CHAPTER 3

SECTION 23

1.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

If $(X, \mathcal{T})$ is disconnected then there is a separation in the topology $\mathcal{T}$ which is also a separation in $\mathcal{T}'$. Equivalently, if $(X, \mathcal{T}')$ is connected then so is $(X, \mathcal{T})$.

2.

Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\cup A_n$ is connected.

If $\cup A_n$ is disconnected then there is a separation (U, V) of the union, then each set of the sequence being connected lies within either U or V (Lemma 23.2). Suppose that $A_1 \subset U$, then, by induction, each $A_n \subset U$, and V is empty. Contradiction.

3.

Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\cup A\alpha)$ is connected.

If there is a separation (U, V) of the union, then A and all A_α lie within U or V (Lemma 23.2). Suppose, $A \subset U$, then each $A_\alpha \subset U$, and V is empty. Contradiction.

Alternatively, the connected (by Theorem 23.3) subspaces $\{A \cup A_\alpha\}$ share a point (apply Theorem 23.3 again).

4.

Show that if X is an infinite set, it is connected in the finite complement topology.

There are no two disjoint nonempty open subsets, as the complement of a nonempty open set is finite, and the only finite open set is the empty set.

Alternatively, closed sets are finite or X, while open sets are $\emptyset$ or infinite.

5.

A space is ***totally disconnected*** if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

If $V \subset X$ and $x \in V$, then $\{x\}$ and $V - \{x\}$ form a separation of V unless $V = \{x\}$. The converse does not hold: see Example 4.

6.

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects $\mathrm{Bd}A$.

If C does not intersect the boundary, then $C \cap \mathrm{Int}A = C \cap \overline{A} \neq \emptyset$ and $C \cap \mathrm{Int}(X - A) = C \cap \overline{X - A} \neq \emptyset$ is a separation of C.

7.

Is the space $\mathbb{R}_l$ connected? Justify your answer.

No, $\mathbb{R} = \overline{\mathbb{R}}_+ \cup \mathbb{R}_-$ is a separation.

8.

Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

Argh! Suppose it is disconnected, so that there is a set A which is both open and closed. For any sequence $\mathbf{x} \in A$, there must be $B = B(\mathbf{x}, \epsilon) \subset A$, and for any $\mathbf{y} \notin A$, there must be $B' = B(\mathbf{y}, \delta) \subset \mathbb{R}^\omega - A$. The set B contains all sequences that differ from $\mathbf{x}$ by no more than $\delta < \epsilon$ in each coordinate, which still have to be in A. And similarly, for B', $\mathbf{y}$ and $\mathbb{R}^\omega - A$. So, for example, $\overline{\mathbb{R}^\infty}$ in the uniform topology, i.e. the set of sequences converging to 0, would not work. At the same time, if A is the set of bounded sequences, and $X - A$ is the set of unbounded sequences, then both seem to be open in the uniform topology. So, the answer is no, it is not connected.

9.

Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Let $M = X \times Y - A \times B$. For any $x \notin A$, $V_x = \cup_{y \in Y}\{(x, y)\} \subset M$ is connected as a set homeomorphic to Y. And for all $y \notin B$, $H_y = \cup_{x \in X}\{(x, y)\} \subset M$ is connected as well. Suppose, $a \notin A$ and $b \notin B$. Then, $C = V_a \cup H_b \subset M$ is connected (the two sets have the point (a, b) in common). Moreover, for every $x \notin A$ and $y \notin B$, V_x and H_y intersect C, and $M = \cup_{x \notin A} V_x \cup \cup_{y \notin B} H_y$. Therefore, by Exercise 3, M is connected.

10.

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of connected spaces; let X be the product space

$$X = \prod_{\alpha\in J} X_\alpha.$$

Let $\mathbf{a} = (a_\alpha)$ be a fixed point of X.

(a) Given any finite subset K of J, let $\mathbf{x}_K$ denote the subspace of X consisting of all points $\mathbf{x} = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show that $\mathbf{x}_K$ is connected.

(b) Show that the union Y of the spaces X_K is connected.

(c) Show that X equals the closure of Y; conclude that X is connected.

(a) It is homeomorphic to a finite product of connected spaces.

(b) They all have a common point ($\mathbf{a}$).

(c) This is the main part of the proof. Take any point $\mathbf{x} \in X$. Any its neighborhood in the product topology has only finite number of coordinates restricted by some proper open subsets of the corresponding coordinate spaces. A point that equals $\mathbf{x}$ at these finite number of coordinates, and equals $\mathbf{a}$ at all others coordinates, belongs to the neighborhood and to Y. Therefore, the closure of Y is the whole space X, and by Theorem 23.4, $X = \overline{Y}$ is connected.

11.

Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Suppose $X = U \cup V$ where U and V are open. Then, each $p^{-1}(\{y\})$ being connected must lie within either U or V. Therefore, U and V are saturated. Then, by a remark after the definition on page 137, $p(U)$ and $p(V)$ are open. Moreover, they are disjoint (being images of disjoint saturated sets), and, since p is surjective, $p(U) \cup p(V) = Y$. Since Y is connected, we conclude that one of the sets U or V must be empty.

12.

Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

It is sufficient to show for one set only, so we show that $Y \cup A$ is connected. Suppose it is not, then let $Y \cup A = C \cup D$ where C and D are nonempty disjoint open subsets of $Y \cup A$. Since Y is a connected subset of $Y \cup A$, it must lie within either C or D (Lemma 23.2), so suppose $Y \subset C$, so that $D \subset A$. We have $X = C \cup D \cup B$. Using Lemma 23.1, no limit point of C can be in D, and no limit point of B can be in $D \subset A$, so that $B \cup C$ is closed, and D is open in X. But no limit point of $D \subset A$ can lie in C or B. So that D is closed in X. Therefore, D is both open and closed in X. Contradiction.

SECTION 24

1.

(a) The three sets in the question are all of different order types, but the question is whether you can construct a homeomorphism of their topologies regardless of the order itself. If we remove a point from each of two homeomorphic spaces they will remain homeomorphic. If one space is connected so is the other. Remove 1 from (0,1], it is still connected, but any point removed from (0,1) makes it disconnected. So that (0,1] is not homeomorphic to (0,1) (for similar reason [0,1) is not homeomorphic to (0,1) — we need it later, neither is [0,1]). The only two points that being removed keep [0,1] connected are 0 and 1, but since (0,1) is not homeomorphic to [0,1) or (0,1], we conclude that (0,1] is not homeomorphic to [0,1].(b) $X=[0,1]$, $Y=(0,1)$, $f(x)=x/2+1/4$, $g(y)=y$.(c) Removing 0 makes $\mathbb{R}$ disconnected but removing any point from $\mathbb{R}^n$ leaves it connected.

2.

Example 6 of §18 proves that S^1 is connected. Let $g: S^1 \to \mathbb{R}$ be the continuous function $g(s)=f(s)-f(-s)$ (§21). For any $s \in S^1 : g(-s)=-g(s)$, therefore, by the Intermediate Value Theorem, there is x such that $g(x)=0$ or $f(-x)=f(x)$.

3.

Suppose $f(0)>0$ and $f(1)<1$. X is connected (in all three cases). Let $g(x)=f(x)-x$. It is continuous (§21) and $g(0)>0$, $g(1)<0$. By the Intermediate Value Theorem there is $x \in (0,1)$ such that $g(x)=0$ or $f(x)=x$. If $X=[0,1)$ or $X=(0,1)$ the problem is that we cannot take the values at both 0 and 1 and apply the theorem. In fact, $f(x)=x/2+1/2$ does not have a fixed point if 1 is not in X .

4.

If there were an empty interval (a,b) then $(-\infty,b)\cup(a,+\infty)$ would be a separation, therefore, for any $a<b$ there is $c: a<c<b$. To show the least upper bound property suppose that A is nonempty and bounded from above. Let B be the set of all upper bounds of A , it is nonempty. All we need to show is that B has the least element. Let $A'=\{x \in X | \exists a \in A : x \leq a\}$. Note that b is an upper bound of A iff for every $a \in A$: $b \geq a$ iff for every $a \in A' : b \geq a$ iff b is an upper bound of A' . Therefore, B is the set of all upper bounds of A' as well. If A' has the greatest element then it is the least element of B , so, suppose it does not. Then A' is open: if $a \in A'$ then, since there is no largest element in A' , there is $a' \in A'$ such that $a \in (-\infty, a') \subseteq A'$. Since X is connected, A' is not closed and there is a limit point b of A' in B . It is an upper bound of A' , and suppose there is $b' \in B$ such that $b'<b$. Then $(b',+\infty)$ is a nonempty open set that contains b but does not intersect A' . This contradicts the assumption that b is a limit point of A' . So b is the least element in B and the least upper bound of A .

5.

(a) Yes, it is a linear continuum. For $(z, x) < (z', y)$ we have a point between them (consider two cases $z = z', x < y$ and $z < z', x < 1$). If A is bounded from above by (z, x) then $\pi_1(A)$ is bounded from above by z and it has the greatest element z' . Then let $y = \sup \pi_2(A \cap (z' \times [0, 1)))$. If $y = 1$ then $(z' + 1, 0)$ is the least upper bound, and if $y < 1$ then (z', y) is the least upper bound. (See also the next exercise.) (b) The other way around it does not work: there is no element between, say, $(0, 1)$ and $(0, 2)$. Also, it is not connected. (c) It is. If $(x, a) < (y, b)$ then $((x + y)/2, (a + b)/2)$ is between them. If A is bounded from above by (x, a) , then $\pi_1(A)$ is bounded from above by x and let $x' = \sup \pi_1(A) \leq x$. If $x' \in \pi_1(A)$ then let $y = \sup \pi_2((x' \times [0, 1]) \cap A) \in [0, 1]$, and then (x', y) is the least upper bound. If $x' \notin \pi_1(A)$ then $(x', 0)$ is the least upper bound. (d) Note that in (c) we needed both points $(x', 0)$ and $(x', 1)$ to be in the space: the first one for the case $x' \notin \pi_1(A)$ and the second one for the case $x' \in \pi_1(A)$ and $y = 1$. This suggests that the set in (d) is not a linear continuum. Indeed, $\{0\} \times [0, 1)$ is bounded from above but does not have the least upper bound.

