NONCOMMUTATIVE GEOMETRY AND THE RIEMANN ZETA FUNCTION

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According to my first teacher Gustave Choquet one does, by openly facing a well known unsolved problem, run the risk of being remembered more by one's failure than anything else. After reaching a certain age, I realized that waiting "safely" until one reaches the end-point of one's life is an equally selfdefeating alternative.

In this paper I shall first look back at my early work on the classification of von Neumann algebras and cast it in the unusual light of André Weil's Basic Number Theory.

I shall then explain that this leads to a natural spectral interpretation of the zeros of the Riemann zeta function and a geometric framework in which the Frobenius, its eigenvalues and the Lefschetz formula interpretation of the explicit formulas continue to hold even for number fields. We shall then prove the positivity of the Weil distribution assuming the validity of the analogue of the Selberg trace formula. The latter remains unproved and is equivalent to RH for all L-functions with Grössencharakter.

1 Local class field theory and the classification of factors

Let K be a *local* field, i.e. a nondiscrete locally compact field. The action of $K^* = GL_1(K)$ on the additive group K by multiplication,

$$(\lambda, x) \to \lambda x \qquad \forall\, \lambda \in K^*\,,\ x \in K\,, \tag{1}$$

together with the uniqueness, up to scale, of the Haar measure of the additive group K, yield a homomorphism,

$$a \in K^* \to |a| \in \mathbb{R}_+^*\,, \tag{2}$$

from K^* to $\mathbb{R}_+^*$, called the *module* of K. Its range

$$\mathrm{Mod}(K) = \{|\lambda| \in \mathbb{R}_+^* \, ; \; \lambda \in K^*\} \tag{3}$$

is a closed subgroup of $\mathbb{R}_+^*$.

The fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (of quaternions) are the only ones with $\mathrm{Mod}(K) = \mathbb{R}_+^*$, they are called Archimedean local fields.

Let K be a non Archimedean local field, then

$$R = \{x \in K \, ; \; |x| \leq 1\} \,, \tag{4}$$

is the unique maximal compact subring of K and the quotient R/P of R by its unique maximal ideal is a finite field $\mathbb{F}_q$ (with $q = p^\ell$ a prime power). One has,

$$\mathrm{Mod}(K) = q^{\mathbb{Z}} \subset \mathbb{R}_+^* \,. \tag{5}$$

Let K be commutative. An extension $K \subset K'$ of finite degree of K is called *unramified* iff the dimension of K' over K is the order of $\mathrm{Mod}(K')$ as a subgroup of $\mathrm{Mod}(K)$. When this is so, the field K' is commutative, is generated over K by roots of unity of order prime to q, and is a cyclic Galois extension of K with Galois group generated by the automorphism $\theta \in \mathrm{Aut}_K(K')$ such that,

$$\theta(\mu) = \mu^q \,, \tag{6}$$

for any root of unity of order prime to q in K'.

The unramified extensions of finite degree of K are classified by the subgroups,

$$\Gamma \subset \mathrm{Mod}(K) \,, \; \Gamma \neq \{1\} \,. \tag{7}$$

Let then $\overline{K}$ be an algebraic closure of K, $K_{\mathrm{sep}} \subset \overline{K}$ the separable algebraic closure, $K_{\mathrm{ab}} \subset K_{\mathrm{sep}}$ the maximal abelian extension of K and $K_{\mathrm{un}} \subset K_{\mathrm{ab}}$ the maximal unramified extension of K, i.e. the union of all unramified extensions of finite degree. One has,

$$K \subset K_{\mathrm{un}} \subset K_{\mathrm{ab}} \subset K_{\mathrm{sep}} \subset \overline{K} \,, \tag{8}$$

and the Galois group $\mathrm{Gal}(K_{\mathrm{un}} : K)$ is topologically generated by θ called the Frobenius automorphism.

The correspondence (7) is given by,

$$K' = \{x \in K_{\rm un}\,;\ \theta_\lambda(x) = x \quad \forall\, \lambda \in \Gamma\}\,, \tag{9}$$

with rather obvious notations so that θ_q is the θ of (6). Let then W_K be the subgroup of $\mathrm{Gal}(K_{\rm ab} : K)$ whose elements induce on $K_{\rm un}$ an integral power of the Frobenius automorphism. One endows W_K with the locally compact topology dictated by the exact sequence of groups,

$$1 \to \mathrm{Gal}(K_{\rm ab} : K_{\rm un}) \to W_K \to \mathrm{Mod}(K) \to 1\,, \tag{10}$$

and the main result of local class field theory asserts the existence of a canonical isomorphism,

$$W_K \xrightarrow{\sim} K^*\,, \tag{11}$$

compatible with the module.

The basic step in the construction of the isomorphism (11) is the classification of finite dimensional central simple algebras A over K. Any such algebra is of the form,

$$A = M_n(D)\,, \tag{12}$$

where D is a (central) division algebra over K and the symbol M_n stands for $n \times n$ matrices.

Moreover D is the crossed product of an unramified extension K' of K by a 2-cocycle on its cyclic Galois group. Elementary group cohomology then yields the isomorphism,

$$\mathrm{Br}(K) \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z}\,, \tag{13}$$

of the Brauer group of classes of central simple algebras over K (with tensor product as the group law), with the group $\mathbb{Q}/\mathbb{Z}$ of roots of 1 in $\mathbb{C}$.

All the above discussion was under the assumption that K is non Archimedian. For Archimedian fields $\mathbb{R}$ and $\mathbb{C}$ the same questions have an idiotically simple answer. Since $\mathbb{C}$ is algebraically closed one has $K = \overline{K}$ and the whole picture collapses. For $K = \mathbb{R}$ the only non trivial value of the Hasse invariant η is

$$\eta(\mathbb{H}) = -1\,. \tag{14}$$

A Galois group G is by construction totally disconnected so that a morphism from K^* to G is necessarily trivial on the connected component of $1 \in K^*$.

Let k be a *global* field, i.e. a discrete cocompact subfield of a (non discrete) locally compact semi-simple commutative ring A. (Cf. Iwasawa *Ann. of Math.* **57** (1953).) The topological ring A is canonically associated to k and called the Adele ring of k, one has,

$$A = \prod_{\text{res}} k_v\,, \tag{15}$$

where the product is the restricted product of the local fields k_v labelled by the places of k.

When the characteristic of k is $p > 1$ so that k is a function field over $\mathbb{F}_q$, one has

$$k \subset k_{\text{un}} \subset k_{\text{ab}} \subset k_{\text{sep}} \subset \overline{k}\,, \tag{16}$$

where, as above $\overline{k}$ is an algebraic closure of k, k_{sep} the separable algebraic closure, k_{ab} the maximal abelian extension and k_{un} is obtained by adjoining to k all roots of unity of order prime to p.

