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ON DISCRETE SUBGROUPS OF LIE GROUPS (II)

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1. This is a continuation of my paper [7] with the same title, which will be referred to as D^{I}, and which has to be supplemented in the manner described below in Appendix I. Indeed, the present paper is nothing else than a combination of the ideas of D^{I} with the method of variation of structure, as applied to special types of semisimple groups by Calabi and by Calabi-Vesentini in their recent work[1].

We shall mainly be concerned with semisimple groups without compact components, i.e., with connected semisimple Lie groups having no connected compact normal subgroup. However, we begin by considering any connected Lie group G having at least one discrete subgroup with compact quotient; this implies that G is unimodular (it would even be enough for this to assume that G has a discrete subgroup H such that G/H has a finite measure, for the measure determined on G/H by a right-invariant measure on G).

Let n be the dimension of G; choose a basis $X_1, \cdots, X_n$ for the space of right-invariant vector-fields on G; we have

$$[X_\lambda, X_\mu] = \sum_{\nu=1}^{n} c_{\lambda\mu}^{\nu} X_\nu \qquad (1 \leqq \lambda, \mu \leqq n)\ , \tag{1}$$

where the $c_{\lambda\mu}^{\nu}$ are the constants of structure of G. The X_λ will be considered, in the usual manner, as differential operators acting on functions on G. To say that they are right-invariant means that they commute with right-translations, so that they are the infinitesimal operators belonging to the group of left-translations on G. The dual basis to the X_λ, in the space of differential forms on G, consists of the forms ω^λ given by $i(X_\lambda)\omega^\mu = \delta_\lambda^\mu$, where i is the interior product and (δ_λ^μ) is the unit-matrix. Then we have, for every function f on G:

$$df = \sum_{\lambda=1}^{n} X_\lambda f \cdot \omega^\lambda\ . \tag{2}$$

The ω^λ are a basis for the right-invariant differential forms on G. On G, we have the right-invariant differential form $\omega^1 \wedge \cdots \wedge \omega^n$; G being oriented so as to make this positive, we can use it as volume-element on G and denote it, as such, by $d\Omega$. As G is unimodular, this must be invariant under left-translations, or, what amounts to the same, under the

[1] Cf. [3], [4] and [1]. I am also greatly indebted to Calabi for permission to make use of notes on seminar lectures given by him on this subject in Princeton in 1958–59.

infinitesimal operators X_λ; in the usual notation, this can be written

$$\theta(X_\lambda)\cdot\omega^1\wedge\cdots\wedge\omega^n = 0 \qquad (1 \leqq \lambda \leqq n)\ .$$

Using H. Cartan's well known identity $\theta(X) = i(X)d + di(X)$, we get:

$$d(\omega^1\wedge\cdots\wedge\omega^{\lambda-1}\wedge\omega^{\lambda+1}\wedge\cdots\wedge\omega^n) = 0\ .$$

Therefore, if H is any discrete subgroup of G such that G/H is compact, we have, by Stokes's formula:

$$\text{(3)}\qquad \int_{G/H} X_\lambda f\cdot d\Omega = \int_{G/H} \pm d(f\cdot\omega^1\wedge\cdots\wedge\omega^{\lambda-1}\wedge\omega^{\lambda+1}\wedge\cdots\wedge\omega^n) = 0$$

for all λ and all functions f on G/H.

2. Now consider a group Γ and a one-parameter family $t\to r_t$ of representations of Γ into G; we assume that r_0 *is injective*, $r_0(\Gamma)$ *is discrete in* G, $G/r_0(\Gamma)$ *is compact*, and also that $t\to r_t(\gamma)$ *is of class* C^∞ *for every* $\gamma\in\Gamma$. We can then apply the results of D^{I}, no. 12, supplemented by Appendix I of this paper; this says that there is an open interval I containing 0, such that, if Γ is made to act on $I\times G$ by

$$(t, x)\gamma = \big(t, x\cdot r_t(\gamma)\big) \qquad (t\in I, x\in G, \gamma\in\Gamma)\ ,$$

it operates freely and properly on $I\times G$. Then the quotient-space $S = (I\times G)/\Gamma$ is a separated manifold of class C^∞, and the mapping $(t, x)\to t$ of $I\times G$ onto I determines on S a proper mapping π of S onto I; for each $t\in I$, the "fibre" $\pi^{-1}(t)$ can be identified in an obvious manner with the compact manifold $G/r_t(\Gamma)$. The vector-fields X_λ determine in an obvious manner n vector-fields on $I\times G$; as these are invariant under Γ, they determine n vector-fields on S; these vector-fields, on $I\times G$ and on S, will also be denoted by X_λ; they are, at every point (t, x) of $I\times G$ (resp. at every point M of S), tangent to the fibre $t\times G$ (resp. $\pi^{-1}(\pi(M))$) through that point, and they satisfy (1).

We can also make G act on $I\times G$ by $s\cdot(t, x) = (t, sx)$; as these operations commute with those of Γ, they determine operations of G on S; the corresponding infinitesimal transformations are the linear combinations of the X_λ with constant coefficients.

3. Now let $\mathfrak{Y}$ be the class of all vector-fields Y of class C^∞ on S such that $Y\pi = 1$. By an obvious identification, this may also be considered as the class of the vector-fields Y of class C^∞ on $I\times G$, invariant under Γ, such that $Yt = 1$ (where t means the function $(t, x)\to t$ on $I\times G$). On $I\times G$, a vector-field Y satisfies $Yt = 1$ if and only if it can be written as $Y = \partial/\partial t + \sum_\lambda \varphi^\lambda X_\lambda$, where the φ^λ are functions on $I\times G$; but this does not show how to choose the φ^λ so that Y may be invariant under Γ. However, it is easily seen that $\mathfrak{F}$ is not empty; in fact, by means of a locally finite covering, put on S a ds^2 of class C^∞, and take for Y, at every point M of S, the vector which is orthogonal to the fibre $\pi^{-1}(\pi(M))$ through M and satisfies $Y\pi = 1$; this belongs to $\mathfrak{F}$. If $Y \in \mathfrak{F}$, the vector-fields in $\mathfrak{F}$ are those of the form $Y + \sum_\lambda \varphi^\lambda X_\lambda$, where the φ^λ are functions of class C^∞ on S.

