Lecture X
Matrix Analysis of Differential and Difference Equations

Key points:

- **Linear Differential Equations:**
  1) Homogeneous and inhomogeneous cases;
  2) Solutions based on matrix exponential.

- **Linear Difference Equations**

- **Extensions: Higher-Order Equations**

Homogeneous Time-Varying Equations

- A linear homogeneous eq. is usually described by:

$$\frac{dx(t)}{dt} = A(t)x(t), \quad x(t) \in \mathbb{R}^n,\ t \geq 0.$$

- Fundamental solution:

It is composed of the solutions $x^i(t)$ with initial value $x^i(0) = e^i$, the i-th column of $I \in \mathbb{R}^{n\times n}$.

In compact notation, $X(t) \triangleq \left[x^1(t), \cdots, x^n(t) \right]$ satisfies

$$\frac{d}{dt} X(t) = A(t)X(t), \quad X(0) \doteq \left[x^1(0), \cdots, x^n(0) \right] = I.$$

Why $X(t)$ Useful?

- Any solution $x(t)$ with initial value $x(0) = c$ can be written as: $x(t) = X(t)c$, or, $X(t)^{-1} x\left(t\right) = c$.

- Derive solutions to inhomogeneous equations. (see next slide)

Fact: $X(t)$ is invertible for all t

First, note that, if $X(t)c = 0$ for some $t = t^* > 0$ *and* $c \neq 0$,

then $X(t)c = 0$ for all t

because of the uniqueness of solutions.

Fact: $X(t)$ is invertible for all t

Second, assume that $X(t^*)$ is singular for some $t^* > 0$. Then, the columns of $X(t)$ are linearly dependent. So, there is some $a \neq 0$ satisfying $X(t^*)a = 0$. Thus, $\tilde{x}(t) = X(t)a$ is an identically zero solution, leading to a contradiction with the fact that $\tilde{x}(0) = X(0)a = a \neq 0$.

Extension to inhomogeneous equations

Consider the inhomogeneous eq.:

$$\frac{dy}{dt} = A(t)y + f(t).$$

Let us consider the new variable $z(t)$:

$$z(t) = X^{-1}(t)y(t), \text{ or, } y(t) = X(t)z(t)$$

Differentiating with respect to time yields:

$$\frac{dy}{dt} = \frac{dX}{dt}z + X\frac{dz}{dt}, \text{ or again,}$$

$$A(t)y + f(t) = A(t)Xz + X\frac{dz}{dt}$$

Reduction to homogeneous equations

$$X\frac{dz}{dt} = f(t) \;\Rightarrow\; \frac{dz}{dt} = X^{-1}(t)f(t)$$

$$\Rightarrow z(t) = z(0) + \int_0^t X^{-1}(s)f(s)ds$$

Finally, we arrive at the

"*Variation of Parameters Formula*":

$$y(t) = X(t)y(0) + X(t)\int_0^t X^{-1}(s)f(s)ds.$$

Comment

If the eq. is time-invariant, *i.e.*, $A(t) = A$, then,

$$y(t) = X(t)y(0) + \int_0^t X(t-s) f(s) ds.$$

Indeed, in this case, by time-invariance and uniqueness of solutions,

$$X(t) X^{-1}(s) = X(t-s).$$

$$X(t)^{-1} x\big(t\big) = x(0).$$

Example

In order to solve

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y + \begin{bmatrix} 0 \\ e^{-t^2} \end{bmatrix},$$

let's first compute the fundamental solution:

$$X(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\Rightarrow X^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Example (con'd)

The, it follows that

$$y(t) = X(t)y(0) + \int_0^t \begin{bmatrix} \sin(t-s) \\ \cos(t-s) \end{bmatrix} e^{-s^2} ds.$$

Question

How to compute the fundamental matrix for a linear, homogeneous, time-invariant equation:

$$\dot{x}(t) = Ax(t), \quad \text{with } A \in \mathbb{R}^{n\times n}$$

Solutions of Homogeneous Eqs.