One defines the Weil group W_k as above as the subgroup of $\text{Gal}(k_{\text{ab}} : k)$ of those automorphisms which induce on k_{un} an integral power of θ,

$$\theta(\mu) = \mu^q \qquad \forall\, \mu \text{ root of 1 of order prime to } p\,. \tag{17}$$

The main theorem of global class field theory asserts the existence of a canonical isomorphism,

$$W_k \simeq C_k = GL_1(A)/GL_1(k)\,, \tag{18}$$

of locally compact groups.

When k is of characteristic 0, i.e. is a number field, one has a canonical isomorphism,

$$\text{Gal}(k_{\text{ab}} : k) \simeq C_k/D_k\,, \tag{19}$$

where D_k is the connected component of identity in the Idele class group $C_k = GL_1(A)/GL_1(k)$, but because of the Archimedean places of k there is no interpretation of C_k analogous to the Galois group interpretation for function fields. According to A. Weil [28], "La recherche d'une interprétation pour C_k si k est un corps de nombres, analogue en quelque manière à l'interprétation par un groupe de Galois quand k est un corps de fonctions, me semble constituer l'un des problèmes fondamentaux de la théorie des nombres à l'heure actuelle ; il se peut qu'une telle interprétation renferme la clef de l'hypothèse de Riemann ...".

Galois groups are by construction projective limits of the finite groups attached to finite extensions. To get connected groups one clearly needs to relax this finiteness condition which is the same as the finite dimensionality of the central simple algebras. Since Archimedian places of k are responsible for the non triviality of D_k it is natural to ask the following preliminary question,

"Is there a non trivial Brauer theory of central simple algebras over $\mathbb{C}$."

As we shall see shortly the *approximately finite dimensional* simple central algebras over $\mathbb{C}$ provide a satisfactory answer to this question. They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* , \tag{20}$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$.

Let us now explain this statement with more care. First we exclude the trivial case $M = M_n(\mathbb{C})$ of matrix algebras. Next $\mathrm{Mod}(M)$ is a virtual subgroup of $\mathbb{R}_+^*$, in the sense of G. Mackey, i.e. an ergodic action of $\mathbb{R}_+^*$. All ergodic flows appear and M_1 is isomorphic to M_2 iff $\mathrm{Mod}(M_1) \cong \mathrm{Mod}(M_2)$.

The birth place of central simple algebras is as the commutant of isotypic representations. When one works over $\mathbb{C}$ it is natural to consider unitary representations in Hilbert space so that we shall restrict our attention to algebras M which appear as commutants of unitary representations. They are called von Neumann algebras. The terms central and simple keep their usual algebraic meaning.

The classification involves three independent parts,

(A) The definition of the invariant $\mathrm{Mod}(M)$ for arbitrary factors (central von Neumann algebras).

(B) The equivalence of all possible notions of approximate finite dimensionality.

(C) The proof that Mod is a complete invariant and that all virtual subgroups are obtained.

The module of a factor M was first defined ([6]) as a closed subgroup of $\mathbb{R}_+^*$ by the equality

$$S(M) = \bigcap_{\varphi} \mathrm{Spec}(\Delta_\varphi) \subset \mathbb{R}_+ \tag{21}$$

where φ varies among (faithful, normal) states on M, i.e. linear forms $\varphi : M \to \mathbb{C}$ such that,

$$\varphi(x^* x) \geq 0 \qquad \forall \, x \in M \, , \; \varphi(1) = 1 \, , \tag{22}$$

while the operator Δ_φ is the *modular operator* ([24])

$$\Delta_\varphi = S_\varphi^* \, S_\varphi \, , \tag{23}$$

which is the *module* of the involution $x \to x^*$ in the Hilbert space attached to the sesquilinear form,

$$\langle x, y \rangle = \varphi(y^* x) \, , \; x, y \in M \, . \tag{24}$$

In the case of local fields the module was a group homomorphism ((2)) from K^* to $\mathbb{R}_+^*$. The counterpart for factors is the group homomorphism, ([6])

$$\delta : \mathbb{R} \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M) \, , \tag{25}$$

from the additive group $\mathbb{R}$ viewed as the dual of $\mathbb{R}_+^*$ for the pairing,

$$(\lambda, t) \to \lambda^{it} \qquad \forall \, \lambda \in \mathbb{R}_+^* \, , \; t \in \mathbb{R} \, , \tag{26}$$

to the group of automorphism classes of M modulo inner automorphisms.

The virtual subgroup,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* \, , \tag{27}$$

is the *flow of weights* ([25] [15] [8]) of M. It is obtained from the module δ as the dual action of $\mathbb{R}_+^*$ on the abelian algebra,

$$C = \text{Center of } M \rtimes_\delta \mathbb{R} \, , \tag{28}$$

where $M \rtimes_\delta \mathbb{R}$ is the crossed product of M by the modular automorphism group δ.

This takes care of (A), to describe (B) let us simply state the equivalence ([5]) of the following conditions

(29) M is the closure of the union of an increasing sequence of finite dimensional algebras.

(30) M is complemented as a subspace of the normed space of all operators in a Hilbert space.

The condition (29) is obviously what one would expect for an approximately finite dimensional algebra. Condition (30) is similar to *amenability* for discrete groups and the implication (30) $\Rightarrow$ (29) is a very powerful tool.

We refer to [5] [15] [12] for (C) and we just describe the actual construction of the central simple algebra M associated to a given virtual subgroup,

$$\Gamma \underset{\sim}{\subset} \mathbb{R}_+^* \,. \tag{31}$$

Among the approximately finite dimensional factors (central von Neumann algebras), only two are not simple. The first is the algebra

$$M_\infty(\mathbb{C}) \,, \tag{32}$$

of all operators in Hilbert space. The second factor is the unique approximately finite dimensional factor of type II_∞. It is

$$R_{0,1} = R \otimes M_\infty(\mathbb{C}) \,, \tag{33}$$

where R is the unique approximately finite dimensional factor with a finite trace τ_0, i.e. a state such that,

$$\tau_0(xy) = \tau_0(yx) \qquad \forall\, x,y \in R \,. \tag{34}$$

The tensor product of τ_0 by the standard semifinite trace on $M_\infty(\mathbb{C})$ yields a semi-finite trace τ on $R_{0,1}$. There exists, up to conjugacy, a unique one parameter group of automorphisms $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$, $\lambda \in \mathbb{R}_+^*$ such that,

$$\tau(\theta_\lambda(a)) = \lambda \tau(a) \qquad \forall\, a \in \mathrm{Domain}\,\tau \,,\ \lambda \in \mathbb{R}_+^* \,. \tag{35}$$

Let first $\Gamma \subset \mathbb{R}_+^*$ be an ordinary closed subgroup of $\mathbb{R}_+^*$. Then the corresponding factor R_Γ with modulo Γ is given by the equality:

$$R_\Gamma = \{x \in R_{0,1} \,;\ \theta_\lambda(x) = x \quad \forall\, \lambda \in \Gamma\} \,, \tag{36}$$

in perfect analogy with (9).