Taking $Y \in \mathfrak{F}$, write now $[X_\lambda, Y]$ as a linear combination of Y and the X_λ. Since we have $Y\pi = 1$ and $X_\lambda \pi = 0$, we have $[X_\lambda, Y]\pi = 0$, so that Y cannot occur in $[X_\lambda, Y]$; this gives:

$$[X_\lambda, Y] = \sum_{\mu=1}^{n} f_\lambda^\mu \cdot X_\mu \,. \tag{4}$$

The Jacobi identity gives:

$$X_\lambda f_\mu^\nu - X_\mu f_\lambda^\nu = \sum_\rho (c_{\lambda\mu}^\rho f_\rho^\nu + c_{\mu\rho}^\nu f_\lambda^\rho - c_{\lambda\rho}^\nu f_\mu^\rho) \,. \tag{5}$$

Also, if we substitute $Y' = Y + \sum_\lambda \varphi^\lambda X_\lambda$ for Y, the f_λ^μ are replaced by

$$f'^\mu_\lambda = f_\lambda^\mu + X_\lambda \varphi^\mu + \sum_\rho c_{\lambda\rho}^\mu \varphi^\rho \,. \tag{6}$$

Suppose now that, for two vector-fields Y, Y' in $\mathfrak{F}$, we have $[X_\lambda, Y' - Y] = 0$ for all λ; since $Y' - Y$ is a linear combination of the X_λ, this means that $Y' - Y$ induces, on each fibre $t \times G$ of $I \times G$, a vector-field which commutes with all the X_λ, i.e., a left-invariant vector-field. This must at the same time be invariant under Γ, i.e., under the right-translations $r_t(\gamma)$ for all $\gamma \in \Gamma$. As right-translations act on the left-invariant vector-fields by the adjoint group of G, this means that, for each t, $Y' - Y$ is a left-invariant vector-field belonging to the subalgebra $\mathfrak{n}_t = \mathfrak{n}(r_t(\Gamma))$ of the Lie algebra $\mathfrak{g}$ of G which consists of the vectors invariant under $\mathrm{Ad}(r_t(\gamma))$ for every $\gamma \in \Gamma$.

4. *We now make the additional assumption that* $\mathfrak{n}_t = \{0\}$ *for all* $t \in I$ (for our immediate purposes, it would be enough to assume that the dimension of $\mathfrak{n}_t$ remains constant for all t in some neighborhood of 0). As shown in Appendix II, this is certainly the case whenever G is semisimple without compact components. Then, if Y, Y', f_λ^μ, f'^μ_λ are as above, $f'^\mu_\lambda = f_\lambda^\mu$ for all λ, μ implies $Y' = Y$.

Now we seek to determine $Y' \in \mathfrak{F}$ so that, for each $t \in I$, the integral

$$\int_{\pi^{-1}(t)} \sum_{\lambda,\mu} (f'^\mu_\lambda)^2 d\Omega$$

is a minimum. Choosing $Y \in \mathfrak{F}$ arbitrarily, and putting $Y' = Y + \sum_\lambda \varphi^\lambda X_\lambda$, we can express this by saying that the φ^λ are to be determined for each t in such a way that the integral

$$\int_{\pi^{-1}(t)} \sum_{\lambda,\mu} (f_\lambda^\mu + X_\lambda \varphi^\mu + \sum_\rho c_{\lambda\rho}^\mu \varphi^\rho)^2 d\Omega$$

should be a minimum. For a given t, this is a variational problem of classical type, and standard techniques show (in view of the fact that $f'^{\mu}_{\lambda} = f^{\mu}_{\lambda}$ implies $Y' = Y$) that this has for each t a unique solution. I owe to Hörmander the proof of the fact that this solution is of class C^{∞}, not only on each fibre $\pi^{-1}(t)$, but even on the manifold S; his proof will be found below in Appendix III. Replacing now Y by Y', we may assume that Y itself is the solution of our variational problem; then we must have, for all choices of the φ^{λ}, the Euler equations

$$\int_{\pi^{-1}(t)} \sum_{\lambda,\mu} f^{\mu}_{\lambda}(X_{\lambda}\varphi^{\mu} + \sum_{\rho} c^{\mu}_{\lambda\rho}\varphi^{\rho})d\Omega = 0 \ ;$$

if we transform this by means of (3), we get:

$$\sum_{\lambda} X_{\lambda} f^{\nu}_{\lambda} + \sum_{\lambda,\mu} c^{\mu}_{\nu\lambda} f^{\mu}_{\lambda} = 0 \ . \tag{7}$$

5. Now let us assume that there is a subgroup G' of G, connected or not, such that the left-translations by elements of G' leave the quadratic forms $\sum_{\lambda} (\omega^{\lambda})^2$ invariant; this is to say that, for every $s \in G'$, $\mathrm{Ad}(s)$ induces on the X_{λ} an orthogonal substitution $X_{\lambda} \to \sum_{\mu} \alpha^{\mu}_{\lambda} X_{\mu}$. Then, if Y_s is the transform of the vector-field Y under the operation $(t, x) \to (t, sx)$ of s, (4) gives

$$[\sum_{\mu} \alpha^{\mu}_{\lambda} X_{\mu},\ Y_s] = \sum_{\mu,\nu} f^{\mu}_{\lambda} \alpha^{\nu}_{\mu} X_{\nu} \ ,$$

which, in view of the orthogonality of (α^{μ}_{λ}), may also be written

$$[X_{\lambda},\ Y_s] = \sum_{\mu,\nu,\rho} f^{\nu}_{\rho} \alpha^{\lambda}_{\rho} \alpha^{\mu}_{\nu} X_{\mu} \ .$$

This shows at once that $\sum_{\lambda,\mu} (f^{\mu}_{\lambda})^2$ is unchanged if Y is replaced by Y_s, so that Y_s is also a solution of the variational problem by which we determined Y. As that solution is unique, this proves that Y is invariant under G'. In particular, if G' is a Lie subgroup of G, Y must be invariant under the infinitesimal transformations of G'; this is the same as to say that $[X, Y] = 0$ for all the vectors X in the Lie algebra of G'. Also, this shows that Y is invariant under the center of G.

6. The above results will now be applied to the case when G is a connected semisimple group without compact components, with the Lie algebra $\mathfrak{g}$; as proved in Appendix II, our assumption $\mathfrak{n}_t = \{0\}$ is verified in that case. Let $\mathfrak{k}$ be the Lie algebra of a maximal compact subgroup of $\mathrm{Ad}(G)$; we can consider it in an obvious manner as a subalgebra of $\mathfrak{g}$; let K be the connected subgroup of G with the Lie algebra $\mathfrak{k}$; let $n - r$ be its dimension. Latin indices $i, j, \cdots$, will range from 1 to r, while the Greek indices $\alpha, \beta, \cdots$ will range from $r + 1$ to n (and the indices $\lambda, \mu, \cdots$ from 1 to n as before). We choose $X_1, \cdots, X_n$ so that the X_{α}, for

$r+1 \leqq \alpha \leqq n$, are a basis of $\mathfrak{k}$ and that the Killing form of G is

$$2\sum_{i=1}^{r}(\omega^i)^2 - 2\sum_{\alpha=r+1}^{n}(\omega^\alpha)^2 .$$

It is known that $X_i \to -X_i$, $X_\alpha \to X_\alpha$ is then an involutory automorphism of $\mathfrak{g}$; therefore, among the $c^\nu_{\lambda\mu}$, only the $c^\alpha_{\beta\gamma}$, c^α_{ij}, $c^i_{\alpha j}$, $c^i_{j\alpha}$ can be $\neq 0$. We shall write $c_{\alpha\beta\gamma}$, $c_{\alpha ij}$ instead of $c^\alpha_{\beta\gamma}$, c^α_{ij}. Writing that the Killing form is invariant under the adjoint representation, one sees that the $c_{\alpha\beta\gamma}$ are alternating in all three indices α, β, γ, and also that we have

$$c^i_{j\alpha} = -c^i_{\alpha j} = -c_{\alpha ij} .$$

This implies that the infinitesimal operators X_α annul the form $\sum_\alpha(\omega^\alpha)^2$, so that K leaves that form invariant; as it leaves the Killing form invariant, the form $\sum_\lambda(\omega^\lambda)^2$ is also invariant under K.