The essential question is to find the fundamental solution $X(t)$, i.e. solving

$$\frac{dX(t)}{dt} = AX(t), \quad X(0) = I.$$

First, define $e^{At} \doteq \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k t^k.$

If this infinite series of matrices converges, then $X(t) = e^{At}$, by direct term-by-term differentiation.

A Basic Result

For all t over any bounded interval, the infinite series of matrices $\sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$ converges uniformly and absolutely.

Proof:

Using the matrix-norm derived from the Euclidean vector-norm, it holds: $\left| A^k \right| \leq \left| A \right|^k, \ \forall k \in \mathbb{N}.$

Therefore, $\left| \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} |A|^k t^k$

$$\leq e^{|A|t} < \infty, \text{ as wished.}$$

Properties of the Matrix Exponential

The matrix exponential $e^{A} \triangleq \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$

has the following properties:

1) $e^{0} = I$.

2) For any matrix $A \in \mathbb{R}^{n \times n}$, $\left(e^{A}\right)^{T} = e^{A^{T}}$.

3) For all $A \in \mathbb{R}^{n \times n}$, and for all $t, \tau \in \mathbb{R}$,

$$e^{(t+\tau)A} = e^{tA} e^{\tau A} = e^{\tau A} e^{tA}.$$

Properties of the Matrix Exponential

4) For all $A, B \in \mathbb{R}^{n\times n}$, and for all $t \in \mathbb{R}$,

$$e^{t(A+B)} = e^{tA}e^{tB} = e^{tB}e^{tA}$$

if and only if $AB = BA$.

5) For all $A \in \mathbb{R}^{n\times n}$, and for all $t \in \mathbb{R}$,

$$\left(e^{tA}\right)^{-1} = e^{-tA}.$$

Properties of the Matrix Exponential

6) For all $A \in \mathbb{R}^{n\times n}$, and for all $t \in \mathbb{R}$,

$$L\left\{e^{tA}\right\} = \left(sI - A\right)^{-1}$$

$$\Rightarrow e^{tA} = L^{-1}\left\{\left(sI - A\right)^{-1}\right\}.$$

Proof

$$L\left\{e^{tA}\right\} = \int_0^\infty e^{-st} e^{tA} dt$$

$$= \int_0^\infty e^{t(-sI)} e^{tA} dt = \int_0^\infty e^{t(A-sI)} dt$$

$= \left(sI - A\right)^{-1}$, using Jordan canonical form of A,

left as an exercise.

Note: $s \in ROC$ of $L\left\{e^{tA}\right\}$

More Remarks on Analytical Close-Form Solutions

Question :

Apart from Laplace transformation methods, what are other methods to find the close-form expression for e^{At}, for a given matrix $A \in \mathbb{R}^{n \times n}$?

Case 1: *A* has *n* linearly independent eigenvectors

Let $\left\{v^i\right\}_{i=1}^{n}$ be n linearly independent eigenvectors associated with eigenvalues $\left\{\lambda^i\right\}_{i=1}^{n}$.

Then, $P = \left(v^1 \ \cdots \ v^n\right)$ transforms A into a canonical diagonal form:

$$P^{-1}AP = diag\left(\lambda_i\right) \doteq \Lambda.$$

Clearly, $e^{At} = Pe^{\Lambda t}P^{-1}$.

Case 1: using canonical diagonal form

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\text{diag}\left(e^{\lambda_i t}\right)P^{-1}.$$

$Therefore,$ noting $P = \left(v^1 \ \cdots \ v^n\right),$

the solutions $x(t)$ of $\dot{x} = Ax$ are in the form:

$$x(t) = e^{At}x(0) = P\text{diag}\left(e^{\lambda_i t}\right)P^{-1}x(0)$$

$$= \alpha_1 e^{\lambda_1 t}v^1 + \cdots + \alpha_n e^{\lambda_n t}v^n$$

where α_{i} 's are scalars, determined by $x(0)$.

