CHAPTER 4

SECTION 30

1 、

(a) Let $\{U_n\}$ be a countable basis at x . Then for each $y \neq x$ there is a neighborhood U of x such that U does not contain y . Therefore, there is U_k that does not contain y . Thus, $\cap_n U_n = \{x\}$.(b) Given any x and $y \neq x$ there must be a neighborhood of x that does not contain y , therefore, the space is a T_1 -space and every one-point set is closed. We can start with some first-countable T_1 -space. By (a), every one-point set in such a space will be a G_δ set. Then, if we take a finer topology, it will still have the property, however, we may loose the first-countability if the space is large enough. For example, if we take R^ω then it is a metric space in the product topology. Therefore, every one-point set is the intersection of the countable collection of balls of radius $1/n$ centered at the point. Now, if we switch to the box topology, then it seems not to satisfy the first-countability axiom (take a countable collection U_n of neighborhoods of a point and construct a neighborhood by taking the product of open proper subsets of U_{nn}).Also, another example would be any countable T_1 -space which is not first-countable. Such a space will also be separable and Lindelöf but not second-countable. One example of such a space I found by solving Exercise 17.

2 、

Let us choose $C_{n,m}$ as in the hint (whenever it is possible). It is a countable subcollection of $\mathcal{C}$ which is a basis: for any open U and $x \in U$ there is an open set $B_m \subseteq U$ containing x , an open set $C \subseteq B_m$ containing x and an open set $B_n \subseteq C$ containing x , thus, $x \in C_{n,m} \subseteq U$ does exist.

3 、

Suppose the number of limit points in A is countable. For each point in A that is not a limit point take a basis neighborhood that contains this point in A only. All such neighborhoods must be different. Contradiction.

4 、

As in the hint: take a covering by $\mathcal{A}_n$ and its finite subcovering. The union over all n is a countable basis. Another way to show this is as follows: compact metric implies Lindelöf metric implies second-countable.

5 、

(a) For each point in the dense take a countable collection of balls of radius $1/n$ centered at the point. Let $x \in U$, U is open. There is a ball $B(x,r) \subseteq U$. Find a dense point d in $B(x,1/n)$ where $n > 2/r$. Then $x \in B(d,1/n) \subseteq B(x,r) \subseteq U$.(b) Similar to exercise 4 (the first part) with the only difference that the subcovering is going to be countable.

6 、

$\mathbb{R}_l$ is separable but not second-countable (proved in the text). The ordered square is compact (it is a linear continuum and a closed interval) but not separable (take the uncountable collection of vertical open intervals, each must have at least one dense point). Note that both are examples of first-countable but not second-countable spaces.

7 、

Both are not separable: any countable subset of S_Ω has an upper bound. Therefore, both are not second-countable. S_Ω is first-countable: for any x such that $\{x\}$ is not open (x has no predecessor) the countable collection of intervals $(a, x+1)$ for all $a < x$ is a basis at x ; but not Lindelöf: (a_0, x) has no countable subcover. $\overline{S}_\Omega$ is not first-countable: the added point has no countable local basis (for the same reason: the supremum of countable numbers of infima of open sets will be less than ω); but is Lindelöf as it is even compact. This illustrates that a subspace of a Lindelöf space may be not Lindelöf.

8 、

It is a metric space, therefore, it is first-countable and all other three properties are equivalent. In fact, they do not hold. We show that it is not second-countable. According to exercise 3, if it was second-countable, then every uncountable set would have a limit point. But if we take a set of all sequences of 0’ s and 1’ s, then it is uncountable but the distance between any two elements is 1.

9 、

This is pretty much the same as for compact spaces: take any open covering, extend open sets to open sets in X, add $X - A$, find a countable subcovering. Now, the second part asks to show by an example that a closed subspace of a separable space need not be separable. Indeed, $\mathbb{R}_l^2$ is separable but its "inverse diagonal" is uncountable and discrete in the subspace topology.

10 、

Fix any $x_i \in X_i$. Consider all points that equals x_i at all but finitely many coordinates and equal some dense points at all others. There is a countable set of finite subsets of coordinates and for each a countable set of points we just defined.

11 、

If f is a continuous map then the dense maps to the dense (each open set contains the image of a dense point in the preimage of the set) and for any covering of the image a subcovering may be obtained as the image of a subcovering of the preimage of the collection. Therefore, if the space is separable, so is the image, and if the space is Lindelöf, so is the image.

12

We just need to show that the image of a basis is a basis of the image of the space. If V is a neighborhood of $f(x)$ then $U = f^{-1}(V)$ is a neighborhood of x , and there is a basis open set containing x , its image is a basis neighborhood of $f(x)$ (as f is open).

13

Each set in the collection must have a dense point and they must be different as the sets are disjoint.

14

Consider any covering of the product by basis open sets $U_\alpha \times V_\alpha$. For every x find a finite subcovering $U_n^x \times V_n^x$ that covers $x \times Y$ (the projection is an open map). Let $U^x = \cap_n U_n^x$. Find a countable subcovering U^{x_k} of X . Then $\{U_n^{x_k} \times V_n^{x_k}\}$ is a countable subcovering of $X \times Y$. Indeed, for any point (x, y) there is k such that $x \in U^{x_k}$ and also n such that $(x_k, y) \in U_n^{x_k} \times V_n^{x_k}$. Since $U^{x_k} \subseteq U_n^{x_k}$, we have $(x, y) \in U_n^{x_k} \times V_n^{x_k}$.

15

It is the space of all continuous on functions with the uniform metric. We need to show that the space is separable (and, therefore, being metric, it is also second-countable). Consider any finite subset of that includes both end points. There are countably many such sets. Now consider all functions that takes rational values at the points of such a finite set and are linear between the points. They are all continuous functions (use the Pasting Lemma from §18 or the fact that the function is bounded, therefore, the image lies within a compact Hausdorff space and the result from §26 tells us that it is continuous iff its graph is closed) well-defined uniquely by the condition above (all points in a finite set are isolated and for each point not in the set there is the closest point below and another above), and there are countably many of such functions. All we need to show that any continuous function can be approximated by functions from this collection. This is quite easy to show using the results from §27. A continuous function from a compact metric space to a metric space is uniformly continuous. For a given find such that if then . Take a finite set of points of such that the distance between a pair of successive point is less than . For each point find a rational number and construct the dense function . Note that . Then for any we have .

16

(a) The idea (similar to exercise 15): any open set is the product such that only finitely many open sets are not equal spaces. We can always separate the indexes of these spaces by some rational points. So let' s take all finite subsets of rationals in $[0,1]$ containing 0 and 1 (countably many), each such a subset generates a finite number of half-open intervals of the form [), and then take products of rational points such that they are the same across each interval (countably many).(b) Define f as in the hint for a fixed interval. Note that all $f(\alpha)$ are different (indeed, for $\alpha \neq \beta : \pi_\alpha^{-1}((a,b)) \cap \pi_\beta^{-1}((b,+\infty))$ must have a dense point but lies within $\pi_\alpha^{-1}((a,b))$ only). Therefore, f is an injection and $|J| \leq |2^D|$.

17

It is not first countable for the reason exactly the same as in Exercise 1(b). Therefore, it is not second-countable. It is countable, therefore, separable and Lindelöf.

18

(Occasionally we use some results from the Supplementary Exercises of Chapter 2.) So, suppose $\{B_n\}$ is a countable basis at e . If G has a countable dense subset D, then we show that $d \cdot B_n$ form a basis. Indeed, for any $x \in U$: $U' = x^{-1} \cdot U$ is a neighborhood of e , there is a symmetric basis neighborhood $V = V^{-1}$ of e such that $V \cdot V \subseteq U'$, there is a dense point d in the neighborhood $x \cdot V$ of x , and $x \in d \cdot V^{-1} = d \cdot V \subseteq x \cdot V \cdot V \subseteq x \cdot U' = U$. Now, suppose that for every covering of G there is a countable subcovering. For each n take the covering of the space by $x \cdot B_n$ and find a countable subcovering. The union over all n forms a countable basis. Indeed, take $x \in U$. Choose a symmetric basis neighborhood V of e as before, find y such that $y \cdot V$ is a basis neighborhood of x , then $y \cdot V \subseteq x \cdot V^{-1} \cdot V \subseteq x \cdot U' = U$.

SECTION 31

1

Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

For x and y take first a neighborhood V of y such that $\overline{V} \subset X - \{x\}$, and then a neighborhood U of x such that $\overline{U} \subset X - \overline{V}$.

2

Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Similar to 1.

3

Show that every order topology is regular.

Let $x \in (a,b)$ where a or b can be infinite. Let c be a point in (a,x) if there is such a point, otherwise $c = a$ (note that in this case $(-\infty, x) \cap (c,b) = \emptyset$, therefore, no points below x are limit points of (c,b)). Similarly choose d in (x,b) or $d = b$ if the interval is empty. Then $x \in (c,d)$ and $\overline{(c,d)} \subset (a,b)$.