The above facts imply that $[X_i, [X_j, X_k]]$ is a linear combination of the X_i; we write, as usual

$$[X_i, [X_j, X_k]] = \sum_{h=1}^{r} R_{hijk}X_h ,$$

where the R_{hijk} are given by

$$R_{hijk} = -\sum_{\alpha=r+1}^{n} c_{\alpha hi}c_{\alpha jk} ; \tag{8}$$

they can be interpreted geometrically as the Riemann curvature tensor for the riemannian symmetric space G/K. They have the obvious symmetry properties

$$R_{hijk} = R_{jkhi} = -R_{ihjk} = -R_{hikj} ; \tag{9}$$

moreover, the Jacobi identity between X_i, X_j, X_k gives the well-known relations

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 . \tag{10}$$

Finally, writing that the coefficient of $\omega^i\omega^j$ in the Killing form is $2\delta_{ij}$, we get

$$\sum_{k=1}^{r}\sum_{\alpha=r+1}^{n} c_{\alpha ik}c_{\alpha jk} = \delta_{ij} ,$$

which can also be written as

$$\sum_{k=1}^{r} R_{ikjk} = -\delta_{ij} . \tag{11}$$

7. Now we come back to the solution Y of the variational problem of no. 4. Applying to K the result of no. 5, we see that $[X_\alpha, Y] = 0$ for all α, i.e., $f^\lambda_\alpha = 0$ for all λ, α. Write f_{ij}, $f_{\alpha i}$ instead of f^i_j, f^α_i; the equations (5), (7) become:

$$X_kf_{ij} - X_jf_{ik} = \sum_\alpha(c_{\alpha ik}f_{\alpha j} - c_{\alpha ij}f_{\alpha k}) \tag{12}$$

$$X_k f_{\alpha j} - X_j f_{\alpha k} = \sum_h (c_{\alpha jh} f_{hk} - c_{\alpha kh} f_{hj}) \tag{13}$$

$$\sum_k X_k f_{ik} + \sum_{\alpha,k} c_{\alpha ik} f_{\alpha k} = 0 \tag{14}$$

$$\sum_j X_j f_{\alpha j} + \sum_{i,j} c_{\alpha ij} f_{ij} = 0 \tag{15}$$

$$X_\alpha f_{ij} = \sum_k (c_{\alpha kj} f_{ik} - c_{\alpha ik} f_{kj}) . \tag{16}$$

(we omit one additional relation which is not needed for our purposes). Now put:

$$\Phi = \frac{1}{2} \sum_{i,j,k} (X_k f_{ij} - X_j f_{ik})^2 ,$$

$$\Psi = \sum_i \left(\sum_k X_k f_{ik}\right)^2 .$$

We want[2] to evaluate the integral $\int (\Phi + \Psi) d\Omega$, taken on the fibre $\pi^{-1}(t)$. Using (12), we see that we have

$$\begin{aligned}\Phi &= \sum_{i,j,k} X_k f_{ij} (X_k f_{ij} - X_j f_{ik}) \\ &= \sum_{\alpha,i,j,k} (c_{\alpha ik} f_{\alpha j} \cdot X_k f_{ij} - c_{\alpha ij} f_{\alpha k} \cdot X_k f_{ij}) ,\end{aligned}$$

and therefore, using (3):

$$\int_{\pi^{-1}(t)} \Phi d\Omega = \int_{\pi^{-1}(t)} \Phi' d\Omega$$

with

$$\Phi' = \sum_{\alpha,i,j,k} (c_{\alpha ij} f_{ij} \cdot X_k f_{\alpha k} - c_{\alpha ik} f_{ij} \cdot X_k f_{\alpha j}) .$$

Similarly, using (14), we can write

$$\Psi = \sum_{i,j} X_j f_{ij} \left(\sum_k X_k f_{ik}\right) = -\sum_{\alpha,i,j,k} c_{\alpha ik} f_{\alpha k} \cdot X_j f_{ij} ,$$

and therefore, as above,

$$\int_{\pi^{-1}(t)} \Psi d\Omega = \int_{\pi^{-1}(t)} \Psi' d\Omega$$

with

$$\Psi' = \sum_{\alpha,i,j,k} c_{\alpha ik} f_{ij} \cdot X_j f_{\alpha k} .$$

Now, using (13) and (15), one finds:

$$\Phi' + \Psi' = -F(f) ,$$

where F is the quadratic form in the f_{ij} given by

$$F(f) = -\sum_{i,j,h,k} (R_{ijhk} f_{ij} f_{hk} + R_{ikjh} f_{ij} f_{hk} + R_{ikhk} f_{ij} f_{hj}) ,$$

[2] This decisive step in our proof is modelled after the work of Calabi on discrete groups of displacements in spaces of constant negative curvature; cf. footnote[1].

which, in view of (9), (10) and (11), can also be written

$$F(f) = \sum_{i,j} (f_{ij})^2 + \sum_{i,j,h,k} R_{ihkj} f_{ij} f_{hk} .$$

8. It will now be proved that $F(f) \geqq 0$; more precisely, the quadratic form F is positive and non-degenerate unless at least one of the simple components of the Lie algebra of G is of dimension 3. In fact, we can write $f = f' + f''$, where $f' = (f'_{ij})$ is symmetric and $f'' = (f''_{ij})$ is alternating in the indices i, j. If we define the bilinear form $F(f', f'')$ by

$$F(f', f'') = F(f' + f'') - F(f') - F(f'') ,$$

we have, in view of (9):

$$\frac{1}{2} F(f', f'') = \sum_{i,j} f'_{ij} f''_{ij} + \sum_{i,j,h,k} R_{ihkj} f'_{ij} f''_{hk} ;$$

as the first sum changes sign if we exchange i, j, and the second sum does so if we exchange i with j and h with k, this is 0. Therefore $F(f) = F(f') + F(f'')$. Also we have, for a similar reason

$$\sum_{i,j,h,k} R_{ihkj} f''_{ij} f''_{hk} = \sum_{i,j,h,k} R_{ikjh} f''_{ij} f''_{hk} ,$$

and therefore, using (8), (9) and (10):

$$\sum_{i,j,h,k} R_{ihkj} f''_{ij} f''_{hk} = -\frac{1}{2} \sum_{i,j,h,k} R_{ijhk} f''_{ij} f''_{hk} = \frac{1}{2} \sum_{\alpha} \left(\sum_{i,j} c_{\alpha ij} f''_{ij}\right)^2 .$$

This proves that $F(f'') \geqq 0$, and also that $F(f'')$ can be 0 only if $f'' = 0$.