6.

Indeed, for any two elements $(x, a) < (y, b)$ there is an element between them (consider two cases $x < y, a < 1$ and $x = y, a < b$). We show the greatest lower bound property. If A is a nonempty set then $\pi_1(A)$ has the least element x and the greatest lower bound of A is $(x, \inf \pi_2((x \times [0, 1)) \cap A))$. By the way, $X \times [0, 1]$ need not be a linear continuum (it is not iff X has an element that has a successor).

7.

(a) "Order preserving" implies "injective". Therefore, f is bijective. Order preserving also implies that f preserves the order topology.(b) It is increasing from 0 to $+\infty$ (this part may require using the intermediate value theorem if it was not proved before), therefore, it is order preserving and surjective and, by (a), homeomorphism. Therefore, its inverse is continuous.(c) f is obviously increasing and surjective. In the order topology on X $[0, +\infty)$ would not be open, while it is open in the subspace topology. Therefore, f would be a homeomorphism had X the order topology, but in the subspace topology there are too many open sets in X which makes f^{-1} not continuous: $[0, +\infty)$ is open in the subspace topology (not in the order topology) but $f([0, +\infty)) = [0, +\infty)$ which is not open in $\mathbb{R}$.

8.

(a) Let $\mathbf{x}$ and $\mathbf{y}$ be two points in the product. Let $t \in [0,1]$ and $\mathbf{z}(t)$ be a point such that $z_i(t) = f_i(t)$ where $f_i : [0,1] \to X_i$ is a path connecting x_i with y_i in X_i . Since any open subbasis set U in the product is an inverse projection $\pi_i^{-1}(U_i)$ of a proper open set $U_i \subset X_i$ for some i , the preimage $\mathbf{z}^{-1}(U)$ is just $f_i^{-1}(U_i)$ which is open in $[0,1]$. (The proof here repeats partially the proof of Theorem 19.6, which can be used here instead.)(b) No, the topologist' s sine curve is an example.(c) Yes, the composition of two continuous functions is a continuous function.(d) Take any two points. If they are in the same set in the collection, there is a path between them. If they are not in the same set in the collection then there is a path connecting the first point to a common point of all sets in the collection and another one connecting the common point to the second point, the joint path is still continuous and is a path connecting the point. I believe a more general statement is also true: if a path connected subspace has a common point with any set in a collection of path connected subspaces, then the union of the set with the union of the sets in the collection is path connected (the proof must be similar — the only difference should be that the constructed path may consist of three parts).

9.

For any point there is an uncountable number of lines passing through it which do not intersect A . For any two points there is a pair of lines that do intersect each other but do not intersect the set A . So, both points are connected to the point of intersection of the lines, and, therefore, are connected.

10.

If $x \in U$ let $A = \{y \in U\}$ such that there is a path between x and y . Using 8(d), $A \subseteq U$ is path connected. If $y \in A$ then there is a basis neighborhood $V \subseteq U$ of y , V is a ball, and y is path connected with any point in V (just take the closed line interval connecting the points). Therefore, $V \subseteq A$ and A is open. For similar reason, if $z \in U - A$ then there is a neighborhood V of z contained in U . If some point in V were path connected to x then so would be z , but $z \notin A$. Therefore, $U - A$ is open. Then A is closed in U . A is not empty as $x \in A$, so if $U - A$ is not empty as well, then there is a separation of U . Therefore, $U = A$ is path connected.

11.

$(0,1)$ is connected in $\mathbb{R}$ and its interior is connected but its boundary is not. Two balls in $\mathbb{R}^2$ touching at a common boundary point is a connected set (as the union of two connected sets with a common point) and its boundary is connected (for the same reason), but its interior is not connected (as the union of two disjoint open sets). At the same time $\mathbb{Q} \cup (-\infty, 0)$ has connected interior and boundary but is not connected.

12.

(a) If $f:[a,c)\to[0,1)$ is a bijection preserving the order, then $f:[a,b)\to[0,f(b))$ is a bijection preserving the order, similar for $[b,c)$. Vice versa, if there are two order preserving bijections from $[a,b)$ and $[b,c)$ to $[0,1/2)$ and $[1/2,1)$ then the joint function is the order preserving bijection needed.(b) $x_0<x_i<x_{i+1}<b$ and $[x_0,b)$ has the order type of $[0,1)$ implies, by applying (a) two times, that $[x_i,x_{i+1})$ has the order type of $[0,1)$. Vice versa, we construct bijective order preserving functions $[x_i,x_{i+1})\to[1-1/2^i,1-1/2^{i+1})$ and the resulting function is an order preserving surjective function onto $[0,1)$.(c) Transfinite induction: we show that if it holds for any $x\in S_a-\{a_0\}$ then it holds for a . If a has an immediate predecessor b then $[a_0\times 0,b\times 0)$ is either empty or has the order type of $[0,1)$, and $[b\times 0,a\times 0)$ has it as well. By (a), the property holds for a . If a has no immediate predecessor, then the set of all elements lower than a has it as the supremum (for any $b<a$ there is $c: b<c<a$) and we can construct a sequence of points converging to a . This is not a trivial fact, I think (and I don' t remember it was covered anywhere before). Indeed, if we consider $\overline{S}_\Omega$, for example, then Ω is a limit point of S_Ω but there is no sequence converging to it (any sequence is a countable subset and has an upper bound in S_Ω). However, in the case of S_Ω we prove it as follows: S_a is countable; let us enumerate all its elements with positive integer indexes: $S_a=\{s_n\}_{n=0,1,\dots}$; then let n_0 be such that $s_{n0}=a_0$, let n_1 be the minimum index greater than n_0 such that $s_{n_1}>s_{n0}$, let n_2 be the minimum index such that $n_2>n_1$ and $s_{n_2}>s_{n1}$, etc. (we can always do this as there is no largest element in S_a , therefore, there is infinite number of elements greater than any $x\in S_a$); note that for every k , $n_k\geq k$ and $s_{n_k}\geq s_m$ for all $m\leq n_k$; if $x\in S_a$ then $x=s_k$ for some k and $s_{n_m}\geq s_{n_k}\geq s_k=x$ for all $m\geq k$. Now, once we have the sequence converging to a , by using (b), $[a_0\times 0,a\times 0)$ has the order type of $[0,1)$. By the principle of transfinite induction, this holds for any $a>a_o$. (d) We show that any point is path connected to $(a_1,0)$ where a_1 is the immediate successor of a_0 . Indeed, (a_0,x) for $x>0$ is path connected to $(a_1,0)$ ($((a_0,0),(a_1,0)]$ is homeomorphic to $(0,1]$) and for $b>a_0$ any point (b,x) , $x\geq 0$, is path connected to $(b,0)$ which is path connected to $(a_1,0)$ (by (a) and (c), $[(a_1,0),(b,0))$ has the same order type as $[0,1)$ and the order topology on a convex subset is the same as the subspace topology). Therefore, L is path connected as the union of path connected subsets having a common point.(e) An open interval $(-\infty,(b,0))$ in L has the order type of an open interval in $\mathbb{R}$. Indeed, by (c), $[(a_0,0),(b,0))$ has the order type of $[0,1)$. Since for convex subsets the order topology is the same as the subspace topology, $((a_0,0),(b,0))$ has the order type of $(0,1)\subseteq\mathbb{R}$. Therefore, for any (a,x) there is an open neighborhood $(-\infty,(a+1,0))$ with the order type of $(0,1)$. Moreover, there is a bijective order preserving function from one set onto the other, and by 7(a), it is a homeomorphism.(f) If L could be embedded into $\mathbb{R}^n$ then L would be homeomorphic to a subspace of $\mathbb{R}^n$ which has a countable basis for the topology. Therefore, there must be a countable basis of the topology of L . We show that there is no. For any a : $((a,0),(a+1,0))$ is open and contains $(a,1/2)$ but does not contains $(b,1/2)$ for $b\neq a$. Therefore, there is a basis element such that it contains $(a,1/2)$ but does not contain $(b,1/2)$ for $b\neq a$. Call it B_a . Note that for $a\neq b$: $B_a\neq B_b$. Since S_ω is uncountable, there must be uncountable number of distinct sets in the basis.

SECTION 25

1.

Suppose $A \subseteq \mathbb{R}_l$ is connected and $x \in A$. $[x, x+\epsilon) \cap A$ is both open and closed in A and nonempty. Therefore, using §23 criterion, $A = A \cap [x, x+\epsilon)$ for any ϵ and $A = \{x\}$. So, each component of $\mathbb{R}_l$ is a singleton. Hence, path components are also singletons. Now, the image of a connected space under a continuous mapping must be connected. Therefore, only constant functions are continuous functions from $\mathbb{R}$ to $\mathbb{R}_l$. Compare to §18, exercise 7(b).

2.

