NONCOMMUTATIVE GEOMETRY AND MOTIVES: THE THERMODYNAMICS OF ENDOMOTIVES

ALAIN CONNES, CATERINA CONSANI, AND MATILDE MARCOLLI

Contents

1. Introduction

A few unexpected encounters between noncommutative geometry and the theory of motives have taken place recently. A first instance occurred in [14], where the Weil explicit formulae acquire the geometric meaning of a Lefschetz trace formula over the noncommutative space of adele classes. The reason why the construction of [14] should be regarded as motivic is twofold. On the one hand, as we discuss in this paper, the adele class space is obtained from a noncommutative space, the Bost–Connes system, which sits naturally in a category of noncommutative spaces extending the category of Artin motives. Moreover, as we discuss briefly in this paper and in full detail in our forthcoming work [15], it is possible to give a cohomological interpretation of the spectral realization of the zeros of the Riemann zeta function of [14] on the cyclic homology of a noncommutative motive. The reason why this construction takes place in a category of (noncommutative) motives is that the geometric space we need to use is obtained as a cokernel of a morphism of algebras, which only exists in a suitable abelian category extending the non-additive category of algebras, exactly as in the context of motives and algebraic varieties.
There are other instances of interactions between noncommutative geometry and motives, some of which will in fact involve noncommutative versions of more complicated categories of motives, beyond Artin motives, involving pure and sometimes mixed motives. One example is the problem of extending the results of [14] to L-functions of algebraic varieties. The latter are, in fact, naturally associated not to a variety but to a motive. Already in [26] it was shown that noncommutative geometry can be used to model the geometry of the fibers at the archimedean places of arithmetic varieties. This suggested the existence of a Lefschetz trace formula for the local L-factors, and at least a semi-local version for the L-function, over a noncommutative space obtained as a construction over the adele class space. In this paper we give such a Lefschetz interpretation for the archimedean local factors introduced by Serre [50]. In order to obtain a similar Lefschetz formula for the case of several places (archimedean and non-archimedean), it seems necessary to develop a theory that combines noncommutative spaces and motives.
A further example of an intriguing interaction between noncommutative geometry and motives appears in the context of perturbative renormalization in quantum field theory. The results of [19], [18] show the existence of a universal group of symmetries that governs the structure of the divergences in renormalizable quantum field theories and contains the renormalization group as a 1-parameter subgroup. This universal group is a motivic Galois group for a category of mixed Tate motives. On the other hand, the treatment of divergent Feynman integrals by dimensional regularization, used in this geometric reformulation of renormalization, is best described in the framework of noncommutative geometry. In fact, using the formalism of spectral triples in noncommutative geometry, one can construct (*cf.* [20]) spaces with dimension a complex number z and describe geometrically the procedure of dimensional regularization as a cup product of spectral triples. Once again it is desirable to develop a common framework for noncommutative geometry and motives, so as to obtain a more direct identification between the data of perturbative quantum field theory and the objects of the category of mixed motives governed by the same Galois group.
In a different but not unrelated perspective, Kontsevich recently found another very interesting link between noncommutative geometry and motives [40], see also [38].

The present paper is structured as follows. We begin by discussing the problem of morphisms for noncommutative spaces. It is well known, in fact, that morphisms of algebras are not sufficient and one needs a larger class of morphisms that, for instance, can account for the phenomenon of Morita equivalence. We argue that there are, in fact, two well developed constructions in noncommutative geometry that make it possible to define morphisms as correspondences, namely Kasparov's bivariant KK-theory on one hand and modules over the cyclic category (and cyclic (co)homology) on the other.

In KK-theory, one extends the notion of morphism to that of (virtual) correspondences given by formal differences of correspondences defined using bimodules. This yields an *additive* category containing the category of C^*-algebras. The product (composition of correspondences) plays a central role in the understanding of K-theory of C^*-algebras. In the approach via cyclic cohomology, one constructs a functor $\mathcal{A} \to \mathcal{A}^\natural$ from the category of algebras and algebra homomorphisms to the *abelian* category of Λ-modules introduced in [11]. Cyclic cohomology and homology appear then as derived functors, namely as the Ext and Tor functors, much as in the case of the absolute cohomology in the theory of motives. Working with the resulting abelian category has the advantage of making all the standard tools of homological algebra available in noncommutative geometry. It is well known that one obtains this way a good de Rham (or crystalline) theory for noncommutative spaces, a general bivariant cyclic theory, and a framework for the Chern-Weil theory of characteristic classes for general Hopf algebra actions [23].

We show, first in the zero-dimensional case of Artin motives and then in a more general setting in higher dimension, that the notion of morphisms as (virtual) correspondences given by elements in KK-theory is compatible with the notion of morphisms as used in the theory of motives, where correspondences are given by algebraic cycles in the product. In fact, we show correspondences in the sense of motives give rise to elements in KK-theory (using Baum's geometric correspondences [4] and [24]), compatibly with the operation of composition.

We focus on the zero-dimensional case of Artin motives. We extend the category $\mathcal{CV}^0_{\mathbb{K},\mathbb{E}}$ of Artin motives over a field $\mathbb{K}$ with coefficients in a field $\mathbb{E}$ to a larger (pseudo)abelian category $\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}$ of *algebraic endomotives*, where the objects are semigroup actions on projective systems of Artin motives. The objects are described by semigroup crossed product algebras $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ over $\mathbb{K}$ and the correspondences extend the usual correspondences of Artin motives given by algebraic cycles, compatibly with the semigroup actions.
We prove then that, in the case of a number field $\mathbb{K}$, taking points over $\bar{\mathbb{K}}$ in the projective limits of Artin motives gives a natural embedding of the category of algebraic endomotives into a (pseudo)abelian category $C^*\mathcal{V}^0_{\mathbb{K}}$ of *analytic endomotives*. The objects are noncommutative spaces given by crossed product C^*-algebras and the morphisms are given by geometric correspondences, which define a suitable class of bimodules. The Galois group G acts as natural transformations of the functor $\mathcal{F}$ realizing the embedding. The objects in the resulting category of analytic endomotives are typically noncommutative spaces obtained as quotients of semigroup actions on totally disconnected compact spaces.
We show that important examples like the Bost–Connes (BC) system of [7] and its generalization for imaginary quadratic fields considered in [21] belong to this class. We describe a large class of examples obtained from iterations of self-maps of algebraic varieties.

We also show that the category of noncommutative Artin motives considered here is especially well designed for the general program of applications of noncommutative geometry to abelian class field theory proposed in [17], [22]. In fact, a first result in the present paper shows that what we called "fabulous states" property in [17], [22] holds for any noncommutative Artin motive that is obtained from the action of a semigroup S by endomorphisms on a direct limit A of finite dimensional algebras that are products of abelian normal field extensions of a number field $\mathbb{K}$. Namely, we show that there is a canonical action of the absolute Galois group G of $\mathbb{K}$ by automorphisms of a C^*-completion $\bar{A}$ of the noncommutative space $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ and, for pure states φ on this C^*-algebra which are induced from A, one has the intertwining property

$$\alpha\varphi(a) = \varphi(\alpha(a))$$

for all $a \in A \rtimes S$ and all $\alpha \in G/[G,G]$.

When passing from commutative to noncommutative algebras, new tools of thermodynamical nature become available and make it possible to construct an action of $\mathbb{R}_+^*$ on cyclic homology. We show that, when applied to the noncommutative space (analytic endomotive) $\mathcal{F}(M)$ associated to an algebraic endomotive M, this representation of $\mathbb{R}_+^*$ combines with a representation of the Galois group G. In the particular case of the endomotive associated to the BC system, the resulting representation of $G \times \mathbb{R}_+^*$ gives the spectral realization of the zeros of the Riemann zeta function and of the Artin L-functions for abelian characters of G. One sees in this example that this construction plays a role analogous to the action of the Weil group on the ℓ-adic cohomology and it can be thought of as a functor from the category of endomotives to the category of representations of the group $G \times \mathbb{R}_+^*$. In fact, here we think of the action of $\mathbb{R}_+^*$ as a "Frobenius in characteristic zero", hence of $G \times \mathbb{R}_+^*$ as the corresponding Weil group.

Notice that it is the type III nature of the BC system that is at the root of the thermodynamical behavior that makes the procedure described above nontrivial. In particular, this procedure only yields trivial results if applied to commutative and type II cases.

The construction of the appropriate "motivic cohomology" with the "Frobenius" action of $\mathbb{R}_+^*$ for endomotives is obtained through a very general procedure, which we describe here in the generality that suffices to the purpose of the present paper. It consists of three basic steps, starting from the data of a noncommutative algebra $\mathcal{A}$ and a state φ, under specific assumptions that will be discussed in detail in the paper. One considers the time evolution $\sigma_t \in \mathrm{Aut}\mathcal{A}$, $t \in \mathbb{R}$ naturally associated to the state φ (as in [10]).

The first step is what we refer to as *cooling*. One considers the space $\mathcal{E}_\beta$ of extremal KMS_β states, for β greater than critical. Assuming these states are of type I, one obtains a morphism

$$\pi : \mathcal{A} \rtimes_\sigma \mathbb{R} \to \mathcal{S}(\mathcal{E}_\beta \times \mathbb{R}_+^*) \otimes \mathcal{L}^1,$$

where $\mathcal{A}$ is a dense subalgebra of a C^*-algebra $\bar{\mathcal{A}}$, and where $\mathcal{L}^1$ denotes the ideal of trace class operators. In fact, one considers this morphism restricted to the kernel of the canonical trace τ on $\bar{\mathcal{A}} \rtimes_\sigma \mathbb{R}$.

The second step is *distillation*, by which we mean the following. One constructs the Λ-module $D(\mathcal{A}, \varphi)$ given by the cokernel of the cyclic morphism given by the composition of π with the trace $\mathrm{Trace} : \mathcal{L}^1 \to \mathbb{C}$.

The third step is the *dual action*. Namely, one looks at the spectrum of the canonical action of $\mathbb{R}_+^*$ on the cyclic homology

$$HC_0(D(\mathcal{A}, \varphi)).$$

This procedure is quite general and applies to a large class of data $(\mathcal{A}, \varphi)$ and produces spectral realizations of zeros of L-functions. In particular, we can apply it to the analytic endomotive describing the BC-system. In this case, the noncommutative space is that of commensurability classes of $\mathbb{Q}$-lattices up to scaling. The latter can be obtained from the action of the semigroup $\mathbb{N}$ on $\hat{\mathbb{Z}}$. However, the algebraic endomotive corresponding to this action is not isomorphic to the algebraic endomotive of the BC system, which comes from the algebra $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. In fact, they are distinguished by the Galois action. This important difference is a reflection of the distinction between Artin and Hecke L-functions.

In this BC system, the corresponding quantum statistical system is naturally endowed with an action of the group $G^{ab} = G/[G,G]$, where G is the Galois group of $\bar{\mathbb{Q}}$. Any Hecke character χ gives rise to an idempotent p_χ in the ring of endomorphisms of the

0-dimensional noncommutative motive X and when applied to the range of p_χ the above procedure gives as a spectrum the zeros of the Hecke L-function L_χ.

We also show that the "dualization" step, *i.e.* the transition from $\mathcal{A}$ to $\mathcal{A} \rtimes_\sigma \mathbb{R}$, is a very good analog in the case of number fields of what happens for a function field K in passing to the extension $K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. In fact, in the case of positive characteristic, the unramified extensions $K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ combined with the notion of places yield the points $C(\bar{\mathbb{F}}_q)$ over $\bar{\mathbb{F}}_q$ of the underlying curve. This has a good parallel in the theory of factors, as we are going to discuss in this paper. In our forthcoming work [15] this analogy will be developed further and will play an important role in developing a setting in noncommutative geometry that parallels the algebro-geometric framework that Weil used in his proof of RH for function fields.

In the results discussed here and in [15], instead of working with the Hilbert space context of [14] the trace formula involved in the spectral realization of [14] is given in suitable nuclear function spaces associated to algebras of rapidly decaying functions, as in the approach of [46].

We also give an overview of some results that will be proved in full detail in [15], where we give a cohomological interpretation of the spectral realization of the zeros of the zeta function as the cyclic homology of a "suitable" noncommutative space $M_{\mathbb{K}}$. The sought for space $M_{\mathbb{K}}$ appears naturally as the cokernel of the restriction morphism of functions on the adele class space $X_{\mathbb{K}}$ to functions on the "cooled down" subspace $C_{\mathbb{K}}$ of idele classes. This operation would not make sense in the usual category of algebras (commutative or not) and algebra homomorphisms, since this category is not additive and one cannot take cokernels, whereas this can be done in a suitable category of motives.

Acknowledgement. This research was partially supported by the third author's Sofya Kovalevskaya Award and by the second author's NSERC grant 7024520. Part of this work was done during a visit of the first and third authors to the Kavli Institute in Santa Barbara, supported in part by the National Science Foundation under Grant No. PHY99-07949, and during a visit of the first two authors to the Max Planck Institute.

2. Correspondences for noncommutative spaces

It is often regarded as a problem in noncommutative geometry to provide a good notion of *morphisms* for noncommutative spaces, which accounts, for instance, for phenomena such as Morita equivalence. Moreover, it is often desirable to apply to noncommutative geometry the tools of homological algebra, which require working with abelian categories. The category of algebras with algebra homomorphisms is not satisfactory for both of the reasons just mentioned. We argue here that the best approach to dealing with a category of noncommutative spaces is an analog of what happens in algebraic geometry, when one considers motives instead of algebraic varieties and defines morphisms as *correspondences.* In the context of noncommutative geometry, there are well developed analogs of correspondences and of motivic cohomology. For the purposes of this paper we consider correspondences given by Kasparov's KK-theory (*cf.* [39]), although we will see later that a more refined version is desirable, where for instance one does not mod out by homotopy equivalence, and possibly based on a relative version in the fibration relating algebraic and topological K-theory (*cf.* [16]).

The analog of motivic and absolute cohomology is played by modules over the cyclic category and cyclic cohomology (*cf.* [11], [12]).

2.1. KK**-theory.** Similarly to what happens in the context of algebraic varieties, C^*-algebras do not form an additive category. However, there is a functor from the category of all separable C^*-algebras, with $*$-homomorphisms, to an *additive* category $\mathcal{KK}$, whose objects are separable C^*-algebras and where, for $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ in $Obj(\mathcal{KK})$, the morphisms are given by $\mathrm{Hom}(\bar{\mathcal{A}},\bar{\mathcal{B}}) = KK(\bar{\mathcal{A}},\bar{\mathcal{B}})$. Here $KK(\bar{\mathcal{A}},\bar{\mathcal{B}})$ is Kasparov's bivariant K-theory ([39], *cf.* also §8 and §9.22 of [5]). Here the notation $\bar{\mathcal{A}}$ for C^*-algebras comes from the fact that, later on in this paper, we will consider C^*-algebras that are C^*-completions of certain natural dense subalgebras $\mathcal{A}$, though this assumption is not needed in this subsection.

In the $\mathcal{KK}$ category morphisms are defined through "correspondences" in the following way. One considers the set $M(\bar{\mathcal{A}},\bar{\mathcal{B}})$ of Kasparov modules, that is, of triples (E,ϕ,F), where E is a countably generated Hilbert module over $\bar{\mathcal{B}}$, ϕ is a $*$-homomorphism of $\bar{\mathcal{A}}$ to bounded linear operators on E (*i.e.* it gives E the structure of an $(\bar{\mathcal{A}},\bar{\mathcal{B}})$-bimodule) and F is a bounded linear operator on E with the properties that the operators $[F,\phi(a)]$, $(F^2-1)\phi(a)$, and $(F^*-F)\phi(a)$ are in $\mathcal{K}(E)$ (compact operators) for all $a \in \bar{\mathcal{A}}$. Recall here that a Hilbert module E over $\bar{\mathcal{B}}$ is a right $\bar{\mathcal{B}}$-module with a positive $\bar{\mathcal{B}}$-valued inner product which satisfies $\langle x, yb\rangle = \langle x,y\rangle b$ for all $x,y \in E$ and $b \in \bar{\mathcal{B}}$, and with respect to which E is complete. Kasparov modules are "Morita type" correspondences that generalize $*$-homomorphisms of C^*-algebras (the latter are trivially seen as Kasparov modules of the form $(\bar{\mathcal{B}},\phi,0)$).

One then considers on $M(\bar{\mathcal{A}},\bar{\mathcal{B}})$ the equivalence relation of homotopy. Two elements are homotopy equivalent $(E_0,\phi_0,F_0) \sim_h (E_1,\phi_1,F_1)$ if there is an element (E,ϕ,F) of $M(\bar{\mathcal{A}},I\bar{\mathcal{B}})$, where $I\bar{\mathcal{B}} = \{f : [0,1] \to \bar{\mathcal{B}} \,|\, f \text{ continuous}\}$, such that $(E\hat{\otimes}_{f_i}\bar{\mathcal{B}}, f_i\circ\phi, f_i(F))$ is unitarily equivalent to (E_i,ϕ_i,F_i), *i.e.* there is a unitary in bounded operators from $E\hat{\otimes}_{f_i}\bar{\mathcal{B}}$ to E_i intertwining the morphisms $f_i\circ\phi$ and ϕ_i and the operators $f_i(F)$ and F_i. Here f_i is the evaluation at the endpoints $f_i : I\bar{\mathcal{B}} \to \bar{\mathcal{B}}$. By definition $KK(\bar{\mathcal{A}},\bar{\mathcal{B}})$ is the set of homotopy equivalence classes of $M(\bar{\mathcal{A}},\bar{\mathcal{B}})$. This is naturally an abelian group.

A slightly different formulation of KK-theory, which simplifies the external tensor product $KK(\bar{\mathcal{A}},\bar{\mathcal{B}})\otimes KK(\bar{\mathcal{C}},\bar{\mathcal{D}}) \to KK(\bar{\mathcal{A}}\otimes\bar{\mathcal{C}},\bar{\mathcal{B}}\otimes\bar{\mathcal{D}})$, is obtained by replacing in the data (E,ϕ,F) the operator F by an *unbounded* regular self-adjoint operator D. The corresponding F is then given by $D(1+D^2)^{-1/2}$ (*cf.* [3]).

The category $\mathcal{KK}$ obtained this way is a universal enveloping additive category for the category of C^*-algebras (*cf.* [5] §9.22.1). Though it is not an abelian category, it is shown in [47] that $\mathcal{KK}$ and the equivariant $\mathcal{KK}_G$ admit the structure of a triangulated category.

In algebraic geometry the category of pure motives of algebraic varieties is obtained by replacing the (non-additive) category of smooth projective varieties by a category whose objects are triples (X,p,m) with X a smooth projective algebraic variety, p a projector and m an integer. The morphisms

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{n-m}(X,Y)p$$

are then given by correspondences (combinations of algebraic cycles in the product $X \times Y$ with rational coefficients) modulo a suitable equivalence relation. (Notice that the idempotent condition $p^2 = p$ itself depends on the equivalence relation.) Depending on the choice of the equivalence relation, one gets categories with rather different properties. For instance, only the numerical equivalence relation, which is the coarsest among the various relations considered on cycles, is known to produce in general an abelian category (*cf.* [37]). The homological equivalence relation, for example, is only known to produce a $\mathbb{Q}$-linear pseudo-abelian category, that is, an additive category in which the ranges of projectors are included among the objects. One of Grothendieck's standard conjectures would imply the equivalence of numerical and homological equivalence.

In the case of (noncommutative) algebras, when we use KK-theory, the analysis of the defect of surjectivity of the assembly map is intimately related to a specific idempotent (the

γ-element) in a KK-theory group. Thus the analogy with motives suggests that one should similarly enlarge the additive category $\mathcal{KK}$ of C^*-algebras with ranges of idempotents in KK. This amounts to passing to the pseudo-abelian envelope of the additive $\mathcal{KK}$ category.

2.2. **The abelian category of Λ-modules and cyclic (co)homology.** We now interpret cyclic cohomology in a setting which is somewhat analogous to the one relating algebraic varieties and motives. The category of pure motives $\mathcal{M}$ (with the numerical equivalence) is a semi-simple abelian $\mathbb{Q}$-linear category with finite dimensional Hom groups. In the theory of motives, to a smooth projective algebraic variety X over a field $\mathbb{K}$ one wants to associate a *motivic cohomology* $H^i_{mot}(X)$, defined as an object in category $\mathcal{M}_{\mathbb{K}}$,*i.e.* $H^i_{mot}(X)$ is a pure motive of weight i (*cf.* [29]). In this way, one views pure motives as a universal cohomology theory for algebraic varieties. Namely, the motivic cohomology has the property that, for any given cohomology theory H, with reasonable properties, there exists a realization functor R_H satisfying

$$H^n(X) = R_H\, H^n_{mot}(X). \tag{2.1}$$

On a category of mixed motives, one would also have *absolute cohomology* groups given by (*cf.* [29])

$$H^i_{abs}(M) = \mathrm{Ext}^i(1, M). \tag{2.2}$$

for M a motive. Here the Ext^i are taken in a suitable triangulated category $\mathcal{D}(\mathbb{K})$ whose heart is the category of mixed motives. For a variety X the absolute cohomology would then be obtained from the motivic cohomology via a spectral sequence

$$E_2^{pq} = H^p_{abs}(H^q_{mot}(X)) \Rightarrow H^{p+q}_{abs}(X). \tag{2.3}$$

In the context of noncommutative geometry, we may argue that the cyclic category and cyclic (co)homology can be thought of as an analog of motivic and absolute cohomology. In fact, the approach to cyclic cohomology via the cyclic category (*cf.* [11]) provides a way to embed the nonadditive category of algebras and algebra homomorphisms in an *abelian* category of modules. The latter is the category of $\mathbb{K}(\Lambda)$-modules, where Λ is the cyclic category and $\mathbb{K}(\Lambda)$ is the group ring of Λ over a given field $\mathbb{K}$. Cyclic cohomology is then obtained as an Ext functor (*cf.* [11]).

The cyclic category Λ has the same objects as the small category Δ of totally ordered finite sets and increasing maps, which plays a key role in simplicial topology. Namely, Δ has one object $[n]$ for each integer n, and is generated by faces $\delta_i : [n-1] \to [n]$ (the injection that misses i), and degeneracies $\sigma_j : [n+1] \to [n]$ (the surjection which identifies j with $j+1$), with the relations

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{for } i < j, \qquad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j \tag{2.4}$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1. \end{cases} \tag{2.5}$$

To obtain the cyclic category Λ one adds for each n a new morphism $\tau_n : [n] \to [n]$, such that

$$\begin{aligned} &\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \quad 1 \leq i \leq n, \quad \tau_n \delta_0 = \delta_n \\ &\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau^2_{n+1} \\ &\tau_n^{n+1} = 1_n. \end{aligned} \tag{2.6}$$

Alternatively, Λ can be defined by means of its "cyclic covering", through the category $\tilde{\Lambda}$. The latter has one object $(\mathbb{Z}, n)$ for each $n \geq 0$ and the morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$

are given by non decreasing maps $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(x+n) = f(x) + m\,, \quad \forall x \in \mathbb{Z}$. One has $\Lambda = \tilde{\Lambda}/\mathbb{Z}$, with respect to the obvious action of $\mathbb{Z}$ by translation.

To any algebra $\mathcal{A}$ over a field $\mathbb{K}$, one associates a module $\mathcal{A}^\natural$ over the category Λ by assigning to each n the $(n+1)$ tensor power $\mathcal{A} \otimes \mathcal{A} \cdots \otimes \mathcal{A}$. The cyclic morphisms correspond to the cyclic permutations of the tensors while the face and degeneracy maps correspond to the algebra product of consecutive tensors and the insertion of the unit (*cf.* [11]). A trace $\varphi : \mathcal{A} \to \mathbb{K}$ gives rise to a morphism $\varphi^\natural$ of $\mathcal{A}^\natural$ to the Λ-module $\mathbb{K}^\natural$ such that

$$\varphi^\natural(a_0 \otimes \cdots \otimes a_n) = \varphi(a_0 \cdots a_n). \tag{2.7}$$

Using the natural isomorphism of Λ with the opposite small category Λ^o one obtains by dualization that any algebra $\mathcal{A}$ gives rise canonically to a module $\mathcal{A}^\sharp$ over the small category Λ, by assigning to each integer $n \geq 0$ the vector space C^n of $n+1$-linear forms $\varphi(x^0, \dots, x^n)$ on $\mathcal{A}$, with the basic operations $\delta_i : C^{n-1} \to C^n$ and $\sigma_i : C^{n+1} \to C^n$ given by

$$\begin{aligned} (\delta_i\varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^i x^{i+1}, \dots, x^n),\ i = 0, 1, \dots, n-1 \\ (\delta_n\varphi)(x^0, \dots, x^n) &= \varphi(x^n x^0, x^1, \dots, x^{n-1}) \\ (\sigma_j\varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^j, 1, x^{j+1}, \dots, x^n),\ j = 0, 1, \dots, n \\ (\tau_n\varphi)(x^0, \dots, x^n) &= \varphi(x^n, x^0, \dots, x^{n-1}). \end{aligned} \tag{2.8}$$

These operations satisfy the relations (2.4) (2.5) and (2.6). This shows that $\mathcal{A}^\sharp$ is a Λ-module.

It is then possible (*cf.* [11], [43]), to interpret the cyclic cohomology HC^n as Ext^n functors, namely

$$HC^n(\mathcal{A}) = \mathrm{Ext}^n(\mathcal{A}^\natural, \mathbb{K}^\natural) = \mathrm{Ext}^n(\mathbb{K}^\sharp, \mathcal{A}^\sharp), \tag{2.9}$$

for $\mathcal{A}$ an algebra over $\mathbb{K}$. Both terms express the derived functor of the functor which assigns to a Λ-module its space of traces.

The formula (2.9) may be regarded as an analog of (2.2), except for the fact that here the Ext^n are taken in the abelian category of $\mathbb{K}(\Lambda)$-modules.

Similarly to (2.9), one also has an interpretation of cyclic homology as Tor_n functors,

$$HC_n(\mathcal{A}) = \mathrm{Tor}_n(\mathbb{K}^\natural, \mathcal{A}^\natural)\,. \tag{2.10}$$

A canonical projective biresolution of $\mathbb{Z}^\natural$ is obtained by considering a bicomplex of Λ-modules $(C^{n,m}, d_1, d_2)$ (for $n, m > 0$) where $C^{n,m} = C^m$ for all n and the component C^k_j of C^k is the free abelian group on the set $\mathrm{Hom}(\Lambda_k, \Lambda_j)$ so that $\mathrm{Hom}_\Lambda(C^k, \mathcal{E}) = \mathcal{E}_k$. See [11] for the differentials $d_1 : C^{n+1,m} \to C^{n,m}$ and $d_2 : C^{n,m+1} \to C^{n,m}$. This projective resolution of $\mathbb{Z}^\natural$ determines a bicomplex computing the cyclic cohomology $\mathrm{Ext}^n(\mathbb{Z}^\natural, \mathcal{E})$ of a Λ-module $\mathcal{E}$ analogous to the (b, B)-bicomplex $C^{n-m}(\mathcal{A})$.

All of the general properties of cyclic cohomology such as the long exact sequence relating it to Hochschild cohomology are shared by Ext of general Λ-modules and can be attributed to the equality of the classifying space $B\Lambda$ of the small category Λ with the classifying space BS^1 of the compact one-dimensional Lie group S^1. One has

$$B\Lambda = BS^1 = \mathbb{P}^\infty(\mathbb{C}) \tag{2.11}$$

This follows from the fact that $\mathrm{Hom}(\Lambda_n, \Lambda_m)$ is nonempty and that $\mathrm{Hom}(\Lambda_0, \Lambda_0) = 1$, which imply that $B\Lambda$ is connected and simply connected, and from the calculation (*cf.* [11]) of $\mathrm{Ext}^*_\Lambda(\mathbb{Z}^\natural, \mathbb{Z}^\natural) = \mathbb{Z}[\sigma]$ as a polynomial ring in the generator σ of degree two. The cohomology ring $H^*(B\Lambda, \mathbb{Z})$ is given by $\mathbb{Z}[\sigma]$ where σ corresponds to a map $f : B\Lambda \to \mathbb{P}^\infty(\mathbb{C})$ which is a homotopy equivalence.

Finally we note that there is a natural way to associate a cyclic morphism to a correspondence viewed as an $\mathcal{A}$–$\mathcal{B}$ bimodule $\mathcal{E}$, provided that the following finiteness holds

Lemma 2.1. *Let $\mathcal{E}$ be an $\mathcal{A}$–$\mathcal{B}$ bimodule, which is finite projective as a right $\mathcal{B}$-module. Then there is an associated cyclic morphism*

$$\mathcal{E}^\natural \in \mathrm{Hom}(\mathcal{A}^\natural, \mathcal{B}^\natural).$$

Proof. Since $\mathcal{E}$ is finite projective it is a direct summand of the right module $\mathcal{B}^n$ for some n, so that $\mathrm{End}_{\mathcal{B}}(\mathcal{E}) \subset M_n(\mathcal{B}) = \mathcal{B} \otimes M_n(\mathbb{C})$. Thus one obtains $\tau^\natural$ from the left action of $\mathcal{A}$ which gives a homomorphism ρ from $\mathcal{A}$ to $\mathrm{End}_{\mathcal{B}}(\mathcal{E})$ and hence to $\mathcal{B} \otimes M_n(\mathbb{C})$. One then takes the composition of $\rho^\natural$ with the trace map from $\mathcal{B} \otimes M_n(\mathbb{C})$ to $\mathcal{B}$ which is also a cyclic morphism as follows from the more general Proposition 4.7 below. $\square$

Remark 2.2. *Note that the above cyclic morphism depends upon the choice of a connection on the right $\mathcal{B}$-module $\mathcal{E}$, which is used in order to write $\mathcal{E}$ as a direct summand of the right module $\mathcal{B}^n$ for some n.*

All of the discussion above, regarding the cyclic module associated to an algebra, extends to the context of locally convex topological algebras. In that context one needs to work with topological tensor products in the construction of $\mathcal{A}^\natural$ and of continuous multilinear forms in the construction of $\mathcal{A}^\sharp$.
Moreover, an important issue arises, since the ranges of continuous linear maps are not necessarily closed subspaces. In order to preserve the duality between cyclic homology and cyclic cohomology we shall define the cokernel of a cyclic map $T : \mathcal{A}^\natural \to \mathcal{B}^\natural$ as the quotient of $\mathcal{B}^\natural$ by the closure of the range of T. In a dual manner, the kernel of the transposed map $T^t : \mathcal{B}^\sharp \to \mathcal{A}^\sharp$ is automatically closed and is the dual of the above.

