Pointwise convergence of Fourier series

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In this paper, we present a new proof of a theorem of Carleson and Hunt: The Fourier series of an L^p function on $[0, 2\pi]$ converges almost everywhere $(p > 1)$. (See [1], [5].) Our proof is very much in the spirit of the classical theorem of Kolmogoroff-Seliverstoff-Plessner [8]. Unlike Carleson's proof, which makes a careful analysis of the structure of an L^2 function f, our arguments essentially ignore f, and concentrate instead on building up a basic "partial sum" operator from simpler pieces. Our methods are (almost) entirely L^2.

Sections 1–7 of this paper contain a proof of pointwise convergence for L^2 functions; Section 8 contains the modifications necessary to handle L^p, and includes various further comments.

1. Preliminaries

Let $S_N f$ denote the N^{th} partial sum of the Fourier series of f. We shall prove the basic estimate

$$(1) \qquad ||\sup_N |S_N f(\cdot)|\,||_1 \leqq C\,||f||_2 \text{ for all } f \in L^2[0, 2\pi]\ .$$

Once we know (1), pointwise convergence follows from a standard argument: Given $f \in L^2$ and $\varepsilon > 0$, take $\varphi \in C^\infty[0, 2\pi]$ with $||f - \varphi||_2 < \varepsilon$. Then

$$||\limsup_{M,N\to\infty} |S_M f(\cdot) - S_N f(\cdot)|\,||_1 \leqq ||\limsup_{M,N\to\infty} |S_M\varphi(\cdot) - S_N\varphi(\cdot)|\,||_1 + ||\limsup_{M,N\to\infty} |S_M(f-\varphi)(\cdot) - S_N(f-\varphi)(\cdot)|\,||_1\ .$$

The first term on the right is identically zero, since $\varphi \in C^\infty$. The second term is dominated by $||2\sup_N |S_N(f-\varphi)(\cdot)|\,||_1 \leqq 2C\,||f-\varphi||_2 < 2C\varepsilon$ by (1). Since $\varepsilon > 0$ was arbitrary, $\limsup_{M,N\to\infty} |S_M f(\cdot) - S_N f(\cdot)| = 0$ a.e., and pointwise convergence follows.

So the problem is to prove estimate (1).

For $f \in L^2$ and $x \in [0, 2\pi]$, let $n(x)$ be the least integer k for which $|S_k f(x)| \geqq (1/2)\sup_N |S_N f(\cdot)|$. Estimate (1) is equivalent to

$$(2) \qquad ||S_{n(\cdot)} f(\cdot)||_1 \leqq C\,||f||_2\ .$$

The Dirichlet formula for $S_n f$ shows that

$$S_n f(x) = c\int_{-\pi}^{\pi} \frac{e^{iny}}{y} f(x-y)\,dy - c\int_{-\pi}^{\pi} \frac{e^{-iny}}{y} f(x-y)\,dy + \text{trivial error terms,}$$

the integrals being evaluated in the principal value sense. So to prove (2), it is enough to show that

$$\left\| \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)dy \right\|_1 \leqq C \|f\|_2 \tag{3}$$

with $N(x) = n(x)$ and with $N(x) = -n(x)$.

Now we change our point of view and regard $N(x)$ as a fixed function of x. Thus, $T^0\colon f \to \int_{-\pi}^{\pi} (e^{iN(x)y}/y)f(x-y)dy$ is a linear operator. Our problem is to prove

$$\| T^0 f \|_1 \leqq C \|f\|_2 \tag{4}$$

with C independent of $N(\cdot)$. (If necessary, we may always assume $N(\cdot)$ bounded, as long as our estimates don't depend on the bound.) As we shall see, much more is true.

2. Notation

In this section, we present enough notation to state the main ideas in our proof of (4). Let $\psi^{(0)}(z)$ be an odd C^∞ function, supported in $\{|z| \leqq 2\pi\}$, and define $\psi_j(z) = 2^j\psi^{(0)}(2^j z)$. By making a suitable choice of $\psi^{(0)}$, we can arrange that $1/z = \sum_{j=0}^{\infty} \psi_j(z)$ for $|z| < \pi$.

On $[0, 2\pi]$ we use normalized Lebesgue measure $dx/2\pi$. Fix a dyadic grid in $[0, 2\pi]$ and in R^1. A *pair* $[\omega, I]$ consists of dyadic intervals $\omega \subseteq R^1$ and $I \subseteq [0, 2\pi]$, with $|\omega| = |I|^{-1}$. Denote the set of all pairs by $\mathfrak{P}$. For every $[\omega, I] \in \mathfrak{P}$, $|I| = 2^{-k}$, define a set $E(\omega, I) \subseteq [0, 2\pi]$ and an operator $T_{[\omega,I]}$ on L^2, by setting $E(\omega, I) = \{x \in I \mid N(x) \in \omega\}$ and

$$\begin{aligned} T_{[\omega,I]}f(x) &= \left[\int_{-\pi}^{\pi} e^{iN(x)y}\psi_k(y)f(x-y)dy\right]\cdot\chi_{E(\omega,I)}(x) \\ &= \left[\left(e^{iN(x)y}\psi_k(y)\right) * f(x)\right]\cdot\chi_{E(\omega,I)}(x) \;. \end{aligned}$$

As $[\omega, I]$ runs through $\{[\omega, I] \in \mathfrak{P} \mid |I| = 2^{-k}\}$ for fixed k, the $E(\omega, I)$ form a partition of all of $[0, 2\pi]$. Therefore,

$$\sum_{\substack{[\omega,I]\in\mathfrak{P} \\ |I|=2^{-k}}} T_{[\omega,I]}f(x) = \int_{-\pi}^{\pi} e^{iN(x)y}\psi_k(y)f(x-y)dy \;.$$

Summing over $k \geqq 0$ yields

$$\begin{aligned} \sum_{[\omega,I]\in\mathfrak{P}} T_{[\omega,I]}f(x) &= \int_{-\pi}^{\pi} \left(e^{iN(x)y}\sum_{k=0}^{\infty} \psi_k(y)\right)f(x-y)dy \\ &= \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)dy = T^0 f(x) \;. \end{aligned}$$

In other words, $T^0 = \sum_{p\in\mathfrak{P}} T_p$. This is our basic decomposition.

Let us compute the norm of T_p. If $p = [\omega, I]$, $|I| = 2^{-k}$, then

$$|T_p f(x)| = \left|\int_{-\pi}^{\pi} e^{iN(x)y}\psi_k(y)f(x-y)dy\right| \cdot \chi_{E(p)}(x)$$
$$\leqq \left(\int_{-\pi}^{\pi} |\psi_k(y)| \cdot |f(x-y)|\,dy\right) \cdot \chi_{E(p)}(x) \leqq \left(C \operatorname{av}_{I^*}|f|\right) \cdot \chi_{E(p)}(x)\ ,$$

where I^* is the double of I, and $\operatorname{av}_{I^*}|f| = (1/|I^*|)\int_{I^*}|f(y)|\,dy$. Therefore, $||T_p||_2 \leqq C(|E(p)|/|I|)^{1/2}$, where $||T||_r$ denotes the norm of T on L^r. Actually, it is easy to see that $||T_p||_2 \sim (|E(p)|/|I|)^{1/2}$. We define $A_0(p) = |E(p)|/|I|$.

3. Sketch of proof of the basic estimate (4)

Break $\mathfrak{P}$ up into $\bigcup_{n=0}^{\infty}\mathfrak{p}_n$, where $\mathfrak{p}_n = \{p \in \mathfrak{P} \mid 2^{-n-1} \leqq A_0(p) < 2^{-n}\}$. Thus, $T^0 = \sum_{n=0}^{\infty} T^{\mathfrak{p}_n}$, where $T^{\mathfrak{p}} \equiv \sum_{p\in\mathfrak{p}} T_p$ for any $\mathfrak{p} \subseteq \mathfrak{P}$. The main point of our proof is that, roughly speaking,

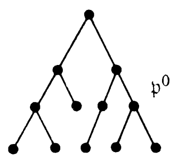
$$||T^{\mathfrak{p}_n}||_2 \sim \max_{p\in\mathfrak{p}_n} ||T_p||_2 \sim 2^{-n/2}\ . \tag{5}$$

Once we know this, we can simply write $||T^0||_2 \leqq \sum_{n=0}^{\infty}||T^{\mathfrak{p}_n}||_2 \sim \sum_{n=0}^{\infty} 2^{-n/2} < \infty$, and (4) follows.

To prove (5), we make a systematic study of subsets $\mathfrak{p} \subseteq \mathfrak{P}$ with the property

$$||T^{\mathfrak{p}}||_2 \sim \max_{p\in\mathfrak{p}} ||T_p||_2\ . \tag{6}$$

We start out with very small, simple subsets, and build up larger and larger $\mathfrak{p}$ satisfying (6), until at last we reach the $\mathfrak{p}_n$. Here is our program: Property (6) is intimately connected with the partial order $<$ on $\mathfrak{P}$, defined by setting $[\omega, I] < [\omega', I']$ if and only if $I \subseteq I'$ and $\omega' \subseteq \omega$. To start off our program, we prove (6) for subsets $\mathfrak{p}^0 \subseteq \mathfrak{P}$ whose structure under $<$ is as in Figure 1.



($\mathfrak{p}^0$ is called a "tree".)

FIGURE 1.

This is not hard, because as it turns out, every $T^{\mathfrak{p}}$ with $\mathfrak{p}$ a tree is essentially of the form $f \to (Hf)\cdot\chi_E$, where H is a (truncated) Hilbert transform, and $E = E(\mathfrak{p})$ is a subset of $[0, 2\pi]$. If $A_0(p)$ is small for every $p \in \mathfrak{p}$, then E will be a very thin set, so that $||T^{\mathfrak{p}}||$ will be small. So (6) holds for trees.

Next we try to put many trees $\mathfrak{p}^1, \mathfrak{p}^2, \cdots$ together, and hope that $\mathfrak{p} = \mathfrak{p}^1 \cup \mathfrak{p}^2 \cup \cdots$ still satisfies (6). Since $T^{\mathfrak{p}} = \sum_i T^{\mathfrak{p}^i} \equiv \sum_i A_i$, (6) amounts to the strong assertion $||\sum_i A_i||_2 \sim \max_i ||A_i||_2$. We use a lemma of Cotlar and Stein.

ORTHOGONALITY LEMMA. *Let $A_1, A_2, \cdots$ be operators on a Hilbert space*

H, all of norm $\leqq M$. Assume that the $\{A_i\}$ are "almost orthogonal", i.e.

(a) $A_i^* A_j = 0$ *for* $i \neq j$.

(b) $\| A_i A_j^* \| \leqq M^2/(|i - j|^2 + j)$.

Then $\|\sum_i A_i\| \leqq 10M$. (See § 4 for the easy proof.)

Roughly speaking, the operators $\{T^{\mathfrak{p}^i}\}$ turn out to be "almost orthogonal" in the sense of (a) and (b), provided that no two pairs $p \in \mathfrak{p}^i$ and $p' \in \mathfrak{p}^j$ $(i \neq j)$ are ever comparable under $<$. Therefore, the orthogonality lemma together with (6) for trees, imply (6) for all subsets $\mathfrak{p} \subseteq \mathfrak{P}$ which look like Figure 2 under $<$.

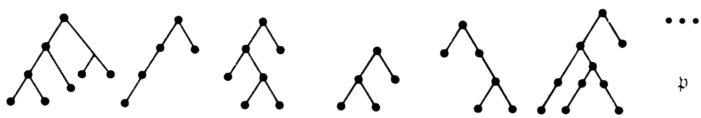


FIGURE 2. ($\mathfrak{p}$ is called a "forest".)