(a) Since $\mathbb{R}$ is path connected, $\mathbb{R}^\omega$ in the product topology is path connected and connected (§24, exercise 8(a)), therefore, there is one component = one path component, i.e. the whole space.(b) $f(\mathbf{x}) = \mathbf{x} - \mathbf{x_0}$ preserves the metric, and therefore, by exercise 2 of §21, it is a homeomorphism. Hence, $\mathbf{x}$ and $\mathbf{y}$ are in the same component iff $\mathbf{z} = \mathbf{x} - \mathbf{y}$ and $\mathbf{0} = (0, 0, \dots)$ are in the same component. We show that it is iff $\mathbf{z}$ is bounded.First, we show that $\mathbf{0}$ is path connected to any other point $\mathbf{x} \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$ where $\epsilon \leq 1$. Indeed, $f(t)_{t\in[0,1]} = t\mathbf{x}$ is a path. To see this, recall that a function to the subspace is continuous iff it is continuous as a function in to the space, and that the topology in a subspace is generated by the same metric as the topology of the space. Moreover, $t'\mathbf{x} \in B_{\bar{\rho}}(t\mathbf{x}, r)$ iff $sup_i\{|t'-t|x_i\} < r$ iff $|t'-t| < r/sup_i\{x_i\}$. Therefore, the preimage is open.Now, two things follow. First, the space is locally path connected. Therefore, two points are connected iff they are path connected. Second, $\mathbf{0}$ and $\mathbf{x}$ are path connected iff $\mathbf{x}$ is bounded. Indeed, according to the exercise 8 of §23, the sets of bounded and unbounded sequences is a separation, therefore, any point connected to $\mathbf{0}$ (which is bounded) must be bounded. The other direction: if a sequence is bounded, then the argument similar to one in the previous paragraph shows that there is a path.(c) If $\mathbf{x} - \mathbf{y} \in \mathbb{R}^\infty$ then $f(t)_{t\in[0,1]} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ is a path from $\mathbf{x}$ to $\mathbf{y}$. Indeed, there is a finite number of nonzero coordinates in $\mathbf{y} - \mathbf{x}$, and, therefore, the preimage of any open basis set of the box topology is a finite intersection of intervals open in $[0, 1]$. On the other hand, if $\mathbf{x} - \mathbf{y} \notin \mathbb{R}^\infty$ then there is an infinite sequence of indexes j such that $x_j \neq y_j$ and the homeomorphism $h(\mathbf{z})_i = z_i - x_i$ if $x_i = y_i$ or $h(\mathbf{z})_i = n(z_i - x_i)/(y_i - x_i)$ otherwise (it is a homeomorphism by exercise 8 of §19), maps $\mathbf{x}$ to $\mathbf{0}$ and y to an unbounded sequence. But the sets of bounded and unbounded sequences are open in the box topology (they are open even in the coarser uniform topology) and form a separation. Therefore, $\mathbf{x}$ and $\mathbf{y}$ must be in different components. BTW, the result also implies that $\mathbb{R}^\omega$ in the box topology is not locally path connected, still components and path components are the same.

3.

Any open interval in the ordered square is a linear continuum, therefore, it is connected. At the same time any open set that contains $(x, 0)$ for $x > 0$ contains some point $(y, 0)$, $y < x$, and the argument similar to Example 6 of §24 shows that it is not path connected. Similar for points $(x, 1)$, $x < 1$. The space is locally path connected at all other points. (Another way to prove that it is not locally path connected is as follows: if it were locally path connected then its components would be the same as its path components, i.e. it would be path connected which contradicts Example 6.) The path components are the vertical segments. By the way, in the subspace topology as a subspace of $\mathbb{R}^2$ in the dictionary topology (which is strictly finer than the order topology of the ordered square, remember exercise 10 of §16?) it is locally path connected but still not path connected (it is not even connected in this topology).

4.

(Exercise 10 of the previous section is a particular case of this general result.) In an open subset U of a locally path connected space X each path component and their unions are open in X and, therefore, open and close in U. Since U is connected, there is only one path component. Another way to show this is as follows. We prove that an open subspace of a locally path connected space is locally path connected (this implies that the components and the path components of the subspace are the same). Indeed, if U is an open subset of a locally path connected space X, $x \in U$ and $V \subseteq U$ is an neighborhood of x in U then V is open in X as well (since U is open) and contains a path connected neighborhood $W \subseteq V \subseteq U$ of x which is also open in U.

5.

(a) Every point is (path) connected to p, hence, the space is (path) connected. For the same reason it is locally (path) connected at p. But for any other point every small enough neighborhood consists of disjoint line segments and is not (path) connected.(b) The idea of the example in (a) is that every path connecting two points on a pair of line segments goes through a point, and, therefore, at any other point a small enough neighborhood that does not contain that point but still intersects an infinite number of line segments is not connected. We can extend the example to require the path for some points to go through two point. Let $X = \{[(q,0),(0,1)] | q \in \mathbb{Q}\} \cup \{[(q,1),(1,0)] | q \in \mathbb{Q}\}$, then X is (path) connected (as the union of the path connected line segment $l = [(0,1),(1,0)]$ and the collection of other path connected line segments, each of which intersects l) but not locally connected at any point (now it is not locally connected at $(0,1)$ or $(1,0)$ and for all other points the argument remains as before).

6.

Here we show as the hint suggested that a component of an open subset of X is open. Let U be open in X and $C \subseteq U$ be its component. For any point $x \in C \subseteq U$ there is a connected subspace $S_x \subseteq X$ and an open neighborhood V_x such that $x \in V_x \subseteq S_x \subseteq U$. Since S_x is a connected subset of U, $S_x \subseteq C$. Therefore, C is the union of V_x for all $x \in C$ and is open. Another way to prove this is to show that the space is locally connected at every point. This way might be better considering the next exercise (in the proof we should somehow use the fact that the space is weakly locally connected at every point, or at least at every point in some neighrborhood of x, to prove that it is locally connected at x). Let U be a neighborhood of x. There is a connected subspace S such that it contains a neighborhood of x. Suppose there is a neighborhood of x such that the space is weakly locally connected at every point in the neighborhood. The intersection of these two neighborhoods of x is a neighborhood V of x such that it is contained in S and the space is weakly locally connected at every point in the neighborhood. If C is a component of V containing x then every point in C is contained in a connected subspace that a) has to be contained in C and b) contains a neighborhood of the point. Therefore, as before, we conclude that C is open in V, and, therefore, in X, contains x and is connected. I.e. the space is locally connected at x. Note that for this to be true we need the space to be weakly locally connected not only at the point x but at any point in some neighborhood of x. The next exercise shows that weak local connectedness at the point only does not imply the local connectedness at the point.

7.

It is not locally connected at p : for any bounded open neighborhood U of p its intersection V with $0x$ is open and bounded, and let v be its $\sup V$ and n be the minimum index such that $a_n < v$. Then U may contain some points a_i for $i >= n$ but it does not contain a_{n-1} . At the same time it does contain some points of $((a_n, 0), (a_{n-1}, 0))$ and, therefore, some points of the line segments $[(a_{n-1}, 0), (a_n, 1/k)]$ for sufficiently large k . Those parts of the set are disconnected and the neighborhood can be separated. (A more formal prove would require a more formal statement of the problem and some technical work.) Now, for any neighborhood of p we may find an open ball B such that its intersection with the set is contained in the neighborhood, then find some m such that $a_m, a_{m+1}, \ldots$ and their line segments are in the ball — this will be the connected subspace, which does contain another neighborhood of p . The reason the previous exercise argument does not work here is that the set is not (weakly) locally connected at some points in any neighborhood of p , and the components can' t be represented as the union of open sets.

8.

Let C be a component of an open set $U \subseteq Y$. Since $p^{-1}(C)$ is saturated, it is enough to show that it is open. Since X is locally connected, every component of every its open subset is open, therefore, it is enough to show that if $x \in p^{-1}(C)$ then $S \subseteq p^{-1}(C)$ where S is the component of x in $p^{-1}(U)$ (which is open). S is connected, therefore, $p(S)$ is connected. It follows that $p(S) \subseteq C$ and $S \subseteq p^{-1}(C)$.

9.

Following the hint. Let $x \in G$. We show that xA is connected iff A is connected iff Ax is connected. Indeed, this follows from the fact that $\alpha \cdot x$ and $x \cdot \alpha$ are homeomorphisms. Therefore, if C is the component of e then each of xC and Cx is the component of G containing x and $xC = Cx$.

10.

(a) It is reflexive and symmetric. To show transitivity suppose that $x \sim y \sim z$ but $x \nsim z$. There is a separation of points x and z, and y is in one of these open sets, i.e. it is either separated from x or z. Contradiction.(b) If for two points in one component there is a separation of these points, then the component is not connected. On the other hand, what can make a component to be a proper subset of a quasi-component? Or, equivalently, when a quasi-component may contain two or more components? Obviously, it cannot be the case if the components are open, as otherwise they would be open and closed and form a separation for points in different components. Therefore, if the space is locally connected, then the components are open, and quasi-components are the same as components.(c) A is not (locally) (path) connected. For any two points with different x -coordinates $x < x'$ consider a point x " such that $x < x$ "$< x'$ and $1/x$ "$\notin \mathbb{Z}$. Then open sets $x < x$ " and $x > x$ " form a separation of the points. Now, each vertical line segment is path connected, therefore, it is a path component, component and quasi-component. The remaining two points are disconnected, therefore, there are two more (path) components. The only question here is whether there is one or two quasi-components. We show that there is only one quasi-component with both points in it. The idea is that any separation of the space must "draw a line" between a pair of vertical lines without intersecting any of them, and therefore, the two points must be in one set in the separation. Suppose there is a separation $A' \cup A$ "$= A$ such that $(0,1) \in A'$ and $(0,0) \in A$ ". For each open set A' and A " take a small open ball-neighborhood of the corresponding point contained in the set. The balls intersect some common vertical line, therefore, A' and A " intersect that line and form its separation. This contradicts the fact that the line is (path) connected.B is connected ($B - \{(0,1)\}$ is path connected and adding a limit point keeps it connected), so that there is one component = quasi-component. But there are two path-components: $(0,1)$ and all other points.C is not locally (path) connected. Its path components are the line segments (so there are infinitely many of them). We show that it is connected and, hence, has one (quasi-)component. Suppose there is a separation $C' \cup C$ "$= C$. Both are open, therefore, they cannot intersect the same line segment. Moreover, they do not contain limit points of each other. Take any point $(-1/n, 0)$ and suppose it is in C'. It is a limit point for the set of line segments $[-1,0] \times K$. Therefore, starting from some k all lines $[-1,0] \times (1/k)$ are in C'. Therefore all points $(-1/n, 0)$ are in C'. Therefore, all line segments $-K \times [-1,0]$ are in C'. This implies all their limit points $(0, -1/n)$ in C'. By continuing this argument we conclude that all points are in C', and C " must be empty. Contradiction.