4

Let X and X' denote a single set under two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; assume that $\mathcal{T}' \supset \mathcal{T}$. If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?

If a space is Hausdorff then it is Hausdorff in a finer topology. There is no such relation for the two other properties. Indeed, $\mathbb{R}$ is normal and regular, while $\mathbb{R}_K$ is not even regular. At the same time every space in the discrete topology is normal.

5

Let $f, g : X \to Y$ be continuous; assume that Y is Hausdorff. Show that $\{x | f(x) = g(x)\}$ is closed in X.

The set is the preimage of the diagonal of $Y \times Y$ (which is closed as Y is Hausdorff) under the continuous (Theorem 19.6) function $x \to (f(x), g(x))$.

6

Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y . [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show there is a neighborhood W of y such that $p^{-1}(W) \subset U$.]

Take a closed set $B \subset Y$ and an open set $V \supset B$. $A = p^{-1}(B) \subset U = p^{-1}(V)$ is a closed set in an open set. Find an open neighborhood W of A such that $\overline{W} \subset U$. $p(X - W)$ is closed, and $B \subset Y - p(X - W) \subset p(W) \subset V$, because $p^{-1}(B) \cap X - W = \emptyset$ and $p(X - W) \cup p(W) = Y$ (p is surjective). So, $B \subset W' \subset V$, where $W' = Y - p(X - W)$ is open. Moreover, $p(\overline{W}) \subset V$ is closed and contains W' . Therefore, $\overline{W'} \subset V$.

For an alternative solution, see §73, Lemma 73.3.

7

Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a *perfect map*.)

(a) Show that if X is Hausdorff, then so is Y.

(b) Show that if X is regular, then so is Y.

(c) Show that if X is locally compact, then so is Y.

(d) Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.]

In Exercise 12 of Section 26 we established that if p is closed and U containing $p^{-1}(y)$ is open, then there is some neighborhood V of y such that $p^{-1}(V) \subset U$. Neither continuity nor surjectivity of p is required for this fact. More generally, we can state the same fact for any subset $B \subset Y$ instead of just one point y (take the union of all such neighborhoods for each point in B).

(a) Take two different points $a, b \in Y$, their preimages are disjoint compact nonempty (surjectivity) subspaces of a Hausdorff space, which can be separated by open neighborhoods U, V. Now, using the fact above, find neighborhoods of a and b such that their preimages are in U and V. Since U and V are disjoint, so are the neighborhoods of a and b (we again use surjectivity here). (How did we use the continuity of p ?)

(b) Let $y \in Y$, and B be a closed subset of Y such that $y \notin B$. Then, $C = p^{-1}(y)$ is compact, and $A = p^{-1}(B)$ is closed (continuity), where both A and C are nonempty (surjectivity). Since X is regular, we can separate each point in C and the set A by some open neighborhoods. Since C is compact, we can take a finite number of such open neighborhoods covering C (let U be their union), and the corresponding finite intersection of open neighborhoods of A is an open neighborhood V of A disjoint from U. Now, we use the same fact above to find neighborhoods of y and B such that their preimages are in U and V, correspondingly. Since p is surjective and U and V are disjoint, the neighborhoods of y and B are disjoint as well.

(c) Let $y \in Y$. $A = p^{-1}(y)$ is a non-empty compact set (surjectivity). Each point of A has a neighborhood that lies within a compact subspace of X. All such neighborhoods cover A, and there is a finite subcovering U, a neighborhood of A. The corresponding finite union C of compact subspaces of X is compact and contains U. Once again, we use the fact above to find a neighborhood V of y such that $p^{-1}(V)$ is in U. Therefore, the neighborhood lies within $p(C)$ (p is surjective), which is compact as the continuous image of a compact set.

(d) Using the hint, let $\{B_n\}$ be a countable basis for X. For each N, a *finite* subset of $\mathbb{Z}_+$, define $B_N = \cup_{n \in N} B_n$, and let V_N be the largest open set in Y such that its preimage is in B_N, i.e. V_N is the union of all open sets of Y having preimages in B_N. Let $U_N = p^{-1}(V_N)$, open by continuity of p. There are countably many sets in $\{V_N\}$. We show that they form a basis for the topology of Y. Take $y \in V$, where V is open in Y. Let $A = p^{-1}(y)$ (compact, nonempty by surjectivity), $U = p^{-1}(V)$ (open by continuity). For each point of A choose a basis neighborhood contained in U, and then, a finite subcovering B_N of A contained in U. Using the fact above, there is a neighborhood of y such that its preimage is contained in B_N, therefore, $y \in V_N$, $A \subset U_N \subset B_N \subset U$, and V_N is a basis neighborhood of y contained in V (again, by surjectivity of p). Done.

8、

Let X be a space; let G be a topological group. An ***action*** of G on X is a continuous map $\alpha : G \times X \to X$ such that, denoting $\alpha(g \times x)$ by $g \cdot x$, one has:

(i) $e \cdot x = x$ for all $x \in X$.

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Define $x \sim g \cdot x$ for all x and g; the resulting quotient space is denoted X/G and called the ***orbit space*** of the action α.

Theorem. Let G be a compact topological group; let X be a topological space; let α be an action of G on X. If X is Hausdorff, or regular, or normal, or locally compact, or second-countable, so is X/G.

[*Hint:* See Exercise 13 of §26.]

Let p be the quotient map. Then $p^{-1}([Gx]) = G \cdot x$, compact as the continuous image of the compact space G. Suppose now, that $A \subset X$ is closed. We show that $G \cdot A$ is closed. This would imply that $p(A) = [GA]$ is closed, therefore, p is a closed quotient map, and we can use the previous two exercises to prove what we are asked to (instead, as suggested in the hint, one could try to use Exercise 13(b) of §26). Let $z \notin G \cdot A$. For every $g \in G$, $g \cdot z \notin A$. Since α is continuous and $X - A$ is open, $\alpha^{-1}(X - A)$ is open, and there is some basis neighborhood $U_g \times V_g$ of $g \times z$ such that $\alpha(U_g \times V_g)$ does not intersect A. The union of all U_g covers G. Since G is compact, take a finite subcovering, the corresponding finite intersection V of V_g is a neighborhood of z such that $\alpha(G \times V)$ does not intersect A. Therefore, V does not intersect $G \cdot A$.

* 9. Let A be the set of all points of $\mathbb{R}_l^2$ of the form $x \times (—x)$, for x rational; let $\mathbf{B}$ be the set of all points of this form for x irrational. If V is an open set of $\mathbb{R}_l^2$ containing B, show there exists no open set U containing A that is disjoint from V, as follows:

(a) Let K_n consist of all irrational numbers x in $[0,1]$ such that $[x, x+1/n) \times [-x, -x+1/n)$ is contained in V. Show $[0,1]$ is the union of the sets K_n and countably many one-point sets.

(b) Use Exercise 5 of §27 to show that some set $\overline{K}_n$ contains an open interval (a, b) of $\mathbb{R}$.

(c) Show that V contains the open parallelogram consisting of all points of the form $x \times (-x+\epsilon)$ for which $a < x < b$ and $0 < \epsilon < 1/n$.

(d) Conclude that if q is a rational number with $a < q < b$, then the point $q \times (-q)$ of $\mathbb{R}_l^2$ is a limit point of V.

(a) For every irrational point $x \in [0,1]$ there is a basis neighborhood $[x, x+\epsilon_x) \times [-x, -x+\delta_x)$ contained in V, therefore, if $n > 1/\min\{\epsilon_x, \delta_x\}$ then $x \in K_n$. The rest of $[0,1]$, i.e. the rational points, is a countable set of points.

(b) According to the result of the Exercise 5 of §27, since $\cup_n \overline{K}_n \cup \cup_{q\in[0,1]\cap\mathbb{Q}} \{q\} = [0,1]$ has non-empty interior in $[0,1] \subset \mathbb{R}$ being compact and Hausdorff, one of the closed sets in the union must have a non-empty interior in $[0,1] \subset \mathbb{R}$ as well. Therefore, there is n such that $\overline{K}_n$ has non-empty interior, and contains a non-empty interval (a, b).

(c) I show a little different result: for n found in (b) there is a non-empty interval $(a', b') \subset [0,1]$ and $0 < \epsilon < 1/n$ such that for every $x \in (a', b')$, $x \times (-x, -x+\epsilon) \subseteq V$. Take $\epsilon < \min\{1/n, (b-a)/2\}$ where n and $a < b$ are such that $(a, b) \subset \overline{K}_n$ (found in (b)). Let $A = (a', b') = (a+\epsilon, b-\epsilon)$. A is not empty. For each $x \in A$ and $0 < \delta < \epsilon$ there is a point $z \in (x-\delta, x) \cap K_n$. Therefore, $(x, -x+\delta) = (z + x - z, -z + \delta - (x-z)) \in [z, z+1/n) \times [-z, -z+1/n) \subset V$.

