1.

$$H-\lambda I = \begin{pmatrix} -\lambda & 2 & -1 \\ 2 & 5-\lambda & -6 \\ -1 & -6 & 8-\lambda \end{pmatrix}$$

$$\begin{aligned}
\det(H-\lambda I) &= -\lambda(5-\lambda)(8-\lambda) - 2\times[2\times(8-\lambda)-6] - [2\times(-6)+(5-\lambda)] + 36\lambda \\
&= -\lambda(5-\lambda)(8-\lambda) + 41\lambda - 13 \\
&= -\lambda^3 + 13\lambda^2 + \lambda - 13 \\
&= -(\lambda-13)(\lambda-1)(\lambda+1) = 0
\end{aligned}$$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 13$

① $\lambda_1 = 1$, $\begin{pmatrix} -1 & 2 & -1 \\ 2 & 4 & -6 \\ -1 & -6 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

② $\lambda_1 = -1$, $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & -6 \\ -1 & -6 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ eigenvector $V_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

③ $\lambda_3 = 13$, $\begin{pmatrix} -13 & 2 & -1 \\ 2 & -8 & 6 \\ -1 & -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ eigenvector $V_3 = \begin{bmatrix} -\frac{1}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix}$

Thus, $Q = \begin{bmatrix} 1 & -3 & -\frac{1}{5} \\ 1 & 2 & -\frac{4}{5} \\ 1 & 1 & 1 \end{bmatrix}$ and $D = Q^{-1}HQ = \mathrm{diag}(1, -1, 13)$

D is a diagonal form for H.

2. We have Hermitian matrix H positive definite,

so there are orthogonal matrix O and diagonal matrix D

that satisfying $H = O^{-1} D O^{-1}$

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Since H is positive definite.

engenvalues $\lambda_1, \lambda_2, \dots, \lambda_n > 0$.

Let $D_1 = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}$ then $D = D_1 \cdot D_1$

Let $P = O^{-1} D_1 O$ and P is positive definite

$$P^2 = O^{-1} D_1 O O^{-1} D_1 O = O^{-1} D_1^2 O = O^{-1} D O = H$$

Therefore, $H = P^2$.