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A THEOREM ON UNITARY REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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We show that a connected semisimple Lie group G none of whose simple constituents is compact (in particular, any connected complex semisimple group) has no nontrivial measurable unitary representations into a finite factor,—i. e. a factor of type $I_n(n \neq \infty)$ or II_1, in the terminology of [3]. This has been known for the case of representations of complex groups into factors of type I_n, but the existing proofs are not applicable either to real groups or to factors of type II_1, and the present proof is therefore necessarily of a different character from the proof for the complex, finite-dimensional case. Our theorem has the relevant consequence that in the reduction of the regular representation of G into factors (see [9] and [5]), those of type I_n or II_1 cannot occur. This is in marked contrast with the situations for compact and discrete groups, only I_n's occurring in the compact case (as is well-known) and only I_n's and II_1's in the discrete case (loc. cit.).

In order to clarify the statement of our theorem, we make the following definitions. A representation U of a locally compact group G by unitary operators on a Hilbert space $\mathcal{H}$ (of arbitrary dimension) is called *measurable* if the inner product $(U(a)x, y)$ is a measurable function of $a \epsilon G$, relative to Haar measure, for all x and y in $\mathcal{H}$. (When $\mathcal{H}$ is separable, such a representation is necessarily continuous in the strong operator topology, as follows from a modification of the proof by the second-named author of a special case of this result; details, as well as a more precise result, are given below.) U is said to be into a factor $\mathcal{F}$ if $\mathcal{F}$ is a factor of which $U(a)$ is an element, for all $a \epsilon G$. Now we state our central result.

THEOREM 1. *Let G be a connected semisimple Lie group none of whose simple constituents is compact. Then the only measurable unitary representation of G into a finite factor is the identity representation.*

The following proof applies also to the case of any weakly continuous representation of G into an algebra of operators on a Hilbert space, on which a weakly continuous trace is defined. To outline briefly the general plan of the proof, the non-compactness of the simple constituents of G is used to show that G must contain one of a certain class of 2- and 3-dimensional solvable Lie groups. A special study of the unitary representations into finite factors of the members of this class of groups shows that such representations must be trivial on certain subgroups. The proof is concluded by showing that G is generated by such subgroups.

An analog of the following lemma is valid for arbitrary topological groups in the large, and can be proved in the same way. A trivial modification of the proof shows also that the same conclusion is valid if G is a cross product of groups A

and B such that b^{a_n} converges to the identity for all $b \in B$ and some fixed sequence $\{a_n\}$ in A. In the statement of the lemma, as well as in the rest of this paper, we use the terminology and notation, as regards Lie groups, of [1a].

LEMMA 1. *Let G be a Lie group, B an invariant analytic subgroup, and let* $\mathbf{G}$ *and* $\mathbf{B}$ *be the Lie algebras of G and B respectively. For any $T \in \mathbf{G}$, let $\tilde{T}$ denote the linear transformation on $\mathbf{B}$, $X \to [X, T]$. Suppose that there exists a sequence of elements $\{Z_n\}$ of $\mathbf{G}$ such that* $\exp(\tilde{Z}_n)W \to 0$ *for all $W \in \mathbf{B}$ ($\mathbf{B}$ being topologized by taking a linear isomorphism between it and an euclidean space to be a homeomorphism). Then every weakly continuous unitary representation of G into a finite factor coincides on B with the identity representation.*

We recall that for any $T \in \mathbf{G}$, $\exp \tilde{T}$ is the differential of the automorphism of G, $a \to (\exp T)\, a(\exp T)^{-1}$. As the image of $\exp V$ in an analytic group homomorphism is the exponential of the image of V under the differential of the homomorphism, $\exp\{\exp(\tilde{Z}_n)W\} = \exp Z_n \exp W(\exp Z_n)^{-1}$ for $W \in \mathbf{G}$. Hence by the assumption about $\{Z_n\}$, $\lim_n \exp Z_n \exp W(\exp Z_n)^{-1} = e$, for $W \in \mathbf{B}$, e being the identity of G. Now the set of all $\exp W$ with $W \in \mathbf{B}$ contains a neighborhood N of the identity in B, and as B is analytic it is connected, so $B = \bigcup_{k=1}^{\infty} N^k$. It is easy to deduce that $\lim_n (\exp Z_n)\, b(\exp Z_n)^{-1} = e$ for all b in B.