A virtual subgroup $\Gamma \underset{\sim}{\subset} \mathbb{R}_+^*$ is by definition an ergodic action α of $\mathbb{R}_+^*$ on an abelian von Neumann algebra A, and the formula (36) easily extends to,

$$R_\Gamma = \{x \in R_{0,1} \otimes A \,;\ (\theta_\lambda \otimes \alpha_\lambda)\, x = x \quad \forall\, \lambda \in \mathbb{R}_+^*\} \,. \tag{37}$$

(This reduces to (36) for the action of $\mathbb{R}_+^*$ on the algebra $A = L^\infty(X)$ where X is the homogeneous space $X = \mathbb{R}_+^*/\Gamma$.)

The pair $(R_{0,1}, \theta_\lambda)$ arises very naturally in geometry from the geodesic flow of a compact Riemann surface (of genus > 1). Let $V = S^*\Sigma$ be the unit cosphere bundle of such a surface Σ, and F be the stable foliation of the geodesic flow. The latter defines a one parameter group of automorphisms of the foliated manifold (V, F) and thus a one parameter group of automorphisms θ_λ of the von Neumann algebra $L^\infty(V, F)$.

This algebra is easy to describe, its elements are random operators $T = (T_f)$, i.e. bounded measurable families of operators T_f parametrized by the leaves f of the foliation. For each leaf f the operator T_f acts in the Hilbert space $L^2(f)$ of square integrable densities on the manifold f. Two random operators are identified if they are equal for almost all leaves f (i.e. a set of leaves whose union in V is negligible). The algebraic operations of sum and product are given by,

$$(T_1 + T_2)_f = (T_1)_f + (T_2)_f \, , \; (T_1 \, T_2)_f = (T_1)_f \, (T_2)_f \, , \tag{38}$$

i.e. are effected pointwise.

One proves that,

$$L^\infty(V, F) \simeq R_{0,1} \, , \tag{39}$$

and that the geodesic flow θ_λ satisfies (35). Indeed the foliation (V, F) admits up to scale a unique transverse measure Λ and the trace τ is given (cf. [4]) by the formal expression,

$$\tau(T) = \int \mathrm{Trace}(T_f) \, d\Lambda(f) \, , \tag{40}$$

since the geodesic flow satisfies $\theta_\lambda(\Lambda) = \lambda\Lambda$ one obtains (35) from simple geometric considerations. The formula (37) shows that most approximately finite dimensional factors already arise from foliations, for instance the unique approximately finite dimensional factor R_∞ such that,

$$\mathrm{Mod}(R_\infty) = \mathbb{R}_+^* \, , \tag{41}$$

arises from the codimension 1 foliation of $V = S^*\Sigma$ generated by F and the geodesic flow.

In fact this relation between the classification of central simple algebras over $\mathbb{C}$ and the geometry of foliations goes much deeper. For instance using cyclic cohomology together with the following simple fact,

(42) "A connected group can only act trivially on a homotopy invariant cohomology theory",

one proves (cf. [4]) that for any codimension one foliation F of a compact manifold V with non vanishing Godbillon-Vey class one has,

$$\text{(43)} \qquad \mathrm{Mod}(M) \text{ has finite covolume in } \mathbb{R}_+^*\,,$$

where $M = L^\infty(V,F)$ and a virtual subgroup of finite covolume is a flow with a finite invariant measure.

2 Global class field theory and spontaneous symmetry breaking

In the above discussion of approximately finite dimensional central simple algebras, we have been working locally over $\mathbb{C}$. We shall now describe a particularly interesting example (cf. [3]) of Hecke algebra intimately related to arithmetic, and defined over $\mathbb{Q}$.

Let $\Gamma_0 \subset \Gamma$ be an almost normal subgroup of a discrete group Γ, i.e. one assumes,

$$\text{(1)} \qquad \Gamma_0 \cap s\,\Gamma_0\,s^{-1} \text{ has finite index in } \Gamma_0 \qquad \forall\, s \in \Gamma\,.$$

Equivalently the orbits of the left action of Γ_0 on Γ/Γ_0 are all finite. One defines the Hecke algebra,

$$\text{(2)} \qquad \mathcal{H}(\Gamma, \Gamma_0)\,,$$

as the convolution algebra of integer valued Γ_0 biinvariant functions with finite support. For any field k one lets,

$$\text{(3)} \qquad \mathcal{H}_k(\Gamma,\Gamma_0) = \mathcal{H}(\Gamma,\Gamma_0) \otimes_{\mathbb{Z}} k\,,$$

be obtained by extending the coefficient ring from $\mathbb{Z}$ to k. We let $\Gamma = P_{\mathbb{Q}}^+$ be the group of 2×2 rational matrices,

$$\text{(4)} \qquad \Gamma = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \;;\; a \in \mathbb{Q}^+\,,\; b \in \mathbb{Q} \right\}\,,$$

and $\Gamma_0 = P_{\mathbb{Z}}^+$ be the subgroup of integral matrices,

$$\text{(5)} \qquad \Gamma_0 = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \;;\; n \in \mathbb{Z} \right\}\,.$$

One checks that Γ_0 is almost normal in Γ.

To obtain a central simple algebra over $\mathbb{C}$ in the sense of the previous section we just take the commutant of the right regular representation of Γ on $\Gamma_0\backslash\Gamma$, i.e. the weak closure of $\mathcal{H}_{\mathbb{C}}(\Gamma,\Gamma_0)$ in the Hilbert space,

$$\ell^2(\Gamma_0\backslash\Gamma)\,, \tag{6}$$

of Γ_0 left invariant function on Γ with norm square,

$$\|\xi\|^2 = \sum_{\gamma\,\in\,\Gamma_0\backslash\Gamma} |\xi(\gamma)|^2\,. \tag{7}$$

This central simple algebra over $\mathbb{C}$ is approximately finite dimensional and its module is $\mathbb{R}_+^*$ so that it is the same as R_∞ of (41).

In particular its modular automorphism group is highly non trivial and one can compute it explicitly for the state φ associated to the vector $\xi_0 \in \ell^2(\Gamma_0\backslash\Gamma)$ corresponding to the left coset Γ_0.