SECTION 26

1.

(a) If $(X, \mathcal{T}')$ is compact then $(X, \mathcal{T})$ is compact.(b) Suppose one is finer than the other. Then the identity mapping from the finer one to the coarser one is a continuous and bijective function that maps a compact space to a Hausdorff space. Therefore, it is a homeomorphism and the topologies are the same.

2.

(a) Any open set in a collection that covers a set covers all but finitely many points of the set. Therefore, a finite subcollection covers it all.(b) No. Let $A_n = [0,1] - \{1/n, 1/(n+1), \dots\}$.

3.

For each subspace in the collection chose a finite subcovering that covers it and then take the finite union of all these finite subcoverings.

4.

It is a Hausdorff, therefore, compact implies closed. If it were not bounded, then for any ball $B(x, n)$ there would be a point outside it, and while the union of all these balls does cover the whole space, there is no finite subcovering for the subspace. See also the proof of Theorem 27.3. Now we must find an example of a space where there is a closed bounded set such that it is not compact. Boundness is not a topological property. Therefore, we can take any non-compact metric space and make it bounded by taking the standard bounded metric. Or, for example, take an infinite space in the discrete topology (with the discrete metric: $d(x, y) = 1$ for $x \neq y$).

5.

We know from Lemma 26.4 that for each $a \in A$ there are open sets $U_a \ni a$ and $V_a \supseteq B$ such that $U_a \cap V_a = \emptyset$. $\cup_a U_a$ covers A and there is a finite subcovering $U = U_{a1} \cup U_{a2} \cup \ldots \supseteq A$. The finite intersection of corresponding sets $V = V_{a1} \cap V_{a2} \cap \ldots$ is open, contains B and does not intersect U . (The proof is also given as a part of the proof of Theorem 32.3.)

6.

Let A be closed in X . Then A is compact. Then $f(A)$ is compact and, since Y is Hausdorff, $f(A)$ is closed.

7.

Let S be closed in $X \times Y$ and suppose $A = \pi_1(S)$. Suppose $x \notin A$ then $x \times Y \not\cap S$ and, since S is closed, $X \times Y - S$ is open and contains a slice $x \times Y$. By the tube lemma there is also a tube about the slice that does not intersect S . Its projection is an open neighborhood of x that does not intersect A .

8.

Theorem. Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the ***graph*** *of f,*

$$G_f = \{x \times f(x) | x \in X\},$$

is closed in $X \times Y$. [*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y — V)$ is closed. Apply Exercise 7.]

Suppose the graph is closed and V is open in Y. If $V \cap f(X) = \emptyset$ then $f^{-1}(V)$ is empty and open. Now suppose that V is a neighborhood of $f(x)$ for some x. Then $X \times V$ is open in the product and $X \times (Y - V)$ is closed and so is its intersection with the graph. Using Exercise 7, its image under the projection on X is closed as well, but the complement of the image is the preimage $f^{-1}(V)$, and it is open. Therefore, f is continuous. Note that we use compactness here, as Exercise 7 requires it.

Now, suppose that f is continuous. We take a point (x, y) outside of the graph and show that there is its open neighborhood that does not intersect the graph. This will show that the complement of the graph is open, and, hence, the graph is closed. For points $f(x)$ and $y \neq f(x)$ take two disjoint open neighborhoods V and V', respectively, using Hausdorff property of Y. Then, since f is continuous, $U = f^{-1}(V)$ is open, $x \in U$, and $U \times V'$ is an open neighborhood of (x, y) such that it does not intersect the graph (as $(a, f(a)) \in U \times V'$ implies $f(a) \in V \cap V'$). Note that this direction does not use compactness of Y.

9.

Suppose X and Y are arbitrary spaces and $B \subseteq Y$ is compact, then if N open in the product contains $x \times B$ then it contains $U_x \times U_B$ for some neighborhood U_x of x and U_B of B. Indeed, each point $z \in x \times B$ has a basis neighborhood $U_z \times V_z$ contained in N. The union of the basis neighborhoods covers the set which is homeomorphic to the compact set B and, therefore, there is a finite subcover. The finite intersection of U_z' s and the finite union of V_z' s are open and their product covers $x \times B$ and is contained in N.Now, for each $a \in A$ consider the basis element $U_a \times V_B$ that contains $a \times B$ and is contained within N. The union covers $A \times B$ and, since it is compact, there is a finite subcover. The finite union of U_a' s and the finite intersection of V_B' s are the open sets in X and Y we are looking for.

10.

(a) For each x let n_x be such that $f(x) - f_{n_x}(x) < \epsilon$. Now, since $f(x) - f_{n_x}(x)$ is continuous, there is an open neighborhood U_x of x such that for each $x' \in U_x$: $f(x') - f_{n_x}(x') < \epsilon$. Since f_n is monotone increasing, for all greater n this still holds. Moreover, all U_x' s cover X and, since it is compact, there is a finite subcover U_{x_i}. Let N be the maximum of n_{x_i}. Then for each $x \in X$ there is some x_i nearby such that $x \in U_{x_i}$ and for all $n \geq N \geq n_{x_i}$: $f(x) - f_n(x) < \epsilon$.(b) The example of exercise 9 of §21 can be restricted to the compact domain $[0, 1]$. $\arctan(x + n)$ is a monotone increasing sequence converging to the constant function $f(x) = 1$, but not uniformly (for any n : $f_n(-n) = 0$).

11.

Just follow the hint. Y is closed. Therefore, both sets C and D are closed in X. Since X is compact, they are compact as well, and, by exercise 5, since X is also Hausdorff, there are open and disjoint U and V containing them. $A - U - V$ are closed, nonempty (otherwise, since U and V are disjoint and both contain a point of Y, there would be a separation of A, which is connected) and nested (here we use the fact that the initial sequence is nested, otherwise the result would not hold), therefore, since X is compact, the intersection is nonempty. That implies that there is a point in Y that is not in $U \cup V$ and, therefore, not in $C \cup D$. Contradiction.

12.

In this exercise we are asked to prove that X is compact given that f is continuous, surjective and closed, Y is compact, and the preimages of all singletons are compact. We prove a little more: if f is closed, and the preimage of each point is compact, then the preimage of any compact subset is compact.

When we can guarantee that the preimage of any compact set is compact? Let us try to collect minimum requirements. A constant function is an example that shows that we may need to require that the preimage of any point is compact. So, suppose it is. Let B be a compact subset of Y, and $A = f^{-1}(B)$. We know nothing about A: it need not even be closed (neither Y is assumed to be Hausdorff, nor f is assumed to be continuous, so far). Let $\{U_\alpha\}$ be an open covering of A. $\cup_\alpha U_\alpha$ covers $\cup_{b \in B} f^{-1}(\{b\})$. We know that for each b there is a finite subcovering $\{U_j(b)\}$ of $f^{-1}(\{b\})$. If we only could group all b ' s in a finite number of groups $\cup_i V(b_i) = B$ where $V(b_i)$ contains b_i and is such that $\{U_j(b_i)\}$ covers not only $f^{-1}(\{b_i\})$ but $f^{-1}(V(b_i))$ as well, then we are done. Take any collection of neighborhoods over all $b \in B$. Since B is compact, there is a finite subcollection that covers B. Therefore, we just need to find a neighborhood $V(b)$ for each b such that $\{U_j(b)\}$ covers $f^{-1}(V(b))$ as well. In other words, what we need for f is the property that if an open set U contains $f^{-1}(\{b\})$, then there is an open neighborhood V of b such that $f^{-1}(V) \subset U$. Now, this property is equivalent to the requirement that any b is an interior point of $Y - f(X - U)$ where U is open and contains $f^{-1}(\{b\})$. The requirement that f is closed is obviously sufficient for this. (How did we use the continuity of f? I think we did not.)

Lemma If $f: X \to Y$ is closed, $y \in Y$, and U is an open subset of X such that $f^{-1}(\{y\}) \subset U$, then there is an open neighborhood V of y such that $f^{-1}(V) \subset U$. In other words, if f is closed, then every open set containing the preimage of a point, contains the preimage of some neighborhood of the point.

Consider $V = Y - f(X - U)$. V is clearly open. Further, since U contains all points mapped to y, $y \notin f(X - U)$, and $y \in V$. Moreover, for every $z \in V$, z is not the image of any point in $X - U$, hence, $f^{-1}(\{z\}) \subset U$.

From the lemma, as discussed above, we conclude the following.

Theorem If $f: X \to Y$ is closed, and for every $y \in Y$, $f^{-1}(\{y\})$ is compact, then the preimage of every compact set is compact. If further $f(X)$ or Y itself is compact, then X is compact.

