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EULER*

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Until the latter part of the seventeenth century, mathematics had sometimes bestowed high reputation upon its adepts but had seldom provided them with the means to social advancement and honorable employment. VIÈTE had made his living as a lawyer, FERMAT as a magistrate; even in Fermat's days, endowed chairs for mathematics were few and far between. In Italy, the University of Bologna ("lo studio di Bologna", as it was commonly called), famous throughout Europe, had indeed counted Scipione del FERRO among its professors in the early sixteenth century; but CARDANO had been active as a physician; BOMBELLI was an engineer, and so was Simon STEVIN in the Netherlands. NAPIER, the inventor of logarithms, was a Scottish laird, living in his castle of Merchiston after coming back from the travels of his early youth. Neighboring disciplines did not fare better. COPERNICUS was an ecclesiastical dignitary. Kepler's teacher MAESTLIN had been a professor in Tübingen, but KEPLER plied his trade as an astrologer and maker of horoscopes. GALILEO's genius, coupled with his domineering personality, earned him, first a professorship in Padova, then an enviable position as a protégé of the Grand-Duke of Tuscany, which saved him from the worst consequences of his disastrous conflict with the Church of Rome; his pupil TORRICELLI succeeded him as "philosopher and mathematician" to the Grand-Duke, while CAVALIERI combined the Bologna chair with the priorate of the Gesuati convent in the same city.

Among Fermat's scientific correspondents, few held professorial rank. ROBERVAL, at the Collège de France (then styled Collège Royal), occupied the chair founded in 1572 from a legacy of the scientist and philosopher Pierre de la Ramée. The Savilian chair at Oxford, created for H. BRIGGS in 1620, was held by WALLIS from 1649 until his death in 1703; but his talented younger friend and collaborator William BROUNCKER, second Viscount, was a nobleman whose career as commissioner of the Navy, and whose amours, are abundantly documented in Pepys's diary. It was only in 1663 that Isaac BARROW became the first Lucasian professor in Cambridge, a position which he relinquished to NEWTON in 1669 to become preacher to Charles II and achieve high reputation as a divine. In the Netherlands, while Descartes's friend and commentator F. SCHOOTEN was a professor in Leiden, René de SLUSE, a mathematician in high esteem among his contemporaries and an attractive personality, was a canon in Liège. DESCARTES, as he tells us, felt himself, by the grace of God ("*graces à Dieu*": *Discours de la Méthode*, *Desc.* VI.9), above the need of gainful employment; so were his friends Constantin HUYGENS and Constantin's son, the great Christian HUYGENS. LEIBNIZ was in the employ of the Hanoverian court; all his life he preserved his love for mathematics, but his friends marveled sometimes that his occupations left him enough leisure to cultivate them.

Whatever their position, the attitude of such men towards mathematics was often what we can describe as a thoroughly professional one. Whether through the printed word or through their correspondence, they took pains to give proper diffusion to their ideas and results and to keep abreast of contemporary progress; for this they relied largely upon a private network of informants. When they traveled, they looked up foreign scientists. At home they were visited by scientifically inclined travelers, busy bees intent on disseminating the pollen picked up here and there. They eagerly sought correspondents with interests similar to their own; letters passed from hand to hand until they had reached whoever might feel concerned. A private library of reasonable size was almost a necessity. Booksellers had standing orders to supply customers with

*These historical paragraphs, as well as the biography of Euler and the proof of a theorem about sums of squares that follow, are excerpts from the author's book *Number Theory* (Birkhäuser Boston, 1984); they appear there on pp. 159–169 and 292–295. The only difference between the book's version and this one is that the detailed references are omitted here. *Editor.*

the latest publications within each one's chosen field. This system, or lack of system, worked fairly well; indeed it subsists down to the present day, supplementing more formalized modes of communication, and its value is undiminished. Nevertheless, even the seventeenth century must have found it increasingly inadequate.

