第一章 常微分方程的数值解法

§1 引论

第二章 椭圆方程的有限差分法

§2 一维差分格式 P67

1. 用有限体积法导出逼近微分方程（2.2.1）的差分方程。

解：(2.1) $Lu = -\frac{d}{dx}\left(p\frac{du}{dx}\right) + r\frac{du}{dx} + qu = f \qquad a < x < b$,

$p, q, r, f \in C(I)$，可以直接积分

在[a，b]内任一小区间[$x^{(1)}, x^{(2)}$]上积分有

$$\int_{x^{(1)}}^{x^{(2)}} -\frac{d}{dx}\left(p\frac{du}{dx}\right)dx + \int_{x^{(1)}}^{x^{(2)}} r\frac{du}{dx}dx + \int_{x^{(1)}}^{x^{(2)}} qu\,dx = \int_{x^{(1)}}^{x^{(2)}} f\,dx$$

即

$$w(x^{(1)}) - w(x^{(2)}) + \int_{x^{(1)}}^{x^{(2)}} r\frac{du}{dx}dx + \int_{x^{(1)}}^{x^{(2)}} qu\,dx = \int_{x^{(1)}}^{x^{(2)}} f\,dx$$

其中 $w(x) = p\frac{du}{dx}$ 在[a，b]上连续。

取[$x^{(1)}, x^{(2)}$]为对偶单元[$x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}$]，则得

$$w(x_{i-\frac{1}{2}}) = - w(x_{i+\frac{1}{2}}) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} r\frac{du}{dx}dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} qu\,dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f\,dx$$

$\because w(x) = p\frac{du}{dx}$

$\therefore \frac{du}{dx} = \frac{w(x)}{p(x)}$

$\therefore \int_{x_{i-1}}^{x_i} \frac{du}{dx}dx = \int_{x_{i-1}}^{x_i} \frac{w(x)}{p(x)}dx$

$u_i - u_{i-1} \approx w(x_{i-\frac{1}{2}}) \int_{x_{i-1}}^{x_i} \frac{1}{p(x)}dx$（中矩形公式）

$w(x_{i-\frac{1}{2}}) \approx (u_i - u_{i-1})\left(\int_{x_{i-1}}^{x_i} \frac{1}{p(x)}dx\right)^{-1} = \frac{u_i - u_{i-1}}{h_i}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} \frac{1}{p(x)}dx\right)^{-1}$

令 $a_i = \left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} \frac{1}{p(x)}dx\right)^{-1}$

$\therefore w_{i-\frac{1}{2}} \approx a_i \frac{u_i - u_{i-1}}{h_i}$

$\therefore w_{i+\frac{1}{2}} \approx a_{i+1} \frac{u_{i+1} - u_i}{h_{i+1}}$

$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} qu\,dx \approx \frac{h_i + h_{i+1}}{2} u_i d_i$ 其中 $d_i = \frac{2}{h_i + h_{i+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)dx$

$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f\,dx \approx \frac{h_i + h_{i+1}}{2} \varphi_i$ ，其中 $\varphi_i = \frac{2}{h_i + h_{i+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx$

$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} r\frac{du}{dx}dx \approx \frac{du}{dx}\Big|_{x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} r(x)dx$（中矩形公式）

$\approx \frac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} r(x)dx$（中心差分）

$= \frac{h_i + h_{i+1}}{2}(u_{i+1} - u_{i-1}) b_i$

其中 $b_i = \frac{2}{(h_i + h_{i+1})^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} r(x)dx$

$\therefore$ 得差分方程

$$-\left[a_{i+1}\frac{u_{i+1} - u_i}{h_{i+1}} - a_i \frac{u_i - u_{i-1}}{h_i}\right] + \frac{h_i + h_{i+1}}{2}(u_{i+1} - u_i) b_i + \frac{h_i + h_{i+1}}{2} d_i u_i = \frac{h_i + h_{i+1}}{2} \varphi_i$$

2. 构造逼近 $\begin{array}{l}(pu'')''+(qu')'+ru=f,\\ u(a)=u'(a)=0,u(a)=u'(a)=0\end{array}$ 的中心差分格式。

解：取 $N+1$ 个节点，$a=x_0<x_1<\dots\dots<x_i<\dots\dots<x_N=b$，

$h_i=x_i-x_{i-1},i=1,2,\dots\dots,N,\qquad h=\max\limits_{1\le i\le N} h_i$

$x_{i-\frac{1}{2}}=\frac{1}{2}(x_{i-1}+x_i)\qquad i=1,2,\dots\dots,N$

则有

1）$$[\frac{d^2u}{dx^2}]_i\approx\frac{[\frac{du}{dx}]_{i+\frac{1}{2}}-[\frac{du}{dx}]_{i-\frac{1}{2}}}{\frac{h_{i+1}+h_i}{2}}\approx\frac{\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}}$$

$$\therefore[p\frac{d^2u}{dx^2}]_i=p_i[\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}]\frac{2}{h_i+h_{i+1}}$$

2）$[\frac{d^2}{dx^2}p\frac{d^2u}{dx^2}]_i$

$$\approx\frac{[\frac{d}{dx}p\frac{d^2u}{dx^2}]_{i+\frac{1}{2}}-[\frac{d}{dx}p\frac{d^2u}{dx^2}]_{i-\frac{1}{2}}}{\frac{h_{i+1}+h_i}{2}}$$

$$\approx(\frac{[p\frac{d^2u}{dx^2}]_{i+1}-[p\frac{d^2u}{dx^2}]_i}{h_{i+1}}-\frac{[p\frac{d^2u}{dx^2}]_i-[p\frac{d^2u}{dx^2}]_{i-1}}{h_i})\cdot\frac{2}{h_i+h_{i+1}}$$

$$=\frac{2}{h_i+h_{i+1}}\{p_{i+1}[\frac{u_{i+2}-u_{i+1}}{h_{i+2}}-\frac{u_{i+1}-u_i}{h_{i+1}}]\cdot\frac{2}{(h_{i+2}+h_{i+1})h_{i+1}}-p_i[\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}]\cdot$$
$$\frac{2}{(h_{i+1}+h_i)h_{i+1}}-p_i[\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}]\cdot\frac{2}{(h_{i+1}+h_i)h_i}+p_{i-1}[\frac{u_i-u_{i-1}}{h_i}-\frac{u_{i-1}-u_{i-2}}{h_{i-1}}]$$
$$\frac{2}{(h_i+h_{i-1})h_{i-1}}\}$$

若网格均匀即 $h_1=h_2=\dots\dots=h_N$，则

$$[\frac{d^2}{dx^2}p\frac{d^2u}{dx^2}]_i\approx\frac{1}{h^4}[p_{i+1}(u_{i+2}-2u_{i+1}+u_i)-2p_i(u_{i+1}-2u_i+u_{i-1})+p_{i-1}(u_i-2u_{i-1}+$$
$$u_{i-2})]$$

$$[p\frac{d^2u}{dx^2}]_i\approx\frac{1}{h^2}p_i(u_{i+1}-2u_i+u_{i-1})$$

∴ 差分方程为

$$L_hu_i\equiv\frac{1}{h^4}[p_{i+1}(u_{i+2}-2u_{i+1}+u_i)-2p_i(u_{i+1}-2u_i+u_{i-1})+p_{i-1}(u_i-2u_{i-1}+$$

$$u_{i-2})+\frac{1}{h^2}q_i(u_{i+1}-2u_i+u_{i-1})+r_iu_i]=f_i\qquad i=1,2,\dots\dots,N$$

$u_0=u'_0=0\qquad u_N=u'_N=0$

其中系数矩阵 A 是一个五对角矩阵。

§3 矩形网的差分格式

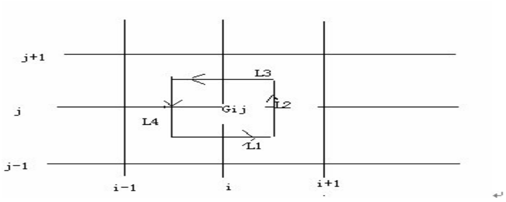
P75

1. 用有限体积法构造逼近方程

$$-\nabla(k\nabla u)=-[\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial u}{\partial y})]=f,(2.3.21)$$

