More information can be found here.

Unfiltered Derivative action

The general expression of a PID is :

$C(p) = G(1 + \frac{1}{Ti*p} + Td * p)$

with :

- G the proportional gain
- Ti the integral time constant
- Td the derivative time constant

The digital version can be derived by changing the complex variable p by another complex variable z^{-1} with

$$p = \frac{1 - z^{-1}}{Ts}$$

with Ts as the sampling period.

This gives :

$$C(z) = G(1 + \frac{1}{Ti * \frac{1 - z^{-1}}{Ts}} + Td * \frac{1 - z^{-1}}{Ts})$$

$$C(z) = G(1 + + \frac{Ts}{Ti * (1 - z^{-1})} + Td * \frac{1 - z^{-1}}{Ts})$$

$$C(z) = G * \left(\frac{(1 - z^{-1})}{(1 - z^{-1})} + \frac{\frac{Ts}{Ti}}{(1 - z^{-1})} + * \frac{\frac{Td}{Ts}(1 - z^{-1})^2}{(1 - z^{-1})} \right)$$

$$C(z) = G * \left(\frac{(1 - z^{-1}) + \frac{Ts}{Ti} + \frac{Td}{Ts}(1 - z^{-1})^2}{(1 - z^{-1})} \right)$$

$$C(z) = G * \left(\frac{1 - z^{-1} + \frac{Ts}{Ti} + \frac{Td}{Ts} - 2\frac{Td}{Ts}z^{-1} + \frac{Td}{Ts}z^2}{(1 - z^{-1})} \right)$$

$$C(z) = \frac{Y}{X} = G * \left(\frac{1 + \frac{Ts}{Ti} + \frac{Td}{Ts} - (1 + 2\frac{Td}{Ts})z^{-1} + \frac{Td}{Ts}z^2}{(1 - z^{-1})} \right)$$

That finally gives :

$$Y(1 - z^{-1}) = GX(1 + \frac{Ts}{Ti} + \frac{Td}{Ts} - (1 + 2\frac{Td}{Ts})z^{-1} + \frac{Td}{Ts}z^2)$$

with :

- Y[i] means $Y\, z^{-i}$
- $A[0] = G\left(1 + \frac{Ts}{Ti} + \frac{Td}{Ts}\right)$, $A[1] = -G\left(1 + 2\frac{Td}{Ts}\right)$, $A[2] = G\frac{Td}{Ts}$
- $B[0] = 1$, $B[1] = 1$

$$Y[0] = Y[1]\frac{B[1]}{B[0]} + X[0]\frac{A[0]}{B[0]} + X[1]\frac{A[1]}{B[0]} + X[2]\frac{A[2]}{B[0]}$$

Filtered derivative action

It can be useful to lessen the noise effect on the derivative action. To do so, we can low-pass filter Td such as :

$$C(p) = G(1 + \frac{1}{Ti*p} + Td*p*\frac{1}{1+\frac{Td}{N}*p}) \;,\; \text{ with } 3 < N < 10.$$

We still use the approximation $p = \frac{1-z^{-1}}{Ts}$

$$C(z) = G(1 + \frac{1}{Ti\left(\frac{1-z^{-1}}{Ts}\right)} + \frac{Td\frac{(1-z^{-1})}{Ts}}{1+\frac{Td}{N}\frac{(1-z^{-1})}{Ts}}) = G(1 + \frac{1}{\frac{Ti}{Ts}(1-z^{-1})} + \frac{\frac{Td}{Ts}(1-z^{-1})}{1+\frac{Td}{NTs}(1-z^{-1})})$$

Let's define : $a = \frac{Ti}{Ts}$, $b = \frac{Td}{Ts}$, $c = \frac{Td}{NTs} = \frac{b}{N}$ and $Z = (1-z^{-1})$, $Z^2 = (1-2z^{-1}+z^{-2})$

$$C(z) = G(1 + \frac{1}{aZ} + \frac{bZ}{1+cZ})$$

$$C(z) = G(1 + \frac{1}{aZ} + \frac{bZ}{1+cZ}) = G\frac{aZ(1+cZ)+(1+cZ)+abZ^2}{aZ(1+cZ)}$$

$$C(z) = G\frac{1+[a+c]Z+[ac+ab]Z^2}{aZ+acZ^2} = G\frac{1+[a+c](1-z^{-1})+[ac+ab](1-2z^{-1}+z^{-2})}{a(1-z^{-1})+ac(1-2z^{-1}+z^{-2})}$$

$$= G\frac{1+[a+c]+[ac+ab]-([a+c]+2[ac+ab])z^{-1}+[ac+ab]z^{-2}}{a(1-z^{-1})+ac(1-2z^{-1}+z^{-2})}$$

$$C(z) = G\frac{1+([a+c]-[a+c]z^{-1})+([ac+ab]-2[ac+ab]z^{-1}+[ac+ab]z^{-2})}{(a-az^{-1})+(ac-2acz^{-1}+acz^{-2})}$$

$$C(z) = G\frac{1+[a+c]+[ac+ab]-\big[a+c+2[ac+ab]\big]z^{-1}+[ac+ab]z^{-2}}{a+ac-(a+2ac)z^{-1}+acz^{-2}}$$

Let's define: $d = [a+c]$, $e = [ac+ab]$,

$$C(z) = \frac{Y}{X} = G\frac{[1+d+e]-[d+2e]z^{-1}+ez^{-2}}{[a+ac]-[a+2ac]z^{-1}+[ac]z^{-2}}$$

$$Y[a+ac] = +[a+2ac]Yz^{-1}-[ac]Yz^{-2}+[1+d+e]X-[d+2E]Xz^{-1}+eXz^{-2}$$

$$Y = \left[G\frac{[1+d+e]}{[a+ac]}X - G\frac{[d+2e]}{[a+ac]}Xz^{-1} + G\frac{e}{[a+ac]}Xz^{-2} + \frac{[a+2ac]}{[a+ac]}Yz^{-1} - \frac{[ac]}{[a+ac]}Yz^{-2}\right]$$

And finally :

$a = \frac{Ti}{Ts}$, $b = \frac{Td}{Ts}$, $c = \frac{Td}{NTs} = \frac{b}{N}$, $d = [a+c]$, $e = [ac+ab]$, $f = [a+ac]$

$A[0] = G\frac{[1+d+e]}{f}$, $A[1] = G\frac{-[d+2e]}{f}$, $A[2] = G\frac{e}{f}$, $B[0] = 1$, $B[1] = \frac{[a+2ac]}{f}$, $B[2] = \frac{-[ac]}{f}$

$$Y[0] = A[0]X[0] + A[1]X[1] + A[2]X[2] + B[1]Y[1] + B[2]Y[2]$$