The modular automorphism group σ_t^φ leaves the dense subalgebra $\mathcal{H}_{\mathbb{C}}$ $(\Gamma,\Gamma_0)\subset R_\infty$ globally invariant and is given by the formula,

$$\sigma_t^\varphi(f)(\gamma) = L(\gamma)^{-it}\,R(\gamma)^{it}\,f(\gamma) \qquad \forall\,\gamma\in\Gamma_0\backslash\Gamma/\Gamma_0 \tag{8}$$

for any $f\in\mathcal{H}_{\mathbb{C}}(\Gamma,\Gamma_0)$. Here we let,

$$\begin{aligned} L(\gamma) &= \text{ Cardinality of the image of } \Gamma_0\,\gamma\,\Gamma_0 \text{ in } \Gamma/\Gamma_0 \\ R(\gamma) &= \text{ Cardinality of the image of } \Gamma_0\,\gamma\,\Gamma_0 \text{ in } \Gamma_0\backslash\Gamma\,. \end{aligned} \tag{9}$$

This is enough to make contact with the formalism of quantum statistical mechanics which we now briefly describe. As many of the mathematical frameworks legated to us by physicists it is characterized “not by this short lived novelty which can too often only influence the mathematician left to his own devices, but this infinitely fecund novelty which springs from the nature of things” (J. Hadamard).

A quantum statistical system is given by,

1) The C^* algebra of observables A,

2) The time evolution $(\sigma_t)_{t\in\mathbb{R}}$ which is a one parameter group of automorphisms of A.

An equilibrium or KMS (for Kubo-Martin and Schwinger) state, at inverse temperature β is a state φ on A which fulfills the following condition,

(10) For any $x,y\in A$ there exists a bounded holomorphic function (continuous on the closed strip), $F_{x,y}(z)$, $0\leq \operatorname{Im} z\leq\beta$ such that

$$\begin{aligned} F_{x,y}(t) &= \varphi(x\,\sigma_t(y)) && \forall\,t\in\mathbb{R} \\ F_{x,y}(t+i\beta) &= \varphi(\sigma_t(y)x) && \forall\,t\in\mathbb{R}\,. \end{aligned}$$

For fixed β the KMS_β states form a Choquet simplex and thus decompose uniquely as a statistical superposition from the pure phases given by the extreme points. For interesting systems with nontrivial interaction, one expects in general that for large temperature T, (i.e. small β since $\beta = \frac{1}{T}$ up to a conversion factor) the disorder will be predominant so that there will exist only one KMS_β state. For low enough temperatures some order should set in and allow for the coexistence of distinct thermodynamical phases so that the simplex K_β of KMS_β states should be non trivial. A given symmetry group G of the system will necessarily act trivially on K_β for large T since K_β is a point, but acts in general non trivially on K_β for small T so that it is no longer a symmetry of a given pure phase. This phenomenon of *spontaneous symmetry breaking* as well as the very particular properties of the critical temperature T_c at the boundary of the two regions are corner stones of statistical mechanics.

In our case we just let A be the C^* algebra which is the *norm* closure of $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in the algebra of operators in $\ell^2(\Gamma_0 \backslash \Gamma)$. We let $\sigma_t \in \mathrm{Aut}(A)$ be the unique extension of the automorphisms σ_t^φ of (8).

For $\beta = 1$ it is tautological that φ is a KMS_β state since we obtained σ_t^φ precisely this way ([24]). One proves ([3]) that for any $\beta \leq 1$ (i.e. for $T = 1$) there exists one and only one KMS_β state.

The compact group G,

$$G = C_{\mathbb{Q}} / D_{\mathbb{Q}} \,, \tag{11}$$

quotient of the Idele class group $C_{\mathbb{Q}}$ by the connected component of identity $D_{\mathbb{Q}} \simeq \mathbb{R}_+^*$, acts in a very simple and natural manner as symmetries of the system (A, σ_t). (To see this one notes that the right action of Γ on $\Gamma_0 \backslash \Gamma$ extends to the action of $P_{\mathcal{A}}$ on the restricted product of the trees of $SL(2, \mathbb{Q}_p)$ where $\mathcal{A}$ is the ring of finite Adeles (cf. [3]).

For $\beta > 1$ this symmetry group G of our system, is spontaneously broken, the compact convex sets K_β are non trivial and have the same structure as K_∞, which we now describe. First some terminology, a KMS_β state for $\beta = \infty$ is called a *ground state* and the KMS_∞ condition is equivalent to *positivity of energy* in the corresponding Hilbert space representation.

Remember that $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ contains $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ so,

$$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \subset A \,. \tag{12}$$

By [3] theorem 5 and proposition 24 one has,

Theorem. *Let $\mathcal{E}(K_\infty)$ be the set of extremal KMS_∞ states.*

a) *The group G acts freely and transitively on $\mathcal{E}(K_\infty)$ by composition, $\varphi \to \varphi \circ g^{-1}$, $\forall\, g \in G$.*
b) *For any $\varphi \in \mathcal{E}(K_\infty)$ one has,*

$$\varphi(\mathcal{H}_{\mathbb{Q}}) = \mathbb{Q}_{\rm ab}\,,$$

and for any element $\alpha \in \mathrm{Gal}(\mathbb{Q}_{\rm ab} : \mathbb{Q})$ there exists a unique extension of $\alpha \circ \varphi$, by continuity, as a state of A. One has $\alpha \circ \varphi \in \mathcal{E}(K_\infty)$.
c) *The map $\alpha \to (\alpha \circ \varphi)\varphi^{-1} \in G = C_k/D_k$ defined for $\alpha \in \mathrm{Gal}(\mathbb{Q}_{\rm ab} : \mathbb{Q})$ is the isomorphism of global class field theory (I.19).*

This last map is independent of the choice of φ. What is quite remarkable in this result is that the existence of the subalgebra $\mathcal{H}_{\mathbb{Q}} \subset \mathcal{H}_{\mathbb{C}}$ allows to bring into action the Galois group of $\mathbb{C}$ on the *values of states*. Since the Galois group of $\mathbb{C} : \mathbb{Q}$ is (except for $z \to \overline{z}$) formed of *discontinuous* automorphisms it is quite surprising that its action can actually be compatible with the characteristic *positivity* of states. It is by no means clear how to extend the above construction to arbitrary number fields k while preserving the three results of the theorem. There is however an easy computation which relates the above construction to an object which makes sense for any global field k. Indeed if we let as above R_∞ be the weak closure of $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0 \backslash \Gamma)$, we can compute the associated pair $(R_{0,1}, \theta_\lambda)$ of section I.

The C^* algebra closure of $\mathcal{H}_{\mathbb{C}}$ is Morita equivalent (cf. M. Laca) to the crossed product C^* algebra,

$$C_0(\mathcal{A}) \rtimes \mathbb{Q}_+^*\,, \tag{13}$$

where $\mathcal{A}$ is the locally compact space of finite Adeles. It follows immediately that,

$$R_{0,1} = L^\infty(\mathbb{Q}_A) \rtimes \mathbb{Q}^*\,, \tag{14}$$

i.e. the von Neumann algebra crossed product of the L^∞ functions on Adeles of $\mathbb{Q}$ by the action of $\mathbb{Q}^*$ by multiplication.