Now we want to show that $F(f') \geqq 0$. In fact, if we make on the X_i any orthogonal substitution $X_i \to \sum_j \xi_{ij} X_j$, this will change f_{ij} into $\sum_{h,k} \xi_{hi} \xi_{kj} f_{hk}$, and therefore the quadratic form $\sum_{i,j} f_{ij} T_i T_j$ into

$$\sum_{h,k} f_{hk} \left(\sum_i \xi_{hi} T_i\right)\left(\sum_j \xi_{kj} T_j\right) ;$$

we may therefore choose that orthogonal substitution in such a way that at a given point, the quadratic form $\sum_{i,j} f_{ij} T_i T_j$ becomes a diagonal form $\sum_i a_i T_i^2$. Then we have, at that point, $f'_{ij} = a_i \delta_{ij}$. By considering the effect of the same substitution on the R_{ijhk}, one sees immediately that it does not change the value of the form $F(f')$. Therefore, in order to show that $F(f') \geqq 0$, it is enough to show that this is so at a point where $f'_{ij} = a_i \delta_{ij}$. Now, at such a point, we have

$$F(f') = \sum_i a_i^2 + \sum_{i,j} r_{ij} a_i a_j ,$$

where we have put

$$r_{ij} = -R_{ijij} = \sum_{\alpha} (c_{\alpha ij})^2 ,$$

so that we have $r_{ij} = r_{ji} \geqq 0$ and also, in view of (11):

$$\textstyle\sum_j r_{ij} = 1 \qquad (1 \leqq i \leqq r)\,. \tag{17}$$

Let ρ be an eigenvalue of the quadratic form $\sum_{i,j} r_{ij} a_i a_j$; if (v_i) is an eigenvector belonging to ρ, we have

$$\rho v_i = \textstyle\sum_j r_{ij} v_j \qquad (1 \leqq i \leqq r)\,,$$

and therefore, in view of (17),

$$|\rho v_i| \leqq \sup_j |v_j| \qquad (1 \leqq i \leqq r)\,,$$

which obviously implies $|\rho| \leqq 1$. Therefore $F(f') \geqq 0$ for all f', as we had asserted.

9. In no. 7, we have proved that

$$\int_{\pi^{-1}(t)} (\Phi + \Psi) d\Omega = -\int_{\pi^{-1}(t)} F(f) d\Omega\,;$$

Φ and Ψ are obviously $\geqq 0$, and we have just proved that $F(f) \geqq 0$. Therefore Φ, Ψ and $F(f)$ are 0. In view of the results of no. 8, this implies that $f'' = 0$, $f = f'$. As $\Phi = 0$, (12) gives $g_{ijk} = g_{ikj}$ for all i, j, k if we put

$$g_{ijk} = \textstyle\sum_\alpha c_{\alpha ij} f_{\alpha k}\,.$$

But we have $g_{ijk} = -g_{jik}$; as the substitutions $(ijk) \to (jik)$, $(ijk) \to (ikj)$ generate the symmetric group in the three letters i, j, k, one sees now at once that g_{ijk} must be both symmetric and alternating in i, j, k, so that it is 0. Put $W_k = \sum_\alpha f_{\alpha k} X_\alpha$; as we have $g_{ijk} = 0$, we have $[W_k, X_i] = 0$ for all i, so that W_k is an element of $\mathfrak{k}$ which commutes with all the X_i. Now one verifies immediately that the vectors W in $\mathfrak{k}$ with that property make up a normal subalgebra $\mathfrak{w}$ of $\mathfrak{g}$; as this is contained in $\mathfrak{k}$, the corresponding subgroup of $\mathrm{Ad}(G)$ is both compact and normal, which contradicts the assumption that G is without compact components unless $\mathfrak{w} = \{0\}$; therefore we have $W_k = 0$ for all k, hence $f_{\alpha k} = 0$ for all α, k.

10. Now we derive some further consequences from $F(f) = 0$ when $f = (f_{ij})$ is symmetric in i, j. If we write $\mathfrak{g}$ as the direct sum of simple algebras, and divide up the ranges of the indices λ, α, i accordingly, the $c^\nu_{\lambda\mu}$ are 0 unless all three indices belong to one and the same simple factor of $\mathfrak{g}$, so that R_{ijhk} is 0 unless all four indices belong to the same simple factor. Therefore $F(f)$ splits up into the similar forms belonging to the simple factors of $\mathfrak{g}$, plus the sum $\sum (f_{ij})^2$ extended to all the pairs (i, j) in which i and j belong to distinct simple factors of $\mathfrak{g}$. As $F(f) = 0$, this shows that $f_{ij} = 0$ unless i, j belong to one and the same simple factor of

$\mathfrak{g}$; and it remains for us to determine when $F(f)$ is 0 in the case of a simple Lie algebra $\mathfrak{g}$; this cannot occur unless -1 is an eigenvalue of the form $\sum_{i,j} r_{ij}a_i a_j$ for some choice of the basis (X_λ) of $\mathfrak{g}$. Let (v_i) be an eigenvector belonging to the eigenvalue -1; we have

$$-v_i = \sum_j r_{ij}v_j \qquad (1 \leqq i \leqq r)\,;$$

in view of (17), this can be written

$$\sum_j r_{ij}(v_i + v_j) = 0 \qquad (1 \leqq i \leqq r)\,.$$

Put $v = \sup_i |v_i|$; then we must have $r_{ij} = 0$ whenever $|v_i| = v$ and $v_i + v_j \neq 0$. Therefore, if $|v_i| = v$, there is a value of j for which $v_j = -v_i$; in other words, the sets of values of i for which $v_i = v$ and for which $v_i = -v$ are not empty. After re-ordering the X_i if necessary, we may now assume that $v_i = v$ for $1 \leqq i \leqq s$, $v_i = -v$ for $s + 1 \leqq i \leqq s + t$, and $|v_i| < v$ for $s + t + 1 \leqq i \leqq r$, with $s > 0$, $t > 0$. Then the matrix (r_{ij}) is of the form

$$\begin{pmatrix} 0 & M & 0 \\ {}^tM & 0 & 0 \\ 0 & 0 & N \end{pmatrix}$$

where M, N are respectively an (s, t)-matrix and a square matrix of order $r - s - t$, and ${}^tM$ is the transpose of M. In view of the values obtained for the r_{ij}, $r_{ij} = 0$ implies $c_{\alpha ij} = 0$, hence also $c^i_{\alpha j} = 0$ for all α. Therefore, in the representation of $\mathfrak{k}$ determined by $X_\alpha \to (c^i_{\alpha j})$, every element X of $\mathfrak{k}$ is mapped onto a matrix of the form

$$\rho(X) = \begin{pmatrix} 0 & A & 0 \\ B & 0 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

It is known that ρ must be irreducible (in fact, if a proper subspace $\mathfrak{m}$ of the space generated by the X_i is stable under ρ, one varifies at once[3] that $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is a normal subalgebra of $\mathfrak{g}$; therefore, as $\mathfrak{g}$ is simple, $\mathfrak{m}$ must be $\{0\}$). In particular, we must have $s + t = r$, so that ρ is of the form

$$\rho(X) = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

If X, X' are in $\mathfrak{k}$, $\rho([X, X'])$ must be of that form; as we have

[3] This argument is taken from E. Cartan's classical *Mémorial* volume, "*La Théorie des Groupes Finis et Continus et l'Analysis Situs*," Paris, 1930. The discerning reader will already have noticed that our whole approach to the subject of semisimple Lie groups, including the notation, is derived from Ch. IV of that volume.

$$\rho([X, X']) = [\rho(X), \rho(X')] = \begin{pmatrix} AB' - A'B & 0 \\ 0 & BA' - B'A \end{pmatrix},$$

this implies that $\rho([X, X']) = 0$. But the kernel of ρ is the same normal subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ which was considered in no. 9; as this is $\{0\}$, we see that $\mathfrak{k}$ is abelian; as it has the faithful irreducible representation ρ, this implies that $r = 2$, $n - r = 1$, and that $\mathfrak{g}$ is the Lie algebra of dimension 3 belonging to the hyperbolic group. Therefore, if this is not so ($\mathfrak{g}$ being still assumed to be simple), $F(f) = 0$ implies $f = 0$.

