1.

H is Hermitian. $\min_{\|u\|=1} \langle Hu, u\rangle = \lambda_n$

where λ_n is the least eigenvalue of H.

H is Hermitian, the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ of H are real and we can find the corresponding orthonormal eigenvalues $u^1, u^2, \cdots u^n$

$\|u^i\| = \sqrt{\langle u^i, u^i\rangle} = 1$

$\langle u^i, u^j\rangle = 0$ for $i \neq j$

$\langle Hu^i, u^i\rangle = \lambda_i$, $\langle Hu^i, u^j\rangle = 0$ for $i \neq j$

For every unit vector u can be written as the linear combination of $u^1, u^2, \cdots u^n$.

$u = c_1 u^1 + c_2 u^2 + \cdots c_n u^n$ where $|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1$

$\langle Hu, u\rangle = \langle H(c_1 u^1 + c_2 u^2 + \cdots + c_n u^n), u\rangle = \sum_i |c_i|^2 \lambda_i \|u^i\|^2 \geq \sum_i |c_i|^2 a_n = \lambda_n$

(a_n is the least eigenvalues)

Thus, $\min_{\|u\|=1} \langle Hu, u\rangle = \lambda_n$

Therefore, $\min_{\|u\|=1} \langle Hu, u\rangle = a_n$

$$\min_{x \neq 0} \left\langle H \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle = \lambda_n$$

$$\Rightarrow \min_{x \neq 0} \frac{\langle Hx, x\rangle}{\langle x, x\rangle} = \lambda_n$$

2.

$H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$, H is a Hermitian matrix

If H is positive-definite

$\Rightarrow \begin{cases} \det\left(\begin{bmatrix} 1 & 2 \\ 2 & \mu \end{bmatrix}\right) > 0 & ① \\ \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}\right) > 0 & ② \end{cases}$

From ①, $\mu - 4 > 0 \Rightarrow \mu > 4$

From ②, $5\mu + 24 + 24 - 9\mu - 16 - 20 > 0$

$\Rightarrow 4\mu < 12, \quad \mu < 3$

thus $\mu > 4$ and $\mu < 3$ contradiction

there is no solution.

3. $|x|_\infty = \max_k |x_k|$, $|x|_1 = \sum_k |x_k|$

Check three properties.

① $x = 0$ $\quad |x|_\infty = \max_k |x_k| = 0$

$x \neq 0$ $\quad |x|_\infty = \max_k |x_k| > 0$

② $\alpha \neq 0$, $|\alpha x|_\infty = \max_k |\alpha x_k| = |\alpha| \cdot \max_k |x_k| = |\alpha| \cdot |x|_\infty$

③ $|x+y|_\infty = \max_k |x_k + y_k| \leq \max_k |x_k| + |y_k| = |x|_\infty + |y|_\infty$

Hence, $|x|_\infty$ is a norm

Check three properties

① $x \neq 0, \quad |x|_1 = \sum_k |x_k| > 0$

$x = 0, \quad |x_1| = \sum_k |x_k| = 0$

② $\alpha \neq 0, \quad |\alpha x|_1 = \sum_k |\alpha x_k| = |\alpha| \sum_k |x_k| = |\alpha| \cdot |x|_1$

③ $|x+y|_1 = \sum_k |x_k + y_k| \leq \sum_k |x_k| + |y_k| = |x|_1 + |y|_1$

Hence, $|x|_1$ is a norm

$|x|_1 = \sum_k |x_k|$

$\|A\|_1 = \max_{x \neq 0} \frac{|Ax|_1}{|x|_1}$, where $A = \begin{bmatrix} a_{11}, a_{12} \cdots a_{1n} \\ a_{21}, a_{22} \cdots a_{2n} \\ \vdots \\ a_{n1}, a_{n2} \cdots a_{nn} \end{bmatrix}$

$$|Ax|_1 = \sum_k |a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n|$$

$$\leq \sum_k (|a_{k1}x_1| + |a_{k2}x_2| + \cdots + |a_{kn}x_n|)$$

$$= \sum_k (|a_{k1}||x_1| + |a_{k2}||x_2| + \cdots + |a_{kn}||x_n|)$$

$$= s_1|x_1| + s_2|x_2| + \cdots + s_n|x_n|$$

where $s_j = \sum_{i=1}^{n} |a_{ij}|$

$$\frac{|Ax|_1}{|x|_1} = \frac{s_1|x_1| + s_2|x_2| + \cdots + s_n|x_n|}{|x_1| + |x_2| + \cdots + |x_n|} \leq \frac{s_m(|x_1| + |x_2| + \cdots + |x_n|)}{|x_1| + |x_2| + \cdots + |x_n|}$$

$s_m = \max\{s_1, \cdots s_n\}$

$$\|A\|_1 = \max_{x \neq 0} \frac{|Ax|_1}{|x|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|$$

$$|x|_\infty = \max_k |x_k|$$

$$\|A\|_\infty = \max_{x \neq 0} \frac{|Ax|_\infty}{|x|_\infty} \quad , \text{where} \quad A = \begin{bmatrix} a_{11}, & a_{12}, & \cdots & a_{1n} \\ a_{21}, & a_{22}, & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1}, & a_{n2}, & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} |Ax|_\infty &= \max_k | a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n | \\ &\le \max_k (|a_{k1}| |x_1| + |a_{k2}| |x_2| + \cdots + |a_{kn}| |x_n|) \\ &\le \max_k (|a_{k1}| + |a_{k2}| + \cdots + |a_{kn}|) \cdot S_m \end{aligned}$$

where $S_m = \max_k |x_k| = |x|_\infty$

$$\frac{|Ax|_\infty}{|x|_\infty} \le \max_k (|a_{k1}| + |a_{k2}| + \cdots + |a_{kn}|) = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$