More information can be found here.

Unfiltered Derevative action

The general expression of a PID is :

$$C(p) = G + \frac{1}{Ti*p} + Td * p$$

with :

- G the proportional gain
- Ti the integral time constant
- Td the derivative time constant

The digital version can be derived by changing the complex variable p by another complex variable z^{-1} with

$$p = \frac{1 - z^{-1}}{Ts}$$

with Ts as the sampling period.

This gives :

$$C(z) = G + \frac{1}{Ti * \frac{1 - z^{-1}}{Ts}} + Td * \frac{1 - z^{-1}}{Ts}$$

$$C(z) = G + \frac{Ts}{Ti * (1 - z^{-1})} + Td * \frac{1 - z^{-1}}{Ts}$$

$$C(z) = \frac{G * (1 - z^{-1})}{(1 - z^{-1})} + \frac{\frac{Ts}{Ti}}{(1 - z^{-1})} + * \frac{\frac{Td}{Ts}(1 - z^{-1})^2}{(1 - z^{-1})}$$

$$C(z) = \frac{G * (1 - z^{-1}) + \frac{Ts}{Ti} + \frac{Td}{Ts}(1 - z^{-1})^2}{(1 - z^{-1})}$$

$$C(z) = \frac{G - Gz^{-1} + \frac{Ts}{Ti} + \frac{Td}{Ts} - 2\frac{Td}{Ts}z^{-1} + \frac{Td}{Ts}z^2}{(1 - z^{-1})}$$

$$C(z) = \frac{Y}{X} = \frac{G + \frac{Ts}{Ti} + \frac{Td}{Ts} - (G + 2\frac{Td}{Ts})z^{-1} + \frac{Td}{Ts}z^2}{(1 - z^{-1})}$$

That finally gives :

$$Y(1 - z^{-1}) = X(G + \frac{Ts}{Ti} + \frac{Td}{Ts} - (G + 2\frac{Td}{Ts})z^{-1} + \frac{Td}{Ts}z^2)$$

$$Y[0] = Y[1] + X[0]A[0] + X[1]A[1] + X[2]A[2]$$

Where :

- Y[i] means $Y\ z^{-i}$
- $A[0] = G + \frac{Ts}{Ti} + \frac{Td}{Ts}$
- $A[1] = -(G + 2\frac{Td}{Ts})$
- $A[2] = \frac{Td}{Ts}$

Filtered derivative action

It can be useful to lessen the variation impact of the derivative action. To do so, we can low-pass filter Td such as :

$C(p) = G + \frac{1}{Ti*p} + \frac{Td*p}{1+\frac{Td}{N}*p}$, with $3 < N < 10.$

We still use the approximation $p = \frac{1-z^{-1}}{Ts}$

$$C(z) = G + \frac{1}{Ti(\frac{1-z^{-1}}{Ts})} + \frac{Td(\frac{1-z^{-1}}{Ts})}{1+\frac{Td}{N}(\frac{1-z^{-1}}{Ts})}$$

$$C(z) = G + \frac{1}{\frac{Ti}{Ts}(1-z^{-1})} + \frac{\frac{Td}{Ts}(1-z^{-1})}{1+\frac{Td}{NTs}(1-z^{-1})}$$

$$C(z) = G + \frac{1}{\frac{Ti}{Ts}(1-z^{-1})} + \frac{\frac{Td}{Ts}(1-z^{-1})}{1+\frac{Td}{NTs}(1-z^{-1})}$$

Let's define : $a = \frac{Ti}{Ts}$, $b = \frac{Td}{Ts}$, $c = \frac{Td}{NTs} = \frac{b}{N}$ and $Z = (1-z^{-1})$, $Z^2 = (1-2z^{-1}+z^{-2})$

$$C(z) = G + \frac{1}{aZ} + \frac{bZ}{1+cZ}$$

$$C(z) = G + \frac{1}{aZ} + \frac{bZ}{1+cZ} = \frac{GaZ(1+cZ)+(1+cZ)+abZ^2}{aZ(1+cZ)}$$

$$C(z) = \frac{1+[aG+c]Z+[acG+ab]Z^2}{aZ+acZ^2} = \frac{1+[aG+c](1-z^{-1})+[acG+ab](1-2z^{-1}+z^{-2})}{a(1-z^{-1})+ac(1-2z^{-1}+z^{-2})}$$

$$C(z) = \frac{1+([aG+c]-[aG+c]z^{-1})+([acG+ab]-2[acG+ab]z^{-1}+[acG+ab]z^{-2})}{(a-az^{-1})+(ac-2acz^{-1}+acz^{-2})}$$

$$C(z) = \frac{1+[aG+c]+[acG+ab]-\big[aG+c+2[acG+ab]\big]z^{-1}+[acG+ab]z^{-2}}{a+ac-(a+2ac)z^{-1}+acz^{-2}}$$

Let's define: $D = [aG+c]$, $E = [acG+ab]$,

$$C(z) = \frac{Y}{X} = \frac{[1+D+E]-[D+2E]z^{-1}+Ez^{-2}}{[a+ac]-[a+2ac]z^{-1}+[ac]z^{-2}}$$

$$Y = X\frac{[1+D+E]}{[a+ac]} - \frac{[D+2E]}{[a+ac]}Xz^{-1} + \frac{E}{[a+ac]}Xz^{-2} + \frac{[a+2ac]}{[a+ac]}Yz^{-1} - \frac{[ac]}{[a+ac]}Yz^{-2}$$

And finally :

$a = \frac{Ti}{Ts}$, $b = \frac{Td}{Ts}$, $c = \frac{Td}{NTs} = \frac{b}{N}$, $D = [aG+c]$, $E = [acG+ab]$

$A[0] = [1+D+E]$, $A[1] = -[D+2E]$, $A[2] = E$, $B[1] = [a+2ac]$, $B[2] = -[ac]$

$$Y[0] = \frac{1}{[a+ac]}(A[0]X[0] + A[1]X[1] + A[2]X[2] + B[1]Y[1] + B[2]Y[2])$$