(d) Given (c), for every point $q \in \mathbb{Q} \cap A$ and every neighborhood W of $(q, -q)$ there is $0 < \delta \leq \epsilon$ such that $q \times (-q, -q+\delta) \subset W \cap V$.

SECTION 32

1

Take two closed subset A, B in the closed subspace $Y \subseteq X$. A and B are closed in X, therefore, they can be separated by disjoint neighborhoods U and V. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of A and B in Y.

2.

Take a point $x \in X_\alpha$ and a point y or a closed set A contained in $X_\alpha - \{x\}$ (if X_α contains one point only, we are done). For every $\beta \neq \alpha$ fix $x_\beta \in X_\beta$. For every $z \in X_\alpha$ let p_z be the product of z with x_β for all $\beta \neq \alpha$. For a subset $S \subseteq X_\alpha$ let $P_S = S \times \sqcap_{\beta \neq \alpha} X_\beta$. If the product is Hausdorff, p_x and p_y can be separated by basis neighborhoods U and V . Then $\pi_\alpha(U)$ and $\pi_\alpha(V)$ are open (the projection is an open map) and disjoint (if there is a common point z then p_z belongs to both U and V) neighborhoods of x and y . Now, if the product is regular, then there are disjoint basis neighborhoods of p_x and P_A , and their projections into X_α are open and disjoint (for the same reason) neighborhoods of x and A . If the product is normal and B is a closed subset of X_α that does not intersect A , then P_A and P_B can be separated by disjoint basis neighborhoods whose projections into X_α separate A and B .

As Yuan says in a comment below, Munkres defines both regular and normal spaces with the additional requirement that singletons are closed. This is actually a common definition now, which guarantees that every normal space is regular etc. So, I missed that point in the original solution. In other words, we need also to show that the space is a T_1- space. I leave this as a very easy exercise to the reader.

> Of course, I am not leaving you alone. Moreover, here is an alternative way to prove almost all statements in this exercise. We can actually use the hereditary properties of spaces. Indeed, each X_α is homeomorphic to a subspace of the product space (which one?), and so if the product is T_1 , Hausdorff or regular, so is X_α . Done. Almost done. Because being a normal space is not a hereditary property, so we still need to show that each X_α is normal if so is the product.

3.

Theorem 29.2: If X is Hausdorff then it is locally compact iff for every x and its neighborhood U there is its neighborhood V such that $\overline{V} \subseteq U$ is compact. Lemma 31.1: If X is T_1 then it is regular iff for every x and its neighborhood U there is its neighborhood V such that $\overline{V} \subseteq U$. Therefore, if X is Hausdorff and locally compact then it is T_1 and regular.

4.

The only difference from the proof of the Theorem 32.1 is that we construct a countable covering using the Lindelöf property of the space instead of its countable basis.

5.

Yes. Yes. Both are metrizable.

6.

$\Rightarrow$ Suppose A, B is a pair of separated subsets of X . Then $Y = X - (\overline{A} \cap \overline{B})$ is an open subset of X that contains both A and B . $\overline{A}_Y \cap \overline{B}_Y = Y \cap \overline{A} \cap \overline{B} = \emptyset$. Thus, $\overline{A}_Y$ and $\overline{B}_Y$ can be separated by open neighborhoods in Y . Since Y is open, these neighborhoods are also open in X .$\Leftarrow$ Take a set $Y \subseteq X$ and two disjoint subsets $A, B \subseteq Y$ closed in Y . $\overline{A}_X \cap B = \overline{A}_X \cap Y \cap B = \overline{A}_Y \cap B = \emptyset$. Similarly, $\overline{B}_X \cap A = \emptyset$. Therefore, A and B can be separated by neighborhoods in X and their intersections with Y separate A and B in Y .

7.

(a) Yes. A subspace of a subspace of a completely normal space is a subspace of the space, and, therefore, is normal.(c) Yes. If a set is well-ordered then every element has a successor and $\{(a, x]\}$ is a basis at x where $a < x$ or $a = -\infty$. Therefore, as in the proof of the Theorem 32.4, for a pair of separated sets we can cover each set with such neighborhoods that do not intersect the other set. Moreover, the neighborhoods belonging to one set do not intersect the neighborhoods belonging to the other set.(For any ordered set such that either each point has a successor or each point has a predecessor the proof above works as well: for the latter case we use basis neighborhoods of the form $[)$. Also, the next exercise states that every linear continuum is normal: a linear continuum is an example of an ordered set such that it satisfies the least upper bound property the same as a well-ordered set does but every point "in the middle" has no successor or predecessor. Note that even though we know that an arbitrary ordered set is normal, its subspace topology is in general finer than the order topology on the subspace, therefore, we cannot immediately conclude that every ordered set is completely normal. However, it is true as well.)(g) Yes. Similarly to (c), we can prove that $\mathbb{R}_l$ is completely normal. Indeed, the proof of the Theorem 32.4 is extremely similar to the one of Example 2 of the previous section: both use the fact that there is a basis at x with sets of the form $[x,)$ or $(, x]$ and both use only the fact that every point in one closed set has such a neighborhood that does not intersect the other set. Then coverings by such basis sets are disjoint automatically. So, as in (c), we obtain the disjoint open neighborhoods for any pair of sets such that neither one contains limit points of the other one.(b) No. $\mathbb{R}_l$ is completely normal (see (g) above), but $\mathbb{R}_l^2$ is not even normal.(d) Yes. Every subspace is metrizable as well, therefore, normal (Theorem 32.2).(e) No. See Example 2.(f) Yes. Every regular second-countable space is normal (Theorem 32.1). Also every its subspace is also regular and second-countable (see Section 31). Therefore, every regular second-countable space is completely normal.

8.

(a) C is non-empty and closed, U is a component of $X-C$. If $C=X$ then U is empty otherwise it is not. So, suppose it is not. Theorem 24.1 tells us that X and all rays and intervals in X are connected. Then x and y are in the same component of $X-C$ iff $[x,y]$ does not intersect C. If C is bounded below by a point not in C, then let c be the greatest lower bound of C. Since C is closed, $c \in C$ (otherwise, let $x \in C$, $c < x$, i.e. c is not the largest element of X, and every basis neighborhood of c has a from (a,b) where $b > c$, but then there is an element $x' \in (c,b) \cap C$, i.e. c cannot be an interior point of $X-C$). Hence, if there is a component containing a lower bound of C then it is of the form $(-\infty, c)$. Similarly, if there is a component containing an upper bound of C then it is of the form $(c, +\infty)$. Now, suppose that U is bounded below and above by points in C. Let c be its greatest lower bound and c' be its least upper bound. Neither c nor c' belongs to U (essentially, for the same reason). But for every $c < x < y < c'$ there are points $c < a < x < y < b < c'$ such that $a, b \in U$, therefore, $x, y \in [a,b] \subseteq U$. Hence, $U = (c, c')$.(b) Suppose x is a limit point of C. If x is in a component then, using (a), the component is a neighborhood of x that contains only one point in C, therefore, since the space is Hausdorff, x must be equal to that point. Suppose now, $x \in A$. Then $x \notin X - A \cup B$ and each neighborhood of x must contain infinite number of points in C. Suppose $a < b < c$ are such points. b lies within an interval with endpoints in A and B that contains neither a nor c. Therefore, x is a limit point of B, but B is closed, so $x \in A \cap B$. This contradicts the assumption that they are disjoint. Similarly, x cannot lie within B. To sum up, only points in C can be limit points of C, i.e. C is closed.(c) Using (a) and (b) we conclude that the components of $X-C$ are open intervals or rays. Then V is an interval that does not contain points in C. Suppose $a \in A \cap V$, $b \in B \cap V$ and $a < b$. Let b' be the greatest lower bound of points in $[a,b] \cap B$. Then, as before, $b' \in B$ and $[a, b')$ contains no points in B. Let a' be the lowest upper bound of points in $[a, b'] \cap A$. Then $a' \in A$, $a' < b'$ and (a', b') has no points in $A \cup B$. Therefore, it is a component of $X - A \cup B$ with the endpoints in both A and B, and must have a point in C. Contradiction.Conclusion. For A choose those components of $X-C$ that contains points in A (they do not contain points in B), and for B choose all other components. Since C intersects neither A nor B, we have open disjoint sets separating A and B.

9.