Now let U be a weakly continuous unitary representation of G into a finite factor $\mathcal{F}$, and let t be the trace function on $\mathcal{F}$, normalized by setting $t(I) = 1$, where I is the identity operator. It results from the unitary invariance of the trace that $t(U((\exp Z_n)\, b(\exp Z_n)^{-1})) = t(U(b))$ for $b \in B$. As the trace is a weakly continuous function it follows, letting $n \to \infty$, that $t(U(b)) = 1$ for all $b \in B$.

The proof of the lemma will be complete when it is shown that the only unitary operator in $\mathcal{F}$ of trace unity is the identity operator. Let V be such an operator, and set $V = R + iS$, where R and S are selfadjoint. Then $R = \frac{1}{2}(V + V^*)$, so $||| R ||| \leqq \frac{1}{2}(||| V ||| + ||| V^* |||) = 1$, and $I - R$ is semi-definite. On the other hand, $t(V) = 1 = t(R) + it(S)$, and as $t(R)$ and $t(S)$ are real, this implies $t(R) = 1$, or $t(I - R) = 0$. Since the trace is definite, $I - R = 0$, so $R = I$. Therefore $V = I + iS$ and $VV^* = I = I + S^2$, which implies $S^2 = 0$, and by virtue of the selfadjointness of S it follows that $S = 0$.

Lemma 1 is needed in connection with the groups described in the following lemma.

LEMMA 2. *Let $S_i (i = 1, 2)$ be a connected Lie group with Lie algebra $\mathbf{S}_i$ as follows: $\mathbf{S}_1$ has a basis $\{X, Y\}$ such that $[X, Y] = X$; $\mathbf{S}_2$ has a basis $\{X, Y, Z\}$ such that $[X, Y] = 0$, $[X, Z] = X - \sigma Y$, and $[Y, Z] = \sigma X + Y$, where σ is nonzero (and real). Then any weakly continuous unitary representation of either one of the S_i into a finite factor is the identity on the subgroup generated by X.*

When $i = 1$, the subgroup generated by X is invariant, for the subalgebra of $\mathbf{S}_1$ spanned by X is an ideal. Setting $Z_n = -nY$ $(n = 1, 2, \cdots)$, plainly $\tilde{Z}_n(X) = -nX$ (in terms of the notation introduced in the statement of Lemma 1), and so $\exp(\tilde{Z}_n)X = e^{-n}X$. The conclusion in this case now follows from Lemma 1. Now suppose $i = 2$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{S}_2$ spanned by X and Y.

It is clear that the matrix $\tilde{Z}'$ of $\tilde{Z}$, relative to the basis $\{X, Y\}$ is $\begin{pmatrix} 1 & -\sigma \\ \sigma & 1 \end{pmatrix} = I + \sigma E$, where $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As I and E commute, $\exp(\alpha \tilde{Z}') = \exp(\alpha I) \exp(\alpha\sigma E) = \exp(\alpha) \exp(\alpha\sigma E)$. Since E is skew-symmetric, $\exp(\alpha\sigma E)$ is orthogonal, and hence bounded as a function of α. Now setting $Z_n = -nZ$, it follows that $\exp(\tilde{Z}_n)W \to 0$ for all $W \in \mathbf{B}$. To conclude the proof it is now sufficient to cite the preceding lemma.

Before stating the next lemma we introduce some terminology. A linear transformation on a finite-dimensional linear space is called *separable* if every invariant subspace of the transformation has a complementary subspace which is also invariant. A *Cartan subalgebra* of a semi-simple Lie algebra Γ is one which is maximal abelian, and all of whose elements map into separable transformations under the adjoint representation on Γ (which takes $X \in \Gamma$ into the linear transformation $Y \to [X, Y]$ on Γ.

We are indebted to I. M. Singer for bringing to our attention an observation of K. Iwasawa which is used in the proof of the following lemma. A different proof has been given by A. A. Albert in lectures at the University of Chicago. This lemma, and also the next one, are due to the first-named author.