Let $B \subset Y$ be compact. Suppose $\{U_\alpha\}$ is an open cover of $f^{-1}(B)$. For every $b \in B$, $f^{-1}(\{b\})$ is compact, hence, there is a finite subcover $\{U_{\alpha_j^b}\}_{j=1}^{n_b}$ of $f^{-1}(\{b\})$. Then, $U^b = \cup_{j=1}^{n_b} U_{\alpha_j^b}$ is an open set containing $f^{-1}(\{b\})$, and, by lemma, there is an open neighborhood V^b of b such that $f^{-1}(V^b) \subset U^b$. $\{V^b\}$ covers B, and there is a finite subcover $\{V^{b_i}\}_{i=1}^{n}$. Therefore, $f^{-1}(B) \subset f^{-1}(\cup_{i=1}^{n} V^{b_i}) = \cup_{i=1}^{n} f^{-1}(V^{b_i}) \subset \cup_{i=1}^{n} \cup_{j=1}^{n_{b_i}} U_{\alpha_j^{b_i}}$, a finite subcover of $\{U_\alpha\}$.

Corollary If $f: X \to Y$ is continuous and closed, X is compact, and Y is a T_1-space, then the preimage of every compact set is compact.

For every $y \in Y$, $\{y\}$ is closed, and $f^{-1}(\{y\})$ is closed, hence, compact. The result follows from the theorem above.

> This corollary generalizes the following well-known fact: if $f: X \to Y$ is a continuous function from a compact space to a Hausdorff space, then it is closed, and the preimage of every compact set is compact. If we assume that Y is just a T_1-space (without additional assumption that f is closed), then we cannot say that every compact subset of Y is closed, and, hence, its preimage is closed and compact in X. So, we need this additional assumption explicitly. Consider, for example, $f: [0,1] \to [0,1]_{fc}$ where the range is in the finite complement topology.

13.

(a) Using exercises 4 and 7(c) of §22, $A \cdot b$ is closed and there are two disjoint open sets U_b and V_b containing given $c \notin A \cdot B$ and $A \cdot b$, respectively. We show there exists some neighborhood W_b of b such that $A \cdot W_b \subseteq V_b$. Indeed, for each $a \in A$ there is a neighborhood K_a of b such that $a \cdot b \in a \cdot K_a \subseteq V_b$ (using the fact that V_b is open and the multiplication is a homeomorphism, exercise 4 of §22), and their W_b union is the neighborhood of b such that $A \cdot b \subseteq A \cdot W_b \subseteq V_b$. Now, $\{W_b\}$ cover B and there is a finite subcovering. The corresponding finite intersection U_b is the neighborhood of c disjoint from $A \cdot B$.(b) If A is closed in G and $xH \notin p(A)$ then $x \notin A \cdot H$ which is closed by (a). And there is a neighborhood U of x such that it does not intersect $A \cdot H$. Then, $p(U)$ is open (exercise 5(c) of §22), contains xH and does not intersect $p(A)$. Therefore, $p(A)$ is closed.(c) p is continuous and closed (by (b)), moreover, $p^{-1}(xH) = xH$ is compact as the multiplication is a homeomorphism. Therefore, by exercise 12, G as the preimage of $G|H$ is compact.

SECTION 27

1.

Let $A \subseteq X$ be nonempty and bounded above by z. Let $a \in A$. $[a, z]$ is closed and compact in the order topology (which is the same at the subspace topology, since the closed interval is convex). Consider the collection $\mathcal{L}$ of all intervals $[a', z'] \subseteq [a, z]$ such that $a' \in A$ and z' is an upper bound of A. $\mathcal{L}$ is nonempty as $[a, z] \in \mathcal{L}$. None of those intervals are empty, and they are all closed in $[a, z]$. Moreover, the intersection of any finite number of intervals in $\mathcal{L}$ is also an interval in $\mathcal{L}$ and, therefore, is a nonempty set. Given that $[a, z]$ is compact, the intersection of all intervals in $\mathcal{L}$ has a point x. Also, we have $x \geq a'$ for all $a' \in A$ (in $[a, z]$ or below) and $x \leq z'$ for all upper bounds z' of A (in $[a, z]$ or above). There cannot be two such points $x < x'$, as in this case both have to upper bounds of A but then $x' \notin [a, x]$. Therefore, there is a unique point x in the intersection which is the least upper bound of A.

2.

(a) $d(x, A) = 0$ iff for any $\epsilon > 0$: $B(x, \epsilon) \cap A \neq \emptyset$ iff $x \in \overline{A}$.(b) A is compact, d is continuous on $X \times X$ and, therefore, continuous in both variables, hence, for a given x : $d(x, a)$ reaches the minimum on A .(c) $x \in U(A, r)$ iff $d(x, a) < r$ for some $a \in A$ iff $x \in \cup_{a \in A} B(a, r)$.(d) For $a_j \in A$ let r_j be such that $B(a_j, 2r_j) \subseteq U$. Balls $B(a_j, r_j)$ cover A, there is a finite subcovering $a_1, \ldots, a_n$, let ϵ be the minimum of corresponding r_i ' s. For $y \in B(a_i, r_i)$: $B(y, \epsilon) \subseteq B(a_i, r_i + \epsilon) \subseteq B(a_i, 2r_i) \subseteq U$.(e) The idea is that there is no finite subcovering and ϵ as the minimum of r_j ' s must be zero. For example, $x \times 1/x$ for $x > 0$ has no ϵ -neighborhood in $\mathbb{R}_+ \times \mathbb{R}_+$.

3.

(a) The topology is strictly finer that the standard topology on $[0, 1]$ which is compact and Hausdorff, therefore, it is not compact. Cool, huh? Another way to show this directly is as follows: $[0, 1] = ([0, 1] - K) \cup \cup_i (1/(i + 1), 1]$ has no finite subcovering. Yet, another way is just to say that K is a closed subspace of $[0, 1]$ in $\mathbb{R}_K$ which is not compact (the set is infinite and all points are isolated), therefore, $[0, 1]$ is not compact.(b) Both topologies induce the same topology on $(-\infty, 0)$ and $(0, +\infty)$ (it is clear for the first one, while for the second a subset U is open in the subtopology of $\mathbb{R}_K$ iff it equals the intersection of the subspace with an open set V in $\mathbb{R}$ minus some points in K iff it equals the intersection of the subspace with $V - \{0\}$ minus those points in K iff it is open in the subtopology of $\mathbb{R}$), therefore, both are connected in either topology. Their closures are connected and share the common point 0, therefore, their union — the whole space — is connected.(c) Suppose there is a path from 0 to 1 in $\mathbb{R}_K$. It is a continuous function from a compact connected space, therefore, the image is compact and connected in $\mathbb{R}_K$. Hence, it must also be connected as a subspace of $\mathbb{R}$ (as it has a coarser topology), i.e. the image is convex and contains the whole interval $[0, 1]$. This implies that $[0, 1]$ is a closed subspace of the compact image, i.e. it is compact in $\mathbb{R}_K$. Contradiction.

4.

Given $x \in (X, d) : d(x, y)$ is a continuous function in y that maps a connected space into $\mathbb{R}_+$, therefore, the image is a connected subspace of $\mathbb{R}_+$ that includes 0 . This implies that it is either $\{0\}$ ($X = \{x\}$) or uncountable.

5.

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\cup A_n$ has empty interior in X. [*Hint*: Imitate the proof of Theorem 27.7.]

Following the idea of Theorem 27.7. If $A \subset X$ is closed, $U \subset X$ is non-empty and open and $U \not\subset A$, then there is a non-empty open set $V \subset X$ such that $\overline{V} \subset U - A$. Indeed, since $U - A \neq \emptyset$, there is a point $x \in U - A$, and there are disjoint open neighborhoods W and V about $A \cup (X - U)$ (closed) and x, respectively (Lemma 26.4), so that $x \in V \subset \overline{V} \subset X - (A \cup (X - U)) = (X - A) \cap U = U - A$.

We take any non-empty open set U_0 and show that it has a point not in the union $\cup_{i \geq 1} A_i$. For A_i, $i \geq 1$, U_{i-1} is non-empty and $U_{i-1} \not\subset A_i$ (A_i has no interior points), hence, there is a non-empty open set $U_i \subset X$ such that $\overline{U}_i \subset U_{i-1} - A_i$. We have a nested sequence of non-empty closed sets $\overline{U}_i$, and their intersection is nonempty as X is compact, where any point in the intersection belongs to $\overline{U}_1 \subset U_0$ but does not belong to $\cup_{i \geq 1} A_i$.

6.

The set A_n consists of closed intervals of length $1/3^n$ and the distance between them at least $1/3^n$. Indeed, this is true for A_1. At every step a closed interval of $1/3^n$ is divided into three parts with two parts in A_{n+1} such that the distance between them is $1/3^{n+1}$.(a) For any two points $x \neq y$ there is n such that they cannot lie in the same closed interval of A_n, and, therefore, there is a point z between them not in the Cantor set, and the set can be separated by $(-\infty, z)$ and $(z, +\infty)$.(b) It' s a closed subspace of a compact space.(c) We have already shown by induction that it is the union of 2^n closed intervals each of length $1/3^n$. By the same argument endpoints of the intervals are never excluded (only interior points are excluded).(d) Consider a sequence of closed intervals containing a given point of C, for each one choose an endpoint which does not equal the point, and all the endpoints are in C, by (c), and the sequence converges to the point. Therefore, it is a limit point of the set of all other points in C.(e) It is nonempty (by (c), for example, or by the fact that it is the intersection of a nested sequence of closed intervals), compact and Hausdorff with no isolated points. By Theorem 27.7, it is uncountable.