Comment

$$x(t) = e^{At}x(0) = \alpha_1 e^{\lambda_1 t} v^1 + \cdots + \alpha_n e^{\lambda_n t} v^n.$$

$*$ The solutions of linear equations take an exponential form.

• When the eigenvalues are in the open left-half plane,
all solutions $x(t)$ go to 0, as $t \to \infty$.

(See "Application to stability of linear systems" for more details.)

Example

Solve the initial-value problem:

$$\dot{x} = \begin{pmatrix} -7 & 4 \\ -8 & 1 \end{pmatrix} x, \quad x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

First, compute the eigenvalues

$$\lambda_1 = -3 + 4i, \;\; \lambda_2 = -3 - 4i$$

with two associated (linearly independent) eigenvectors:

$$v^1 = col\left(1 \;\; 1+i\right), \quad v^2 = col\left(1 \;\; 1-i\right).$$

Example (cont'd)

Therefore, using the formula, the solution is

$$x(t) = 2e^{-3t}\begin{bmatrix} \cos 4t - 2\sin 4t \\ -\cos 4t - 3\sin 4t \end{bmatrix}, \ \forall t \geq 0.$$

Case 2: using Jordan form

Consider $\dot{x} = Ax, \quad x(0) = x_o \in \mathbb{R}^n$.

If A does not have n linearly independent eigenvectors, then it is transformed into a Jordan form:

$$P^{-1}AP = diag\left(\Lambda_i\right) \doteq J, \quad 1 \le i \le s.$$

So, the solutions $x(t) = Py(t)$ where $y(t)$ are solutions to $\quad \dot{y} = Jy, \;\; y(0) = P^{-1}x_o$.

Case 2: using Jordan form

$$\dot{y} = Jy, \quad y(0) = P^{-1}x_o$$

$$\Leftrightarrow$$

$$\frac{dy^{(i)}(t)}{dt} = \Lambda_i y^{(i)}(t), \quad y(t) = \begin{bmatrix} y^{(1)}(t) \\ \vdots \\ y^{(s)}(t) \end{bmatrix}$$

Decoupled differential equations!

Example

Solve the differential eq. in Jordan form:

$$\dot{y} = Jy, \quad J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

By inspection,

$$\Lambda_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}.$$

Example (cont'd)

$$Then,\quad y=\begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix},\ y^{(1)}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ y^{(2)}=\begin{bmatrix} y_3 \\ y_4 \end{bmatrix}.$$

So the original y-equation becomes two decoupled (!) systems:

$$\dot{y}_1=2y_1,\quad \dot{y}_2=2y_2$$

and

$$\dot{y}_3=3y_3,\quad \dot{y}_4=y_3+3y_4.$$

Example (cont'd)

With initial value $y(0) = col\left[1\ 2\ 3\ 4\right]$, we have

$y_1(t) = e^{2t}$, $y_2(t) = 2e^{2t}$,

$y_3(t) = 3e^{3t}$, $y_4(t) = \left(4 + 3t\right)e^{3t}$. ("top down")

It is important to note that not every solution is in exponential form. That is very typical with Jordan form.

More on using Jordan form

Consider $\dot{x} = Ax, \quad x(0) = x_o \in \mathbb{R}^n.$

Using Jordan form:

$$P^{-1}AP = diag\left(\Lambda_i\right) \doteq J, \quad 1 \le i \le s.$$

the solutions $x(t) = e^{At}x_o$ becomes

$$x(t) = e^{tPJP^{-1}}x_o = Pe^{tJ}P^{-1}x_o$$

$$= P \times \text{blockdiag}\left(e^{t\Lambda_i}\right) \times P^{-1}x_o$$

Question :

What is the close-form of $e^{t\Lambda_i}$?

More on using Jordan form

To compute the close-form of $e^{t\Lambda_i}$, notice that

$$\Lambda_i = \begin{pmatrix} \lambda_i & 1 & \dots & & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix} = \lambda_i I + N$$

Clearly, $\lambda_i I$ and N commute, and N is an nilpotent matrix of degree p, *i.e.*, $N^p = 0$, while $N^{p-1} \neq 0$.