LEMMA 3. *A real semisimple Lie algebra* Γ *has a Cartan subalgebra, and the complexification of such a subalgebra is a Cartan subalgebra of the complexification* Γ^c *of* Γ.

A theorem of Cartan asserts the existence of an automorphism A of a compact real form Γ' of Γ^c (i.e. a real subalgebra of Γ^c whose complex form is Γ^c, and whose fundamental quadratic form is negative definite) with the following properties: 1) A is of period 2, 2) if $\{X'_j\,;\, j = 1, \cdots, n\}$ is a basis of Γ' such that $A(X'_j) = \varepsilon_j X'_j (j = 1, \cdots, n)$, where $\varepsilon_j = 1$ for $j \leqq m$ and $\varepsilon_j = -1$ for $j > m$, then the elements X_j of Γ^c are a real basis for Γ, where $X_j = X'_j$ for $j \leqq m$ and $X_j = iX'_j$ for $j > m$. Now the lemma is known to be true for the case in which Γ has a negative definite quadratic form; in fact any maximal abelian subalgebra of Γ is then a Cartan subalgebra and its complexification is a Cartan subalgebra of Γ^c. Iwasawa has noted (in a proof ascribed by him to Chevalley; [2], p. 526) that there exists a maximal abelian subalgebra, say Δ', of Γ', which is invariant under A. Hence the above basis $\{X'_j\}$ for Γ' can be assumed to have the property that a subset of it is a basis for Δ'; let this subset be $\{X'_j\,;\, j \in J\}$. Now the $\{X_j; j \in J\}$ evidently span an abelian subalgebra Δ of Γ whose complexification Δ^c is the same as that of Δ', and hence a Cartan subalgebra of Γ^c.

Putting R^c and R for the adjoint representations of Γ^c and Γ, for any $Y \in \Delta^c$, $R^c(Y)$ is separable. If $Y \in \Delta$, the contraction of $R^c(Y)$ to Γ, which is then invariant under $R^c(Y)$, is clearly $R(Y)$, and so $R(Y)$ is separable (the contraction of a separable linear transformation to an invariant subspace being evidently also separable). To see that Δ is maximal abelian in Γ, we observe that if it were not, then, as is easily seen, Δ^c could not be maximal abelian in Γ^c.

LEMMA 4. *Let Γ be a real simple Lie algebra whose fundamental quadratic form is not negative definite. Then either Γ contains a 2-dimensional subalgebra with basis elements X and Y such that $[X, Y] = X$, or Γ contains a 3-dimensional subalgebra with basis elements X, Y, and Z such that $[X, Y] = 0$, $[X, Z] = X - \sigma Y$,* and *$[Y, Z] = \sigma X + Y$, where σ is real and nonzero.*

Let Δ be a Cartan subalgebra of Γ whose complexification Δ^c is a Cartan subalgebra of the complexification Γ^c of Γ; this exists by the preceding lemma. For any element Y in Γ^c with $Y = Y_1 + iY_2$ and $Y_j \in \Gamma(j = 1, 2)$, we denote $Y_1 - iY_2$ by $\bar{Y}$; then the mapping $Y \to \bar{Y}$ is an automorphism of Γ^c as a real Lie algebra, and leaves Γ and Δ^c invariant. Let α be an arbitrary root of Γ^c with respect to Δ^c, so that $[X, D] = \alpha(D)X$ for some nonzero $X \in \Gamma^c$ and all $D \in \Delta^c$. It follows that $[\bar{X}, \bar{D}] = \overline{\alpha(D)}\bar{X}$ or replacing D by $\bar{D}$, $[\bar{X}, D] = \overline{\alpha(\bar{D})}\bar{X}$. This means that if α^* is the function on Δ^c defined by the equation $\alpha^*(D) = \overline{\alpha(\bar{D})}$, then α^* is a root to which $\bar{X}$ belongs. We now break down the situation into various cases which we treat separately; these cases are not mutually exclusive, but include all possibilities.