SECTION 28

1.

Interesting. $A = \{x_i = (0, \dots, 0, 1, 0, \dots)\}$ has no limit point (if x has a coordinate in $(0,1)$ then a small enough ball does not contain any other point in A, otherwise the distance to any point in A is either 0 or 1).

2.

$\{1 - 1/n\}$ does not have a limit point (if there were a limit point it would be a limit point in the standard topology as well, i.e. 1, but in the given topology $\{1\}$ is open).

3.

(a) No. If X is compact the image is compact, so if we believe the statement is wrong, we need to look for X which is limit point compact but not compact. Consider X in the Example 1. The projection on the first coordinate is continuous (it can also be thought of as the quotient space obtained by identifying all points in $z \times Y$) but maps the limit point compact space to the not limit point compact set $\mathbb{Z}_+$.(b) Yes. An infinite subset of A has a limit point in X which is a limit point of A as well, i.e. it is in A .(c) No. Once again for the counterexample we need a limit point compact space which is not compact. Now the Example 2 works better: $S_\Omega \subset \overline{S}_\Omega$. Note that $\overline{S}_\Omega$ not only Hausdorff but also compact (and, hence, limit point compact).

4.

Limit point compactness implies countable compactness for a T_1 -space. Let X be a limit point compact T_1 -space and $\{U_n\}$ be a countable open covering of X such that there is no finite subcovering of X. Let $V_n = U_1 \cup \ldots \cup U_n$. Note that for every n , V_n does not cover X , but for every $x \in X$ there is minimal n_x such that $x \in V_{n_x}$. Let $x_0 \in X$. For each $n \geq 1$ let $x_n \in X - V_{n_{x_{n-1}}}$. This defines an infinite subset of X that must have a limit point a . But then the neighborhood V_{n_a} of a contains only finite number of elements in the sequence. And for each of them different from a we can find a neighborhood of a that does not contain it (here we use the T_1 -property — consider Example 1 to see why it does not work in general). The finite intersection of all these neighborhoods with V_{n_a} is a neighborhood of a that does not contain any point of the sequence different from a . This contradicts the fact that a is a limit point of the sequence.*Countable compactness implies limit point compactness.* Suppose X is a countably compact space and Y is an infinite subset of X . There is a countably infinite subset $Z \subseteq Y$ and every limit point of Z is a limit point of Y as well. If no point in Z is a limit point of Z then every point in Z has a neighborhood that does not contain any other points in Z , and the countable collection of such neighborhoods covers Z . Since each set in the collection covers one point of Z only and Z is infinite, there is no finite subcollection covering Z . Therefore, Z is not closed (a closed subset of a countably compact space is countably compact: add the complement of the subset to any its covering) and there is a limit point of Z .

5.

If X is countably compact, then since $\{X - C_i\}$ has no finite subcovering of X , it does not cover X . The other direction: suppose that there is a countable covering $\{U_n\}$ that has no finite subcovering, then $\{X - \cup_{i<n} U_i\}$ is a nested sequence of nonempty closed sets and it has a nonempty intersection. This contradicts the assumption that $\{U_n\}$ is a covering of X . Note that the second part of the proof works for countable coverings only. For uncountable coverings there is no general way to order them in a sequence so that the sequence will contain all the sets in the collection and the complements of the partial unions of the sequence will generate a nested sequence of closed sets. Instead, we use the finite intersection property.

6.

f is injective because if $x \neq x'$, $d(f(x), f(x')) = d(x, x') > 0$. We show it is surjective. Suppose not. Then, there is some $y \in X - f(X)$. Since f is continuous (Theorem 21.1) and X is compact, $f(X)$ is compact, and, hence, closed. Therefore, there exists an ϵ -neighborhood of y that does not intersect $f(X)$. Now, using the hint, let $x_1 = y$, and $x_{n+1} = f(x_n)$. Then, $d(x_1, x_n) \geq \epsilon$ for all $n > 1$, because $x_1 = y$ and $x_n \in f(X)$. Further, for $n > m > 1$, $d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) \quad = d(x_{n-1}, x_{m-1}) \quad = \cdots \quad = d(x_{n-m+1}, x_1) \quad \geq \epsilon$. Therefore, we have an infinite sequence of pairwise distinct points such that no subsequence converges (if there was a converging subsequence, it would be within $\frac{\epsilon}{2}$ -ball of some point starting from some index, and pairwise distances between the points of the subsequence starting from this index would be less than ϵ). This means that X is not sequentially compact, but X is a compact metric space. This fact contradicts Theorem 28.2. Hence, there is no such point y , and f is surjective. Also, by Exercise 2 of §21, f is a homeomorphism.

> If X is not compact, the result is not true. For example, consider $\mathbb{Z}_+$ with discrete topology ($d(m, n) = 1$ for $m \neq n$), and $f(n) = n + 1$, which is clearly an isometry. While f is an (isometric) imbedding, it is not bijective, and, hence, not homeomorphism.

7.

(a) and (b) Step 1. If f is a shrinking map then there is at most one fixed point. Suppose there are two fixed points x and y . Then $d(x, y) = d(f(x), f(y)) < d(x, y)$ .Step 2. We construct a closed compact set C that will be proved to have a fixed point. Let $C_0 = X$ and $C_n = f^n(X)$. f is a continuous map from a compact space to Hausdorff space, therefore, it is a closed map. Then, by induction, every set in the sequence is closed. Moreover, since the image of a subset lies within the image of a superset, the sequence is a nested sequence of nonempty closed sets that has a nonempty closed and compact intersection C .Step 3. We show that $f(C) \subseteq C$. Indeed, suppose that for $z \in C$: $f(z) \notin C$. Hence, there is $n \geq 1$ such that $f(z) \in C_n - C_{n+1}$. This implies that $z \notin C_n$ which contradicts the fact that $c \in C$ .Step 4. We show that C has only one point, this implies that the point is the fixed point. If f is a contraction we obtain the result immediately. Indeed, the diameter of C_n decreases exponentially and there cannot be more than 1 point in the intersection. More generally, we want to show that any $z \in C$ is the image of some point in C . For each $n : z \in C_n$, i.e. there exists $x_n \in X$ such that $f^n(x_n) = z$. A compact metric space is sequentially compact, hence, there is a subsequence of $\{y_n = f^{n-1}(x_n)\}$ that converges to some point a . Any neighborhood of a contains infinitely many members of the sequence, therefore, it is in the closure of every C_n , but C_n ' s are closed, therefore, $a \in C$. Moreover, since $f(y_n) = z$ for all n and f is continuous, $f(a) = z$. We conclude that $C \subseteq f(C) \subseteq C$, i.e. $C = f(C)$. The distance on C is a continuous function from the compact product $C \times C$ to the ordered set $\mathbb{R}$. Therefore, there is a pair of points $x, y \in C$ with the maximum distance between them. Let $x = f(a), y = f(b)$ where $a, b \in C$. Then if $x \neq y$: $d(x, y) = d(f(a), f(b)) < d(a, b) \leq d(x, y)$. This contradiction implies that there is only one point in C , the fixed point of f .(c) f is strictly increasing from 0 to 1/2. $f(y) - f(x) = (y - x)(1 - (y + x)/2)$ implies it is a shrinking map (the unique fixed point is 0). $f(1) - f(x) = (1 - x)^2/2$ implies it is not a contraction.(d) f is strictly increasing, moreover $f(x) > (x + |x|)/2 \geq x$ for all x . So, there is no fixed point. $(f(y) - f(x))/(y - x) = [1 + (y + x)/(\sqrt{y^2 + 1} + \sqrt{x^2 + 1})]/2 < 1$ but can be made as close to 1 as desired when $x, y \to +\infty$. Therefore, it is a shrinking map which is not contraction and has no fixed point.

SECTION 29

1.

$[a, b] \cap \mathbb{Q}$ are not compact as we may take a sequence converging to an irrational number (in $\mathbb{R}$) and no subsequence converges to a point in $\mathbb{Q}$ (sequential compactness is equivalent to compactness for metric spaces). Suppose some compact (and, therefore, closed) subset S of $\mathbb{Q}$ contains an open subset of $\mathbb{Q}$. Then it contains an interval $[a, b] \cap \mathbb{Q}$. The interval is closed in S and, therefore, compact. Contradiction. Therefore, there are no compact subsets of $\mathbb{Q}$ that contain any open subset. Hence, $\mathbb{Q}$ is not locally compact.

2.

(a) The projection is an open continuous map, therefore, we may use the next exercise to argue that all X_α are locally compact. A compact subspace of the product containing an open set has all but finitely many projections equal to the whole corresponding space, since the projection is continuous, these spaces must be compact.(b) Assuming the Tychonoff lemma all we need to prove is that the product of two locally compact spaces is locally compact. For any $x \times y$ find the corresponding compact subsets and neighborhoods in both spaces and take their products.

3.

What we need from the map is that it preserves both compactness and openness. The continuity guarantees that if x is contained in a compact subspace then its image is contained in a compact subspace as well. The openness of the map guarantees that a neighborhood of x within the compact subspace maps into a neighborhood of the image within the compact subspace containing it. If a continuous function is not open, in general, we cannot guarantee the second property. To find a counterexample we need to consider some space which is locally compact but not compact (otherwise the continuity will be enough). Moreover, it must be mapped onto a space that is not locally compact. By now I know three types of spaces which are not locally compact: $\mathbb{R}^\omega$ (infinite products), $[0, 1] \cap \mathbb{Q}$ (infinite totally disconnected) and $[0, 1] \subset \mathbb{R}_l$ (strictly finer than a compact Hausdorff topology). I think the second one should work. So we should take a locally compact but not compact space, for example, a locally compact unbounded subset of $\mathbb{R}$ and construct a continuous function that would map it onto $[0, 1] \cap \mathbb{Q}$. Let $A = \cup_n (n, n + 1)$ and $[0, 1] \cap \mathbb{Q} = \{q_1, q_2, \dots\}$. Let $f((n, n + 1)) = q_n$. Then f is a continuous (the preimage of any set is the union of open intervals) non-open (the image of a bounded open interval is not open in the range) map defined on a locally compact set A whose image is exactly $[0, 1] \cap \mathbb{Q}$ which is not locally compact. Cool!