的第一边值问题的五点差分格式，这里$k=k(x,y)\geq k_{\min}>0.$

解：1）正则内点



于 G_{ij} 上积分（3.21）式，

$$-\iint_{G_{ij}} \nabla\cdot(k\nabla u)\,dxdy=\iint_{G_{ij}} f\,dxdy$$

$$-\iint_{G_{ij}}\left[\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)\right]dxdy=\iint_{G_{ij}} f\,dxdy$$

由Green第一公式得：

$$-\int_{\partial G_{ij}}\frac{\partial u}{\partial n}k\,ds=\iint_{G_{ij}} f\,dxdy$$

$$-\left(\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}\right)\frac{\partial u}{\partial n}k\,ds=\iint_{G_{ij}} f\,dxdy$$

$$\int_{L_1}\frac{\partial u}{\partial n}k\,ds=-\frac{\partial u}{\partial y}k\Big|_{i,j-\frac{1}{2}}h_1=k_{i,j-\frac{1}{2}}\frac{u_{i,j-1}-u_{i,j}}{h_2}h_1$$

$$\int_{L_2}\frac{\partial u}{\partial n}k\,ds=\frac{\partial u}{\partial x}k\Big|_{i+\frac{1}{2},j}h_2=k_{i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h_1}h_2$$

$$\int_{L_3}\frac{\partial u}{\partial n}k\,ds=\frac{\partial u}{\partial y}k\Big|_{i,j+\frac{1}{2}}h_1=k_{i,j+\frac{1}{2}}\frac{u_{i,j+1}-u_{i,j}}{h_2}h_1$$

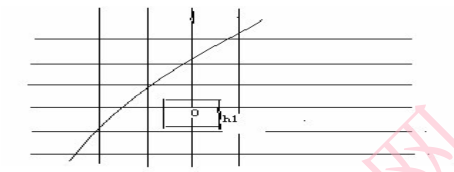
$$\int_{L_4}\frac{\partial u}{\partial n}k\,ds=\frac{\partial u}{\partial x}k\Big|_{i-\frac{1}{2},j}h_2=k_{i-\frac{1}{2},j}\frac{u_{i-1,j}-u_{i,j}}{h_1}h_2$$

∴综上有：

$$-\left[k_{i,j-\frac{1}{2}}\frac{u_{i,j-1}-u_{i,j}}{h_2^2}+k_{i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h_1^2}+k_{i,j+\frac{1}{2}}\frac{u_{i,j+1}-u_{i,j}}{h_2^2}+k_{i-\frac{1}{2},j}\frac{u_{i-1,j}-u_{i,j}}{h_1^2}\right]$$

$=\varphi_{i,j}$，其中 $\varphi_{i,j}=\frac{1}{h_1h_2}\iint_{G_{ij}} f\,dxdy\approx f_{i,j}$

1）非正则内点



在节点 o 处，于 G_{ij} 对（3。21）积分，设点 o 为（x_i，y_j）

$$-\iint_{G_{ij}} \nabla\cdot(k\nabla u)\,dxdy=\iint_{G_{ij}} f\,dxdy$$

$$-\int_{\partial G_{ij}}\frac{\partial u}{\partial n}k\,ds=\iint_{G_{ij}} f\,dxdy$$

$$-\left(\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}\right)\frac{\partial u}{\partial n}k\,ds=\iint_{G_{ij}} f\,dxdy$$

$$\int_{L_1}\frac{\partial u}{\partial n}k\,ds=-\frac{\partial u}{\partial y}k\Big|_{i,j-\frac{1}{2}}\frac{h_1^-+h_1}{2}=k_{i,j-\frac{1}{2}}\frac{u_4-u_0}{h_2}\cdot\frac{h_1^-+h_1}{2}\approx k_0\frac{u_4-u_0}{h_2}\bar{h}_1$$

$$\int_{L_2}\frac{\partial u}{\partial n}k\,ds=k_{i+\frac{1}{2},j}\frac{u_1-u_0}{h_1}h_2\approx k_0\frac{u_1-u_0}{h_1}h_2$$

$$\int_{L_3}\frac{\partial u}{\partial n}k\,ds=k_{i,j+\frac{1}{2}}\frac{u_2-u_0}{h_2}\frac{h_1^-+h_1}{2}\approx k_0\frac{u_2-u_0}{h_2}\bar{h}_1$$

$$\int_{L_4}\frac{\partial u}{\partial n}k\,ds=k_{i-\frac{1}{2},j}\frac{u_3-u_0}{h_1^-}\frac{h_1^-+h_1}{2}\approx k_0\frac{u_3-u_0}{h_1^-}\bar{h}_1$$

其中 $\bar{h}_1=\frac{h_1^-+h_1}{2}$

∴综上有

$$-k_0\left[\frac{u_4-2u_0+u_2}{h_2^2}+\frac{1}{\bar{h}_1}\left(\frac{u_1-u_0}{h_1}+\frac{u_3-u_0}{h_1^-}\right)\right]=\frac{1}{h_2\bar{h}_1}\iint_{G_{ij}} f\,dxdy$$

即有：

$$-k_0\left[\frac{u_4-2u_0+u_2}{h_2^2}+\frac{1}{\bar{h}_1}\left(\frac{u_1-u_0}{h_1}+\frac{u_3-u_0}{h_1^-}\right)\right]=f_0$$

为保持五点差分格式的正定性，可用下式代替上式。

$$-k_0\left[\frac{u_4-2u_0+u_2}{h_2^2}+\frac{1}{h_1}\left(\frac{u_1-u_0}{h_1}+\frac{u_3-u_0}{h_1^-}\right)\right]=f_0$$

2）界点

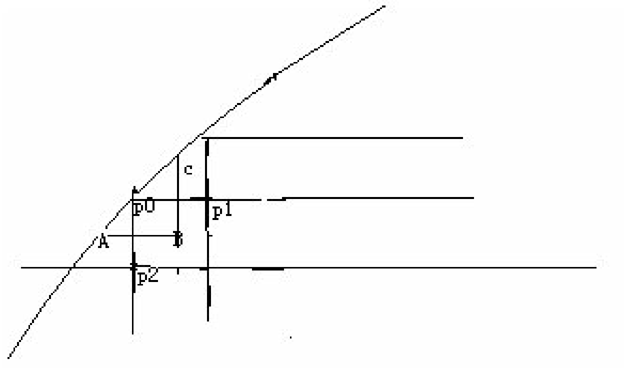
当（$\bar{x}_i$，$\bar{y}_j$）$\in\Gamma_k\subset\Gamma$ 时，有 $u_{i,j}=\alpha$（$\bar{x}_i$，$\bar{y}_j$）

2. 用有限体积法构造逼近方程（2.3.21）的第二边值问题的五点差分格式。

解：1）正则内点，同第一题中 1）

2）非正则内点，同第一题中 2）

3）界点



在界点Γ_0处于曲边三角形 ABC 上对（3。21）式积分，得：

$$-\iint_{\Delta ABC} \nabla(k\nabla u)\,dxdy = \iint_{\Delta ABC} f\,dxdy$$

$$-\iint_{\hat{A}\hat{B}\hat{C}\hat{A}} \frac{\partial u}{\partial n} k\,ds = \iint_{\Delta ABC} f\,dxdy$$

$$-\left(\int_{\hat{A}\hat{B}} + \int_{\hat{B}\hat{C}} + \int_{\hat{C}\hat{A}}\right) \frac{\partial u}{\partial n} k\,ds \approx \iint_{\Delta ABC} f\,dxdy$$

$$\int_{\hat{A}\hat{B}} \frac{\partial u}{\partial n} k\,ds \approx k_{p0} \frac{u_{p2}-u_{p0}}{h_2} \overline{AB}$$

$$\int_{\hat{B}\hat{C}} \frac{\partial u}{\partial n} k\,ds \approx k_{p0} \frac{u_{p1}-u_{p0}}{h_1} \overline{BC}$$

$$\int_{\hat{C}\hat{A}} \frac{\partial u}{\partial n} k\,ds \approx \int_{\hat{C}\hat{A}} \beta(x,y)\,k\,ds = \beta_{p0} k_{p0} \overline{CA}$$