3. Artin Motives and noncommutative spaces

In this section we show how certain categories of motives can be embedded faithfully into categories of noncommutative spaces. We first recall some general facts about motives.
Historically, the roots of the idea of motives introduced by Grothendieck can be found in the L-functions $L(X, s)$ of algebraic varieties, which can be written as an alternating product

$$L(X,s) = \prod_{i=0}^{n} L(H^i(X), s)^{(-1)^{i+1}}. \tag{3.1}$$

Typically, such combined L-function as (3.1) tends to have infinitely many poles, corresponding to the zeros of the factors in the denominator, whereas in isolating a single factor $L(H^i(X), s)$ one avoids this problem and also gains the possibility of having a nice functional equation. It is convenient to think of such a factor $L(H^i(X), s)$ as the L-function, not of a variety, but of a pure motive of weight i, viewed as a direct summand of the variety X.
In algebraic geometry, the various constructions of categories of pure motives of algebraic varieties are obtained by replacing the (non additive) category $\mathcal{V}_{\mathbb{K}}$ of algebraic varieties over a field $\mathbb{K}$ by a (pseudo)abelian category $\mathcal{M}_{\mathbb{K}}$ constructed in three steps (*cf.* [45]): (1) Enlarging the set of morphisms of algebraic varieties to correspondences by algebraic cycles (modulo a suitable equivalence relation); (2) Adding the ranges of projectors to the collection of objects; (3) Adding the “non-effective” Tate motives.

3.1. **Complex coefficients.** Since in the following we deal with reductions of motives by Hecke characters which are complex valued, we need to say a few words about coefficients. In fact, we are interested in considering categories of motives $\mathcal{M}_{\mathbb{K},\mathbb{E}}$ over a field $\mathbb{K}$ with coefficients in an extension $\mathbb{E}$ of $\mathbb{Q}$. This may sometime be a finite extension, but we are also interested in treating the case where $\mathbb{E} = \mathbb{C}$, *i.e.* the category $\mathcal{M}_{\mathbb{K},\mathbb{C}}$ of motives over $\mathbb{K}$ with complex coefficients.
By this we mean the category obtained as follows. We let $\mathcal{M}_{\mathbb{K}}$ denote, as above the (pseudo)abelian category of pure motives over $\mathbb{K}$. We then define $\mathcal{M}_{\mathbb{K},\mathbb{C}}$ as the (pseudo)abelian envelope of the additive category with $Obj(\mathcal{M}_{\mathbb{K}}) \subset Obj(\mathcal{M}_{\mathbb{K},\mathbb{C}})$ and morphisms

$$\mathrm{Hom}_{\mathcal{M}_{\mathbb{K},\mathbb{C}}}(M,N) = \mathrm{Hom}_{\mathcal{M}_{\mathbb{K}}}(M,N) \otimes \mathbb{C}. \tag{3.2}$$

Notice that, having modified the morphisms to (3.2) one in general will have to add new objects to still have a (pseudo)abelian category.

Notice that the notion of motives with coefficients that we use here is compatible with the apparently different notion often adopted in the literature, where motives with coefficients are defined for $\mathbb{E}$ a *finite* extension of $\mathbb{K}$, *cf.* [53] and [28] §2.1. In this setting one says that a motive M over $\mathbb{K}$ has coefficients in $\mathbb{E}$ if there is a homomorphism $\mathbb{E} \to \mathrm{End}(M)$. This description is compatible with defining, as we did above, $\mathcal{M}_{\mathbb{K},\mathbb{E}}$ as the smallest (pseudo)abelian category containing the pure motives over $\mathbb{K}$ with morphisms $\mathrm{Hom}_{\mathcal{M}_{\mathbb{K},\mathbb{E}}}(M,N) = \mathrm{Hom}_{\mathcal{M}_{\mathbb{K}}}(M,N) \otimes \mathbb{E}$. This is shown in [28] §2.1: first notice that to a motive M over $\mathbb{K}$ one can associate an element of $\mathcal{M}_{\mathbb{K},\mathbb{E}}$ by taking $M \otimes \mathbb{E}$, with the obvious $\mathbb{E}$-module structure. If M is endowed with a map $\mathbb{E} \to \mathrm{End}(M)$, then the corresponding object in $\mathcal{M}_{\mathbb{K},\mathbb{E}}$ is the range of the projector onto the largest direct summand on which the two structures of $\mathbb{E}$-module agree. Thus, while for a finite extension $\mathbb{E}$, both notions of "motives with coefficients" make sense and give rise to the same category $\mathcal{M}_{\mathbb{K},\mathbb{E}}$, for an infinite extension like $\mathbb{C}$ only the definition we gave above makes sense.

3.2. **Artin motives.** Since our main motivation comes from the adèle class space of [14], we focus on zero-dimensional examples, like the Bost–Connes system of [7] (whose dual system gives the adèle class space). Thus, from the point of view of motives, we consider mostly the simpler case of Artin motives. We return to discuss some higher dimensional cases in Sections 6 and 7.
Artin motives are motives of zero-dimensional smooth algebraic varieties over a field $\mathbb{K}$ and they provide a geometric counterpart for finite dimensional linear representations of the absolute Galois group of $\mathbb{K}$.
We recall some facts about the category of Artin motives (*cf.* [1], [53]). We consider in particular the case where $\mathbb{K}$ is a number field, with an algebraic closure $\bar{\mathbb{K}}$ and with the absolute Galois group $G = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$. We fix an embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ and write $X(\mathbb{C})$ for $\sigma X(\mathbb{C})$.
Let $\mathcal{V}^0_{\mathbb{K}}$ denotes the category of zero-dimensional smooth projective varieties over $\mathbb{K}$. For $X \in Obj(\mathcal{V}^0_{\mathbb{K}})$ we denote by $X(\bar{\mathbb{K}})$ the set of algebraic points of X. This is a finite set on which G acts continuously. Given X and Y in $\mathcal{V}^0_{\mathbb{K}}$ one defines $M(X,Y)$ to be the finite dimensional $\mathbb{Q}$-vector space

$$M(X,Y) := \mathrm{Hom}_G(\mathbb{Q}^{X(\bar{\mathbb{K}})}, \mathbb{Q}^{Y(\bar{\mathbb{K}})}) = (\mathbb{Q}^{X(\bar{\mathbb{K}}) \times Y(\bar{\mathbb{K}})})^G. \tag{3.3}$$

Thus, a (virtual) correspondence $Z \in M(X,Y)$ is a formal linear combination of connected components of $X \times Y$ with coefficients in $\mathbb{Q}$. These are identified with $\mathbb{Q}$-valued G-invariant functions on $X(\bar{\mathbb{K}}) \times Y(\bar{\mathbb{K}})$ by $Z = \sum a_i Z_i \mapsto \sum a_i \chi_{Z_i(\bar{\mathbb{K}})}$, where χ is the characteristic function, and the Z_i are connected components of $X \times Y$.

We denote by $\mathcal{CV}^0_{\mathbb{K}}$ the (pseudo)abelian envelope of the additive category with objects $Obj(\mathcal{V}^0_{\mathbb{K}})$ and morphisms as in (3.3). Thus, $\mathcal{CV}^0_{\mathbb{K}}$ is the smallest (pseudo)abelian category with $Obj(\mathcal{V}^0_{\mathbb{K}}) \subset Obj(\mathcal{CV}^0_{\mathbb{K}})$ and morphisms $\mathrm{Hom}_{\mathcal{CV}^0_{\mathbb{K}}}(X,Y) = M(X,Y)$, for $X, Y \in Obj(\mathcal{V}^0_{\mathbb{K}})$. In this case it is in fact an abelian category.
Moreover, there is a fiber functor

$$\omega : X \mapsto H^0(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}^{X(\bar{\mathbb{K}})}, \tag{3.4}$$

which gives an identification of the category $\mathcal{CV}^0_{\mathbb{K}}$ with the category of finite dimensional $\mathbb{Q}$-linear representations of $G = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$.

Notice that, in the case of Artin motives, the issue of different properties of the category of motives depending on the different possible choices of the equivalence relation on cycles (numerical, homological, rational) does not arise. In fact, in dimension zero there is no equivalence relation needed on the cycles in the product, *cf. e.g.* [1]. Nonetheless, one sees that passing to the category $\mathcal{CV}^0_{\mathbb{K}}$ requires adding new objects in order to have a (pseudo)abelian category. One can see this in a very simple example, for $K = \mathbb{Q}$. Consider the field $\mathbb{L} = \mathbb{Q}(\sqrt{2})$. Then consider the one dimensional non-trivial representation of the Galois group G that factors through the character of order two of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$. This representation does not correspond to an object in $Obj(\mathcal{V}^0_{\mathbb{Q}})$ but can be obtained as the range of the projector $p = (1-\sigma)/2$, where σ is the generator of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$. Namely it is a new object in $Obj(\mathcal{CV}^0_{\mathbb{Q}})$.

3.3. **Endomotives.** We now show that there is a very simple way to give a faithful embedding of the category $\mathcal{CV}^0_{\mathbb{K}}$ of Artin motives in an additive category of noncommutative spaces. The construction extends to arbitrary coefficients $\mathcal{CV}^0_{\mathbb{K},\mathbb{E}}$ (in the sense of §3.1) and in particular to $\mathcal{CV}^0_{\mathbb{K},\mathbb{C}}$. The objects of the new category are obtained from projective systems of Artin motives with actions of semigroups of endomorphisms. For this reason we will call them *endomotives*. We construct a category $\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}$ of algebraic endomotives, which makes sense over any field $\mathbb{K}$. For $\mathbb{K}$ a number field, this embeds in a category $C^*\mathcal{V}^0_{\mathbb{K}}$ of analytic endomotives, which is defined in the context of C^*-algebras.

One can describe the category $\mathcal{V}^0_{\mathbb{K}}$ as the category of reduced finite dimensional commutative algebras over $\mathbb{K}$. The correspondences for Artin motives given by (3.3) can also be described in terms of bimodules (*cf.* (3.44) and (3.45) below). We consider a specific class of noncommutative algebras to which all these notions extend naturally in the zero-dimensional case.

The noncommutative algebras we deal with are of the form

$$\mathcal{A}_{\mathbb{K}} = A \rtimes S, \tag{3.5}$$

where A is an inductive limit of reduced finite dimensional commutative algebras over the field $\mathbb{K}$ and S is a unital abelian semigroup of algebra endomorphisms $\rho : A \to A$. The algebra A is unital and, for $\rho \in S$, the image $e = \rho(1) \in A$ is an idempotent. We assume that ρ is an isomorphism of A with the compressed algebra eAe.

The crossed product algebra $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ is obtained by adjoining to A new generators U_ρ and U_ρ^*, for $\rho \in S$, with algebraic rules given by

$$\begin{array}{lll} U_\rho^* U_\rho = 1, & U_\rho U_\rho^* = \rho(1), & \forall \rho \in S \\ U_{\rho_1 \rho_2} = U_{\rho_1} U_{\rho_2}, & U^*_{\rho_2 \rho_1} = U^*_{\rho_1} U^*_{\rho_2}, & \forall \rho_j \in S \\ U_\rho \, x = \rho(x) \, U_\rho, & x \, U_\rho^* = U_\rho^* \, \rho(x), & \forall \rho \in S, \, \forall x \in A. \end{array} \tag{3.6}$$

Since S is abelian these rules suffice to show that $A \rtimes S$ is the linear span of the monomials $U_{\rho_1}^* \, a \, U_{\rho_2}$ for $a \in A$ and $\rho_j \in S$. In fact one gets

$$U_{\rho_1}^* \, a \, U_{\rho_2} \, U_{\rho_3}^* \, b \, U_{\rho_4} = U_{\rho_1\rho_3}^* \, \rho_3(a) \, \rho_2\rho_3(1) \, \rho_2(b) \, U_{\rho_2\rho_4} \,, \quad \forall \, \rho_j \in S \,, a, b \in A \,. \tag{3.7}$$

By hypothesis, A is an inductive limit of reduced finite dimensional commutative algebras A_α over $\mathbb{K}$. Thus, the construction of the algebraic semigroup crossed product $\mathcal{A}_{\mathbb{K}}$ given above is determined by assigning the following data: a projective system $\{X_\alpha\}_{\alpha \in I}$ of varieties in $\mathcal{V}^0_{\mathbb{K}}$, with morphisms $\xi_{\alpha,\beta} : X_\beta \to X_\alpha$, and a countable indexing set I. The graphs $\Gamma(\xi_{\alpha,\beta})$ of these morphisms are G-invariant subsets of $X_\beta(\bar{\mathbb{K}}) \times X_\alpha(\bar{\mathbb{K}})$. We denote by X the projective limit, that is, the pro-variety

$$X = \varprojlim_{\alpha} X_\alpha, \tag{3.8}$$

with maps $\xi_\alpha : X \to X_\alpha$.

The endomorphisms ρ give isomorphisms

$$\tilde{\rho} : X \to X^e$$

of X with the subvariety X^e associated to the idempotent $e = \rho(1)$, *i.e.* corresponding to the compressed algebra eAe.

The noncommutative space described by the algebra $\mathcal{A}_{\mathbb{K}}$ of (3.5) is the quotient of $X(\bar{\mathbb{K}})$ by the action of S, *i.e.* of the $\tilde{\rho}$.

By construction, the Galois group G acts on $X(\bar{\mathbb{K}})$ by composition. Thus, if we view the elements of $X(\bar{\mathbb{K}})$ as characters, *i.e.* as $\mathbb{K}$-algebra homomorphisms $\chi : A \to \bar{\mathbb{K}}$, we can write the Galois action of G as

$$\alpha(\chi) = \alpha \circ \chi : A \to \bar{\mathbb{K}}, \quad \forall \alpha \in G = \mathrm{Aut}_{\mathbb{K}}(\bar{\mathbb{K}}), \quad \forall \chi \in X(\bar{\mathbb{K}}) \,. \tag{3.9}$$

This action commutes with the maps $\tilde{\rho}$. In fact, given a character $\chi \in X(\bar{\mathbb{K}})$ of A, one has $\chi(\rho(1)) \in \{0,1\}$ (with $\chi(\rho(1)) = 1 \Leftrightarrow \chi \in X^e$) and $(\alpha \circ \chi) \circ \rho = \alpha \circ (\chi \circ \rho)$.

Thus, the whole construction is G-equivariant. Notice that this does not mean that G acts by automorphisms on $\mathcal{A}_{\mathbb{K}}$, but it does act on $X(\bar{\mathbb{K}})$, and on the noncommutative quotient $X(\bar{\mathbb{K}})/S$.

The purely algebraic construction of the crossed product algebra $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ and of the action of G and S on $X(\bar{\mathbb{K}})$ given above makes sense also when $\mathbb{K}$ has positive characteristic. One can extend the basic notions of Artin motives in the above generality. For the resulting crossed product algebras $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ one can define the set of correspondences $M(\mathcal{A}_{\mathbb{K}}, \mathcal{B}_{\mathbb{K}})$ using $\mathcal{A}_{\mathbb{K}}$–$\mathcal{B}_{\mathbb{K}}$-bimodules (which are finite and projective as right modules). One can then use Lemma 2.1 to obtain a realization in the abelian category of $\mathbb{K}(\Lambda)$-modules. This may be useful in order to consider systems like the generalization of the BC system of [7] to rank one Drinfeld modules recently studied by Benoit Jacob [36].

In the following, we concentrate mostly on the case of characteristic zero, where $\mathbb{K}$ is a number field. We fix as above an embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ and we take $\bar{\mathbb{K}}$ to be the algebraic closure of $\sigma(\mathbb{K}) \subset \mathbb{C}$ in $\mathbb{C}$. We then let

$$A_{\mathbb{C}} = A \otimes_{\mathbb{K}} \mathbb{C} = \varinjlim_{\alpha} A_\alpha \otimes_{\mathbb{K}} \mathbb{C}, \quad \mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{C} = A_{\mathbb{C}} \rtimes S. \tag{3.10}$$

We define an embedding of algebras $A_{\mathbb{C}} \subset C(X)$ by

$$a \in A \to \hat{a}, \quad \hat{a}(\chi) = \chi(a) \quad \forall \chi \in X. \tag{3.11}$$

Notice that A is by construction a subalgebra $A \subset C(X)$ but it is not in general an involutive subalgebra for the canonical involution of $C(X)$. It is true, however, that the $\mathbb{C}$-linear span $A_{\mathbb{C}} \subset C(X)$ is an involutive subalgebra of $C(X)$ since it is the algebra of locally constant functions on X. The C^*-completion $R = C(X)$ of $A_{\mathbb{C}}$ is an abelian AF

C^*-algebra. We then let $\bar{\mathcal{A}} = C(X) \rtimes S$ be the C^*-crossed product of the C^*-algebra $C(X)$ by the semi-group action of S. It is the C^*-completion of the algebraic crossed product $A_{\mathbb{C}} \rtimes S$ and it is defined by the algebraic relations (3.6) with the involution which is apparent in the formulae (*cf.* [41] [42] and the references there for the general theory of crossed products by semi-groups).

It is important, in order to work with cyclic (co)homology as we do in Section 4 below, to be able to restrict from C^*-algebras $C(X) \rtimes S$ to a canonical dense subalgebra

$$\mathcal{A} = C^\infty(X) \rtimes_{alg} S \subset \bar{\mathcal{A}} = C(X) \rtimes S \tag{3.12}$$

where $\mathcal{A} = C^\infty(X) \subset C(X)$ is the algebra of locally constant functions, and the crossed product $C^\infty(X) \rtimes S$ is the algebraic one. It is to this category of (smooth) algebras, rather than to that of C^*-algebras, that cyclic homology applies.

Proposition 3.1. *1) The action* (3.9) *of G on $X(\bar{\mathbb{K}})$ defines a canonical action of G by automorphisms of the C^*-algebra $\bar{\mathcal{A}} = C(X) \rtimes S$, preserving globally $C(X)$ and such that, for any pure state φ of $C(X)$,*

$$\alpha\,\varphi(a) = \varphi(\alpha^{-1}(a))\,, \quad \forall\, a \in A\,, \quad \alpha \in G. \tag{3.13}$$

2) When the Artin motives A_α are abelian and normal, the subalgebras $A \subset C(X)$ and $\mathcal{A}_{\mathbb{K}} \subset \bar{\mathcal{A}} = C(X) \rtimes S$ are globally invariant under the action of G and the states φ of $R \rtimes S$ induced by pure states of R fulfill

$$\alpha\,\varphi(a) = \varphi(\theta(\alpha)(a))\,, \quad \forall\, a \in \mathcal{A}_{\mathbb{K}}\,, \quad \theta(\alpha) = \alpha^{-1}\,, \quad \forall\, \alpha \in G^{ab} = G/[G,G] \tag{3.14}$$

Proof. 1) The action of the Galois group G on $C(X)$ is given, using the notations of (3.9), by

$$\alpha(h)(\chi) = h(\alpha^{-1}(\chi))\,, \quad \forall\, \chi \in X(\bar{\mathbb{K}})\,, \quad h \in C(X)\,. \tag{3.15}$$

For $a \in A \subset C(X)$ and φ given by the evaluation at $\chi \in X(\bar{\mathbb{K}})$ one gets

$$\alpha\,\varphi(a) = (\alpha \circ \chi)(a) = \alpha(\chi)(a) = \hat{a}(\alpha(\chi)) = \varphi(\alpha^{-1}(\hat{a}))$$

using (3.15). Since the action of G on X commutes with the action of S, it extends to the crossed product $C(X) \rtimes S$.

2) Let us show that $A \subset C(X)$ is globally invariant under the action of $\alpha \in G$. By construction $A = \cup A_\beta$ where the finite dimensional $\mathbb{K}$-algebras A_β are of the form

$$A_\beta = \prod \mathbb{L}_i$$

where each $\mathbb{L}_i$ is a normal abelian extension of $\mathbb{K}$. In particular the automorphism of $\mathbb{L}_i$ defined by

$$\tilde{\alpha} = \chi^{-1} \circ \alpha \circ \chi$$

is independent of the choice of the embedding $\chi : L_i \to \bar{\mathbb{K}}$. These assemble into an automorphism $\tilde{\alpha}$ of A_β such that for any $\chi \in X$ one has $\chi \circ \tilde{\alpha} = \alpha \circ \chi$. One has by construction for $a \in A$ with $\hat{a}$ given by (3.11), and $b = \tilde{\alpha}(a)$, the equality

$$\hat{b}(\chi) = \chi(b) = \chi\tilde{\alpha}(a) = \alpha\chi(a) = \alpha^{-1}(\hat{a})(\chi)$$

so that $b = \alpha^{-1}(\hat{a})$ and one gets the required global invariance for A. The invariance of $\mathcal{A}_{\mathbb{K}}$ follows. Finally note that $\theta(\alpha) = \alpha^{-1}$ is a group homomorphism for the abelianized groups. ∎

The intertwining property (3.14) is (part of) the *fabulous state* property of [17], [21], [22]. The subalgebra $\mathcal{A}_{\mathbb{K}}$ then qualifies as a rational subalgebra of $\bar{\mathcal{A}} = C(X) \rtimes S$ in the normal abelian case. In the normal non-abelian case the global invariance of A no longer holds. One can define an action of G on A but only in a non-canonical manner which depends on the choices of base points. Moreover in the non-abelian case one should not confuse the contravariant intertwining stated in (3.13) with the covariant intertwining of [17].

Example 3.2. The prototype example of the data described above is the Bost–Connes system. In this case we work over $\mathbb{K} = \mathbb{Q}$, and we consider the projective system of the $X_n = \mathrm{Spec}(A_n)$, where $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ is the group ring of the abelian group $\mathbb{Z}/n\mathbb{Z}$. The inductive limit is the group ring $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of $\mathbb{Q}/\mathbb{Z}$. The endomorphism ρ_n associated to an element $n \in S$ of the (multiplicative) semigroup $S = \mathbb{N} = \mathbb{Z}_{>0}$ is given on the canonical basis $e_r \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$, $r \in \mathbb{Q}/\mathbb{Z}$, by

$$\rho_n(e_r) = \frac{1}{n} \sum_{ns=r} e_s \tag{3.16}$$

The direct limit C^*-algebra is $R = C^*(\mathbb{Q}/\mathbb{Z})$ and the crossed product $\bar{\mathcal{A}} = R \rtimes S$ is the C^*-algebra of the BC-system. One is in the normal abelian situation, so that proposition 3.1 applies. The action of the Galois group G of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ factorizes through the cyclotomic action and coincides with the symmetry group of the BC-system. In fact, the action on $X_n = \mathrm{Spec}(A_n)$ is obtained by composing a character $\chi : A_n \to \bar{\mathbb{Q}}$ with an element g of the Galois group G. Since χ is determined by the n-th root of unity $\chi(e_{1/n})$, we just obtain the cyclotomic action. The subalgebra $\mathcal{A}_{\mathbb{Q}} \subset \bar{\mathcal{A}}$ coincides with the rational subalgebra of [7]. Finally the fabulous state property of extreme KMS_∞ states follows from Proposition 3.1 and the fact (*cf.* [7]) that such states are induced from pure states of R.

Remark 3.3. Fourier transform gives an algebra isomorphism

$$C^*(\mathbb{Q}/\mathbb{Z}) \sim C(\hat{\mathbb{Z}})$$

which is compatible with the projective system $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, while the endomorphisms ρ_n are given by division by n. It is thus natural to consider the example given by $Y_n = \mathrm{Spec}(B_n)$ where B_n is simply the algebra of functions $C(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$, with the corresponding action of $S = \mathbb{N}$. It is crucial to note that the corresponding system is *not isomorphic* to the one described above in Example 3.2. Indeed what happens is that the corresponding action of the Galois group G is now trivial. More generally, one could construct examples of that kind by taking S to be the ring of integers $\mathcal{O}$ of a number field and consider its action on the quotients $\mathcal{O}/J$, by ideals J. The action is transitive at each finite level, though the action of $\mathcal{O}$ on the profinite completion $\hat{\mathcal{O}}$ is not. The resulting C^*-algebra is $C(\hat{\mathcal{O}}) \rtimes \mathcal{O}$. The basic defect of these examples is that they do not yield the correct Galois action.

Example 3.4. As a corollary of the results of [21] one can show that given an imaginary quadratic extension $\mathbb{K}$ of $\mathbb{Q}$ there is a canonically associated noncommutative space of the form (3.5), with the correct non-trivial Galois action of the absolute Galois group of $\mathbb{K}$. The normal abelian hypothesis of Proposition 3.1 is fulfilled. The rational subalgebra obtained from Proposition 3.1 is the same as the rational subalgebra of [21] and the structure of noncommutative Artin motive is uniquely dictated by the structure of inductive limit of finite dimensional $\mathbb{K}$-algebras of the rational subalgebra of [21]. The construction of the system (*cf.* [21]) is much more elaborate than in Remark 3.3.

We now describe a large class of examples of noncommutative spaces (3.5), obtained as semigroup actions on projective systems of Artin motives, that arise from self-maps of algebraic varieties.

We consider a pointed algebraic variety (Y, y_0) over $\mathbb{K}$ and a countable unital abelian semigroup S of finite endomorphisms of (Y, y_0), unramified over $y_0 \in Y$. Finite endomorphisms have a well defined degree. For $s \in S$ we let $\deg s$ denote the degree of s. We construct a projective system of Artin motives X_s over $\mathbb{K}$ from these data as follows. For $s \in S$, we set

$$X_s = \{y \in Y : s(y) = y_0\}. \tag{3.17}$$

For a pair $s, s' \in S$, with $s' = sr$, the map $\xi_{s,s'} : X_{sr} \to X_s$ is given by

$$X_{sr} \ni y \mapsto r(y) \in X_s. \tag{3.18}$$

This defines a projective system indexed by the semigroup S itself with partial order given by divisibility. We let $X = \varprojlim_s X_s$.
Since $s(y_0) = y_0$, the base point y_0 defines a component Z_s of X_s for all $s \in S$. Let $\xi^{-1}_{s,s'}(Z_s)$ be the inverse image of Z_s in $X_{s'}$. It is a union of components of $X_{s'}$. This defines a projection e_s onto an open and closed subset X^{e_s} of the projective limit X.

Proposition 3.5. *The semigroup S acts on the projective limit X by partial isomorphisms $\beta_s : X \to X^{e_s}$ defined by the property that*

$$\xi_{su}(\beta_s(x)) = \xi_u(x), \ \forall u \in S, \forall x \in X. \tag{3.19}$$

Proof. The map β_s is well defined as a map $X \to X$ by (3.19), since the set $\{su : u \in S\}$ is cofinal and $\xi_u(x) \in X_{su}$. In fact, we have $su\xi_u(x) = s(y_0) = y_0$. The image of β_s is in X^{e_s}, since by (3.19) we have $\xi_s(\beta_s(x)) = \xi_1(x) = y_0$. We show that β_s is an isomorphism of X with X^{e_s}. The inverse map is given by

$$\xi_u(\beta_s^{-1}(x)) = \xi_{su}(x), \ \forall x \in X^{e_s}, \forall u \in S. \tag{3.20}$$

In fact, for $x \in X^{e_s}$, we have $\xi_{su}(x) \in X_u$. □

The corresponding algebra morphisms ρ_s are then given by

$$\rho_s(f)(x) = f(\beta_s^{-1}(x)), \ \forall x \in X^{e_s}, \quad \rho_s(f)(x) = 0, \ \forall x \notin X^{e_s}. \tag{3.21}$$

In this class of examples one has an "equidistribution" property, by which the uniform normalized counting measures μ_s on X_s are compatible with the projective system and define a probability measure on the limit X. Namely, one has

$$\xi_{s',s}\mu_s = \mu_{s'}, \ \forall s, s' \in S. \tag{3.22}$$

This follows from the fact that the number of preimages of a point under $s \in S$ is equal to $\deg s$.

Example 3.6. Let Y be the affine group scheme $\mathbb{G}_m$ (the multiplicative group). Let S be the semigroup of non-zero endomorphisms of $\mathbb{G}_m$. These correspond to maps of the form $u \mapsto u^n$ for some non-zero $n \in \mathbb{Z}$. In fact one can restrict to $\mathbb{N} \subset \mathbb{Z} \setminus \{0\}$.