We show that $X=(\mathbb{Z}_+)^J$, where J is uncountable, is not normal.(a) B is a finite subset of J , $x\in X$, $U(x,B)=\{y\in X|y(j)=x(j), j\in B\}=\pi_B(x)\times\sqcap_{j\notin B}X_j$. Every $U(x,B)$ is open ($\mathbb{Z}_+$ is in the discrete topology). If $x\in U$, U is open in X , then there is a finite set B of indexes such that $j\notin B$ implies $U_j=X_j$. Therefore, $U(x,B)\subseteq U$ and they form a basis.(b) Let $P_n=\{x\in X|x_j\neq n\neq x_k\Rightarrow x_j\neq x_k\}$, i.e. x may have many coordinates equal to n but all other coordinates different. $\{P_n\}$ are closed and disjoint. Indeed, if $y\notin P_n$ then $y_j=y_k\neq n$ and $U(y,\{j,k\})$ does not intersect P_n . For $z\in P_n$ let $A=\{j\in J|z_j\neq n\}$. Since z is injective on A , A is countable. Therefore, there is uncountable set of indexes B such that $z_j=n$ for $j\in B$, i.e. z cannot be a point in any other P_m .Suppose U and V separate P_1 and P_2 and derive a contradiction.(c) Essentially, what we do here: we want to construct a sequence of points in P_1 such that they "converge" eventually to a point of the form: 1 for all indexes except a countable subset of indexes for which it takes values 1, 2, 3, ... We take the first point x^1 equal to 1, then for the first n_1 indexes in a countable sequence α of indexes we define values from 1 to n_1 (x^2), then do the same for the first $n_2>n_1$ indexes (x^3), etc. Moreover, we want to define the sequence α of indexes and the sequence $\{n_i\}$ of "steps" such that $U(x^i,B_i)=U(x^i,\{\alpha_1,\ldots,\alpha_{n_i}\})\subseteq U$. Note that x_i takes values different from 1 on the set $\{\alpha_1,\ldots,\alpha_{n_{i-1}}\}$ while B_i includes the next step as well. Ok, so, $x^1_j=1$ for all $j\in J$ and $B_0=\emptyset$. Given x^n and B_{n-1} we first define B_n such that $U(x^n,B_n)\subseteq U$ (this is possible because $x^n\in P_1\subseteq U$ and we can define $B_n\supseteq B_{n-1}$) and then just define x^{n+1} by the expression.(d) Now, given the sequence of points α and the sequence of points x^i , we define a point y in P_2 equal to 2 for all points not in α and j for α_j . $y\in V$, so there is a finite set B such that $U(y,B)\subseteq V$. Since B is finite, there is i such that $B\cap\alpha\subseteq B_i=\{\alpha_1,\ldots,\alpha_{n_i}\}$. Let $z_j=j$ for $j\in B_i$, $z_j=1$ for $j\in B_{i+1}-B_i$ and $z_j=2$ for all other indexes. Then for all $j\in B$ either $j\in B_i$ and $z_j=j=y_j$ or $j\notin\alpha$ and $z_j=2=y_j$. For all $j\in B_{i+1}$ either $j\in B_i$ and $z_j=j=x^{i+1}_j$ or $j\in B_{i+1}-B_i$ and $z_j=1=x^{i+1}_j$. Therefore, $z\in U(y,B)\cap U(x^{i+1},B_{i+1})\subseteq V\cap U$.

10.

In the Supplementary Exercises (exercise 7(c)) of Chapter 2 we have shown that G is regular. The previous exercises shows that it does not have to be normal.

SECTION 33

1.

$x\in\overline{U}_r\Rightarrow f(x)\leq r$, $x\notin U_r\Rightarrow f(x)\geq r$. So, if $f(x)=r$, for $p>r$, $x\in U_p$, for $q<r$, $x\notin\overline{U}_q$. The other direction. If $x\in U_p$ for all $p>r$ then $f(x)\leq r$. If $x\notin U_p$ for $p<r$ then $f(x)\geq r$.

2.

(a) Follows from (b), but, in fact, we will prove (b) based on (a). For any completely separable space: take two points, a continuous function that separates them, if some point r in the image is missing, can separate: $f(x)<r$ and $f(x)>r$.(b) Countable regular implies countable Lindelöf regular implies countable normal. By (a), if it has at least two points, it is disconnected.

3.

The distance between a point and a set was defined in §27. It was shown that it is continuous, and $d(x, A) = 0$ iff $x \in \overline{A}$ (exercise 2 of §27). Therefore, for two closed sets the expression is well-defined (the denominator is never 0), continuous and takes 0 on A, 1 on B.

4.

Suppose such f exists. $A = f^{-1}(\{0\})$ must be closed, and, using exercise 1, it is the countable intersection of open sets. The other direction. If A is closed and there is a countable collection of open sets V_n such that their intersection is A, then let $U_1 = V_1$ and for every n if $U_{1/n}$ is defined let $A \subseteq U_{1/(n+1)} \subseteq \overline{U}_{1/(n+1)} \subseteq U_{1/n} \cap V_{n+1}$. This is possible because the space is normal and A is closed. We need to slightly modify the construction in the proof of the Urysohn lemma. Namely, we do not define U_0, we define a sequence of point of $\mathbb{Q} \cap [0, 1] - \{0\}$ starting from 1 such that no U_p is defined before $U_{1/n}$ if $p < 1/n$. So, first we define $U_1 = V_1$, then $U_{1/2}$, then some U_p for $1/2 < p < 1$, then $U_{1/3}$, then some U_p for $1/3 < p < 1$, etc. This way we can define U_p for all rational points in $(0, 1]$ with the properties required by the proof. Moreover, $A \subseteq f^{-1}(0) \subseteq \cap_{p>0} U_p = A$.

5.

Suppose such function exists. For A and f, and for B and $1 - f$, apply the previous exercise. Suppose now we have a pair of disjoint closed G_δ sets. Using the previous exercise we construct f such that $g^{-1}(\{0\}) = A$ and $h^{-1}(\{0\}) = B$. Then $f = g/(g + h)$ is what we need. (Compare to exercise 3: for any closed A, $d(x, A)$ satisfies all the properties of a function constructed in exercise 4.)

6.

(a) The intersection of $1/n$ -neighborhoods of a closed set (see exercise 2 of §27) is the set itself: since the set is closed, for a point outside it there is a neighborhood that does not intersect the set. Therefore, a metric space being normal is perfectly normal as well.(b) Let A, B be a pair of separated sets, f and g be two continuous functions that vanish precisely on $\overline{A}$ and $\overline{B}$, respectively. Let $h = f - g$. It is continuous, $A \subseteq f^{-1}(\mathbb{R}_-)$ and $B \subseteq f^{-1}(\mathbb{R}_+)$.(c) We need some completely normal space that is not perfectly normal. This should be a normal space such that there is a closed set which is not G_δ set. What types of completely normal spaces we know so far: metric, ordered and regular second-countable. Metric spaces are perfectly normal. In fact, I looked up ahead, and have realised that the Urysohn Theorem states that a regular second-countable space is metrizable. So, this leaves us with the option to find an ordered space with a closed set that is not G_δ set. The obvious example is $\overline{S}_\Omega$. Its largest element is the one that causes all the problems with the space: it is not separable, it is not Lindelöf, it is not second-countable — all because of the basis at ω. And, I believe, for the same reason it is not perfectly normal. For each set in any countable collection of neighborhoods of ω find a basis neighborhood $(a_n, \omega]$ contained in the neighborhood. Then $\{a_n\}$ has an upper bound, and the intersection cannot be $\{\omega\}$.

7.

X is a subspace of a compact Hausdorff space (§29, one-point compactification), which is normal (§32) and, hence, completely regular (§33, corollary from the Urysohn Lemma), therefore, X is completely regular (Theorem 33.2).

8.

So, what it says is that a completely regular space is "normal" on pairs of closed sets at least one of which is compact. Each point x of the compact set A can be separated from the closed set B by a continuous function f_x (here $f_x(x) = 0$ and $f_x(B) = 1$). Then for some fixed $0 < \epsilon < 1$, $U_x = f_x^{-1}([0; \epsilon))$ defines an open neighborhood of x that does not intersect B . $\{U_x\}$ covers A . We can find a finite subcover $\{U_{xk}\}$, $k = 1 \ldots n$, and take the corresponding finite product of continuous functions $f = f_{x1} \cdot \ldots \cdot f_{x_n}$. Now, for each $x \in A$: $x \in U_{x_k}$ for some k , and $f_{x_k}(x) < \epsilon$, therefore, $f(x) \in [0; \epsilon)$. For every $x \in B$ we have $f(x) = 1$. All we need to do now is to transform f : find continuous $h : [0, 1] \to [0; 1]$ such that for $g = h \circ f : g(x) = 0$ for $x \in A$ and $g(x) = f(x) = 1$ for $x \in B$. This will separate A and B . Indeed, the finite product of continuous functions is continuous and the composition of continuous functions is continuous. For example, $h(z) = (\max\{z, \epsilon\} - \epsilon)/(1 - \epsilon)$ will do.

9.