CASE A: *There exists a nonzero root α which is real-valued on Δ.* If X belongs to α and $X = X_1 + iX_2$ with $X_j \in \Gamma(j = 1, 2)$ then for $D \in \Delta$, $[X_1, D] + i[X_2, D] = \alpha(D)X_1 + i\alpha(D)X_2$, which implies that $[X_1, D] = \alpha(D)X_1$ and $[X_2, D] = \alpha(D)X_2$. Since $\alpha \neq 0$ there exists an element D_0 in such that $\alpha(D_0) \neq 0$. Now evidently either $X_1 \neq 0$ or $X_2 \neq 0$. If $X_1 \neq 0$, we set $X = X_1$; otherwise we set $X = X_2$; and in either case we set $Y = (\alpha(D_0))^{-1}D_0$. It is clear that $[X, Y] = X$ so that Γ contains the 2-dimensional algebra described in the lemma.

CASE B: *There exists a nonzero root α whose values on Δ are neither all real nor all pure imaginary.*

We observe first that there exists an element D_0 in Δ such that both real and imaginary parts of $\alpha(D_0)$ are nonzero. For assuming that for all $D \in \Delta$, either $\alpha(D)$ is real or $\alpha(D)$ is pure imaginary, a contradiction is obtained as follows. Let $D_j(j = 1, 2)$ be elements of Δ such that $\alpha(D_j)$ has nonvanishing real or imaginary part according as $j = 1$ or $j = 2$. Then $D_1 + D_2 \in \Delta$ and $\alpha(D_1 + D_2)$ has nonvanishing real and imaginary parts. Now let D_0 be an element of Δ such that $\alpha(D_0) = \alpha_1 + i\alpha_2$, where α_j is real and nonzero $(j = 1, 2)$. Then if X belongs to and $X = X_1 + iX_2$ with $X_j \in \Gamma(j = 1, 2)$, it is easily seen that $[X_1, D_0] = \alpha_1 X_1 - \alpha_2 X_2$ and $[X_2, D_0] = \alpha_2 X_1 + \alpha_1 X_2$. If either X_1 or X_2 vanishes, it is plain that Γ contains a 2-dimensional subalgebra of the type described in the lemma. If neither X_1 nor X_2 vanishes, then setting $X = X_1$, $Y = X_2$, and $Z = \alpha_1^{-1}D_0$, the elements X, Y, and Z have the commutation relations stated in the lemma.

CASE C: *All roots are pure imaginary on Δ.* Let X_α belong to the nonzero root α; we assume that X_α has been normalized in the fashion described by Weyl, i.e. 1) $[X_{-\alpha}, X_\alpha] = D_\alpha$, where D_α is the unique element of Δ^c such that for all $D \in \Delta^c$, $\alpha(D) = \text{tr}\,(\hat{D}_\alpha \hat{D})$, where for any $W \in \Gamma^c$, $\hat{W}$ denotes the linear transformation on Γ^c, $V \to [V, W](V \in \Gamma^c)$, and tr is the usual trace function; 2) if α, β, and $\alpha + \beta$ are nonzero roots, then $[X_\alpha, X_\beta] = \omega_{\alpha,\beta} X_{\alpha+\beta}$ with $\omega_{\alpha\beta}$ nonzero and real, and $\omega_{-\alpha,-\beta} = -\omega_{\alpha,\beta}$. Now from the assumption that a root α is pure imaginary on Δ it is readily deduced that for $D \in \Delta^c$, $\overline{\alpha(\bar{D})} = -\alpha(D)$. Taking complex conjugates of both sides of the equation $[X_\alpha, D] = \alpha(D)X_\alpha$ shows that $[\bar{X}_\alpha, \bar{D}] = \overline{\alpha(D)}\bar{X}_\alpha$ or $[\bar{X}_\alpha, D] = -\alpha(D)\bar{X}_\alpha$, $D \in \Delta^c$. Thus $\bar{X}_\alpha$ belongs to $-\alpha$ and hence $\bar{X}_\alpha = \eta_\alpha X_{-\alpha}$ for some nonzero complex number η_α. It follows that $[X_\alpha, \bar{X}_\alpha] = \eta_\alpha[X_\alpha, X_{-\alpha}] = -\eta_\alpha D_\alpha$. Now it is known that all the roots are real-valued on D_α, and it results that $iD_\alpha \in \Delta$. If $X_\alpha = X_1 + iX_2$, with $X_j \in \Gamma$ $(j = 1, 2)$, then $[X_\alpha, \bar{X}_\alpha] = -2i[X_1, X_2]$, and so $i[X_\alpha, \bar{X}_\alpha] \in \Gamma$. Clearly $i[X_\alpha, \bar{X}_\alpha] = -\eta_\alpha iD_\alpha$, so that $U = \eta_\alpha V$ for some nonzero U and V in Γ, and from this it is easy to conclude that η_α is real.

