Lecture 2, Fall 2023

$$a^{\log_b x} \stackrel{?}{=} x^{\log_b a} \leftarrow \log_b(\quad)$$

$$\log_b\left(a^{\log_b x}\right) = \log_b x \log_b a$$

$$\log_b\left(x^{\log_b a}\right) = \log_b a \log_b x$$

$$\log_b x = \frac{\log_a x}{\log_a b} \Leftrightarrow \log_a b \cdot \log_b x = \log_a x$$

$$a^{(?)}$$

$$a^{(\log_a b \cdot \log_b x)} = \left(a^{\log_a b}\right)^{\log_b x} = b^{\log_b x} = x$$

$$a^{(\log_a x)} = x$$

$$b^{\log_a a}$$

$$x(1+x+x^2 \cdots +x^n) = S \cdot x$$

$$\underbrace{x+x^2+x^3 \cdots +x^n}+x^{n+1} = S \cdot x$$

$$S-1+x^{n+1} = S \cdot x$$

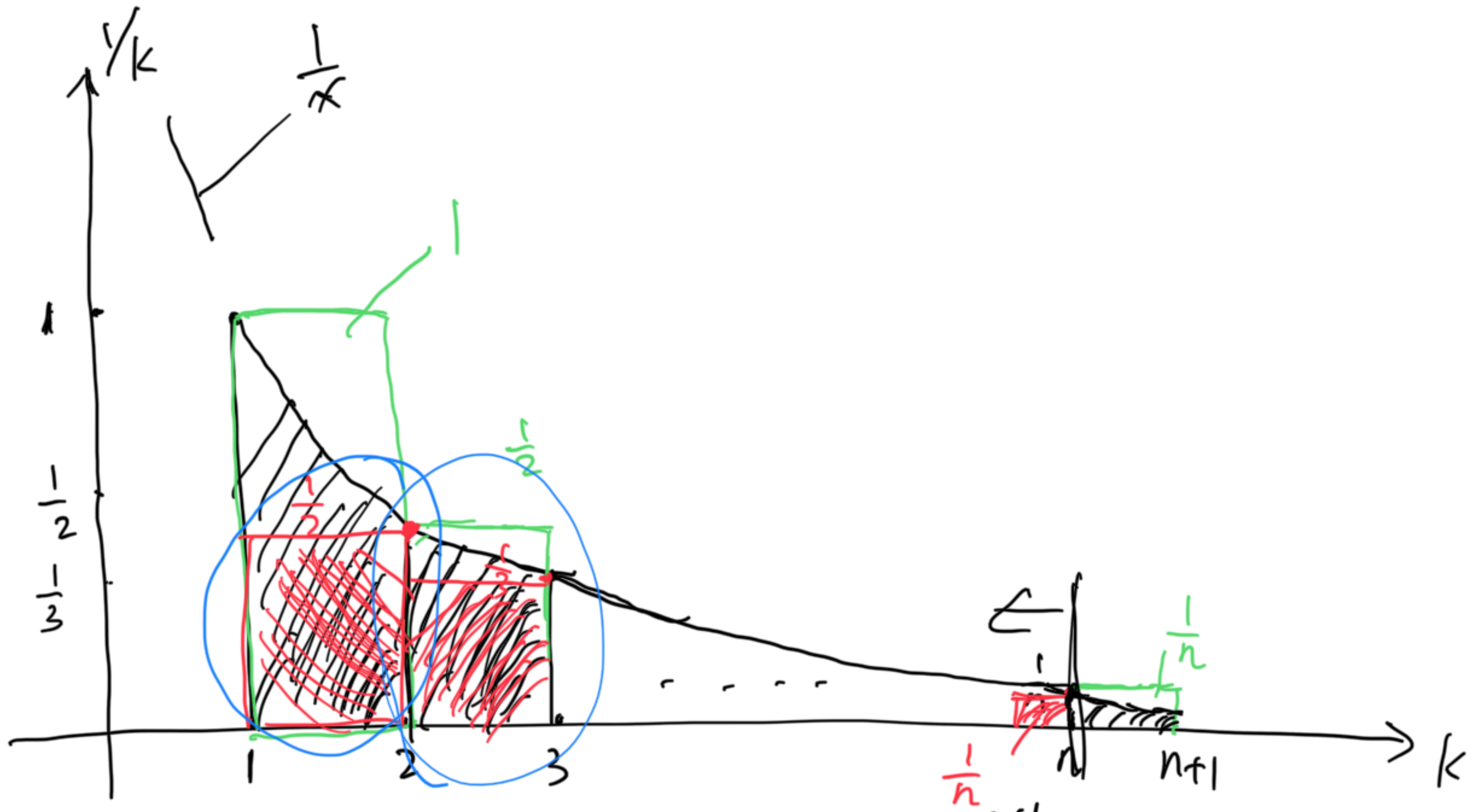
$$x^{n+1}-1 = S(x-1)$$

$$S = \frac{x^{n+1}-1}{x-1}$$

$|x|<1$, $n \to \infty$ $\qquad x^{n+1} \to 0$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{if } |x| < 1$$

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n}} = \sum_{k=1}^{n} \frac{1}{k}$$



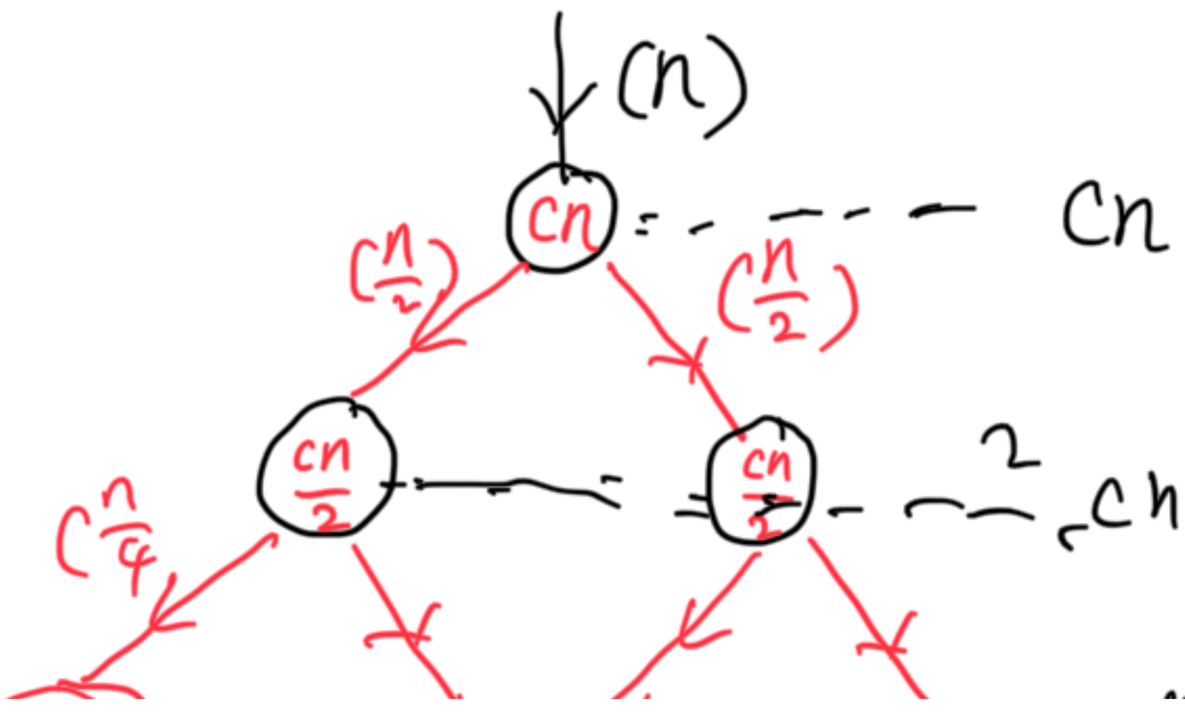
$$> \int^{n+1} \frac{1}{x}\, dx$$

$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$

$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \leq 1+\int_1^{n} \frac{1}{x}\,dx = 1+\ln(n)$