Proposition 3.7. *The construction of Example 3.6, applied to the pointed algebraic variety $(\mathbb{G}_m(\mathbb{Q}), 1)$, gives the BC system.*

Proof. We restrict to the semi-group $S = \mathbb{N}$ and determine X_n for $n \in \mathbb{N}$. One has by construction $X_n = \operatorname{Spec}(A_n)$ where A_n is the quotient of the algebra $\mathbb{Q}[u(n), u(n)^{-1}]$ of Laurent polynomials by the relation $u(n)^n = 1$. For $n|m$ the map $\xi_{m,n}$ from X_m to X_n is given by the algebra homomorphism that sends the generators $u(n)^{\pm 1} \in A_n$ to

$u(m)^{\pm a} \in A_m$ with $a = m/n$. Thus, one obtains an isomorphism of the inductive limit algebra A with the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of the form

$$\iota \,:\, \varinjlim A_n \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]\,, \quad \iota(u(n)) = e_{\frac{1}{n}}\,. \tag{3.23}$$

Let us check that the partial isomorphisms ρ_n (3.16) of the group ring correspond to those given by (3.19) under the isomorphism ι. We identify the projective limit X with the space of characters of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ with values in $\bar{\mathbb{Q}}$. The projection $\xi_m(x)$ is simply given by the restriction of $x \in X$ to the subalgebra A_m. Let us compute the projections of the composition of $x \in X$ with the endomorphism ρ_n (3.16). One has

$$x(\rho_n(e_r)) = \frac{1}{n} \sum_{ns=r} x(e_s)\,.$$

This is non-zero iff the restriction $x|_{A_n}$ is the trivial character, *i.e.* iff $\xi_n(x) = 1$. Moreover, in that case one has

$$x(\rho_n(e_r)) = x(e_s)\,, \quad \forall\, s\,, \quad ns = r\,,$$

and in particular

$$x(\rho_n(e_{\frac{1}{k}})) = x(e_{\frac{1}{nk}}). \tag{3.24}$$

For $k|m$ the inclusion $X_k \subset X_m$ is given at the algebra level by the homomorphism

$$j_{k,m} \,:\, A_m \to A_k\,, \quad j_{k,m}(u(m)) = u(k).$$

Thus, one can rewrite (3.24) as

$$x \circ \rho_n \circ j_{k,nk} = x|_{A_{nk}}\,, \tag{3.25}$$

that is, one has

$$\xi_{nk}(x) = \xi_k(x \circ \rho_n),$$

which, using (3.20), gives the desired equality of the ρ's of (3.16) and (3.21). $\square$

The construction of Example 3.6 continues to make sense for $\mathbb{G}_m(\mathbb{K})$ for other fields, *e.g.* for a field of positive characteristic. In this case one obtains new systems, different from BC.

Example 3.8. Let Y be an elliptic curve defined over $\mathbb{K}$. Let S be the semigroup of non-zero endomorphisms of Y. This gives rise to an example in the general class described above. When the elliptic curve has complex multiplication, this gives rise to a system which, in the case of a maximal order, agrees with the one constructed in [21]. In the case without complex multiplication, this provides an example of a system where the Galois action does not factor through an abelian quotient.

In general, we say that a system (X_α, S) with X_α a projective system of Artin motives and S a semigroup of endomorphisms of the limit X is *uniform* if the normalized counting measures μ_α on X_α satisfy

$$\xi_{\beta,\alpha}\mu_\alpha = \mu_\beta \tag{3.26}$$

as above.

In order to define morphisms in the category of algebraic endomotives, it is best to encode the datum (X_α, S) by the algebraic groupoid obtained as the crossed product $\mathcal{G} = X \rtimes S$. We let $\tilde{S}$ be the (Grothendieck) group of the abelian semigroup S and may assume, using the injectivity of the partial action of S, that S embedds in $\tilde{S}$. Then the action of S on X extends to a partial action of $\tilde{S}$. For $s = \rho_1/\rho_2 \in \tilde{S}$ the projections

$$E(s) = \rho_1^{-1}(\rho_2(1)\rho_1(1)) \quad \text{and} \quad F(s) = \rho_2^{-1}(\rho_2(1)\rho_1(1)) \tag{3.27}$$

only depend on s and the map $(\rho_2)^{-1}\rho_1$ is an isomorphism of the reduced algebras

$$s = \rho_2^{-1}\rho_1 \,:\, A_{E(s)} \to A_{F(s)}\,. \tag{3.28}$$

One checks that

$$E(s^{-1}) = F(s) = s(E(s))\,, \quad F(s\,s') \geq F(s)s(F(s'))\,, \quad \forall s,s' \in \tilde{S}\,. \tag{3.29}$$

The algebraic groupoid $\mathcal{G}$ is the disjoint union

$$\mathcal{G} = \sqcup_{s\in\tilde{S}}\, X^{F(s)} \tag{3.30}$$

which corresponds to the algebra $\oplus_{s\in\tilde{S}}A_{F(s)}$, which is the commutative algebra direct sum of the reduced algebras $A_{F(s)}$. The range and source maps are given by the natural projection from $\mathcal{G}$ to X and by its composition with the antipode $\mathbf{S}$ which is given, at the algebra level, by

$$\mathbf{S}(a)_s = s(a_{s^{-1}})\,, \quad \forall s \in \tilde{S}\,. \tag{3.31}$$

The composition in the groupoid corresponds to the product of monomials

$$a\,U_s\,b\,U_t = \, a\,s(b)\,U_{st}\,. \tag{3.32}$$

Given algebraic endomotives (X_α, S) and $(X'_{\alpha'}, S')$, with associated groupoids $\mathcal{G} = \mathcal{G}(X_\alpha, S)$, $\mathcal{G}' = \mathcal{G}(X'_\alpha, S')$ a geometric correspondence is given by a $\mathcal{G}-\mathcal{G}'$-space Z where the right action of $\mathcal{G}'$ fulfills a suitable *etale* condition which we now describe. Given a space such as $\mathcal{G}$, *i.e.* a disjoint union $Z = \operatorname{Spec} C$ of zero dimensional pro-varieties over $\mathbb{K}$, a *right action* of $\mathcal{G}$ on Z is specified by a map $g : Z \to X$ and a collection of partial isomorphisms

$$z \in g^{-1}(F(s)) \mapsto z\cdot s \in g^{-1}(E(s)) \tag{3.33}$$

fulfilling the obvious rules for a partial action of the abelian group $\tilde{S}$. More precisely one requires that

$$g(z\cdot s) = g(z)\cdot s\,, \quad z\cdot(ss') = (z\cdot s)\cdot s' \text{ on } g^{-1}(F(s)\cap s(F(s'))) \tag{3.34}$$

where we denote by $x \mapsto x\cdot s$ the given partial action of $\tilde{S}$ on X.
One checks that such an action gives to the $\mathbb{K}$-linear space C (for $Z = \operatorname{Spec} C$) a structure of right module over $\mathcal{A}_{\mathbb{K}}$.

Definition 3.9. *We say that the action of $\mathcal{G}$ on Z is* etale *if the corresponding module C is finite and projective over $\mathcal{A}_{\mathbb{K}}$.*

Given algebraic endomotives (X_α, S) and $(X'_{\alpha'}, S')$, an etale correspondence is a $\mathcal{G}(X_\alpha, S)-\mathcal{G}(X'_{\alpha'}, S')$ space Z such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is etale. We define the $\mathbb{Q}$-linear space of (virtual) correspondences $\operatorname{Corr}((X_\alpha, S), (X'_{\alpha'}, S'))$ as formal linear combinations $U = \sum_i a_i Z_i$ of etale correspondences Z_i modulo isomorphism and the equivalence $Z \sqcup Z' \sim Z + Z'$ where $Z\sqcup Z'$ is the disjoint union. The composition of correspondences is given by the fiber product over a groupoid. Namely, for algebraic endomotives (X_α, S), $(X'_{\alpha'}, S')$, $(X''_{\alpha''}, S'')$ and correspondences Z and W, their composition is given by

$$Z \circ W = Z \times_{\mathcal{G}'} W, \tag{3.35}$$

which is the fiber product over the groupoid $\mathcal{G}' = \mathcal{G}(X'_{\alpha'}, S')$. We can then form a category of algebraic endomotives in the following way.

Definition 3.10. *The category $\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}$ of* algebraic endomotives *with coefficients in $\mathbb{E}$ is the (pseudo)abelian category generated by the following objects and morphisms. The objects are uniform systems (X_α, S) as above, with X_α Artin motives over $\mathbb{K}$. For $M = (X_\alpha, S)$ and $M' = (X'_{\alpha'}, S')$ we set*

$$\operatorname{Hom}_{\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}}(M, M') = \operatorname{Corr}((X_\alpha, S), (X'_{\alpha'}, S')) \otimes \mathbb{E}. \tag{3.36}$$

When taking points over $\bar{\mathbb{K}}$, the algebraic endomotives described above yield 0-dimensional singular quotient spaces $X(\bar{\mathbb{K}})/S$, which can be described through locally compact etale groupoids $\mathcal{G}(\bar{\mathbb{K}})$ (and the associated crossed product C^*-algebras $C(X(\bar{\mathbb{K}})) \rtimes S$). This gives rise to a corresponding category of analytic endomotives given by locally compact etale groupoids and geometric correspondences. This category provides the data for the construction in Section 4 below.

Let X be a totally disconnected compact space, S an abelian semigroup of homeomorphisms $X^s \to X$, $x \mapsto x \cdot s$, with X^s a closed and open subset of X, and μ a probability measure on X with the property that the Radon–Nikodym derivatives

$$\frac{ds^*\mu}{d\mu} \tag{3.37}$$

are locally constant functions on X. We let $\mathcal{G} = X \rtimes S$ be the corresponding etale locally compact groupoid, constructed in the same way as in the above algebraic case. The crossed product C^*-algebra $C(X(\bar{\mathbb{K}})) \rtimes S$ coincides with the C^*-algebra $C^*(\mathcal{G})$ of the groupoid G.

The notion of right (or left) action of $\mathcal{G}$ on a totally disconnected locally compact spaces $\mathcal{Z}$ is defined as in the algebraic case by (3.33) and (3.34). A right action of $\mathcal{G}$ on $\mathcal{Z}$ gives on the space $C_c(\mathcal{Z})$ of continuous functions with compact support on $\mathcal{Z}$ a structure of right module over $C_c(\mathcal{G})$ as in the algebraic case. When the fibers of the map g are discrete (countable) subsets of $\mathcal{Z}$ one can define on $C_c(\mathcal{Z})$ an inner product with values in $C_c(\mathcal{G})$ by

$$\langle \xi, \eta \rangle (x, s) = \sum_{z \in g^{-1}(x)} \bar{\xi}(z)\, \eta(z \circ s) \tag{3.38}$$

We define the notion of *etale* action by

Definition 3.11. *A right action of $\mathcal{G}$ on $\mathcal{Z}$ is* etale *if and only if the fibers of the map g are discrete and the identity is a compact operator in the right C^*-module $\mathcal{E}_{\mathcal{Z}}$ over $C^*(\mathcal{G})$ given by* (3.38).

An etale correspondence is a $\mathcal{G}(X_\alpha, S) - \mathcal{G}(X'_{\alpha'}, S')$ space $\mathcal{Z}$ such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is etale. We consider the $\mathbb{Q}$-vector space

$$\mathrm{Corr}((X, S, \mu), (X', S', \mu'))$$

of linear combinations of etale correspondences $\mathcal{Z}$ modulo the equivalence relation $\mathcal{Z} \cup \mathcal{Z}' = \mathcal{Z} + \mathcal{Z}'$ for disjoint unions.
For $M = (X, S, \mu)$, $M' = (X', S', \mu')$, and $M'' = (X'', S'', \mu'')$, the composition of correspondences

$$\mathrm{Corr}(M, M') \times \mathrm{Corr}(M', M'') \to \mathrm{Corr}(M, M'')$$

is given as above by the fiber product over $\mathcal{G}'$. In fact, a correspondence, in the sense above, gives rise to a bimodule $\mathcal{M}_{\mathcal{Z}}$ over the algebras $C(X) \rtimes S$ and $C(X') \rtimes S'$ and the composition of correspondences translates into the tensor product of bimodules.

Definition 3.12. *The category $C^*\mathcal{V}^0_{\mathbb{E}}$ of* analytic endomotives *is the (pseudo)abelian category generated by objects of the form (X, S, μ) with the properties listed above and morphisms given as follows. For $M = (X, S, \mu)$ and $M' = (X', S', \mu')$ we set*

$$\mathrm{Hom}_{C^*\mathcal{V}^0_{\mathbb{E}}}(M, M') = \mathrm{Corr}(M, M') \otimes \mathbb{E}. \tag{3.39}$$

It would also be possible to define a category where morphisms are given by KK-classes $KK(C^*(\mathcal{G}), C^*(\mathcal{G}')) \otimes \mathbb{E}$. The definition we gave above for the category $C^*\mathcal{V}^0_{\mathbb{E}}$ is more refined. In particular, we do not divide by homotopy equivalence. The associated KK-class $k(\mathcal{Z})$ to a correspondence $\mathcal{Z}$ is simply given by taking the equivalence class of the triple (E, ϕ, F) with (E, ϕ) given by the bimodule $\mathcal{M}_{\mathcal{Z}}$ with the trivial grading $\gamma = 1$ and the zero endomorphism $F = 0$. Since the correspondence is etale any endomorphism of the right C^*-module $\mathcal{M}_{\mathcal{Z}}$ on $C(\mathcal{X}') \rtimes S'$ is compact. Thus in particular $F^2 - 1 = -1$ is compact. This is exactly the reason why one had to require that the correspondence is etale. There is a functor $k : C^*\mathcal{V}^0_{\mathbb{E}} \to \mathcal{KK} \otimes \mathbb{E}$ for the category $\mathcal{KK}$ described in Section 2 above, which extends by $\mathbb{E}$-linearity the map $\mathcal{Z} \mapsto k(\mathcal{Z})$. The version of Definition 3.12 is preferable to the KK-formulation, since we want morphisms that will act on the cohomology theory that we introduce in Section 4 below.

Theorem 3.13. *The categories introduced above are related as follows.*

(1) *The map $\mathcal{G} \mapsto \mathcal{G}(\bar{\mathbb{K}})$ determines a tensor functor*

$$\mathcal{F} : \mathcal{EV}^0_{\mathbb{K},\mathbb{E}} \to C^*\mathcal{V}^0_{\mathbb{E}} \tag{3.40}$$

from algebraic to analytic endomotives.

(2) *The Galois group $G = \mathrm{Gal}(\overline{\sigma(\mathbb{K})}/\sigma(\mathbb{K}))$ acts by natural transformations of $\mathcal{F}$.*

(3) *The category $\mathcal{CV}^0_{\mathbb{K},\mathbb{E}}$ of Artin motives embeds as a full subcategory of $\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}$.*

(4) *The composite functor*

$$k \circ \mathcal{F} : \mathcal{EV}^0_{\mathbb{K},\mathbb{E}} \to \mathcal{KK} \otimes \mathbb{E} \tag{3.41}$$

maps the full subcategory $\mathcal{CV}^0_{\mathbb{K},\mathbb{E}}$ of Artin motives faithfully to the category $\mathcal{KK}_{G,\mathbb{E}}$ of G-equivariant KK-theory with coefficients in $\mathbb{E}$.

Proof. (1) The functor $\mathcal{F}$ defined by taking points over $\bar{\mathbb{K}}$ maps

$$\mathcal{F} : (X_\alpha, S) \mapsto (\mathcal{X}, S) \tag{3.42}$$

where $\mathcal{X} = X(\bar{\mathbb{K}}) = \varprojlim_\alpha X_\alpha(\bar{\mathbb{K}})$. If (X_α, S) is a uniform algebraic endomotive then $\mathcal{F}(X_\alpha, S) = (X, S, \mu)$ is a measured analytic endomotive with μ the projective limit of the normalized counting measures. On correspondences the functor gives

$$\mathcal{F} : Z \mapsto \mathcal{Z} = Z(\bar{\mathbb{K}}). \tag{3.43}$$

The obtained correspondence is etale since the corresponding right C^*-module is finite and projective as an induced module of the finite projective module of the algebraic situation. This is compatible with composition of correspondences since in both cases the composition is given by the fiber product over the middle groupoid (*cf.* (3.35)).

(2) We know from (3.9) that the group G acts by automorphisms on $\mathcal{F}(X_\alpha, S)$. This action is compatible with the morphisms. This shows that G acts on the functor $\mathcal{F}$ by natural transformations.

(3) The category of Artin motives is embedded in the category $\mathcal{EV}^0_{\mathbb{K},\mathbb{E}}$ by the functor $\mathcal{J}$ that maps an Artin motive M to the system with $X_\alpha = M$ for all α and S trivial. Morphisms are then given by the same geometric objects.

(4) Let $X = \mathrm{Spec}\ A$, $X' = \mathrm{Spec}\ B$ be 0-dimensional varieties. Given a component $Z = \mathrm{Spec}\ C$ of $X \times X'$ the two projections turn C into an $A-B$-bimodule $k(Z)$. Moreover, the composition of correspondences translates into the tensor product of bimodules *i.e.* one has

$$k(Z) \otimes_B k(L) \simeq k(Z \circ L)\,. \tag{3.44}$$

We can then compose k with the natural functor $A \to A_{\mathbb{C}} = A \otimes_K \mathbb{C}$ which associates to an $A - B$-bimodule $\mathcal{E}$ the $A_{\mathbb{C}} - B_{\mathbb{C}}$-bimodule $\mathcal{E}_{\mathbb{C}} = \mathcal{E} \otimes_{\mathbb{K}} \mathbb{C}$. Thus, we view the result as an element in KK.

Recall that we have, for any coefficients $\mathbb{E}$, the functor $\mathcal{J}$ which to a variety $X \in Obj(\mathcal{V}^0_{\mathbb{K},\mathbb{E}})$ associates the system (X_α, S) where the system X_α consists of a single X and the semi-group S is trivial. Let $U \in \mathrm{Hom}_{\mathcal{C}\mathcal{V}^0_{\mathbb{K},\mathbb{E}}}(X, X')$ be given by a correspondence $U = \sum a_i \chi_{Z_i}$, with coefficients in $\mathbb{E}$. We consider the element in KK (with coefficients in $\mathbb{E}$) given by the sum of bimodules

$$k(U) = \sum a_i k(Z_i). \tag{3.45}$$

Using the isomorphism (3.11) we thus obtain a G-equivariant bimodule for the corresponding C^*-algebras, and hence owing to finite dimensionality a corresponding class in KK_G. Thus and for any coefficients $\mathbb{E}$ we get a faithful functor

$$k \circ \mathcal{F}\left(\mathrm{Hom}_{\mathcal{C}\mathcal{V}^0_{\mathbb{K},\mathbb{E}}}(X, X')\right) \subset KK_G(k\mathcal{F}(X), k\mathcal{F}(X')) \otimes \mathbb{E} \tag{3.46}$$

compatibly with the composition of correspondences.
Since a correspondence $U = \sum a_i \chi_{Z_i}$, with coefficients in $\mathbb{E}$, is uniquely determined by the corresponding map

$$K_0(A_{\mathbb{C}}) \otimes \mathbb{E} \ \to \ K_0(B_{\mathbb{C}}) \otimes \mathbb{E}$$

we get the required faithfulness of the restriction of $k \circ \mathcal{F}$ to Artin motives. $\square$

The fact of considering only a zero dimensional setting in Theorem 3.13 made it especially simple to compare the composition of correspondences between Artin motives and noncommutative spaces. This is a special case of a more general result that holds in higher dimension as well and which we discuss in Section 6 below. The comparison between correspondences given by algebraic cycles and correspondences in KK-theory is based in the more general case on the description given in [24] of correspondences in KK-theory.

The setting described in this section is useful in order to develop some basic tools that can be applied to a class of "zero dimensional" noncommutative spaces generalizing Artin motives, which inherit natural analogs of arithmetic notions for motives. It is the minimal one that makes it possible to understand the intrinsic role of the absolute Galois group in dynamical systems such as the BC-system. This set-up has no pretention at defining the complete category of noncommutative Artin motives. This would require in particular to spell out exactly the finiteness conditions (such as the finite dimensionality of the center) that should be required on the noncommutative algebra $\mathcal{A}$.

We now turn to a general procedure which should play in characteristic zero a role similar to the action of the Frobenius on the ℓ-adic cohomology. Its main virtue is its generality. It makes essential use of *positivity i.e.* of the key feature of C^*-algebras. It is precisely for this reason that it was important to construct the above functor $\mathcal{F}$ that effects a bridge between the world of Artin motives and that of noncommutative geometry.

4. Scaling as Frobenius in characteristic zero

In this section we describe a general cohomological procedure which starting from a noncommutative space given by a pair $(\mathcal{A}, \varphi)$ of a unital involutive algebra $\mathcal{A}$ over $\mathbb{C}$ and a state φ on $\mathcal{A}$ yields a representation of the multiplicative group $\mathbb{R}^*_+$. We follow all the steps in a particular example: the BC-system described above in Example 3.2. We show that, in this case, the spectrum of the representation is the set of non-trivial zeros of Hecke L-functions with Grössencharakter.
What is striking is the generality of the procedure and the analogy with the role of the Frobenius in positive characteristic. The action of the multiplicative group $\mathbb{R}^*_+$ on the cyclic homology HC_0 of the distilled dual system plays the role of the action of the Frobenius on the ℓ-adic cohomology $H^1(C(\bar{\mathbb{F}}_q), \mathbb{Q}_\ell)$. We explain in details that passing to the dual

system is the analog in characteristic zero of the transition from $\mathbb{F}_q$ to its algebraic closure $\bar{\mathbb{F}}_q$. We also discuss the analog of the intermediate step $\mathbb{F}_q \to \mathbb{F}_{q^n}$. What we call the "distillation" procedure gives a Λ-module $D(\mathcal{A}, \varphi)$ which plays a role similar to a motivic H^1_{mot} (*cf.* (2.3) above). Finally, the dual action of $\mathbb{R}^*_+$ on cyclic homology $HC_0(D(\mathcal{A}, \varphi))$ plays a role similar to the action of the Frobenius on the ℓ-adic cohomology.
While here we only illustrate the procedure in the case of the BC-system, in the next section we discuss it in the more general context of global fields and get in that way a cohomological interpretation of the spectral realization of zeros of Hecke L-functions with Grössencharakter of [14].
There is a deep relation outlined in Section 4.5 between this procedure and the classification of type III factors [10]. In particular, one expects interesting phenomena when the noncommutative space $(\mathcal{A}, \varphi)$ is of type III and examples abound where the general cohomological procedure should be applied, with the variations required in more involved cases. Other type III_1 examples include the following.

- The spaces of leaves of Anosov foliations and their codings.
- The noncommutative space $(\mathcal{A}, \varphi)$ given by the restriction of the vacuum state on the local algebra in the Unruh model of black holes.

One should expect that the analysis of the thermodynamics will get more involved in general, where more that one critical temperature and phase transitions occur, but that a similar distillation procedure will apply at the various critical temperatures.
The existence of a "canonically given" time evolution for noncommutative spaces (see [10] and subsection 4.5 below) plays a central role in noncommutative geometry and the procedure outlined below provides a very general substitute for the Frobenius when working in characteristic zero.

4.1. **Preliminaries.** Let $\mathcal{A}$ be a unital involutive algebra over $\mathbb{C}$. We assume that $\mathcal{A}$ has a countable basis (as a vector space over $\mathbb{C}$) and that the expression

$$||x|| = \sup ||\pi(x)|| \tag{4.1}$$

defines a *finite* norm on $\mathcal{A}$, where π in (4.1) ranges through all unitary Hilbert space representations of the unital involutive algebra $\mathcal{A}$.
We let $\bar{\mathcal{A}}$ denote the completion of $\mathcal{A}$ in this norm. It is a C^*-algebra.
This applies in particular to the unital involutive algebras $\mathcal{A} = C^\infty(X) \rtimes_{alg} S$ of (3.12) and the corresponding C^*-algebra is $\bar{\mathcal{A}} = C(X) \rtimes S$.
A state φ on $\mathcal{A}$ is a linear form $\varphi : \bar{\mathcal{A}} \to \mathbb{C}$ such that

$$\varphi(x^* x) \geq 0, \quad \forall\, x \in \bar{\mathcal{A}} \quad \text{and} \quad \varphi(1) = 1\,.$$

One lets $\mathcal{H}_\varphi$ be the Hilbert space of the Gelfand–Naimark–Segal (GNS) construction, *i.e.* the completion of $\mathcal{A}$ associated to the sesquilinear form

$$\langle x, y\rangle = \varphi(y^* x)\,,\ x, y \in \mathcal{A}\,.$$

Definition 4.1. *A state φ on a C^*-algebra $\bar{\mathcal{A}}$ is* regular *iff the vector $1 \in \mathcal{H}_\varphi$ is cyclic for the commutant of the action of $\bar{\mathcal{A}}$ by left multiplication on $\mathcal{H}_\varphi$.*

Let M be the von-Neumann algebra weak closure of the action of $\bar{\mathcal{A}}$ in $\mathcal{H}_\varphi$ by left multiplication.
The main result of Tomita's theory [51] is that, if we let Δ_φ be the *modular operator*

$$\Delta_\varphi = S^*_\varphi\, S_\varphi\,, \tag{4.2}$$

which is the *module* of the involution S_φ, $x \to x^*$, then we have

$$\Delta^{it}_\varphi\, M\, \Delta^{-it}_\varphi = M\,, \quad \forall\, t \in \mathbb{R}\,.$$

The corresponding one parameter group $\sigma_t^\varphi \in \mathrm{Aut}(M)$ is called the modular automorphism group of φ and it satisfies the KMS_1 condition relative to φ (*cf.* [51]). In general, given a unital involutive algebra $\mathcal{A}$, a one parameter group of automorphisms $\sigma_t \in \mathrm{Aut}(\mathcal{A})$ and a state φ on $\mathcal{A}$, the KMS_β-condition is defined as follows.

Definition 4.2. *A triple $(\mathcal{A}, \sigma_t, \varphi)$ satisfies the Kubo-Martin-Schwinger (KMS) condition at inverse temperature $0 \leq \beta < \infty$, if the following holds. For all $x, y \in \mathcal{A}$, there exists a holomorphic function $F_{x,y}(z)$ on the strip $0 < \mathrm{Im}(z) < \beta$, which extends as a continuous function on the boundary of the strip, with the property that*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}. \tag{4.3}$$

We also say that φ is a KMS_β state for $(\mathcal{A}, \sigma_t)$. The set Σ_β of KMS_β states is a compact convex Choquet simplex [8, II §5] whose set of extreme points $\mathcal{E}_\beta$ consists of the factor states. One can express any KMS_β state uniquely in terms of extremal states, because of the uniqueness of the barycentric decomposition of a Choquet simplex.

4.2. **The cooling morphism.** Let $(\mathcal{A}, \varphi)$ be a pair of an algebra and a state as above. We assume that the modular automorphism group σ_t^φ associated to the state φ leaves the algebra $\mathcal{A}$ globally invariant and we denote by $\sigma_t \in \mathrm{Aut}(\mathcal{A})$ its restriction to $\mathcal{A}$. By construction the state φ is a KMS_1 state for the quantum statistical system $(\bar{\mathcal{A}}, \sigma)$. Here $\bar{\mathcal{A}}$ is the C^*-completion in (4.1) as above.

The process of cooling down the system consists of investigating KMS_β-states for $\beta > 1$ (*i.e.* at lower temperatures).

The general theory of quantum statistical mechanics shows that the space Σ_β of KMS_β-states is a Choquet simplex. We let $\mathcal{E}_\beta$ denote the set of its extremal points. Each $\varepsilon \in \mathcal{E}_\beta$ is a factor state on $\mathcal{A}$.

We require that, for sufficiently large β, all the $\epsilon \in \mathcal{E}_\beta$ are type I_∞ factor states.

In general, the original state φ will be of type III (it is of type III_1 for the BC-system), so that the type I_∞ property we required means that, under cooling, the system simplifies and becomes "commutative". In fact, under this assumption, we obtain a "cooled down dual algebra" $C(\tilde{\Omega}_\beta, \mathcal{L}^1)$, using the set Ω_β of regular extremal KMS_β states (*cf.* (4.24) below), that is Morita equivalent to a commutative algebra.