Suppose now that η_α is positive for some α, say for $\alpha = \beta$. As X_β belongs to β, $[X_\beta, D_\beta] = \beta(D_\beta)X_\beta$, and it is known that $\beta(D_\beta) > 0$. Putting $\beta(D_\beta) = \rho$, and setting $X_\beta = Z_1 + iZ_2$ with $Z_j \in \Gamma (j = 1, 2)$ and $iD_\beta = Z_3$, then $Z_3 \in \Gamma$ and it is clear that $[Z_1 + iZ_2, Z_3] = i\rho(Z_1 + iZ_2)$. From the last equation it is easy to deduce that $[Z_1, Z_3] = -\rho Z_2$ and $[Z_2, Z_3] = \rho Z_1$. By the preceding paragraph, $-2i[Z_1, Z_2] = -\eta_\beta D_\beta$, so $[Z_1, Z_2] = -\eta_{\beta_2} Z_3$, or setting $\eta_\beta/2 = \tau$, $[Z_1, Z_2] = -\tau Z_3$, where $\tau > 0$. Putting $Z_1' = (\rho\tau)^{-\frac{1}{2}}Z_1$, $Z_2' = (\rho\tau)^{-\frac{1}{2}}Z_2$, and $Z_3' = \rho^{-1}Z_3$, it is easy to compute that $[Z_1', Z_2'] = -Z_3'$; $[Z_1', Z_3'] = -Z_2'$; $[Z_2', Z_3'] = Z_1'$. Now putting $X = Z_2' + Z_3'$ and $Y = Z_1'$, it is easily seen that $[X, Y] = X$, so that Γ contains a 2-dimensional subalgebra of the type described in the lemma.

It remains only to consider the case in which the η_α are all negative. We shall show that this case can be excluded because the fundamental quadratic form of Γ is then negative definite. If U and V belong to α and $-\alpha$ respectively, then $[V, U] = \operatorname{tr}(\hat{U}\hat{V})D_\alpha$; hence in particular $[\bar{X}_\alpha, X_\alpha] = \operatorname{tr}(\hat{X}_\alpha\hat{\bar{X}}_\alpha)D_\alpha$. It follows from the first paragraph treated under Case C that $\operatorname{tr}(\hat{X}_\alpha\hat{\bar{X}}_\alpha) = \eta_\alpha$. Now taking complex conjugates of the relations $[X_{-\alpha}, X_\alpha] = D_\alpha$ and $[X_\alpha, X_\beta] = \omega_{\alpha,\beta}X_{\alpha+\beta}$ it is not difficult to deduce (recalling that all roots are real-valued on D_α so that $\bar{D}_\alpha = -D_\alpha$) that $\eta_\alpha\eta_{-\alpha} = 1$ and $\eta_\alpha\eta_\beta = -\eta_{\alpha+\beta}(\alpha, \beta$, and $\alpha + \beta$ being nonzero roots). Now defining $X_\alpha' = (-\eta_\alpha)^{-\frac{1}{2}}X_\alpha$, it is clear that $\bar{X}_\alpha' = (-\eta_\alpha)^{-\frac{1}{2}}\bar{X}_\alpha = (-\eta_\alpha)^{-\frac{1}{2}}\eta_\alpha X_{-\alpha} = -(-\eta_\alpha)^{\frac{1}{2}}X_{-\alpha} = -(-\eta_{-\alpha})^{-\frac{1}{2}}X_{-\alpha} = -X_\alpha'$. It is further readily verified that $[X_{-\alpha}', X_\alpha'] = D_\alpha$ and that $[X_\alpha', X_\rho'] = \omega_{\alpha,\beta}X_{\alpha+\beta}'$. Hence the X_α' are normalized just as the X_α were, and so (dropping the primes) we may now assume that $\bar{X}_\alpha = -X_{-\alpha}$. As the X_α and the H_α span Γ^c relative to the complex field, the $i(X_\alpha + X_{-\alpha})$, the $X_\alpha - X_{-\alpha}$, and iH_α span Γ relative to the real field. But it is known that these elements span a real form of Γ^c whose fundamental quadratic form is negative definite, contrary to our original assumption, so this case cannot arise.