Going back now to the general case of a semisimple group, we see that, *unless* $\mathfrak{g}$ *has a simple factor of dimension* 3, *all the* f^{μ}_{λ} *are* 0, *so that the vector-field* Y *on* $I \times G$ *satisfies* $[X_\lambda, Y] = 0$ *for all* λ. In other words, the vector-field Y is then invariant under *all* left-translation $(t, x) \to (t, sx)$ in $I \times G$.

11. Consider now the differential systems, in S and in $I \times G$, whose solutions are the curves which, at every point, are tangent to Y. As the vector-field Y on S is nowhere tangent to the fibre $\pi^{-1}(t)$, and as these fibres are compact, one sees at once that, in S, every such curve cuts each fibre once and only once; therefore the same is true in $I \times G$, so that, in $I \times G$, every solution of our differential system is a cross-section of $I \times G$, i.e., the image of I by a mapping $t \to (t, \xi(t))$ of class C^∞. We shall denote by $t \to (t, \xi(t, s))$ the uniquely determined solution of that system in $I \times G$ which goes through the point $(0, s)$. As Y is invariant under Γ operating on $I \times G$ to the right in the manner explained in no. 2, we have, for all $t \in I$, $s \in G$ and $\gamma \in \Gamma$:

$$\xi(t, sr_0(\gamma)) = \xi(t, s)r_t(\gamma)\ . \tag{18}$$

On the other hand, if G' is a group of left-translations leaving Y invariant, we have, for all $t \in I$, $s \in G$ and $s' \in G'$:

$$\xi(t, s's) = s'\xi(t, s)\ . \tag{19}$$

Let us assume first, as at the end of no. 10, that $\mathfrak{g}$ has no simple factor of dimension 3. In that case, as we have seen there, Y is invariant under all left-translations by G, so that (19) gives $\xi(t, s) = s\xi(t, e)$; then (18) gives, for all γ and t:

$$r_t(\gamma) = \xi(t, e)^{-1}r_0(\gamma)\xi(t, e)\ .$$

This can be expressed by saying that r_t differs from r_0 only by an inner automorphism of G.

In D^{I}, we agreed to denote by $\mathfrak{R}_0(\Gamma, G)$ the space of the injective representations r of Γ into G such that $r(\Gamma)$ is discrete and $G/r(\Gamma)$ is compact. In particular, let Γ be a discrete subgroup of G such that G/Γ is compact, so that the identity mapping r_0 of Γ belongs to $\mathfrak{R}_0(\Gamma, G)$; from now on, *the connected component of* r_0 *in* $\mathfrak{R}_0(\Gamma, G)$ *will then be denoted by* $\mathfrak{R}_1(\Gamma, G)$. With this notation, let r, r' be any two points in $\mathfrak{R}_1(\Gamma, G)$; as shown in D^{I}, they can be joined together by a succession of analytic arcs contained in $\mathfrak{R}_1(\Gamma, G)$. We have just proved that, in a sufficiently small neighborhood of any point on such an arc, all points belong to representations which differ only by inner automorphisms. An obvious compactness argument gives now:

THEOREM 1. *Let G be a connected semisimple Lie group without compact components, whose Lie algebra has no simple factor of dimension* 3. *Let Γ be a discrete subgroup of G such that G/Γ is compact. Then $\mathfrak{R}_1(\Gamma, G)$ consists of the representations of Γ into G induced on Γ by the inner automorphisms of G.*

12. In order to get a more complete result, we need additional information on the nature of the solution (f_{ij}) of our variational problem in the case of a simple group G of dimension 3; as shown in no. 5, this must be invariant under the center of G, so that it is enough to consider the case when G is its own adjoint group, i.e., when it is the quotient of $SL(2, \mathbf{R})$ by its center $\{\pm \mathbf{1}_2\}$. Here we have $r = 2$, $n = 3$; it is easily seen that the $c^{\nu}_{\lambda\mu}$ are uniquely determined by the conventions in no. 6, except for the sign, and that this may be chosen so that we have

$$[X_1, X_2] = X_3 \,, \qquad [X_1, X_3] = X_2 \,, \qquad [X_2, X_3] = -X_1 \,.$$

Then K is the compact subgroup of dimension 1 with the infinitesimal transformation X_3. It is well-known that the space H of left-cosets of K in G can be identified with the half-plane $\mathrm{Im}(z) > 0$ in the plane of a complex variable z; we do this so that K is the coset corresponding to the point $i = \sqrt{-1}$ in H (we shall no longer need i as an index, so that this notation will cause no confusion). Then, if s is the image in G of the element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbf{R})$, s acts on H by

$$z \to zs = \frac{az + c}{bz + d}\,,$$

and the coset Ks in G corresponds to the point $\tau = (ai + c)/(bi + d)$ of H.

Now, if we combine the results of no. 9 with (13), we see that, for any solution of our variational problem, we must have $f_{12} = f_{21}$ and $f_{11} + f_{22} = 0$. Put $F = f_{11} + i \cdot f_{12}$; then (14), (16) give $X_2F = -i \cdot X_1F$, $X_3F = 2i \cdot F$. An easy calculation, which we omit, shows that the most general solution for these differential equations is

$$F(s) = (bi + d)^{-4}\Phi(\tau) ,$$

where Φ is any holomorphic function in H; F is invariant under a right-translation $s \to ss_0$ if and only if the "quadratic differential" $\Phi(\tau)d\tau^2$ is invariant under s_0 acting on H as we have said. For a given $\Phi \neq 0$, it is clear that the subgroup G' of G which leaves $\Phi(\tau)d\tau^2$ invariant is closed. Assume that G' is not discrete; then there is a one-parameter subgroup G_1 of G leaving $\Phi(\tau)d\tau^2$ invariant; therefore, in a suitable neighborhood of any point of H where Φ is not 0, the holomorphic differential $\Phi^{1/2}d\tau$ is invariant under G_1. It is now easily seen that, in any neighborhood of a point of H, every holomorphic differential which is invariant under a one-parameter subgroup of G must be either 0 or of the form $d\tau/(A\tau^2 + B\tau + C)$, where A, B, C are constants. Therefore, when we assume that $\Phi \neq 0$ and that G' is not discrete, $\Phi d\tau^2$ must be of the form

$$\Phi d\tau^2 = (A\tau^2 + B\tau + C)^{-2}d\tau^2 .$$

This is invariant under an element s of G, corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbf{R})$, if and only if the binary form $Au^2 + Buv + Cv^2$ is invariant under $(u, v) \to (au + cv, bu + dv)$. One finds at once that G' has then at most two connected components, and that G/G' cannot be compact. Therefore, *if* $\Phi \neq 0$ *and* G/G' *is compact,* G' *must be discrete in* G.