Lemma 4.3. *Let φ be a regular extremal KMS_β state on the C^*-algebra $\bar{\mathcal{A}}$ with the time evolution σ_t. Assume that the corresponding factor M_φ obtained through the GNS representation $\mathcal{H}_\varphi$ is of type I_∞. Then there exists an irreducible representation π_φ of $\bar{\mathcal{A}}$ in a Hilbert space $\mathcal{H}(\varphi)$ and an unbounded self-adjoint operator H acting on $\mathcal{H}(\varphi)$, such that $e^{-\beta H}$ is of trace class. Moreover, one has*

$$\varphi(x) = \mathrm{Trace}(\pi_\varphi(x)\, e^{-\beta H})/\mathrm{Trace}(e^{-\beta H}), \quad \forall x \in \bar{\mathcal{A}}, \tag{4.4}$$

and

$$e^{itH}\, \pi_\varphi(x)\, e^{-itH} = \pi_\varphi(\sigma_t(x)). \tag{4.5}$$

Proof. Let $1_\varphi \in \mathcal{H}_\varphi$ be the cyclic and separating vector associated to the regular state φ in the GNS representation. The factor M_φ is of type I_∞ hence one has a factorization

$$\mathcal{H}_\varphi = \mathcal{H}(\varphi) \otimes \mathcal{H}', \quad M_\varphi = \{T \otimes 1 | T \in \mathcal{L}(\mathcal{H}(\varphi))\}. \tag{4.6}$$

The restriction of the vector state 1_φ to M_φ is faithful and normal, hence it can be uniquely written in the form

$$\langle (T \otimes 1) 1_\varphi, 1_\varphi \rangle = \mathrm{Trace}(T\rho), \tag{4.7}$$

for a density matrix ρ, that is, a positive trace class operator with $\mathrm{Trace}(\rho) = 1$. This can be written in the form $\rho = e^{-\beta H}$. It then follows that the vector state of (4.7) satisfies the KMS_β condition on M_φ, relative to the 1-parameter group implemented by e^{itH}.
The factorization (4.6) shows that the left action of $\bar{\mathcal{A}}$ in the GNS representation $\mathcal{H}_\varphi$ is of the form $a \mapsto \pi_\varphi(a) \otimes 1$. The representation π_φ on $\mathcal{H}(\varphi)$ is irreducible.
Let $\tilde{H}$ be the generator of the 1-parameter group of unitary operators σ_t acting on $\mathcal{H}_\varphi$. Since M_φ is the weak closure of $\bar{\mathcal{A}}$ in $\mathcal{L}(\mathcal{H}_\varphi)$, it follows that the extension of the state φ to M_φ fulfills the KMS_β condition with respect to the 1-parameter group implemented by $e^{it\tilde{H}}$. This extension of φ is given by the vector state (4.7). This shows that the 1-parameter groups implemented by $e^{it\tilde{H}}$ and e^{itH} agree on M_φ. This proves (4.5). Equation (4.4) follows from (4.7), which implies in particular that the operator H has discrete spectrum bounded below. ☐

In general, we let $\Omega_\beta \subset \mathcal{E}_\beta$ be the set of regular (in the sense of Definition 4.1) extremal KMS_β states of type I_∞.
The equation (4.5) does not determine H uniquely, but only up to an additive constant. For any choice of H, the corresponding representation of the crossed product algebra $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ is given by

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) = \int \pi_\varepsilon(x(t))\, e^{itH}\, dt. \tag{4.8}$$

Let us define carefully the crossed product algebra $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ by specifying precisely which maps $t \to x(t) \in \mathcal{A}$ qualify as elements of $\hat{\mathcal{A}}$. One defines the Schwartz space $\mathcal{S}(\mathbb{R}, \mathcal{A})$ as the direct limit of the Schwartz space $\mathcal{S}(\mathbb{R}, V)$ over *finite dimensional* subspaces $V \subset \mathcal{A}$:

$$\mathcal{S}(\mathbb{R}, \mathcal{A}) = \cup_V\, \mathcal{S}(\mathbb{R}, V). \tag{4.9}$$

Notice then that, since the vector space $\mathcal{A}$ has countable basis and the action of σ_t is unitary, one can find a linear basis of $\mathcal{A}$ whose elements are eigenvectors for σ_t and such that the corresponding characters are unitary. It follows that the Schwartz space $\mathcal{S}(\mathbb{R}, \mathcal{A})$ is an algebra under the product

$$(x \star y)(s) = \int x(t)\, \sigma_t(y(s-t))\, dt\,. \tag{4.10}$$

We denote the algebra obtained in this way by

$$\hat{\mathcal{A}} = \mathcal{A} \rtimes_{\sigma_t} \mathbb{R} = \mathcal{S}(\mathbb{R}, \mathcal{A}). \tag{4.11}$$

The dual action $\theta_\lambda \in \mathrm{Aut}(\hat{\mathcal{A}})$ of $\mathbb{R}_+^*$ is given by

$$\theta_\lambda(\int x(t)\, U_t\, dt) = \int \lambda^{it}\, x(t)\, U_t\, dt. \tag{4.12}$$

By construction, the set $\tilde{\Omega}_\beta$ of pairs (ε, H) forms the total space of a principal $\mathbb{R}$-bundle over Ω_β, where the action of $\mathbb{R}$ is simply given by translation of H *i.e.* in multiplicative terms, one has a principal $\mathbb{R}_+^*$-bundle with

$$\lambda\, (\varepsilon, H) = (\varepsilon, H + \log \lambda)\,, \quad \forall \lambda \in \mathbb{R}_+^*, \tag{4.13}$$

which gives the fibration

$$\mathbb{R}_+^* \to \tilde{\Omega}_\beta \to \Omega_\beta\,. \tag{4.14}$$

Notice that (4.14) admits a natural section given by the condition

$$\mathrm{Trace}(e^{-\beta\, H}) = 1\,, \tag{4.15}$$

and a natural splitting

$$\tilde{\Omega}_\beta \sim \Omega_\beta \times \mathbb{R}_+^* \,. \tag{4.16}$$

The states $\varepsilon \in \Omega_\beta$ are in fact uniquely determined by the corresponding representation π_ε, thanks to the formula (4.4), which does not depend on the additional choice of H fulfilling (4.5). The property (4.4) also ensures that, for any function $f \in \mathcal{S}(\mathbb{R})$ with sufficiently fast decay at ∞, the operator $f(H)$ is of trace class.

We let $\hat{\mathcal{A}}_\beta$ be the linear subspace of $\hat{\mathcal{A}}$ spanned by elements of the form $\int x(t)\, U_t\, dt$ where $x \in \mathcal{S}(I_\beta, \mathcal{A})$. Namely, $x \in \mathcal{S}(\mathbb{R}, \mathcal{A})$ admits an analytic continuation to the strip $I_\beta = \{z \,:\, \Im z \in [0, \beta]\}$, which restricts to Schwartz functions on the boundary of I_β. One checks that $\hat{\mathcal{A}}_\beta$ is a subalgebra of $\hat{\mathcal{A}}$. Indeed the product (4.10) belongs to $\mathcal{S}(I_\beta, \mathcal{A})$ provided that $x \in \mathcal{S}(\mathbb{R}, \mathcal{A})$ and $y \in \hat{\mathcal{A}}_\beta$. One then gets the following.

Proposition 4.4. (1) *For $(\varepsilon, H) \in \tilde{\Omega}_\beta$, the representations $\pi_{\varepsilon,H}$ are pairwise inequivalent irreducible representations of $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$.*

(2) *Let $\theta_\lambda \in \mathrm{Aut}(\hat{\mathcal{A}})$ be the dual action* (4.12). *This satisfies*

$$\pi_{\varepsilon,H} \circ \theta_\lambda = \pi_{\lambda(\varepsilon,H)}, \quad \forall (\varepsilon, H) \in \tilde{\Omega}_\beta. \tag{4.17}$$

(3) *For $x \in \hat{\mathcal{A}}_\beta$, one has*

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) \in \mathcal{L}^1(\mathcal{H}(\varepsilon)). \tag{4.18}$$

Proof. (1) By construction the representations π_ε of $\mathcal{A}$ are already irreducible. Let us show that the unitary equivalence class of the representation $\pi_{\varepsilon,H}$ determines (ε, H). First the state ε is uniquely determined from (4.4). Next the irreducibility of the restriction to $\mathcal{A}$ shows that only scalar operators can intertwine $\pi_{\varepsilon,H}$ with $\pi_{\varepsilon,H+c}$, which implies that $c = 0$.
(2) One has $\pi_{\varepsilon,H}(U_t) = e^{itH}$, hence $\pi_{\varepsilon,H}(\theta_\lambda(U_t)) = \lambda^{it}\, e^{itH} = e^{it(H+\log\lambda)}$, as required.
(3) Let us first handle the case of scalar valued $x(t) \in \mathbb{C}$ for simplicity. One then has

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) = g(H) = \sum g(u) e(u),$$

where the $e(u)$ are the spectral projections of H and g is the Fourier transform in the form

$$g(s) = \int x(t)\, e^{its}\, dt \,.$$

Since $x \in \mathcal{S}(\mathbb{R})$ admits analytic continuation to the strip $I_\beta = \{z \,:\, \Im z \in [0, \beta]\}$ and since we are assuming that the restriction to the boundary of the strip I_β is also in Schwarz space, one gets that the function $g(s)\, e^{\beta s}$ belongs to $\mathcal{S}(\mathbb{R})$ and the trace class property follows from $\mathrm{Trace}(e^{-\beta H}) < \infty$. In the general case with $x \in \mathcal{S}(I_\beta, \mathcal{A})$, with the notation above, one gets

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) = \int \pi_\varepsilon(x(t)) \sum e^{its} e(s)\, dt = \sum \pi_\varepsilon(g(s))\, e(s)$$

where, as above, $e(s)$ is the spectral projection of H associated to the eigenvalue $s \in \mathbb{R}$. Again $||g(s)\, e^{\beta s}|| < C$ independently of s and $\sum e^{-\beta s}\,\mathrm{Trace}(e(s)) < \infty$, which gives the required trace class property. $\square$

In the case of the BC-system the above construction is actually independent of $\beta > 1$. Indeed, what happens is that the set of irreducible representations of $\hat{\mathcal{A}}$ thus constructed does not depend on $\beta > 1$ and the corresponding map at the level of the extremal KMS_β states is simply obtained by changing the value of β in formula (4.4). In general the size of Ω_β is a non-decreasing function of β :

Corollary 4.5. *For any $\beta' > \beta$ the formula* (4.4) *defines a canonical injection*

$$c_{\beta',\beta} : \Omega_\beta \to \Omega_{\beta'} \tag{4.19}$$

which extends to an $\mathbb{R}_+^*$-equivariant map of the bundles $\tilde{\Omega}_\beta$.

Proof. First $\mathrm{Trace}(e^{-\beta H}) < \infty$ implies $\mathrm{Trace}(e^{-\beta' H}) < \infty$ for all $\beta' > \beta$ so that the map $c_{\beta',\beta}$ is well defined. Notice that the obtained state given by (4.4) for β' is $\mathrm{KMS}_{\beta'}$ extremal (since it is factorial), regular and of type I. The map $c_{\beta',\beta}$ is injective since one recovers the original KMS_β state by the reverse procedure. The latter in general only makes sense for those values of β for which $\mathrm{Trace}(e^{-\beta H}) < \infty$. This set of values is a half line with lower bound depending on the corresponding element of $\Omega_\infty = \cup_\beta \Omega_\beta$. ☐

We fix a separable Hilbert space $\mathcal{H}$ and let, as in [32], $\mathrm{Irr}\hat{\mathcal{A}}$ be the space of irreducible representations of the C^*-algebra $\hat{\mathcal{A}}$ in $\mathcal{H}$ endowed with the topology of pointwise weak convergence. Thus

$$\pi_\alpha \to \pi \iff \langle \xi, \pi_\alpha(x)\eta \rangle \to \langle \xi, \pi(x)\eta \rangle , \quad \forall\, \xi, \eta \in \mathcal{H},\, x \in \hat{\mathcal{A}} . \tag{4.20}$$

This is equivalent to pointwise strong convergence by [32] 3.5.2. By [32] Theorem 3.5.8 the map

$$\pi \in \mathrm{Irr}\hat{\mathcal{A}} \mapsto \ker \pi \in \mathrm{Prim}\,\hat{\mathcal{A}} \tag{4.21}$$

is continuous and open. Indeed by Definition 3.1.5 of [32] the topology on the spectrum, *i.e.* the space of equivalence classes of irreducible representations, is *defined* as the pull back of the Jacobson topology of the primitive ideal space. By Proposition 4.4 each $\pi_{\varepsilon,H}$ defines an irreducible representation of $\hat{\mathcal{A}}$ whose range is the elementary C^*-algebra of compact operators in $\mathcal{H}(\varepsilon)$. Thus, by Corollaries 4.10 and 4.11 of [32], this representation is characterized by its kernel which is an element of $\mathrm{Prim}\hat{\mathcal{A}}$. The map $(\varepsilon, H) \mapsto \ker \pi_{\varepsilon,H}$ gives an $\mathbb{R}_+^*$-equivariant embedding

$$\tilde{\Omega}_\beta \subset \mathrm{Prim}\,\hat{\mathcal{A}} . \tag{4.22}$$

We assume that the representations $\pi_{\varepsilon,H}$, for $(\varepsilon, H) \in \tilde{\Omega}_\beta$ can be continuously realized as representations $\tilde{\pi}_{\varepsilon,H}$ in $\mathcal{H}$ *i.e.* more precisely that the injective map (4.22) is lifted to a continuous map from $\tilde{\Omega}_\beta$ to $\mathrm{Irr}\hat{\mathcal{A}}$ such that

$$(\varepsilon, H) \mapsto \tilde{\pi}_{\varepsilon,H}(\hat{f}(H)) \tag{4.23}$$

is a continuous map to $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{H})$ for any $f \in \mathcal{S}(I_\beta)$.

Corollary 4.6. *Under the above conditions, the representations $\pi_{\varepsilon,H}$, for $(\varepsilon, H) \in \tilde{\Omega}_\beta$, assemble to give an $\mathbb{R}_+^*$-equivariant morphism*

$$\pi \ : \ \hat{\mathcal{A}}_\beta \to C(\tilde{\Omega}_\beta, \mathcal{L}^1) \tag{4.24}$$

defined by

$$\pi(x)(\varepsilon, H) = \ \pi_{\varepsilon,H}(x) , \quad \forall (\varepsilon, H) \in \tilde{\Omega}_\beta . \tag{4.25}$$

This continuity can be checked directly in the case of the BC-system where $\Omega_\beta = \mathcal{E}_\beta$ is compact for the weak topology and where all the representations $\pi_{\varepsilon,H}$ take place in the same Hilbert space.

4.3. **The distilled Λ-module $D(\mathcal{A},\varphi)$.** The above cooling morphism of (4.24) composes with the trace

$$\mathrm{Trace} : \mathcal{L}^1 \to \mathbb{C}$$

to yield a cyclic morphism from the Λ-module associated to $\hat{\mathcal{A}}_\beta$ to one associated to a commutative algebra of functions on $\tilde{\Omega}_\beta$. Our aim in this section is to define appropriately the cokernel of this cyclic morphism, which gives what we call the distilled Λ-module $D(\mathcal{A},\varphi)$.

A partial trace gives rise to a cyclic morphism as follows.

Proposition 4.7. *Let $\mathcal{B}$ be a unital algebra. The equality*

$$\mathrm{Trace}((x_0\otimes t_0)\otimes(x_1\otimes t_1)\otimes\ldots\otimes(x_n\otimes t_n)) = x_0\otimes x_1\otimes\ldots\otimes x_n\,\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n)$$

defines a map of Λ-modules from $(\mathcal{B}\otimes\mathcal{L}^1)^\natural$ to $\mathcal{B}^\natural$.

Proof. We perform the construction with the Λ-module of the inclusion

$$\mathcal{B}\otimes\mathcal{L}^1 \subset \mathcal{B}\otimes\tilde{\mathcal{L}}^1$$

and obtain by restriction to $(\mathcal{B}\otimes\mathcal{L}^1)^\natural$ the required cyclic morphism. The basic cyclic operations are given by

$$\delta_i(x^0\otimes\ldots\otimes x^n) = x^0\otimes\ldots\otimes x^i x^{i+1}\otimes\ldots\otimes x^n\,, \quad 0\le i\le n-1,$$
$$\delta_n(x^0\otimes\ldots\otimes x^n) = x^n x^0\otimes x^1\otimes\ldots\otimes x^{n-1},$$
$$\sigma_j(x^0\otimes\ldots\otimes x^n) = x^0\otimes\ldots\otimes x^j\otimes 1\otimes x^{j+1}\otimes\ldots\otimes x^n, \quad 0\le j\le n,$$
$$\tau_n(x^0\otimes\ldots\otimes x^n) = x^n\otimes x^0\otimes\ldots\otimes x^{n-1}\,.$$

One has, for $0\le i\le n-1$,

$$\begin{gathered}\mathrm{Trace}\;\delta_i((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)) = \\ x_0\otimes\ldots\otimes x_i x_{i+1}\otimes\ldots\otimes x_n\,\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n) = \\ \delta_i\;\mathrm{Trace}((x_0\otimes t_0)\otimes(x_1\otimes t_1)\otimes\ldots\otimes(x_n\otimes t_n)).\end{gathered}$$

Similarly,

$$\begin{gathered}\mathrm{Trace}\;\delta_n((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)) = \\ x_n\,x_0\otimes\ldots\otimes x_{n-1}\,\mathrm{Trace}(t_n\,t_0\,t_1\,\cdots\,t_{n-1}) = \\ \delta_n(x_0\otimes\ldots\otimes x_n)\,\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n) = \\ \delta_n\;\mathrm{Trace}((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)).\end{gathered}$$

One has, for $0\le j\le n$,

$$\begin{gathered}\mathrm{Trace}\;\sigma_j((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)) = \\ x_0\otimes\ldots\otimes x_j\otimes 1\otimes x_{j+1}\otimes\ldots\otimes x_n\;\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n) = \\ \sigma_j(x_0\otimes\ldots\otimes x_n)\,\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n) = \\ \sigma_j\;\mathrm{Trace}((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)).\end{gathered}$$

Finally, one has

$$\begin{gathered}\mathrm{Trace}\;\tau_n((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)) = \\ x_n\otimes x_0\otimes\ldots\otimes x_{n-1}\,\mathrm{Trace}(t_0\,t_1\,\cdots\,t_n) = \\ \tau_n\;\mathrm{Trace}((x_0\otimes t_0)\otimes\ldots\otimes(x_n\otimes t_n)).\end{gathered}$$

∎

When dealing with non-unital algebras $\mathcal{A}$, the maps $x \mapsto x \otimes 1$ in the associated Λ-module $\mathcal{A}^\natural$ are not defined. Thus one uses the algebra $\tilde{\mathcal{A}}$ obtained by adjoining a unit, and defines $\mathcal{A}^\natural$ as the submodule of $\tilde{\mathcal{A}}^\natural$ which is the kernel of the augmentation morphism,

$$\varepsilon^\natural : \tilde{\mathcal{A}}^\natural \to \mathbb{C}$$

associated to the algebra homomorphism $\varepsilon : \tilde{\mathcal{A}} \to \mathbb{C}$.
We take this as a definition of $\mathcal{A}^\natural$ for non unital algebras. One obtains a Λ-module which coincides with $\mathcal{A}$ in degree 0 and whose elements in degree n are tensors

$$\sum a_0 \otimes \cdots \otimes a_n\,, \quad a_j \in \tilde{\mathcal{A}}\,, \quad \sum \varepsilon(a_0) \cdots \varepsilon(a_n) = 0$$

While the algebra $\tilde{\mathcal{A}}$ corresponds to the "one point compactification" of the noncommutative space $\mathcal{A}$, it is convenient to also consider more general compactifications. Thus if $\mathcal{A} \subset \mathcal{A}^{comp}$ is an inclusion of $\mathcal{A}$ as an essential ideal in a unital algebra $\mathcal{A}^{comp}$ one gets a Λ-module $(\mathcal{A}, \mathcal{A}^{comp})^\natural$ which in degree n is given by tensors

$$\sum a_0 \otimes \cdots \otimes a_n\,, \quad a_j \in \mathcal{A}^{comp}\,,$$

where for each simple tensor $a_0 \otimes \cdots \otimes a_n$ in the sum, at least one of the a_j belongs to $\mathcal{A}$. One checks that there is a canonical cyclic morphism

$$\mathcal{A}^\natural \to (\mathcal{A},\, \mathcal{A}^{comp})^\natural$$

obtained from the algebra homomorphism $\tilde{\mathcal{A}} \to \mathcal{A}^{comp}$.
Thus, we can apply the construction of Proposition 4.7 to the case $\mathcal{B} = C(\tilde{\Omega}_\beta)$ under the hypothesis of Corollary 4.6. We shall briefly discuss in Remark 4.14 below how to adapt the construction in the general case.

The next step is to analyze the decay of the obtained functions on $\tilde{\Omega}_\beta$. Let us look at the behavior on a fiber of the fibration (4.14). We first look at elements $\int x(t)\, U_t\, dt \in \hat{\mathcal{A}}_\beta$ with scalar valued $x(t) \in \mathbb{C}$ for simplicity. One then has, with $g(s) = \int x(t)\, e^{its}\, dt$,

$$\mathrm{Trace}(\pi_{\varepsilon, H+c}(\int x(t)\, U_t\, dt)) = \mathrm{Trace}(g(H+c))$$

and we need to understand the behavior of this function when $c \to \pm\infty$. Recall from the above discussion that both g and the function $g(s)\, e^{\beta s}$ belong to $\mathcal{S}(\mathbb{R})$.

Lemma 4.8. *(i) For all $N > 0$, one has $|\mathrm{Trace}(g(H+c))| = O(e^{-\beta\, c}\, c^{-N})$, for $c \to +\infty$.*
(ii) For any element $x \in \hat{\mathcal{A}}_\beta$, one has

$$|\mathrm{Trace}(\pi_{\varepsilon, H+c}(x))| = O(e^{-\beta\, c}\, c^{-N})\,, \quad c \to +\infty\,.$$

Proof. (i) Let $g(s) = e^{-\beta\, s}\, h(s)$ where $h \in \mathcal{S}(\mathbb{R})$. One has $|h(s+c)| \leq C\, c^{-N}$ for $s \in \mathrm{Spec}\, H$, hence $|g(s+c)| \leq C\, e^{-\beta\, c}\, c^{-N}\, e^{-\beta\, s}$, for $s \in \mathrm{Spec}\, H$, so that $|\mathrm{Trace}(g(H+c))| \leq C\, e^{-\beta\, c}\, c^{-N}\, \mathrm{Trace}(e^{-\beta\, H})$.
(ii) In the general case with $x \in \mathcal{S}(I_\beta, \mathcal{A})$ and the notation as in the proof of Proposition 4.4, one gets

$$\mathrm{Trace}(\pi_{\varepsilon, H+c}(x)) = \sum \mathrm{Trace}(\pi_\varepsilon(g(s+c))\, e(s)),$$

whose size is controlled by $\sum ||g(s+c)||\, \mathrm{Trace}(e(s))$. As above one has $||g(s+c)|| \leq C\, e^{-\beta\, c}\, c^{-N}\, e^{-\beta\, s}$, for $s \in \mathrm{Spec}\, H$, so that using $\sum \mathrm{Trace}(e(s))\, e^{-\beta\, s} < \infty$ one gets the required estimate. $\Box$

Let us now investigate the behavior of this function when $c \to -\infty$. To get some feeling, we first take the case of the BC-system. Then the spectrum of H is the set $\{\log(n) : n \in \mathbb{N} = \mathbb{Z}_{>0}\}$, so that $\mathrm{Trace}(g(H+c)) = \sum_{n\in\mathbb{N}^*} g(\log(n)+c)$. Let $f(x) = g(\log(x))$ be extended to $\mathbb{R}$ as a continuous even function. Then one has $f(0) = 0$ and, for $\lambda = e^c$,

$$\mathrm{Trace}(g(H+c)) = \frac{1}{2}\sum_{n\in\mathbb{Z}} f(\lambda\, n)\,.$$

The Poisson summation formula therefore gives, with the appropriate normalization of the Fourier transform $\hat{f}$ of f,

$$\lambda \sum_{n\in\mathbb{Z}} f(\lambda\, n) = \sum_{n\in\mathbb{Z}} \hat{f}(\lambda^{-1}\, n).$$

This shows that, for $c \to -\infty$, *i.e.* for $\lambda \to 0$, the leading behavior is governed by

$$\sum_{n\in\mathbb{Z}} f(\lambda\, n) \sim \lambda^{-1}\int f(x)\, dx.$$

This was clear from the Riemann sum approximation to the integral. The remainder is controlled by the decay of $\hat{f}$ at ∞, *i.e.* by the smoothness of f. Notice that the latter is not guaranteed by that of g because of a possible singularity at 0. One gets in fact the following result.

Lemma 4.9. *In the BC case, for any $\beta > 1$ and any element $x \in \hat{\mathcal{A}}_\beta$, one has*

$$\mathrm{Trace}(\pi_{\varepsilon,H+c}(x)) = e^{-c}\,\tau(x) + O(|c|^{-N})\,, \quad c \to -\infty\,, \quad \forall N > 0, \tag{4.26}$$

where τ is the canonical dual trace on $\hat{\mathcal{A}}$.

We give the proof after Lemma 4.17. Combining Lemmata 4.8 and 4.9 one gets, in the BC case and for $x \in \hat{\mathcal{A}}_\beta$, $\tau(x) = 0$, that the function

$$f_\varepsilon(c) = \mathrm{Trace}(\pi_{\varepsilon,H+c}(x)) \tag{4.27}$$

has the correct decay for $c \to \pm\infty$ in order to have a Fourier transform (in the variable c) that is holomorphic in the strip I_β. Indeed one has, for all $N > 0$,

$$\begin{aligned} |f_\varepsilon(c)| &= O(e^{-\beta\, c}\,|c|^{-N}), &&\text{for}\quad c \to +\infty \\ |f_\varepsilon(c)| &= O(|c|^{-N}) &&\text{for}\quad c \to -\infty\,. \end{aligned} \tag{4.28}$$

This implies that the Fourier transform

$$F(f_\varepsilon)(u) = \int f_\varepsilon(c)\, e^{-icu}\, dc \tag{4.29}$$

is holomorphic in the strip I_β, bounded and smooth on the boundary. In fact, one has $F(f_\varepsilon) \in \mathcal{S}(I_\beta)$, as follows from Lemma 4.10 below, based on the covariance of the map $\mathrm{Trace}\circ\pi$.

We let $\mathrm{Hol}(I_\beta)$ be the algebra of multipliers of $\mathcal{S}(I_\beta)$, *i.e.* of holomorphic functions h in the strip I_β such that $h\,\mathcal{S}(I_\beta) \subset \mathcal{S}(I_\beta)$. For such a function h its restriction to the boundary component $\mathbb{R}$ is a function of tempered growth and there is a unique distribution $\hat{h}$ such that

$$\int \hat{h}(\lambda)\,\lambda^{it}\, d^*\lambda = h(t).$$

One then has in full generality the following result.

Lemma 4.10. *1) For $h \in \mathrm{Hol}(I_\beta)$ the operator $\theta(\hat{h}) = \int \hat{h}(\lambda)\,\theta_\lambda\, d^*\lambda$ acts on $\hat{\mathcal{A}}_\beta$. This turns $\hat{\mathcal{A}}_\beta$ into a module over the algebra $\mathrm{Hol}(I_\beta)$.*
2) The map $F\circ\mathrm{Trace}\circ\pi$ is an $\mathrm{Hol}(I_\beta)$-module map.

Proof. 1) One has

$$\theta(\hat h)(\int x(t)\,U_t\,dt) = \int\int \hat h(\lambda)\,\lambda^{it}\,x(t)\,U_t\,dt\,d^*\lambda = \int h(t)\,x(t)\,U_t\,dt$$

and $h\,x \in \mathcal{S}(I_\beta, \mathcal{A})$.
2) The equivariance of π, *i.e.* the equality (*cf.* (4.17))

$$\pi_{\varepsilon,H}\circ\theta_\lambda = \pi_{(\varepsilon,H+\log\lambda)}\,,\quad \forall\,(\varepsilon,H)\in\tilde\Omega_\beta,$$

gives

$$(\mathrm{Trace}\circ\pi\circ\theta_\lambda)(x)_{(\varepsilon,H)} = (\mathrm{Trace}\circ\pi)(x)_{(\varepsilon,H+\log\lambda)}.$$

Thus, with f_ε as in (4.27), one gets

$$(\mathrm{Trace}\circ\pi\circ\theta(\hat h))(x)_{(\varepsilon,H)} = \int \hat h(\lambda)\,f_\varepsilon(\log\lambda)\,d^*\lambda.$$

Replacing H by $H+c$ one gets that the function associated to $\theta(\hat h)(x)$ by (4.27) is given by

$$g_\varepsilon(c) = \int \hat h(\lambda)\,f_\varepsilon(\log\lambda + c)\,d^*\lambda.$$

One then gets

$$F(g_\varepsilon)(u) = \int g_\varepsilon(c)\,e^{-icu}\,dc = \int\int \hat h(\lambda)\,f_\varepsilon(\log\lambda+c)\,e^{-icu}\,dc\,d^*\lambda$$
$$= \int \hat h(\lambda)\,\lambda^{iu}\,d^*\lambda \int f_\varepsilon(a)\,e^{-iau}\,da = h(u)\,F(f_\varepsilon)(u)\,.$$

□

This makes it possible to improve the estimate (4.28) for the BC-system immediately. Indeed, the function $h(s)=s$ belongs to $\mathrm{Hol}(I_\beta)$ and this shows that (4.28) holds for all derivatives of f_ε. In other words it shows the following.

Corollary 4.11. *In the BC case, for any $\beta>1$ and any element $x\in\hat{\mathcal{A}}_\beta\cap\mathrm{Ker}\,\tau$, one has*

$$F\circ\mathrm{Trace}\circ\pi(x)\in C(\Omega_\beta,\mathcal{S}(I_\beta))\,.$$

Thus, by Lemma 4.10 we obtain an $\mathrm{Hol}(I_\beta)$-submodule of $C(\Omega_\beta,\mathcal{S}(I_\beta))$ and as we will see shortly that this contains all the information on the zeros of L-functions with Grössencharakters over the global field $\mathbb{Q}$.
When expressed in the variable $\lambda = e^c$, the regularity of the function $\mathrm{Trace}\circ\pi(x)$ can be written equivalently in the form $\mathrm{Trace}\circ\pi(x)\in C(\Omega_\beta,\mathcal{S}_\beta(\mathbb{R}_+^*))$ where, with $\mu(\lambda)=\lambda$, $\forall\lambda\in\mathbb{R}_+^*$, we let

$$\mathcal{S}_\beta(\mathbb{R}_+^*) = \cap_{[0,\beta]}\,\mu^{-\alpha}\,\mathcal{S}(\mathbb{R}_+^*) \tag{4.30}$$

be the intersection of the shifted Schwartz spaces of $\mathbb{R}_+^*\sim\mathbb{R}$.
We let $\mathcal{S}^\natural(\tilde\Omega_\beta)$ be the cyclic submodule of $C(\tilde\Omega_\beta)^\natural$ whose elements are functions with restriction to the main diagonal that belongs to $C(\Omega_\beta,\mathcal{S}_\beta(\mathbb{R}_+^*))$. We then obtain the following result.

Proposition 4.12. *Assuming* (4.26)*, the map* $\mathrm{Trace}\circ\pi$ *defines a cyclic morphism*

$$\delta\;:\;\hat{\mathcal{A}}^\natural_{\beta,0}\to\mathcal{S}^\natural(\tilde\Omega_\beta)\,. \tag{4.31}$$

The subscript 0 in $\hat{\mathcal{A}}^\natural_{\beta,0}$ of (4.31) means that we are considering the cyclic module defined as in (5.1) below. The proof of Proposition 4.12 follows from Proposition 4.7.