$T(n) = 2T\left(\frac{n}{2}\right) + \cancel{\theta(n)}\ cn$

$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{cn}{2}$



$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{cn}{4}$$

$\vdots$

$$T(1) = C_0$$

$$T(n) = \Theta(n \log n) \;?$$

$$\Updownarrow$$

$$\exists\, C_1, C_2, n_0, \quad C_2 n \log n \le T(n) \le C_1 n \log n, \;\forall n \ge n_0$$

$S(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + Cn$$

hypo. if $T(k) \le C_1 k \log k \quad \forall k < n$

induction: we need to show $T(n) \le c_1 n \log n$

$$<n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

$$\le \cancel{2}\, c_1 \frac{n}{\cancel{2}} \log\frac{n}{2} + cn$$

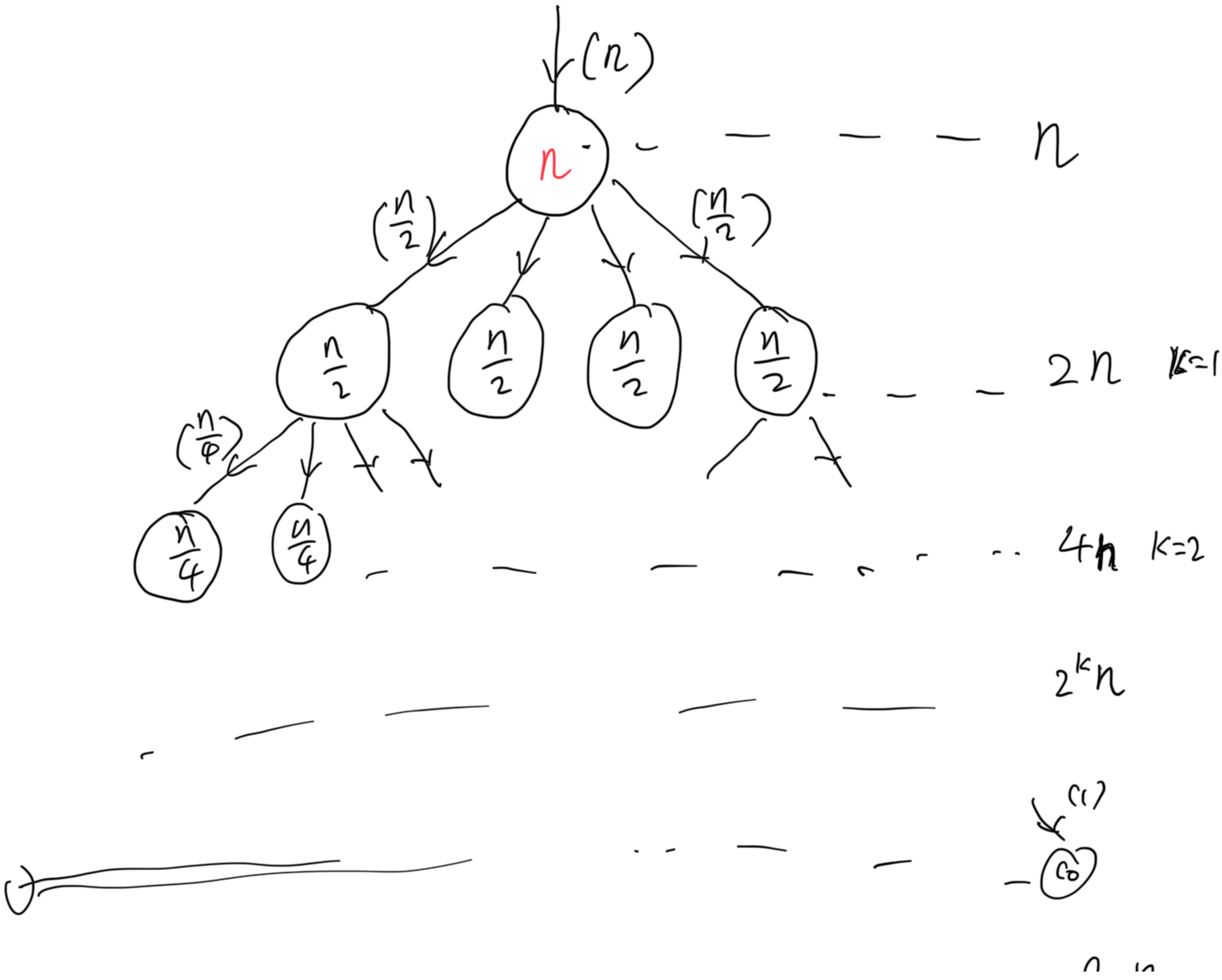
$$= c_1 n(\log n - \log 2) + cn$$

$$= \cancel{c_1 n \log n} + n(c - c_1 \log 2) \le \cancel{c_1 n \log n}$$

