15.094J/1.142J Robust Modeling, Optimization and Computation

Problem Set 1

Due February 22

Exercise 1.1

Consider the minimum-cost flow problem, where we have a graph $G = (V, E)$, $V = \{1, \ldots, n\}$, and must send a unit of flow from node 1 to node n. Each edge $(i, j) \in E$ has capacity C_{ij} and a per-unit flow cost of c_{ij}. We can formulate this as the following linear optimization problem:

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(1,i)\in E} x_{1i} = 1 \\
& \sum_{(i,n)\in E} x_{in} = 1 \\
& \sum_{(i,j)\in E} x_{ij} = \sum_{(j,k)\in E} x_{jk} \quad \forall j \in V \setminus \{1, n\} \\
& 0 \leq x_{ij} \leq C_{ij},
\end{aligned}
$$

where the decision variable x_{ij} gives the flow sent from node i to node j.

(a) Consider the case that the flow costs c_{ij} are not known exactly, but instead are uncertain. Formulate the robust optimization problem for each of the following uncertainty sets and find the corresponding robust counterpart:

(i) **Box uncertainty:** $\mathcal{U} = \{\mathbf{c} \mid \mu_{ij} - \delta_{ij}\gamma_{ij} \leq c_{ij} \leq \mu_{ij} + \delta_{ij}\gamma_{ij}, \|\boldsymbol{\gamma}\|_\infty \leq \Gamma\}$

(ii) **Polyhedral uncertainty:** $\mathcal{U} = \{\mathbf{c} \mid \mu_{ij} - \delta_{ij}\gamma_{ij} \leq c_{ij} \leq \mu_{ij} + \delta_{ij}\gamma_{ij}, \|\boldsymbol{\gamma}\|_1 \leq \Gamma\}$

(iii) **Ellipsoidal uncertainty:** $\mathcal{U} = \{\mathbf{c} \mid \mu_{ij} - \delta_{ij}\gamma_{ij} \leq c_{ij} \leq \mu_{ij} + \delta_{ij}\gamma_{ij}, \|\boldsymbol{\gamma}\|_2 \leq \Gamma\}$

(b) Suppose G is a fully connected graph, with all edge capacities set to $\frac{1}{2}$. Suppose the nominal costs μ_{ij} are drawn from $Uniform(0, 10)$, and the deviations δ_{ij} are drawn from $Uniform(0, \mu_{ij})$. Generate a single instance of the problem ($\boldsymbol{\mu}$ and $\boldsymbol{\delta}$). Implement and solve the problem using JuMPeR for each of the three uncertainty sets (along with the nominal model where $\mathbf{c} = \boldsymbol{\mu}$). Evaluate the solutions using simulation (you could simulate costs with $c_i \sim Normal(\mu_i, \delta_i^2)$ using the $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ you generated earlier), and discuss the differences between the results of each approach. You should explore how the results are affected by the choice of n and Γ.

Exercise 1.2

The typical mantra of robust optimization is that "the robust analogue of (Optimization Problem of Type X) is an (Optimization Problem of Type X) of comparable size." We will explore the validity of such a statement for robust LPs. Modern LP solvers can solve massive scale LPs (with millions of variables and constraints). It is possible to solve even larger problems when the systems of inequalities are well-structured or sparse. Sparse LPs are ones for which many of the coefficients in the constraints are zeros. The focus of the question is, *what happens to sparsity properties of an LP under robustification?*

(a) Consider the constraint $\mathbf{a}^T\mathbf{x} \leq b,\ \forall \mathbf{a} \in \mathcal{U}$, where $\mathcal{U} = \{\mathbf{a} : \|\mathbf{a}-\hat{\mathbf{a}}\|_\infty \leq \epsilon\}$ (here $\hat{\mathbf{a}} \in \mathbb{R}^n$ and $\epsilon > 0$ are fixed; $\|\cdot\|_\infty$ denotes the ℓ_∞ norm, with $\|\mathbf{a}\|_\infty = \max_i |a_i|$). Rewrite the semi-infinite constraint as a finite number of linear inequality constraints. *Hint: you will likely need to use auxiliary variables.*

(b) Comment on the sparsity of the new representation in part (a). If $\hat{\mathbf{a}}$ has mostly zero entries (in which case we would call the constraint $\hat{\mathbf{a}}^T\mathbf{x} \leq b$ "sparse"), are the constraints in the new representation of the uncertain constraint still sparse? Please be as specific as possible.

(c) Repeat parts (a) and (b) with a different uncertainty set:

$$\mathcal{U} = \{\mathbf{a} : \|\mathbf{a} - \hat{\mathbf{a}}\|_1 \leq \epsilon\}.$$

(Here $\|\mathbf{a}\|_1 = \sum_i |a_i|$.)

(d) What differences, if any, did you observe between parts (b) and (c)? Offer an explanation for why there are (or are not) differences. Does this say anything about how to choose uncertainty sets?