By the time of Euler's birth in 1707, a radical change had taken place; its first signs had become apparent even before Fermat's death. The *Journal des Sçavans* was started in January 1665, just in time to carry an obituary on Fermat ("*ce grand homme*", as he is called there). Louis XIV's far-sighted minister Colbert had attracted Huygens to Paris in 1666, and the astronomer CASSINI in 1669, awarding to each a royal pension of the kind hitherto reserved to literati. In 1635 Richelieu had founded the *Académie françoise*; the more practical-minded Colbert, realizing the value of scientific research (pure no less than applied) for the prosperity of the realm, set up the *Académie des sciences* in 1666 around a nucleus consisting largely of Fermat's former correspondents; Fermat's great friend and former colleague Carcavi was entrusted with its administration and became its first secretary. In England, some degree of political stability had been restored in 1660 by the recall of Charles II; in 1662 the group of amateurs ("*virtuosi*") who had for some time held regular meetings in Gresham College received the charter which made of them the *Royal Society*, with Brouncker as its first president; in 1665 they started the publication of the *Philosophical Transactions*, which has been continued down to the present day. In 1698 the French academy followed suit with a series of yearly volumes, variously entitled *Histoire* and *Mémoires de l'Académie des Sciences*. In 1682 Leibniz was instrumental in creating, not yet an academy, but at least a major scientific journal, the *Acta Eruditorum* of Leipzig, to whose early issues he contributed the articles by which he was giving birth to the infinitesimal calculus.

Soon universities and academies were competing for scientific talent and sparing neither effort nor expense in order to attract it. Jacob BERNOULLI had become a professor in his native city of Basel in 1687; as long as he lived, this left little prospect to his younger brother and bitter rival Johann of finding academic employment there; at first he had to teach Leibniz's infinitesimal calculus to a French nobleman, the Marquis de l'HÔPITAL, even agreeing to a remarkable contract whereby the latter acquired an option upon all of BERNOULLI's mathematical discoveries. In 1695, however, Joh. Bernoulli became a professor in Groningen, eventually improving his position there by skilfully playing Utrecht against Groningen; finally he settled down in Basel in 1705 after his brother's death. No wonder, then, that in 1741 we find him congratulating Euler on the financial aspects of his Berlin appointment and suggesting at the same time that he would be willing (for a moderate yearly stipend, "*pro modico subsidio annuo*") to enrich the memoirs of the Berlin Academy with regular contributions of his own. In short, scientific life, by the turn of the century, had acquired a structure not too different from what we witness to-day.

Euler's father, Paul EULER, was a parish priest established in Riehen near Basel; he had studied theology at the university of Basel, while at the same time attending the lectures of Jacob Bernoulli; he had planned a similar career for his son, but placed no obstacle in the way of young Leonhard's inclinations when they became manifest. Clearly, by that time, a bright future was in store for any young man with exceptional scientific talent.

In 1707, when EULER was born, Jacob Bernoulli was dead, and Johann had succeeded him; Johann's two sons Nicolas (born in 1695) and Daniel (born in 1700) were following the family tradition, except that, in contrast to their father and uncle, they loved each other dearly, as they took pains to make known. Euler became their close friend and Johann's favorite disciple; in his old age, he liked to recall how he had visited his teacher every Saturday and laid before him the difficulties he had encountered during the week, and how hard he had worked so as not to bother him with unnecessary questions.

Three monarchs came to play a decisive role in Euler's career: Peter the Great, Frederic the Great, and the Great Catherine. Peter, a truly great czar perhaps, died in 1725; but he had had time to found Saint Petersburg, erect some of its most impressive buildings, and, most important of all for our story, to make plans for an Academy of Sciences modelled on what he had seen in the West; those plans were faithfully carried out by his widow. In 1725 the two younger BERNOULLIS, Nicolas and Daniel, were called there. Nicolas died the next year, apparently of appendicitis. About the same time an offer went to Euler to join the Petersburg Academy. He was not quite twenty years old; he had just won a prize for an essay on ship-building, never having seen a sea-going ship in his life. He had no early prospects at home. He accepted with alacrity.

From Basel he sailed down the Rhine to Mainz, then traveled to Lübeck, mostly on foot, visiting Christian Wolff on the way; this was a philosopher and follower of Leibniz, banished from Berlin (as he told Euler) by a king with little understanding for philosophy; his hobby-horse was Leibniz's theory of monads, and Euler was clearly not impressed. From Lübeck a ship took the young mathematician to Petersburg.

In those days academies were well-endowed research institutions, provided with ample funds and good libraries. Their members enjoyed considerable freedom; their primary duty was to contribute substantially to the academy's publications and keep high its prestige in the international scientific world. At the same time they were the scientific advisers to the monarch and to state authorities, always on hand for such tasks, congenial or not, as the latter might find fit to assign to them; had it not been so, no state would have undergone the high expense of maintaining such institutions, as Euler once acknowledged to Catherine. In 1758, at the height of his fame, Euler (who had acquired in Petersburg a good command of the Russian language) did not find it beneath him, nor inconsistent with his continuing close relations with the Petersburg Academy, to translate for king Frederic some dispatches seized during military operations against the Russian army.