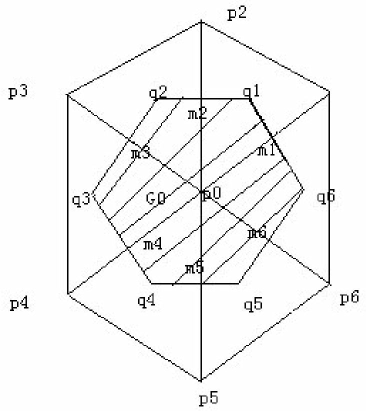
∴ 综上有：

$$-k_{p0}\left(\frac{u_{p2}-u_{p0}}{h_2}\overline{AB} + \frac{u_{p1}-u_{p0}}{h_1}\overline{BC} + \beta_{p0}\overline{CA}\right) = \iint_{\Delta ABC} f\,dxdy$$

§4 三角形网的差分格式 P80

2. 构造逼近方程 $-\nabla(k\nabla u) = -[\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial u}{\partial y})] = f$,(2.3.21) 的三角网差分格式。

解:



如图,设 p_0 是内点, $p_1,\cdots,p_6$ 是和 p_0 相邻的节点, q_i 为三角形 $p_0p_ip_{i+1}$ 的外心, m_i

是 $\overline{p_0p_i}$ 的中点, G_0 是由六边形 $q_1,\cdots,q_6$ 围成的对偶单元, 在子域 G_0 积分得

$$-\iint_{G_0}[\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial u}{\partial y})]dxdy=\iint_{G_0}fdxdy,$$

由 Green 公式得:

$$-\int_{\partial G_0}k\frac{\partial u}{\partial n}ds=\iint_{G_0}fdxdy,$$

即

$$\int_{\partial G_0}k\frac{\partial u}{\partial n}ds=\sum_{i=1}^{6}\int_{q_iq_{i+1}}k\frac{\partial u}{\partial n}ds$$
$$=\sum_{i=1}^{6}(\overline{q_iq_{i+1}}/\overline{p_0p_{i+1}})k_{\frac{i}{2}}[u(p_{i+1})-u(p_6)]+m(G_0)R_{G_0}(u),$$

其中 $k_{\frac{i}{2}}$ 即为 k(x,y) 在 m_i 中点的值, $m(G_0)$ 是 G_0 的面积, $R_{G_0}(u)$ 是截断误差,

∴ 得点 p_0 的差分方程为:

$$-\sum_i k_{\frac{1}{2}}(\overline{q_iq_{i+1}}/\overline{p_0p_{i+1}})(u_{i+1}-u_i)=\iint_{G_0}fdxdy=m(G_0)\varphi_0,$$

其中 $\varphi_0=\frac{1}{m(G_0)}\iint_{G_0}fdxdy$, $k_{\frac{i}{2}}$ 是 k 在 $\overline{p_0p_i}$ 中点的值,

其次建立界点的差分方程:设 p_0 是界点,则 $u_{p_0}=\alpha(p_0)$

∴ 三角网格差分格式为:

在 p_0 点: $-\sum_i k_{\frac{1}{2}}(\overline{q_iq_{i+1}}/\overline{p_0p_{i+1}})(u_{i+1}-u_i)=\iint_{G_0}fdxdy=m(G_0)\varphi_0$,

若 p_0 为界点: $u_{p_0}=\alpha(p_0)$

第三章 抛物型方程的有限差分法

§1 最简差分格式

P112 2 题

解：$E_j^k u = L_k u(x_j,t_k) - [Lu]_j^k$

$$= \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} - \frac{a}{h^2}[\theta(u(x_{j+1},t_{k+1}) - 2u(x_j,t_{k+1})$$
$$+u(x_{j-1},t_{k+1})) + (1-\theta)(u(x_{j+1},t_k)$$
$$-2u(x_j,t_k)+u(x_{j-1},t_k)] - \frac{\partial u(x_j,t_k)}{\partial t} + a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}$$

*

(1) $\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}$

$$= \frac{1}{\tau}[u(x_j,t_k) + \tau\frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\tau^3) - u(x_j,t_k)]$$

$$= \frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\tau^2)$$

(2) $\frac{1}{h^2}[u(x_{j+1},t_k) - 2u(x_j,t_k) + u(x_{j-1},t_k)]$

$$= \frac{1}{h^2}[u(x_j,t_k) + h\frac{\partial u(x_j,t_k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial x^2}$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + \frac{h^5}{5!}\frac{\partial^5 u(x_j,t_k)}{\partial x^5} + O(h^6) + u(x_j,t_k) - h\frac{\partial u(x_j,t_k)}{\partial x}$$
$$+ \frac{h^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial x^2} \quad - \frac{h^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial x^2} \quad + \frac{h^4}{4!}\frac{\partial^4 u(x_j,t_k)}{\partial x^4}$$
$$- \frac{h^5}{5!}\frac{\partial^5 u(x_j,t_k)}{\partial x^5} + O(h^6) - 2u(x_j,t_k)$$

$$= \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4)$$

(3) $\frac{1}{h^2}[u(x_{j+1},t_{k+1}) - 2u(x_j,t_{k+1}) + u(x_{j-1},t_{k+1})]$

$$= \frac{\partial^2 u(x_j,t_{k+1})}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_{k+1})}{\partial x^4} + O(h^4)$$
$$= [\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \tau\frac{\partial}{\partial t}(\frac{\partial^2 u(x_j,t_k)}{\partial x^2}) + O(\tau^2)] + \frac{h^2}{12}[\frac{\partial^4 u(x_j,t_k)}{\partial x^4}$$
$$+ \tau\frac{\partial}{\partial t}(\frac{\partial^4 u(x_j,t_k)}{\partial x^4}) + O(\tau^2)] + O(h^4)$$

$$= [\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \tau a\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2)] + \frac{h^2}{12}[\frac{\partial^4 u(x_j,t_k)}{\partial x^4}$$
$$+ O(\tau^2)] + O(h^4)$$

$$= [\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + (\frac{h^2}{12} + a\tau)\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2) + O(\tau^2 h^2) + O(h^4)$$

将(1)(2)(3)代入*式，得

$$E_j^k u = \frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\tau^2) - a[\theta(\frac{\partial^2 u(x_j,t_k)}{\partial x^2}$$
$$+ (\frac{h^2}{12} + a\tau)\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2) + O(\tau^2 h^2) + O(h^4))$$
$$+ (1-\theta)(\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4))]$$
$$- \frac{\partial u(x_j,t_k)}{\partial t} + a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}$$

$$= [\frac{\tau}{2}a^2 - (a\theta(\frac{h^2}{12} + a\tau) + a(1-\theta)\frac{h^2}{12})]\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2)$$
$$+ O(\tau^2 h^2) + O(h^4)$$

$$= (\frac{\tau}{2}a^2 - a^2\tau\theta - a\frac{h^2}{12})\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2) + O(\tau^2 h^2) + O(h^4)$$

$$= a[a\tau(\frac{1}{2} - \theta) - \frac{h^2}{12}]\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^2) + O(\tau^2 h^2) + O(h^4)$$

当 $a\tau(\frac{1}{2} - \theta) - \frac{h^2}{12} = 0$ 即 $\theta = \frac{1}{2} - \frac{1}{12r}$ 时，截断误差的阶最高

