2(t) = \begin{bmatrix} \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$

$x_h(t) = \Phi(t)\, x(0)$

$\quad = \begin{bmatrix} \Phi_1(t) & \Phi_2(t) \end{bmatrix} x(0)$

$$= \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} & \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^2 - \frac{6}{5}(t+1)^{-3} & \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X_n(t) = \begin{bmatrix} \frac{4}{5}(t+1)^3 + \frac{1}{5}(t+1)^{-2} \\ \frac{12}{5}(t+1)^2 - \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$

$$X(t) = \mathbb{X}(t)X(0) + \mathbb{X}(t)\int_0^t \mathbb{X}^{-1}(s) f(s)\,ds$$

$$= X_n(t) + \begin{bmatrix} \frac{2}{5}(t+1)^3 + \frac{3}{5}(t+1)^{-2} & \frac{1}{5}(t+1)^3 - \frac{1}{5}(t+1)^{-2} \\ \frac{6}{5}(t+1)^3 - \frac{6}{5}(t+1)^{-3} & \frac{3}{5}(t+1)^2 + \frac{2}{5}(t+1)^{-3} \end{bmatrix}$$

$$\int_0^t \begin{bmatrix} \frac{3}{5}(s+1)^2 + \frac{2}{5}(s+1)^{-3} & -\frac{1}{5}(s+1)^3 + \frac{1}{5}(s+1)^{-2} \\ -\frac{6}{5}(s+1)^2 + \frac{6}{5}(s+1)^{-3} & \frac{2}{5}(s+1)^3 + \frac{3}{5}(s+1)^{-2} \end{bmatrix} \begin{bmatrix} e^{-s} \\ \sqrt{s} \end{bmatrix} ds$$

$$= X_n(t) + \mathbb{X}(t)\int_0^t \begin{bmatrix} e^{-s}\left(\frac{3}{5}(s+1)^2 + \frac{2}{5}(s+1)^{-3}\right) + \sqrt{s}\left(-\frac{1}{5}(s+1)^3 + \frac{1}{5}(s+1)^{-2}\right) \\ e^{-s}\left(-\frac{6}{5}(s+1)^2 + \frac{6}{5}(s+1)^{-3}\right) + \sqrt{s}\left(\frac{2}{5}(s+1)^3 + \frac{3}{5}(s+1)^{-2}\right) \end{bmatrix} ds$$

2.

$$\frac{dy(t)}{dt} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} y(t), \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

let $y(t) = [y_1(t) \ \ y_2(t) \ \ y_3(t) \ \ y_4(t) \ \ y_5(t)]^T$

Then,
$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \\ \dot{y}_5(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix}$$

$$\begin{cases} \dot{y}_1(t) = 2y_1(t) \\ y_2(t) = y_1(t) + 2y_2(t) \end{cases} \quad ①$$

$$\begin{cases} \dot{y}_3(t) = 3y_3(t) \\ \dot{y}_4(t) = y_3(t) + 3y_4(t) \\ y_5(t) = y_4(t) + 3y_5(t) \end{cases} \quad ②$$

$y(0) = [y_1(0) \ \ y_2(0) \ \ y_3(0) \ \ y_4(0) \ \ y_5(0)]^T = [1 \ \ 2 \ \ 3 \ \ 4 \ \ 5]^T$

From ①, $\begin{cases} y_1(t) = e^{2t} \\ y_2(t) = (2+t)e^{-t} \end{cases}$

From ②. $\begin{cases} y_3(t) = 3e^{3t} \\ y_4(t) = (4+3t)e^{3t} \\ y_5(t) = (5+4t+\frac{3}{2}t^2)e^{3t} \end{cases}$

$$\Rightarrow \begin{cases} y_1(t) = e^{2t} \\ y_2(t) = (2+t)e^{-t} \\ y_3(t) = 3e^{3t} \\ y_4(t) = (4+3t)e^{3t} \\ y_5(t) = (5+4t+\frac{3}{2}t^2)e^{3t} \end{cases}$$