By now, we have built up enough sets with property (6), to prove the basic estimate (4). As above, set $\mathfrak{p}_n = \{p \in \mathfrak{P} \mid 2^{-n-1} < A_0(p) \leqq 2^{-n}\}$, and recall that $T^0 = \sum_{n=0}^{\infty} T^{\mathfrak{p}_n}$. We shall prove by an elementary combinatorial argument, that $\mathfrak{p}_n$ may be written as a disjoint union of at most (roughly) $n + 1$ forests, $\mathfrak{p}_{n0} \cup \mathfrak{p}_{n1} \cup \mathfrak{p}_{n2} \cup \cdots \cup \mathfrak{p}_{nn}$. For each forest $\mathfrak{p}_{nr}$, (6) shows that

$$\| T^{\mathfrak{p}_{nr}} \|_2 \sim \max_{p \in \mathfrak{p}_{nr}} \| T_p \|_2 \sim \max_{p \in \mathfrak{p}_{nr}} (A_0(p))^{1/2} \sim 2^{-n/2} .$$

Therefore

$$\| T^0 \|_2 \leqq \sum_{n=0}^{\infty} \| T^{\mathfrak{p}_n} \|_2 \leqq \sum_{n=0}^{\infty} \sum_{r=0}^{n} \| T^{\mathfrak{p}_{nr}} \|_2 \sim \sum_{n=0}^{\infty} \sum_{r=0}^{n} 2^{-n/2}$$
$$= \sum_{n=0}^{\infty} (n + 1) 2^{-n/2} < \infty ,$$

which completes the proof of the basic estimate (4). This is the plan of our proof.

4. Proof of the orthogonality lemma

Hypothesis (a) means that $(A_i x, A_j y) = (x, A_i^* A_j y) = 0$ for all $x, y \in H$ $(i \neq j)$. In other words, Range $(A_i) \perp$ Range (A_j). Consequently,

$$\sum_i \| \pi_i x \|^2 \leqq \| x \|^2 , \tag{7}$$

where π_i denotes the orthogonal projection from H to $\overline{\text{Range}\,(A_i)}$.

For each i, H splits into the orthogonal direct sum $\overline{\text{Range}\,(A_i)} \oplus \ker A_i^*$, which implies that $A_i^* x = A_i^*(\pi_i x)$ for $x \in H$. Therefore

$$\| \textstyle\sum_i A_i^* x \|^2 = \| \sum_i A_i^*(\pi_i x) \| = \sum_{i,j} \big(A_i^*(\pi_i x), A_j^*(\pi_j x)\big) = \sum_{i,j} \big(\pi_i x, A_i A_j^*(\pi_i x)\big)$$
$$\leqq \sum_{i,j} \frac{M^2}{|i - j|^2 + 1} \| \pi_i x \| \cdot \| \pi_j x \| \leqq 10M^2 (\textstyle\sum_i \| \pi_i x \|^2) \text{ by Cauchy-Schwarz,}$$
$$\leqq 10M^2 \| x \|^2, \text{ by (7).}$$

Hence, $\|\sum_i A_i\| = \|\sum_i A_i^*\| \leqq 10M$. The orthogonality lemma is proved.

5. Unfortunate technicalities

For any dyadic interval $\omega \subseteq R^1$, let $\tilde{\omega}$ be the next larger dyadic interval containing ω and let ω^* be the double of ω. We would like to say that $\omega^* \subseteq \tilde{\tilde{\omega}}$, but in general, not even $\omega^* \subseteq \tilde{\tilde{\vdots\tilde{\omega}}}$ holds. This causes technical problems, which we surmount by making the following definitions: ω is *central* if $\omega^* \subseteq \tilde{\tilde{\omega}}$; $[\omega, I] \in \mathfrak{P}$ is *admissible* if ω is central. Recall that $T^0 = \sum_{p \in \mathfrak{P}} T_p$; define $T \equiv \sum_{p \text{ admissible}} T_p$.

Instead of proving our basic estimate (4) directly, we shall prove

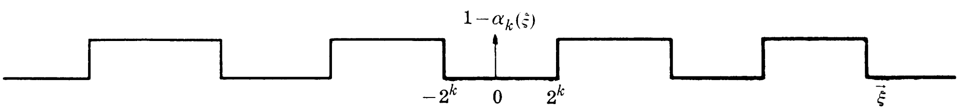
$$\|Tf\|_1 \leqq C\,\|f\|_2\,. \tag{8}$$

In the paragraphs below, we prove that (8) implies (4).

The definitions of "admissible", "central" and "T" all depend on the choice of a dyadic grid $\mathcal{G}_0$, to determine dyadic ω's. By translating $\mathcal{G}_0$ by $\xi \in R^1$, we arrive at a new grid $\mathcal{G}_\xi$, which gives rise to a new modified T^0 — call it T_ξ. Equivalently, T_ξ arises by conjugating T by the isometry $U_\xi\colon f(x) \to e^{i\xi x} f(x)$ and then changing $N(\cdot)$ to $N(\cdot) - \xi$. Since (8) is to hold for all $N(\cdot)$, (8) implies $\|T_\xi\|_1 \leqq C\,\|f\|_2$, with C independent of ξ. Now (4) follows at once from

LEMMA 0. $T^0 f(x) = c \cdot \lim_{M\to\infty} (1/2M) \int_{-M}^{M} T_\xi f(x)\,d\xi$.

Proof of Lemma 0. For $n \in R^1$, set $\alpha_k(n) = 1$ if the dyadic interval ω_k of the grid $\mathcal{G}_\xi$, with $|\omega_k| = 2^k$ and $n \in \omega_k$, is central in $\mathcal{G}_\xi$; $\alpha_k(n, \xi) = 0$ otherwise. Then $\alpha_k(n, \xi) = \alpha_k(n - \xi)$, where α_k is the function graphed below.



Now

$$\sum_{\substack{p=[\omega,I] \text{ admissible} \\ |\omega|=2^k \text{ for } \mathcal{G}_\xi}} T_p f(x) = \alpha_k(N(x), \xi)\cdot\big([e^{iN(x)y}\psi_k(y)] * f(x)\big)\,,$$

so that

$$T_\xi f(x) = \sum_{p \text{ admissible for } \mathcal{G}_\xi} T_p f(x) = \sum_{k=0}^{\infty} \alpha_k(N(x), \xi)\cdot[e^{iN(x)y}\psi_k(y)] * f(x)\,.$$

Hence,

$$\lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} T_\xi f(x)\,d\xi$$
$$= \sum_{k=0}^{\infty}\Big(\lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M}\alpha_k(N(x), \xi)\,d\xi\Big)\cdot[e^{iN(x)y}\psi_k(y)] * f(x)\,.$$

(The interchanges of sums, limits, and integrals are routine.) However,

$$\lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M}\alpha_k(N(x),\,\xi)d\xi = \lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M}\alpha_k(N(x)-\xi)d\xi = \frac{1}{2}\,,$$

as one sees at once from Figure 3. Therefore,

$$\lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M}T_\xi f(x)d\xi = \frac{1}{2}\cdot\sum_{k=0}^{\infty}\left[e^{iN(x)y}\psi_k(y)\right]*f(x) = \frac{1}{2}T^0 f(x)\;.$$

So Lemma 0 holds with $c = 2$. Q.E.D.

Henceforth, "pair" means "admissible pair", and $\mathfrak{P}$ stands for the set of admissible pairs. We mention a convenient property of central dyadic intervals. If $\omega_0 \subsetneqq \omega_1 \subsetneqq \cdots \subsetneqq \omega_{M+1}$ with all the ω's central, then distance $(\xi, \omega_0) < 2^{M-1}|\omega_0|$ implies $\xi\in\omega_{M+1}$. Equivalently, $\xi\notin\omega_{M+1}$ implies distance $(\xi, \omega_0) \geqq 2^{M-1}|\omega_0|$. The easy proof is left to the reader.

6. Main sequence of lemmas

In this section, we build large $\mathfrak{p}$ with property (6). Instead of $A_0(p)$, we shall use

$$A([\omega, I]) = \sup_{\substack{p'=[\omega',I']\\ I\subseteq I'}}\frac{|E(p')|}{|I'|}\cdot\left(\frac{\text{distance}\,(\omega,\omega')+|\omega|}{|\omega|}\right)^{-2000}.$$

LEMMA 1. *Let $\mathfrak{p}$ be a set of pairs, no two of which are comparable under $<$. Suppose that $A(p) < \delta$ for $p\in\mathfrak{p}$, and that distance$(n_0, \omega) \leqq \delta^{-\varepsilon}|\omega|$ for all $[\omega, I]\in\mathfrak{p}$ (n_0 is fixed). Then*

$$\|T\|_r \leqq C_r\delta^{1/r-5000\varepsilon} \qquad (r>1)\;.$$

Proof. Consider a "maximal operator" $f\to Mf$, defined as follows. We are given a partition $\{I_j\}$ of $[0, 2\pi]$ into intervals, and a family of sets $E_j\subseteq I_j$ with the property $|E_j|/|I_j| < 2\delta^{1-5000\varepsilon}$. For $x\in I_j$, define $Mf(x) = \sup_{I_j\subseteq I}(1/|I|)\int_I |F(y)|\,dy$ if $x\in E_j$; $Mf(x) = 0$ if $x\notin E_j$. Then

$$\text{(9)}\qquad \|Mf\|_r \leqq C_r\delta^{(1-5000\varepsilon)/r}\|f\|_r \ \text{ for } f\in L^r\;.$$

To see this, we set $M_0 f(x) = \sup_{I_j\subseteq I}(1/|I|)\int_I |f(y)|\,dy$ for all $x\in I_j$. $\|M_0 f\|_r \leqq C_r\|f\|_r$ by the maximal theorem. On the other hand, $M_0 f$ is constant on each I_j, so that

$$\begin{aligned}\int_0^{2\pi}(Mf(x))^r dx &= \sum_j\int_{E_j}(M_0 f(x))^r dx \leqq 2\delta^{1-5000\varepsilon}\sum_j\int_{I_j}(M_0 f(x))^r dx\\ &= 2\delta^{1-5000\varepsilon}\|M_0 f\|_r^r \leqq C_r\delta^{1-5000\varepsilon}\|f\|_r^r\;.\end{aligned}$$

Equation (9) follows.

To prove Lemma 1, we shall dominate $T^{\mathfrak{p}}f$ pointwise by Mf, for suitable

$\{I_j\}$, $\{E_j\}$, which we construct: Let $I_1, I_2, \cdots$ be the maximal dyadic intervals satisfying

$$\frac{|\{x \in I \mid |N(x) - n_0| \leqq \delta^{-2\varepsilon}|I|^{-1}\}|}{|I|} > \delta^{1-1000\varepsilon} . \tag{10}$$

The $\{I_j\}$ form a partition of $[0, 2\pi]$. For, they are clearly pairwise disjoint; and given any $x \in [0, 2\pi]$, (10) holds for sufficiently small I containing x, so that x belongs to some I_j. Define $\tilde{E}_j = \{x \in \tilde{I}_j \mid |N(x) - n_0| \leqq \delta^{-2\varepsilon}|\tilde{I}_j|^{-1}\}$, and put $E_j = \tilde{E}_j \subsetneqq I_j$. Since (10) does not hold for $\tilde{I}_j$, we have $|E_j| \leqq |\tilde{E}_j| \leqq \delta^{1-10000\varepsilon}|\tilde{I}_j| = 2\delta^{1-10000\varepsilon}|I_j|$. (We may speak of $\tilde{I}_j$, since I_j is not all of $[0, 2\pi]$, as the proof of (α') below shows.) Thus $\{I_j\}$, $\{E_j\}$ are as in (9).