（$O(\tau^2 + h^4)$）。

P112　 3 题

解：　$E_j^k u = L_k u(x_j,t_k) - [Lu]_j^k$

$$= \frac{u(x_j,t_{k+1}) - u(x_j,t_{k-1})}{2\tau} - a\frac{u(x_{j+1},t_k) - u(x_j,t_{k+1}) - u(x_j,t_{k-1}) + u}{h^2}$$

$$-[\frac{\partial u(x_j,t_k)}{\partial t} - a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}]$$

（1）　$\frac{u(x_j,t_{k+1}) - u(x_j,t_{k-1})}{2\tau}$

$$= \frac{1}{2\tau}[(u(x_j,t_k) + \tau\frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial t^3}$$
$$+ O(\tau^4))$$

$$-(u(x_j,t_k) - \tau\frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} - \frac{\tau^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial t^3}$$
$$+ O(\tau^4))]$$

$$= \frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau^2}{6}\frac{\partial^3 u(x_j,t_k)}{\partial t^3} + O(\tau^3)$$

$$= \frac{\partial u(x_j,t_k)}{\partial t} + O(\tau^2)$$

（2）　$\frac{1}{h^2}[u(x_{j+1},t_k) - u(x_j,t_{k+1}) - u(x_j,t_{k-1}) + u(x_{j-1},t_k)]$

$$= \frac{1}{h^2}[u(x_{j+1},t_k) + u(x_{j-1},t_k) - u(x_j,t_{k+1}) - u(x_j,t_{k-1})]$$

$$= \frac{1}{h^2}[(u(x_j,t_k) + h\frac{\partial u(x_j,t_k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial x^3}$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + \frac{h^5}{5!}\frac{\partial^5 u(x_j,t_k)}{\partial x^5} + O(h^6) + u(x_j,t_k)$$

$$- h\frac{\partial u(x_j,t_k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial x^3}$$

$$+ \frac{h^4}{4!}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} - \frac{h^5}{5!}\frac{\partial^5 u(x_j,t_k)}{\partial x^5} + O(h^6) - (2u(x_j,t_k) + \tau^2\frac{\partial^2 u(x_j}{\partial t^2}$$

$$+ \frac{\tau^4}{12}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(\tau^6))]$$

$$= \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4) - \frac{\tau^2}{h^2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\frac{\tau^4}{h^2})$$

$$= \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + (\frac{h^2}{12} - \frac{a^2\tau^2}{h^2})\frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4) + O(\frac{\tau^4}{h^2})$$

$$= \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + O(h^2) + O(\frac{\tau^2}{h^2})$$

代入得　　$E_j^k u = O(\tau^2) + O(h^2) + O(\frac{\tau^2}{h^2})$

P112　　4 题

解：$E_j^k u = L_k u(x_j,t_k) - [Lu]_j^k$

$$= (1+\theta)\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} - \theta\frac{u(x_j,t_k)-u(x_j,t_{k-1})}{\tau}$$

$$-\frac{a}{h^2}[u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})$$

$$+u(x_{j-1},t_{k+1})] - [\frac{\partial u(x_j,t_k)}{\partial t} - a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}]$$

（1）$\frac{1}{\tau}[u(x_j,t_{k+1})-u(x_j,t_k)]$

$$= \frac{1}{\tau}[u(x_j,t_k)+\tau\frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^3)-u(x_j,t_k)]$$

$$= \frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^2)$$

（2）$\frac{1}{\tau}[u(x_j,t_k)-u(x_j,t_{k-1})]$

$$= \frac{1}{\tau}[u(x_j,t_k)-(u(x_j,t_k)-\tau\frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^3)$$

$$= \frac{\partial u(x_j,t_k)}{\partial t}-\frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^2)$$

(3)　由第 3 题知

$$\frac{1}{h^2}[u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})]$$

$$= [\frac{\partial^2 u(x_j,t_k)}{\partial x^2}+(\frac{h^2}{12}+a\tau)\frac{\partial^4 u(x_j,t_k)}{\partial x^4}+O(\tau^2)+O(\tau^2h^2)+O(h^4)$$

代入得

$$E_j^k u = (1+\theta)[\frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^2)] - \theta[\frac{\partial u(x_j,t_k)}{\partial t}$$

$$-\frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^2)] - a[\frac{\partial^2 u(x_j,t_k)}{\partial x^2}+(\frac{h^2}{12}+a\tau)\frac{\partial^4 u(x_j,t_k)}{\partial x^4}$$

$$+O(\tau^2)+O(\tau^2h^2)+O(h^4)] \quad -[\frac{\partial u(x_j,t_k)}{\partial t}-a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}]$$

$$= a[a\tau(\theta-\frac{1}{2})-\frac{h^2}{12}]\frac{\partial^4 u(x_j,t_k)}{\partial x^4}+O(\tau^2)+O(\tau^2h^2)+O(h^4)$$

当 $a\tau(\theta-\frac{1}{2})-\frac{h^2}{12}=0$　即　$\theta=\frac{1}{2}+\frac{1}{12r}$ 时，截断误差的阶最高为

（$O(\tau^2+h^4)$）。

§2 稳定性与收敛性　　P121　　1 题

证明：

$$\frac{u_j^{k+1}-u_j^k}{\tau}=\frac{a}{h^2}[\theta(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1})+(1-\theta)(u_{j+1}^k-2u_j^k+u_{j-1}^k)]$$

(1.13)

$$u_j^{k+1}-u_j^k=r[\theta(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1})+(1-\theta)(u_{j+1}^k-2u_j^k+u_{j-1}^k)]$$

$$-r\theta u_{j+1}^{k+1}+(1+2r\theta)u_j^{k+1}-r\theta u_{j-1}^{k+1}=r(1-\theta)u_{j+1}^k-(1-2r(1-\theta))u_j^k$$
$$+r(1-\theta)u_{j-1}^k$$

$$AU^{k+1}=BU^k \quad \text{即} \quad U^{k+1}=CU^k$$

$$A=(1+2r\theta)I-r\theta S$$

$$B=[1-2r(1-\theta)]I+r(1-\theta)S$$

$$C=[(1+2r\theta)I-r\theta S]^{-1}[(1-2r(1-\theta))I+r(1-\theta)S]$$

$$\lambda_j^c=[(1+2r\theta)-r\theta\lambda_j^s]^{-1}\cdot[1-2r(1-\theta)+r(1-\theta)\lambda_j^s]$$

$$=\frac{1-2r(1-\theta)+2r(1-\theta)\cos j\pi h}{1+2r\theta-2r\theta\cos j\pi h}$$

$$=\frac{1-4r(1-\theta)\sin^2\frac{j\pi h}{2}}{1+4r\theta\sin^2\frac{j\pi h}{2}}$$

（1）　当 $\frac{1}{2}\le\theta\le 1$ 时，恒有 $1-4r(1-\theta)\sin^2\frac{j\pi h}{2}\le 1+4r\theta\sin^2\frac{j\pi h}{2}$

及

$$4r(1-\theta)\sin^2\frac{j\pi h}{2}-1\le 1+4r\theta\sin^2\frac{j\pi h}{2}$$

$\therefore\ \left|\lambda_j^c\right|\le 1$　　　$\therefore$ 此时(1.13)恒稳定。

（2）　当 $0\le\theta<\frac{1}{2}$ 时

若(1.13)稳定时　$-1-M\tau\le\lambda_j^c\le 1+M\tau$

$$-1-M\tau\le\frac{1-4r(1-\theta)\sin^2\frac{j\pi h}{2}}{1+4r\theta\sin^2\frac{j\pi h}{2}}\le 1+M\tau$$

右不等式恒成立

$$-1-4r\theta\sin^2\frac{j\pi h}{2}-M\tau(1+4r\theta\sin^2\frac{j\pi h}{2})\le 1-4r(1-\theta)\sin^2\frac{j\pi h}{2}$$

$$4r(1-2\theta)\sin^2\frac{j\pi h}{2}\le 2+M\tau(1+4r\theta\sin^2\frac{j\pi h}{2})$$