In 1727, however, the political situation had changed by the time Euler reached Petersburg. Under a new czar all academic appointments were in abeyance. On the strength of his prize essay, Euler was commissioned into the Russian navy, but not for long. Soon he was a salaried member of the academy, at first with the junior rank of "adjunct". When his friend Daniel left for Basel in 1733, he was appointed in his place; thus he could afford to marry, naturally into the local Swiss colony, and to buy for himself a comfortable house. His bride was the daughter of the painter GSELL; in due course she was to give birth to thirteen children, out of whom only three sons survived Euler; little is recorded of her otherwise. The eldest son Johann Albert, born in 1734, was to become one of his father's collaborators, and later a leading member of the academy.

Once Euler was thus well established in Petersburg, his productivity exceeded all expectations, in spite of the comparative isolation in which he had been left by Daniel Bernoulli's departure. It was hardly interrupted by a severe illness in 1735 and the subsequent loss of his right eye. He had beyond doubt become the most valuable member of the Academy, and his reputation had been growing by leaps and bounds, when two events in the higher spheres of European politics brought about a major change in his peaceful life. In Petersburg the death of the czarina in 1740, a regency, and the ensuing turmoils, seemed to threaten the very existence of the Academy. Just at this juncture Frederic II succeeded his father (the same king who had so cavalierly thrown Chr. Wolff out of Berlin) on the throne of Prussia; he immediately took steps directed towards the establishment of an academy under his patronage, for which he sought out the most famous names in European science; naturally Euler was on the list. A munificent offer from Frederic, coupled with a fast deteriorating situation in Petersburg, brought Euler to Berlin in July 1741, after a sea-voyage of three weeks on the Baltic during which he alone among his family (or so he claimed) had been free from sea-sickness. In the following year, to his great satisfaction, he was able to purchase an excellent house, well situated, and, by special royal order, exempt from requisition. There he lived for the next 24 years, with apparently the sole interruption of seasonal visits to the country estate he acquired in 1752 in Charlottenburg, and of a family trip to Frankfurt in 1750 to meet his widowed mother who was coming from Basel to live in Berlin with him; his father, who had been disappointed in his hope of getting Euler's visit in Basel had died in 1745.

With Euler's change of residence one might have expected that the steady flow of his publications would be diverted from Petersburg to Berlin; but far from it! He was not only allowed to keep his membership in the Petersburg Academy, but his pension from Petersburg was continued, and he was intent upon giving his former colleagues value for their money. Well might his great-grandson P.-H. Fuss describe Euler's Berlin period as "twenty-five years of prodigious activity". More than 100 memoirs sent to Petersburg, 127 published in Berlin on all possible topics in pure and applied mathematics were the products of those years, side by side with major treatises on analysis, but also on artillery, ship-building, lunar theory; not to mention the prize-winning essays sent to the Paris academy (whose prizes brought substantial cash rewards in addition to high reputation); to which one has to add the *Letters to a German Princess* (one of the most successful popular books on science ever written) and even a defense of christianity (*Rettung der göttlichen Offenbarung...*) which did nothing to ingratiate its author with the would-be philosopher-king Frederic. At the same time Euler was conducting an increasingly heavy correspondence, scientific, personal and also official since the administrative burdens of the academy tended to fall more and more upon his shoulders.

As years went by, Euler and Frederic became disenchanted with each other. The king was not unaware of the lustre that Euler was throwing upon his academy, but French literati stood far higher in his favor. He was seeking to attract d'ALEMBERT to Berlin and was expected to put him above Euler as head of the Academy. Euler was spared this blow to his self-esteem; d'Alembert, forewarned perhaps by Voltaire's unpleasant experience with Frederic in 1753, enjoyed basking in the king's favor for the time of a short visit but valued his freedom far too highly to alienate it more durably. Nevertheless, as early as 1763, Euler's thoughts started turning again towards Russia.

Fortunately another political upheaval had just taken place there. In 1762 the czar's German wife had seized power as Catherine II after ridding herself and Russia of her husband. One of her first projects was to restore the Petersburg academy to its former glory. This was almost synonymous with bringing Euler back. Negotiations dragged on for three years. Finally, in 1766, the Russian ambassador in Berlin was instructed to request Euler to write his own contract. Frederic, realizing too late the magnitude of this loss, had tried to put obstacles in the way; he soon found that he could not afford to displease the imperial lady. In the same year Euler was back in Petersburg, after a triumphal journey through Poland where Catherine's former lover, king Stanislas, treated him almost like a fellow-sovereign.