More on using Jordan form

So, $e^{t\Lambda_i} = e^{\lambda_i t} e^{tN}$, with

$$e^{tN} = I + tN + \frac{t^2}{2!} N^2 + \cdots + \frac{t^{p-1}}{(p-1)!} N^{p-1}$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{p-1}}{(p-1)!} \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Interpolation Method

Based on Cayley-Hamilton Theorem, all powers of a *nxn* matrix *A* greater than *n*-1 can be expressed as Linear combinations of A^k for *k*=0,1,…, n-1.

Define $f(\lambda) = e^{t\lambda}$, so $f(A) = e^{tA}$.

Also define $g(\lambda) = \alpha_0 + \alpha_1 \lambda + \cdots + \alpha_{n-1}\lambda^{n-1}$.

Using the following linear equations to determine the values of α_j, for $j = 0,1,\ldots,n-1$:

$$g^{(k)}(\lambda_i) = f^{(k)}(\lambda_i), \text{ for } k = 0,1, \ldots, n_i - 1$$

where each λ_i is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ with algebraic multiplicity n_i.

Interpolation Method

$$\text{Thus, } e^{tA} = f(A)$$
$$= g(A)$$
$$= \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}.$$

Exercise :

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Linear Difference Equations

For a linear difference equation described by

$$x[k+1] = Ax[k], \quad x[0] = x_0,$$

the solution $x[k], k \geq 0,$ is given by:

$$x[k] = A^k x_0.$$

Proof. It follows via direct substitution.

Linear Difference Equations

For an inhomogeneous linear difference equation:

$$x[k+1] = Ax[k] + Bu[k], \quad x[0] = x_0,$$

the solution $x[k],\ k \geq 0,$ is given by:

$$x[k] = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu[j].$$

Again, the proof follows upon direct substitution.

Higher-Order Equations

$$\frac{d^n y(t)}{dt^n} + a_1(t)\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy(t)}{dt} + a_n(t)y(t) + a_{n+1}(t) = 0$$

with initial conditions

$$y(0) = c_0,\ \dot{y}(0) = c_1,\ \cdots,\ y^{(n-1)}(0) = c_{n-1}.$$

Question :

How to transform it into a system of first-order equations:

$$\frac{dx(t)}{dt} = A(t)x(t) + f\left(t\right),\ x \in \mathbb{R}^n.$$

Transformation to First-Order Eqs

Define $x_1 = y,\ x_2 = \dot{y},\ \ldots,\ x_n = y^{(n-1)}$. Then,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = y^{(n)} = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n - a_{n+1} \end{cases}$$

Example

A higher-order differential equation can be put into a system of first-order differential equations:

$$\ddot{y}(t) + \left(\sin \omega t\right) \dot{y}(t) + y(t) = 0$$

$\Rightarrow$

$$\dot{x}(t) \doteq \frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -\sin \omega t \end{pmatrix} x(t)$$

with $x(t) = col\left(y(t),\ \dot{y}(t)\right)$.

Exercise

Try to put two (joint) higher-order differential eqs. into a system of first-order differential eqs.:

$$\ddot{y}_1 + 3\dot{y}_2 + 4y_1 + y_2 = 8t,$$

$$\ddot{y}_1 - \dot{y}_2 + y_1 + y_2 = \cos t.$$

Higher-Order Difference Equations

Can you transform a higher-order difference equation

$$y[k+n]+a_{n-1}y[k+n-1]+\cdots+a_1 y[k+1]+a_0 y[k]=\phi_k$$

into a family of first-order difference equations?

Homework 9

1. Solve the initial-value problem

$$\dot{x}_1 = x_2 + e^{-t}, \qquad x_1(0) = 1,$$

$$\dot{x}_2 = 6(t+1)^{-2} x_1 + \sqrt{t}, \quad x_2(0) = 2.$$

Homework 9

2. Solve the initial-value problem

$$\frac{dy(t)}{dt} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} y(t), \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$