It remains to show that Mf dominates $T^{\mathfrak{p}}f$ pointwise. This amounts to proving

$$|T^{\mathfrak{p}}f(x)| \leqq C\cdot\sup_{I_j \subseteq I} \frac{1}{|I|}\int_I |f(y)|\,dy \quad \text{for } x \in I_j , \tag{α}$$

and

$$T^{\mathfrak{p}}f(x) = 0 \quad \text{for } x \in I_j - E_j . \tag{β}$$

Now, $T^{\mathfrak{p}}f(x) = \sum_{p \in \mathfrak{p}} T_p f(x)$. Since $p \not< p'$ for any $p, p' \in \mathfrak{p}$, it follows that $\{E(p)\}$, the supports of $\{T_p f\}$, are pairwise disjoint. Therefore,

$$|T^{\mathfrak{p}}f(x)| = \max_{\substack{p = [\omega, I] \in \mathfrak{p} \\ x \in E(p)}} |T_p f(x)| \leqq \max_{\substack{p = [\omega, I] \in \mathfrak{p} \\ x \in E(p)}} \left(\frac{C}{|I^*|}\int_{I^*} |f(y)|\,dy\right)$$

by the trivial estimate for T_p (see §2). So (α) and (β) are simple consequences of

$$[\omega, I] \in \mathfrak{p} , \quad I \cap I_j \neq \varnothing \ \text{ imply } \ I_j \subseteq \tilde{I} \tag{α'}$$

$$[\omega, I] \in \mathfrak{p} , \quad I \cap I_j \neq \varnothing \ \text{ imply } \ I_j \cap E(\omega, I) \subseteq E_j . \tag{β'}$$

Proof of (α'). Either $\tilde{I}_j \subseteq I$ or $I \subseteq I_j$. Assume the latter. By (10), at least one of the $\delta^{-2\varepsilon}$ dyadic intervals ω' of length $|I_j|^{-1}$ and distance $\leqq \delta^{-2\varepsilon}|I_j|^{-1}$ from n_0 must satisfy $|E(\omega', I_j)|/|I_j| \geqq \delta^{1-10000\varepsilon}$. Also

$$\frac{\text{distance}\,(\omega, n_0)}{|\omega|} , \ \frac{\text{distance}\,(\omega', n_0)}{|\omega'|} < \delta^{-2\varepsilon} , \qquad |\omega'| < |\omega|$$

imply $\big((\text{distance}\,(\omega, \omega') + |\omega|)/|\omega|\big)^{-2000} > \delta^{4000\varepsilon}$. Therefore $A(\omega, I) \geqq |E([\omega', I'])|/|I'|$.

$$\left(\frac{\text{distance}\,(\omega, \omega') + |\omega|}{|\omega|}\right)^{-2000} \geqq \delta^{1-10000\varepsilon+4000\varepsilon} > \delta ,$$

contradicting $[\omega, I] \in \mathfrak{p}$.

Proof of (β'). $I_j \cap E(\omega, I) = \{x \in I_j \mid N(x) \in \omega\}$ (since $I_j \subseteq I$ by (α')) $\subseteq \{x \in I_j \mid |N(x) - n_0| < \delta^{-2\varepsilon}|\omega|\} \subseteq \{x \in I_j \mid |N(x) - n_0| \leqq \delta^{-2\varepsilon}|\tilde{I}_j|^{-1}\} = E_j$, since we know $|\omega| \leqq |\tilde{I}_j|^{-1}$ from (α'). Q.E.D.

LEMMA 2. *Let $\mathfrak{p}$ be a set of pairs, no two of which are comparable under $<$. Assume that $A(p) \leqq \delta$ for all $p \in \mathfrak{p}$. Then $\|T^{\mathfrak{p}}\|_2 \leqq C_\eta \delta^{1/2-\eta}$ (any $\eta > 0$).*

Proof. Let us compute

$$(11)\qquad \begin{aligned} \int_0^{2\pi} |T^{\mathfrak{p}*}f(x)|^2\,dx &= \sum_{p,p'\in\mathfrak{p}} \int_0^{2\pi} T_p^* f(x)\overline{T_{p'}^* f(x)}dx \\ &= \sum_{\substack{[\omega,I],[\omega',I']\in\mathfrak{p}\\ |I|\leqq|I'|}} \int_0^{2\pi} T_{[\omega,I]}^* f(x)\overline{T_{[\omega',I']}^* f(x)}dx \\ &\quad + \sum_{\substack{[\omega,I],[\omega',I']\in\mathfrak{p}\\ |I|>|I'|}} \int_0^{2\pi} T_{[\omega,I]}^* f(x)\overline{T_{[\omega',I']}^* f(x)}dx\,. \end{aligned}$$

The two terms are similar. Let us look at the first; we want to estimate

$$(12)\qquad \begin{aligned} \sum_{p'=[\omega',I]} \int_0^{2\pi} \overline{T_{[\omega',I']}^* f(x)} \cdot \Big[\sum_{\substack{p=[\omega,I]\\ |I|\leqq|I'|}} T_p^* f(x)\Big]dx \\ = \sum_{p'} \int_0^{2\pi} \overline{T_{p'}^* f(x)} \cdot \Big[\sum_{p\in\mathcal{A}(p')} T_p^* f(x)\Big]dx \\ + \sum_{p'} \int_0^{2\pi} \overline{T_{p'}^* f(x)} \cdot \Big[\sum_{p\in\mathcal{B}(p')} T_p^* f(x)\Big]dx\,, \end{aligned}$$

where

$\mathcal{A}(p') = \{p = [\omega, I] \in \mathfrak{p}\,,\ \ |I| \leqq |I'|\,|\,\text{distance}\,(2, \omega') \leqq \delta^{-\varepsilon/2}|\omega|$ and $I \subseteq I'^*\}$,

$\mathcal{B}(p') = \{p = [\omega, I] \in \mathfrak{p}\,,\ \ |I| \leqq |I'|\,|\,\text{distance}\,(\omega, \omega') > \delta^{-\varepsilon/2}|\omega|$ or $I \not\subseteq I'^*\}$.

To estimate the $\mathcal{A}(p')$ term in (12), we use the trivial estimate $|T_{p'}^* f(x)| \leqq C\cdot\Big((1/|I'^*|)\int_{E(p')} |f(y)|\,dy\Big)\cdot\chi_{I'^*}(x)$. Thus,

$$\begin{aligned} \Big|\int_0^{2\pi} T_{p'}^* f(x)\Big[\sum_{p\in\mathcal{A}(p')} T_p^* f(x)\Big]\,dx\Big| \\ \leqq C\cdot\Big(\frac{1}{|I'^*|}\int_{E(p')} |f(y)|\,dy\Big)\cdot\int_{I'^*} \Big|\sum_{p\in\mathcal{A}(p')} T_p^* f(x)\Big|\,dx \\ \leqq C\Big(\int_{E(p')} |f(y)|\,dy\Big)\cdot\Big(\frac{1}{|I'^*|}\int_{I'^*} \Big|\sum_{p\in\mathcal{A}(p')} T_p^* f(x)\Big|^r\,dx\Big)^{1/r} \\ \leqq C_r\delta^{1/r-5000\varepsilon}\Big(\int_{E(p')} |f(y)|\,dy\Big)\cdot\Big(\frac{1}{|I'^*|}\int_{I'^*} |f(x)|^r\,dx\Big)^{1/r} \end{aligned}$$

$(r > 1)$, because $\mathcal{A}(p')$ satisfies the hypotheses of Lemma 1. Moreover, the right-hand side of this inequality is at most $C_r\delta^{1/r-5000\varepsilon}\int_{E(p')} |f(y)|\,f_r^*(y)dy$, simply because $\Big((1/|I'^*|)\int_{I'^*} |f(x)|^r\,dx\Big)^{1/r} \leqq f_r^*(y)$ for $y \in E(p') \subseteq I'$. (Here f_r^* denotes the maximal function $f_r^*(x) = \sup_{x\in I}\Big((1/|I|)\int_I |f(y)|^r dy\Big)^{1/r}$.) So

$$(13)\qquad \left|\int_0^{2\pi} T^*_{p'}f(x)\left[\sum_{p\in\mathfrak{C}(p')} T^*_p f(x)\right]dx\right| \leqq C_{r\varepsilon}\delta^{1/r-5000\varepsilon}\int_{E(p')}|f(y)|\cdot[f^*_r(y)]dy \qquad (r>1)\,.$$

This estimates the first term in (12).

To estimate the second term, we write

$$(14)\qquad \left|\int_0^{2\pi}\overline{T^*_{p'}f(x)}\cdot\left[\sum_{p\in\mathfrak{B}(p')} T^*_p f(x)\right]dx\right| = \left|\int_{E(p')}\overline{f(x)}\cdot\left[\sum_{p\in\mathfrak{B}(p')} T_{p'}T^*_p f(x)\right]dx\right|\,.$$

Trivial estimates show that $|T_{p'}T^*_p f(x)| \leqq (\delta^{10}/|I'^*|)\int_{E(p)}|f(y)|dy$ if distance $(\omega, \omega') > \delta^{-\varepsilon/2}|\omega|$, $T_{p'}T^*_p f = 0$ if $I \not\subseteq I'^*$. (Recall: We're assuming $|I| \leqq |I'|$.) Since also the $E(p)$'s are pairwise disjoint, we can continue (14),

$$\leqq \int_{E(p')}|f(x)|\cdot\left(\frac{C\delta^{10}}{|I'^*|}\int_{\bigcup\{E(p):\,p\in\mathfrak{B}(p'),I\subseteq I'^*\}}|f(y)|\,dy\right)dx$$

$$\leqq C\delta^{10}\int_{E(p')}|f(x)|\cdot\left(\frac{1}{|I'^*|}\int_{I'^*}|f(y)|\,dy\right)dx \leqq C\delta^{10}\int_{E(p')}|f(x)|\cdot f^*(x)dx\,.$$

Putting this and (13) into (12), we find that

$$\left|\int_0^{2\pi}\overline{T^*_{p'}f(x)}\cdot\left[\sum_{\substack{p=[\omega,I]\in\mathfrak{p}\\|I|\leqq|I'|}} T^*_p f(x)\right]dx\right| \leqq C_{r\varepsilon}\delta^{1/r-5000\varepsilon}\int_{E(p')}|f(x)|\cdot f^*_r(x)dx \qquad (r>1)\,.$$

Now we sum over $p'\in\mathfrak{p}$, remembering that the $E(p')$ are pairwise disjoint, to obtain

$$|\text{FIRST TERM IN (11)}| \leqq C_{r\varepsilon}\delta^{1/r-\varepsilon}\int_0^{2\pi}|f(x)|\,f^*_r(x)dx \leqq C_{r\varepsilon}\delta^{1/r-5000\varepsilon}\,||f||_2^2$$

by the maximal theorem and Cauchy-Schwarz, $(1 < r < 2)$. The second term in (11) is exactly analogous to the first (we simply switch the roles of I and I'). So

$$|\text{SECOND TERM IN (11)}| \leqq C_{r\varepsilon}\delta^{1/r-5000\varepsilon}\,||f||_2^2$$

also. Consequently, (11) implies that $||T^{\mathfrak{p}*}f||_2^2 \leqq C_{r\varepsilon}\delta^{1/r-5000\varepsilon}\,||f||_2^2$ $(1 < r < 2$, $\varepsilon > 0)$. Lemma 2 follows if we take $\varepsilon > 0$ small and r close to i. Q.E.D.

Lemma 2 will handle some technical error terms in what follows.

