1. $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda = 0$ or $\lambda = 2$

when $\lambda = 0$

$A v_{\lambda=0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_{\lambda=0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

when $\lambda = 2$

$A v_{\lambda=2} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_{\lambda=2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the singular matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ have two independent eigenvectors

2. Since $\det(A) = \det(A^T)$

$(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$\det(A-\lambda I) = \det\left((A-\lambda I)^T\right) = \det(A^T - \lambda I)$

Thus, A and A^T have the same eigenvalues.

3. $\det(AB-\lambda I) = \det(ABAA^{-1} - \lambda AA^{-1})$

$= \det(A(BA-\lambda I)A^{-1})$

$= \det A \cdot \det(BA-\lambda I) \cdot \det(A^{-1})$

$= \det(BA-\lambda I)$ since $AA^{-1} = I$ and

$\det A \cdot \det(A^{-1}) = 1$

$$\lambda^2 - 3\lambda + 1 \qquad \frac{3 \pm \sqrt{9-4}}{2}$$

4. $A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(A_1 - \lambda I) = (2-\lambda)(1-\lambda) - 1 = 0 \Rightarrow \lambda_1 = \frac{3+\sqrt{5}}{2}$ $\lambda_2 = \frac{3-\sqrt{5}}{2}$

$\det(A_2 - \lambda I) = (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 1$ $\lambda_2 = 2$

Since both A_1 and A_2 have 2 distinct eigenvalues, it is similar to a diagonal matrix which is $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$