1.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 0 & 1 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix} = 0 \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 2$$

$$(\lambda_i I - A)^g p = 0$$

for $\lambda = 1$, $(I - A)^2 = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^1 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^1 = 0 \Rightarrow p^1 = \begin{bmatrix} \alpha \\ 0 \\ \gamma \end{bmatrix}$$

for $\lambda = 2$, $(2I - A)^2 = 0$

$$\Leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} p^2 = 0 \Rightarrow p^2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

2.

i) $$x = \begin{bmatrix} \sqrt{2} \\ -9 \\ 84 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 84\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ii) $$x = \begin{bmatrix} 0 \\ 9.3 \\ 0 \end{bmatrix} = 9.3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.

$$\lambda I - A = \begin{bmatrix} 0 & -\lambda & -\alpha \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{null}(P_1) = (1, 0, 0)^T$$

$$(\lambda I - A)^2 = \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{null}(P_2) = (1,0,0)^T,\ (0,1,0)^T$$

$$(\lambda I - A)^3 = \begin{bmatrix} 0 & 0 & \lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\lambda & -\lambda \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{null}(P_3) = (1,0,0)^T,\ (0,1,0)^T,\ (0,0,1)^T$$

$$v^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v^2 = (A - \lambda I)v^1 = \begin{bmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix}$$

$$v^3 = (A - \lambda I)v^2 = \begin{bmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$P = \begin{bmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & -\frac{1}{\lambda^2} & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{\lambda} & 0 \\ \frac{1}{\lambda^2} & -\frac{1}{\lambda^2} & 0 \end{bmatrix} \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & \lambda & \lambda^2 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$