$$n(c - c_1 \log 2) \le 0$$

$$c_1 \ge \frac{c}{\log 2}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$



(n)
n
$(\frac{n}{2})$
$\frac{n}{2}$
$(\frac{n}{4})$
$\frac{n}{4}$
n
2n k=1
4n k=2
$2^k n$
(1)

$\log n$

$$2^{(\log_2 n)} \cdot n = c_0 n^2$$

$$n + 2n + 4n + \cdots n^2$$

$$n(1+2+4 \cdots n) \qquad \frac{2n-1}{2-1} = 2n-1$$

$$\Theta(n^2)$$

$$c_2 n^2 \le T(n) \le c_1 n^2, \quad \forall n \ge n_0 \qquad S(n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(k) \le c_1 k^2, \quad \forall k < n \qquad \frac{n}{2} < n$$

$$T(n) \le 4 \cdot c_1 \left(\frac{n}{2}\right)^2 + n$$

$$= \quad c_1 n^2 + n \quad \le c_1 n$$

$$T(n) = \Theta(n^2) = \underline{\Theta(n^2 - n)}$$

$$T(k) \le c_1 (k^2 - k)$$

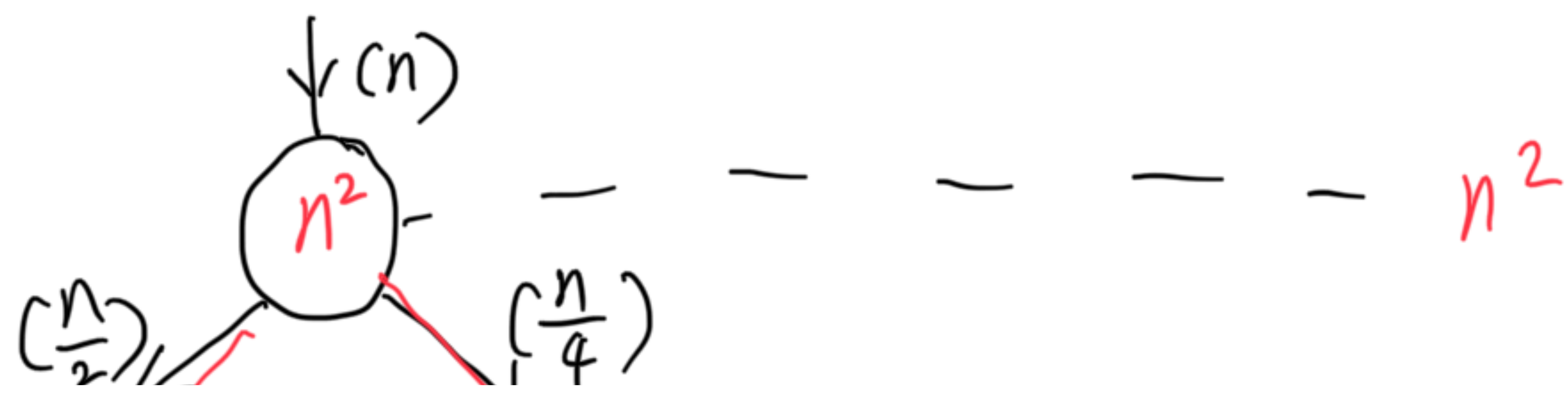
$$T(n) \le 4 c_1 \left(\frac{n^2}{4} - \frac{n}{2}\right) + n$$