It is sufficient to show that $\mathbf{0}$ can be separated by a continuous function from a closed set A that does not contain it. There is a neighborhood $W = \sqcap_j(-r_j, r_j)$ that does not intersect A . Consider a linear transformation $h_j : x_j \to x_j/r_j$. We have shown back in the exercises of Section 19 that the linear transformation is a homeomorphism in both the product and box topology on $\mathbb{R}^\omega$. It seems a more general result is true: a product of continuous functions is continuous (so, that the product of homeomorphisms is a homeomorphism). If $\mathbf{y} = h(\mathbf{x}) \in V$, V is open in the box topology, then there is a basis neighborhood $\mathbf{y} \in V' = \sqcap_j V_j' \subseteq V$ and $h(\mathbf{z}) \in V'$ iff for all j : $h_j(z_j) \in V_j'$ iff for all j : $z_j \in h_j^{-1}(V_j')$. Therefore, for all $\mathbf{z} \in \sqcap_j h_j^{-1}(V_j')$ which is open in the box topology if all h_j are continuous, $h(\mathbf{z}) \in V' \subseteq V$. Anyway, it is true for the linear transformation, and if could separate $\mathbf{0}$ from $h(A)$ by a continuous function f , then $f \circ h$ would separate $\mathbf{0}$ and A . Note, that $h(A)$ is closed and does not have any points in $W' = \sqcap_j(-1, 1)$ for if it had some than A would have a point in W . Now, since $\mathbb{R}^J_{uniform}$ is completely separable (as a metric space), there is a continuous function f that separates $\mathbf{0}$ and $X - B(\mathbf{0}, 1) \subseteq W'$ (remember, that W' is not open in the uniform metric: exercise 6 of Section 20). Now, f is continous in the finer box topology as well, and separates $\mathbf{0}$ from $h(A)$.

10.

In the exercise 10 of the previous section we have mentioned that a topological group must be regular but there is an example of a topological group that is not normal. So, the only question is whether it has to be completely regular or not. Here we show that the answer is positive. Recall (Supplementary exercises, Chapter 2) that there is a self-homeomorphism that maps any element to e . Therefore, we only need to show that we can separate e from any closed set that does not contain it by a continuous function. Also, for any neighborhood U of e there is a symmetric neighborhood V of e such that $V \cdot V \subseteq U$. So, for every dyadic rational in $[0,1]$ define U_p by the recursive procedure given in the exercise. Note that the product of any two open sets is open (as the union of open sets of the form a point $\times$ an open set), and that the product of two neighborhoods of e is a neighborhood of e .Let $U(0)=\{e\}$ (it is not open but just for convenience to prove the relations below, after that we assume $U(0)=\emptyset$). Then $U(0) \subseteq U(p)$ for all $p>0$ ($U(0) \subseteq V_n$ for all n , and therefore, $U(0) \subseteq V_n \times U$ for any neighborhood U of e). Now, $U(0/2^n)$, $U(1/2^n)$, $U(2/2^n)$, $U(3/2^n)$, $U(4/2^n), \ldots$ = $U(0/2^{n-1})$, $V_n \cdot U(0/2^{n-1})$, $U(1/2^{n-1})$, $V_n \cdot U(1/2^{n-1})$, $U(2/2^{n-1}), \ldots$. We see immediately that $V_n \cdot U(2k/2^n) = U((2k+1)/2^n)$. At the same time, $V_n \cdot U((2k+1)/2^n) = V_n \cdot V_n \cdot U(k/2^{n-1}) \subseteq V_{n-1} \cdot U(k/2^{n-1})$. And (by induction) the latter term is a subset of $U((k+1)/2^{n-1}) = U((2k+2)/2^n)$. Therefore, for every n and k , $V_n \subseteq U(k/2^n) \subseteq U((k+1)/2^n)$ (this obviously holds for $U(0)=\emptyset$ and outside $[0,1]$ as well).Now, given a closed set $A \subseteq G-\{0\}$ take a neighborhood $V_0 = G-A$ of e , define $U(p)$ as above, and $f(g)=\inf\{p=k/2^n | g \in U(p)\}$. The only difference from the proof of the Urysohn lemma is in that we define $U(p)$ on a dense subset of set $\mathbb{Q}$ instead of the whole set $\mathbb{Q}$, and for $p<q$ in the dense we proved there is some n such that $V_n \cdot U(p) \subseteq U(q)$ instead of $\overline{U}(p) \subseteq U(q)$. The first difference does not matter. Now, let $g \notin U(q)$. For $v \in V_n$, since V_n is symmetric, $v \cdot g \notin U(p)$. Therefore, $g \notin \overline{U}(p)$ and $\overline{U}(p) \subseteq U(q)$.

11.

(a) The sets cover X ((iii) and (iv) is already enough). Now, the non-empty intersections: (i) and (i) is of type (i), (i) and (ii) is (i), (i) and (iii) is (i), (i) and (iv) is (i), (ii) and (ii) is (ii), (ii) and (iii) is (ii), (ii) and (iv) is (ii), (iii) and (iii) is (iii), (iii) and (iv) contains (ii), (iv) and (iv) is (iv) (note, once again, the intersection is assumed to be non-empty).(b) $f^{-1}(c) = \cap_n f^{-1}((c-1/n, c+1/n))$, $f^{-1}(f(p_{n,k}))$ and its intersection with $C_{n,k}$ is a G_δ set containing $p_{n,k}$. Therefore, there are only countably many points (given by the intersection of (ii)-type neighborhoods) in $S_{n,k}$, the set of $p \in C_{n,k}$ such that $f(p) \neq f(p_{n,k})$. If $y = d$ does not intersect any of $S_{n,k}$ (countably many points) then for every $x \times d \in C_{n,k}$, $f(x \times d) = f(p_{n,k})$. Since f is continuous and every neighborhood of a point in L_m contains the intersection of some horizontal interval containing the point with X , $f((n-1) \times d) = f((n+1) \times d)$. Now, every open set containing a , contains the whole bunch of sets L_m , and the same is true for open sets containing b . Therefore, both points a limit points of $\{m \times d\}$ and $f(a) = f(b)$.(c) It is not completely regular because a and b cannot be separated (and we will show that one-point sets are closed). Now, it is a T_1 space: consider points a , b , two points in L_m and a point in L_n and two points in $C_{n,k}$ and a point in $C_{n',k'}$. For each point in the list we can easily find a neighborhood that does not contain another point in the list. Now, we show it is regular: every neighborhood of a point contains the closure of another neighborhood.Let L_m^- and L_m^+ denote all points to the left (including a) and to the right (including b) of L_m . Then $L_m \cup L_m^-$, $L_m \cup L_m^+$ and L_m are closed. Therefore, any basis neighborhood of a (or b) contains the closure of another neighborhood.$C_{n,k}$ is closed (for any point not in it find a neighborhood that does not intersect $C_{n,k}$). If a point is in $C_{n,k} - p_{n,k}$ then the one-point set is open (the intersection of a small enough horizontal interval and $C_{n,k}$) and closed. Every basis neighborhood of $p_{n,k}$ contains all but finite number of points in $C_{n,k}$, but it is closed as well.If a point is in L_m then its basis neighborhood is either $L_{m'}^-$ for some $m' > m$ or $L_{m''}^+$ for some $m'' < m$ or the intersection of a horizontal open interval with X . The first two types of neighborhoods contain a neighborhood of the third type. But the third type neighborhood (small enough so that they do not contain other limit points) is closed as well (the rest of the space is the union of horizontal intersections, some neighborhoods of a and b , and sets $C_{n,k}$ minus finite number of points).

SECTION 34

1.

$\mathbb{R}_K$ was given before as an example of a space which is Hausdorff but not regular (K is closed but cannot be separated from 0). It is also second-countable. Since it is not regular, it is not metrizable.

2.

For a metric space being separable and second-countable is equivalent, at the same time if a space is second-countable and regular then it is metrizable, so, an example of a space which is completely normal, first-countable, separable and Lindelöf but not second-countable is necessary and sufficient to answer the question. $\mathbb{R}_l$ was shown to be completely normal (in fact, it is perfectly normal) and to satisfy all the countability axioms but one: it is not second-countable.

3.

A compact metric space is second-countable (exercise 4 of §30). A second-countable compact Hausdorff space is second-countable and normal (Theorem 32.3), therefore, metrizable (Theorem 34.1).

4.

If a space is locally compact Hausdorff then it is completely regular and being second-countable implies being metrizable. At the same time a locally compact metric space does not have to be second-countable (at least I do not remember we proved anything like that). For a metric space being separable, Lindelöf or second-countable is equivalent. So, we need an example of a locally compact metric space which is neither one (it can still be first-countable). I know couple spaces that are first-countable but not separable, Lindelöf or second-countable. Both are Hausdorff and locally compact. The first is S_Ω , not metrizable (§28, page 181: it is limit point compact but not compact). The second is a discrete uncountable space, metrizable.

5.