13. Now we go back to the problem discussed in nos. 3–10, and we begin by assuming that G is a product of simple groups $G^{(\rho)}$, none of which is compact. As the f^μ_λ satisfy (5) and (7), they also satisfy the equations

$$\begin{aligned}\textstyle\sum_\lambda (X_\lambda)^2 f^\nu_\mu = \sum_{\lambda,\rho} (c^\lambda_{\rho\mu} + c^\rho_{\lambda\mu}) X_\lambda f^\nu_\rho + \sum_{\lambda,\rho} c^\nu_{\mu\rho} X_\lambda f^\rho_\lambda \\ \textstyle - \sum_{\lambda,\rho} c^\nu_{\lambda\rho} X_\lambda f^\rho_\mu - \sum_{\lambda,\rho} c^\rho_{\nu\lambda} X_\mu f^\rho_\lambda ,\end{aligned}$$

obtained by applying X_λ to (5), using summation with respect to λ, and applying (1) and (7). For each value of $t \in I$, this is an elliptic system on the fibre $\pi^{-1}(t)$. Therefore, if, for a sequence (t_m) of values of t, tending to 0, we have, on $\pi^{-1}(t_m)$, a solution $\bar{f}^\mu_\lambda(t_m)$ of the system (5), (7), and if we assume for instance that we have, for each t_m:

$$\int_{\pi^{-1}(t_m)} \textstyle\sum_{\lambda,\mu} \bar{f}^\mu_\lambda(t_m)^2 d\Omega = 1 , \tag{20}$$

the known *a priori* estimates for solutions of elliptic systems (cf. [5], [6]) show that a suitable subsequence of the sequence $\bar{f}^\mu_\lambda(t_m)$ converges towards a solution $\bar{f}^\mu_\lambda$ of the same system on $\pi^{-1}(0)$, and that the convergence is uniform in the $\bar{f}^\mu_\lambda(t_m)$ and their derivatives up to any order.

Now we choose a basis of the Lie algebra of G consisting of bases $X^{(\rho)}_\lambda$ for the Lie algebras of the simple factors $G^{(\rho)}$ of G; Y being as before, we know that $[X_\lambda^{(\rho)}, Y]$ is 0 unless $G^{(\rho)}$ has the dimension 3, and also that, for each ρ, $[X_\lambda^{(\rho)}, Y]$ is a linear combination of the $X_\mu^{(\rho)}$ corresponding to the same value of ρ; therefore we may write

$$[X_\lambda^{(\rho)}, Y] = \sum_\mu f_{\mu\lambda}^{(\rho)} X_\mu^{(\rho)} .$$

Applying (12) and (16), we see that $X_\nu^{(\rho)} f_{\mu\lambda}^{(\rho')} = 0$ for $\rho \neq \rho'$; therefore, for each ρ, the $f_{\mu\lambda}^{(\rho)}$, considered as functions on $I \times \prod_\rho G^{(\rho)}$, depend upon the I-coordinate and the $G^{(\rho)}$-coordinate alone.

For each ρ, put:

$$a_\rho(t) = \int_{\pi^{-1}(t)} \sum_{\lambda,\mu} (f_{\mu\lambda}^{(\rho)})^2 d\Omega \ ;$$

we have $a_\rho(t) = 0$ for all $t \in I$ unless $G^{(\rho)}$ is of dimension 3. Let ρ be such that $a_\rho(t)$ is not identically 0 in any neighborhood of $t = 0$, and let (t_m) be a sequence of values of t, tending to 0, such that $a_\rho(t_m) \neq 0$. Put now

$$\bar{f}_{\mu\lambda}^{(\rho\rho)}(t_m) = a_\rho(t_m)^{-1/2} f_{\mu\lambda}^{(\rho)}(t_m) ,$$
$$\bar{f}_{\mu\lambda}^{(\rho'\rho'')}(t_m) = 0 \qquad (\rho' \neq \rho \quad \text{or} \quad \rho'' \neq \rho) \ ;$$

this defines, for each t_m, a solution of the system (5), (7) which satisfies (20); as we have seen, it must (after the sequence (t_m) has been replaced by a suitable subsequence) converge towards a solution $(\bar{f})$ of the system (5), (7) on $\pi^{-1}(0)$, other than 0, such that $\bar{f}_{\mu\lambda}^{(\rho'\rho'')} = 0$ unless $\rho' = \rho'' = \rho$; this determines a solution of the same system on G, invariant under right-translations by elements of $r_0(\Gamma)$. But then the functions $\bar{f}_{\mu\lambda}^{(\rho\rho)}$ define a solution of the corresponding system on $G^{(\rho)}$, other than 0, which is invariant under the projection of $r_0(\Gamma)$ on $G^{(\rho)}$. Let G' be the group of all the right-translations in $G^{(\rho)}$ which leave that solution invariant. Then $r_0(\Gamma)$ is contained in the group of the elements of G whose $G^{(\rho)}$-coordinate is in G'; as $G/r_0(\Gamma)$ is compact, and as we have seen in no. 12 that G' is closed in $G^{(\rho)}$, this implies that $G^{(\rho)}/G'$ is compact, and therefore, by the final result in no. 12, that G' is discrete. This proves that, *unless the projection of $r_0(\Gamma)$ on $G^{(\rho)}$ is discrete in $G^{(\rho)}$, all the $f_{\mu\lambda}^{(\rho)}$ must be 0 in some neighborhood of $t = 0$ in I.*

14. For convenience, we shall identify Γ with $r_0(\Gamma)$ from now on. Collect into a partial product G'' of G all the simple factors $G^{(\rho)}$ of G of dimension 3 such that the projection of Γ on $G^{(\rho)}$ is discrete; let G' be the product of all the other simple factors of G. We know that, if $G^{(\rho)}$ is any one of the factors in G', all the $f_{\mu\lambda}^{(\rho)}$ are 0 for t in a suitable neighborhood of 0, which we may assume to be I. Then we have $[X, Y] = 0$ for every X in the Lie algebra of G', so that Y is invariant under left-translations by elements of G'; therefore, if $\xi(t, s)$ is as in no. 11, (19) is valid for all $t \in I$, $s \in G$, $s' \in G'$.

Call Γ', Γ'' the projections of Γ on G', G''; and call Δ', Δ'' its intersections with G', G'' considered as subgroups of G. The definition of G'' implies that Γ'' is discrete in G''; therefore we can apply corollary 3 of Appendix II. This shows in particular that Γ' is discrete in G', that Δ' has finite index in Γ', and that G'/Δ' is compact. Now, if we apply (18) and (19) to any $\delta' \in \Delta'$, we get

$$\xi(t, s\delta') = \xi(t, s)r_t(\delta') , \qquad \xi(t, \delta's) = \delta'\xi(t, s) . \tag{21}$$