The one parameter group of automorphisms, $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$, is obtained as the restriction to,

$$D_{\mathbb{Q}} = \mathbb{R}_+^*\,, \tag{15}$$

of the obvious action of the Idele class group $C_{\mathbb{Q}}$,

$$(g, x) \to g\,x \qquad \forall\, g \in C_{\mathbb{Q}}\,,\ x \in A_{\mathbb{Q}}/\mathbb{Q}^*\,, \tag{16}$$

on the space $X = A_{\mathbb{Q}}/\mathbb{Q}^*$ of Adele classes.

Our next goal will be to show that the latter space is intimately related to the *zeros* of the Hecke L-functions with Grössencharakter.

(We showed in [3] that the partition function of the above system is the Riemann zeta function.)

3 Weil positivity and the Trace formula

Global fields k provide a natural context for the Riemann Hypothesis on the zeros of the zeta function and its generalization to Hecke L-functions. When the characteristic of k is non zero this conjecture was proved by A. Weil. His proof relies on the following dictionary (put in modern language) which provides a geometric meaning, in terms of algebraic geometry over finite fields, to the function theoretic properties of the zeta functions. Recall that k is a function field over a curve Σ defined over $\mathbb{F}_q$,

Algebraic Geometry	Function Theory
Eigenvalues of action of Frobenius on $H^1_{\mathrm{et}}(\overline{\Sigma}, \mathbb{Q}_\ell)$	Zeros of ζ
Poincaré duality in ℓ-adic cohomology	Functional equation
Lefschetz formula for the Frobenius	Explicit formulas
Castelnuovo positivity	Riemann Hypothesis

We shall describe a third column in this dictionary, which will make sense for any global field. It is based on the geometry of the Adele class space,

$$X = A/k^* \,,\ A = \text{Adeles of } k\,. \tag{1}$$

This space is of the same nature as the space of leaves of the horocycle foliation (section I) and the same geometry will be used to analyse it.

Our spectral interpretation of the zeros of zeta involves Hilbert space. The reasons why Hilbert space (apparently invented by Hilbert for this purpose) should be involved are manifold, let us mention three,

(A) Let $N(E)$ be the number of zeros of the Riemann zeta function satisfying $0 < \mathrm{Im}\rho < E$, then ([22])

$$N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E)\,, \tag{2}$$

where the smooth function $\langle N(E) \rangle$ is given by

$$\langle N(E) \rangle = \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right) + \frac{7}{8} + o(1)\,, \tag{3}$$

while the oscillatory part is

$$N_{\mathrm{osc}}(E) = \frac{1}{\pi}\, \mathrm{Im} \log \zeta \left(\frac{1}{2} + i\,E \right)\,. \tag{4}$$

The numbers $x_j = \langle N(\rho_j) \rangle$ where ρ_n is the imaginary part of the n^{th} zero are of average density one and behave like the eigenvalues of a random Hermitian matrix. This was discovered by H. Montgomery [18] who conjectured (and proved for suitable test functions) that when $M \to \infty$, with $\alpha, \beta > 0$,

$$\#\left\{ (i,j) \in \{1, \ldots, M\}^2\,;\ x_i - x_j \in [\alpha, \beta] \right\} \sim M \int_\alpha^\beta 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 du \tag{5}$$

which is exactly what happens in the Gaussian Unitary Ensemble. Numerical tests by A. Odlyzko [20] and further theoretical work by Katz-Sarnak [17] and J. Keating give overwhelming evidence that zeros of zeta should be the eigenvalues of a hermitian matrix.

(B) The equivalence between RH and the positivity of the Weil distribution on the Idele class group C_k shows that Hilbert space is implicitly present.

(C) The deep arithmetic significance of the work of A. Selberg on the spectral analysis of the Laplacian on $L^2(G/\Gamma)$ where Γ is an arithmetic subgroup of a semi simple Lie group G.

Direct atempts (cf. [2]) to construct the Polya-Hilbert space giving a spectral realization of the zeros of ζ using quantum mechanics, meet the following $-$ sign problem: Let H be the Hamiltonian of the quantum mechanical system obtained by quantizing the classical system,

$$(X, F_t) \tag{6}$$

where X is phase space and $t \in \mathbb{R} \to F_t$ the Hamiltonian flow. Let $N(E)$ be the number of eigenvalues λ of H such that $0 \leq \lambda \leq E$. Then, as for ζ,

$$N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E)\,, \tag{7}$$

where $\langle N(E)\rangle$ is essentially a volume in phase space, while the oscillatory part admits a heuristic asymptotic expansion (cf. [2]) of the form,

$$N_{\rm osc}(E) \sim \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{1}{m} \, \frac{1}{2sh\left(\frac{m\lambda_\gamma}{2}\right)} \, \sin(T_\gamma^{\#} \, m \, E) \tag{8}$$

where the γ are the periodic orbits of the flow F, the $T_\gamma^{\#}$ are their periods and the λ_γ the unstability exponents of these orbits.

One can compare ([2]) (8) with the equally heuristic asymptotic expansion of (4) using the Euler product of ζ which gives, using $-\log(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m}$,

$$N_{\rm osc}(E) \simeq -\frac{1}{\pi} \sum_{p} \sum_{m=1}^{\infty} \frac{1}{m} \, \frac{1}{p^{m/2}} \, \sin((\log p) \, m \, E) \, . \tag{9}$$

Comparing (8) and (9) one gets precious information on the hypothetical "Riemann flow" of M. Berry. The periodic orbits γ should be labelled by the primes p, the periods should be the $\log p$ as well as the unstability exponents λ_p. Also, in order to avoid duplication of orbits, the flow should not be "time reversal symmetric", i.e. non isomorphic to the time reversed:

$$(X, F_{-t}) \, . \tag{10}$$

There is however a fundamental mismatch between (8) and (9) which is the overall $-$ sign in front of (9) and no adjustment of Maslov phases can account for it.