Proof of theorem. As a simply-connected semisimple Lie group is a direct product of simple Lie groups, we may assume that G is simple (as a Lie group). As a semisimple Lie group is known to be compact if and only if its fundamental quadratic form is negative definite, it follows from Lemma 4 that G contains an analytic subgroup which is isomorphic either to an S_1 or to an S_2 (in the notation of Lemma 2), and hence by Lemma 2 G contains a one-parameter subgroup, say H, on which every weakly continuous unitary representation of G into a finite factor is trivial. It follows that every such representation of G is trivial on all the conjugates of H under inner automorphisms of G. The adjoint representation R of G (for $a \in G$, $R(a)$ is the linear transformation on the Lie algebra $\mathbf{G}$ of G which is the differential of the automorphism of G, $x \rightarrow axa^{-1}$) leaves no subspace of $\mathbf{G}$ invariant, for such a subspace constitutes an ideal and $\mathbf{G}$ is simple. Hence, if $X \in \mathbf{G}$ generates H, the $R(a)X$ span $\mathbf{G}$, as a ranges over G. It follows that if n is the dimension of G, there exist elements $a_i (i = 1, 2, \cdots, n)$ of G such that the $\{R(a_i)X; i = 1, 2, \cdots, n\}$ span $\mathbf{G}$. Now there is a neighborhood N of the identity in G in which every element has the form exp $(\alpha_1 R(a_1)X)$ exp $(\alpha_2 R(a_2)X) \cdots$ exp $(\alpha_n R(a_n)X)$ for some real $\alpha_1, \cdots, \alpha_n$; and exp $(\alpha_1 R(a_1)X) = a_i$ exp $(\alpha_i X)a_i^{-1}$. Hence, if U is a weakly continuous unitary representation of G into a finite factor, U is the identity on N. As G is connected, $G = \bigcup_{k=1}^{\infty} N^k$, and so U is trivial on all of G. This completes the proof, except for the demonstration of the fact that a measurable unitary representation of G on a separable Hilbert space is weakly continuous, which fact is contained in Theorem 2.

Corollary 1. *A connected complex semisimple Lie group has no measurable unitary representations into a finite factor except the identity representation.*

Such a group is locally the direct product of complex simple groups, and hence it suffices to consider the latter type of group. If G is such a group it is noncompact, for as shown by Montgomery and Bochner [1] a compact complex analytic group is necessarily abelian. The Corollary will plainly follow if it is shown that G is simple as a real Lie group. To show this, assume that $\mathbf{I}$ is a real ideal in the Lie algebra $\mathbf{G}$ of G such that $\mathbf{I}$ is neither $\mathbf{G}$ nor $\{0\}$. Let $\mathbf{M}$ be a maximal real ideal in $\mathbf{G}$ which contains $\mathbf{I}$. Then $\mathbf{M}$ cannot be a complex ideal, and so there exists an element $X \in \mathbf{M}$ such that $iX \notin \mathbf{M}$. The set of all $\alpha iX + Y$ with α real and $Y \in \mathbf{M}$ is plainly a real linear subspace of $\mathbf{G}$, and it is a real ideal, for if $W \in \mathbf{G}$, $[W, \alpha iX + Y] = [iW, \alpha X] + [W, Y]$, which is in $\mathbf{M}$ because $\mathbf{M}$ is an ideal. Therefore the quotient real Lie algebra $\mathbf{G}/\mathbf{M}$ is 1-dimensional. But $\mathbf{G}$ is real semisimple, for its fundamental quadratic form is the same as that of $\mathbf{G}$ as a complex algebra, and its quotient algebras are therefore semisimple, in contradiction with the abelian character of $\mathbf{G}/\mathbf{M}$.

It is easy to see that the following corollary can be expressed in a form which gives a purely algebraic condition on the Lie algebra of a semisimple Lie group for the corresponding connected group to be compact.