If X is locally compact and Hausdorff then Y is compact and Hausdorff. Y is metrizable iff it is second-countable. So, if Y is metrizable then X is second-countable (note that Y being metrizable is a stronger condition than X being metrizable: the discrete uncountable topology is an example, this is why it did not work in the previous exercise). If X is second-countable then it is metrizable (see the previous exercise), but we need to check whether Y is metrizable. For this we need only to check whether it is second-countable. A countable basis U_n for the topology of X will do for $X \subset Y$ as well. We only need to find a countable basis at ∞ . Take an open neighborhood $Y - C$ where C is compact in X . C is compact in X iff it is closed in X . Therefore, $X - C$ isopen in X and must contain some basis neighborhoods U_n . But we do not know whether $X - U_n$ is compact for any of these basis sets, therefore, we cannot guarantee that $U_n \cup \{\infty\}$ is open in Y for some n . At the same time, we may instead build a countable family of compact sets such that every compact set is contained in a set from the family. We use the fact that C is compact and the space is Hausdorff and locally compact. Consider the countable family $\mathcal{B}$ of all basis open sets such that their closures are compact. Since the space is Hausdorff and locally compact, every point has a neighborhood in $\mathcal{B}$ (Theorem 29.2). Therefore, $\mathcal{B}$ covers C and some its finite subset covers C as well. The corresponding finite union of closures is compact (and closed) and contains C . Therefore, we may take as basis neighborhoods of ∞ the complements of all finite union of compact closures of basis sets of X . Something like that.Summary of 4 and 5: a one-point compactification of a locally compact Hausdorff space X is metrizable iff X is second-countable.

6.

F defined as in the proof of the Theorem 34.1 maps X to $\mathbb{R}^J$. It is continuous as the range is in the product topology. It is injective because X is a T_1 -space (for a pair of points there is a function in the family such that it maps the points to different values). We need to show that F maps open sets in X to open sets in the image. For $x \in U$ find an index α such that $f_\alpha(x) \neq 0 = f_\alpha(X - U)$. Then the set of points $y \in \mathbb{R}^J$ such that $y_\alpha \neq 0$ is an open neighborhood of $F(x)$ in $\mathbb{R}^J$ and its intersection W with the image of F is open in the image. Moreover, if $x' \notin U$ then $f_\alpha(x') = 0$ and $F(x') \notin W$, hence, $W \subseteq F(U)$.

7.

Let $x \in X$. Let U be an open set containing x such that it is metrizable in the subspace topology. Then U is locally compact (Corollary 29.3) and Hausdorff. Let V be a neighborhood of x in U such that $\overline{V}_U$ is compact. Then $\overline{V}_U = \overline{V} \cap U$ is a compact Hausdorff metrizable subspace containing x . Using Exercise 3, we conclude that the subspace topology of $\overline{V} \cap U$ is generated by a countable basis. $V \subseteq \overline{V} \cap U$ is open in X and second-countable as well (in the subspace topology). Now cover X with such neighborhoods for all points and find a finite subcovering V_n . Consider the finite union $\mathcal{B}$ of all countable subspace bases. Suppose $y \in U$. Then there is V_n containing y . $V_n \cap U$ is a neighborhood of y in V_n , and it contains a basis sub-neighborhood that is open in X and belongs to $\mathcal{B}$. So, X is second-countable, therefore, according to Exercise 3, it is metrizable.

8.

As in the previous exercise we want to cover X with a countable collection of open subspaces such that each one has a countable basis. For x take open metrizable U containing x , find open V such that $x \in V \subseteq \overline{V} \subseteq U$. $\overline{V}$ is a closed subspace of a Lindelöf space, i.e. Lindelöf. It is also metrizable, therefore, being Lindelöf is equivalent to being second-countable. Hence, V is an open second-countable (in the subspace topology) neighborhood of V . Now cover X with such neighborhoods, find a countable subcovering, and prove that the countable union of countable bases of all open sets in the subcovering is a countable basis of X . Therefore, X is regular and second-countable. Hence, metrizable. We used the regularity twice: to find a neighborhood with the closure within a given neighborhood, and to argue that the space being regular and second-countable is metrizable. If the space is Hausdorff, Lindelöf and locally metrizable then it is not necessarily metrizable. To find a counterexample we need a space which is Hausdorff but not regular. $\mathbb{R}_K$ is such a space. It is also Lindelöf but not metrizable. The only question is whether it is locally metrizable. The subspace topology on $(-\infty, 0)$ and $(0, +\infty)$ is the same as the standard sub-topology, therefore, metrizable. The only question is now whether 0 has a metrizable neighborhood. But $\mathbb{R} - K$ is its neighborhood with the same topology as the standard one, therefore, it is metrizable.

9.

Each subspace is compact, Hausdorff and metrizable. Therefore, second-countable (Exercise 3). According to the same exercise, we only need to show that X is second-countable. If X_1 and X_2 were disjoint, they would be open as well, and we could take the union of their bases: if $x \in X_i$ then every its neighborhood U in X has a sub-neighborhood $U \cap X_i$, open in X_i and X , and, therefore, it would contain a basis neighborhood open in X_i and X . If they are not disjoint, then we do not even know that a basis set in X_i is open in X . Let $\{U_n^i\}$ be a basis in X_i . Suppose U is a neighborhood of $x \in X$. If $x \in X_i - X_j$ then $U \cap X_i$ is open in X_i and there is a basis neighborhood U_n^i containing x . Moreover, $x \in U_n^i \cap X - X_j$ is open in $X - X_j$ which is open in X , therefore, $U_n^i \cap X - X_j$ is open in X as well. This suggests that, first, we take all intersections $\{U_n^i \cap X_i - X_j\}$ (they are all open in X). Note that if X_1 and X_2 were disjoint this would be exactly the union of their bases. In either case, this already provides bases for all points not in $X_1 \cap X_2$. Now, suppose $x \in X_1 \cap X_2$. $U \cap X_i$ has a basis sub-neighborhood U_{ni}^i of x in X_i . $U_{ni}^i = X_i \cap V_{ni}^i$ where V_{ni}^i is open in X . $x \in V_{n1}^1 \cap V_{n2}^2 = (V_{n1}^1 \cap V_{n2}^2 \cap X_1) \cup (V_{n1}^1 \cap V_{n2}^2 \cap X_2) = (U_{n1}^1 \cap V_{n2}^2) \cup (V_{n1}^1 \cap U_{n2}^2) \subseteq U$. So, for every pair of intersecting basis sets in X_1 and X_2 choose some open sets in X that intersect with the subspaces at these basis sets and take their intersection. We have a countable basis for points in $X_1 \cap X_2$.

SECTION 35

1.

Take the continuous function on the union of two disjoint closed sets equal to 1 for one set and 0 for the other set (it is continuous because both sets are closed and, therefore, open in the union) and extend it continuously on X.

2.

When we approximate a function $f_0 : A \to [-r, r]$, we construct a function $g_0 : X \to [-ar, ar]$ such that on A it differs from f no more than by

$$\max\{2ar, (1-a)r\} = r\max\{2a, 1-a\} \\ := rb.$$

By taking the difference $f_0 - g_0$ on A we obtain a new function $f_1 : A \to [-rb, rb]$. And we approximate it on A by $g_1 : X \to [-rab, rab]$ such that the difference on A between the two functions is not greater than rb^2 .

Continuing this way we need to ensure that

1. $g = \sum_{k\geq 0} g_k$ is well-defined. This holds as long as

   $$\sum_{k\geq 0} rab^k < \infty,$$

   or $b < 1$. This means

   $$0 < a < \tfrac{1}{2}.$$

2. $g = f$ on A . This holds as long as $rb^k \to +0$ as $k \to +\infty$, or $b < 1$. And we have the same condition.

> Note that the choice $a = \frac{1}{3}$ is optimal in the sense that b as a function of a reaches its minimum at $a = \frac{1}{3}$, which ensures the fastest convergence of the approximation by the partial sums (not that it is important for the result itself).

3.

For a metric space (iii) is equivalent to being compact and implies (i) (a compact metric space is bounded) and (ii) (the continuous image of a compact set is compact). (ii) obviously implies (i) (fix a point, the distance from the point is continuous). Now, (ii) also implies (iii): if an infinite subspace has no limit points then it is closed and discrete; a surjective function from it on the set of positive integers is continuous and can be extended to a continuous unbounded function (the space is metrizable, therefore, normal). We need only to show that (i) implies (ii) or (iii). We show (i) implies (ii). The hint was ambiguous for me so I figured out an alternative proof.Remember from Section 20 that the topology generated by a metric is the coarsest topology such that the metric as a function is continuous. This implies also that if a metric d' is continuous relative to (X, d) then (X, d') is coarser than (X, d) . We prove the following lemma.*Lemma.* Suppose that $(X, \mathcal{T})$ is metrizable and every metric that generates the topology is bounded. Then every metric d' continuous relative to $\mathcal{T}$ is bounded as well.*Proof.* Suppose $(X, d) = (X, \mathcal{T})$. d is bounded. Let $m = d + d'$. m is a metric (easily verifiable) and continuous relative to (X, d) , therefore, (X, m) is coarser than (X, d) . We show that, in fact, $(X, m) = (X, d)$. For every $y \in B_d(x, r)$, suppose $z \in B_m(y, r - d(y, x))$, then $d(z, x) \leq d(x, y) + d(y, z) \leq d(x, y) + m(y, z) < r$, therefore, for every point in $B_d(x, r)$ there is a neighborhood in (X, m) contained in the ball, and (X, m) is finer than (X, d) . This implies that m is bounded, and so is d' .Now, take any continuous ϕ . And suppose d is a metric on X . Then $d(x, y) + |\phi(x) - \phi(y)|$ is a metric that is continuous relative to the topology on X . Therefore, it is bounded, and so is ϕ .