Definition 4.13. *We define the distilled Λ-module $D(\mathcal{A},\varphi)$ as the cokernel of the cyclic morphism δ.*

Remark 4.14. In the general case *i.e.* without the lifting hypothesis of Corollary 4.6, one is dealing (instead of the representations $\tilde{\pi}_{\varepsilon,H}$ in a fixed Hilbert space) with the continuous field of elementary C^*-algebras $C_{\varepsilon,H} = \pi_{\varepsilon,H}(\mathcal{A})$ on $\tilde{\Omega}_\beta$. Even though the algebras $C_{\varepsilon,H}$ are isomorphic to the algebra of compact operators in $\mathcal{H}$ (or in a finite dimensional Hilbert space) the Dixmier-Douady invariant (*cf.* [32] Theorem 10.8.4) gives an obstruction to the Morita equivalence between the algebra $C(\tilde{\Omega}_\beta, C)$ of continuous sections of the field and the algebra of continuous functions on $\tilde{\Omega}_\beta$. When this obstruction is non-trivial it is no longer possible to use the cyclic morphism coming from the Morita equivalence and one has to work with $C(\tilde{\Omega}_\beta, C)$ rather than with the algebra of continuous functions on $\tilde{\Omega}_\beta$. The cooling morphism is now the canonical map

$$\begin{aligned} &\pi : \hat{\mathcal{A}}_\beta \to C(\tilde{\Omega}_\beta, C) \\ &\pi(x)(\varepsilon, H) = \tilde{\pi}_{\varepsilon,H}(x), \quad \forall (\varepsilon, H) \in \tilde{\Omega}_\beta. \end{aligned} \tag{4.32}$$

which defines an $\mathbb{R}_+^*$-equivariant algebra homomorphism. The cokernel of this morphism as well as its cyclic cohomology continue to make sense but are more complicated to compute. Moreover for a general pair $(\mathcal{A},\varphi)$ the decay condition (4.26) will not be fulfilled. One then needs to proceed with care and analyze the behavior of these functions case by case. One can nevertheless give a rough general definition, just by replacing $\mathcal{S} \subset C(\tilde{\Omega}_\beta)$ above by the subalgebra generated by $\mathrm{Trace} \circ \pi(\mathrm{Ker}\,\tau)$.

4.4. **Dual action on cyclic homology.** By construction, the cyclic morphism δ is equivariant for the dual action θ_λ of $\mathbb{R}_+^*$ (*cf.* (4.17)). This makes it possible to consider the corresponding representation of $\mathbb{R}_+^*$ in the cyclic homology group, *i.e.*

$$\theta(\lambda) \in \mathrm{Aut}(HC_0(D(\mathcal{A},\varphi))), \quad \forall\, \lambda \in \mathbb{R}_+^*. \tag{4.33}$$

The spectrum of this representation defines a very subtle invariant of the original noncommutative space $(\mathcal{A},\varphi)$.
Notice that, since the Λ-module $\mathcal{S}^\natural(\tilde{\Omega}_\beta)$ is commutative, so is the cokernel of δ (it is a quotient of the above). Thus, the cyclic homology group $HC_0(D(\mathcal{A},\varphi))$ is simply given by the cokernel of δ in degree 0.
It might seem unnecessary to describe this as HC_0 instead of simply looking at the cokernel of δ in degree 0, but the main reason is the Morita invariance of cyclic homology which wipes out the distinction between algebras such as B and $B \otimes \mathcal{L}^1$.
We now compute the invariant described above in the case of the BC-system described in Example 3.2 of Section 3. Recall that one has a canonical action of the Galois group $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the system, which factorizes through the action on the cyclotomic extension $\mathbb{Q}^{ab}$, yielding an action of $G^{ab} = \hat{\mathbb{Z}}^*$ as symmetries of the BC-system. Thus, one obtains a decomposition as a direct sum using idempotents associated to Grössencharakters.

Proposition 4.15. *Let χ be a character of the compact group $\hat{\mathbb{Z}}^*$. Then the expression*

$$p_\chi = \int_{\hat{\mathbb{Z}}^*} g\, \chi(g)\, dg$$

determines an idempotent p_χ in $\mathrm{End}_\Lambda D(\mathcal{A},\varphi)$.

Notice that the idempotent p_χ already makes sense in the category $\mathcal{EV}^0_{K,\mathbb{C}}$ introduced above in Section 3.
The idempotents p_χ add up to the identity, namely

$$\sum_\chi p_\chi = \mathrm{Id}, \tag{4.34}$$

and we get a corresponding direct sum decomposition of the representation (4.33).

Theorem 4.16. *The representation of* $\mathbb{R}_+^*$ *in*

$$\mathcal{M} = HC_0(p_\chi\, D(\mathcal{A},\varphi)) \tag{4.35}$$

gives the spectral realization of the zeros of the L-function L_χ. *More precisely, let* $z \in I_\beta$ *be viewed as a character of* $\mathrm{Hol}(I_\beta)$ *and* $\mathbb{C}_z$ *be the corresponding one dimensional module. One has*

$$\mathcal{M} \otimes_{\mathrm{Hol}(I_\beta)} \mathbb{C}_z \neq \{0\} \Longleftrightarrow L_\chi(-i\,z) = 0. \tag{4.36}$$

One can give a geometric description of the BC-system in terms of $\mathbb{Q}$-lattices in $\mathbb{R}$ (*cf.* [17]). The $\mathbb{Q}$-lattices in $\mathbb{R}$ are labeled by pairs$(\rho,\lambda) \in \hat{\mathbb{Z}} \times \mathbb{R}_+^*$ and the commensurability equivalence relation is given by the orbits of the action of the semigroup $\mathbb{N} = \mathbb{Z}_{>0}$ given by $n\,(\rho,\lambda) = (n\,\rho, n\,\lambda)$. The $\mathbb{Q}$-lattice associated to the pair (ρ,λ) is $(\lambda^{-1}\mathbb{Z}, \lambda^{-1}\rho)$ where ρ is viewed in $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$. The BC-system itself corresponds to the noncommutative space of commensurability classes of $\mathbb{Q}$-lattices up to scale, which eliminates the λ (*cf.* [17]). The $\mathbb{Q}$-lattices up to scale are labelled by $\rho \in \hat{\mathbb{Z}}$ and the commensurability equivalence relation is given by the orbits of the multiplicative action of the semigroup $\mathbb{N}$. The algebra $\mathcal{A}$ of the BC-system is the crossed product $C^\infty(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ where $C^\infty(\hat{\mathbb{Z}})$ is the algebra of locally constant functions on $\hat{\mathbb{Z}}$. By construction every $f \in C^\infty(\hat{\mathbb{Z}})$ has a level *i.e.* can be written as

$$f = h \circ p_N\,, \quad p_N\ :\ \hat{\mathbb{Z}} \to \mathbb{Z}/N\mathbb{Z}. \tag{4.37}$$

General elements of $\mathcal{A}$ are finite linear combinations of monomials, as in (3.7). Equivalently one can describe $\mathcal{A}$ as a subalgebra of the convolution algebra of the étale groupoid G_{bc} associated to the partial action of $\mathbb{Q}_+^*$ on $\hat{\mathbb{Z}}$. One has $G_{bc} = \{(k,\rho) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} \,|\, k\,\rho \in \hat{\mathbb{Z}}\}$ and $(k_1,\rho_1)\circ(k_2,\rho_2) = (k_1k_2,\rho_2)$ if $\rho_1 = k_2\,\rho_2$. The product in the groupoid algebra is given by the associative convolution product

$$f_1 * f_2\,(k,x) = \sum_{s\in\mathbb{Q}_+^*} f_1(k\,s^{-1}, s\,x) f_2(s,x), \tag{4.38}$$

and the adjoint is given by $f^*(k,x) = \overline{f(k^{-1},k\,x)}$. The elements of the algebra $\mathcal{A}$ are functions $f(k,\rho)$ with finite support in $k \in \mathbb{Q}_+^*$ and finite level (*cf.* (4.37)) in the variable ρ. The time evolution is given by

$$\sigma_t(g)(k,\rho) = k^{it}\, g(k,\rho)\,, \quad \forall\, g \in \mathcal{A}\,. \tag{4.39}$$

The dual system of the BC-system corresponds to the noncommutative space of commensurability classes of $\mathbb{Q}$-lattices in $\mathbb{R}$ (*cf.* [17]). Thus, the dual algebra $\hat{\mathcal{A}}$ for the BC-system is a subalgebra of the convolution algebra of the étale groupoid $\hat{G}_{bc}$ associated to the partial action of $\mathbb{Q}_+^*$ on $\hat{\mathbb{Z}}\times\mathbb{R}_+^*$. The product in the groupoid algebra is given by the same formula as (4.38) with $x = (\rho,\lambda) \in \hat{\mathbb{Z}} \times \mathbb{R}_+^*$.
Let $x = \int x(t)\,U_t\,dt \in \hat{\mathcal{A}}$. Then the corresponding function $f = \iota(x)$ on $\hat{G}_{bc}$ is given explicitly by

$$f(k,\rho,\lambda) = \int x(t)(k,\rho)\,\lambda^{it}\,dt. \tag{4.40}$$

One checks that ι is an algebra homomorphism using (4.39). Using the definition (4.9) of the Schwartz space $\mathcal{S}(\mathbb{R},\mathcal{A})$, one gets that elements of $\iota\hat{\mathcal{A}}$ are functions $f(k,\rho,\lambda)$ with finite support in k, finite level in ρ, and rapid decay (Schwartz functions) on $\mathbb{R}_+^* \sim \mathbb{R}$. Similarly, elements of $\iota\hat{\mathcal{A}}_\beta$ are functions $f(k,\rho,\lambda)$ with finite support in k, finite level in ρ, and such that the finitely many scalar functions

$$f_{k,\rho}(\lambda) = f(k,\rho,\lambda)\,, \quad \forall\,\lambda\in\mathbb{R}_+^*$$

are in the Fourier transform of the space $\mathcal{S}(I_\beta)$,

$$f\in\iota\hat{\mathcal{A}}_\beta \iff \tilde{F}(f_{k,\rho})\in\mathcal{S}(I_\beta)\,, \quad \forall\,(k,\rho). \tag{4.41}$$

Here the Fourier transform $\tilde{F}$ in the variable $\lambda\in\mathbb{R}_+^*$ corresponds to (4.29) in the variable $c=\log(\lambda)$, *i.e.*

$$\tilde{F}(g)(u) = \int g(\lambda)\,\lambda^{-iu}\,d^*\lambda. \tag{4.42}$$

Up to Morita equivalence $\hat{G}_{bc}$ is the same as the groupoid of the partial action of $\mathbb{Q}^*$ on $\hat{\mathbb{Z}}\times\mathbb{R}^*$. It is not quite the same as the adele class space *i.e.* the action of $\mathbb{Q}^*$ on $\mathbb{A}_\mathbb{Q}$, not because of the strict inclusion $\hat{\mathbb{Z}}\subset\mathbb{A}_f$ of $\hat{\mathbb{Z}}$ in finite adeles, which is a Morita equivalence, but because we are missing the compact piece $\hat{\mathbb{Z}}\times\{0\}\subset\mathbb{A}_\mathbb{Q}$. Modulo this nuance, one can compare the restriction map δ with the map E of [14] given (up to normalization by $|j|^{1/2}$) by

$$E(f)(j) = \sum_{q\in\mathbb{Q}^*} f(q\,j)\,, \quad j\in C_\mathbb{Q}\,, \quad f\in\mathcal{S}(\mathbb{A}_\mathbb{Q})\,, \tag{4.43}$$

where $\mathbb{A}_\mathbb{Q}$ are the adeles of $\mathbb{Q}$ and $C_\mathbb{Q}$ the idele class group.

Lemma 4.17. *1) For $\beta>1$ one has a canonical isomorphism $\tilde{\Omega}_\beta\sim\hat{\mathbb{Z}}^*\times\mathbb{R}_+^*\sim C_\mathbb{Q}$ of $\tilde{\Omega}_\beta$ with the space of invertible $\mathbb{Q}$-lattices.*
2) Let $x\in\hat{\mathcal{A}}$ and $f=\iota(x)$ be the corresponding function on $\hat{G}_{bc}\subset\mathbb{Q}_+^*\times\hat{\mathbb{Z}}\times\mathbb{R}_+^*$. Then

$$\delta(x)(j) = \sum_{n\in\mathbb{N}^*} f(1,n\,u,n\,\lambda)\,, \quad \forall\,j=(u,\lambda)\in\hat{\mathbb{Z}}^*\times\mathbb{R}_+^* = C_\mathbb{Q}\,.$$

Proof. 1) Let us give explicitly the covariant irreducible representation of $\mathcal{A}$ associated to a pair $(u,\lambda)\in\hat{\mathbb{Z}}^*\times\mathbb{R}_+^*$. The algebra acts on the fixed Hilbert space $\mathcal{H}=\ell^2(\mathbb{N}^*)$ with

$$(\pi_{(u,\lambda)}(x)\,\xi)(n) = \sum_{m\in\mathbb{N}^*} x(n\,m^{-1},\,m\,u)\,\xi(m)\,, \quad \forall\,x\in\mathcal{A},$$

while $\pi_{(u,\lambda)}(H)$ is the diagonal operator D_λ of multiplication by $\log n+\log\lambda$.
The classification of KMS$_\beta$ states of the BC system ([17]) shows that the map $(u,\lambda)\to\pi_{(u,\lambda)}$ is an isomorphism of $\hat{\mathbb{Z}}^*\times\mathbb{R}_+^*$ with $\tilde{\Omega}_\beta$ for $\beta>1$.
2) By construction, the function $\delta(x)(j)$ is given, for $j=(u,\lambda)\in\hat{\mathbb{Z}}^*\times\mathbb{R}_+^*$, by $\delta(x)(j)=\mathrm{Trace}(\pi_{(u,\lambda)}(x))$. One has

$$\pi_{(u,\lambda)}(x) = \int \pi_{(u,\lambda)}(x(t))\,e^{itD_\lambda}\,dt$$

and the trace is the sum of the diagonal entries of this matrix. Namely, it is given by

$$\sum_{n\in\mathbb{N}^*}\int x(t)(1,nu)\,e^{it(\log n+\log\lambda)}\,dt = \sum_{n\in\mathbb{N}^*} f(1,n\,u,n\,\lambda),$$

where we used (4.40). $\square$

Proof of Lemma 4.9. Using Lemma 4.17 we need to show that for any $u \in \hat{\mathbb{Z}}^*$ and any function $f(k,\rho,\lambda)$ in $\iota\hat{\mathcal{A}}_\beta$, one has when $\lambda \to 0$,

$$\sum_{n\in\mathbb{N}^*} f(1,n\,u,n\,\lambda) = \lambda^{-1}\int f(1,\rho,v)\,d\rho\,dv + O(|\log(\lambda)|^{-N}) \tag{4.44}$$

for all $N > 0$. One checks indeed that the dual trace (*cf.* [52]) is given explicitly by the additive Haar measure

$$\tau(f) = \int f(1,\rho,v)\,d\rho\,dv$$

so that (4.44) implies Lemma 4.9. We only consider the function of two variables $f(\rho,v) = f(1,\rho,v)$ and we let $\tilde{f}$ be its unique extension to adeles by 0 by setting $\tilde{f}$ equal to zero outside of $\hat{\mathbb{Z}}\times\mathbb{R}^+ \subset \mathbb{A}_\mathbb{Q}$. Thus, the left hand side of (4.44) is given by

$$\sum_{n\in\mathbb{N}^*} f(1,n\,u,n\,\lambda) = \sum_{q\in\mathbb{Q}^*} \tilde{f}(q\,j)\,, \quad j=(u,\lambda). \tag{4.45}$$

If the extended function $\tilde{f}$ were in the Bruhat-Schwartz space $\mathcal{S}(\mathbb{A}_\mathbb{Q})$, then one could use the Poisson summation formula

$$|j|\sum_{q\in\mathbb{Q}} h(q\,j) = \sum_{q\in\mathbb{Q}} \widehat{h}(q\,j^{-1}),$$

with $\widehat{h}$ the Fourier transform of h, in order to get a better estimate with a remainder in $O(|\lambda|^N)$. However, in general one has $\tilde{f}\notin\mathcal{S}(\mathbb{A}_\mathbb{Q})$, because of the singularity at $\lambda=0$. At least some functions f with non-zero integral do extend to an element $\tilde{f}\in\mathcal{S}(\mathbb{A}_\mathbb{Q})$ and this allows us to restrict to functions of two variables $f(\rho,v)$ such that

$$\int f(\rho,v)\,d\rho\,dv = 0\,. \tag{4.46}$$

We want to show that, if $f(\rho,v) = g(p_n(\rho),v)$ with g a map from $\mathbb{Z}/n\mathbb{Z}$ to $\mathcal{S}(\mathbb{R}_+^*)$, and if (4.46) holds, then one has

$$E(\tilde{f})(j) = \sum_{q\in\mathbb{Q}^*} \tilde{f}(q\,j) = O(|\log|j|\,|^{-N}) \tag{4.47}$$

for $|j|\to 0$ and for all $N>0$. Furthermore, we can assume that there exists a character χ_n of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ such that

$$g(m\,a,v) = \chi_n(m)\,g(a,v)\,, \quad \forall\,a\in\mathbb{Z}/n\mathbb{Z}\,, \quad m\in(\mathbb{Z}/n\mathbb{Z})^*\,.$$

With $\chi = \chi_n\circ p_n$, one then has

$$\tilde{f}(m\,a,v) = \chi(m)\,\tilde{f}(a,v)\,, \quad \forall\,a\in\hat{\mathbb{Z}}\,, \quad m\in\hat{\mathbb{Z}}^*\,. \tag{4.48}$$

In particular, one gets

$$E(\tilde{f})(m\,j) = \chi(m)\,E(\tilde{f})(j) \quad \forall m\in\hat{\mathbb{Z}}^*,\ \forall j\in C_\mathbb{Q}\,. \tag{4.49}$$

This shows that, in order to prove (4.47), it is enough to prove the estimate

$$|h(\lambda)| = O(|\log|\lambda|\,|^{-N}), \quad \text{for } \lambda\to 0, \tag{4.50}$$

where

$$h(\lambda) = \int_{\hat{\mathbb{Z}}^*} E(\tilde{f})(m\,j)\,\chi_0(m)\,d^*m\,, \quad \forall\,j \quad \text{with} \quad |j|=\lambda,$$

and with $\chi_0=\bar{\chi}$, the conjugate of χ. Using (4.42), the Fourier transform of h (viewed as a function on $\mathbb{R}_+^*$) is given by

$$\tilde{F}h(t) = \int h(\lambda)\,\lambda^{-it}\,d^*\lambda = \int_{\mathbb{A}_\mathbb{Q}^*} \tilde{f}(j)\,\chi_0(j)\,|j|^s\,d^*j\,, \quad \text{with} \quad s=-it\,, \tag{4.51}$$

where d^*j is the multiplicative Haar measure on the ideles and χ_0 is extended as a unitary character of the ideles. More precisely, we identify $\mathbb{A}_{\mathbb{Q}}^* = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^* \times \mathbb{Q}^*$ where $\mathbb{Q}^*$ is embedded as principal ideles. We extend χ_0 by 1 on $\mathbb{R}_+^* \times \mathbb{Q}^* \subset \mathbb{A}_{\mathbb{Q}}^*$. Thus χ_0 becomes a unitary Grössencharakter. By Weil's results on homogeneous distributions (*cf.* [14] Appendix I, Lemma 1), one has, with $\tilde{f}$ as above, the equality

$$\int_{\mathbb{A}_{\mathbb{Q}}^*} \tilde{f}(j)\,\chi_0(j)\,|j|^s\,d^*j = L(\chi_0,s)\,\langle D'(s),\tilde{f}\rangle\,, \quad \text{for } \Re(s)>1, \tag{4.52}$$

where $D' = \otimes D'_v$ is an infinite tensor product over the places of $\mathbb{Q}$ of homogeneous distributions on the local fields $\mathbb{Q}_v$ and $L(\chi_0,s)$ is the L-function with Grössencharakter χ_0. This equality is a priori only valid for $\tilde{f} \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$, but it remains valid for $\tilde{f}$ by absolute convergence of both sides. In fact, we can assume that $f(\rho,v) = f_0(\rho)f_\infty(v)$, where $f_\infty \in \mathcal{S}(\mathbb{R}_+^*)$. One then gets $\tilde{f} = \tilde{f}_0 \otimes \tilde{f}_\infty$ and

$$\langle D'(s),\tilde{f}\rangle = \langle D'_0(s),\tilde{f}_0\rangle\,\langle D'_\infty(s),\tilde{f}_\infty\rangle,$$

where by construction (*cf.* [14]) one has

$$\langle D'_\infty(s),\tilde{f}_\infty\rangle = \int_{\mathbb{R}_+^*} f_\infty(\lambda)\,\lambda^s d^*\lambda\,, \quad d^*\lambda = d\lambda/\lambda. \tag{4.53}$$

Using (4.41), one gets $\tilde{F}(f_\infty) \in \mathcal{S}(I_\beta)$. Thus, the function $t \mapsto \langle D'_\infty(-it),\tilde{f}_\infty\rangle$ is in $\mathcal{S}(I_\beta)$. The term $\langle D'_0(s),\tilde{f}_0\rangle$ is simple to compute explicitly and the function $t \mapsto \langle D'_0(-it),\tilde{f}_0\rangle$ belongs to $\mathrm{Hol}(I_\beta)$.

Indeed, it is enough to do the computation when f_0 is a finite tensor product $f_0 = \otimes_{v\in S} f_p$ over a finite set S of primes, while f_p is a locally constant function on $\mathbb{Q}_p$ vanishing outside $\mathbb{Z}_p$ and such that $f_p(ma) = \chi(m)\,f(a)$, for all $a \in \mathbb{Z}_p$, $m \in \mathbb{Z}_p^*$, using (4.48) for the restriction of χ to $\mathbb{Z}_p^*$. One has

$$\langle D'_0(s),\tilde{f}_0\rangle = \prod_S \langle D'_p(s),\tilde{f}_p\rangle.$$

One can assume that S contains the finite set P of places at which χ is ramified, *i.e.* where the restriction of χ to $\mathbb{Z}_p^*$ is non-trivial. For $p \in P$ one gets that f_p vanishes in a neighborhood of $0 \in \mathbb{Z}_p$, while

$$\langle D'_p(s),\tilde{f}_p\rangle = \int_{\mathbb{Q}_p^*} f_p(x)\,\chi_0(x)\,|x|^s\,d^*x,$$

which, as a function of s, is a finite sum of exponentials p^{ks}.

For $p \notin P$, one has the normalization condition

$$\langle D'_p(s), 1_{\mathbb{Z}_p}\rangle = 1$$

and the function f_p is a finite linear combination of the functions $x \to 1_{\mathbb{Z}_p}(p^k\,x)$. It follows from homogeneity of the distribution $D'_p(s)$ that $\langle D'_p(s),\tilde{f}_p\rangle$, as a function of s, is a finite sum of exponentials p^{ks}.

Similarly, if χ_0 is non-trivial, the function $L(\chi_0,-it)$ belongs to $\mathrm{Hol}(I_\beta)$ and $\zeta(-it) = L(1,-it)$ is the sum of $1/(-it-1)$ with an element of $\mathrm{Hol}(I_\beta)$. The presence of the pole at $t=i$ for $\chi_0 = 1$ is taken care of by the condition (4.46). All of this shows that

$$\tilde{F}h \in \mathcal{S}(I_\beta)\,, \quad \tilde{F}h(z) = 0 \quad \forall z \in I_\beta\,, \quad \text{with } L(\chi_0,-iz) = 0. \tag{4.54}$$

This shows in particular that $h \in \mathcal{S}(\mathbb{R}_+^*)$, which gives the required estimate (4.50) on $|h(\lambda)|$, for $\lambda \to 0$. This ends the proof of Lemma 4.9.

Proof of Theorem 4.16.

It follows from (4.54) that all elements h of the range of δ in degree zero have Fourier transform $\tilde{F}h \in \mathcal{S}(I_\beta)$ that vanishes at any z such that $L(\chi_0, -iz) = 0$. Moreover, explicit choices of the test function show that, if we let $L_{\mathbb{Q}}$ be the complete L-function (with its Archimedean Euler factors), then, for χ_0 non-trivial, $L_{\mathbb{Q}}(\chi_0, -iz)$ is the Fourier transform of an element of the range of δ. When χ_0 is trivial, the range of δ contains $\xi(z - \frac{i}{2})$, where ξ is the Riemann ξ function [48]

$$\xi(t) := -\frac{1}{2}\left(\frac{1}{4} + t^2\right)\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\pi^{-\frac{1}{4} - \frac{it}{2}}\zeta\left(\frac{1}{2} + it\right).$$

Thus, we see that, for h in the range of δ in degree zero, the set Z_{χ_0} of common zeros of the $\tilde{F}h$ is given by

$$Z_{\chi_0} = \{z \in I_\beta \mid L_{\mathbb{Q}}(\chi_0, -iz) = 0\}.$$

Using Lemma 4.10 one gets the required spectral realization.
Indeed, this lemma shows that the $\mathrm{Hol}(I_\beta)$-module $\mathcal{M}$ is the quotient of $\mathcal{S}(I_\beta)$ by the $\mathrm{Hol}(I_\beta)$-submodule $\mathcal{N} = \{\tilde{F}h \,|\, h \in \delta(\hat{\mathcal{A}}_\beta)\}$. Let $z \in I_\beta$ with $L_{\mathbb{Q}}(\chi_0, -iz) = 0$. Then the map $\mathcal{S}(I_\beta) \ni f \mapsto f(z)$ vanishes on $\mathcal{N}$ and yields a non-trivial map of $\mathcal{M} \otimes_{\mathrm{Hol}(I_\beta)} \mathbb{C}_z$ to $\mathbb{C}$. Let $z \in I_\beta$ with $L_{\mathbb{Q}}(\chi_0, -iz) \neq 0$. Any element of $\mathcal{M} \otimes_{\mathrm{Hol}(I_\beta)} \mathbb{C}_z$ is of the form $f \otimes 1$, with $f \in \mathcal{S}(I_\beta)$. There exists an element $\xi = g \otimes 1 \in \mathcal{N}$ such that $g(z) \neq 0$. Thus, we can assume, since we work modulo $\mathcal{N}$, that $f(z) = 0$. It remains to show that $f \otimes 1$ is in $\mathcal{S}(I_\beta) \otimes_{\mathrm{Hol}(I_\beta)} \mathbb{C}_z$. To see this, one writes $f(t) = (t - z)\, f_1(t)$ and checks that $f_1 \in \mathcal{S}(I_\beta)$, while

$$f \otimes 1 = (t - z)\, f_1 \otimes 1 \sim f_1 \otimes (t - z) = 0.$$

4.5. **Type III factors and unramified extensions.** In this section we briefly recall the classification of type III factors (*cf.* [10]) and we explain the analogy between the reduction from type III to type II and automorphisms, first done in [10], and the use by Weil of the unramified extensions $K \subset K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ of global fields of positive characteristic.
Central simple algebras over a (local) field $\mathbb{L}$ naturally arise as the commutant of isotypic representations of semi-simple algebraic structures over $\mathbb{L}$. When one works over $\mathbb{L} = \mathbb{C}$ it is natural to consider unitary representations in Hilbert space and to restrict to the algebras M (called von Neumann algebras) which appear as commutants of unitary representations. The central von Neumann algebras are called *factors*.
The module of a factor M was first defined in [10] as a closed subgroup of $\mathbb{R}_+^*$ by the equality

$$S(M) = \bigcap_{\varphi} \mathrm{Spec}(\Delta_\varphi) \subset \mathbb{R}_+, \tag{4.55}$$

where φ varies among (faithful, normal) states on M, while the operator Δ_φ is the modular operator (4.2).
The central result of [10], which made it possible to perform the classification of factors, is that the 1-parameter family

$$\sigma_t^\varphi \in \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$$

is *independent* of the choice of the state φ. Here $\mathrm{Inn}(M)$ is the group of inner automorphisms of M. This gives a canonical one parameter group of automorphism classes

$$\delta : \mathbb{R} \to \mathrm{Out}(M). \tag{4.56}$$

This implies, in particular, that the crossed product $\hat{M} = M \rtimes_{\sigma_t^\varphi} \mathbb{R}$ is *independent* of the choice of φ and one can show (*cf.* [25]) that the dual action θ_λ is also independent of the choice of φ.
The classification of *approximately finite dimensional* factors involves the following steps.

- The definition of the invariant $\mathrm{Mod}(M)$ for arbitrary factors (central von Neumann algebras).
- The equivalence of all possible notions of approximate finite dimensionality.
- The proof that Mod is a complete invariant[1] and that all virtual subgroups are obtained.

In the general case,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* . \tag{4.57}$$

was defined in [25] (see also [10]) and is a virtual subgroup of $\mathbb{R}_+^*$ in the sense of G. Mackey, i.e. an ergodic action of $\mathbb{R}_+^*$. All ergodic flows appear and M_1 is isomorphic to M_2 iff $\mathrm{Mod}(M_1) \cong \mathrm{Mod}(M_2)$.
The period group $T(M) = \mathrm{Ker}\,\delta$ of a factor was defined in [10] and it was shown that, for $T \in T(M)$, there exists a faithful normal state on M whose modular automorphism is periodic with $\sigma_T^\varphi = 1$. The relevance in the context described above should now be clear since for such a *periodic state* the discussion above regarding the distilled module simplifies notably. In fact, the multiplicative group $\mathbb{R}_+^*$ gets replaced everywhere by its discrete subgroup

$$\lambda^{\mathbb{Z}} \subset \mathbb{R}_+^* , \quad \lambda = e^{-2\pi/T}.$$

This situation happens in the context of the next section, for function fields, where one gets $\lambda = \frac{1}{q}$ in terms of the cardinality of the field of constants $\mathbb{F}_q$. In that situation the obtained factor is of type $\mathrm{III}_{\frac{1}{q}}$.
It is an open question to find relevant number theoretic examples dealing with factors of type III_0.
We can now state a proposition showing that taking the cross product by the modular automorphism is an operation entirely analogous to the unramified extension

$$\mathbb{K} \subset \mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$$

of a global field $\mathbb{K}$ of positive characteristic with field of constant $\mathbb{F}_q$.