4.

Suppose $\mathbf{0} \in U \subseteq S$ where U is open and S is compact, then there are two balls centered at $\mathbf{0}$ such that $\overline{B} = \overline{B}(\mathbf{0}, \epsilon) \subseteq B(\mathbf{0}, \delta) \subseteq U$. But then $\overline{B}$ as a closed subset of compact set must be compact, but it is not (similar to Exercise 1 of the previous section).

5.

Let $f(\infty_1) = \infty_2$, then it is bijective and U is open in $\overline{X}_1$ iff $U \subseteq X_1$ is open in X_1 or $X_1 - U$ is compact iff $f(U) \subseteq X_2$ is open in X_2 or $X_2 - f(U)$ is compact.

6.

The circle without a point is homeomorphic to the real line. Now using two facts, first, that the one-point compactification is unique up to a homeomorphism, and, second, that the compactification of the punctured circle is the whole circle (in fact, showing this is quite similar to the next exercise), we get the result.

7.

$S_\Omega \subset \overline{S}_\Omega$: if we show that the latter space is compact and Hausdorff we are done. It is Hausdorff, and compact as any covering contains an open set containing Ω, which in its order contains an interval $(a, +\infty)$. But the rest of the space $[a_0, a]$ is compact (as the ordering satisfies the least upper bound property).

8.

$\mathbb{Z}_+$ is homeomorphic to the set $A = \{1/n | n \in \mathbb{Z}\}$ in the discrete topology, which is equivalent to the topology inherited from the standard topology of the real line. $A \cup \{0\}$ is a compact and Hausdorff space, therefore, it is a one-point compactification of the subspace A.

9.

I believe the quotient map from G onto $G|H$ is open, therefore, $G|H$ is locally compact (exercise 3). Yes, right: see the supplementary exercises of the previous chapter.

10.

Let $x \in W \subseteq S$ where W is open and S is compact. $x \in U \cap W \subseteq S$ and $U \cap W$ is open. $S - (U \cap W)$ is closed in S which is compact (and closed), therefore, $S - (U \cap W)$ is closed in X and compact. We can separate x and $S - (U \cap W)$ by neighborhoods V' and V''. It follows that $V = V' \cap U \cap W$ is an open neighborhood of x in X such that $\overline{V}$ is a closed (and compact) subset of U.

11.

* (a) Z *is not locally compact.* Example 7 of §22. Let p be a quotient map on $X = \mathbb{R}$ that identifies all positive integers to a point b. Let $Z = \mathbb{Q}$ (it is not locally compact but Hausdorff). Then $\pi = p \times \iota_Z$ is not a quotient map. The open set U in the example has the property that for any $\delta > 0$ and $\epsilon > 0$ there is a large enough $n \in p^{-1}(b)$ such that $(n + \epsilon, \delta)$ is not in U. In other words, there is no saturated neighborhood U_1 of, let say, point 1 in $\mathbb{R}$ for which there exists neighborhood V of 0 in $\mathbb{Q}$ such that there product $U_1 \times V$ is contained in U. Any such neighborhood U_1 being saturated would contain all points n and some their local basis neighborhoods $W_n = (n - \epsilon_n, n + \epsilon_n)$. Any V would contain an interval $(-\delta, +\delta) \cap \mathbb{Q}$. Then for a large enough n such that $c_n < \delta$ there would be a point in $(n - \epsilon_n, n + \epsilon_n) \times (-\delta, +\delta) \cap \mathbb{Q}$ not in U. Note that U is a neighborhood such that it contains $n \times \mathbb{Q}$ for any n but does not contain any tube about it. This would not be possible if $\mathbb{Q}$ were compact.Z *is not Hausdorff.* Exercise 6 of §22 is an example of a quotient map from a non-locally compact Hausdorff set onto a compact non-Hausdorff space such that the product of it with itself is not a quotient map. What I want is similar to the previous example with the only difference that Z is locally compact but not Hausdorff. Let us take a one point compactification of $\mathbb{Q}$ (in fact, the one point compactification was defined for locally compact Hausdorff spaces only, $\mathbb{Q}$ is not locally compact, but it seems that the Hausdorff property is enough to construct the one-point compactification, and local compactness is needed for the Hausdorff property of the constructed space only). Therefore, the constructed space $\mathbb{Q}'$ is (locally) compact but not Hausdorff: the open sets containing the new point are those having bounded and closed in $\mathbb{R}$ complement, such sets have no interior points in $\mathbb{Q}$. Now, if we take the same example, then U is still saturated and open, and its image is not open. The difference between this case and the previous is that here we made the space compact at the cost of being Hausdorff, hence, we still cannot guarantee that any neighborhood would contain a compact closure of another neighborhood — something that we need for the proof.Now the proof. π is continuous (see §18). Let $B = p^{-1}(A)$ be open. We show that A is open. Let $(x, z) = p^{-1}((y, z))$. Since B is open and Z is locally compact there are neighborhoods U_1 of x in X and V of z in Z such that $U_1 \times \overline{V} \subseteq B$. If we knew that U_1 is saturated (or, alternatively, if we knew that p is an open map) we would immediately conclude that (y, z) has a neighborhood contained in A. Since we don' t know that, we must proceed in finding a saturated neighborhood U of x such that $U \times V \subseteq B$. Since B is saturated, we have $U_1 \times \overline{V} \subseteq p^{-1}(p(U_1)) \times \overline{V} \subseteq B$. In the first example, suppose we take point (1,0), then find a "rectangle neighborhood" about it, then the image of this rectangle has point b, therefore, the preimage of the image has all points $(n, 0)$ (the preimage of the image of the rectangle neighborhood contains the original rectangle about (1,0) plus all other points $(n, 0)$ with vertical intervals). Now we use the fact that $\overline{V}$ is compact. For each point in $p^{-1}(p(U_1))$ we take a neighborhood such that its product with $\overline{V}$ is still in B. We call the union of all this neighborhoods U_2. This is exactly what we could not do in the first example: for any $\overline{V}$ there was a large enough point n such that we could not find the desired neighborhood containing it due to the fact that no $\overline{V}$ were compact. Now we proceed this way constructing $U_3, U_4, \ldots$. Note that for each n, $U_n \subseteq p^{-1}(p(U_n)) \subseteq U_{n+1}$ and $U_n \times \overline{V} \subseteq B$. Let $U = \cup_n U_n$. Then $U \times V \subseteq B$ is open. If we show that U is saturated, we are done. Indeed, $U \subseteq p^{-1}(p(U)) = \cup_n p^{-1}(p(U_n)) \subseteq \cup_n U_{n+1} = U$, hence $U = p^{-1}(p(U))$ is saturated.(b) This is the easier part. Using (a), we conclude that the composition of two quotient maps $p \times q = (\iota_B \times q) \circ (p \times \iota_C)$ is a quotient map given that both B and C are locally compact and Hausdorff.

Supplementary Exercises

1.

Given two elements the one that is greater than both is: (a) the greater one (any pair of elements are comparable), (b), (d) the union of a pair of subsets, (c) the intersection of a pair of subsets.

2.

Take two elements in K, there is an element $\alpha \in J$ greater than both (J is directed), and an element $\beta \in K$ greater than α (K is cofinal).

3.

If $J = \mathbb{Z}_+$, which is ordered and, therefore, by 1, is a directed set, then the net f is merely a sequence of points x_n and the definition of convergence is as follows: $x_n \to x$ iff for any neighborhood U of x there is N such that $n \geq N$ implies $x_n \in U$.

4.

Every neighborhood of $x \times y$ contains a basis neighborhood $U \times V$. There are α and β such that $x_j \in U$ and $y_k \in V$ for all $\alpha \preceq j$ and $\beta \preceq k$. Since J is a net, there is γ such that $\gamma \preceq j$ implies both $\alpha \preceq j$ and $\beta \preceq j$. Therefore, "starting" from γ all points $x_j \times y_j$ are in $U \times V$.

5.

Separate two points by two non-intersecting neighborhoods $x \in X$ and $y \in Y$. There are α and β such that $x_j \in U$ and $x_k \in V$ for $\alpha \preceq j$ and $\beta \preceq k$. There is γ such that $\gamma \preceq j$ implies both $\alpha, \beta \preceq j$ and $x_j \in U \cap V$ which contradicts the fact that $U \cap V = \emptyset$.

6.

If there is a net in A converging to x then for any neighborhood of x there is a point in A, therefore, $x \in \overline{A}$. Now, suppose $x \in \overline{A}$. Using 1(c), consider the collection $\mathcal{U}_x$ of all neighborhoods of x partially ordered by the reverse inclusion (the "finer" is the set, the "greater" it is). Now, for each neighborhood $U \in \mathcal{U}_x$ take a point $x_U \in U \cap A \neq \emptyset$. Then, (x_U) is a net of points of A converging to x. Indeed, given any neighborhood U of x and x_V for $U \preceq V$: $x_V \in V \subseteq U$.

7.

