Problem 1

$$A=\begin{bmatrix}3&0&0\\0&4&-1\\0&1&2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&0\\0&4&-1\\0&1&2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&0\\0&4&-1\\0&0&6\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Problem 2

(a) True

Proof:

There exist invertible matrices $P_{n\times n}$, $Q_{m\times n}$ such that

$$PMQ=\begin{pmatrix}I_{r\times r} & N_{(n-r)\times(m-r)}\\ O_{(n-r)\times r} & O_{(n-r)\times(m-r)}\end{pmatrix}$$

Then

$$M=P^{-1}\begin{pmatrix}I & N\\0&0\end{pmatrix}Q^{-1}=P^{-1}\begin{pmatrix}I\\0\end{pmatrix}(I,N)Q^{-1}$$

Let

$$A=P^{-1}\begin{pmatrix}I\\0\end{pmatrix},\quad B=(I,N)Q^{-1}$$

and then $M = AB$.

(b)

False

Counterexample: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(c) True

Proof:

Observe that there exists an $n \times n$ invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad (**)$$

This is an upper triangular matrix and diagonal entries are eigenvalues. Since the determinant of an upper triangular matrix is the product of diagonal entries. we have

$$\prod_{i=1}^{n} \lambda_i = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$

$$= \det(P)^{-1}\det(A)\det(P) = \det(A)$$

We take the trace of both sides of $(**)$ and get

$$\sum_{i=1}^{n} \lambda_i = tr(P^{-1}AP) = tr(A)$$

Problem 3

1. Proof:

First, multiply the equation $Au = \alpha u$ by v^T to the left:

$$v^T(Au) = v^T(\alpha u)$$

Using the associativity of matrix multiplication

$$(v^T A)u = \alpha(v^T u)$$

Since $v^T A = \beta v^T$, we can substitute βv^T:

$$\beta(v^T u) = \alpha(v^T u)$$

Then $(\beta - \alpha)(v^T u) = 0$

Since $\alpha \neq \beta$,

if $\beta - \alpha \neq 0$, then $v^T u = 0$

2. Proof:

To prove $A = T\begin{bmatrix} \alpha & 0 \\ 0 & N \end{bmatrix}T^{-1}$ that satisfies the condition described,

we have to prove $AT = T\begin{bmatrix} \alpha & 0 \\ 0 & N \end{bmatrix}$

that is $v^T \cdot A \cdot T = v^T T \begin{bmatrix} \alpha & 0 \\ 0 & N \end{bmatrix}$

Since $v^T A = \beta v^T = \alpha v^T$

We have to prove $\alpha \cdot v^T \cdot T = v^T \cdot T \cdot \alpha \begin{bmatrix} 1 & 0 \\ \alpha & \frac{N}{\alpha} \end{bmatrix}$

that is $\begin{bmatrix} 1 & 0 \\ 0 & \frac{N}{\alpha} \end{bmatrix} = I_n$

So when $\frac{N}{\alpha}$ is identity matrix in $(n-1)$ order, there exists T

such that $A = T\begin{bmatrix} \alpha & 0 \\ 0 & N \end{bmatrix}T^{-1}$

$$(T^{-1})^T = (T^T)^{-1} = \left[\frac{v}{u^T v}, M_1\right]$$

$$T^T = [u, T_1]^T = \begin{bmatrix} u^T \\ T_1^T \end{bmatrix}$$

$$T^T \cdot (T^T)^{-1} = \begin{bmatrix} u^T \\ T_1^T \end{bmatrix} \left[\frac{v}{u^T v}, M_1\right] = \begin{bmatrix} \frac{u^T v}{u^T v}, & u^T M_1 \\ \frac{v T_1^T}{u^T v}, & T_1^T M_1 \end{bmatrix} = I_n$$

So $\frac{u^T v}{u^T v} = 1$, $u^T M_1 = 0$ $\frac{v T_1^T}{u^T v} = 0$

$T_1^T \cdot M_1$ is identity matrix in $(n-1)$ order

Therefore when $u^T M_1 = 0$, $v T_1^T = 0$ and $T_1^T \cdot M_1 = I_{n-1}$,
there exists an invertible matrix $T = [u, T_1]$ with
$(T^{-1})^T = \left[\frac{v}{u^T v}, M_1\right]$