Proposition 4.18. *Let M be a factor of type III_1.*
1) For $\lambda \in]0,1[$ and $T = -\frac{2\pi}{\ln \lambda}$, the algebra

$$M_T = M \rtimes_{\sigma_T} \mathbb{Z}$$

is a factor of type III_λ with

$$\mathrm{Mod}(M_T) = \lambda^{\mathbb{Z}} \subset \mathbb{R}_+^*.$$

2) One has a natural inclusion $M \subset M_T$ and the dual action

$$\theta \;:\; \mathbb{R}_+^* / \lambda^{\mathbb{Z}} \to \mathrm{Aut}\, M_T$$

of $U(1) = \mathbb{R}_+^* / \lambda^{\mathbb{Z}}$ admits M as fixed points,

$$M = \{x \in M_T \;;\; \theta_\mu(x) = x\,, \quad \forall \mu \in \mathbb{R}_+^* / \lambda^{\mathbb{Z}}\}.$$

In our case, the value of λ is given, in the analogy with the function field case, by $\lambda = \frac{1}{p}$. Thus, we get

$$T = \frac{2\pi}{\ln p}\,, \quad \mathrm{Mod}(M_T) = p^{\mathbb{Z}} \subset \mathbb{R}_+^*$$

We can summarize some aspects of the analogy between global fields and factors in the set-up decsribed above in the following table.

[1] we exclude the trivial case $M = M_n(\mathbb{C})$ of matrix algebras, also $\mathrm{Mod}(M)$ does not distinguish between type II_1 and II_∞

Global field $\mathbb{K}$	**Factor** M
$\mathrm{Mod}\,\mathbb{K} \subset \mathbb{R}_+^*$	$\mathrm{Mod}\,M \subset \mathbb{R}_+^*$
$\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$	$M \to M \rtimes_{\sigma_T} \mathbb{Z}$
$\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$M \to M \rtimes_\sigma \mathbb{R}$

Notice also the following. For a global field $\mathbb{K}$ of positive characteristic, the union of the fields $\mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, *i.e.* the field $\mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, is the field of rational functions from the associated curve C to $\bar{\mathbb{F}}_q$. In a similar manner, the crossed product $M \rtimes_\sigma \mathbb{R}$ corresponds, at the geometric level, to the space of commensurability classes of one dimensional $\mathbb{Q}$-lattices (*cf.* [17]) closely related to the noncommutative adele class space $X_{\mathbb{Q}}$ to which we now turn our attention.
It is of course crucial to follow the above analogy in the specific example of the BC-system. We shall show in §5.3 that there is a natural way to associate to a noncommutative space X not only its set of "classical points" but in a more subtle manner the set of all its classical points that can be defined over an unramified extension, through the following basic steps:

$$X \stackrel{Dual\,System}{\longrightarrow} \hat{X} \stackrel{Periodic\,Orbits}{\longrightarrow} \cup \hat{X}_v \stackrel{Classical\,Points}{\longrightarrow} \cup \Xi_v = \Xi$$

When applied to the BC-system this procedure yields a candidate for an analogue in characteristic zero of the geometric points $C(\bar{\mathbb{F}}_q)$ of the curve over $\bar{\mathbb{F}}_q$ in the function field case.

5. Geometry of the adele class space

We apply the cohomological construction of the previous section in the general context of global fields and we obtain a cohomological interpretation of the spectral realization of zeros of Hecke L-functions of [14]. This also provides a geometric framework in which one can begin to transpose the main steps of Weil's proof of RH in the case of positive characteristic to the case of number fields.
We give in this section a brief summary of these results, which will be dealt with in details in our forthcoming paper [15].

5.1. **The spectral realization on $\mathcal{H}^1_{\mathbb{K},\mathbb{C}}$.** We let $\mathcal{A} = \mathcal{S}(\mathbb{A}_{\mathbb{K}}) \rtimes \mathbb{K}^*$ be the noncommutative algebra of coordinates on the adele class space $X_{\mathbb{K}}$. It is the crossed product of the Bruhat-Schwartz algebra $\mathcal{S}(\mathbb{A}_{\mathbb{K}})$ by the action of $\mathbb{K}^*$. We let $\mathcal{G}_{\mathbb{K}} = \mathbb{K}^* \ltimes \mathbb{A}_{\mathbb{K}}$ be the locally compact etale groupoid associated to the action of $\mathbb{K}^*$ on $\mathbb{A}_{\mathbb{K}}$ and write $\mathcal{A} = \mathcal{S}(\mathcal{G}_{\mathbb{K}})$. We then consider the cyclic module

$$\mathcal{A}_0^\natural = \cap \,\mathrm{Ker}\, \varepsilon_j^\natural \tag{5.1}$$

where the two cyclic morphisms

$$\varepsilon_j^\natural \;:\; \mathcal{A}^\natural \to \mathbb{C}, \quad \text{for} \;\; j = 0, 1,$$

are given by the traces on $\mathcal{A}$ associated, respectively, to the evaluation at $0 \in \mathbb{A}_{\mathbb{K}}$ for ε_0 and to the integral on $\mathbb{A}_{\mathbb{K}}$ for ε_1. The integral is taken with respect to the additive Haar measure, which is $\mathbb{K}^*$-invariant and hence gives a trace on $\mathcal{A} = \mathcal{S}(\mathbb{A}_{\mathbb{K}}) \rtimes \mathbb{K}^*$. In other words, one has

$$\varepsilon_0(\sum f_g\, U_g) = f_1(0)\,, \qquad \varepsilon_1(\sum f_g\, U_g) = \int f_1(a)\, da, \tag{5.2}$$

and in higher degree

$$\varepsilon_j^\natural(x^0 \otimes \cdots \otimes x^n) = \varepsilon_j(x^0 \cdots x^n).$$

In this context, as we saw in the proof of Theorem 4.16, the cooling morphism of Section 4 is simply given by the restriction map to the idele class group $C_{\mathbb{K}} \subset X_{\mathbb{K}}$. In other words, it is given by the restriction map

$$\rho(\sum f_g\, U_g) = \sum f_g|_{\mathbb{A}_{\mathbb{K}}^*}\; U_g\,, \tag{5.3}$$

where $f|_{\mathbb{A}_{\mathbb{K}}^*} \in C(\mathbb{A}_{\mathbb{K}}^*)$ is the restriction of $f \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})$ to the ideles $\mathbb{A}_{\mathbb{K}}^* \subset \mathbb{A}_{\mathbb{K}}$. The action of $\mathbb{K}^*$ on $\mathbb{A}_{\mathbb{K}}^*$ is free and proper and the corresponding groupoid $\mathbb{K}^* \ltimes \mathbb{A}_{\mathbb{K}}^*$ is Morita equivalent to $C_{\mathbb{K}}$. We use the exact sequence of locally compact groups

$$1 \to \mathbb{K}^* \to \mathbb{A}_{\mathbb{K}}^* \xrightarrow{p} C_{\mathbb{K}} \to 1 \tag{5.4}$$

to parameterize the orbits of $\mathbb{K}^*$ as the fibers $p^{-1}(x)$ for $x \in C_{\mathbb{K}}$. By construction the Hilbert spaces

$$\mathcal{H}_x = \ell^2(p^{-1}(x))\,, \quad \forall\, x \in C_K \tag{5.5}$$

form a continuous field of Hilbert spaces over $C_{\mathbb{K}}$. We let $\mathcal{L}^1(\mathcal{H}_x)$ be the Banach algebra of trace class operators in $\mathcal{H}_x$, these form a continuous field over $C_{\mathbb{K}}$.

Proposition 5.1. *The restriction map ρ of* (5.3) *extends to an algebra homomorphism*

$$\rho : \mathcal{S}(\mathcal{G}_{\mathbb{K}}) \to C(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))\,. \tag{5.6}$$

Proof. Each $p^{-1}(x)$ is globally invariant under the action of $\mathbb{K}^*$ so the crossed product rules in $C_\rho(\mathbb{A}_{\mathbb{K}}^*) \rtimes \mathbb{K}^*$ are just multiplication of operators in $\mathcal{H}_x$. To show that the obtained operators are in $\mathcal{L}^1$ we just need to consider monomials $f_k\, U_k$. In that case the only non-zero matrix elements correspond to $k = xy^{-1}$. It is enough to show that, for any $f \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})$, the function $k \mapsto f(k\, b)$ is summable. This follows from the discreteness of $b\,\mathbb{K} \subset \mathbb{A}_{\mathbb{K}}$ and the construction of the Bruhat–Schwartz space $\mathcal{S}(\mathbb{A}_{\mathbb{K}})$, *cf.* [9], [14]. In fact the associated operator is of finite rank when f has compact support. In general what happens is that the sum will look like the sum over $\mathbb{Z}$ of the values $f(nb)$ of a Schwartz function f on $\mathbb{R}$. □

We let, in the number field case,

$$\mathbf{S}\,(C_{\mathbb{K}}) = \cap_{\beta \in \mathbb{R}}\, \mu^\beta \mathcal{S}(C_{\mathbb{K}}), \tag{5.7}$$

where $\mu \in C(C_{\mathbb{K}})$ is the module morphism from $C_{\mathbb{K}}$ to $\mathbb{R}_+^*$. In the function field case one can simply use for $\mathbf{S}\,(C_{\mathbb{K}})$ the Schwartz functions with compact support.

Definition 5.2. *We define $\mathbf{S}^{\natural}(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))$ to be the cyclic submodule of the cyclic module $C(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))^\natural$, whose elements are continuous functions such that the trace of the restriction to the main diagonal belongs to $\mathbf{S}\,(C_{\mathbb{K}})$.*

Note that for $T \in C(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))^\natural$ of degree n, $T(x_0, \ldots, x_n)$ is an operator in $\mathcal{H}_{x_0} \otimes \ldots \otimes \mathcal{H}_{x_n}$. On the diagonal, $x_j = x$ for all j, the trace map corresponding to $\mathrm{Trace}^\natural$ is given by

$$\mathrm{Trace}^\natural(T_0 \otimes T_1 \otimes \ldots \otimes T_n) = \mathrm{Trace}(T_0\, T_1\, \ldots\, T_n)\,. \tag{5.8}$$

This makes sense since on the diagonal all the Hilbert spaces $\mathcal{H}_{x_j}$ are the same.

Definition 5.3. *We define $\mathcal{H}^1_{\mathbb{K},\mathbb{C}}$ as the cokernel of the cyclic morphism*

$$\rho^\natural \;:\; \mathcal{S}(\mathcal{G}_{\mathbb{K}})^\natural_0 \to \mathbf{S}^{\,\natural}(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))$$

We can use the result of [11], describing the cyclic (co)homology in terms of derived functors in the category of cylic modules, to write the cyclic homology as

$$HC_n(\mathcal{A}) = \mathrm{Tor}_n(\mathbb{C}^\natural, \mathcal{A}^\natural). \tag{5.9}$$

Thus, we obtain a cohomological realization of the cyclic module $\mathcal{H}^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$ by setting

$$H^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}}) := \mathrm{Tor}(\mathbb{C}^\natural, \mathcal{H}^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})). \tag{5.10}$$

We think of this as an H^1 because of its role as a relative term in a cohomology exact sequence of the pair $(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$.

We now show that $H^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$ carries an action of $C_{\mathbb{K}}$, which we can view as the abelianization $W^{ab}_{\mathbb{K}} \sim C_{\mathbb{K}}$ of the Weil group. This action is induced by the multiplicative action of $C_{\mathbb{K}}$ on $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ and on itself. This generalizes to global fields the action of $C_{\mathbb{Q}} = \hat{\mathbb{Z}}^* \times \mathbb{R}^*_+$ on $HC_0(D(\mathcal{A}, \varphi))$ for the Bost–Connes endomotive.

Proposition 5.4. *The cyclic modules $\mathcal{S}(\mathcal{G}_{\mathbb{K}})^\natural_0$ and $\mathbf{S}^{\,\natural}(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))$ are endowed with an action of $\mathbb{A}^*_{\mathbb{K}}$ and the morphism $\rho^\natural$ is $\mathbb{A}^*_{\mathbb{K}}$-equivariant. This induces an action of $C_{\mathbb{K}}$ on $H^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$.*

Proof. For $\gamma \in \mathbb{A}^*_{\mathbb{K}}$ one defines an action by automorphisms of the algebra $\mathcal{A} = \mathcal{S}(\mathcal{G}_{\mathbb{K}})$ by setting

$$\vartheta_a(\gamma)(f)(x) := f(\gamma^{-1}x), \quad \text{for } f \in \mathcal{S}(\mathbb{A}_{\mathbb{K}}), \tag{5.11}$$

$$\vartheta_a(\gamma)\Big(\sum_{k\in\mathbb{K}^*} f_k\, U_k\Big) := \sum_{k\in\mathbb{K}^*} \vartheta_a(\gamma)(f_k)\, U_k\,. \tag{5.12}$$

This action is inner for $\gamma \in \mathbb{K}^*$ and induces an outer action

$$C_{\mathbb{K}} \to \mathrm{Out}(\mathcal{S}(\mathcal{G}_{\mathbb{K}}))\,. \tag{5.13}$$

Similarly, the continuous field $\mathcal{H}_x = \ell^2(p^{-1}(x))$ over $C_{\mathbb{K}}$ is $\mathbb{A}^*_{\mathbb{K}}$-equivariant for the action of $\mathbb{A}^*_{\mathbb{K}}$ on $C_{\mathbb{K}}$ by translations, and the equality

$$(V(\gamma)\xi)(y) := \xi(\gamma^{-1}\, y)\,, \quad \forall\, y \in p^{-1}(\gamma x)\,,\ \xi \in \ell^2(p^{-1}(x))\,, \tag{5.14}$$

defines an isomorphism $\mathcal{H}_x \xrightarrow{V(\gamma)} \mathcal{H}_{\gamma x}$. One obtains then an action of $\mathbb{A}^*_{\mathbb{K}}$ on $C(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))$ by setting

$$\vartheta_m(\gamma)(f)(x) := V(\gamma)\, f(\gamma^{-1}\, x)\, V(\gamma^{-1}), \quad \forall f \in C(C_{\mathbb{K}}, \mathcal{L}^1(\mathcal{H}_x))\,. \tag{5.15}$$

The morphism ρ is $\mathbb{A}^*_{\mathbb{K}}$-equivariant, so that one obtains an induced action on the cokernel $\mathcal{H}^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$. This action is inner for $\gamma \in \mathbb{K}^*$ and thus induces an action of $C_{\mathbb{K}}$ on $H^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$. ∎

We denote by

$$C_{\mathbb{K}} \ni \gamma \mapsto \underline{\vartheta}_m(\gamma) \tag{5.16}$$

the induced action on $H^1(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*, C_{\mathbb{K}})$.

Theorem 5.5. *The representation of $C_{\mathbb{K}}$ on*

$$H^1_{\mathbb{K},\mathbb{C}_\chi} = HC_0(\mathcal{H}^1_{\mathbb{K},\mathbb{C}_\chi})$$

gives the spectral realization of the zeros of the L-function with Grössencharakter χ.

This is a variant of Theorem 1 of [14]. We use the same notation as in [14] and, in particular, we take a non canonical isomorphism

$$C_{\mathbb{K}} \sim C_{\mathbb{K},1} \times N,$$

where $N \subset \mathbb{R}_+^*$ is the range of the module. For number fields one has $N = \mathbb{R}_+^*$, while for fields of nonzero characteristic $N \sim \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}_+^*$, where $q = p^\ell$ is the cardinality of the field of constants.
Given a character χ of $C_{\mathbb{K},1}$, we let $\widetilde{\chi}$ be the unique extension of χ to $C_{\mathbb{K}}$ which is equal to 1 on N.
The map E used in [14] involved a slightly different normalization since the result of the summation on $\mathbb{K}^*$ was then multiplied by $|x|^{1/2}$. Thus in essence the representation W given in [14] is related to (5.16) above by

$$\underline{\vartheta}_m(\gamma) = |\gamma|^{1/2}\, W(\gamma).$$

Instead of working at the Hilbert space level we deal with function spaces dictated by the algebras at hand. This implies in particular that we no longer have the restriction of unitarity imposed in [14] and we get the spectral side of the trace formula in the following form.

Theorem 5.6. *For any function $f \in \mathbf{S}\,(C_{\mathbb{K}})$, the operator*

$$\underline{\vartheta}_m(f) = \int \underline{\vartheta}_m(\gamma)\, f(\gamma)\, d^*\gamma$$

acting on $H^1_{\mathbb{K},\mathbb{C}_\chi}$ is of trace class, and its trace is given by

$$\text{Trace }\,(\underline{\vartheta}_m(f)|_{H^1}) = \sum_{L(\widetilde{\chi},\rho)=0\;,\;\rho\in\mathbb{C}/N^\perp} \widehat{f}(\widetilde{\chi},\rho),$$

where the Fourier transform $\widehat{f}$ of f is defined by

$$\widehat{f}(\widetilde{\chi},\rho) = \int_{C_{\mathbb{K}}} f(u)\, \widetilde{\chi}(u)\, |u|^\rho\, d^*u\,.$$

This is a variant of Corollary 2 of [14]. At the technical level the work of Ralf Meyer [46] is quite relevant and Theorem 5.6 follows directly from [46].

5.2. **The Trace Formula.** A main result in [14] is a trace formula, called the S-local trace formula, for the action of the class group on the simplified noncommutative space obtained by considering only finitely many places $v \in S$ of the global field $\mathbb{K}$. The remarkable feature of that formula is that it produces exactly the complicated principal values which enter in the Riemann-Weil explicit formulas of number theory. It is then shown in [14] that, in the Hilbert space context, the global trace formula is in fact equivalent to the Riemann hypothesis for all L-functions with Grössencharakter.
It was clear from the start that relaxing from the Hilbert space framework to the softer one of nuclear spaces would eliminate the difficulty coming from the potential non-critical zeros,

so that the trace formula would reduce to the Riemann-Weil explicit formula. However, it was not obvious how to obtain a direct geometric proof of this formula from the S-local trace formula of [14]. This was done in [46], showing that the noncommutative geometry framework makes it possible to give a geometric interpretation of the Riemann-Weil explicit formula. While the spectral side of the trace formula was given in Theorem 5.6, the geometric side is given as follows.

Theorem 5.7. *Let $h \in \mathbf{S}\,(C_{\mathbb{K}})$. Then the following holds:*

$$\text{Trace}\;(\underline{\vartheta}_m(h)|_{H^1}) = \widehat{h}(0) + \widehat{h}(1) - \Delta \bullet \Delta\; h(1) - \sum_v \int'_{(\mathbb{K}_v^*, e_{\mathbb{K}_v})} \frac{h(u^{-1})}{|1-u|}\, d^*u\,. \tag{5.17}$$

We used the following notation. For a local field $\mathbb{L}$ one chooses a preferred additive character $e_{\mathbb{L}}$ with the following properties.

- $e_{\mathbb{R}}(x) = e^{-2\pi\, ix}\,, \quad \forall\, x \in \mathbb{R}$
- $e_{\mathbb{C}}(z) = e^{-2\pi\, i(z+\bar{z})}\,, \quad \forall\, z \in \mathbb{C}$
- for $\mathbb{L}$ non-archimedian with maximal compact subring $\mathcal{O}$ the character $e_{\mathbb{L}}$ fulfills the condition $\operatorname{Ker} e_{\mathbb{L}} = \mathcal{O}$.

One lets $d^*\lambda$ be the multiplicative Haar measure normalized by

$$\int_{1 \le |\lambda| \le \Lambda} d^*\lambda \sim \log \Lambda \qquad \text{when } \Lambda \to \infty$$

For $\mathbb{L} = \mathbb{K}_v$, the definition of the finite value

$$\int'_{(\mathbb{K}_v^*, \beta)} \frac{h(u^{-1})}{|1-u|}\, d^*u$$

relative to a given additive character β of $\mathbb{L}$ is obtained as follows [14]. Let ϱ_β be the unique distribution extending d^*u at $u = 0$ whose Fourier transform relative to β *i.e.*

$$\int \varrho(x)\, \beta(xy) dx = \hat{\varrho}(y).$$

vanishes at 1, $\widehat{\varrho}(1) = 0$. Notice that this does not depend upon the normalization of the additive Haar measure dx of $\mathbb{K}_v$. One then has by definition

$$\int'_{(\mathbb{L}, \beta)} \frac{h(u^{-1})}{|1-u|}\, d^*u = \langle \varrho_\beta,\, g \rangle\;, \quad g(\lambda) = h((\lambda+1)^{-1})\, |\lambda + 1|^{-1}\,.$$

The slight shift of notation with [14] comes from the choice of the additive characters $e_{\mathbb{K}_v}$ instead of characters $\alpha_v(x) = e_{\mathbb{K}_v}(a_v\, x)$ such that $\mathbb{K} \subset \mathbb{A}_{\mathbb{K}}$ be self-dual relative to $\prod \alpha_v$. The idele $a = (a_v)$ is called a *differental idele.* This introduces the term $\Delta \bullet \Delta$ given by

$$\Delta \bullet \Delta = \log |a| \tag{5.18}$$

This coincides, up to the overall factor $\log q$, with the Euler characteristic in the function field case and with $-\log |D| \le 0$ in the case of number fields, D being the discriminant.

5.3. **The "periodic classical points" of $X_{\mathbb{K}}$.** The origin (*cf.* [14]) of the terms in the geometric side of the trace formula (Theorem 5.7) comes from the Lefschetz formula by Atiyah-Bott [2] and its adaptation by Guillemin-Sternberg (*cf.* [33]) to the distribution theoretic trace for flows on manifolds, which is a variation on the theme of [2]. For the action of $C_{\mathbb{K}}$ on the adele class space $X_{\mathbb{K}}$ the relevant periodic points are

$$P = \{(x, u) \in X_{\mathbb{K}} \times C_{\mathbb{K}} \,|\, u\, x = x\} \tag{5.19}$$

and one has (*cf.* [14])

Proposition 5.8. *Let $(x,u) \in P$, with $u \neq 1$. There exists a place $v \in \Sigma_{\mathbb{K}}$ such that*

$$x \in X_{\mathbb{K},v} = \{x \in X_{\mathbb{K}} \,|\, x_v = 0\} \tag{5.20}$$

The isotropy subgroup of any $x \in X_{\mathbb{K},v}$ contains the cocompact subgroup

$$\mathbb{K}_v^* \subset C_{\mathbb{K}}\,, \quad \mathbb{K}_v^* = \{(k_w) \,|\, k_w = 1 \; \forall w \neq v\} \tag{5.21}$$

The spaces $X_{\mathbb{K},v}$ are noncommutative spaces but they admit classical points distilled (in the sense of Section 4). One then obtains the following description: For each place $v \in \Sigma_{\mathbb{K}}$, one lets $[v]$ be the adele

$$[v]_w = 1\,, \quad \forall\, w \neq v\,, \quad [v]_v = 0\,. \tag{5.22}$$

Definition 5.9. *Let $\mathbb{K}$ be a global field and $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ be the adele class space of $\mathbb{K}$. The set of* periodic classical *points of $X_{\mathbb{K}}$ is*

$$\Xi_{\mathbb{K}} = \cup\, C_{\mathbb{K}}\,[v]\,, \quad v \in \Sigma_{\mathbb{K}}. \tag{5.23}$$

The reason why the space $\Xi_{\mathbb{K},\,v} = C_{\mathbb{K}}\,[v]$ inside the periodic orbit $X_{\mathbb{K},v}$ can be regarded as its set of classical points will be discussed more in detail in [15]. In essence, one shows first that each $X_{\mathbb{K},v}$ is a noncommutative space of type III with a canonical time evolution σ_t^v. While points $x \in X_{\mathbb{K},v}$ yield irreducible representations π_x of the corresponding crossed product algebra, one shows that this representation has positive energy (*i.e.* the generator of σ_t^v has positive spectrum) if and only if $x \in \Xi_{\mathbb{K},\,v} \subset X_{\mathbb{K},v}$.

While the structure of the space $\Xi_{\mathbb{K}}$ only comes from its being embedded in the ambient noncommutative space $X_{\mathbb{K}} \smallsetminus C_{\mathbb{K}}$, it is still worthwhile, in order to have a mental picture by which to translate to standard geometric language, to describe $\Xi_{\mathbb{K}}$ explicitly both in the function field case and in the case $\mathbb{K} = \mathbb{Q}$.

The reduction to eigenspaces of Grössencharakters corresponds to induced bundles over the quotient $X_{\mathbb{K}}/C_{\mathbb{K},1}$ so that we may as well concentrate on the quotient space $\Xi_{\mathbb{K}}/C_{\mathbb{K},1}$ and see how it looks like under the action of the quotient group $C_{\mathbb{K}}/C_{\mathbb{K},1}$.

In the function field case, one has a *non-canonical* isomorphism of the following form.

Proposition 5.10. *Let $\mathbb{K}$ be the function field of an algebraic curve C over $\mathbb{F}_q$. Then the action of the Frobenius on $Y = C(\bar{\mathbb{F}}_q)$ is isomorphic to the action of $q^{\mathbb{Z}}$ on the quotient*

$$\Xi_{\mathbb{K}}/C_{\mathbb{K},1}.$$

In the case $\mathbb{K} = \mathbb{Q}$ the space $\Xi_{\mathbb{Q}}/C_{\mathbb{Q},1}$ appears as the union of periodic orbits of period $\log p$ under the action of $C_{\mathbb{Q}}/C_{\mathbb{Q},1} \sim \mathbb{R}$ (*cf.* Figure 1).

5.4. **Weil positivity.** It is a well known result of A. Weil (*cf.* [55], [6]) that RH is equivalent to the positivity of the distribution entering in the explicit formulae. This can be stated as follows.

Theorem 5.11. *The following two conditions are equivalent.*

- *All L-functions with Grössencharakter on $\mathbb{K}$ satisfy the Riemann Hypothesis.*
- Trace $\underline{\vartheta}_m(f \star f^\sharp)|_{H^1} \geq 0$, *for all $f \in \mathbf{S}\,(C_{\mathbb{K}})$.*

Here we used the notation

$$f = f_1 \star f_2\,, \quad \text{with} \;\; (f_1 \star f_2)(g) = \int f_1(k)\, f_2(k^{-1}\, g)\, d^*g \tag{5.24}$$

for the convolution of functions, using the multiplicative Haar measure d^*g, and for the adjoint

$$f \to f^\sharp\,, \quad f^\sharp(g) = |\,g|^{-1}\, \bar{f}(g^{-1}). \tag{5.25}$$

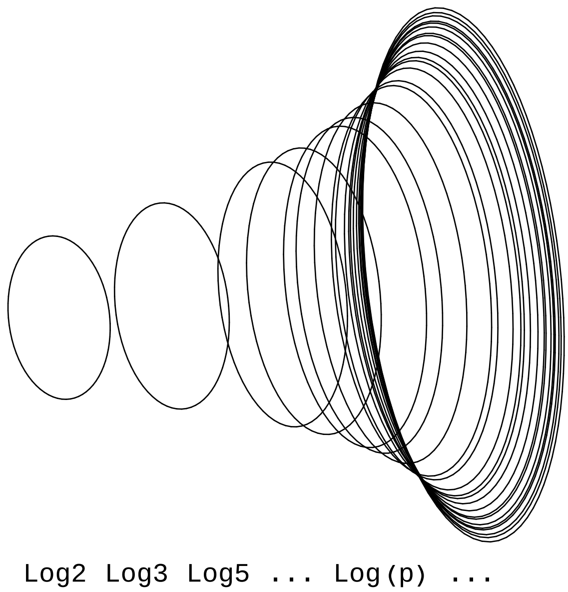

FIGURE 1. The classical points of the adeles class space

The role of the specific correspondences used in Weil's proof of RH in positive characteristic is played by the test functions $f \in \mathbf{S}(C_{\mathbb{K}})$. More precisely the scaling map which replaces $f(x)$ by $f(g^{-1}x)$ has a graph, namely the set of pairs $(x, g^{-1}x) \in X_{\mathbb{K}} \times X_{\mathbb{K}}$, which we view as a correspondence Z_g. Then, given a test function f on the ideles classes, one assigns to f the linear combination

$$Z(f) = \int f(g) Z_g d^* g \tag{5.26}$$

of the above graphs, viewed as a "divisor" on $X_{\mathbb{K}} \times X_{\mathbb{K}}$.
The analogs of the degrees $d(Z)$ and codegrees $d^{'}(Z) = d(Z')$ of correspondences in the context of Weil's proof are given, for the degree, by

$$d(Z(h)) = \widehat{h}(1) = \int h(u)\, |u|\, d^* u, \tag{5.27}$$

so that the degree $d(Z_g)$ of the correspondence Z_g is equal to $|g|$. Similarly, for the codegree one has

$$d^{'}(Z(h)) = d(Z(\bar{h}^{\sharp})) = \int h(u)\, d^* u = \widehat{h}(0), \tag{5.28}$$

so that the codegree $d^{'}(Z_g)$ of the correspondence Z_g is equal to 1.