The very same $-$ sign appears in the Riemann-Weil explicit formula,

$$\sum_{L(\chi,\rho)=0} \widehat{h}(\chi,\rho) - \widehat{h}(0) - \widehat{h}(1) = -\sum_{v} \int_{k_v^*}' \frac{h(u^{-1})}{|1-u|} \, d^*u \, , \tag{11}$$

where h is a test function on the Idele class group C_k, $\widehat{h}$ is its Fourier transform,

$$\widehat{h}(x,z) = \int_{C_k} h(u) \, \chi(u) \, |u|^z \, d^*u \, , \tag{12}$$

and the finite values $\int'$ are suitably normalized. If we use the above dictionary when $\mathrm{char}(k) \neq 0$, the geometric origin of this $-$ sign becomes clear, the formula (11) is the Lefschetz formula,

$$\# \text{ of fixed points of } \varphi = \mathrm{Trace}\, \varphi/H^0 - \mathrm{Trace}\, \varphi/H^1 + \mathrm{Trace}\, \varphi/H^2 \tag{13}$$

in which the space $H^1_{\rm et}(\overline{\Sigma}, \mathbb{Q}_\ell)$ which provides the spectral realization of the zeros appears with a $-$ sign. This indicates that the spectral realization of zeros of zeta should be of cohomological nature or to be more specific, that the Polya-Hilbert space should appear as the last term of an exact sequence of Hilbert spaces,

$$0 \to \mathcal{H}_0 \xrightarrow{T} \mathcal{H}_1 \to \mathcal{H} \to 0 \,. \tag{14}$$

The example we have in mind for (14) is the assembled Euler complex for a Riemann surface, where $\mathcal{H}_0$ is the *codimension* 2 *subspace* of differential forms of even degree orthogonal to harmonic forms, where $\mathcal{H}_1$ is the space of 1-forms and where $T = d + d^*$ is the sum of the de Rham coboundary with its adjoint d^*.

Since we want to obtain the spectral interpretation not only for zeta functions but for all L-functions with Grössencharakter we do not expect to have only an action of $\mathbb{Z}$ for $\mathrm{char}(k) > 0$ corresponding to the Frobenius, or of the group $\mathbb{R}_+^*$ if $\mathrm{char}(k) = 0$, but to have the equivariance of (14) with respect to a natural action of the Idele class group $C_k = GL_1(A)/k^*$.

Let $X = A/k^*$ be the Adele class space. Our basic idea is to take for $\mathcal{H}_0$ a suitable completion of the codimension 2 subspace of functions on X such that,

$$f(0) = 0 \,, \ \int f \, dx = 0 \,, \tag{15}$$

while $\mathcal{H}_1 = L^2(C_k)$ and T is the restriction map coming from the inclusion $C_k \to X$, multiplied by $|a|^{1/2}$,

$$(Tf)(a) = |a|^{1/2} \, f(a) \,. \tag{16}$$

The action of C_k is then the obvious one, for $\mathcal{H}_0$

$$(U(g)f)(x) = f(g^{-1} \, x) \qquad \forall \, g \in C_k \tag{17}$$

using the action II.15 of C_k on X, and similarly the regular representation V for $\mathcal{H}_1$.

This idea works but there are two subtle points; first since X is a delicate quotient space the function spaces for X are naturally obtained by starting with function spaces on A and moding out by the "gauge transformations"

$$f \to f_q \,, \ f_q(x) = f(xq) \,, \quad \forall \, q \in k^* \,. \tag{18}$$

Here the natural function space is the Bruhat-Schwarz space $\mathcal{S}(A)$ and by (15) the codimension 2 subspace,

$$\mathcal{S}(A)_0 = \left\{ f \in \mathcal{S}(A)\,;\ f(0) = 0\,,\ \int f\,dx = 0 \right\}. \tag{19}$$

The restriction map T is then given by,

$$T(f)(a) = |a|^{1/2} \sum_{q \in k^*} f(aq) \qquad \forall\, a \in C_k\,. \tag{20}$$

The corresponding function $T(f)$ belongs to $\mathcal{S}(C_k)$ and all functions $f - f_q$ are in the kernel of T.

The second subtle point is that since C_k is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Polya-Hilbert space cannot be a subrepresentation (or unitary quotient) of V. There is an easy way out (which we shall improve shortly) which is to replace $L^2(C_k)$ by $L^2_\delta(C_k)$ using the polynomial weight $(\log^2|a|)^{\delta/2}$, i.e. the norm,

$$\|\xi\|_\delta^2 = \int_{C_k} |\xi(a)|^2\,(1+\log^2|a|)^{\delta/2}\,d^*a\,. \tag{21}$$

Let $\mathrm{char}(k) = 0$ so that $\mathrm{Mod}\,k = \mathbb{R}_+^*$ and $C_k = K \times \mathbb{R}_+^*$ where K is the compact group $C_{k,1} = \{a \in C_k\,;\ |a| = 1\}$.

Theorem. *Let $\delta > 1$, $\mathcal{H}$ be the cokernel of T in $L_\delta(C_k)$ and W the quotient representation of C_k. Let χ be a character of K, $\widetilde{\chi} = \chi \times 1$ the corresponding character of C_k. Let $\mathcal{H}_\chi = \{\xi \in \mathcal{H}\,;\ W(g)\,\xi = \chi(g)\,\xi \quad \forall\, g \in K\}$ and $D_\chi = \lim\limits_{\epsilon\to 0} \frac{1}{\epsilon}\,(W(e^\epsilon) - 1)$. Then D_χ is an unbounded closed operator with discrete spectrum, $\mathrm{Sp}\,D_\chi \subset i\,\mathbb{R}$ is the set of imaginary parts of zeros of the L function with Grössencharakter $\widetilde{\chi}$ which have real part $1/2$. Moreover the spectral multiplicity of ρ is the largest integer $n < \frac{1+\delta}{2}$ in $\{1, \ldots,$ multiplicity as a zero of $L\}$.*

A similar result holds for $\mathrm{char}(k) > 0$. This allows to compute the character of the representation W as,

$$\mathrm{Trace}(W(h)) = \sum_{\substack{L\left(\chi, \frac{1}{2}+\rho\right) = 0 \\ \rho \in i\,\mathbb{R}/N^\perp}} \widehat{h}(\chi, \rho) \tag{22}$$

where $N = \mathrm{Mod}(k)$, $W(h) = \int W(g)\,h(g)\,d^*g$, $h \in \mathcal{S}(C_k)$, $\widehat{h}$ is defined in (12) and the multiplicity is counted as in the theorem.

