Applied Matrix Theory

ECE-GY 5253 Midterm

Fall 2023

Due: Friday, November 4, 11 am (US Eastern Time)

Problem 1 (30 pts)

Transform the following matrix into one of the canonical forms.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution
The characteristic polynomial is given as:

$$p(\lambda) = (3-\lambda)(\lambda^2 - 6\lambda + 9) = 0 \tag{1}$$

Thus, the eigenvalues are $\lambda = 3$ with multiplicity 3.

Observe that $(A - \lambda I)x = 0$ has a null space of dimension two, which means $\exists$ two linearly independent eigenvectors to $(A - \lambda I)x = 0$. These are obtained as:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{2}$$

Hence, there is only one $(n-2)$ linearly independent principal vector, which can be obtained as:

$$(A - \lambda I)^2 x_3 = 0 \implies x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

Thus,

$$P = [x_1 \ x_2 \ x_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

Then $A = PJP^{-1}$, where

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}. \tag{5}$$

Problem 2 (30 pts)

Are the following statements true or false ? If true, provide a proof; if false, give a counter-example.

(a) Let $M \in \mathbb{R}^{n \times m}$ and $r = rank(M)$. There exist $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times m}$ such that $M = AB$.

(b) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. A is orthogonal if and only if its eigenvalues all have the absolute value of one.

(c) If $A \in \mathbb{R}^{n\times n}$, then the sum of the n eigenvalues of A is the trace of A and the product of the n eigenvalues is the determinant of A.

Solution
(a) True.
By the definition of rank, we have

$$\dim(\mathrm{Im}(M)) = \mathrm{rank}(M) = r. \tag{6}$$

Let $a_1, \cdots, a_r$ be the basis of the $\mathrm{Im}(M)$, and $c_1, \cdots, c_m$ be the columns of M. For any $i = 1, \cdots, m$, $c_i \in \mathrm{Im}(M)$. Therefore, there exist scalars $b_{i,1}, \cdots, b_{i,r}$, such that

$$c_i = b_{i,1}a_1 + \cdots + b_{i,r}a_r = \begin{bmatrix} a_1 & \cdots & a_r \end{bmatrix} \begin{bmatrix} b_{i,1} \\ \vdots \\ b_{i,r} \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_r \end{bmatrix} b_i. \tag{7}$$

Consequently, we have

$$M = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_r \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} = AB. \tag{8}$$

(b) True.
$(\Rightarrow)$ Since A is symmetric and orthogonal, we have $A = A^T = A^{-1}$. Let $u \in \mathbb{R}^n$ be the nonzero eigenvalue of A associated with the eigenvalue $\lambda \in \mathbb{R}$, (the eigenvalues of symmetric matrices are all real), that is

$$Au = \lambda u. \tag{9}$$

Since $A = A^T = A^{-1}$, it follows that

$$\lambda Au = \lambda A^T u = u. \tag{10}$$

Combining (9) and (10), we can obtain

$$\lambda^2 u = u. \tag{11}$$

Because $u \neq 0$, $\lambda^2 = 1$, and $|\lambda| = 1$.
$(\Leftarrow)$ Since A is symmetric, there exist an invertible matrix $O \in \mathbb{R}^n$ and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, such that

$$A = O^{-1}\Lambda O, \quad A^T = A = O^{-1}\Lambda O. \tag{12}$$

Since $\Lambda^2 = I_n$, we have

$$A^T A = O^{-1}\Lambda O O^{-1}\Lambda O = O^{-1}\Lambda^2 O = O^{-1}O = I_n. \tag{13}$$

Similarly, $AA^T = I_n$, and therefore, A is orthogonal.
(c) True.
Let $A \in \mathbb{R}^{n\times n}$ whose different eigenvalues are $\lambda_1, \cdots, \lambda_s$ with multiplicity $m_1, \cdots, m_s$, i.e.

$$\det(\lambda I_n - A) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{m_i}. \tag{14}$$