Corollary 2. *A connected simple Lie group is compact if and only if it contains no analytic subgroup isomorphic to either of the groups $S_i (i = 1, 2)$ defined in Lemma 2.*

If G is a connected simple Lie group which is not compact, then by Lemma 4 it contains either S_1 or an S_2. On the other hand, if G is compact, then by the Peter-Weyl theorem it has a complete set of continuous unitary finite-dimensional representations. It follows from Lemma 2 that G can contain neither of the S_i as analytic subgroups.

The following corollary was proved in [8] for the case of the real unimodular group. We recall that a group is called minimally almost periodic if every almost periodic function on the group is a constant.

COROLLARY 3. *A connected semisimple Lie group none of whose simple constituents is compact is minimally almost periodic.*

By a theorem of van der Waerden [6] a finite-dimensional unitary representation of a semisimple Lie group is necessarily continuous. The corollary follows immediately from this fact together with Theorem 1 and the expansion theorem of [8].

COROLLARY 4. *A semisimple analytic subgroup of a compact Lie group is closed.*

Let G be a compact Lie group, H a semisimple analytic subgroup, and H_0 a simple constituent of H, which we regard as an analytic subgroup of H. By the Peter-Weyl theorem, G has a complete set of continuous finite-dimensional unitary representations. As the identity map of H into G is continuous, so also does H have such representations; and as the identity map of H_0 into H is continuous, also also does H_0. It follows from this together with Theorem 1 that H_0 is compact. Now a connected semisimple Lie group H all of whose simple constituents H_i are compact is itself compact. For if $\hat{H}_i$ is the universal covering group of H_i, the direct product P of the $\hat{H}_i$ is a simply connected group whose Lie algebra is isomorphic to that of H, so that H is a quotient group of P. Now a theorem of Weyl asserts that the universal covering group of a compact semisimple group is also compact, and hence the $\hat{H}_i$ are compact, which implies that P, and hence that H, is compact. By the compactness of a continuous image of a compact set, H is also compact as a relative space of G, and therefore is closed in G.

REMARK. A proof of Theorem 1 for the case when G is a complex semisimple Lie group can be given which is considerably shorter than the proof for the real case, largely because Lemmas 3 and 4 are then not required. In outline, this shorter proof is as follows. If α is any nonzero root of the Lie algebra G of G with respect to a Cartan subalgebra $\mathbf{K}$, and if X belongs to α, then $[X, K] = \alpha(K)X(K \in \mathbf{K})$. Hence, if $\alpha(K) \neq 0$, the real subgroup generated by X and $K(\alpha(K))^{-1}$ is an $\mathbf{S}_1$, in the notation of Lemma 2. Hence by Lemma 2 (or by a direct argument on the affine group of the line, which is the unique connected Lie group with Lie algebra $\mathbf{S}_1$, along the lines of the proof of Lemma 1), any weakly continuous unitary representation of G is the identity representation on the subgroup generated by X. As a complex semisimple Lie group is locally a direct product of connected complex simple Lie groups, it suffices to consider the case in which G is simple as a complex Lie group. But it is then simple as a real Lie group. The remainder of the proof is then just as in the Proof of Theorem 1.

We conclude this paper by showing that a measurable unitary representation of a locally compact group on a Hilbert space is the direct sum of a strongly continuous representation, and a "singular" representation which vanishes when the representation space is separable. Our method is an adaptation of that used in [8], p. 479, for the special case of the characters of the additive group of the reals. A weaker result, which however is equivalent to ours in the case of a separable representation space, is due to Neumark [4]. We first give an example to show that singular representations actually occur, a representation U of a locally compact group G being defined to be singular when $(U(a)x, y) = 0$ nearly everywhere (i.e. almost everywhere on every measurable set of finite measure) on G, relative to Haar measure, for all x and y in the representation space.

EXAMPLE. Let $\mathcal{H}$ be the Hilbert space of all complex-valued functions $f(x)$ on the reals which vanish except on a denumerable set and such that $\sum_x |f(x)|^2$ is finite, the inner product of any two elements f and g being defined as $\sum_x f(x)\overline{g(x)}$. Let G be the additive group of the reals, and let U be the mapping on G to the linear operators on $\mathcal{H}$ given by the equation $(U(a)f)(x) = f(x + a)$, $f \in \mathcal{H}$. It it easy to see that $U(a)$ is unitary and that $(U(a)f, g) = 0$ unless $a \in S(f) \ominus S(g)$, where $S(h)$ is the set of reals on which the function h does not vanish, and $\ominus$ indicates group subtraction. Hence $(U(a)f, g) = 0$, except on a denumerable set.