$j=1,2,\cdots,N-1$

$\therefore\ 4r(1-2\theta)\le 2$

$\therefore\ r\le\frac{1}{2}(1-2\theta)$

此时(1.13)稳定的充要条件是　$r\le\frac{1}{2}(1-2\theta)$

$$-1-4r\theta\sin^2\frac{j\pi h}{2}-M\tau(1+4r\theta\sin^2\frac{j\pi h}{2})\le 1-4r(1-\theta)\sin^2\frac{j\pi h}{2}$$

$$4r(1-2\theta)\sin^2\frac{j\pi h}{2}\le 2+M\tau(1+4r\theta\sin^2\frac{j\pi h}{2})$$

$j=1,2,\cdots,N-1$

$\therefore\ 4r(1-2\theta)\le 2$

$\therefore\ r\le\frac{1}{2}(1-2\theta)$

此时(1.13)稳定的充要条件是　$r\le\frac{1}{2}(1-2\theta)$

P121　　2 题

证明：$\frac{1}{12}(u_{j+1}^{k+1}-u_{j+1}^{k})+\frac{5}{6}(u_{j}^{k+1}-u_{j}^{k})+\frac{1}{12}(u_{j-1}^{k+1}-u_{j-1}^{k})$

$$=\frac{r}{2}(u_{j+1}^{k+1}-2u_{j}^{k+1}+u_{j-1}^{k+1}+u_{j+1}^{k}-2u_{j}^{k}+u_{j-1}^{k})$$

$$(\frac{1}{12}-\frac{r}{2})u_{j+1}^{k+1}+(\frac{5}{6}+r)u_{j}^{k+1}+(\frac{1}{12}-\frac{r}{2})u_{j-1}^{k+1}=(\frac{1}{12}+\frac{r}{2})u_{j}^{k+1}+(\frac{5}{6}-r)u_{j}^{k}$$

$$+(\frac{1}{12}+\frac{r}{2})u_{j-1}^{k}$$

$AU^{k+1}=BU^{k}$　　即 $U^{k+1}=CU^{k}$

其　　中　　$A=(\frac{1}{12}-\frac{r}{2})S+(\frac{5}{6}+r)I$

$$B=(\frac{1}{12}+\frac{r}{2})S+(\frac{5}{6}-r)I$$

$$C=[(\frac{5}{6}+r)I+(\frac{1}{12}-\frac{r}{2})S]^{-1}[(\frac{5}{6}-r)I+(\frac{1}{12}+\frac{r}{2})S]$$

$$\lambda_j^c=[(\frac{5}{6}+r)+(\frac{1}{12}-\frac{r}{2})\lambda_j^s]^{-1}[(\frac{5}{6}-r)+(\frac{1}{12}+\frac{r}{2})\lambda_j^s]$$

$$=\frac{(\frac{5}{6}-r)+(\frac{1}{12}+\frac{r}{2})2\cos j\pi h}{(\frac{5}{6}+r)+(\frac{1}{12}-\frac{r}{2})2\cos j\pi h}$$

$$=\frac{\frac{5}{6}+\frac{1}{6}\cos j\pi h-2r\sin^2\frac{j\pi h}{2}}{\frac{5}{6}+\frac{1}{6}\cos j\pi h+2r\sin^2\frac{j\pi h}{2}}$$

$\because \forall r>0$　有

$$\frac{5}{6}+\frac{1}{6}\cos j\pi h-2r\sin^2\frac{j\pi h}{2}\le\frac{5}{6}+\frac{1}{6}\cos j\pi h+2r\sin^2\frac{j\pi h}{2}$$ 及

$$2r\sin^2\frac{j\pi h}{2}-(\frac{5}{6}+\frac{1}{6}\cos j\pi h)\le(\frac{5}{6}+\frac{1}{6}\cos j\pi h)+2r\sin^2\frac{j\pi h}{2}$$

$\therefore \left|\lambda_j^c\right|\le 1$　　$\therefore$ 格式(2.21)恒稳定。

§3 Fourier 方法　P127　1 题

证明：$u_j^{k+1}-u_j^k=r[\theta(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1})+(1-\theta)(u_{j+1}^k-2u_j^k+u_{j-1}^k)]$

令 $u_j^k=v^k e^{i\alpha jh}$ 得

$$v^{k+1}e^{i\alpha jh}-v^k e^{i\alpha jh}=r[\theta v^{k+1}(e^{i\alpha(j+1)h}-2e^{i\alpha jh}+e^{i\alpha(j-1)h})$$

$$+(1-\theta)v^k(e^{i\alpha(j+1)h}-2e^{i\alpha jh}+e^{i\alpha(j-1)h})]$$

$$v^{k+1}-v^k=r[\theta v^{k+1}(e^{i\alpha h}-2+e^{-i\alpha h})+(1-\theta)v^k(e^{i\alpha h}-2+e^{-i\alpha h})]$$

$$v^{k+1}-v^k=2r\theta(\cos\alpha h-1)v^{k+1}+2r(1-\theta)(\cos\alpha h-1)v^k$$

$$[1-2r\theta(\cos\alpha h-1)]v^{k+1}=[1+2r(1-\theta)(\cos\alpha h-1)]v^k$$

$$v^{k+1}=\frac{1-4r(1-\theta)\sin^2\frac{\alpha h}{2}}{1+4r\theta\sin^2\frac{\alpha h}{2}}v^k$$

$$G(ph,\tau)=\frac{1-4r(1-\theta)\sin^2\frac{\alpha h}{2}}{1+4r\theta\sin^2\frac{\alpha h}{2}}$$

(1.13)稳定$\Leftrightarrow$ $|G(ph,\tau)|\le 1+M\tau$

$$\therefore\ \left|\frac{1-4r(1-\theta)\sin^2\frac{\alpha h}{2}}{1+4r\theta\sin^2\frac{\alpha h}{2}}\right|\le 1$$

$$-1-4r\theta\sin^2\frac{\alpha h}{2}\le 1-4r(1-\theta)\sin^2\frac{\alpha h}{2}\le 1+4r\theta\sin^2\frac{\alpha h}{2}$$

右不等式显然恒成立

$\therefore$ 当 $0\le\theta\le\frac{1}{2}$ 时 $4r\sin^2\frac{\alpha h}{2}[1-\theta-\theta]\le 2$ 的充要条件是

$r\le\frac{1}{2}(1-2\theta)^{-1}$

当$\frac{1}{2}<\theta<1$时 $4r\sin^2\frac{\alpha h}{2}(1-2\theta)\le 2$恒成立即 $|G(ph,\tau)|\le 1+M\tau$
恒成立

$\therefore$ 当 $0\le\theta\le\frac{1}{2}$时(1.13)稳定的充要条件是$r\le\frac{1}{2}(1-2\theta)^{-1}$。

§4 判别差分格式稳定性的代数准则

P132 3 题

证明：

$$\begin{cases}(1+ar)u_j^{k+1}-aru_{j-1}^{k+1}=aru_{j+1}^k+(1-ar)u_j^k\\ -aru_{j+1}^{k+2}+(1+ar)u_j^{k+2}=(1-ar)u_j^{k+1}+aru_{j-1}^{k+1}\end{cases}$$

$$\begin{cases}(1+ar)u_j^{k+1}-aru_{j-1}^{k+1}=aru_{j+1}^k+(1-ar)u_j^k\\ -arw_{j+1}^{k+1}+(1+ar)w_j^{k+1}=(1-ar)w_j^k+arw_{j-1}^k\ (w_j^k=u_j^{k+1})\end{cases}$$