For $s = e$, these relations show that, if we modify r_t by the inner automorphism of G determined by $\xi(t, e)$, i.e., if we replace it by $\xi(t,e)\cdot r_t\cdot\xi(t,e)^{-1}$, r_t induces the identity on Δ'. Then, if $\gamma = (\gamma', \gamma'') \in \Gamma$, the projection of $r_t(\gamma)$ on G'' depends only upon t and γ'' and may be written as $r''_t(\gamma'')$, where r''_t is a one-parameter family of representations of Γ'' into G'' such that r''_0 is the identity. Now, applying (21) to any $s = s'' \in G''$, we see, in view of the fact that $r_t(\delta') = \delta'$ and $s''\delta' = \delta's''$, that $\xi(t, s'')$ commutes with every $\delta' \in \Delta'$; therefore, for every $s'' \in G''$, the projection of $\xi(t, s'')$ on G' is in the normalizer $N(\Delta')$ of Δ' in G'; as that projection is e for $t = 0$, and $N(\Delta')$ is discrete by corollary 1 of Appendix II, this shows that $\xi(t, s'')$ is in G'' for all $t \in I$ and $s'' \in G''$. That being so, if we take $s = e$ and $\gamma \in \Delta''$ in (18), we see that $r_t(\gamma) \in G''$ for all $t \in I$ and $\gamma \in \Delta''$; therefore, if $\gamma = (\gamma', \gamma'') \in \Gamma$, the projection of $r_t(\gamma)$ on G' depends only upon t and γ' and may be written as $r'_t(\gamma')$, where r'_t is a one-parameter family of representations of Γ' into G' such that r'_0 is the identity. As r'_t is the identity on Δ', corollary 2 of Appendix II shows that it is the identity on Γ'. This gives, for all $\gamma = (\gamma', \gamma'') \in \Gamma$:

$$r_t((\gamma', \gamma'')) = (\gamma', r''_t(\gamma'')) .$$

If now we apply the results of D^{I} just as in no. 11, and if we use the notation $\mathfrak{R}_1(\Gamma, G)$ in the manner explained there, we see that we have proved the following theorem:

THEOREM 2. *Let G be a product of connected non-compact simple Lie groups. Let Γ be a discrete subgroup of G such that G/Γ is compact. Let G'' be the product of those simple factors $G^{(\rho)}$ of G which are of dimension 3 and such that the projection of Γ on $G^{(\rho)}$ is discrete in $G^{(\rho)}$, and let G' be the product of all the other simple factors of G. Then the projections Γ', Γ'' of Γ on G', G'' are discrete in G' and in G'', respectively; G'/Γ' and G''/Γ'' are compact; and Γ is of finite index in $\Gamma' \times \Gamma''$. Moreover, if j is the injection mapping of Γ into $\Gamma' \times \Gamma''$, and if $\mathfrak{R}'$ is the set of the representations of Γ' into G' induced on Γ' by the inner automorphisms of G', then $(r', r'') \to (r' \times r'') \circ j$ is a homeomorphism of*

$\mathfrak{R}' \times \mathfrak{R}_1(\Gamma'', G'')$ *onto* $\mathfrak{R}_1(\Gamma, G)$.

Here we have denoted by $r' \times r''$ the representation of $\Gamma' \times \Gamma''$ into $G' \times G''$ which maps (γ', γ'') onto $(r'(\gamma'), r''(\gamma''))$.

15. In order to obtain a complete result for a product of simple non-compact groups, we still have to deal with the case $G = G''$. The result is as follows:

THEOREM 3. *Let G be a product of connected non-compact simple Lie groups G_ρ of dimension* 3. *Let Γ be a discrete subgroup of G such that G/Γ is compact and that, for every ρ, the projection Γ_ρ of Γ on G_ρ is discrete in G_ρ. Then, for each ρ, G_ρ/Γ_ρ is compact, and Γ is of finite index in $\prod_\rho \Gamma_\rho$. Moreover, if j is the injection mapping of Γ into $\prod_\rho \Gamma_\rho$, $(r_\rho) \to (\prod_\rho r_\rho) \circ j$ is a homeomorphism of $\prod_\rho \mathfrak{R}_1(\Gamma_\rho, G_\rho)$ onto $\mathfrak{R}_1(\Gamma, G)$.*

Here we have denoted by $\prod_\rho r_\rho$ the representation of $\prod_\rho \Gamma_\rho$ into G which maps (γ_ρ) onto $(r_\rho(\gamma_\rho))$.

The first assertion in our theorem follows at once from corollary 3 of Appendix II by induction on the number of the factors in G. The second part will be proved in our usual manner by dealing first with a one-parameter family r_t of representations of Γ into G. Notations being the same as before, we shall write, on $I \times G$:

$$Y = \frac{\partial}{\partial t} + \sum\nolimits_{\rho,\lambda} \varphi_\lambda^{(\rho)} X_\lambda^{(\rho)} ;$$

then the $\varphi_\lambda^{(\rho)}$ satisfy the equations (6) with f, f' replaced by $0, f$; this can be written

$$\begin{aligned} &X_\lambda^{(\rho)} \varphi_\lambda^{(\rho)} + \sum\nolimits_\nu c_{\mu\lambda\nu}^{(\rho)} \varphi_\nu^{(\rho)} = f_{\mu\lambda}^{(\rho)} \\ &X_\lambda^{(\rho')} \varphi_\mu^{(\rho)} = 0 \qquad\qquad (\rho' \neq \rho) . \end{aligned}$$

The latter equations show that $\varphi_\mu^{(\rho)}$, as a function on $I \times \prod_\rho G_\rho$, depends only upon the I-coordinate and the G_ρ-coordinate, so that we can write

$$Y = \frac{\partial}{\partial t} + \sum\nolimits_{\rho,\lambda} \varphi_\lambda^{(\rho)}(t, x_\rho) \cdot X_\lambda^{(\rho)}$$

at the point $(t, (x_\rho))$ of $I \times \prod_\rho G_\rho$. This implies that the solutions $(t, \xi(t, s))$ of the differential system determined by Y on $I \times G$ are of the form

$$\xi(t, (s_\rho)) = ((\xi_\rho(t, s_\rho)) \tag{22}$$

where, for each ρ, $(t, \xi_\rho(t, s_\rho))$ is the solution through $(0, s_\rho)$ of the differential system determined in $I \times G_\rho$ be the vector-field

$$Y_\rho = \frac{\partial}{\partial t} + \sum\nolimits_\lambda \varphi_\lambda^{(\rho)}(t, x_\rho) \cdot X_\lambda^{(\rho)} .$$

If we again identify Γ with $r_0(\Gamma)$, we see now at once, by combining (18) with (22), that the G_ρ-coordinate of $r_t(\gamma)$, for $\gamma = (\gamma_\rho) \in \Gamma$, depends only upon t and γ_ρ and may be written as $r_t^{(\rho)}(\gamma_\rho)$, where $r_t^{(\rho)}$ is a one-parameter family of representations of Γ_ρ into G_ρ. The conclusion of Theorem 3 follows now by making use of the results of D^I in the same way as in no. 11.

16. Now take for G any connected semisimple group without compact components, and let Z be its center. Notations being as before, it has been shown in no. 5 that Y is invariant under Z, so that (19) gives $\xi(t, zs) = z\xi(t, s)$ for $z \in Z$; if now $z = r_0(\gamma)$ for some $\gamma \in \Gamma$, a comparison with (18) shows that $z = r_t(\gamma)$ for all $t \in I$. Let now Γ be a discrete subgroup of G such that G/Γ is compact; by making use of the results of D^I in the same way as in no. 11, we see now that, whenever r, r' are in $\mathfrak{R}_1(\Gamma, G)$ and an element γ of Γ is such that $r(\gamma) = z \in Z$, then $r'(\gamma) = z$. Therefore every $r \in \mathfrak{R}_1(\Gamma, G)$ can be extended to an injective representation r^* of the group $\Gamma^* = Z \cdot \Gamma$ into G by putting $r^*(z\gamma) = z \cdot r(\gamma)$ for $z \in Z$, $\gamma \in \Gamma$. Moreover, as $Z \cdot \Gamma$ is contained in the normalizer $N(\Gamma)$ of Γ in G, corollary 1 of Appendix II shows that it is discrete in G; as G/Γ is compact, G/Γ^* is so; for similar reasons, the group $r^*(\Gamma^*) = Z \cdot r(\Gamma)$ is discrete in G, and $G/r^*(\Gamma^*)$ is compact, for every $r \in \mathfrak{R}_1(\Gamma, G)$. Thus we have proved that every element of $\mathfrak{R}_1(\Gamma, G)$ induces the identity on $\Gamma \cap Z$, and that $r \to r^*$ is a homeomorphism of $\mathfrak{R}_1(\Gamma, G)$ onto $\mathfrak{R}_1(\Gamma^*, G)$.