4.

This is something from algebraic topology, as I understand (and the index suggests that retractions are covered mainly in the second part of the book). But, in fact, there was a fact about retractions (and that was the reason to look up the index) in the section on the quotient maps. A retraction (a continuous map that preserves all points in the image) is a quotient map. (a) Let $x \in Z - Y$, let $y = r(x)$. There are disjoint neighborhoods U and V of x and y. $V \cap Y$ is open in Y and contains y. $W = r^{-1}(V \cap Y)$ is open in Z and contains x and y. If $z \in Y \cap W$ then $z = r(z) \in V$ and $z \notin U$. Therefore, for every $z \in U \cap W$, $z \notin Y$.(b) The preimage of each point must be open and non-empty, and the union must be the whole space, which is connected.(c) For x take $x/|x|$. It is well-defined for all non-zero points, and continuous as a composition of continuous functions. Moreover, it preserves all points on the unit circle. As for the second question, it seems a little more trickier: whether S^1 is a retract of $\mathbb{R}^2$. Suppose a continuous function f maps $\mathbb{R}^2$ continuously onto the unit disc (not necessarily unit circle) and preserves all points on the circle. The restriction of the map onto the unit disc is a continuous map from the disc to the disc that preserves all the points on the circle. It seems that such a function must map some point to the origin (as well as to any other point in the disc). In other words, it must map the disc surjectively onto the disc. How to show this based on what we know? Another way is to show that a continuous map from the unit disc to itself must have a fixed point. Then we take the composition of r with the rotation and the new continuous map does not have fixed points on the boundary, therefore, there is no continuous retraction onto the boundary. Yet, another way is just to try to prove that every circle centered at 0 maps to the entire boundary (under the retraction). This way when the circle vanishes to the origin, the origin must maps to all points on the boundary. This seems more promising. Let r be the infimum of the set of radii such that every circle centered at 0 with the given radius or greater maps to the entire boundary. If $r > 0$ then we show that r is in the set and find ϵ such that $r - \epsilon$ is in the set. To find ϵ we, probably, need something like the uniform convergence.

5.

(a) $f : A \to \mathbb{R}$ can be extended to a continuous $g : X \to \mathbb{R}$, then $h : X \to \mathbb{R}^J$ such that $h_j(x) = g(x)$ is continuous (in the product topology).(b) If $f : A \to Y$ is continuous, and $h : Y \to B$ is a homeomorphism such that B is a retract of $\mathbb{R}^J$ then there is a retraction $r : \mathbb{R}^J \to B$ and $h \circ f$ can be extended to a continuous function $g : X \to \mathbb{R}^J$. Then $h^{-1} \circ r \circ g : X \to Y$ is continuous.

6.

(a) $f : A \to \mathbb{R}$ can be extended to a continuous $g : X \to \mathbb{R}$, then $h : X \to \mathbb{R}^J$ such that $h_j(x) = g(x)$ is continuous (in the product topology).(b) If $f : A \to Y$ is continuous, and $h : Y \to B$ is a homeomorphism such that B is a retract of $\mathbb{R}^J$ then there is a retraction $r : \mathbb{R}^J \to B$ and $h \circ f$ can be extended to a continuous function $g : X \to \mathbb{R}^J$. Then $h^{-1} \circ r \circ g : X \to Y$ is continuous.

7.

(a) The quotient space of $A=\mathbb{R}^2-\{(0,0)\}$ obtained by identifying points with the same distance to the origin is homeomorphic to $\mathbb{R}$ and to $C'=C-\{(0,0)\}$. This gives us a continuous function f' from A to C' . The explicit expression for the function will be something like this: $f'(x,y)=(e^t\cos t, e^t\sin t)$ where $t=\sqrt{x^2+y^2}$. Every sequence of points converging to the origin maps to a sequence converging to the origin as well, therefore, we can extend f' to a continuous function $f:\mathbb{R}^2\to C$.(b) K is homeomorphic to $\mathbb{R}$, which is normal. Therefore, using the two previous exercises, it has the universal extension property and is an absolute retract. Being closed in $\mathbb{R}^3$, it is a retract of $\mathbb{R}^3$.

8.

One direction: 6(a). The other direction: if a normal space is an absolute retract then it has the universal extension property. The idea, as I understand, is as follows: we consider a continuous function f from a closed subspace A of X to Y ; we know that whenever Y is homeomorphic to a closed subspace of a normal space, the subspace is a retract, i.e. there exists a continuous map from the space onto the subspace that preserves all points of the subspace; but we need a continuous function from X to Y that extends f ; we consider the space which is the union of the spaces X and Y and group some points into equivalence classes such that every point in Y belongs to a separate class that also contains the preimage of the point; the union of all classes for all points in Y is a closed subspace of the quotient space being homeomorphic to Y ; we show the quotient space is normal and consider a retraction from X onto the set of classes of points in Y ; now we can construct a continuous extension of f , as we shall see.Construct the adjunction space Z_f as hinted: the quotient space obtained from $Z=X\cup Y$ by identifying each point $y\in Y$ with points in $f^{-1}(y)$ (for this we, first, need to specify the topology on Z : a set U is open in Z iff $U\cap X$ is open in X and $U\cap Y$ is open in Y). Let p be the quotient map. Let $B=X-A$ and denote elements of $Z_f=[B]\cup[Y]$ as follows: $[x]\in[B]$ where $x\in B$ and $[y]\in[Y]$ where $y\in Y$ (if $y\notin f(A)$ then $[y]$ contains only one element).First, we show that Z_f is normal. Let $[S_0]$ and $[S_1]$ be closed in Z_f . Let $[T_i]=[S_i]\cap[Y]$ and $[R_i]=[S_i]\cap[B]$. $[T_i]$ are closed in $[Y]$ ($[Y]$ itself is closed as it contains all elements from Y and A , both closed in Z), therefore, T_i are closed in Y and we can separate them by a continuous function $g(T_i)=i$ (Urysohn lemma). Now, $g'=g\circ f$ is continuous on A and $g'(f^{-1}(T_i))=i$. We can extend it onto R_i : $g'(R_i)=i$. It is still continuous: it is continuous on $R_i\cup f^{-1}(T_i)$ and A (the pasting lemma?). Now we can extend it onto X (Tietze theorem). Note that the extended g' is constant on $p^{-1}([z])$ for all $z\in Z$. Therefore, it induces a continuous function g " on Z_f (Section 22) such that it separates $[S_0]$ and $[S_1]$. Hence, Z_f is normal.Now, we show that Y is homeomorphic to a closed subset of Z_f , namely, to the set $[Y]$. Indeed, $h=p|Y:Y\to[Y]$ is continuous and bijective. It maps closed $T\subseteq Y$ to $[T]=p(T\cup f^{-1}(T))$ which is the image of a closed saturated subset of Z under the quotient map ($f^{-1}(T)$ is closed in A which is closed in X).Finally, since $[Y]$ is a closed subset of the normal space Z_f homeomorphic to Y , and Y is an absolute retract, there is a retraction $r:Z_f\to[Y]$ such that for every $y\in Y:r([y])=[y]$. Let $f'=h^{-1}\circ r\circ p$. f' is continuous, and maps $X\cup Y$ to Y . For every $a\in A$: $f'(a)=h^{-1}(r([f(a)]))=h^{-1}([f(a)])=f(a)$. Therefore, $f'|X$ extends f .

9.