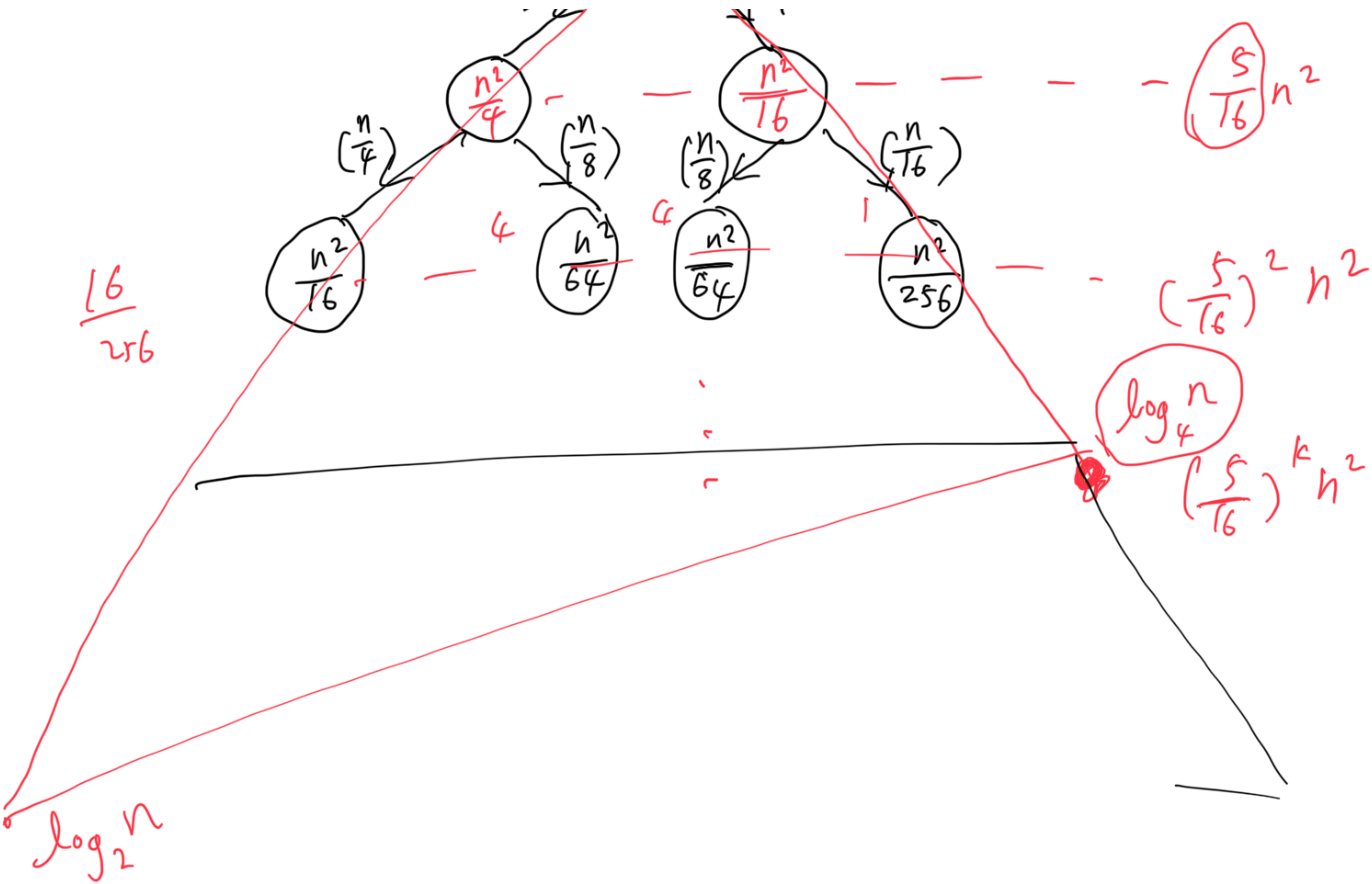
$$= \cancel{c_1 n^2} - 2c_1 n + n \quad \le \cancel{c_1 (n^2 - n)}$$

$$-2c_1 n + n \le -c_1 n$$

$$c_1 \ge 1$$

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + n^2$$




n²/4
n²/16
(n/4)
(n/8)
(n/8)
(n/16)
n²/16
n²/64
n²/64
n²/256
(5/16) n²
(5/16)² n²
log₄ n
(5/16)^k n²
16/256
log₂ n