We begin the program outlined in the Sketch of Proof. A *tree* $\mathfrak{p}$ with *top* $p^0 = [\omega^0, I^0]$ is simply a set of pairs with the properties: (a) $p < p' < p''$, p' admissible, and $p, p''\in\mathfrak{p}$, imply $p'\in\mathfrak{p}$. (b) $p < p^0$ for all $p\in\mathfrak{p}$. This is slightly more general than the notion of "tree" in §3.

LEMMA 3. *Let* $\mathfrak{p}$ *be a tree with* top $p^0 = [\omega^0, I^0]$, *and suppose that* $A(p) \leqq \delta$ *for all* $p\in\mathfrak{p}$. *Then* $||T^{\mathfrak{p}}||_2 \leqq C\delta^{1/2}$.

Proof. Our first step is to decode the notation and see what $T^{\mathfrak{p}}$ really looks like. Claim:

$$T^{\mathfrak{p}}f(x) = \sum_{\substack{K_0(x) \leqq k \leqq K_1(x) \\ k \in J}} \left(e^{iN(x)y}\psi_k(y)\right) * f(x)$$

$\left(\text{or } Tf(x) = 0, \text{ depending on } x\right)$ where $K_0(\cdot)$ and $K_1(\cdot)$ are fixed functions of x, and J is a set of positive integers. *Proof*: Pick a point $\xi_0 \in \omega^0$. For $I \subseteq [0, 2\pi]$, $|I| = 2^{-k}$, set $\omega_I = \omega(k) =$ the dyadic interval of length 2^k containing ξ_0. Then $\mathfrak{p}$ consists entirely of pairs $[\omega_I, I]$, because $[\omega, I] \in \mathfrak{p}$ implies $[\omega, I] < [\omega^0, I^0]$ so that $\xi_0 \in \omega^0 \subseteq \omega$. Let $J = \{k \geqq 0 \mid \omega(k) \text{ is central}\}$.

Now

$$Tf(x) = \sum_{p \in \mathfrak{p}} T_p f(x) = \sum_{\substack{p \in [\omega, I] \in \mathfrak{p} \\ x \in E(p) \\ |I| = 2^{-k}}} \left(e^{iN(x)y}\psi_k(y)\right) * f(x)$$

$$= \sum_{k \in A(x) \cap B(x)} \left(e^{iN(x)y}\psi_k(y)\right) * f(x) ,$$

where $A(x) = \{k \geqq 0 \mid [\omega_I, I] \in \mathfrak{p}$ with $x \in I$ and $|I| = 2^{-k}\}$ and $B(x) = \{k \geqq 0 \mid N(x) \in \omega(k)\}$. For fixed x, the pairs $[\omega_I, I]$ with $x \in I$ are totally ordered under $<$. So by (a) above, $A(x) = \{k \in J \mid K(x) \leqq k \leqq K'(x)\}$ or $\varnothing$. Since $B(x)$ clearly has the form $\{k \geqq K''(x)\}$, it follows that $A(x) \cap B(x)$ has the form $\{k \in J \mid K_0(x) \leqq k \leqq K_1(x)\}$ or $\varnothing$, and the claim is proved.

As in Lemma 1, we shall work with a partition of I^0: Let $I_1, I_2, \cdots$ be the maximal dyadic subintervals of I^0 for which $|E(\omega_I, I)|/|I| > \delta$. Set $\widetilde{E}_j = E(\omega_{\widetilde{I}_j}, \widetilde{I}_j)$ and $E_j = \widetilde{E}_j \cap I_j$. As in Lemma 1, $\{I_j\}$ is a non-trivial partition of I^0, $|E_j|/|I_j| \leqq C\delta$, and if $[\omega, I] \in \mathfrak{p}$, $I \cap I_j \neq \varnothing$, then $\widetilde{I}_j \subseteq I$ and $E(p) \cap I_j \subseteq E_j$.

It is convenient to assume $\xi_0 = 0$. This is no loss of generality, since we may simply conjugate $T^{\mathfrak{p}}$ by $U_{\xi_0}: f(x) \rightarrow e^{i\xi_0 x}f(x)$ as in Lemma 0. Alternatively, the reader is invited to supply factors of $e^{i\xi_0 x}$ to adapt the equations below to the case $\xi_0 \neq 0$.

Now we can analyze $T^{\mathfrak{p}}$. First, we write $T^{\mathfrak{p}}$ as a "constant coefficient" operator plus an error:

$$(15) \quad T^{\mathfrak{p}}f(x)\Big(\sum_{\substack{K_0(x) \leqq k \leqq K_1(x) \\ k \in J}} \psi_k\Big) * f(x) + \Big[\sum_{K_0(x) \leqq k \leqq K_1(x)} \left(e^{iN(x)y}\psi_k(y) - \psi_k(y)\right)\Big] * f(x)$$

as follows from the claim. The error term is small. For, by definition of $K_0(x)$, $x \in E(\omega_{I_x}, I_x)$ for some $[\omega_{I_x}, I_x] \in \mathfrak{p}$ with $|I_x| = 2^{-K_0(x)}$. In particular, $N(x) \in \omega_{I_x}$ and $0 = \xi_0 \in \omega_{I_x}$, so that $|N(x)| \leqq |\omega_{I_x}| = 2^{K_0(x)}$. So

$$\Big|\sum_{\substack{K_0(x) \leqq k \leqq K_1(x) \\ k \in J}} \left(e^{iN(x)y}\psi_k(y) - \psi_k(y)\right)\Big| \leqq |N(x)y| \sum_{k \geqq K_0(x)} |\psi_k(y)|$$

$$\leqq C \cdot 2^{K_0(x)} \chi_{[-2^{-K_0(x)}, 2^{-K_0(x)}]}(y)$$

which means that the second term in (15) is dominated by

$$C \cdot 2^{K_0(x)} \int_{|z| \leqq 2^{-K_0(x)}} |f(x - z)|\, dz \leqq \frac{C}{|I_x^*|} \int_{I_x^*} |f(y)|\, dy .$$

On the other hand, if $x \in I_j$, say, then we know that $I_j \subseteq I_x$, since $[\omega_{I_x}, I_x] \in \mathfrak{p}$ and $x \in I_x \cap I_j$. Therefore, $(1/|I_x^*|)\int_{I_x^*} |f(y)|\, dy \leqq M_0(f)(x)$, as defined in Lemma 1, so that $|$ERROR TERM IN (15)$| \leqq CM_0(f)(x)$ pointwise.

We turn to the "constant coefficient" term in (15). That term is a slight variant of the maximal Hilbert transform, and we treat it accordingly. Let $R(y) = \sum_{k \in J} \psi_k(y)$. Elementary calculations show that

$$\left|\sum_{\substack{k \leqq K \\ k \in J}} \psi_k(y) - 2^{K-1}\int_{-2^{-K}}^{2^{-K}} R(y+z)\, dz\right| \leqq C\cdot\frac{2^{-K}}{\left(|y|^2 + 2^{-2K}\right)}$$
$$\left(\text{any } K \geqq 0\ ,\quad y \in [0, 2\pi]\right)\,,$$

which implies

$$\left|\Big(\sum_{\substack{k \leqq K \\ k \in J}} \psi_k\Big) * f(x) - 2^{K-1}\int_{-2^{-K}}^{2^{-K}} R * f(x+z) dz\right|$$
$$\leqq C \cdot \sup_{h > 2^{-K}} \left(\frac{1}{2h}\int_{-h}^{h} |f(x+y)|\, dy\right)\,,$$

for all $f \in L^2$. Therefore,

$$\Big|\Big(\sum_{\substack{k \leqq K \\ k \in J}} \psi_k\Big) * f(x)\Big| \leqq C\cdot\Big[\sup_{h > 2^{-K}}\left(\frac{1}{2h}\int_{-h}^{h} |R * f(x+z)|\, dz\right)$$
$$+ \sup_{h > 2^{-K}}\left(\frac{1}{2h}\int_{-h}^{h} |f(x+y)|\, dy\right)\Big]$$

for all $f \in L^2$ and $K \geqq 0$. We apply this inequality both to $K = K_1(x)$ and to $K = K_0(x) - 1$, to obtain

$$\tag{16} \begin{aligned} \Big|\Big(\sum_{\substack{K_0(x) \leqq k \leqq K_1(x) \\ k \in J}} \psi_k\Big) * f(x)\Big| \geqq C\cdot\Big[&\sup_{k > 2^{-K_1(x)}}\left(\frac{1}{2h}\int_{-h}^{h} |R * f(x+z)|\, dz\right) \\ &+ \sup_{h > 2^{-K_1(x)}}\left(\frac{1}{2h}\int_{-h}^{h} |f(x+y)|\, dy\right)\Big]\,. \end{aligned}$$

Let $x \in I_j$. By definition, $K_1(x)$ corresponds to a pair $[\omega_{I(x)}, I(x)] \in \mathfrak{p}$ with $x \in E(\omega_{I(x)}, I(x))$ and $|I(x)| = 2^{-K_1(x)}$. Since $x \in I(x) \cap I_j$, we must have $I_j \subseteq I(x)$, so that $|I_j| \leqq |I(x)| = 2^{-K_1(x)}$. Therefore, the right-hand side of (16) is dominated by

$$C\cdot\left(\sup_{I_j \subseteq I} \frac{1}{|I|}\int_I |f(y)|\, dy + \sup_{I_j \subseteq I}\frac{1}{|I|}\int_I |R * f(y)|\, dy\right)$$
$$\equiv CM_0(f)(x) + CM_0(R * f)(x)\,.$$

This means that $|$CONSTANT COEFFICIENT TERM IN (15)$| \leqq CM_0(f) + CM_0(R * f)$ pointwise.

Combining our estimates for the two right-hand terms in (15), we have $|Tf(x)| \leqq CM_0(f)(x) + CM_0(R * f)(x)$ for all $x \in [0, 2\pi]$. On the other hand,

$T^{\mathfrak{p}}f$ is supported entirely in $\bigcup_{p\in\mathfrak{p}} E(p) = \bigcup_j \big(\bigcup_{p\in\mathfrak{p}} E(p) \cap I_j\big) \subseteq \bigcup_j E_j$. So in fact $|T^{\mathfrak{p}}f(x)| \leqq CM(f)(x) + CM(R*f)(x)$ for all x, with M defined as in Lemma 1. Consequently, $\|T^{\mathfrak{p}}f\|_2 \leqq C\|Mf\|_2 + C\|M(R*f)\|_2 \leqq C\delta^{1/2}\big(\|f\|_2 + \|R*f\|_2\big) \leqq C\delta^{1/2}\|f\|_2$ since $\hat{R} \in L^\infty$. Q.E.D.

We remark that trivial changes have to be made in the above if $\delta = 1$. Actually $\delta = 1$ is by far the easiest case of Lemma 3, and is left to the reader.

The next step is to study the almost orthogonality of $T^{\mathfrak{p}}$'s, for trees $\mathfrak{p}$. This is the most technical part of our proof.

Fix numbers $\delta > 0$ and $K > 10$. A tree $\mathfrak{p}$ with top $p^0 = [w^0, I^0]$ is *normal* if for $[\omega, I] \in \mathfrak{p}$ we have $|I| < (\delta^{1000}/K)|I^0|$, distance $(I, \partial I_0) > 2(\delta^{100}/K)$. ($\partial I_0$ is the boundary of I_0.) One advantage of normal trees $\mathfrak{p}$ is that $T^{\mathfrak{p}*}f$ lives entirely in I^0; in fact in $\{x \in I^0 \mid \text{distance}\,(x, \partial I^0) > \delta^{100}/K\}$.

