ECE 5253 - Applied Matrix Theory — Problem Set 8
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Problem 1. *If λ_n is the least eigenvalue of a Hermitian matrix H, show that* $\lambda_n = \min_{x\neq 0} \frac{<Hx,x>}{<x,x>}$

Solution. See the prove below.
First show that Rayleigh principle. H is Hermitian.

$$\min_{||u||=1} < Hu, u >= \lambda_n$$

where λ_n is the least eigenvalue of H.
H is Hermitian. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of H are real and we can find the corresponding orthonormal eigenvectors $u^1, u^2, \ldots, u^n$.

$$\begin{aligned}
&||u^i|| = \sqrt{< u^i, u^i >} = 1 \\
&< u^i, u^j >= 0 \quad \text{for } i \neq j \\
&< Hu^i, u^i >= \lambda_i, \quad < Hu^i, u^j >= 0 \quad \text{for } i \neq j
\end{aligned}$$

for every unit vector u can be written as the linear combination of $u^1, u^2, \ldots, u^n$.

$$u = c_1u^1 + c_2u^2 + \cdots + c_nu^n$$

where $|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1$. So,

$$\begin{aligned}
< Hu, u > &=< H(c_1u^1 + c_2u^2 + \cdots + c_nu^n), u > \\
&= \sum_i |c_i|^2\lambda_i||u^i||^2 \geq \sum_i |c_i|^2\lambda_n = \lambda_n
\end{aligned}$$

where λ_n is the least eigenvalue.
Thus,

$$\min_{||u||=1} < Hu, u >= \lambda_n \tag{1}$$

Show that $\lambda_n = \min_{x\neq 0} \frac{<Hx,x>}{<x,x>}$. $x \neq 0$ at $u = \frac{x}{||x||}$ be a unit vector. From equation 1,

$$\begin{aligned}
&\min_{||u||=1} < Hu, u >= \lambda_n \\
&\min_{||u||=1} < H\frac{x}{||x||}, \frac{x}{||x||} >= \lambda_n \\
&\min_{||u||=1} < \frac{< Hx, x >}{< x, x >} >= \lambda_n
\end{aligned}$$

Problem 2. *Find all possible values of μ guaranteeing the positive-definiteness of*

$$H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Solution. See the prove below.
Since,

$$H = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

, H is a Hermitian matrix.
If H is positive definite

$$det(\begin{bmatrix} 1 & 2 \\ 2 & \mu \end{bmatrix}) > 0 \tag{2}$$

$$det(\begin{bmatrix} 1 & 2 & 3 \\ 2 & \mu & 4 \\ 3 & 4 & 5 \end{bmatrix}) > 0 \tag{3}$$

From equation 2, $\mu - 4 > 0$, $\mu > 4$.
From equation 3, $\mu < 3$.
From above result, $\mu > 4$ and $\mu < 3$ are contradiction.
Therefore, there is no solution of μ.

Problem 3. *Show that $|x| = \max_k |x_k|$, denoted as $|x|_\infty$, and $|x| = \sum_k |x_k|$, denoted as $|x|_1$, are both norms. What are their associated matrix norms?*

Solution. See the prove below.

$$|x|_\infty = \max_k |x_k|, \quad |x|_1 = \sum_k |x_k|$$

(1) Show that $|x|_\infty$ is a norm.
Check three properties,
1.

$$x = 0, \quad |x|_\infty = \max_k |x_k| = 0$$
$$x \neq 0, \quad |x|_\infty = \max_k |x_k| > 0$$

2.

$$\alpha \neq 0, \quad |\alpha x|_\infty = \max_k |\alpha x_k| = |\alpha| \cdot \max_k |x_k| = |\alpha| \cdot |x|_\infty$$

3.

$$|x + y|_\infty = \max_k |x_k + y_k| \leq \max_k |x_k| + |y_k| = |x|_\infty + |y|_\infty$$

Hence, $|x|_\infty$ is a norm.
(2) Show that $|x|_1$ is a norm
Check three properties,
1.

$$x = 0, \quad |x|_1 = \sum_k |x_k| = 0$$
$$x \neq 0, \quad |x|_1 = \sum_k |x_k| > 0$$

2.

$$\alpha \neq 0, \quad |\alpha x|_1 = \sum_k |\alpha x_k| = |\alpha| \cdot \sum_k |x_k| = |\alpha| \cdot |x|_1$$

3.

$$|x+y|_1 = \sum_k |x_k + y_k| \leq \sum_k |x_k| + |y_k| = |x|_1 + |y|_1$$

Hence, $|x|_1$ is a norm.

Find the matrix norm for $|x|_1$. Since,

$$|x|_1 = \sum_k |x_k|$$

$$||A||_1 \max_{x \neq 0} \frac{|Ax|_1}{|x|_1}$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

So,

$$\begin{aligned} |Ax|_1 &= \sum_k |a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n| \\ &\leq \sum_k (|a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n|) \\ &= \sum_k |a_{k1}||x_1| + |a_{k2}||x_2| + \cdots + |a_{kn}||x_n| \\ &= S_1|x_1| + S_2|x_2| + \cdots + S_n|x_n| \end{aligned}$$

where, $S_j = \sum_{i=1}^n |a_{ij}|$

Hence,

$$\frac{|Ax|_1}{|x|_1} = \frac{S_1|x_1| + S_2|x_2| + \cdots + S_n|x_n|}{|x_1| + |x_2| + \cdots + |x_n|} \leq \frac{S_M(|x_1| + |x_2| + \cdots + |x_n|)}{|x_1| + |x_2| + \cdots + |x_n|}$$

where, $S_M = \max\{S_1, S_2, \dots, S_n\}$.

Therefore,

$$||A||_1 = \max_{x \neq 0} \frac{|Ax|_1}{|x|_1} = \max_j \sum_{i=1}^n |a_{ij}|$$