The role of principal divisors is to be found, but already one can see that there is an interesting subspace $\mathcal{V}$ of the linear space of correspondences described above. It is given by the range of the map E or, more precisely, since there is a small shift in the normalization, by the subspace

$$\mathcal{V} \subset \mathbf{S}(C_{\mathbb{K}}), \quad \mathcal{V} = \{\, g(x) = \sum \xi(k\, x) \,|\, \xi \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \}, \tag{5.29}$$

where the subspace $\mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \subset \mathcal{S}(\mathbb{A}_{\mathbb{K}})$ is defined by the two boundary conditions

$$\xi(0) = 0\,, \qquad \int \xi(x)\,dx \,=\, 0.$$

Lemma 5.12. *For any $f \in \mathcal{V} \subset \mathbf{S}\,(C_{\mathbb{K}})$, one has*

$$\underline{\vartheta}_m(f)|_{H^1} \,=\, 0.$$

This shows that the Weil pairing of Theorem 5.11 admits a huge radical given by all functions which extend to adeles. Thus, one is led anyway to divide by this radical and hence to work with the cohomology $H^1_{\mathbb{K},\mathbb{C}}$ defined above. We will show in a forthcoming paper [15] that several of the steps of Weil's proof can be transposed in the framework described above. This constitutes a clear motivation to develop noncommutative geometry much further.

6. Higher dimensional virtual correspondences

In section 3 we have compared correspondences for motives, given by algebraic cycles in the product $X \times Y$, and correspondences for noncommutative spaces, given by bimodules (elements in KK-theory), in the very special zero dimensional case of Artin motives.

It is interesting to also consider higher dimensional cases. For example, it appears from the recent work [21], [22], [34] that the categories of motives of abelian varieties should be enriched by adding degenerations of abelian varieties to higher dimensional analogs of noncommutative tori, so that the resulting moduli spaces are the noncommutative spaces considered in [17], [22] and [34], having the classical Shimura varieties as the set of classical points. Another compelling reason for considering a unified setting for motives and noncommutative spaces in higher dimensions is given by the results on the Lefschetz formula for archimedean local factors of L-functions of motives discussed in Section 7 below.

Thus, it is useful to discuss the compatibility of morphisms given by correspondences in the higher dimensional setting. To this purpose, we compare correspondences defined by algebraic cycles with the topological correspondences of [4], [24], by reformulating the case of algebraic cycles as a particular case of the KK correspondences of [24].

6.1. **Geometric correspondences and KK-theory.** In the topological case, one considers smooth manifolds X and Y (in fact in [24] one only assumes Y a smooth manifold while X is only a locally compact parameter space). A topological (geometric) correspondence is given by data (Z, E, f_X, g_Y) where Z is a smooth manifold with continuous maps $f_X : Z \to X$ and $g_Y : Z \to Y$. One assumes f_X continuous and proper and g_Y continuous and K-oriented (orientation in K-homology).

The remaining piece of data E is a complex vector bundle over Z. Notice that in this setting one does not require that $Z \subset X \times Y$, while, on the other hand, one has the extra piece of data given by the vector bundle E. To the data (Z, E, f_X, g_Y) one associates an element $k(Z, E, f_X, g_Y)$ in $KK(X, Y)$ as in [24], defined as

$$k(Z, E, f_X, g_Y) = (f_X)_*((E) \otimes (g_Y)!), \tag{6.1}$$

where (E) is the class of E in $KK(Z, Z)$. The element $(g_Y)!$ in KK-theory satisfies the following property.

Given X_1 and X_2 smooth manifolds and a continuous oriented map $f : X_1 \to X_2$, the element $f! \in KK(X_1, X_2)$ gives the Grothendieck Riemann–Roch formula

$$\mathrm{ch}(F \otimes f!) = f_!(\mathrm{Td}(f) \cup \mathrm{ch}(F)), \tag{6.2}$$

for all $F \in K^*(X_1)$, with $\mathrm{Td}(f)$ the Todd genus

$$\mathrm{Td}(f) = \mathrm{Td}(TX_1)/\mathrm{Td}(f^*TX_2). \tag{6.3}$$

The composition of two correspondences (Z_1, E_1, f_X, g_Y) and (Z_2, E_2, f_Y, g_W) is given in this setting by taking the fibered product $Z = Z_1 \times_Y Z_2$ and the bundle $E = \pi_1^* E_1 \times \pi_2^* E_2$, with $\pi_i : Z \to Z_i$ the projections. One needs to assume a transversality conditions on the maps g_Y and f_Y in order to ensure that the fibered product is a smooth manifold. Theorem 3.2 of [24] shows that the Kasparov product in KK-theory is given by this product of correspondences, namely

$$k(Z_1, E_1, f_X, g_Y) \circ k(Z_2, E_2, f_Y, g_W) = k(Z, E, f_X, g_W) \quad \in KK(X, W). \tag{6.4}$$

6.2. **Cycles and K-theory.** In the algebro-geometric setting, the formulation of correspondences that comes closest to the setting described above is obtained by considering maps induced by algebraic cycles on K-theory (*cf.* [44]).

Let us assume that X and Y are given smooth projective algebraic varieties. Also we consider an algebraic cycle $Z = \sum_i n_i Z_i$ in $X \times Y$, namely an element of the free abelian group generated by closed irreducible algebraic subvarieties Z_i of $X \times Y$. We may assume, for the purpose of this discussion that $Z = Z_i$ is an irreducible subvariety of $X \times Y$. We denote by p_X and p_Y the projections of $X \times Y$ onto X and Y, respectively, and we assume that they are proper. We denote by $f_X = p_X|_Z$ and $g_Y = p_Y|_Z$ the restrictions.

To an irreducible subvariety $T \stackrel{i}{\hookrightarrow} Y$ we associate the coherent $\mathcal{O}_Y$-module $i_* \mathcal{O}_T$. For simplicity of notation we write it as $\mathcal{O}_T$. We use a similar notation for the coherent sheaf $\mathcal{O}_Z$, for $Z \hookrightarrow X \times Y$ as above. The pullback

$$p_Y^* \mathcal{O}_T = p_Y^{-1} \mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{X \times Y}. \tag{6.5}$$

is naturally a $\mathcal{O}_{X \times Y}$-module.

We consider the map on sheaves that corresponds to the cap product on cocycles. This is given by

$$Z : \mathcal{O}_T \mapsto (p_X)_* \left(p_Y^* \mathcal{O}_T \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_Z \right), \tag{6.6}$$

where the result is a coherent sheaf, since p_X is proper. Using (6.5), we can write equivalently

$$Z : \mathcal{O}_T \mapsto (p_X)_* \left(p_Y^{-1} \mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_X} \mathcal{O}_Z \right). \tag{6.7}$$

Recall that $f_!$ is right adjoint to f^*, *i.e.* $f^* f_! = id$, and that it satisfies the Grothendieck Riemann–Roch formula

$$\mathrm{ch}(f_!(F)) = f_!(\mathrm{Td}(f) \cup \mathrm{ch}(F)). \tag{6.8}$$

Using this fact, we can equally compute the intersection product of (6.6) by first computing

$$\mathcal{O}_T \otimes_{\mathcal{O}_X} (p_1)_! \mathcal{O}_Z \tag{6.9}$$

and then applying p_Y^*. Using (6.8) and (6.2) we know that we can replace (6.9) by $\mathcal{O}_T \otimes_{\mathcal{O}_X} (\mathcal{O}_Z \otimes (p_1)!)$ with the same effect on K-theory.

Thus, to a correspondence in the sense of (6.6) given by an algebraic cycle $Z \subset X \times Y$ we associate the geometric correspondence (Z, E, f_X, g_Y) with $f_X = p_X|_Z$ and $g_Y = p_Y|Z$ and with the bundle $E = \mathcal{O}_Z$.

Now we consider the composition of correspondences. Suppose given smooth projective varieties X, Y, and W, and (virtual) correspondences $U = \sum a_i Z_i$ and $V = \sum c_j Z'_j$, with $Z_i \subset X \times Y$ and $Z'_j \subset Y \times W$ closed reduced irreducible subschemes. The composition of correspondences is then given in terms of the intersection products

$$U \circ V = (\pi_{13})_* ((\pi_{12})^* U \bullet (\pi_{23})^* V), \tag{6.10}$$

with the projection maps $\pi_{12} : X \times Y \times W \to X \times Y$, $\pi_{23} : X \times Y \times W \to Y \times W$, and $\pi_{13} : X \times Y \times W \to X \times W$. Under an assumption analogous to the transversality requirement for the topological case, we obtain the following result.

Proposition 6.1. *Suppose given smooth projective varieties X, Y, and W and correspondences U given by a single $Z_1 \subset X \times Y$ and V given by a single $Z_2 \subset Y \times W$. Assume that $(\pi_{12})^* Z_1$ and $(\pi_{23})^* Z_2$ are in general position in $X \times Y \times W$. Then assigning to a cycle Z the topological correspondence $\mathcal{F}(Z) = (Z, E, f_X, g_Y)$ satisfies*

$$\mathcal{F}(Z_1 \circ Z_2) = \mathcal{F}(Z_1) \circ \mathcal{F}(Z_2), \tag{6.11}$$

where $Z_1 \circ Z_2$ is the product of algebraic cycles and $\mathcal{F}(Z_1) \circ \mathcal{F}(Z_2)$ is the Kasparov product of the topological correspondences.

Proof. Recall the following facts. Assume that $T_1, T_1 \subset X$ are closed subschemes of X, with $\mathcal{O}_{T_i}$ the corresponding structure sheaves. If T_1 and T_2 are in general position, then their product in $K_*(X)$ agrees with the intersection product ([44], Theorem 2.3)

$$[\mathcal{O}_{T_1}] \cdot [\mathcal{O}_{T_2}] = [\mathcal{O}_{T_1 \bullet T_2}]. \tag{6.12}$$

Using (6.12) we write the composition (6.10) in the form

$$Z_1 \circ Z_2 = (\pi_{13})_*([\pi_{12}^* \mathcal{O}_{Z_1}] \cdot [\pi_{23}^* \mathcal{O}_{Z_2}]). \tag{6.13}$$

Notice that, for $Z_1 \subset X \times Y$ and $Z_2 \subset Y \times W$ the intersection in $X \times Y \times W$ is exactly the fiber product considered in [24], $\pi_{12}^{-1} Z_1 \cap \pi_{23}^{-1} Z_2 = Z = Z_1 \times_Y Z_2$, with $\mathcal{O}_Z = \pi_{12}^* \mathcal{O}_{Z_1} \otimes \pi_{23}^* \mathcal{O}_{Z_2}$. Moreover, we have $\pi_{13} = (f_X, g_W) : Z \to X \times W$, where both f_X and g_W are proper, so that we can identify $(f_X)_*(\mathcal{O}_Z \otimes (g_W)!)$ with $(\pi_{13})_*(\pi_{12}^* \mathcal{O}_{Z_1} \otimes \pi_{23}^* \mathcal{O}_{Z_2})$ as desired. In fact, Theorem 3.2 of [24] shows that $(f_X)_*(\mathcal{O}_Z \otimes (g_W)!)$ represents the Kasparov product. □

Notice that, while in the topological (smooth) setting transversality can always be achieved by a small deformation (*cf.* §III, [24]), in the algebro-geometric case one needs to modify the above construction in the case of the product of cycles that are not in general position. In this case the formula (6.12) is modified by Tor corrections and one obtains ([44], Theorem 2.7)

$$[\mathcal{O}_{T_1}] \cdot [\mathcal{O}_{T_2}] = \sum_{i=0}^{n} (-1)^i \left[\mathrm{Tor}_i^{\mathcal{O}_X} (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}) \right]. \tag{6.14}$$

6.3. **Algebraic versus topological K-theory.** We have considered so far only correspondences in KK-theory and the cohomological realization given by an absolute cohomology, defined for noncommutative spaces by the cyclic category and cyclic cohomology. However, we found in the work [15] that the “primary” invariants obtained by the Chern character map from K-theory to cyclic cohomology is trivial, for the simple reason that we are looking at an action of a continuous group (the scaling action). In such cases, one wants to be able to consider a refined setting in which secondary invariants appear and a more refined version of K-theory is considered, which combines algebraic and topological K-theory.

It is known from the results of [16] that, by viewing both algebraic and topological K-theory in terms of homotopy groups of corresponding classifying spaces, one obtains a fibration and a corresponding long exact sequence. This sequence is related to the long exact sequence for Hochschild and cyclic cohomology of [12] through the Chern character and a regulator map, so that one has a commutative diagram with exact rows

$$\begin{tikzcd}
K^{top}_{n+1}(A) \arrow[r] \arrow[d, "ch_{n+1}"] & K^{rel}_n(A) \arrow[r] \arrow[d, "ch^{rel}_n"] & K^{alg}_n(A) \arrow[r] \arrow[d, "D_n/n!"] & K^{top}_n(A) \arrow[r] \arrow[d, "ch_n"] & K^{rel}_{n-1}(A) \arrow[d, "ch^{rel}_{n-1}"] \\
HC_{n+1}(A) \arrow[r, "S"] & HC_{n-1}(A) \arrow[r, "B/n"] & HH_n(A) \arrow[r, "I"] & HC_n(A) \arrow[r, "S"] & HC_{n-2}(A).
\end{tikzcd}$$

7. A Lefschetz formula for archimedean local factors

As we mentioned already at the beginning of Section 3 we can consider the L-functions $L(H^m(X), s)$ of a smooth projective algebraic variety over a number field $\mathbb{K}$. This is written as an Euler product over finite and archimedean places v of $\mathbb{K}$,

$$L(H^m(X), s) = \prod_v L_v(H^m(X), s). \tag{7.1}$$

The local L-factors of (7.1) at finite primes encode the action of the geometric Frobenius on the inertia invariants $H^m(\bar{X}, \mathbb{Q}_\ell)^{I_v}$ of the étale cohomology in the form (*cf.* [50])

$$L_v(H^m(X), s) = \det\left(1 - Fr_v^* N(v)^{-s} | H^m(\bar{X}, \mathbb{Q}_\ell)^{I_v}\right)^{-1}, \tag{7.2}$$

with N the norm map.

Serre showed in [50] that the local factors one needs to consider at the archimedean primes, dictated by the expected form of the functional equation for the L-function $L(H^m(X), s)$, depend upon the Hodge structure

$$H^m(X_v(\mathbb{C})) = \oplus_{p+q=m} H^{p,q}(X_v(\mathbb{C})), \tag{7.3}$$

where $v : \mathbb{K} \hookrightarrow \mathbb{C}$ is the archimedean place and $X_v(\mathbb{C})$ is the corresponding complex algebraic variety.

The archimedean local factors have an explicit formula (*cf.* [50]) given in terms of Gamma functions, with shifts in the argument and powers that depend on the Hodge numbers. Namely, one has a product of Gamma functions according to the Hodge numbers $h^{p,q}$ of the form

$$L(H^*, s) = \begin{cases} \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min(p,q))^{h^{p,q}} \\ \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}} \end{cases} \tag{7.4}$$

where the two cases correspond, respectively, to the complex and the real embeddings, $\mathbb{K}_v = \mathbb{C}$ and $\mathbb{K}_v = \mathbb{R}$. Here $h^{p,\pm}$ is the dimension of the $\pm(-1)^p$-eigenspace of the involution on $H^{p,p}$ induced by the real structure and

$$\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := 2^{-1/2}\pi^{-s/2}\Gamma(s/2). \tag{7.5}$$

It is clear that it is desirable to have a reformulation of the formulae for the local factors in such a way that the archimedean and the non-archimedean cases are treated as much as possible on equal footing. To this purpose, Deninger in [30], [31] expressed both (7.2) and (7.4) in the form of zeta-regularized infinite determinants. Here we propose a different approach. Namely, we reinterpret the archimedean local factors as a Lefschetz Trace formula over a suitable geometric space. The main idea is that we expect a global (or at least semi-local) Lefschetz trace formula to exist for the L-functions $L(H^m(X), s)$, over a noncommutative space which will be a noncommutative generalization of the pure motive $h^m(X)$ and should be obtained as an extension of $h^m(X)$ by a suitable modification of the adèles class space (by certain division algebras instead of the local fields $\mathbb{K}_v$ at the places v of bad reduction and at the real archimedean place). Here we give some evidence for this idea, by giving the local formula for a single archimedean place v. One sees already in the case of a single real place that the underlying geometric space for the trace formula is obtained by passing to the division algebra of quaternions.

As we recalled already in Section 5 above, it was shown in [14] that the noncommutative space of adele classes over a global field $\mathbb{K}$ provides both a spectral realization of zeros of L-functions and an interpretation of the explicit formulas of Riemann-Weil as a Lefschetz formula. The corresponding trace formula was proved in [14] in the semilocal case (finitely many places). It is natural then to ask whether a similar approach can be applied to the L-functions of motives, also in view of the fact that the results of [26], [27] showed that noncommutative geometry can be employed to describe properties of the fibers at archimedean places (and at places of totally degenerate reduction) of arithmetic varieties.

In this section we consider the archimedean factors of the Hasse-Weil L-function attached to a non-singular projective algebraic variety X defined over a number field $\mathbb{K}$. We obtain the real part of the logarithmic derivative of the archimedean factors of the L-function $L(H^m, z)$ on the critical line as a trace formula for the action of a suitable Weil group on a complex manifold attached to the archimedean place.

In §7.1 we make some preliminary calculations. First we show (Lemma 7.1) that for $\mathbb{K}_v = \mathbb{C}$ or $\mathbb{R}$, one obtains the logarithmic derivative of the imaginary part of the Gamma functions $\Gamma_{\mathbb{K}_v}$ (7.5) on the critical line as the principal value on $\mathbb{K}_v^*$ of a distribution

$$\frac{|u|^{\frac{1}{2}+i\,s}}{|1-u|}.$$

We then show (Lemma 7.3) that the shift by $\min\{p,q\}$ in the argument of $\Gamma_{\mathbb{C}}$ appears when one considers the principal value on $\mathbb{C}^*$ of

$$\frac{u^{-p}\overline{u}^{-q}|u|_{\mathbb{C}}^{z}}{|1-u|_{\mathbb{C}}}.$$

This is sufficient to obtain in Theorem 7.4 of §7.2 the logarithmic derivative of the archimedean local factor as a trace formula for the action of $\mathbb{C}^*$ on a space with base $\mathbb{C}$ and fiber $H^m(X,\mathbb{C})$ with the representation

$$\pi(H^m, u)\,\xi = u^{-p}\overline{u}^{-q}\xi \tag{7.6}$$

of $\mathbb{C}^*$ on the cohomology. The real case is more delicate and it is treated in §7.3. In this case, to obtain a similar result, one needs to consider a space that has base the quaternions $\mathbb{H}$ and fiber the cohomology. We then obtain the logarithmic derivative of the local factor as a trace formula for an action on this space of the Weil group, with representation

$$\pi(H^m, wj)\,\xi = i^{p+q}\,w^{-p}\overline{w}^{-q}F_\infty(\xi) \tag{7.7}$$

on the cohomology, with F_∞ the linear involution induced by complex conjugation. The properties of the L-functions of algebraic varieties defined over a number field $\mathbb{K}$ are mostly conjectural ([50]). In particular one expects from the functional equation ([50], C_9) that the zeros of $L(H^m(X), s)$ are located on the critical line $\Re(s) = \frac{1+m}{2}$. By analogy with the case of the Riemann zeta function, one expects that the number of non-trivial zeros

$$N_s(E) = \#\{\rho \,|\, L(H^m(X),\rho) = 0, \ \text{ and } \ -E \le \Im(\rho) \le E\}\,, \tag{7.8}$$

can be decomposed as the sum of an average part and an oscillatory part

$$N_s(E) = \langle N_s(E)\rangle + N_{\rm osc}(E)\,, \tag{7.9}$$

where the average part is given in terms of the Archimedean local factors described above

$$\langle N_s(E)\rangle = \sum_{v|\infty} \frac{1}{\pi}\int_{-E}^{E} \frac{d}{ds}\Im \log L_v(H^m(X), \frac{1+m}{2} + is)ds\,. \tag{7.10}$$

We shall show in Theorems 7.4 and 7.6 that the key ingredient of this formula can be expressed in the form of a Lefschetz contribution, using the above representations (7.6) and (7.7) of the local Weil group W_v

$$\frac{1}{\pi}\frac{d}{ds}\Im\log L_v(H^m(X),\frac{1+m}{2}+is)=-\frac{1}{2\pi}\int'_{W_v}\frac{\mathrm{Trace}(\pi_v(H^m,u))|u|^z_{W_v}}{|1-u|_{\mathbb{H}_v}}\,d^*u \tag{7.11}$$

where $z=\frac{1+m}{2}+is$. The local Weil group W_v is a modulated group. It embeds in the algebra $\mathbb{H}_v$ which is the algebra $\mathbb{H}$ of quaternions for v a real place and the algebra $\mathbb{C}$ for a complex place. We shall describe below, in more detail, in §7.2 and 7.3 the natural representation $\pi_v(H^m,u)$ of W_v.
In §7.4 we describe the problem of a semilocal trace formula involving several archimedean places and explain how the conjectured trace formula combines with (7.11) to suggest that the (non-trivial) zeros of the L-function appear as an absorption spectrum as in [14]. In §7.5 we address the problem of considering simultaneously archimedean and nonarchimedean places and in particular the fact that in the usual approaches one is forced to make a choice of embeddings of $\mathbb{Q}_\ell$ in $\mathbb{C}$. We suggest that, at least in the case of a curve X, one may obtain a different approach by considering, as the fiber of the space one would like to construct for a semi-local Lefschetz trace formula, the adele class space of the function field of the curve over the residue field k_v at the place v.

7.1. **Weil form of logarithmic derivatives of local factors.** We begin by checking the formulae ([56], [14]) that will relate the Fourier transform of the local Lefschetz contribution (viewed as a distribution on the multiplicative group) with the derivative of the imaginary part of the logarithm of the archimedean local factor $\Gamma_{\mathbb{K}_v}$, for $\mathbb{K}_v=\mathbb{R}$ (v a real archimedean place) or $\mathbb{K}_v=\mathbb{C}$ (v a complex archimedean place).
We use the notations of [14] for the principal values. We recall the following basic facts.

Lemma 7.1. *For $\mathbb{K}_v=\mathbb{R}$ or $\mathbb{K}_v=\mathbb{C}$ and for s real, one has*

$$\int'_{\mathbb{K}_v^*}\frac{|u|^{\frac12+i\,s}}{|1-u|}d^*u=-2\,\frac{d}{ds}\Im\log\Gamma_{\mathbb{K}_v}\left(\frac12+i\,s\right). \tag{7.12}$$

Proof. First notice that, for s real, one has

$$2i\,\Im\log\Gamma_{\mathbb{K}_v}\left(\frac12+i\,s\right)=\log\Gamma_{\mathbb{K}_v}\left(\frac12+i\,s\right)-\log\Gamma_{\mathbb{K}_v}\left(\frac12-i\,s\right), \tag{7.13}$$

since $\Gamma_{\mathbb{K}_v}$ is a "real" function, *i.e.* it fulfills $f(\overline{z})=\overline{f(z)}$.
Thus, one can rewrite the equality above in the form

$$\int'_{\mathbb{K}_v^*}\frac{|u|^{\frac12+i\,s}}{|1-u|}d^*u=-\left(\frac{\Gamma'_{\mathbb{K}_v}}{\Gamma_{\mathbb{K}_v}}\left(\frac12+i\,s\right)+\frac{\Gamma'_{\mathbb{K}_v}}{\Gamma_{\mathbb{K}_v}}\left(\frac12-i\,s\right)\right). \tag{7.14}$$

The i in $i\,s$ brings up the real part of $\frac{\Gamma'_{\mathbb{K}_v}}{\Gamma_{\mathbb{K}_v}}$ which is again a "real" function.
Notice also that (7.14) now holds without the restriction $s\in\mathbb{R}$, since both sides are analytic functions of $s\in\mathbb{C}$.
Let us first take $\mathbb{K}_v=\mathbb{R}$. In this case we have

$$\Gamma_{\mathbb{R}}(x)=2^{-1/2}\pi^{-x/2}\Gamma(x/2), \tag{7.15}$$

hence the equality takes the explicit form in terms of the usual Γ-function

$$\int'_{\mathbb{R}^*}|u|^{is}\frac{|u|^{1/2}}{|1-u|}d^*u=\log\pi-\frac12\left(\frac{\Gamma'}{\Gamma}\left(\frac14+i\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac14-i\frac{s}{2}\right)\right). \tag{7.16}$$

which can be deduced from [14], Appendix 2. Let us then take $\mathbb{K}_v=\mathbb{C}$. In this case we have

$$\Gamma_{\mathbb{C}}(x)=(2\,\pi)^{-x}\Gamma(x), \tag{7.17}$$

and again, in terms of the usual Γ-function, the equality takes the form

$$\int'_{\mathbb{C}^*} |u|^{is} \frac{|u|^{1/2}}{|1-u|} d^*u = 2 \log 2\pi - \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + i\,s\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - i\,s\right)\right) . \tag{7.18}$$

This can be extracted from [14] (Appendix 2) and will be proved as a special case of Lemma 7.3 below. □

The next step consists of obtaining a similar formula for

$$\frac{d}{ds} \Im \log \Gamma_{\mathbb{K}_v} \left(\frac{1}{2} + \frac{|n|}{2} + i\,s\right) , \tag{7.19}$$

where $n \in \mathbb{Z}$ is an integer.
Notice that the function $\Gamma_{\mathbb{K}_v}(\frac{|n|}{2} + z)$ is still real.
We take $\mathbb{K}_v = \mathbb{C}$ and we let

$$f_0(\nu) = \inf(\nu^{1/2}, \nu^{-1/2}), \qquad \forall \nu \in \mathbb{R}_+^* . \tag{7.20}$$

We then obtain the following result.

Lemma 7.2. *For $n \in \mathbb{Z}$ and $\rho \in \mathbb{R}_+^*$ with $\rho \neq 1$, one has*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{in\theta}}{|1 - e^{i\theta}\rho|^2}\, d\theta = \frac{f_0(\nu)^{|n|}}{|1-\nu|} , \qquad \nu = \rho^2 . \tag{7.21}$$

Proof. Let us first consider the case $\rho < 1$. Then we have

$$\frac{1-\rho^2}{|1 - e^{i\theta}\rho|^2} = \frac{1}{1-\rho e^{i\theta}} + \frac{1}{1-\rho e^{-i\theta}} - 1 , \tag{7.22}$$

whose Fourier coefficients are $\rho^{|n|}$. This gives the equality in the case $\rho < 1$, with $f_0(\nu) = \rho$. One then checks that both sides of (7.21) fulfill

$$f(\rho^{-1}) = \rho^2 f(\rho) ,$$

which gives the desired equality for $\rho > 1$. □

We then obtain the following result.

Lemma 7.3. *Let $\mathbb{K}_v = \mathbb{C}$ and suppose given $p, q \in \mathbb{N}$ with $m = p + q$. For $z = \frac{1+m}{2} + is$, with $s \in \mathbb{R}$, one obtains*

$$\int'_{\mathbb{C}^*} \frac{u^{-p}\overline{u}^{-q}|u|_{\mathbb{C}}^{z}}{|1-u|_{\mathbb{C}}} d^*u = -2\, \frac{d}{ds} \Im \log \Gamma_{\mathbb{C}}\left(z - \min(p,q)\right) . \tag{7.23}$$

Proof. Let $n = p - q$. One has $\min(p,q) = \frac{m}{2} - \frac{|n|}{2}$. Since $|u|_{\mathbb{C}} = u\overline{u}$, one has $u^{-p}\overline{u}^{-q} = e^{-in\theta}\,|u|_{\mathbb{C}}^{\frac{-m}{2}}$, where θ is the argument of u. One can then rewrite the desired equality in the form

$$\int'_{\mathbb{C}^*} \frac{e^{-in\theta}\,|u|_{\mathbb{C}}^{\frac{1}{2}+i\,s}}{|1-u|_{\mathbb{C}}} d^*u = -2\, \frac{d}{ds} \Im \log \Gamma_{\mathbb{C}}\left(\frac{1}{2} + i\,s + \frac{|n|}{2}\right) . \tag{7.24}$$

Recall now that $\int'$ coincides with the Weil principal value Pfw of the integral which is obtained as

$$Pfw \int_{\mathbb{C}^*} \varphi(u)\, d^*u = PF_0 \int_{\mathbb{R}_+^*} \psi(\nu)\, d^*\nu , \tag{7.25}$$

where $\psi(\nu) = \int_{|u|_{\mathbb{C}}=\nu} \varphi(u)\, d_\nu u$ is obtained by integrating φ over the fibers, while one has

$$PF_0 \int \psi(\nu) d^*\nu = 2\log(2\pi)\, c + \lim_{t\to\infty} \left(\int (1 - f_0^{2t})\, \psi(\nu)\, d^*\nu - 2c\log t \right) . \tag{7.26}$$

Here one assumes that $\psi - c\, f_1^{-1}$ is integrable on $\mathbb{R}_+^*$, with $f_1 = f_0^{-1} - f_0$. In our case we have

$$\varphi(u) = \frac{e^{-in\theta}\, |u|_{\mathbb{C}}^{\frac{1}{2}+i\,s}}{|1-u|_{\mathbb{C}}}.$$

By Lemma 7.2, integration over the fibers gives

$$\psi(\nu) = \frac{\nu^{\frac{1}{2}+i\,s}\, f_0(\nu)^{|n|}}{|1-\nu|} .$$

We have $c = 1$ independently of s and n.
One has the identity ([14], Appendix 2, (46)),

$$PF_0 \int f_0\, f_1^{-1} d^*\nu = 2(\log 2\pi + \gamma) .$$

This allows one to check the equality for $n = 1$ and $s = 0$. Indeed, in that case one gets $\psi(\nu) = f_0\, f_1^{-1}$, while

$$-2\left(\frac{\Gamma'_{\mathbb{C}}}{\Gamma_{\mathbb{C}}} \left(\frac{1+n}{2} \right) \right) = -2(-\log(2\pi) - \gamma)$$

Having checked the result for some value while c is independent of both n and s one can now use any regularization to compare other values of

$$\int_{\mathbb{R}_+^*} \frac{\nu^{\frac{1}{2}+i\,s}\, f_0(\nu)^{|n|}}{|1-\nu|}\, d^*\nu . \tag{7.27}$$

We write the integral as $\int_0^1 + \int_1^\infty$ and use the minimal substraction as regularization *i.e.* the substraction of the pole part in ε after replacing the denominator $|1-\nu|$ by $|1-\nu|^{1-\varepsilon}$. The first integral gives

$$\int_0^1 \nu^{\left(\frac{1}{2}+is+\frac{|n|}{2}\right)} (1-\nu)^{-1+\varepsilon} \frac{d\nu}{\nu} = B\left(\frac{1}{2} + is + \frac{|n|}{2}, \varepsilon \right) = \frac{\Gamma\left(\frac{1}{2}+is+\frac{|n|}{2}\right)\Gamma(\varepsilon)}{\Gamma\left(\frac{1}{2}+is+\frac{|n|}{2}+\varepsilon\right)} .$$

The residue at $\varepsilon = 0$ is equal to one and the finite part gives

$$-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + is + \frac{|n|}{2} \right) - \gamma . \tag{7.28}$$

The other integral $\int_1^\infty$ gives the complex conjugate,

$$-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - is + \frac{|n|}{2} \right) - \gamma . \tag{7.29}$$

Thus, having checked the additive constant term, we get

$$\int'_{\mathbb{C}^*} \frac{e^{-in\theta}\, |u|_{\mathbb{C}}^{\frac{1}{2}+i\,s}}{|1-u|_{\mathbb{C}}} d^*u = -2\, \frac{d}{ds} \Im \log \Gamma_{\mathbb{C}} \left(\frac{1}{2} + i\,s + \frac{|n|}{2} \right) , \tag{7.30}$$

and the required equality follows. ∎

7.2. **Lefschetz formula for complex places.** We now look at a complex archimedean place with the local L-factor $L_{\mathbb{C}}(H^m, z)$. We let $\pi(H^m, u)$ be the canonical representation ([28]) of $\mathbb{C}^*$ on $H^m = \oplus H^{p,q}$,

$$\pi(H^m, u)\,\xi = u^{-p}\overline{u}^{-q}\xi\,, \quad \forall \xi \in H^{p,q}\,, \quad \forall u \in \mathbb{C}^*\,. \tag{7.31}$$

We obtain the following formula for the archimedean local factor at a complex place.