令 $u_j^k=v_1^k e^{i\alpha jh},w_j^k=v_2^k e^{i\alpha jh}$

$$\begin{cases}(1+ar)v_1^{k+1}e^{i\alpha jh}-arv_1^{k+1}e^{i\alpha(j-1)h}=arv_1^k e^{i\alpha(j+1)h}+(1-ar)v_1^k e^{i\alpha jh}\\ -arv_2^k e^{i\alpha(j+1)h}+(1+ar)v_2^{k+1}e^{i\alpha jh}=(1-ar)v_2^k e^{i\alpha jh}+arv_2^k e^{i\alpha(j-1)h}\end{cases}$$

$$\begin{cases}(1+ar)v_1^{k+1}-arv_1^{k+1}e^{-i\alpha h}=arv_1^k e^{i\alpha h}+(1-ar)v_1^k\\ -arv_2^{k+1}e^{i\alpha h}+(1+ar)v_2^{k+1}=(1-ar)v_2^k+arv_2^k e^{-i\alpha h}\end{cases}$$

$$\begin{cases}v_1^{k+1}=\dfrac{(1-ar)+are^{i\alpha h}}{(1+ar)-are^{-i\alpha h}}\cdot v_1^k\\ v_2^{k+1}=\dfrac{(1-ar)+are^{-i\alpha h}}{(1+ar)-are^{i\alpha h}}\cdot v_2^k\end{cases}$$

$$\begin{bmatrix}v_1^{k+1}\\ v_2^{k+1}\end{bmatrix}=\begin{bmatrix}\dfrac{(1-ar)+are^{i\alpha h}}{(1+ar)-are^{-i\alpha h}} & 0\\ 0 & \dfrac{(1-ar)+are^{-i\alpha h}}{(1+ar)-are^{i\alpha h}}\end{bmatrix}\begin{bmatrix}v_1^k\\ v_2^k\end{bmatrix}$$

$$|\lambda E-G|=\begin{vmatrix}\lambda-\dfrac{(1-ar)+are^{i\alpha h}}{(1+ar)+are^{-i\alpha h}} & 0\\ 0 & \lambda-\dfrac{(1-ar)+are^{-i\alpha h}}{(1+ar)-are^{i\alpha h}}\end{vmatrix}=0$$

解得 $\lambda_1=\dfrac{(1-ar)+are^{i\alpha h}}{(1+ar)-are^{-i\alpha h}}$， $\lambda_2=\dfrac{(1-ar)+are^{-i\alpha h}}{(1+ar)-are^{i\alpha h}}$

$$\begin{aligned}|\lambda_1|^2&=\left|\frac{1-ar(1-\cos\alpha h)+iar\sin\alpha h}{1+ar(1-\cos\alpha h)+iar\sin\alpha h}\right|^2\\ &=\left|\frac{1-2ar\sin^2(\alpha h/2)+ari\sin\alpha h}{1+2ar\sin^2(\alpha h/2)+ari\sin\alpha h}\right|^2\\ &=\frac{(1-2ar\sin^2(\alpha h/2))^2+a^2r^2\sin^2\alpha h}{(1+2ar\sin^2(\alpha h/2))^2+a^2r^2\sin^2\alpha h}<1\end{aligned}$$

$$\begin{aligned}|\lambda_2|^2&=\left|\frac{1-2ar\sin^2(\alpha h/2)-iar\sin\alpha h}{1+2ar\sin^2(\alpha h/2)+iar\sin\alpha h}\right|^2\\ &=\frac{(1-2ar\sin^2(\alpha h/2))^2+a^2r^2\sin^2\alpha h}{(1+2ar\sin^2(\alpha h/2))^2+a^2r^2\sin^2\alpha h}<1\end{aligned}$$

$\therefore\ \rho(G)\le 1+m\tau$

$\therefore$ 由一致对角化定理知，上述差分格式稳定（这里 H=I）.

第四章 双曲型方程的有限差分法

§1 波动方程的差分逼近 P158 1 题

解：$\dfrac{\partial^2 u}{\partial t^2}=a^2(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2})$

$$\frac{u_{jk}^{n+1}-2u_{jk}^{n}+u_{jk}^{n-1}}{\tau^2}=a^2(\frac{u_{j+1,k}^{n}-2u_{jk}^{n}+u_{j-1,k}^{n}}{h^2}+\frac{u_{j,k+1}^{n}-2u_{jk}^{n}+u_{j,k-1}^{n}}{h^2})$$

（*）

即 $u_{jk}^{n+1}=r^2(u_{j+1,k}^{n}-2u_{jk}^{n}+u_{j-1,k}^{n})+r^2(u_{j,k+1}^{n}-2u_{jk}^{n}+u_{j,k-1}^{n})+2u_{jk}^{n}-u_{jk}^{n-1}$，

其中 $r=a\tau/h$

则有

$$\begin{cases}u_{jk}^{n+1}=r^2(u_{j+1,k}^{n}-2u_{jk}^{n}+u_{j-1,k}^{n})+r^2(u_{j,k+1}^{n}-2u_{jk}^{n}+u_{j,k-1}^{n})+2u_{jk}^{n}-w_{jk}^{n}\\ w_{jk}^{n+1}=u_{jk}^{n}\end{cases}$$

（1）

令 $u_{jk}^{n}=v_1^{n}e^{i\alpha jh}e^{i\beta kh}$， $w_{jk}^{n}=v_2^{n}e^{i\alpha jh}e^{i\beta kh}$

代入（1）得

$$\begin{cases}v_1^{n+1}=r^2(2\cos\alpha h-2)v_1^{n}+r^2(2\cos\beta h-2)v_1^{n}+2v_1^{n}-v_2^{n}\\ v_2^{n+1}=v_1^{n}\end{cases}$$

得 $\begin{bmatrix}v_1^{n+1}\\ v_2^{n+1}\end{bmatrix}=\begin{bmatrix}2(1-2r^2\sin^2(\alpha h/2)-2r^2\sin^2(\beta h/2)) & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_1^{n}\\ v_2^{n}\end{bmatrix}$

$$G(\alpha h,\beta h)=\begin{bmatrix}2(1-2r^2\sin^2(\alpha h/2)-2r^2\sin^2(\beta h/2)) & -1\\ 1 & 0\end{bmatrix}$$

$$\begin{aligned}|\lambda E-G|&=[\lambda-2(1-2r^2\sin^2(\alpha h/2)-2r^2\sin^2(\beta h/2))]\lambda+1\\ &=\lambda^2-(2-C_1^2-C_2^2)\lambda+1\end{aligned}$$

令 $|\lambda E-G|=0$，得

$$\lambda^2-(2-C_1^2-C_2^2)\lambda+1=0 \qquad (2)$$

其中 $C_1=2r\sin(\alpha h/2)$， $C_2=2r\sin(\beta h/2)$

$\therefore$（2）的根按模 ≤ 1 的充要条件是

$|2-C_1^2-C_2^2|\le 2$

$\Rightarrow C_1^2+C_2^2\le 4$

即 $4r^2(\sin^2(\alpha h/2)+\sin^2(\beta h/2))\le 4$

$\Rightarrow r\le\dfrac{\sqrt{2}}{2}$ 是差分格式稳定的必要条件

因此 当 $r\le\dfrac{\sqrt{2}}{2}$ 时，$\{G^k(\theta)\}$ 一致有界；

当 $r\le\dfrac{\sqrt{2}}{2}$ 时，$\{G^k(\theta)\}$ 不一致有界

从而当 $r<\dfrac{\sqrt{2}}{2}$ 时，差分格式（*）是稳定的。

P158　2 题

解：当 $\theta=\dfrac{1}{4}$时，（1.20）如下

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\tau^2}=a^2[\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{4h^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{2h^2}$$

$$+\frac{u_{j-1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{4h^2}]$$

此时差分格式等价于

$$\begin{cases}\dfrac{v_j^{n+1}-v_j^n}{\tau}=a\dfrac{w_{j+\frac{1}{2}}^n-w_{j-\frac{1}{2}}^n+w_{j+\frac{1}{2}}^{n+1}-w_{j-\frac{1}{2}}^{n+1}}{2h}\\ \dfrac{w_{j-\frac{1}{2}}^{n+1}-w_{j-\frac{1}{2}}^{n-1}}{\tau}=a\dfrac{v_j^{n+1}-v_{j-1}^{n+1}+v_j^n-v_{j-1}^n}{2h}\end{cases}$$