By then Euler was losing his eyesight. He had lost the use of his right eye during his first stay in Petersburg. About the time when he left Berlin, or shortly thereafter, a cataract developed in his left eye. In 1770, in answer to a letter from Lagrange on number theory, he described his condition as follows: "*Je me suis fait lire toutes les opérations que vous avez faites sur la formule* $101 = pp - 13qq$ *et je suis entièrement convaincu de leur solidité; mais, étant hors d'état de lire ou d'écrire moi-même, je dois vous avouer que mon imagination n'a pas été capable de saisir le fondement de toutes les déductions que vous avez été obligé de faire et encore moins de fixer dans mon esprit la signification de toutes les lettres que vous y avez introduites. Il est bien vrai que de semblables recherches ont fait autrefois mes délices et m'ont coûté bien du tems; mais à présent je ne saurois plus entreprendre que celles que je suis capable de dévellopper dans ma tête et souvent je suis obligé de recourir à un ami pour exécuter les calculs que mon imagination projette*" ["I have had all your calculations read to me, concerning the equation $101 = p^2 - 13q^2$, and I am fully persuaded of their validity; but, as I am unable to read or write, I must confess that my imagination could not follow the reasons for all the steps you have had to take, nor keep in mind the meaning of all your symbols. It is true that such investigations have formerly been a delight to me and that I have spent much time on them; but now I can only undertake what I can carry out in my head, and often I have to depend upon some friend to do the calculations which I have planned"].

An operation was attempted in 1771 and was successful at first, but the eye soon became infected, and total or near-total blindness ensued. Except for this misfortune, and for a fire which destroyed his house in 1771 among many others in Petersburg, he lived on in comfort, greatly honored and respected; neither old age nor blindness could induce him to take a well-deserved rest. He had assistants, one of whom was his own son; others were sent to him from Basel, with the co-operation of his old friend Daniel Bernoulli; to one of them, N. FUSS, who had come to Petersburg in 1773 and later married a granddaughter of Euler's, we owe a vivid description of Euler's method of work in the last decade of his life. Hundreds of memoirs were written during that period; enough, as Euler had predicted, to fill up the academy publications for many years to come. He died suddenly on 18 September 1783, having preserved excellent general health and his full mental powers until that very day.

AN ELEMENTARY PROOF FOR SUMS OF SQUARES

In 1751, Euler, having proved that every integer is a sum of (at most) four squares "*in fractis*", i.e. in rational numbers, wrote as follows:

"*In Analysi quidem Diophantea pro certo assumi solet nullum numerum integrum in quatuor quadrata fracta dispertiri posse, nisi eius resolutio in quatuor quadrata integra vel pauciora constet...Verum nusquam adhuc eiusmodi demonstrationem inveni...*" ["In diophantine analysis one usually takes for granted that no integer can be split into four rational squares unless it has an expression as a sum of four integral squares or less... but so far I have nowhere found a proof of this..."]. This is indeed a question which must occur naturally, concerning sums of 2, 3, or 4 squares, to any reader of Diophantus, and Fermat had already given it some thought, apparently without success, at the beginning of his career as a number-theorist. Euler himself had raised it, more particularly concerning sums of two squares, in 1745, and again repeatedly, concerning sums of four squares, in his correspondence with Goldbach.

Here we shall reproduce a proof due to L. Aubry (*Sphinx-Œdipe* 7 (1912), pp. 81–84) which applies equally well to sums of 2, 3, or 4 squares and a few other quadratic forms. We first describe it in geometric terms for sums of 3 squares.

Points in $\mathbf{R}^3$ will be called *rational* (resp. *integral*) if they have rational (resp. integral) coordinates. To each point $a = (x, y, z)$ there is an integral point $a' = (x', y', z')$ at a euclidean distance less than 1 from a; for instance one can take for x', y', z' the integers respectively closest to x, to y and to z, in which case the distance from a to a' is $\leqslant \sqrt{3}/2$.

Now assume that an integer N is a sum of three rational squares; this is the same as to say that there is a rational point a_0 on the sphere S given by $N = x^2 + y^2 + z^2$. Let a'_0 be the integral point (or one of the integral points, if there are two or more) closest to a_0; its distance from a_0 is < 1. Call a_1 the second intersection of S with the straight line joining a_0 to a'_0; it is rational. Let a'_1 be an integral point closest to a_1; let a_2 be the intersection of S with the line joining a_1 to a'_1; etc. It will now be shown that, for some n, a_n is an integral point.