Two trees, $\mathfrak{p}$ with top $[\omega^0, I^0]$, and $\mathfrak{p}'$ with top $[\omega^1, I^1]$ are *separated* if either $I^0 \cap I^1 = \varnothing$, or else

(α) $[\omega, I] \in \mathfrak{p}$, $I \subseteq I^1$ imply distance $(\omega, \omega^1) > \delta^{-1}|\omega|$,

and

(β) $[\omega', I'] \in \mathfrak{p}'$, $I' \subseteq I^0$ imply distance $(\omega', \omega^0) > \delta^{-1}|\omega'|$.

This is somewhat stronger than saying that no two $p \in \mathfrak{p}$, $p' \in \mathfrak{p}'$ are comparable under $<$.

LEMMA 4. *Let $\mathfrak{p}$ with top $[\omega^0, I^0]$ and $\mathfrak{p}'$ with top $[\omega^1, I^0]$ be separated trees. Then $\|T^{\mathfrak{p}'}T^{\mathfrak{p}*}\|_2 \leqq C_M\delta^M$ (any $M > 0$). Equivalently, $|(T^{\mathfrak{p}*}f, T^{\mathfrak{p}'*}g)| \leqq C_M\delta^M\|f\|_2\|g\|_2$ for any f, $g \in L^2$.*

Proof. Roughly speaking, $\widehat{T^*_{[\omega,I]}f}$ is concentrated near ω. (See the definition of T_p in §2.) So if $\mathfrak{p}$ and $\mathfrak{p}'$ are separated, $\widehat{T^{\mathfrak{p}*}f}$ and $\widehat{T^{\mathfrak{p}'*}g}$ should, in effect, live on disjoint sets. We express this precisely by showing that $T^{\mathfrak{p}*}f = \varphi * (T^{\mathfrak{p}*}f) + \mathcal{E}(f)$, $T^{\mathfrak{p}'*}g = \varphi' * (T^{\mathfrak{p}'*}g) + \mathcal{E}'(g)$, where

$$\|\mathcal{E}\|_2 \leqq C_M\delta^M , \tag{17}$$

$$\|\mathcal{E}'\|_2 \leqq C_M\delta^M , \tag{18}$$

and $\hat{\varphi}$, $\hat{\varphi}'$ are "bump" functions which live far apart. More specifically, let $d = \min\{|I| \mid [\omega, I] \in \mathfrak{p}\}$ and $d' = \min\{|I'| \mid [\omega', I'] \in \mathfrak{p}'\}$. Pick a C_0^∞ function φ on R^1 satisfying

(i) φ is supported in $\{|x| \leqq \delta^{1/2}d\}$, $\|\varphi\|_1 \leqq C_M$.

(ii) $|\hat{\varphi}(\xi)| \leqq C_M\big(\delta^{1/2}d\,|\xi - \xi_0|\big)^{-2M}$ for all ξ with $|\xi - \xi_0| > \delta^{-1/2}d^{-1}$, ($\xi_0$ = midpoint of ω^0).

(iii) $|\hat{\varphi}(\xi) - 1| \leqq C_M\big(\delta^{1/2}d\,|\xi - \xi_0|\big)^{2M}$ for all ξ with $|\xi - \xi_0| < \delta^{-1/2}d^{-1}$.

Similarly, pick a φ' corresponding to $\mathfrak{p}'$. Note that $\|\hat{\varphi}\hat{\varphi}'\|_\infty \leqq C_M \delta^M$, by (ii) and its analogue for φ', and the fact that $\mathfrak{p}$, $\mathfrak{p}'$ are separated.

Now define $\mathcal{E}(f) = T^{\mathfrak{p}*}f - \varphi*(T^{\mathfrak{p}*}f)$ and $\mathcal{E}'(g) = T^{\mathfrak{p}'*}g - \varphi*(T^{\mathfrak{p}'*}g)$. To prove (17), we verify the dual statement

$$\|T^{\mathfrak{p}}f - T^{\mathfrak{p}}(\varphi*f)\|_2 \leqq C_M\delta^M \|f\|_2 \; . \tag{19}$$

We have

$$\begin{aligned} &|T^{\mathfrak{p}}f(x) - T^{\mathfrak{p}}(\varphi*f)(x)| \\ &= \left|\sum\nolimits_{\text{certain } k} \left(e^{iN(x)y}\psi_k(y)\right)*f(x) - \sum\nolimits_{\text{certain } k} \left(e^{iN(x)y}\psi_k(y)\right)*(\varphi*f)(x)\right| \\ &\leqq \left(\sum\nolimits_{\text{certain } k} \left|e^{iN(x)y}\psi_k(y) - \left(e^{iN(x)y}\psi_k(y)\right)*\varphi(y)\right|\right)*|f|(x) \; . \end{aligned} \tag{20}$$

The k's arising in this sum correspond to $[\omega, I] \in \mathfrak{p}$ with $|I| = 2^{-k}$ and $x \in E(\omega, I)$. Now $|I| \geqq d$ by definition of d; also $N(x) \in \omega$ and $\xi_0 \in \omega^0 \subseteq \omega$, so that $|N(x) - \xi_0| \leqq |\omega| = 2^k \leqq d^{-1}$. Since $\hat{\varphi}(\xi)$ is so close to 1 for all ξ reasonably near ξ_0 (iii), it follows that

$$\left|e^{iN(x)y}\psi_k(y) - \left(e^{iN(x)y}\psi_k(y)\right)*\varphi(y)\right| \leqq C_M(\delta^{1/2}\cdot 2^k d)^{2M}\cdot 2^k\chi_{[-2^{-k},2^{-k}]}(y) \; .$$

Therefore

$$\sup_x \left(\sum\nolimits_{\text{certain } k} \left|e^{iN(x)y}\psi_k(y) - \left(e^{iN(x)y}\psi_k(y)\right)*\varphi(y)\right|\right) \leqq \mathcal{R}(y) \; ,$$

where

$$\mathcal{R}(y) = \sum\nolimits_{2^k \leqq d^{-1}} C_M(\delta^{1/2}\cdot 2^k d)^{2M}\cdot 2^k\chi_{[-2^{-k},2^{-k}]}(y) \; .$$

Since $\|\mathcal{R}\|_1 \leqq C_M\delta^M$, the right-hand side of (20) has L^2-norm at most $C_M\delta^M\|f\|_2$, which proves (19) and so also (17). Inequality (18) is proved similarly.

From (17) and (18), Lemma 4 is trivial. We write

$$\begin{aligned} (T^{\mathfrak{p}*}f, T^{\mathfrak{p}*}g) = \left(\varphi*(T^{\mathfrak{p}*}f), \varphi'*(T^{\mathfrak{p}'*}g)\right) + \left(\mathcal{E}(f), \varphi'*(T^{\mathfrak{p}'*}g)\right) \\ + \left(\varphi*(T^{\mathfrak{p}*}f), \mathcal{E}'(g)\right) + \left(\mathcal{E}(f), \mathcal{E}'(g)\right) \; . \end{aligned}$$

The last three terms are smaller than $C_M\delta^M \|f\|_2 \|g\|_2$ by (17), (18), and Lemma 3 (with $\delta = 1$, to show that $\|T^{\mathfrak{p}*}\|_2, \|T^{\mathfrak{p}'*}\|_2 \leqq C$).

The "main" term is

$$\begin{aligned} \left(\varphi*(T^{\mathfrak{p}*}f), \varphi'*(T^{\mathfrak{p}'*}g)\right) = \left((\varphi'*\varphi)*(T^{\mathfrak{p}*}f), (T^{\mathfrak{p}'*}g)\right) \\ \leqq \|\widehat{\varphi'*\varphi}\|_\infty \|T^{\mathfrak{p}*}f\|_2 \|T^{\mathfrak{p}'*}g\|_2 \leqq C_M\delta^M \|f\|_2 \|g\|_2 \; . \end{aligned}$$

Thus $|(T^{\mathfrak{p}*}f, T^{\mathfrak{p}'*}g)| \leqq C_M\delta^M \|f\|_2 \|g\|_2$. Q.E.D.

A *row* is a union $\mathfrak{p} = \mathfrak{p}^1 \cup \mathfrak{p}^2 \cup \cdots$ of normal trees $\mathfrak{p}^j$ with tops $[\omega_0^j, I_0^j]$, where the $\{I^j\}$ are pairwise disjoint.

LEMMA 5. *Let $\mathfrak{p}$ be a row as above, let $\mathfrak{p}'$ be a tree with top $[\omega_0', I_0']$, and* *suppose for each k that $I_0^k \subseteq I_0'$ and $\mathfrak{p}^k$, $\mathfrak{p}'$ are separated. Then* $\|T^{\mathfrak{p}'}T^{\mathfrak{p}*}\|_2 \leqq C_M\delta^M$ (*any* $M > 0$).

Proof. The conclusion is equivalent to $|\sum_k (T^{\mathfrak{p}'*}g, T^{\mathfrak{p}^k*}f)| \leqq C_M\delta^M \|f\|_2 \|g\|_2$. We begin by examining a single term $(T^{\mathfrak{p}'*}g, T^{\mathfrak{p}^k*}f)$. Break up $\mathfrak{p}'$ into $\mathfrak{p}_k' \cup \mathfrak{p}_k'' \cup \mathfrak{p}_k'''$, where $\mathfrak{p}_k'' = \{[\omega', I'] \in \mathfrak{p}' \mid |I'| > (\delta^{1000}/10K)\,|I_0^k|\}$, $\mathfrak{p}_k' = \{[\omega', I'] \in \mathfrak{p}' \mid |I'| \leqq (\delta^{1000}/10K)\,|I_0^k|$, $I' \subseteq I_0^k$, and distance$(I', \partial I_0^k) > (\delta^{100}/K)\,|I_0^k|\}$, $\mathfrak{p}_k''' = \{$all other $p' \in \mathfrak{p}'\}$. Now $(T^{\mathfrak{p}_k'''*}g, T^{\mathfrak{p}_k'*}f) = 0$, since the two factors have disjoint supports. $\mathfrak{p}_k'$ and $\mathfrak{p}_k''$ are trees with top $[\omega_0', I_0']$. $\mathfrak{p}_k'$ is also a normal tree with top $[\omega_k', I_0^k]$, for an ω_k' containing ω_0'.

We have to estimate

$$(T^{\mathfrak{p}'*}g,\, T^{\mathfrak{p}^k*}f) = (T^{\mathfrak{p}_k'*}g,\, T^{\mathfrak{p}^k*}f) + (T^{\mathfrak{p}_k''*}g,\, T^{\mathfrak{p}^k*}f)\ . \tag{21}$$

Lemma 4 applies to the first term, since $\mathfrak{p}_k'$ and $\mathfrak{p}^k$ both have top [something, I_0^k]. So that term is at most $C_M\delta^M \|f\|_{L^2(I_0^k)} \|g\|_{L^2(I_0^k)}$. To estimate the second term, we construct φ_k and $\mathcal{E}_k$ as in the proof of Lemma 4, using $\mathfrak{p}^k$ for "$\mathfrak{p}$". In addition to (i)–(iii) above, we can assume that

(iv) $\varphi_k(\xi_0') = 0$, where ξ_0' is the midpoint of ω_0'.