For any matrix $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, such that

$$A = P^{-1}JP, \quad \Lambda_i = \begin{bmatrix} \lambda_i & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \lambda_i & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_i & 0 \\ 0 & 0 & \cdots & \cdots & 1 & \lambda_i \end{bmatrix}, \quad J = \mathrm{blockdiag}(\Lambda_i). \tag{15}$$

Then, $\det(A) = \det(P^{-1})\det(J)\det(P) = \det(J) = \prod_{i=1}^{s}(\lambda_i)^{m_i}$; $\mathrm{trace}(A) = \mathrm{trace}(P^{-1}JP) = \mathrm{trace}(JPP^{-1}) = \mathrm{trace}(J) = \sum_{i=1}^{s} m_i\lambda_i$.

Problem 3 (40 pts)

Let $A \in \mathbb{R}^{n\times n}$. Suppose that $u \in \mathbb{R}^n$ ($v \in \mathbb{R}^n$) is nonzero right (left) eigenvector corresponding to the eigenvalue $\alpha \in \mathbb{C}$ ($\beta \in \mathbb{C}$), i.e.

$$Au = \alpha u, \quad v^T A = \beta v^T.$$

Prove that

1. If $\alpha \neq \beta$, then $v^T u = 0$.
2. If $\alpha = \beta$ and $v^T u \neq 0$, then there exists an invertible matrix $T = [u, T_1]$ with $(T^{-1})^T = [v/(u^T v), M_1]$, where $T_1, M_1 \in \mathbb{R}^{n\times(n-1)}$, such that

$$A = T \begin{bmatrix} \alpha & 0 \\ 0 & N \end{bmatrix} T^{-1}, \quad N \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Solution
1. Since u and v are right and left eigenvectors of A, respectively, we have

$$\begin{aligned} v^T Au &= v^T(\alpha u) = \alpha v^T u \\ &= (\beta v^T)u. \end{aligned} \tag{16}$$

Since $\alpha \neq \beta$, $\alpha v^T u = \beta v^T u$ if and only if $v^T u = 0$.

2. Without losing generality, assume that $v^T u = 1$ (we can replace v by $v/(v^T u)$). Let the columns of T_1 be any basis for the orthogonal complement of v (so $v^T T_1 = 0$) and consider $T = [u, T_1]$. Let $w = [w_1, \zeta^T]^T$ with $\zeta \in \mathbb{R}^{n-1}$ and suppose that $Tw = 0$. Then

$$0 = v^T Tw = v^T(w_1 u + T_1\zeta) = w_1 v^T u + v^T T_1 \zeta = w_1. \tag{17}$$

Therefore, $w_1 = 0$ and $0 = Tw = T_1\zeta$, which implies that $\zeta = 0$, since T_1 is full column rank. We conclude that T is nonsingular.

Partition $(T^{-1})^T = [\eta, M_1]$ with $\eta \in \mathbb{R}^n$ and compute

$$I_n = T^{-1}T = \begin{bmatrix} \eta^T \\ M_1^T \end{bmatrix} [u, T_1] = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1}. \end{bmatrix} \tag{18}$$

The identity $\eta^T T_1 = 0$ implies that η is orthogonal to the orthogonal complement of v, so $\eta = av$ for some $a \in \mathbb{R}$. The identity $\eta^T u = 1$ tells us that

$$\eta^T u = a v^T u = 1, \tag{19}$$

so $\eta = v$. Using the identities $\eta^T T_1 = v^T T_1 = 0$ and $M_1^T u = 0$ as well as the eigenvector properties of u and v, compute the similarity

$$\begin{aligned} T^{-1}AT &= \begin{bmatrix} v^T \\ M_1^T \end{bmatrix} A \begin{bmatrix} u & T_1 \end{bmatrix} = \begin{bmatrix} v^T Au & v^T AT_1 \\ M_1^T Au & M_1^T AT_1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha v^T u & \alpha v^T T_1 \\ \alpha M_1^T u & M_1^T AT_1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & M_1^T AT_1 \end{bmatrix}. \end{aligned} \tag{20}$$