Put $a_0 = (x, y, z)$; assume that it is not an integral point, and call m the lowest common denominator for x, y, z; write $x = n/m$, $y = p/m$, $z = q/m$; we have

$$Nm^2 = n^2 + p^2 + q^2.$$

Let $a'_0 = (n', p', q')$ be an integral point closest to a_0, and put

$$r = n - mn',\ s = p - mp',\ t = q - mq',$$
$$N' = n'^2 + p'^2 + q'^2,\ M = 2(nn' + pp' + qq').$$

The squared distance between a_0 and a'_0 is then

$$\frac{1}{m^2}(r^2 + s^2 + t^2) = N + N' - \frac{M}{m}$$

which can be written as m'/m, where m' is an integer; as it is < 1, we have $0 < m' < m$, and at the same time:

$$r^2 + s^2 + t^2 = mm',\ M = m(N + N') - m'.$$

Now the line joining a_0 to a'_0 consists of the points

$$(n' + \lambda r, p' + \lambda s, q' + \lambda t);$$

here the point a_1 is given by

$$\begin{aligned} 0 &= (n' + \lambda r)^2 + (p' + \lambda s)^2 + (q' + \lambda t)^2 - N \\ &= mm'\lambda^2 + (M - 2mN')\lambda + N' - N \\ &= (m\lambda - 1)(m'\lambda + N - N'). \end{aligned}$$

The root $\lambda = 1/m$ corresponds to the point a_0, so that the other root $\lambda = (N' - N)/m'$ corresponds to a_1. Thus m' is a common denominator for the coordinates of a_1; as it is $< m$, this shows that the lowest common denominators for the coordinates of a_0, a_1, a_2, etc. make up a decreasing sequence of positive integers, which proves our assertion.

The only property of the quadratic form

$$F(X, Y, Z) = X^2 + Y^2 + Z^2$$

which has been used in this proof is that, to every nonintegral point (x, y, z), there is an integral point (x', y', z') such that

$$0 < |F(x - x', y - y', z - z')| < 1.$$

This applies equally well, for instance, to the forms $X^2 + Y^2$, $X^2 \pm 2Y^2$, $X^2 - 3Y^2$. Now, modifying the notation in an obvious manner, we will show how the same proof can be applied to some quadratic forms $F(x)$ in $\mathbf{R}^n$, taking integral values at all integral vectors x and such that, to every nonintegral vector x in $\mathbf{R}^n$, there is an integral vector x' for which $0 < |F(x - x')| \leqslant 1$; this will be the case for instance for $X^2 + 3Y^2$, $X^2 + Y^2 + 2Z^2$, $X^2 + Y^2 + Z^2 + T^2$. Define the bilinear form $B(x, y)$ by

$$F(\lambda x + \mu y) = \lambda^2 F(x) + \lambda\mu B(x, y) + \mu^2 F(y).$$

As before, take a rational point a_0 such that $F(a_0) = N$, and an integral point a'_0 such that

$$0 < |F(a_0 - a'_0)| \leqslant 1.$$

Let m be the smallest integer such that $n = ma_0$ is an integral point; put $a'_0 = n'$, $r = n - mn'$, $N' = F(n')$, $M = B(n, n')$. We have

$$F(a_0 - a'_0) = F(m^{-1}r) = F(m^{-1}n - n') = N + N' - \frac{M}{m};$$

writing this as m'/m, we have $0 < m' \leqslant m$, and m' is an integer.

The line joining a_0 to a'_0 consists of the points $n' + \lambda r$, and its intersection a_1 with the hypersurface $F(x) = N$ is given by

$$0 = F(n' + \lambda r) - N = mm'\lambda^2 + (M - 2mN')\lambda + N' - N = (m\lambda - 1)(m'\lambda + N - N').$$

As before, a_1 corresponds to the root $\lambda = (N' - N)/m'$, so that $m'a_1$ is an integral point, and m' is a common denominator for the coordinates of a_1; however, we have now to take into account the case $|m'| = m$, i.e. $|F(m^{-1}r)| = 1$. Taking for F one of the three forms listed above, we may assume that we have taken for a'_0 the point, or one of the points, whose coordinates are the integers closest to those of a_0, so that the coordinates of $m^{-1}r$ are $\leqslant \frac{1}{2}$ in absolute value; then, for $F(m^{-1}r)$ to be 1, they must all be $\pm \frac{1}{2}$, and $2a_0$ must be integral, so that $m = 2$. In that case there are 2^n possible choices for a'_0 and this choice can be made so that $N - N'$ is even; since at the same time m' must be 1 or 2, a_1 is then integral.

The above proof would have been easily understood by Euler; perhaps, with a little more effort, it would have been understood by Fermat, whose algebraic skills still fell somewhat short of the required level. That it was discovered so late may serve as an encouragement to those who seek elementary proofs for supposedly sophisticated results.