(a) Let $\{U_j\}$ be a collection of open subsets of X. Then for each i, $\cap_j U_j \cap X_i = \cap_j (U_j \cap X_i)$ is open in X_i if there are finitely many open sets, and $\cup_j U_j \cap X_i = \cup_j (U_j \cap X_i)$ is open in X_i as well. Now we need to show that X_i is a (closed) subspace of X. From the definition of the coherent topology on X it follows that the topology on X_i is finer than the subspace topology on X_i. We need to show that if U_0 is open in X_i then there is some set $V \subseteq X$ such that $V \cap X_i = U_0$ and $V \cap X_n$ is open in X_n for every n. X_n is a subspace of X_{n+1} for every n. For $k \geq 1$ let U_k be a set open in X_{i+k} such that $U_k \cap X_{i+k-1} = U_{k-1}$. Note that for $k < k'$, $U_{k'} \cap X_{i+k} = U_{k'} \cap X_{i+k'-1} \cap \ldots \cap X_{i+k} = U_k$ and for $k \geq k'$, $U_{k'} \cap X_{i+k} = U_{k'}$. Let $U = \cup_k U_k$. Then, for $n \geq i$, $U \cap X_n = (\cup_k U_k) \cap X_n = \cup_k (U_k \cap X_n) = \cup_{k<n-i} U_k \cup_{k \geq n-i} U_{n-i} = U_{n-i}$ is open in X_n. For $n < i$, $U \cap X_n = U \cap X_i \cap X_n = U_0 \cap X_n$ is open in X_n. Therefore, the subspace topology is finer than the topology on X_i, and, overall, we have that the topologies are equal. To show that X_i is a closed subspace of X we can take any point $x \in X - X_i$ and construct its neighborhood in X disjoint from X_i by taking the union of $X_n - X_i$ for all $n > i$.(b) In the past we had the following fact: if a function is continuous on every space in a collection of subspaces covering the space and either all subspaces in the collection are open or all subspaces are closed and the collection is finite, then the function is continuous on the whole space. Here we have either the infinite union of closed subspaces or the infinite union of open subspaces $X_{n+1} - X_n$ union one closed subspace X_1. If we knew that $X_1 \subseteq U$ where U is open in X and there is some n such that $U \subseteq X_n$ then we could apply the result stated above. However, it might be that every neighborhood of some point in X_1 intersects all spaces X_n. For example, suppose $X_n = \{1, \ldots, n\}$ and U is open in X_n iff for every $x \in U$ all $y > x$ also in U. Note that every X_n is closed in X_{n+1}. Then X is $\mathbb{Z}_+$ but there is no open set in X such that it is contained in any X_n. So we show the result directly.Let V be open in Y. Then for every $i \geq 1$, $f^{-1}(V) \cap X_i = (f|X_i)^{-1}(V)$ which is open in X_i. Therefore, $f^{-1}(V)$ is open in X.(c) Let A and B be closed disjoint subsets of X. Let $X_0 = \emptyset$ (it is a closed subspace of X_1) and f_0 be defined on $A \cup B$ such that $f_0(A) = 0$ and $f_0(B) = 1$. We assume that $f_0 : A \cup B \to [0, 1]$. Note that both A and B are closed and open in $A \cup B$ and f_0 is continuous. For every $i \geq 1$, given a continuous function $f_{i-1} : X_{i-1} \cup A \cup B \to [0, 1]$ such that $f_{i-1}(A) = 0$ and $f_{i-1}(B) = 1$, we extend it to a continuous function $f_i : X_i \cup A \cup B \to [0, 1]$ which satisfies the same property: $f_i(A) = 0$ and $f_i(B) = 1$. We cannot directly apply the Tietze Extension Theorem because $X_{i-1} \cup A \cup B$ may be not normal. However, X_{i-1} is normal. So we can extend $f_{i-1}|X_{i-1}$ to a continuous function $g : X_i \to [0, 1]$ as follows: first, we say that $g(A \cap X_i) = 0$ and $g(B \cap X_i) = 1$, the function is still continuous (A and B are closed in X, hence, their intersection with X_i is closed in X_i and the continuity follows by the pasting lemma of Section 18), then, we extend it from the closed subset $(A \cup B \cup X_{i-1}) \cap X_i$ onto X_i. Now, we define f_i to be equal to g on X_i, 0 on A and 1 on B (it is still continuous by the pasting lemma). For every $x \in X - A - B$ there is the minimal n_x such that $x \in X_{n_x}$. We define $f(x) = f_{n_x}(x)$. Note that $f_i(x)$ is not defined for $i < n_x$ and agrees with $f_{n_x}(x)$ for $i > n_x$. For $x \in A$ we define $f(x) = 0$ and for $x \in B$ we define $f(x) = 1$. Note, again, that this definition agrees with all f_i. This implies that $f|X_i = f_i$ which is continuous, and by (b), f is continuous. Since $f(A) = 0$ and $f(B) = 1$, f separates A and B. Hence, X is normal.

SECTION 36

1.

Let $x \in U$, open subset of the manifold X. There is a neighborhood V of x homeomorphic to an open euclidean space. Then $W = U \cap V$ is open in V and is homeomorphic to an open subset of the euclidean space. Let h be the homeomorphism. There is a neighborhood $h(A)$ of $h(x)$ such that $\overline{h(A)} \subseteq h(W)$. Then A is a neighborhood of x such that $\overline{A} \subseteq W \subseteq U$. Now, important! This does not show yet that the space is regular. Consider the following example. Let us take the standard topology on $\mathbb{R}$ and make a copy of the origin, a new point a. The basis for the topology is the collection of open intervals $(a, b) \subseteq \mathbb{R}$ and open intervals containing 0 with a substituted for 0 : $(c, d) - \{0\} \cup \{a\}$ where $c < 0 < d$. Obviously, this space is a manifold (if we do not require the Hausdorff condition in the definition), however, it is not Hausdorff: we can not separate 0 and a (even though for every point x and its neighborhood U there is a neighborhood V of x such that $\overline{V} \subseteq U$, i.e. it is "almost regular"). So, we do need require a manifold be Hausdorff to complete our proof that it is regular.

2.

The space is compact and Hausdorff, therefore, normal. Cover it with a finite number of open sets $\{U_n\}$ such that each can be imbedded into $\mathbb{R}^{k_n}$, let $\{f_n\}$ be the corresponding collection of imbeddings. By Theorem 36.1, there is a partition of unity $\{g_n\}$ dominated by $\{U_n\}$. Similar to the proof of Theorem 36.2, we construct $F = (g_1, \dots, g_n, f_1 \cdot g_1, \dots, f_n \cdot g_n) : X \to \mathbb{R}^{n+k_1+\dots+k_n}$ so that it is continuous and injective.

3.

Using the exercise 2, X can be imbedded into a second-countable space. Hence, it is second-countable and, therefore, an m -manifold.

4.

In the proof of the Theorem 36.1 the first step was to show the result of this question for finite families of open set covering X. The essential part of the proof is to construct each V_i such that $X - \cup_{j\neq i} V_j \subseteq V_i \subseteq \overline{V}_i \subseteq U_i$. We need the first inclusion to ensure that it covers X, and the last one for later construction of a partition of unity. In the finite case we construct V_i ' s one-by-one such that V_i contains $X - \cup_{j<i} V_j \cup_{j>i} U_j$. Note that at the last step V_n covers all of $X - \cup_{j<n} V_j$, therefore, all points of X are covered. If the covering is countable, and we use the same way to construct sets V_i ' s, then, in general, we cannot guarantee that each point will be contained in some set V_i. However, if every point x is contained in a finite number of sets U_i, then for some n, $U_{n+1} \cup U_{n+2} \cup ...$ does not contain x and, since V_n contains $X - V_1 \cup V_2 \cup ... \cup V_{n-1} \cup U_{n+1} \cup U_{n+2} \cup ...$, x must be contained in $V_1 \cup ... \cup V_n$. Therefore, the collection of V_i ' s constructed the same way as in the theorem, does cover all of X.[To construct a counterexample we would need a space such that it is a) normal, b) there is a countable cover by open subsets U_n (not point-finite) such that there is no other cover V_n such that $\overline{V}_n \subseteq U_n$. I have been thinking about the example for a while, but have not figured out one yet. The fact that the space is normal implies, in particular, that the construction above is possible. On the other hand, it seems that the construction is, in some sence, necessary... What I mean is that for any set V_n, a) V_i must cover $X - \cup_{j\neq i} V_j$, therefore, it must cover $X - \cup_{j<i} V_j \cup_{j>i} U_j$, and b) it must be such that $\overline{V}_j \subseteq U_j$. So, if there are V_i ' s for U_i ' s, then they can be constructed using the procedure above. Our only hope is to find a space such that whatever sequence V_n we take using the procedure above, there will be a point such that it is not covered. In particular, when we define V_1 such that $X - \cup_{j>1} U_j \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1$ there must be some points in U_1 not covered by V_1 such that each is contained in an infinite number of sets U_j.]

5.

(a), (b) It is better to think about the space as described in the solution for the exercise 1: we just add a point and new basis neighborhoods of the point. The intersection of a new basis neighborhood with an old one is either an interval or the union of two intervals or empty. Also, this way it is immediate that the space without the new point is homeomorphic to the real line, and if we substitute the new point for 0, then it is also homeomorphic to the set of real numbers.(c) It satisfies the T_1 axiom: in fact, any pair of points except (p, q) can be separated by two neighborhoods, while these two points cannot be separated by two disjoint neighborhoods, though each one has a neighborhood not containing the other one.(d) $X - \{p\}$ and $X - \{q\}$ are both open, metrizable, and each point belongs to at least one of these open sets. Also, since $\mathbb{R}$ is second-countable, so is X, as $(-q, q)$ where $q \in \mathbb{Q}$ is a basis at the new point.