1.
$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ax = 0 \Rightarrow \begin{matrix} x_1 + 4x_2 + 7x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -4x_2 - 7x_3 = x_3 \\ x_2 = -2x_3 \end{matrix} \text{ with } x_3 \text{ free}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \mathrm{Nul}\, A = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\dim(\mathrm{Nul}(A)) = 1 \qquad \mathrm{rank}(A) = n - r = 2$$

2.
$$\det(AB) = \sum_i (-1)^{i+1} \Big(\sum_j A_1^j B_j^i \Big) \det(AB)_j^i$$
$$= \sum_j \sum_i (-1)^{i+1} A_1^j B_j^i \det(AB)_1^i$$
$$= \sum_j \sum_i (-1)^{i+1} A_1^j B_j^i \det(A_1^j B_j^i)$$
$$= \sum_j (-1)^{j+1} A_1^j \det A_1^j \sum_i (-1)^{i+j} B_j^i \det B_j^i$$
$$= \det A \det B$$

Similarly $\det(BA) = \det B \det A = \det A \det B \Rightarrow \det(AB) = \det(BA) = \det A \det B$

3.
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Therefore, $AB \neq BA$

4. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & \lambda_1 & \lambda_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & \frac{\lambda_1}{2} & \frac{\lambda_2}{2} \end{bmatrix}$$

$$\begin{aligned} & x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ & x_1 + 2x_2 + \frac{\lambda_1}{2}x_3 + \frac{\lambda_2}{2}x_4 = 0 \end{aligned} \Rightarrow 3x_3 + 4x_4 = \frac{\lambda_1}{2}x_3 + \frac{\lambda_2}{2}x_4$$

$$\Rightarrow \lambda_1 = 6 \quad \lambda_2 = 8$$