If f is continuous, $x_\alpha \to x$ and V is a neighborhood of $f(x)$, then $U = f^{-1}(V)$ is an open neighborhood of x and for some β, $\beta \preceq \alpha$ implies $x_\alpha \in U$ and $f(x_\alpha) \in V$ .The other direction. Let V be open, $U = f^{-1}(V)$, and $x \in U$. We need to show that there is an open neighborhood of x contained in U. Suppose there is no such neighborhood. Then $x \in \overline{X - U}$. Using the previous exercise, there is a net (x_α) of points of $X - U$ converging to x. But in this case V contains no points of $(f(x_\alpha))$ which contradicts the assumption that the image of any net converging to x converges to $f(x)$.

8.

If $x_\alpha \to x$ then for any neighborhood U of x there is β such that $\beta \preceq \alpha$ implies $x_\alpha \in U$. Since $g(K)$ is cofinal in J, there is $\gamma \in K$ such that $\beta \preceq g(\gamma)$. For all $\gamma \preceq \delta \in K$, $\beta \preceq g(\delta)$, hence, $x_{g(\delta)} \in U$.

9.

If there is a subnet $g: K \to J$ converging to x then for any neighborhood U of x there is $\gamma \in K$ such that $\gamma \preceq \alpha$ implies $x_{g(\alpha)} \in U$. Let $L = \{g(\alpha)|\gamma \preceq \alpha\}$. L is cofinal in J. Indeed, for any $\beta \in J$ there is $\delta \in K$ such that $g(\delta) \succeq \beta$ and $\alpha \in K$ such that α is greater than both δ and γ. Then $g(\alpha) \in L$ and $\beta \preceq g(\alpha)$. Now, the set of all points from (x_α) such that $x_\alpha \in U$ contains L, therefore, it is cofinal in J.The other direction. For any neighborhood U of x let $J_U \subseteq J$ be the cofinal subset of those α such that $x_\alpha \in U$. We need to define a directed set K and $g: K \to J$ such that $k \preceq k'$ implies $g(k) \preceq g(k')$, $g(K)$ is cofinal in J and for any neighborhood U of x there is $k \in K$ such that $k \preceq \alpha$ implies $x_{g(\alpha)} \in U$. Consider two different neighborhoods U and V of x and their intersection W. Note that $J_W = J_U \cap J_V$. However, we want to distinguish all indexes in J_W from the same indexes in J_V and J_U. This suggests to take K as the union of all pairs (α, U) where $\alpha \in J_U$ and define $g((\alpha, U)) = \alpha$. Moreover, for a given $(\alpha, U) \preceq (\beta, V)$ we want both: $g((\alpha, U)) = \alpha \preceq g((\beta, V)) = \beta$ and $x_{g((\beta,V))} \in U$. Therefore, we define the partial order on K as follows: $(\alpha, U) \preceq (\beta, V)$ iff $\alpha \preceq \beta$ and $V \subseteq U$. This is a partial order on K (this can be easily checked even in general for the partial order defined this way on the product of two partially ordered sets). Now, for any pair of elements of K, (α, U) and (β, V): find $\gamma \succeq \alpha, \beta$, then $\delta \in J_{U \cap V}$ such that $\delta \succeq \gamma$ (here we use the fact that all J_U are cofinal), then we have $(\delta, U \cap V) \succeq (\alpha, U), (\beta, V)$. Therefore, K is a directed set. Moreover, $k = (\alpha, U) \preceq k' = (\beta, V)$ implies $g(k) = \alpha \preceq g(k') = \beta$ and $g(K) = J$ is cofinal in J. For any neighborhood U of x the set J_U is not empty, and $(\alpha, U) \preceq (\beta, V)$ implies $x_{g((\beta,V))} = x_\beta \in V \subseteq U$.

10.

Suppose the space is compact. Take any net $x_\alpha = f(\alpha)$. For each α let $B_\alpha = \{\beta|\alpha \preceq \beta\}$. Then, given that the index set is directed, the collection of sets $f(B_\alpha)$ satisfies the finite intersection property. Let x be the intersection of the closures of these sets and U be its neighborhood. Since x lies in the closure of each $f(B_\alpha)$, for each α there is $\beta \in B_\alpha$ such that $x_\beta \in U$. Therefore, x is an accumulation point of the net, and, using the previous exercise, there is a subnet converging to x.The other direction. Suppose that every net in X has a convergent subnet. Let B_j, $j \in J$ be a collection of closed sets satisfying the finite intersection property. Consider the set K of all finite subsets of J: $K = \{\alpha \subseteq J||\alpha| < \infty\}$. It is a directed set with the partial order given by the reverse inclusion. Moreover, since each finite intersection has a point, for each $\alpha \in K$ we can take $x_\alpha \in \cap_{j\in\alpha} B_j$. This is a net in X, and it must have an accumulation point x. In particular, for every $k \in J$ and neighborhood U of x there must be $\beta \succeq \{k\}$ such that $x_\beta \in U$. Given that $x_\beta \in \cap_{j\in\beta} B_j$ and $\beta \subseteq \{k\}$ we conclude that $U \cap B_k \neq \emptyset$. Thus, x lies within the closure of B_k which is equal to B_k. Therefore, $\cap_j B_j$ is not empty.

11.

Roughly speaking, if $x \in \overline{A \cdot B}$ then there is a net in $A \cdot B$ converging to x, there is a subnet of points in B converging to some $b \in B$ (B is compact and closed as G is Hausdorff), and the corresponding subnet of points in A converges to a point in A which must be $a = x \cdot b^{-1} \in A$ (A is closed, the operations are continuous).

12.

If we omit condition (2) in the definition of the directed set, then it is the same as to assume that there could be classes of different but "equivalent" indexes. Therefore, this does not restrict the previous definition, and all the examples in the Exercise 1 are still nets. The exercise 2 showed that a cofinal subset of a directed set is a directed set itself, and the proof remains valid as it does not depend on the property (2). The Exercise 3 shows that a sequence is a net, and since the new definition of a net enlarges the set of all nets, the fact still holds. In the exercise (4) we showed that a pointwise convergence of a finite product of nets implies the convergence of the product and, once again, the proof does not depend on the property (2). Exercise 5 tells us that in a Hausdorff space a net cannot converge to two different points at the same time, and the result is still valid as the proof depends on the fact for a given pair of elements there is another one greater than both whether the two are equivalent or not.Now, we claim that for any net $(x_\alpha)_{\alpha \in J}$ satisfying the new definition there is a subnet that is also a net according to the initial definition. Indeed, consider the new index set $[J]$ that consists of all equivalence classes $[\alpha]$ of J with the partial order given by the previous relation (i.e. $[\alpha] \preceq [\beta]$ iff $\alpha \preceq \beta$). It is straightforward to see that this new relation is well-defined and is a (strict) partial order on the set of all equivalence classes. Take the map $f : [J] \to J$ by taking $f([\alpha])$ to be any element from $[\alpha]$. This map preserves the relation, and its image is cofinal in J as $f([\alpha]) \succeq \alpha$ for all α . Let us call a subnet constructed this way $(x_{[\alpha]})$ (there may be many as the choice of $f([\alpha])$ is arbitrary).If (x_α) converges to some point x , then $(x_{[\alpha]})$ converges to the same point. In fact, this holds more generally, namely, if a net converges to a point, then every its subnet converges to the same point. This was proved in Exercise 8, and the proof remains true for the new definition as well. Using this and the construction of a subnet satisfying the initial definition given above, we conclude that the theorem of Exercise 6 (a point is in the closure of a set iff there is a net in the set converging to the point) still holds.For Exercise 7 we need a more general result. Suppose that for a net (x_α) that satisfies the new definition every its subnet that satisfies the initial definition converges to x . Then $x_\alpha \to x$. [Suppose not. Then there is a neighborhood U of x such that for every α there is $\gamma(\alpha) \succeq \alpha$ such that $x_{\gamma(\alpha)} \notin U$. For each $[\alpha]$ define $f([\alpha])$ to be an element $\beta \in [\alpha]$ such that $x_\beta \notin U$ if such an element exists or any arbitrary element in $[\alpha]$ otherwise. Then this defines a subnet of the type $x_{[\alpha]}$ the same way as before, i.e. it satisfies the initial definition. For any $[\alpha]$: $x_{f([\gamma(\alpha)])} \notin U$ and $[\alpha] \preceq [\gamma(\alpha)]$. This contradicts the assumption.] Given this result, we conclude that if f is continuous and a net (in the new definition) converges to x then every its subnet satisfying the initial definition converges to x and its image converges to $f(x)$, therefore, the image of the net converges to $f(x)$ as well. The reverse implication is immediate.The lemma of Exercise 9 tells us that x is an accumulation point of the net (x_α) iff its a limit of a subnet. Note, that x is an accumulation point of (x_α) iff for any neighborhood U of x the set of all indexes α such that $x_\alpha \in U$ is cofinal in J iff the set of all indexes $[\alpha]$ such that there is $\beta \in [\alpha]$ such that $x_\beta \in U$ is cofinal in $[J]$ iff there is a subnet $(x_{[\alpha]})$ that has an accumulation point x . This implies that there is a subnet converging to x . The other direction: if there is a subnet $x_{g(\beta)}$ converging to x then we can leave the proof that it is an accumulation point the same as before.For the Exercise 10 suppose, first, that every net has a convergent subnet. Then every net according to the initial definition has a convergent subnet that has a convergent subnet satisfying the initial definition. Therefore, as was proved before, the space is compact. Now, suppose that the space is compact. Then for a net there is a subnet satisfying the initial definition that has another subnet converging to a point.Finally, the proof of the Exercise 11 does not depend on the definition of the net but rather on the previous results which we have already proved to hold under the new definition.