（1）

$$G(\theta)=\begin{bmatrix}\dfrac{1-c^2/4}{1+c^2/4} & \dfrac{ic}{1+c^2/4}\\ \dfrac{ic}{1+c^2/4} & \dfrac{1-c^2/4}{1+c^2/4}\end{bmatrix},\quad c=2r\sin\theta$$

由 $|\lambda E-G|=0$得

$$\lambda=\frac{1-c^2/4}{1+c^2/4}\pm\frac{ic}{1+c^2/4}$$

$$|\lambda|^2=(\frac{1-c^2/4}{1+c^2/4})^2+\frac{c^2}{(1+c^2/4)^2}=1$$

可见 $G(\theta)$ 的特征值按绝对值等于 1，且 G 是酉矩阵，因此 $\|G\|_0=1$，

从而矩阵族 $\{G^n(\theta)\}$ 一致有界，即（1）绝对稳定。

§3 初值问题的差分逼近

P174　3 （4.3.32）

第五章 边值问题的变分形式与 Ritz-Galerkin 法

§1 二次函数的极值 P185 1 题

证： 充分性：$\Phi(\lambda)=J(x_0)+\lambda(Ax_0-b,x)+\frac{\lambda^2}{2}(Ax,x)$

$\Phi'(\lambda)=(Ax_0-b,x)+\lambda(Ax,x)$

$\Phi'(0)=(Ax_0-b,x)$

若$\Phi'(0)=0$，即$(Ax_0-b,x)=0\quad\forall x\in R^n$

$Ax_0-b=0$ 即 $Ax=b$

则 x_0是方程$Ax=b$的解

必要性： 若x_0是方程$Ax=b$的解

则$Ax_0-b=0\qquad(Ax_0-b,x)=0$

$\Phi'(0)=(Ax_0-b,x)=0$

所以x_0是$J(x)$的驻点

§3 两点边值问题 P198 1 题

证明： 令 $u(x)=w(x)+v(x)$ 其中 $w(x)=\alpha+(x-a)\beta$ $w(a)=\alpha$

$w'(b)=\beta$

$v(a)=0$ $v'(b)=0$

所以

$$Lu=-\frac{d}{dx}(p\frac{du}{dx})+qu=f$$

$$=-\frac{d}{dx}[p(\frac{dw}{dx}+\frac{dv}{dx})]+q(w+v)=f$$

令 $Lu=-\frac{d}{dx}(p\frac{dv}{dx})+qv=f-(-\frac{d}{dx}p\frac{dw}{dx}+qw)=f_1$

所以（1） 的等价的形式

$$Lu=-\frac{d}{dx}(p\frac{dv}{dx})+qv=f_1$$

$u(a)=\alpha$ $u'(b)=\beta$

其中 $f_1=f-(-\frac{d}{dx}p\frac{dw}{dx}+qw)$

则由定理 2.2 知，v_*是边值问题（2） 的解的充要条件是 $t_*\in H_E^1$ 且满足变分方程

$a(v_*,t)-(f_1,t)=0$ $\forall v\in H_E^1$

又 $a(v_*,t)-(f_1,t)$

$$=\int_a^b(Lv_*-f_1)tdx+p(b)v_*'(b)t(b) \qquad (3)$$

$\Phi(\lambda)=J(u)=J(u_*+\lambda t)$

$$=\frac{1}{2}a(u_*+\lambda t,u_*+\lambda t)-(f,u_*+\lambda t)-p(b)\beta[u_*(b)+\lambda t(b)]$$

$$=J(u_*)+\lambda[a(u_*,t)-(f,t)-p(b)\beta t(b)]+\frac{\lambda^2}{2}a(t,t)$$

$a(u_*,t)-(f,t)-p(b)\beta t(b)$

$$=\int_a^b[p\frac{du_*}{dx}\frac{dt}{dx}+qu_*t-ft]dx-p(b)\beta t(b)$$

$$=\int_a^b(Lu_*-f)tdx+p(b)u_*'(b)t(b)-p(b)\beta t(b) \qquad (4)$$

（3） $\Rightarrow$ （4） 所以可证得。

必要性：若 u_* 是边值问题（1）的解。则 $Lu_*-f=0$ $u_*'(b)=\beta$

所以 $a(u_*,t)-(f,t)-p(b)\beta t(b)=0$ $\Phi'(0)=0$ 且 $\Phi(\lambda)>=\Phi(0)$

u_* 使得

$$J(u_*)=\min_{\substack{u\in H\\ u(a)=\alpha}}J(u)$$

充分行： 若 $\Phi'(0)=0$ 即 $a(u_*,t)-(f,t)-p(b)\beta t(b)=0$

$u_*\in H_E^1\cap C^2$

即 $\int_a^b(Lu_*-f)tdx+p(b)u_*'(b)t(b)-p(b)\beta t(b)=0$

不妨取 $\forall t(x)\in C_0^\infty(I)$ 则 $t(b)=0$

所以 $\int_a^b(Lu_*-f)tdx=0$

由引理知道 $Lu_*-f=0$

取 $t(x)=x-a$ 则有

$p(b)(b-a)(u_*'(b)-\beta)=0$ $p(b)>0$ $b-a>0$

所以 $u_*'(b)=\beta$

所以 u_*是边值问题（1）的解。得证

P198　3 题

解：设 $\forall v \in H_E^1$ 且 $v(a) = v'(a) = 0 \quad v(b) = v'(b) = 0$

则 $\int_a^b (Lu - f)v dx = \int_a^b \frac{d^4u}{dx^4} v dx + \int_a^b uv - fv dx$

$\int_a^b \frac{d^4u}{dx^4} v dx$

$= \frac{d^3u}{dx^3} v \Big|_a^b - \int_a^b \frac{d^3u}{dx^3} \frac{dv}{dx} dx$

$= u^{(3)}(b)v(b) - u''(b)v'(b) + u''(a)v'(a) + \int_a^b \frac{d^2u}{dx^2} \frac{d^2v}{dx^2} dx$

= ……………………………………………

$= u^{(3)}(b)v(b) - u''(b)v'(b) + u''(a)v'(a) + \int_a^b \frac{d^4u}{dx^4} u dx$

又 $v(a) = v'(a) = v(b) = v'(b) = 0$

所以 $\int_a^b \frac{d^4u}{dx^4} v dx = \int_a^b u \frac{d^4v}{dx^4} dx$

所以 $\int_a^b (Lu - f)v dx = \int_a^b u \frac{d^4v}{dx^4} + \int_a^b uv - fv dx$

$= b(u,v) - (f,v) = 0$

其中 $b(u,v) = \int_a^b u \frac{d^4v}{dx^4} + uv) dx$ 是一个双线性泛函。

所以边值问题的变分问题为

求 $u \in H_E^1$ 使得

$b(u,v) - (f,v) = 0$ ，　$\forall v \in H_E^1$　且　$v(a) = v'(a) = 0$

$u(b) = u'(b) = 0$

$J(u) = \frac{1}{2}(Lu,u) - (f,u)$

§4 二阶椭圆边值问题 P205　3 题

解:取一特定函数 $u_0 \in C^2(\bar{u})$　$\frac{\partial u_0}{\partial n} + au_0|_\gamma = \beta$　令 $v = u - u_0$ 则