Thus, the second term in (21) is equal to

$$\begin{aligned}\big(T^{\mathfrak{p}_k''*}g,\, \varphi_k * (T^{\mathfrak{p}^k*}f)\big) + \big(T^{\mathfrak{p}_k''*}g,\, \mathcal{E}_k(f)\big) &= \big(g,\, T^{\mathfrak{p}_k''}(\varphi_k * T^{\mathfrak{p}*}f)\big)\\ +\big(T^{\mathfrak{p}'*},\, \mathcal{E}_k(f)\big) - \big(T^{\mathfrak{p}_k'*}g,\, \mathcal{E}_k(f)\big) &\equiv A + B - C\ .\end{aligned} \tag{22}$$

(Note that $\big(T^{\mathfrak{p}_k'''*}g, \mathcal{E}_k(f)\big) = 0$ since the factors have disjoint supports.) B and C are the easy terms. We have $|B| \leqq C_M\delta^M \|T^{\mathfrak{p}'*}g\|_{L^2(I_0^k)} \|f\|_{L^2(I_0^k)}$ by (17), and

$$|C| \leqq C_M\delta^M \|T^{\mathfrak{p}_k'*}g\|_{L^2(I_0^k)} \|f\|_{L^2(I_0^k)} \leqq C_M\delta^M \|g\|_{L^2(I_0^k)} \|f\|_{L^2(I_0^k)}\ .$$

Only the main term A in (22) remains. We shall prove that

$$|T^{\mathfrak{p}_k''}(\varphi_k * F)(x)| \leqq C_M\delta^M \Phi_k * |F|\,(x)\ , \tag{23}$$

where $\Phi_k(x) = |I_0^k|\,\delta^{1000}/(|x| + |I_0^k|\,\delta^{1000})^2$. Assuming this for the moment, we put $F = T^{\mathfrak{p}^k*}f$ to obtain $|A| \leqq (|g|,\, C_M\delta^M\Phi_k * |T^{\mathfrak{p}^k*}f|) \leqq C_M\delta^M(\Phi_k * |g|,\, |T^{\mathfrak{p}^k*}f|) \leqq C_M\delta^M(g^*,\, |T^{\mathfrak{p}^k*}f|) \leqq C_M\delta^M \|g^*\|_{L^2(I_0^k)} \|f\|_{L^2(I_0^k)}$. Together, our estimates for A, B, and C yield

$$|(T^{\mathfrak{p}'*}g,\, T^{\mathfrak{p}^k*}f)| \leqq C_M\delta^M \|f\|_{L^2(I_0^k)}\big(\|g^*\|_{L^2(I_0^k)} + \|T^{\mathfrak{p}'*}g\|_{L^2(I_0^k)}\big)\ .$$

If we sum over k, we find that

$$\begin{aligned}|(T^{\mathfrak{p}'*}g,\, T^{\mathfrak{p}*}f)| &\leqq C_M\delta^M \textstyle\sum_k \|f\|_{L^2(I_0^k)}\big(\|g^*\|_{L^2(I_0^k)} + \|T^{\mathfrak{p}'*}g\|_{L^2(I_0^k)}\big)\\ &\leqq C_M\delta^M\big(\textstyle\sum_k \|f\|^2_{L^2(I_0^k)}\big)^{1/2}\cdot\big[\big(\textstyle\sum_k \|g^*\|^2_{L^2(I_0^k)}\big)^{1/2} + \big(\textstyle\sum_k \|T^{\mathfrak{p}'*}g\|^2_{L^2(I_0^k)}\big)^{1/2}\big]\\ &\leqq C_M\delta^M \|f\|_2(\|g^*\|_2 + \|T^{\mathfrak{p}'*}g\|_2 \leqq C_M\delta^M \|f\|_2 \|g\|_2\ ,\end{aligned}$$

by the maximal theorem and Lemma 3 with $\delta = 1$. This is obviously equivalent to the conclusion of Lemma 5.

Let us return to (23), which we left open in the above.

$$(24) \qquad |T^{\mathfrak{p}''_k}(\varphi_k * F)(x)| = \left|\sum\nolimits_{\text{certain } j} \left(e^{iN(x)y}\psi_j(y)\right) * (\varphi_k * F)(x)\right| .$$

The j's arising in this sum correspond to $[\omega', I'] \in \mathfrak{p}''_k$ with $x \in E(\omega', I')$ and $|I'| = 2^{-j} \geqq (\delta^{1000}/10K)\,|I^k_0|$, by definition of $\mathfrak{p}''_k$. As before, $N(x) \in \omega'$, $\xi'_0 \in \omega'_0 \subseteq \omega'$, so that $|N(x) - \xi'_0| \leqq |\omega'| = 2^j = |I'|^{-1} \leqq 10K\delta^{-1000}\,|I^k_0|^{-1}$. By property (iv) of φ_k and the fact that

$$\left|\frac{\partial}{\partial\xi}\varphi_k(\xi'_0)\right| \leqq C_M(\delta^{1/2}d_k)\cdot\left(|\xi'_0 - \xi^0_k|\right)^{-2M} \leqq C_M\delta^M\frac{|I^k_0|}{K} ,$$

we have

$$\left|\left(e^{iN(x)y}\psi_j(y)\right) * \varphi_k(z)\right| \leqq \left(\frac{2^j}{\delta^{-1000}\,|I^k_0|^{-1}}\right)\cdot C_M\delta^{M-1000}\Psi_j(z) ,$$

where $\Psi_j(z) = 2^j\chi_{[-1,1]}(2^jz)$. Summing over j yields

$$\left|\sum\nolimits_{\text{certain } j}\left(e^{iN(x)y}\psi_j(y)\right) * \varphi_k(z)\right| \leqq C_M\delta^M\Phi_k(z) .$$

If we substitute this in (24), then (23) follows at once. Q.E.D.

COROLLARY. *Let $\mathfrak{p} = \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots$ and $\mathfrak{p}' = \mathfrak{p}'_1 \cup \mathfrak{p}'_2 \cup \cdots$ be rows, with tops $[\omega^0_k, I^0_k]$ for $\mathfrak{p}_k$, and $[\omega^1_k, I^1_k]$ for $\mathfrak{p}'_k$. Suppose that each I^0_k is contained in an $I^1_{k'}$, with $\mathfrak{p}_k$ and $\mathfrak{p}'_{k'}$ separated. Then $||T^{\mathfrak{p}'}T^{\mathfrak{p}*}||_2 \leqq C_M\delta^M$ (any $M > 0$).*

Proof. Regard $L^2[0, 2\pi]$ as $\sum_{k'} \oplus L^2(I^1_{k'}) \oplus L^2(F)$, where $F = [0, 2\pi] - \bigcup_{j'} I_{j'}$. $T^{\mathfrak{p}'}T^{\mathfrak{p}*}$ acts on each component individually — it is zero on $L^2(F)$, and it has norm $\leqq C_M\delta^M$ on each $L^2(I^1_{k'})$ by Lemma 5. Q.E.D.

Similarly, if $A(p) \leqq \delta$ for $p \in \mathfrak{p}$ above, then $||T^{\mathfrak{p}}||_2 \leqq C\delta^{1/2}$, by Lemma 3.

MAIN LEMMA. *Let $\{\mathfrak{p}_j\}$ be a family of trees with tops $[\omega^0_j, I^0_j]$. Assume that $[\omega^0_j, I^0_j] \in \mathfrak{p}_j$ for each j, and that*

(a) *$A(p) \leqq \delta$ for $p \in$ any $\mathfrak{p}_j$.*

(b) *$p \not\leqq p'$ for any $p \in \mathfrak{p}_j$, $p' \in \mathfrak{p}_{j'}$, $j \neq j'$.*

(c) *No point of $[0, 2\pi]$ belongs to more than $K\delta^{-20}$ of the I^0_j. Then there is a set $F \subseteq [0, 2\pi]$, $|F| \leqq C\delta^{100}/K$, with the property*

$$\left|\left|\sum\nolimits_j T^{\mathfrak{p}_j}f\right|\right|_{L^2(^cF)} \leqq C_\eta(\log K)\delta^{1/2-\eta}\,||f||_2$$

for all $f \in L^2[0, 2\pi]$.

Roughly speaking, then, $\bigcup_j \mathfrak{p}_j$ satisfies (6). This completes the second part of the program of § 3.

Proof of the Main Lemma. F will be

$$\bigcup_j\left\{x \in I^0_j \mid \text{distance}\,(x, \partial I^0_j) \leqq \frac{10\delta^{200}}{K}\,|I^0_j|\right\} \equiv \bigcup_j F_j .$$

$|F| \leqq \sum_j |F_j| = (20\delta^{200}/K) \sum_j |I_j^0| = (20\delta^{200}/K) \cdot \int_0^{2\pi}$ (NUMBER OF I_j^0 CONTAINING $x)dx \leqq C\delta^{100}/K$, by (c).

We prepare to apply the corollary to Lemma 5, and the orthogonality lemma. First, we have to remove a few pairs from the $\mathfrak{p}_j$.

From $\mathfrak{p} = \bigcup_j \mathfrak{p}_j$, we skim off the top: Let $\mathfrak{p}^+ = \{p \in \mathfrak{p} \mid$ there are no strictly ascending sequences $p \lneqq p_1 \lneqq \cdots \lneqq p_{M+1}$ with all $p_j \in \mathfrak{p}\}$. Here, $M = \log(K^{10000}\delta^{-10000}) \leqq C(\log K)\delta^{-\varepsilon}$. By definition, $\mathfrak{p}^+$ contains no ascending chains of length $M+1$. An easy induction on M shows that $\mathfrak{p}^+$ can be partitioned into $\mathfrak{p}^+_{(1)} \cup \cdots \cup \mathfrak{p}^+_{(M)}$, where no two pairs in any $\mathfrak{p}^+_{(i)}$ are comparable. Lemma 2 shows that $\|T^{\mathfrak{p}^+_{(i)}}\|_2 \leqq C\delta^{1/2-\eta}$, which implies that $\|T^{\mathfrak{p}^+}\|_2 \leqq \sum_{i=1}^M \|T^{\mathfrak{p}^+_{(i)}}\|_2 \leqq MC\delta^{1/2-\eta} \leqq C(\log K)\delta^{1/2-\eta-\varepsilon}$. Therefore, it is enough to prove the lemma for $\mathfrak{p}^0 = \mathfrak{p} - \mathfrak{p}^+$. From $\mathfrak{p}^0$ we remove the bottom: Let $\mathfrak{p}^- = \{p \in \mathfrak{p}^0 \mid$ there are no strictly ascending chains $p_1 \lneqq p_2 \lneqq \cdots \lneqq p_{M+1} \lneqq p$, all $p_i \in \mathfrak{p}^0\}$. As above, $\|T^{\mathfrak{p}^-}\|_2 \leqq C(\log K)\delta^{1/2-\eta-\varepsilon}$, so that is enough to prove the lemma for $\mathfrak{p}^\# = \mathfrak{p} - \mathfrak{p}^-$.

Write $\mathfrak{p}^\# = \bigcup \mathfrak{p}_j^0$, where $\mathfrak{p}_j^0 = \mathfrak{p}_j \cap \mathfrak{p}^\#$ is still a tree with top $[\omega_j^0, I_j^0]$. The $\mathfrak{p}_j^0$ are much better behaved than the original $\mathfrak{p}_j$. In particular,

(i) $[\omega, I] \in \mathfrak{p}_j^0$ implies $|I| \leqq (\delta^{1000}/K)\,|I_j^0|$.

(ii) For $j \neq j'$, $\mathfrak{p}_j^0$ and $\mathfrak{p}_{j'}^0$ are separated. In fact (α) and (β) hold with δ replaced by $\delta' = K^{-10000}\delta$.