This result is only preliminary because of the unwanted parameter δ which artificially restricts the multiplicities. The restriction $\mathrm{Re}\,\rho = \frac{1}{2}$ involves the same $\frac{1}{2}$ as in (16), and this has a natural meaning. Indeed the natural Hilbert space norm for $L^2(X)$, namely $\|\xi\|^2 = \int_X |\xi(x)|^2\, dx$ is naturally given upstairs on $\mathcal{S}(A)_0$ by:

$$\|f\|^2 = \int_D |\Sigma\, f(xq)|^2\, |x|\, d^*x\,,\ \forall\, f \in \mathcal{S}(A)_0\,, \tag{23}$$

where D is a fundamental domain for k^* acting on Ideles. For a local field one has indeed the equality

$$dx = |x|\, d^*x\,, \tag{24}$$

(up to normalization) between the additive Haar measure and the multiplicative one. In the global case one has,

$$dx = \lim_{\epsilon \to 0}\ \epsilon\, |x|^{1+\epsilon}\, d^*x\,, \tag{25}$$

and (23) ignores the divergent normalization constant which plays no role in the computation of traces or of adjoint operators. The exponent $\frac{1}{2}$ in (16) turns T into an isometry,

$$T : L^2(X)_0 \to L^2(C_k)\,. \tag{26}$$

The analogue of the Hodge $*$ operation is given on $\mathcal{H}_0$ by the Fourier transform,

$$(Ff)(x) = \int_A f(y)\, \alpha(xy)\, dy \qquad \forall\, f \in \mathcal{S}(A)_0 \tag{27}$$

which, because we take the quotient by (18), is independent of the choice of additive character α of A such that $\alpha \neq 1$ and $\alpha(q) = 1 \quad \forall\, q \in k$. Note also that $F^2 = 1$ on the quotient. On $\mathcal{H}_1$ the Hodge $*$ is given by,

$$(*\,\xi)(a) = \xi(a^{-1}) \qquad \forall\, a \in C_k\,. \tag{28}$$

The Poisson formula means exactly that T commutes with the $*$ operation. This is just a reformulation of the work of Tate and Iwasawa on the proof of the functional equation, but we shall now see that if we follow the proof by Atiyah-Bott ([1]) of the Lefschetz formula we do obtain a clear geometric meaning for the Weil distribution. One can of course as in [10] define inner products on function spaces on C_k using the Weil distribution, but as long as the latter is put by hands and does not appear naturally one has very little chance to understand why it should be positive. Now, let φ be a diffeomorphism of a smooth manifold Σ and assume that the graph of φ is transverse to the diagonal, one can then easily define and compute (cf. [1]) the distribution theoretic trace of the permutation U of functions on Σ associated to φ,

$$(U\xi)(x) = \xi(\varphi(x)) \qquad \forall \, x \in \Sigma \, . \tag{29}$$

One has "Trace" $(U) = \int k(x,x) \, dx$, where $k(x,y) \, dy$ is the Schwarz kernel associated to U, i.e. the distribution on $\Sigma \times \Sigma$ such that,

$$(U\xi)(x) = \int k(x,y) \, \xi(y) \, dy \, . \tag{30}$$

Now near the diagonal and in local coordinates one has,

$$k(x,y) = \delta(y - \varphi(x)) \, , \tag{31}$$

where δ is the Dirac distribution. One then obtains,

$$\text{“Trace”} \ (U) = \sum_{\varphi(x) = x} \frac{1}{\vert 1 - \varphi'(x) \vert} \, , \tag{32}$$

where φ' is the Jacobian of φ and $\vert \ \vert$ stands for the absolute value of the determinant.

With more work ([11]) one obtains a similar formula for the distributional trace of the action of a flow,

$$(U_t \, \xi)(x) = \xi(F_t(x)) \qquad \forall \, x \in \Sigma \, , \ t \in \mathbb{R} \, . \tag{33}$$

It is given, under suitable transversality hypothesis, by

$$\text{“Trace”} \ (U(h)) = \sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{\vert 1 - (F_u)_* \vert} \, d^* u \, , \tag{34}$$

where $U(h) = \int h(t) \, U(t) \, dt$, h is a test function on $\mathbb{R}$, the γ labels the periodic orbits of the flow, including the fixed points, I_γ is the corresponding isotropy subgroup, and $(F_u)_*$ is the tangent map to F_u on the transverse space to the orbits, and finally $d^* u$ is the unique Haar measure on I_γ which is of covolume 1 in $(\mathbb{R}, dt)$.

Now it is truly remarkable that when one analyzes the periodic orbits of the action of C_k on X one finds that not only it qualifies as a Riemann flow in the above sense, but that (34) becomes,

$$\text{(35)} \qquad \text{"Trace" } (U(h)) = \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u \, .$$

Thus, the isotropy subgroups I_γ are parametrized by the places v of k and coincide with the natural cocompact inclusion $k_v^* \subset C_k$ which relates local to global in class field theory. The denominator $|1-u|$ is for the module of the local field k_v and the u^{-1} in $h(u^{-1})$ comes from the discrepancy between notations (16) and (28). It turns out that if one normalizes the Haar measure d^*u of modulated groups as in Weil [27], by,

$$\text{(36)} \qquad \int_{1 \leq |u| \leq \Lambda} d^*u \sim \log \Lambda \qquad \text{for } \Lambda \to \infty \, ,$$

one gets the same covolume 1 condition as in (34).

The transversality condition imposes the condition $h(1) = 0$. The distributional trace for the action of C_k on C_k by translations vanishes under the condition $h(1) = 0$.

Remembering that $\mathcal{H}_0$ is the codimension 2 subspace of $L^2(X)$ determined by the condition (15) and computing the characters of the corresponding 1-dimensional representations gives,

$$\text{(37)} \qquad h \to \widehat{h}(0) + \widehat{h}(1) \, .$$

Thus equating the alternate sum of traces on $\mathcal{H}_0$, $\mathcal{H}_1$ with the trace on the cohomology should thus provide the geometric understanding of the Riemann-Weil explicit formula (11) and in fact of RH using (21) if it could be justified for some value of δ.

The trace of permutation matrices is positive and this explains the Hadamard positivity,

$$\text{(38)} \qquad \text{"Trace" } (U(h)) \geq 0 \qquad \forall \, h \, , \; h(1) = 0 \, , \; h(u) \geq 0 \quad \forall \, u \in C$$

(not to be confused with Weil postivity).

To eliminate the artificial parameter δ and give rigorous meaning, as a Hilbert space trace, to the distribution "trace", one proceeds as in the Selberg trace formula [23] and introduces a cutoff. In physics terminology the divergence of the trace is both infrared and ultraviolet as is seen in the simplest case of the action of K^* on $L^2(K)$ for a local field K. In this local case one lets,

$$R_\Lambda = \widehat{P}_\Lambda \, P_\Lambda \,,\ \Lambda \in \mathbb{R}_+ \,, \tag{39}$$

where P_Λ is the orthogonal projection on the subspace,

$$\{\xi \in L^2(K) \,;\ \xi(x) = 0 \qquad \forall \, x, |x| > \Lambda\} \,, \tag{40}$$

while $\widehat{P}_\Lambda = F \, P_\Lambda \, F^{-1}$, F the Fourier transform.

