ECE 5253 - Applied Matrix Theory
 Problem Set 6
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Problem 1. *Transform the following Hermitian matrix*

$$H = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 5 & -6 \\ -1 & -6 & 8 \end{bmatrix}$$

into a diagonal form.

Solution. First we have to find the eigenvalues and eigenvectors.
So,

$$det(\lambda I - H) = 0, \qquad \text{where} \quad \begin{bmatrix} 0 & 2 & -1 \\ 2 & 5 & -6 \\ -1 & -6 & 8 \end{bmatrix}$$

Then, we can find,

$$\lambda = \pm 1, 13$$

Hence, each eigenvector of eigenvalue is

$$\lambda = 1, \quad p_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\lambda = -1, \quad p_2 = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix}$$

$$\lambda = 13, \quad p_3 = \begin{bmatrix} \frac{-1}{\sqrt{42}} \\ \frac{-4}{\sqrt{42}} \\ \frac{5}{\sqrt{42}} \end{bmatrix}$$

Therefore, the complete normalized eigenvector is

$$O = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{5}{\sqrt{42}} \end{bmatrix}$$

Then,

$$O^{-1} = O^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} \\ \frac{-1}{\sqrt{42}} & \frac{-4}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix}$$

Therefore,

$$O^T H O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

Problem 2. *If a (real) Hermitian matrix H is positive definite, prove that $H = P^2$, for a positive definite matrix P.*

Solution. Since H is Hermitian and positive definite, there exists a unitary matrix such that,

$$U^*HU = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_3 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_4 \end{bmatrix}, \qquad \text{where} \quad U^*U = I$$

where $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigenvalues of H and $\lambda_1, \lambda_2 \dots \lambda_n$ are all positive at,

$$\Lambda = \Lambda_1^2, \qquad \text{where} \quad \Lambda_1 = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_3} & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{\lambda_4} \end{bmatrix}$$

So,

$$\begin{aligned} U^*HU &= \Lambda_1^2 = \Lambda_1\Lambda_1^2 = \Lambda_1 I \Lambda_1 = \Lambda_1 UU^*\Lambda_1 \\ H &= U\Lambda_1 U^* U \Lambda_1 U^* = (U\Lambda_1 U^*)^2 \\ &\text{where} \quad P = U\Lambda_1 U^* = U_1^*\Lambda_1 U_1, \quad U_1^*U_1 = UU^* = I \end{aligned}$$

U_1 is a unitary matrix, and the diagonal element of Λ_1 are all greater than zero (Λ_1 is a diagonal matrix, therefore, P is a positive definite matrix).
Hence,

$$H = P^2$$