$\frac{\partial v}{\partial n} + av|_\gamma = 0$

则得（3.3）（3.31）的等价问题

$-\Delta v = f + \Delta u_0 = F$

$\frac{\partial v}{\partial n} + av|_\gamma = 0$

所以 $\bar{J}(v) = \frac{1}{2}(-\Delta v, v) - (F, v)$

$= \frac{1}{2}(v, v) - (F, v)$

$= \frac{1}{2}\iint_u [(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2]dxdy + \frac{1}{2}\int_\gamma av^2 ds - \iint_u Fvdxdy$

$$= \begin{array}{l} \frac{1}{2}\iint_u [(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 dxdy - \iint_u [\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y}]dxdy \\ + \frac{1}{2}\int_\gamma au^2 ds - \int_\gamma auu_0 ds - \iint_u fudxdy - \iint_u \Delta u_0 udxdy + \end{array} \text{常数}$$

$= \bar{J}(u) - \iint_u [\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y}]dxdy + \int_\gamma (\frac{\partial u_0}{\partial n} - \beta)ds - \iint_u \Delta u_0 udxdy$

+常数

其中 $\bar{J}(u) - \iint_u [\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y}]dxdy + \int_\gamma au^2 ds - \iint_u fudxdy$

又由格林第一公式知道

$\iint_u -\Delta u_0 udxdy = \iint_u [\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y}]dxdy - \int_\gamma \frac{\partial u_0}{\partial n} ds$

原问题的变分问题的为

求 $u_* \in H_E^1(u)$ 使得

$J(u_*) = \min\limits_{\frac{\partial u}{\partial n} + au|_\gamma = \beta} J(u)$

$$J(u) = \frac{1}{2}\iint_u [\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y}]dxdy + \frac{1}{2}\int_\gamma au^2 ds - \iint_u fudxdy - \int_\gamma \beta uds$$

P205　4 题

解：(1) 极小位能原理：

设 $u_0 \in C^2(\bar{G})$ 为一特定函数，$u_0|_\Gamma = g$ 令 $v = u - u_0$

则得（3.32）的等价问题：

$$\begin{cases} -\nabla(k\nabla v) + \sigma v = F = f + \frac{\partial}{\partial y}(k\frac{\partial u_0}{\partial y}) - \sigma u_0 \\ v|_\Gamma = 0 \end{cases}$$

$$\begin{aligned}\hat{J}(v) &= \frac{1}{2}(-\nabla(k\nabla v) + \sigma v, v) - (F, v) \\ &= \frac{1}{2}\iint_G -\nabla(k\nabla v)v\,dxdy + \frac{1}{2}\iint_G \sigma v^2\,dxdy - \iint_G Fv\,dxdy\end{aligned}$$

$$\because \iint_G -\nabla(k\nabla v)v\,dxdy$$

$$= -\iint_G [\frac{\partial}{\partial x}(k\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial v}{\partial y})]v\,dxdy$$

$$= -\iint_G [\frac{\partial k}{\partial x}\frac{\partial v}{\partial x}v + k\frac{\partial^2 v}{\partial x^2}v + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y}v + k\frac{\partial^2 v}{\partial y^2}v]dxdy$$

$$= -\iint_G [\frac{\partial k}{\partial x}\frac{\partial v}{\partial x}v + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y}v]dxdy + [-\iint_G (\frac{\partial^2 v}{\partial x^2}kv + \frac{\partial^2 v}{\partial y^2}kv)dxdy]$$

$$\overset{\text{Green第一公式}}{=} -\iint_G [\frac{\partial k}{\partial x}\frac{\partial v}{\partial x}v + \frac{\partial k}{\partial y}\frac{\partial v}{\partial y}v]dxdy + [\iint_G (\frac{\partial v}{\partial x}\frac{\partial kv}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial kv}{\partial y})dxdy - \int_\Gamma \frac{\partial v}{\partial n}kv\,ds]$$

$$= \iint_G k[(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2]dxdy \qquad (1)$$

$$\therefore 令 a(u,v) = \iint_G k(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y})dxdy + \iint_G \sigma uv\,dxdy$$

则 $\hat{J}(v) = \frac{1}{2}a(v,v) - (F,v)$

下面回到原问题

$$\begin{aligned}\hat{J}(v) &= \frac{1}{2}a(v,v) - (F,v) \\ &= \frac{1}{2}\iint_G [k(\frac{\partial u}{\partial x} - \frac{\partial u_0}{\partial x})^2 + k(\frac{\partial u}{\partial y} - \frac{\partial u_0}{\partial y})^2]dxdy + \frac{1}{2}\iint_G \sigma(u-u_0)^2\,dxdy \\ &\quad - \iint_G [f + \frac{\partial}{\partial x}(k\frac{\partial u_0}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial u_0}{\partial y}) - \sigma u_0](u-u_0)dxdy \\ &= \frac{1}{2}\iint_G [k(\frac{\partial u}{\partial x})^2 + k(\frac{\partial u}{\partial y})^2]dxdy - \iint_G k(\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y})dxdy + \iint_G \sigma u^2\,dxdy\end{aligned}$$

$$- \iint_G \sigma u u_0\,dxdy - \iint_G fu\,dxdy - \iint_G [\frac{\partial}{\partial x}(k\frac{\partial u_0}{\partial x})$$

$$- \frac{\partial}{\partial y}(k\frac{\partial u_0}{\partial y})]u\,dxdy + \iint_G \sigma u u_0\,dxdy + 常$$

$$\because \; -\iint_G [\frac{\partial}{\partial x}(k\frac{\partial u_0}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial u_0}{\partial y})]u\,dxdy$$

$$= \iint_G k(\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y})dxdy - \int_\Gamma \frac{\partial u_0}{\partial n}ku\,ds$$

$$= \iint_G k(\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y})dxdy - \int_\Gamma \frac{\partial u_0}{\partial n}kg\,ds$$

$\therefore \quad \hat{J}(v) = J(u) + 常数$

其中 $J(u) = \frac{1}{2}a(u,u) - (f,u)$

∴ 原问题的变分问题为：

求 $u_* \in H^1(G)$ 使得 $J(u_*) = \min\limits_{\substack{u\in H^1(G)\\ u|_\Gamma = g}} J(u)$

$$= \iint_G k(\frac{\partial u}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_0}{\partial y})dxdy - \int_\Gamma \frac{\partial u_0}{\partial n}kg\,ds$$

$\therefore \quad \hat{J}(v) = J(u) + 常数$

其中 $J(u) = \frac{1}{2}a(u,u) - (f,u)$

∴ 原问题的变分问题为：

求 $u_* \in H^1(G)$ 使得 $J(u_*) = \min\limits_{\substack{u\in H^1(G)\\ u|_\Gamma = g}} J(u)$

(2) 虚功原理

两边同乘 v　$v|_\Gamma = 0$

$$\iint_G [-\nabla(k\nabla u) + \sigma u]v - \iint_G fv\,dxdy$$

$$= \iint_G -\nabla(k\nabla u)v\,dxdy + \iint_G \sigma uv\,dxdy - \iint_G fv\,dxdy = 0 \qquad (2)$$

$$\because \iint_G -\nabla(k\nabla u)v\,dxdy$$

$$= -\iint_G [\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x})v + \frac{\partial}{\partial y}(k\frac{\partial u}{\partial y})v]dxdy$$

$$= \iint_G (k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - k\frac{\partial u}{\partial y}\frac{\partial v}{\partial y})dxdy - \int_\Gamma k\frac{\partial u}{\partial n}v\,ds$$

又 $\because v|_\Gamma = 0$

$$\therefore \iint_G -\nabla(k\nabla u)v\,dxdy = \iint_G (k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + k\frac{\partial u}{\partial y}\frac{\partial v}{\partial y})dxdy$$

令 $a(u,v) = \iint_G (k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - k\frac{\partial u}{\partial y}\frac{\partial v}{\partial y})dxdy + \iint_G \sigma uv\,dxdy$

∴ (2)可写成　$a(u,v) - (f,v) = 0$

∴ 原问题的变分问题为：

求 $u \in H^1(G)$ 且 $u|_\Gamma = g$　使得对 $\forall v \in H_0^1(G)$

有　$a(u,v) = (f,v)$