THEOREM 2. *Let U be a measurable unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Then there exist complementary invariant closed linear manifolds $\mathcal{K}$ and $\mathcal{L}$ in $\mathcal{H}$ such that if P and Q are the projections of $\mathcal{H}$ onto $\mathcal{K}$ and $\mathcal{L}$ respectively, PU is a strongly continuous representation of G on $\mathcal{K}$ and QU is a singular representation of G on $\mathcal{L}$. If $\mathcal{H}$ is separable, then $\mathcal{L} = 0$.*

If $f \in L_1(G)$, where G is considered as a measure space relative to Haar measure, then for any x and y in $\mathcal{H}$ the integral $\int (U(a)x, y)f(a)\, da$ exists and defines a Hermitian bilinear form in x and y which is bounded, and so has the form (Tx, y) for some bounded linear operator T on $\mathcal{H}$. We denote T by the symbol, $\int U(a)f(a)\, da$ and observe that $|||\, T\, ||| \leqq \|f\|_1$. Now if y is in the range of T, say $y = Tz$, then $U(a)y$ is a continuous function of a, for if x is arbitrary in $\mathcal{H}$, $(U(a)y - U(a_0)y, x)$ is readily seen to equal $\int (U(b)z, x)(f_a(b) - f_{a_0}(b))\, db$, where $f_a(x) = f(a^{-1}x)$, so that $|(U(a)y - U(a_0)y, x)| \leqq \|x\|\,\|z\|\,\|f_a - f_{a_0}\|_1$, whence $\|U(a)y - U(a_0)y\| \leqq \|z\|\,\|f_a - f_{a_0}\|_1$, and it is known that $\|f_a - f_{a_0}\| \to 0$ as $a \to a_0$.

It follows that $U(a)y$ is a continuous function of a for any y in the linear span of the ranges of all T's arising from f's in $L_1(G)$. It is easy to deduce from this, together with the fact that a uniform limit of continuous functions is continuous, that $U(a)y$ is continuous as a function of $a \in G$ for all y in the closed linear manifold $\mathcal{K}$ spanned by the ranges of the T's. It is easily seen that $U(a) \int U(b)f(b)\, db = \int U(ab)f(b)\, db = \int U(b)f_a(b)\, db$, which implies that $\mathcal{K}$ is invariant under the $U(a)$.

It follows that the orthogonal complement $\mathcal{L}$ of $\mathcal{K}$ is invariant. Now if $x \in \mathcal{L}$, then $Tx \in \mathcal{L}$ for any T, since $y \in \mathcal{K}$ implies that $(Tx, y) = \int (U(a)x, y)f(a)\, da =$

$\int (x, U(a^{-1})y)f(a)\, da = 0$, where $T = \int U(a)f(a)\, da$. Hence $Tx \in \mathcal{K} \cap \mathfrak{L}$, so $Tx = 0$ and $(Tx, y) = 0$ for $x \in \mathfrak{L}$ and $y \in \mathcal{K}$. That is, $\int (U(a)x, y)f(a)\, da = 0$, which by the arbitrary character of f implies that $(U(a)x, y) = 0$ nearly everywhere on G. Now if $\mathcal{K}$ is separable, so also is $\mathfrak{L}$, and if $\{x_i\}$ is a denumerable dense subset of $\mathfrak{L}$, the equations $(U(a)x_i, x_j) = 0$ nearly everywhere for each i and j imply that $(U(a)x_i, x_j) = 0$ hold simultaneously for all i and j, nearly everywhere, from which it follows that $(U(a)x, y) = 0$ holds simultaneously for all x and y in $\mathfrak{L}$, nearly everywhere. If a_0 is any particular element in G such that $(U(a_0)x, y) = 0$ for all x and y in $\mathcal{K}$, putting $y = U(a_0)x$ shows that $U(a_0)x = 0$, which implies $x = 0$ and $\mathfrak{L} = \{0\}$.

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