Theorem 7.4. *Let $\mathbb{K}_v = \mathbb{C}$. For $m \in \mathbb{N}$, let $\pi(H^m, u)$ be the canonical representation of $\mathbb{C}^*$ on H^m. Then, for $z = \frac{1+m}{2} + is$ with $s \in \mathbb{R}$, we have*

$$\int'_{\mathbb{C}^*} \frac{\mathrm{Trace}(\pi(H^m, u))|u|_{\mathbb{C}}^z}{|1-u|_{\mathbb{C}}} d^*u = -2\,\frac{d}{ds}\Im \log L_{\mathbb{C}}(H^m, z)\,. \tag{7.32}$$

Proof. This follows directly from Lemma 7.3 and the formula (7.4) (*cf.* [50]) that expresses $L_{\mathbb{C}}(H^m, z)$ as a product of powers of the $\Gamma_{\mathbb{C}}(z - \min(p, q))$.

7.3. **Lefschetz formula for real places.** We now look at a real archimedean place with the local L-factor $L_{\mathbb{R}}(H^m, z)$. We let W be the Weil group [56] which, in this case, is the normalizer of $\mathbb{C}^*$ in $\mathbb{H}^*$ where $\mathbb{H}$ here denotes the division algebra of quaternions $\mathbb{H} = \mathbb{C} + \mathbb{C}j$, with $j^2 = -1$ and $j\,w\,j^{-1} = \overline{w}$, $\forall w \in \mathbb{C}$. Elements of W are of the form

$$u = w\,j^{\epsilon}\,, \quad w \in \mathbb{C}^*\,, \qquad \epsilon \in \{0, 1\}\,,$$

where we use the notation $j^0 = 1$, $j^1 = j$.
We let $\pi(H^m, u)$ be the canonical representation ([28]) of the Weil group in $H^m = \oplus H^{p,q}$. The subgroup $\mathbb{C}^* \subset W$ acts as above, while elements in $\mathbb{C}^* j$ act by

$$\pi(H^m, wj)\,\xi = i^{p+q}\,w^{-p}\overline{w}^{-q}F_{\infty}(\xi)\,, \quad \forall \xi \in H^{p,q}\,. \tag{7.33}$$

Here F_∞ is the linear involution associated to the geometric action of complex conjugation (once translated in the cohomology with complex coefficients) as in Serre ([50]).
One has to check that π is a representation, and in particular that $\pi(H^m, j)^2 = \pi(H^m, -1)$. This follows for $\xi \in H^{p,q}$ from

$$\pi(H^m, j)^2\xi = (-1)^{p+q}F_{\infty}^2(\xi) = (-1)^{p+q}\xi\,.$$

One checks in the same way that $\pi(H^m, j)\,\pi(H^m, w) = \pi(H^m, \overline{w})\,\pi(H^m, j)$.
We now investigate the integral

$$\int'_{W} \frac{\mathrm{Trace}(\pi(H^m, u))|u|_{\mathbb{H}}^z}{|1-u|_{\mathbb{H}}} d^*u\,, \tag{7.34}$$

in which we follow the conventions of Weil [56]. Thus, for $u \in W$, the module $|u|_{\mathbb{H}}$ is the same as the natural module $|u|_W$ of the Weil group and $|1-u|_{\mathbb{H}}$ is the reduced norm in quaternions.
Let us start by the case m odd, since then the subtle term involving $\Gamma_{\mathbb{R}}$ does not enter. Then the action of W on $H^{p,q} \oplus H^{q,p}$ will give the same trace as in the complex case for elements $u = w\,j^0 = w \in \mathbb{C}^*$, but it will give a zero trace to any element $u = w\,j^1 = w\,j$. In fact, in this case the action is given by an off diagonal matrix. This gives an overall factor of $\frac{1}{2}$ in the above expression, since when one integrates on the fibers of the module map $u \mapsto |u|$ from W to $\mathbb{R}_+^*$ the only contribution will come from the $u = w\,j^0 = w$. This will give the same answer as for the complex case, except for an overall factor $\frac{1}{2}$ due to the normalization of the fiber measure as a probability measure.
Thus, things work when m is odd, since then the local factor $L_{\mathbb{R}}(H^m, z)$ is really the square root of what it would be when viewed as complex.
When $m = 2p$ is even and $h^{p+} = h^{p-}$, the same argument does apply irrespectively of the detailed definition of the representation of W. Indeed, elements of W of the form $u = w\,j$

have pairs of eigenvalues of opposite signs $\pm w^{-p}\overline{w}^{-p}$. One thus gets the required result using (7.4) and the duplication formula

$$\Gamma_{\mathbb{R}}(z)\,\Gamma_{\mathbb{R}}(z+1) = \Gamma_{\mathbb{C}}(z)\,. \tag{7.35}$$

When $h^{p+} \neq h^{p-}$, we use the detailed definition of the action of j. Notice that the computation will not be reducible to the previous ones, since one uses the quaternions to evaluate

$$|1-u|_{\mathbb{H}} = 1+|w|^2\,, \quad u = wj\,,$$

The action of j is $(-1)^p$ times the geometric action of complex conjugation (once translated in the cohomology with complex coefficients) on the space $H^{p,p}$.
We compute in the following lemma the relevant integral.

Lemma 7.5. *For $z = \frac{1}{2} + is$ with $s \in \mathbb{R}$, one has*

$$\int_{\mathbb{R}_+^*} \frac{u^z}{1+u} d^*u = -2\frac{d}{ds}\Im\log(\Gamma_{\mathbb{R}}(z)/\Gamma_{\mathbb{R}}(z+1))\,. \tag{7.36}$$

Proof. First for $z \in \mathbb{C}$ with positive real part, one has

$$\int_0^1 \frac{u^z}{1+u} d^*u = \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{z+1}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{z}{2}\right)\right)\,. \tag{7.37}$$

Let us prove (7.36). The factors in $\pi^{-z/2}$ in (7.15) for $\Gamma_{\mathbb{R}}(z)$ do not contribute to the right hand side, which one can replace by

$$-\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{s}{2}\right)\right)+\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}+i\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}-i\frac{s}{2}\right)\right)\,,$$

The equality follows from (7.37), using the symmetry of the integral to integrate from 1 to ∞. $\square$

We then obtain the following formula for the archimedean local factor at a real place.

Theorem 7.6. *Let $\mathbb{K}_v = \mathbb{R}$. For $m \in \mathbb{N}$, let $\pi(H^m, u)$ be the representation of the Weil group W on H^m described above. Then, for $z = \frac{1+m}{2} + is$ with $s \in \mathbb{R}$, we have*

$$\int_W' \frac{\mathrm{Trace}(\pi(H^m,u))|u|_{\mathbb{H}}^z}{|1-u|_{\mathbb{H}}} d^*u = -2\,\frac{d}{ds}\Im\log L_{\mathbb{R}}(H^m, z)\,. \tag{7.38}$$

Proof. The discussion above shows that we are reduced to the case of even $m = 2p$ and the subspace $H^{p,p}$. Let then $k = h^{p+} - h^{p-}$. When $k = 0$ the discussion above already gives the result. When $k \neq 0$ the left hand side gets an additional term from the trace of elements of the form $u = wj \in W$. Their contribution is given by

$$\frac{k}{2}\int_{\mathbb{R}_+^*} \frac{v^{z-p}}{1+v} d^*v,$$

where the factor $\frac{1}{2}$ comes from the normalization (as a probability measure) of the fiber measure of the module $W \mapsto \mathbb{R}_+^*$. The term in v^{-p} comes from the action of the scalar part $w \in \mathbb{C}^*$ with $w\overline{w} = v$ in $u = wj$. The denominator comes from the equality

$$|1-u|_{\mathbb{H}} = 1+|w|^2\,, \quad u = wj\,.$$

The other side changes in the same way using (7.36). Indeed one has

$$h^{p+}\,\log\Gamma_{\mathbb{R}}(z-p) + h^{p-}\,\log\Gamma_{\mathbb{R}}(z-p+1) =$$

$$\frac{h^{p+}+h^{p-}}{2}\,\log(\Gamma_{\mathbb{R}}(z-p)\Gamma_{\mathbb{R}}(z-p+1)) + \frac{k}{2}\,\log(\Gamma_{\mathbb{R}}(z-p)/\Gamma_{\mathbb{R}}(z-p+1))$$

$$= \frac{h^{p+} + h^{p-}}{2} \log \Gamma_{\mathbb{C}}(z-p) + \frac{k}{2} \log(\Gamma_{\mathbb{R}}(z-p)/\Gamma_{\mathbb{R}}(z-p+1)) \,.$$

This completes the proof.

□

The space which gives the formula of theorem 7.6 as a Lefschetz local contribution is obtained by taking the quaternions $\mathbb{H}$ as the base, but endowing them with an additional structure, namely their complex structure when viewed as a right vector space over $\mathbb{C}$. We will use the Atiyah-Bott Lefschetz formula (*cf.* [2]) applied to the $\bar{\partial}$-complex, which generates the crucial term of theorem 7.6, that is,

$$\frac{1}{|1-u|_{\mathbb{H}}}\,, \tag{7.39}$$

with the reduced norm $|u|_{\mathbb{H}}$ as above, while a more naive approach without the use of an elliptic complex would involve the square of the reduced norm. The Atiyah-Bott Lefschetz formula involves a numerator $\chi(u)$ which yields,

$$\frac{\chi(u)}{|1-u|_{\mathbb{H}}^2} = \frac{1}{|1-u|_{\mathbb{H}}}\,. \tag{7.40}$$

In fact, the use of the $\bar{\partial}$-complex brings in the powers $\wedge^j T_{\mathbb{C}}(\mathbb{H})$ of the complex tangent space $T_{\mathbb{C}}(\mathbb{H})$ of the complex manifold $\mathbb{H}$ and the alternating sum

$$\chi(u) = \sum_j (-1)^j \, \chi_j(u) = \sum_j (-1)^j \operatorname{trace}(\wedge^j(u)). \tag{7.41}$$

This gives the determinant of the quaternion $1-u \in \mathbb{H}$ viewed as a two by two complex matrix (even if some complex conjugates are involved) and

$$\chi(u) = |1-u|_{\mathbb{H}}. \tag{7.42}$$

7.4. **The question of the spectral realization.** A more general problem will be to obtain for archimedian local factors a trace formula analogous to the semilocal case of Theorem 4 of [14]. This means that one considers a number field K, with S be the set of all archimedean places, and X a non-singular projective variety over $\mathbb{K}$. One would like to obtain the real part of the logarithmic derivative of the full archimedian factor $L_S(H^m, z) = \prod_{v\in S} L_{\mathbb{K}_v}(H^m, z)$ on the critical line as a trace formula for the action of a suitable Weil group on a complex space.
For a single place, one can work with a vector bundle over a base space that is $B = \mathbb{C}$ for v complex and $B = \mathbb{H}$ for v real, with fiber the $\mathbb{Z}$-graded vector space $E^{(m)}$ given by the cohomology $H^m(X_v, \mathbb{C})$, with X_v the compact Kähler manifold determined by the embedding $v : \mathbb{K} \hookrightarrow \mathbb{C}$. One expects a trace formula of the following sort, analogous to the semilocal case of [14], modelled on the trace formula of [33].

Problem 7.7. *Let $h \in \mathcal{S}(\mathbb{R}_+^*)$ with compact support be viewed as an element of $\mathcal{S}(W)$ by composition with the module. Then, for $\Lambda \to \infty$, one has*

$$\operatorname{Trace}(R_\Lambda \, \vartheta_a(h)) = 2h(1)B_m \log \Lambda + \sum_{v\in S} \int'_{W_v} \frac{h(|u|^{-1}) \operatorname{Trace}(\pi_v(H^m, u))}{|1-u|_{\mathbb{H}_v}} \, d^*u + o(1) \tag{7.43}$$

where B_m is the m-th Betti number of X, and $\int'$ is uniquely determined by the pairing with the unique distribution on $\mathbb{K}_v$ which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.

For example, the above formula can be proved in simplest case of a single complex place. The bundle $E = \oplus_m E^{(m)}$ comes endowed with a representation of $\mathbb{C}^*$, by

$$\lambda : (z, \omega) \mapsto (\lambda z, \lambda^{-p}\bar{\lambda}^{-q}\omega), \tag{7.44}$$

for $\omega \in H^{p,q}(X_v, \mathbb{C})$. We then let $\mathcal{H} = L^2(B, E^{(m)})$ be the Hilbert space of L^2-sections of $E^{(m)}$ (for its hermitian metric of trivial bundle). The action of $W = \mathbb{C}^*$ on $\mathcal{H}$ is given by

$$(\vartheta_a(\lambda)\xi)(b) = \lambda^p \bar{\lambda}^q \xi(\lambda^{-1}b)\,, \quad \forall \xi \in L^2(B, H^{p,q})\,, \tag{7.45}$$

where we identify sections of $E^{(m)}$ with $H^m(X_v, \mathbb{C})$ valued functions on B.
One can use a cutoff as in [14], by taking orthogonal projection P_Λ onto the subspace

$$P_\Lambda = \{\xi \in L^2(B, E^{(m)})\,;\ \xi(b) = 0\,,\ \forall b \in B\,,\ |b|_{\mathbb{C}} > \Lambda\}\,. \tag{7.46}$$

Thus, P_Λ is the multiplication operator by the function ρ_Λ, where $\rho_\Lambda(b) = 1$ if $|b|_{\mathbb{C}} \leq \Lambda$, and $\rho_\Lambda(b) = 0$ for $|b|_{\mathbb{C}} > \Lambda$. This gives an infrared cutoff and to get an ultraviolet cutoff we use $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$ where F is the Fourier transform which depends upon the choice of the basic character α_v for the place v. We let

$$R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda\,.$$

Proposition 7.8. *For the set of places S consisting of a single complex place the trace formula* (7.43) *holds.*

Proof. Both sides of the formula are additive functions of the representation of $\mathbb{C}^*$ in $H^m(X_v, \mathbb{C})$. We can thus assume that this representation corresponds to a one dimensional $H^{p,q}$. Let then $h_1(\lambda) = \lambda^p \bar{\lambda}^q\, h(|\lambda|)$, one has

$$h_1(u^{-1}) = h(|u|^{-1})\,\mathrm{Trace}(\pi_v(H^m, u))\,, \qquad \forall u \in \mathbb{C}^*$$

while by (7.45) we get that $\vartheta_a(h)$ is the same operator as $U(h_1)$ in the notations of [14]. Thus applying Theorem 4 of [14] to h_1 gives the desired result. □

In discussing here the case of a single complex place v, we have taken the trivial bundle over $\mathbb{C}$ with fiber the cohomology $H^m(X_v, \mathbb{C})$. Already in order to treat the case of several complex places, one needs to use the following result ([49], proof of Proposition 12): the integers $h^{p,q}(X_v)$ are independent of the archimedean place $v \in S$. This suggests that it is in fact more convenient to think of the fiber $E^{(m)}$ as the motive $h^m(X)$ rather than its realization. This leads naturally to the further question of a semi-local trace formula where the finite set of places S involves both archimedean and non-archimedean places. We discuss in section 7.5 below, in the case of curves, how one can think of a replacement for the ℓ-adic cohomology at the non-archimedean places, using noncommutative geometry.

Remark 7.9. The representation ϑ_a of Problem 7.7 is not unitary but the product

$$u \mapsto |u|_W^{-(1+m)/2}\vartheta_a(u) \tag{7.47}$$

should be unitary. In particular the spectral projection N_E, for the scaling action, associated to the interval $[-E, E]$ in the dual group $\mathbb{R}$ of $\mathbb{R}_+^*$ is then given by

$$N_E = \vartheta_a(h_E^{(m)}), \quad \text{with } h_E^{(m)}(u) = |u|_W^{-(1+m)/2}\frac{1}{2\pi}\int_{-E}^{E}|u|_W^{is}ds\,. \tag{7.48}$$

Applying the conjectured formula (7.43) to the function $h_E^{(m)}$ the left hand side gives the counting of quantum states, $\mathrm{Trace}(R_\Lambda\, N_E)$ as in [14]. The first term in the right hand side of (7.43) gives the contribution of the regular representation of the scaling group. Finally, using (7.11) and (7.10), the last terms of (7.43) combine to give

$$\sum_{v|\infty}\int'_{W_v}\frac{h_E^{(m)}(u^{-1})\,\mathrm{Trace}(\pi_v(H^m,u))}{|1-u|_{\mathbb{H}_v}}\,d^*u=\sum_{v|\infty}\frac{1}{2\pi}\int_{-E}^{E}\int'_{W_v}\frac{\mathrm{Trace}(\pi_v(H^m,u))|u|_{W_v}^{\frac{1+m}{2}+is}}{|1-u|_{\mathbb{H}_v}}\,d^*u\,ds$$

$$-\sum_{v|\infty}\frac{1}{\pi}\int_{-E}^{E}\frac{d}{ds}\Im\log L_v(H^m(X),\frac{1+m}{2}+is)ds=-\langle N_s(E)\rangle$$

which shows that one should expect that the zeros of the L-function appear as an absorption spectrum as in [14]. Unlike in the case of the Riemann zeta function this remains conjectural in the above case of L-functions of arithmetic varieties.

7.5. **Local factors for curves.** In the previous part of this section, we showed how to write the archimedean local factors of the L-function of an algebraic variety X in the form of a Lefschetz trace formula. Eventually, one would like to obtain a semi-local formula, like the one conjectured above, not just for the archimeden places, but for the full L-function. We are only making some rather speculative remarks at this point, and we simply want to illustrate the type of construction one expects should give the geometric side of such a Lefschetz trace formula. We just discuss the case where X is a curve for simplicity. In this case we can concentrate on the L-function for $H^1(X)$.

Let X be a curve over a number field $\mathbb{K}$. The local Euler factor at a place v has the following description (*cf.* [50]):

$$L_v(H^1(X),s)=\det\left(1-Fr_v^*N(v)^{-s}|H^1(\bar X,\mathbb{Q}_\ell)^{I_v}\right)^{-1}. \tag{7.49}$$

Here Fr_v^* is the geometric Frobenius acting on ℓ-adic cohomology of $\bar X=X\otimes\mathrm{Spec}(\bar{\mathbb{K}})$, with $\bar{\mathbb{K}}$ an algebraic closure and ℓ a prime with $(\ell,q)=1$, where q is the cardinality of the residue field k_v at v. We denote by N the norm map. The determinant is evaluated on the inertia invariants $H^1(\bar X,\mathbb{Q}_\ell)^{I_v}$ at v (this is all of $H^1(\bar X,\mathbb{Q}_\ell)$ when v is a place of good reduction). The L-function has then the Euler product description

$$L(H^1(X),s)=\prod_v L_v(H^1(X),s),$$

where for v a non-archimedean place the local factor is of the form (7.49) and if v is a complex or real archimedean place then the local factor is given by the corresponding Γ-factor as discussed in the previous part of this section.

Usually, in using the expression (7.49) for the local factors, one makes use of a choice of an embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$. In our setting, if one wants to obtain a semi-local trace formula for the L-function $L(H^1(X),s)$, one needs a geometric construction which does not depend on such choices and treats the archimedean and non-archimedean places on equal footing. One expects that a geometric space on which the geometric side of the desired Lefschetz trace formula will concentrate should be obtained as a fibration, where the base space should be a noncommutative space obtained from the adele class space of the number field $\mathbb{K}$, modified by considering, at the places of bad reduction and at the real archimedean places, suitable division algebras over the local field. The fiber should also be a noncommutative space in which the special fiber embeds (at least at the places of good reduction). This will have the effect of replacing the use of the ℓ-adic cohomology and the need for a choice of an embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$.

In the case of a curve X, one can obtain this by embedding the special fiber X_v at a place v, which is a curve over the residue field k_v of cardinality q, in the adele class space of the function field of X_v. Indeed by Theorem 5.6 applied to the global field $\mathbb{K}$ of functions on X_v, one obtains the local factor $L_v(H^1(X),s)$ directly over $\mathbb{C}$. Thus, at least in the case of curves, the adele class spaces of the global fields of functions on the curves X_v should be essential ingredients in the construction. This would then make it possible to work everywhere with a cohomology defined over $\mathbb{C}$, in the form of cyclic (co)homology.

References

[1] Y. André, *Une introduction aux motifs*, Panoramas et Synthèses, Vol.17, Société mathématique de France, 2005.

[2] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: I*, Annals of Math, Vol.86 (1967) 374–407.

[3] S. Baaj, P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens.* C. R. Acad. Sci. Paris Sr. I Math. 296 (1983), no. 21, 875–878.

[4] P. Baum, *Cycles, cocycles and K-theory*, unpublished manuscript.

[5] B. Blackadar, *K-theory for operator algebras*, Second edition, MRSI Publications, 1998.

[6] E. Bombieri, *Problems of the Millenium: The Riemann Hypothesis*, Clay mathematical Institute (2000).

[7] J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory.* Selecta Math. (N.S.) 1 (1995), no. 3, 411–457.

[8] O. Bratteli, D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics* I and II, Springer, New York 1979 and 1981.

[9] F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques.* Bull. Soc. Math. France, 89 (1961), 43–75.

[10] A. Connes, *Une classification des facteurs de type* III. Ann. Sci. École Norm. Sup. (4) 6 (1973), 133–252.

[11] A. Connes, *Cohomologie cyclique et foncteurs* Ext^n. C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 23, 953–958.

[12] A. Connes, *Noncommutative differential geometry.* Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[13] A. Connes, *Noncommutative geometry.* Academic Press, 1994.

[14] A. Connes, *Trace formula in noncommutative geometry and zeros of the Riemann zeta function*, Selecta Math. (N.S.) 5 (1999), no. 5, 29–106.

[15] A. Connes, C. Consani, M. Marcolli, *The Weil proof and the geometry of the adeles class space*, in preparation.

[16] A. Connes, M. Karoubi, *Caractère multiplicatif d'un module de Fredholm.* K-Theory 2 (1988), no. 3, 431–463.

[17] A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry. Part I. Quantum statitical mechanics of $\mathbb{Q}$-lattices*, preprint math.NT/0404128.

[18] A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry. Part II. Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory*, hep-th/0411114.

[19] A. Connes, M. Marcolli, *Renormalization and motivic Galois theory*, IMRN (2004) N.76, 4073–4092.

[20] A. Connes, M. Marcolli, *Dimensional regularization, anomalies, and noncommutative geometry*, in preparation.

[21] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, to appear in Selecta Math.

[22] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication. Part II*, to appear in the Abel Symposium Volume.

[23] A. Connes, H. Moscovici, *Hopf Algebras, Cyclic Cohomology and the Transverse Index Theorem*, Commun. Math. Phys., 198, (1998), 199–246.

[24] A. Connes, G. Skandalis, *The longitudinal index theorem for foliations*, Publ. RIMS Kyoto Univ. 20 (1984) 1139–1183.

[25] A. Connes, M. Takesaki, *The flow of weights on factors of type III.* Tohoku Math. J., 29, (1977) 473–575.

[26] C. Consani, M. Marcolli, *Noncommutative geometry, dynamics, and ∞-adic Arakelov geometry.* Selecta Math. (N.S.) 10 (2004), no. 2, 167–251.

[27] C. Consani, M. Marcolli, *New perspectives in Arakelov geometry.* Number theory, 81–102, CRM Proc. Lecture Notes, 36, Amer. Math. Soc., Providence, RI, 2004.

[28] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Symp. Pure Math. Vol.33 (1979) part II, 313–346.

[29] P. Deligne, *À quoi servent les motifs?* in "Motives" (Seattle, WA, 1991), 143–161, Proc. Sympos . Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[30] C. Deninger, *On the Γ-factors attached to motives*, Invent. Math. 104 (1991) 245–261.
[31] C. Deninger, *Motivic L-functions and regularized determinants*, in "Motives", Proceedings of Symposia in Pure Mathematics, Vol. 55 (1994) Part I, 707–743.
[32] J. Dixmier, *Les C^*-algèbres et leurs représentations.* Gauthier-Villars, Paris, 1964.
[33] V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J., Vol.44 (1977) N.3, 485–517.
[34] E. Ha, F. Paugam, *A Bost–Connes–Marcolli system for Shimura varieties*, IMRP 5 (2005) 237–286.
[35] M. Hilsum and G. Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov.* Ann. Sci. École Norm. Sup. (4) 20 (1987), 325-390.
[36] B. Jacob, *Quantum statistical mechanics of sign normalized rank one Drinfeld modules*, talk at MPI, October 2005.
[37] U. Jannsen, *Motives, numerical equivalence, and semi-simplicity.* Invent. Math. 107 (1992), no. 3, 447–452.
[38] D. Kaledin, *Non-commutative Cartier operator and Hodge-to-de Rham degeneration*, preprint math.AG/0511665.
[39] G.G. Kasparov, *The operator K-functor and extensions of C^*-algebras.* Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719.
[40] M. Kontsevich, *Noncommutative motives*, talk at Princeton, October 2005.
[41] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal., **139** (1996), 415–440.
[42] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. (2) **61** (2000), no. 3, 893–904.
[43] J.L. Loday, *Cyclic homology.* Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998. xx+513 pp.
[44] Yu.I. Manin, *Lectures on the K-functor in algebraic geometry.* Uspehi Mat. Nauk 24 1969 no. 5 (149), 3–86.
[45] Yu.I. Manin, *Correspondences, motifs, and monoidal transformations*, Mat. Sb. (N.S.) 77 (1968) 119, 475–507.
[46] R. Meyer, *On a representation of the idele class group related to primes and zeros of L-functions*, preprint arXiv math.NT/0311468.
[47] R. Meyer, R. Nest, *The Baum-Connes conjecture via localization of categories.* Lett. Math. Phys. 69 (2004), 237–263.
[48] B. Riemann, *Mathematical Works*, Dover, New York, 1953.
[49] J. P. Serre, *Géométrie algébrique et géométrie analytique.* Annales Inst. Fourier, Grenoble 6, (1956), 1-42.
[50] J. P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures).* Sém. Delange-Pisot-Poitou, exp. 19, 1969/70.
[51] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications.* Lecture Notes in Math., 28, Springer, 1970.
[52] M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math. (131) (1973), 249-310.
[53] J. Tate, *Number theoretic background*, in "Automorphic forms, representations, and L-functions", Proc. Symp. Pure Math. Vol.33, Part 2, 1979, pp.3–26.
[54] A. Weil, *Sur la théorie du corps de classes*, J. Math. Soc. Japan, 3 (1951) 1–35.
[55] A. Weil, *Sur les formules explicites de la théorie des nombres premiers*, Oeuvres compltes, Vol. 2, 48–62.
[56] A. Weil, *Sur les formules explicites de la théorie des nombres*, Izv. Mat. Nauk., (Ser. Mat.) Vol.36 (1972) 3–18.

A. Connes: Collège de France, 3, rue d'Ulm, Paris, F-75005 France, I.H.E.S. and Vanderbilt University
E-mail address: alain@connes.org

C. Consani: Mathematics Department, Johns Hopkins University, Baltimore, MD 21218 USA
E-mail address: kc@math.jhu.edu

M. Marcolli: Max–Planck Institut für Mathematik, Vivatsgasse 7, Bonn, D-53111 Germany
E-mail address: marcolli@mpim-bonn.mpg.de