To prove (i), we note that $[\omega, I] \notin \mathfrak{p}^+$, so that $[\omega, I] < p_1 \lneqq \cdots \lneqq p_{m+1} = [\omega_{M+1}, I_{M+1}]$ for some p_i's in $\mathfrak{p}$. By hypothesis (b), all the p_i's belong to $\mathfrak{p}_j$, so that $[\omega_{M+1}, I_{M+1}] \leqq [\omega_j^0, I_j^0]$, and $|I| \leqq 2^{-(M+1)}\,|I_{M+1}| \leqq 2^{-(M+1)}\,|I_j^0| \leqq (\delta^{1000}/K)\,|I_j^0|$. To prove (ii), assume, say, $[\omega, I] \in \mathfrak{p}_j^0$ with $I \subseteq I_{j'}^0$. Since $[\omega, I] \notin \mathfrak{p}^-$, we have $[\omega_1, I_1] = p_1 \lneqq p_2 \lneqq \cdots \lneqq p_{M+1} \lneqq [\omega, I]$, with all p_j's in $\mathfrak{p}$. Again by (b), all the p_j's belong to $\mathfrak{p}_j$. In particular, $I_I \subseteq I \subseteq I_{j'}^0$, which implies that $\omega_1 \cap \omega_{j'}^0 = \emptyset$, since otherwise $[\omega_1, I_1] < [\omega_{j'}^0, I_{j'}^0] \in \mathfrak{p}_{j'}$, contradicting (b). Since all the p_i's are admissible, the "convenient property of central dyadic intervals" mentioned in §5, shows that distance$(\omega, \omega_{j'}^0) \geqq 2^M\,|\omega| \geqq K^{10000}\delta^{-1}\,|\omega|$. This is part ($\alpha$) of the definition of "$\mathfrak{p}_j^0$ and $\mathfrak{p}_{j'}^0$ are separated", with δ replaced by δ'. Part (β) is exactly analogous, and (ii) is proved.

Before applying the orthogonality lemma, we make one further modification in $\mathfrak{p}^\#$. Decompose $\mathfrak{p}_j^0 = \mathfrak{p}_j^\# \cup \mathfrak{p}_j^\flat$, where $\mathfrak{p}_j^\# = \{[\omega, I] \in \mathfrak{p}_j^0 \mid I \not\subseteq F_j\}$ and $\mathfrak{p}_j^\flat = \{[\omega, I] \in \mathfrak{p}_j^0 \mid I \subseteq F_j\}$. Note that $\mathfrak{p}_j^\#$ is a normal tree, by (i).

Now $T^{\mathfrak{p}_j^\flat}$ is supported entirely on F_j. Therefore, on ${}^cF$, $T^{\mathfrak{p}_j^\#}f = \sum_j T^{\mathfrak{p}_j^\#}f$, and the conclusion of the lemma reduces to

$$\Big\|\sum_j T^{\mathfrak{p}_j^\#}\Big\|_2 \leqq C(\log K)\delta^{1/2-\eta}\,. \tag{25}$$

We prepare to apply the orthogonality lemma. By hypothesis (c), $\bigcup_j \mathfrak{p}_j^{\#}$ breaks up as a union of at most $K\delta^{-20}$ rows, $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_{K\delta^{-20}}$. To see this, let $I^1, I^2, \cdots$ be all the maximal dyadic intervals among the $\{I_j^0\}$, each I^s appearing only once in the list. For each I^s, pick one $[\omega^0_{j(s)}, I^0_{j(s)}]$ with $I_j^0 = I^s$. Set $\mathcal{R}_1 = \bigcup_s \mathfrak{p}^{\#}_{j(s)}$; $\mathcal{R}_1$ is a row. Delete all the $\mathfrak{p}^{\#}_{j(s)}$ from $\{\mathfrak{p}_j^{\#}\}$, and apply the same process as above to our new set of trees, to form a row $\mathcal{R}_2$. Continue to form $\mathcal{R}_3$, $\mathcal{R}_4$, etc., until we run out of trees. Hypothesis (c) guarantees that this will happen within $K\delta^{-20}$ steps, since the I_j^0 corresponding to $\mathcal{R}_{m+1}$ are nested in those for $\mathcal{R}_m$.

We know that

(A) $||T^{\mathcal{R}_i}||_2 \leqq C\delta^{1/2}$ by (a) and the comment after the corollary to Lemma 5.

(B) $||T^{\mathcal{R}_i}T^{\mathcal{R}_{i'}*}||_2 \leqq C_M K^{-M}\delta^M$ $(i \neq i')$, by (ii) and the corollary to Lemma 5 (with δ replaced by δ').

(C) $T^{\mathcal{R}_i*}T^{\mathcal{R}_{i'}} = 0$ $(i \neq i')$. For, $T^{\mathcal{R}_i}f$ and $T^{\mathcal{R}_{i'}}g$ live on $\bigcup\{E(p) \mid p \in$ some $\mathfrak{p}_j^{\#} \subseteq \mathcal{R}_i\}$ and $\bigcup\{E(p') \mid p' \in$ some $\mathfrak{p}^{\#}_{j'} \subseteq \mathcal{R}_{i'}\}$, respectively. Since $p \in \mathfrak{p}_j^{\#} \subseteq \mathcal{R}_i$ and $p' \in \mathfrak{p}^{\#}_{j'} \subseteq \mathcal{R}_{i'}$ are not comparable, these two sets are disjoint.

The orthogonality lemma implies at once that $||\sum_j T^{\mathfrak{p}_j^{\#}}||_2 = ||\sum_i T^{\mathcal{R}_i}||_2 \leqq C\delta^{1/2}$, which is even stronger than (25). Q.E.D.

COROLLARY. *Let $\mathfrak{p}$ be a set of pairs. Assume:*

(a) *$A(p) \leqq \delta$ for all $p \in \mathfrak{p}$*

(b) *If p, p'' belong to $\mathfrak{p}$ and $p < p' < p''$, then $p' \in \mathfrak{p}$.*

(c) *If $p, p', p'' \in \mathfrak{p}$ and $p < p'$, $p < p''$, then either $p' < p''$ or $p'' < p$.*

(d) *For any point $x \in [0, 2\pi]$, there are at most $K\delta^{-20}$ mutually incomparable $[\omega_i, I_i] \in \mathfrak{p}$ with $x \in I_i$.*

Then for a small set $F \subseteq [0, 2\pi]$, $|F| < C\delta^{100}/K$, we have

$$||T^{\mathfrak{p}}f||_{L^2(^cF)} \leqq C_\gamma(\log K)\delta^{(1/2)-\gamma}\,||f||_2 \qquad \textit{for all } f \in L^2[0, 2\pi]\ .$$

Proof. Let $\{[\omega_j^0, I_j^0]\}$ be the maximal pairs in $\mathfrak{p}$, and let $\mathfrak{p}_j = \{p \in \mathfrak{p} \mid p < [\omega_j^0, I_j^0]\}$. Each $\mathfrak{p}_j$ is a tree with top $[\omega_j^0, I_j^0]$ by (b), and $[\omega_j^0, I_j^0] \in \mathfrak{p}_j$. Clearly, $\mathfrak{p} = \bigcup_j \mathfrak{p}_j$. Moreover, for $j \neq j'$, $\mathfrak{p}_j$ and $\mathfrak{p}_{j'}$ have no pairs in common — in fact if $p \in \mathfrak{p}_j$ and $p' \in \mathfrak{p}_{j'}$, then $p \not\leqq p'$. For otherwise, $p < [\omega_j^0, I_j^0]$ and $p < p' < [\omega^0_{j'}, I^0_{j'}]$, and since $[\omega_j^0, J_j^0]$ and $[\omega^0_{j'}, I^0_{j'}]$ are incomparable (both being maximal), we have contradicted (c).

Thus, $T^{\mathfrak{p}} = \sum_j T^{\mathfrak{p}_j}$, where the $\{\mathfrak{p}_j\}$ satisfy hypotheses (a) and (b) of the main lemma. (c) of the main lemma is immediate from our present hypothesis (d). Q.E.D.

A set $\mathfrak{p}$ satisfying (a)–(d) above is called a "forest".

7. Proof of pointwise convergence

We are trying to prove the basic estimate (8): $\|Tf\|_1 \leqq C\|f\|_2$. Break up $\mathfrak{p}$ into $\bigcup_{n=0}^{\infty}\mathfrak{p}_n$, where $\mathfrak{p}_n = \{p \in \mathfrak{P} \mid 2^{-n-1} < A(p) \leqq 2^{-n}\}$. Thus, $T = \sum_{n=0}^{\infty} T^{\mathfrak{p}_n}$. We shall remove a few pairs from $\mathfrak{p}_n$, and decompose the rest of $\mathfrak{p}_n$ into a small number of forests by means of a combinatorial trick. Then we can simply apply the corollary to the main lemma to each forest, put the resulting inequalities together to estimate $T^{\mathfrak{p}_n}$, and combine the estimates for $T^{\mathfrak{p}_n}$ into an estimate for $T^{\mathfrak{p}}$. The result is something even stronger than (8).

Let $\{\bar{p}_k\}$, $\bar{p}_k = [\bar{\omega}_k, \bar{I}_k]$, be the set of maximal pairs with $|E(\omega, I)|/|I| \geqq 2^{-n-1}$. We begin removing pairs from $\mathfrak{p}_n$ — first we skim off the top. Let $\mathfrak{p}_n^+ = \{p \in \mathfrak{p}_n \mid$ there are no ascending chains $p \lneqq p_1 \lneqq \cdots \lneqq p_{n+1}$, with all $p_i \in \mathfrak{p}_n\}$. $\mathfrak{p}_n^+$ contains no ascending chains of length $n+2$, and so breaks up as a disjoint union of at most $n+1$ sets $p_{n1}^+ \cup \mathfrak{p}_{n2}^+ \cup \cdots \cup \mathfrak{p}_{n(n+1)}^+$, with no two pairs in the same $\mathfrak{p}_{ni}^+$ comparable $\|T^{\mathfrak{p}_{ni}^+}\|_2 \leqq C\cdot 2^{-n(1/2-\eta)}$ by Lemma 2, so that

$$\|T^{\mathfrak{p}_n^+}\|_2 \leqq \textstyle\sum_i \|T^{\mathfrak{p}_{ni}^+}\|_2 \leqq C(n+1)2^{-n(1/2-\eta)} \leqq C'2^{-n(1/2-\eta')} . \tag{26}$$

We are left with $\mathfrak{p}_n^0 = \mathfrak{p}_n - \mathfrak{p}_n^+$. Claim: Every $p \in \mathfrak{p}_n^0$ satisfies $p < \bar{p}_j$ for some j. For, by definition of $A(p)$, given $[\omega, I] \in \mathfrak{p}_n$, there must be some $p' = [\omega', I']$ with $I \subseteq I'$, distance $(\omega, \omega') \leqq 2^{n/1000}|\omega|$, and $|E(\omega', I')|/|I'| \geqq 2^{-n-1}$. p' must satisfy $p' < \bar{p}_j$ for some j, so that $I \subseteq \bar{I}_j$, distance$(\omega, \bar{\omega}_j) \leqq 2^{n/2}|\omega|$. Now let $p \in \mathfrak{p}_n^0$ be given; there is a chain $p \lneqq p_1 \lneqq p_2 \lneqq \cdots \lneqq p_{n+1} = [\omega_{n+1}, I_{n+1}]$, with all the $p_i \in \mathfrak{p}_n$. Since $p_{n+1} \in \mathfrak{p}_n$, there is a $\bar{p}_j$ for which $I_{n+1} \subseteq \bar{I}_j$ and distance$(\omega_{n+1}, \bar{\omega}_j) \leqq 2^{n/2}|\omega_{n+1}|$. By the "convenient property of central dyadic intervals" in §5, $p < \bar{p}_j$. The claim is proved.