One proves ([9]) in this local case the following analogue of the Selberg trace formula,

$$\mathrm{Trace}\,(R_\Lambda \, U(h)) = 2\, h(1) \log'(\Lambda) + \int' \frac{h(u^{-1})}{|1-u|} \, d^* u + o(1) \tag{41}$$

where $h \in \mathcal{S}(K^*)$ has compact support, $2 \log'(\Lambda) = \int_{\lambda \in K^*, \, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^* \lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

As it turns out this principal value agrees with that of Weil for the choice of F associated to the standard character of K.

Let k be a global field and let first S be a finite set of places of k containing all the infinite places. To S corresponds the following localized version of the action of C_k on X. One replaces C_k by

$$C_S = \prod_{v \in S} k_v^* / O_S^* \,, \tag{42}$$

where $O_S^* \subset k^*$ is the group of S-units. One replaces X by

$$X_S = \prod_{v \in S} k_v / O_S^* \,. \tag{43}$$

The Hilbert space $L^2(X_S)$, its Fourier transform F and the orthogonal projection P_Λ, $\widehat{P}_\Lambda = F \, P_\Lambda \, F^{-1}$ continue to make sense, with

$$\mathrm{Im}\, P_\Lambda = \{\xi \in L^2(X_S) \,;\ \xi(x) = 0 \quad \forall \, x \,,\ |x| > \Lambda\} \,. \tag{44}$$

As soon as S contains more than 3 elements, (e.g. $\{2, 3, \infty\}$ for $k = \mathbb{Q}$) the space X_S is an extremely delicate quotient space. It is thus quite remarkable that the *trace formula* holds,

Theorem. *For any $h \in \mathcal{S}_c(C_S)$ one has, with $R_\Lambda = \widehat{P}_\Lambda \, P_\Lambda$,*

$$\textit{Trace}\,(R_\Lambda \, U(h)) = 2\log'(\Lambda)\, h(1) + \sum_{v \in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^* u + o(1)$$

where the notations are as above and the finite values $\int'$ depend on the additive character of Πk_v defining the Fourier transform F. When $\mathrm{Char}\,(k) = 0$ the projectors P_Λ, $\widehat{P}_\Lambda$ commute on L^2_χ for Λ large enough so that one can replace R_Λ by the orthogonal projection Q_Λ on $\mathrm{Im}\, P_\Lambda \cap \mathrm{Im}\, \widehat{P}_\Lambda$. The situation for $\mathrm{Char}\,(k) = 0$ is more delicate since P_Λ and $\widehat{P}_\Lambda$ do not commute (for Λ large) even in the local Archimedean case. But fortunately [21] these operators commute with a specific second order differential operator, whose eigenfunctions, the Prolate Spheroidal Wave functions provide the right filtration Q_Λ. This allows to replace R_Λ by Q_Λ and to state the global trace formula

$$\mathrm{Trace}\,(Q_\Lambda \, U(h)) = 2\log'(\Lambda)\, h(1) + \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^* u + o(1)\,. \tag{45}$$

Our final result is that the validity of this trace formula implies (in fact is equivalent to) the positivity of the Weil distribution, i.e. *RH* for all L-functions with Grössencharakter. Moreover the filtration by Q_Λ allows to define the Adelic cohomology and to complete the dictionary between the function theory and the geometry of the Adele class space.

Function Theory	Geometry
Zeros and poles of Zeta	Eigenvalues of action of C_k on Adelic cohomology
Functional Equation	$*$ operation
Explicit formula	Lefschetz formula
RH	Trace formula

References

[1] M.F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: I, *Annals of Math,* **86** (1967), 374-407.

[2] M. Berry, Riemann's zeta function: a model of quantum chaos, *Lecture Notes in Physics,* **263**, Springer (1986).

[3] J.-B. Bost and A. Connes, Hecke Algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Mathematica, New Series* **1**, n. 3 (1995), 411-457.

[4] A. Connes, Noncommutative Geometry, Academic Press (1994).

[5] A. Connes, Classification of injective factors, *Ann. of Math.,* **104**, n. 2 (1976), 73-115.

[6] A. Connes, Une classification des facteurs de type III, *Ann. Sci. Ecole Norm. Sup.,* **6**, n. 4 (1973), 133-252.

[7] A. Connes, Formule de trace en Géométrie non commutative et hypothèse de Riemann, *C.R. Acad. Sci. Paris Ser. A-B* (1996)

[8] A. Connes and M. Takesaki, The flow of weights on factors of type III, *Tohoku Math. J.,* **29** (1977), 473-575.

[9] A. Connes, Trace formula in Noncommutative Geometry and the zeros of the Riemann zeta function. To appear in Selecta Mathematica.

[10] D. Goldfeld, A spectral interpretation of Weil's explicit formula, *Lecture Notes in Math.,* **1593**, Springer Verlag (1994), 135-152.

[11] V. Guillemin, Lectures on spectral theory of elliptic operators, *Duke Math. J.,* **44**, n. 3 (1977), 485-517.

[12] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1, *Acta Math.,* **158** (1987), 95-148.

[13] S. Haran, Riesz potentials and explicit sums in arithmetic, *Invent. Math.,* **101** (1990), 697-703.

[14] B. Julia, Statistical theory of numbers, *Number Theory and Physics, Springer Proceedings in Physics,* **47** (1990).

[15] W. Krieger, On ergodic flows and the isomorphism of factors, *Math. Ann.,* **223** (1976), 19-70.

[16] N. Katz and P. Sarnak, Random matrices, Frobenius eigenvalues and Monodromy, (1996) , Book, to appear.

[17] N. Katz and P. Sarnak, Zeros of zeta functions, their spacings and spectral nature, (1997), to appear.

[18] H. Montgomery, The pair correlation of zeros of the zeta function, *Analytic Number Theory,* AMS (1973).

[19] M.L. Mehta, Random matrices, Academic Press,(1991).

[20] A. Odlyzko, On the distribution of spacings between zeros of zeta functions, *Math. Comp.* **48** (1987), 273-308.

[21] D. Slepian and H. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty I, *Bell Syst. Tech. J.* **40** (1961).

[22] B. Riemann, Mathematical Werke, Dover, New York (1953).

[23] A.Selberg, *Collected papers*, Springer (1989).

[24] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Math. bf 128, Springer (1989).

[25] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.* **131** (1973), 249-310.

[26] A. Weil, Basic Number Theory, Springer, New York (1974).

[27] A. Weil, Sur les formules explicites de la théorie des nombres, *Izv. Mat. Nauk.,* (Ser. Mat.) **36**, 3-18.

[28] A. Weil, Sur la théorie du corps de classes, *J. Math. Soc. Japan*, **3**, (1951).

[29] D. Zagier, Eisenstein series and the Riemannian zeta function, *Automorphic Forms, Representation Theory and Arithmetic,* Tata, Bombay (1979), 275-301.