Let us turn to the $\bar{p}_j$. The $\{E(\bar{p}_j)\}$ are pairwise disjoint, since $\bar{p}_j \not< \bar{p}_{j'}$. So of course, $\sum_j |E(\bar{p}_j)| \leqq |[0, 2\pi]| = 1$. On the other hand, the $\bar{p}_j$ were defined to satisfy $|E(\bar{p}_j)|/|\bar{I}_j| \geqq 2^{-n-1}$, which implies that

$$\int_0^{2\pi} (\text{NUMBER OF } \bar{I}_j \text{ CONTAINING } x)dx = \textstyle\sum_j |\bar{I}_j| \leqq 2^{n+1}\sum_j |E(\bar{p}_j)| \leqq 2^{n+1} .$$

Therefore, $G_n = \{x \in [0, 2\pi] \mid x$ is contained in more than $(K/2)\cdot 2^{2n}$ of the $\bar{I}_j\}$ has measure $|G_n| \leqq C/(2^n K)$. Using G_n we can delete more pairs from $\mathfrak{p}_n^0$: let $\mathfrak{p}_n^\# = \{[\omega, I] \in \mathfrak{p}_n^0 \mid I \not\subseteq G_n\}$. Since $T_{[\omega,I]}f$ lives on $I \subseteq G_n$ for $[\omega, I] \in \mathfrak{p}_n^0 - \mathfrak{p}_n^\#$, we have

$$T^{\mathfrak{p}_n^\#}f(x) = T^{\mathfrak{p}_n^0}f(x) \qquad \text{for all } f \in L^2 \text{ and } x \in {}^cG_n . \tag{27}$$

We shall make no attempt to estimate $T^{\mathfrak{p}_n^0}f$ on G_n. After deleting all $[\bar{\omega}_j, \bar{I}_j]$ with $\bar{I}_j \subseteq G_n$ from our list of $\bar{p}_j$, we are in the following situation: $\mathfrak{p}_n^\#$ is a set of pairs p for which $A(p) \leqq 2^{-n}$. $\bar{p}_1, \bar{p}_2, \cdots$ is a list of pairs $[\bar{\omega}_j, \bar{I}_j]$. Every

$p \in \mathfrak{p}_n^\#$ satisfies $p < \bar{p}$ for some j. No $x \in [0, 2\pi]$ belongs to more than $(K/2)\cdot 2^n$ of the $\bar{I}_j$'s.

Now we shall prove that $\mathfrak{p}_n^\#$ decomposes as a disjoint union of at most $M = 2n \log K + 1$ forests $\mathfrak{p}_{n0} \cup \mathfrak{p}_{n1} \cup \cdots \cup \mathfrak{p}_{n(M-1)}$, where each $\mathfrak{p}_{nj}$ satisfies the hypotheses of the corollary to the main lemma. In order to make the decomposition, we use the index $B(p) =$ (number of j for which $p < \bar{p}_j$), defined for $p \in \mathfrak{p}_n^\#$. We know that $1 \leqq B(p) \leqq (K/2)\cdot 2^{2n} = 2^{M-2}$ for any $p = [\omega, I] \in \mathfrak{p}_n^\#$, since $p < \bar{p}_j$ implies $I \subseteq \bar{I}_j$. Our forests are simply defined as $\mathfrak{p}_{ns} = \{p \in \mathfrak{p}_n^\# \mid 2^s \leqq B(p) < 2^{s+1}\}$. Now it is clear that $\mathfrak{p}_n^\# = \mathfrak{p}_{n0} \cup \mathfrak{p}_{n1} \cup \cdots \cup \mathfrak{p}_{n(M-1)}$, but it is not yet clear that $\mathfrak{p}_{ns}$ is a forest. To prove that it is a forest, we use a simple combinatorial property of B: Suppose $p, p', p'' \in \mathfrak{p}_n^\#$, $p < p'$, $p < p''$, but p', p'' are not comparable under $<$. Then $B(p) \geqq B(p') + B(p'')$. This is easy to see. Let $B(p') = s$ and $B(p'') = t$. Then $p' < \bar{p}_{j_1}, \bar{p}_{j_2}, \cdots, \bar{p}_{j_s}$, and $p'' < \bar{p}_{k_1}, \bar{p}_{k_2}, \cdots, \bar{p}_{k_t}$. Now $\bar{p}_{j_l} \neq \bar{p}_{k_m}$; since otherwise $p < p'$, $p'' < p''' = \bar{p}_{j_l} = \bar{p}_{k_m}$, and it is a simple property of $<$ that $p < p'$, $p'' < p'''$ implies p', p'' comparable, for any pairs p, p', p'', p'''. Since $p < \bar{p}_{j_1}, \cdots, \bar{p}_{j_s}, \bar{p}_{k_1}, \cdots, \bar{p}_{k_t}$, we have $G(p) \geqq s + t = B(p') + B(p'')$.

Now we can show that $\mathfrak{p}_{ns}$ is a forest. We have to check conditions (a)–(d) in the corollary of the main lemma, with $\delta = 2^{-n}$. (a), (b), and (d) are obvious — they were true even for $\mathfrak{p}_n^\#$ — details are left to the reader. (c) is the interesting part: Let $p, p', p'' \in \mathfrak{p}_{ns}$, and suppose $p < p', p''$. If p', p'' were not comparable under $<$, then $B(p) \geqq B(p') + B(p'') \geqq 2^s + 2^s = 2^{s+1}$, contradicting $p \in \mathfrak{p}_{ns}$.

Thus, the corollary to the main lemma shows that

$$||T^{\mathfrak{p}_{ns}}f||_{L^2({}^cF_{ns})} \leqq C(\log K)2^{-n(1/2-\gamma)}\,||f||_2\ ,$$

for a small set F_{ns} of measure $|F_{ns}| \leqq C/2^nK$. Let us put these estimates together, and see what happens. We have

$$||T^{\mathfrak{p}_n^\#}f||_{L^2({}^cF_n)} \leqq \sum_{s=0}^{2n\log K} ||T^{\mathfrak{p}_{ns}}f||_{L^2(F_{ns})} \leqq C(n+1)(\log K)^2 2^{-n(1/2-\gamma)}\,||f||_2\ ,$$

where $F_n = \bigcup_{s=0}^{2n\log K} F_{ns}$ has measure $|F_n| \leqq \sum_s |F_{ns}| \leqq C(n+1)\log K/(2^nK)$. Therefore, by (27),

$$||T^{\mathfrak{p}_n^0}f||_{L^2({}^cE_n)} \leqq Cn(\log K)^2 2^{-n(1/2-\gamma)}\,||f||_2\ ,$$

where $E_n = F_n \cup G_n$ has measure $|E_n| \leqq |F_n| + |G_n| \leqq \big(C(n+2)/2^n\big)\cdot(\log K/K)$. By (26), the same estimate holds for $T^{\mathfrak{p}_n} = T^{\mathfrak{p}_n^0} + T^{\mathfrak{p}_n^+}$. Summing on n, we obtain $||Tf||_{L^2({}^cE)} \leqq C(\log K)^2\,||f||_2$, where $E = \bigcup_{n=0}^\infty E_n$ and $|E| \leqq \sum_{n=0}^\infty |E_n| \leqq C(\log K/K)$. What does this say about the size of Tf? Given $\alpha > 0$, we have

$$|\{|Tf(x)| > \alpha\}| \leqq \frac{\|Tf\|^2_{L^2({}^cE)}}{\alpha^2} + |E| \leqq C(\log K)^4 \frac{\|f\|_2^2}{\alpha^2} + C\frac{\log K}{K}.$$

This inequality holds for all $K > 10$, with C independent of K. If we simply pick K to minimize the right-hand side, we find that $|\{|Tf(x)| > \alpha\}| \leqq C_\varepsilon \cdot (\|f\|_2/\alpha)^{2-\varepsilon}$ for any $\varepsilon > 0$, which means that $\|Tf\|_p \leqq C_p \|f\|_2$ for any $p < 2$. The basic estimate (8) is merely the case $p = 1$. Q.E.D.

8. Remarks

(a) It is known that $\|Tf\|_2 \leqq C\|f\|_2$. (See Hunt [5].) The above proof can be modified to show this, I think, but only at the expense of great complexity.

(b) We indicate briefly how to adapt the proof in §§1–7 to L^p $(1 < p < 2)$. The idea is to prove L^p-analogues of Lemmas 2, 3, 5, the main lemma, and and finally the argument in §7, by interpolating between our L^2 results and standard estimates for L^1. For example, we have

LEMMA 2′. *Under the hypotheses of Lemma* 2, $\|T^{\mathfrak{p}}\|_r \leqq C\delta^{1/r-\eta}$ $(\eta > 0, 1 < r < 2)$.

Proof. We know that $\|T^{\mathfrak{p}}\|_2 \leqq C\delta^{1/2-\eta}$. On the other hand, the proof of Lemma 1 shows that $|T^{\mathfrak{p}}f(x)| \leqq Cf^*(x)$ a.e., so that $\|T^{\mathfrak{p}}\|_{1+\varepsilon} \leqq C$ by the maximal theorem. Lemma 2′ follows by interpolation. Q.E.D.

Only two points in the L^2 argument require non-trivial changes:

(1) In the routine L^p analogue of the main lemma, the hypothesis has to be strengthened and the conclusion weakened. Instead of (c), we need to assume

(c′) No point in $[0, 2\pi]$ belongs to more than $K \cdot \delta^{1+\rho}$ of the I_j^0, where ρ is a small positive number depending only on r.

The conclusion should read $|F| \leqq C\delta^\rho/K^M$ and

$$\|\textstyle\sum_j T^j f\|_{L^r({}^cF)} \leqq CK^a\delta^{1/2-\eta}\|f\|_r$$

for all $f \in L^r$. (Note that the factor $(\log K)$ in the main lemma is now degraded to K^a with $0 < a < 1$. This will require us to know more precise estimates for the sets G_n which we throw away.)

(2) To obtain a better estimate for $|G_n|$ in §7, we set $h_j = \chi_{E(\bar{\mathfrak{p}}_j)}$. It is easy to see that $\|\sum_j |h_j|\|_\infty = 1$, while

$$\textstyle\sum_j |h_j^*(x)|^{1+\varepsilon} \geqq 2^{-n(1+\varepsilon)} \cdot (\text{NUMBER OF } \bar{I}_j \text{ CONTAINING } x) \qquad \text{for } x \in [0, 2\pi].$$

It follows from results in [4] that $|G_n| \leqq C2^{-n}/K^M$, any $M > 10$. With these two changes made, everything carries over routinely to L^p.

(c) However, our proof is very inefficient near L^1. Carleson's construction can be pushed down as far as $L \log L(\log \log L)$ (See Sjölin [7].), but our proof seems unavoidably restricted to $L(\log L)^M$ for some large M.

(d) One way to get some feeling for our proof is to analyze operators

$$T^0_{N(\cdot)}f(x) = \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)dy$$

for particular functions $N(\cdot)$, using the techniques in §§1–7. This is how I discovered the proof myself. Here are some amusing examples: $N(x) = [\lambda x]$, $N(x) = [\lambda x \bmod \mu]$, $N(x) = \nu\cdot[\lambda x \bmod \mu]$, λ, μ, ν stand for large numbers, $[\cdot]$ is the greatest integer function, and $(x \bmod y) = x - \big(y\cdot[x/y]\big)$.

(e) The viewpoint of §§1–7 is well-suited to multi-dimensional problems. Compare (d) above, with the counterexample in [3].

(f) Our "orthogonality lemma" is watered down from a lemma in Knapp and Stein [6], first proved by Cotlar [2] in the case of commuting operators. It would be interesting to prove the full Cotlar lemma of [6] by geometrical arguments like those in §4.

Finally, we urge the reader not to take too seriously the complicated technicalities in §6.

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