From now on, assume that Γ contains Z. Let Z_1 be a subgroup of Z; put $G' = G/Z_1$; call p the canonical homomorphism of G onto G'; and put $\Gamma' = p(\Gamma)$, so that $\Gamma = p^{-1}(\Gamma')$. As every $r \in \mathfrak{R}_1(\Gamma, G)$ induces the identity on Z, the relation $p \circ r = r' \circ p$ determines a mapping $r \to r'$ of $\mathfrak{R}_1(\Gamma, G)$ into $\mathfrak{R}_1(\Gamma', G')$, which is obviously continuous. Choose now (as in D^I, no. 11) a finite set (γ_i) of generators of Γ, such that Γ is defined by a finite set of relations $R_\lambda(\gamma_i) = e$ between the γ_i; choose a finite set (ζ_j) of generators for Z_1; and, for each ζ_j, choose an expression $\zeta_j = F_j(\gamma_i)$ of ζ_j in terms of the γ_i; then, if we put $\gamma'_i = p(\gamma_i)$, Γ' has the generators γ'_i and is defined by the relations $R_\lambda(\gamma'_i) = e'$, $F_j(\gamma'_i) = e'$ between them. Take any $r'_0 \in \mathfrak{R}_1(\Gamma', G')$; put $s'_i = r'_0(\gamma'_i)$; and, for each i, choose $s_i \in G$ such that $p(s_i) = s'_i$. Let V be a neighborhood of e in G, such that $V \cap Z_1 = \{e\}$; choose an open neighborhood U of e in G such that $R_\lambda(u_i s_i) \in V \cdot R_\lambda(s_i)$ and $F_j(u_i s_i) \in V \cdot F_j(s_i)$ for all λ, j whenever all the u_i are in U, and also such that p induces on U a homeomorphism of U onto its image U' in G'; call φ the inverse of that homeomorphism. Let $\mathfrak{U}'$ be the open neighborhood of r'_0 in $\mathfrak{R}_1(\Gamma', G')$, consisting of the representations r' such that $r'(\gamma'_i) \in U's'_i$ for all i. Then it is easy to see that if, for any $r' \in \mathfrak{U}'$, there is an $r \in \mathfrak{R}_1(\Gamma, G)$ such that $p \circ r = r' \circ p$, this must be given by formulas

$$r(\gamma_i) = z_i^{-1} \cdot \varphi\big(r'(\gamma'_i)s_i'^{-1}\big) \cdot s_i \ ,$$

where the z_i are elements of Z_1 satisfying the relations

$$R_\lambda(z_i) = R_\lambda(s_i) \ , \qquad F_j(z_i) = \zeta_j \cdot F_j(s_i) \ .$$

As these relations are independent of the choice of r' in $\mathfrak{U}'$, one concludes immediately from this, and from obvious continuity considerations, that the inverse image of $\mathfrak{U}'$ under the mapping $r \to r'$ is the union of neighborhoods of the points in the inverse image of r'_0, and that this mapping determines, on each one of these neighborhoods, a homeomorphism onto $\mathfrak{U}'$. In other words, for this "natural" mapping, $\mathfrak{R}_1(\Gamma, G)$ is a covering space of $\mathfrak{R}_1(\Gamma', G')$[4].

In particular, we can apply this to the case $Z_1 = Z$; G' is then the adjoint group of G and is a product of simple groups, so that the structure of $\mathfrak{R}_1(\Gamma', G')$ is fully determined by Theorems 2 and 3. In view of the known facts on fuchsian groups, this shows for instance that $\mathfrak{R}_1(\Gamma, G)$ is a manifold. Alternatively, we may write $G = G_1/Z_1$, where G_1 is the simply connected group with the same infinitesimal structure as G, and Z_1 is a subgroup of the center of G_1; then, if Γ_1 is the inverse image of Γ in G_1, we see that $\mathfrak{R}_1(\Gamma_1, G_1)$ is a covering space of $\mathfrak{R}_1(\Gamma, G)$; here again G_1 is a product of simple groups, so that the structure of $\mathfrak{R}_1(\Gamma_1, G_1)$ is again given by Theorems 2 and 3.

Appendix I

Through an oversight, the main theorem of D^{I}, as formulated there in no. 1, is not quite strong enough for the application which is made of it in D^{I}, no. 12; this is to be corrected now. We wish to show that the theorem in question is valid with the following addition:

Let G, Γ, r_0, $\mathfrak{R}$, $\mathfrak{U}$ be as in that theorem; then, for every $g \in G$, there is a neighborhood W of g in G, and a neighborhood $\mathfrak{U}'$ of r_0 in $\mathfrak{U}$, such that the union of the sets $r^{-1}(W)$, for all $r \in \mathfrak{U}'$, is a finite set.

Assume first that G is simply connected. Let all assumptions and notations be as in nos. 7–10 of D^{I}; put $W = s_0^{-1}U'^{-1}s_0 g$. As f maps $U' \times S \times \Gamma$ onto G, we can write $s_0 g = us\gamma$, with $u \in U'$, $s \in S$, $\gamma \in \Gamma$. If the representation $\gamma \to \bar{\gamma}$ is close enough to the identity, the point $\bar{u}$ determined by

$$us\gamma = \bar{u}s\delta(s)^{-1}\bar{\delta}(s)\bar{\gamma}$$

[4] If G has no component of dimension 3, these two spaces are actually isomorphic. This follows from Theorem 1 and from a result of Borel [2, Corollary 4.4]. If the same could be proved for groups of dimension 3, then Theorems 2 and 3 would show that it remains true for all semisimple groups.

will be in U'. Now assume that, for such a representation, and for some $\gamma' \in \Gamma$, $\bar{\gamma}'$ is in W; this means that we have

$$\bar{u}' s_0 \bar{\gamma}' = s_0 g = \bar{u} s \delta(s)^{-1} \bar{\delta}(s) \bar{\gamma}$$

with some $\bar{u}' \in U'$. This can also be written as

$$\bar{f}(\bar{u}', s_0, \gamma'\gamma^{-1}) = \bar{f}(\bar{u}, s, e) ,$$

which amounts to saying that $(\bar{u}', s_0, \gamma'\gamma^{-1})$ and $(\bar{u}, s, e)$ are equivalent for $\bar{R}$; this implies that $\{s_0, s\} \in N(S)$ and $\gamma'\gamma^{-1} = \gamma(s_0, s)$. So γ' must have one of the finitely many values $\gamma(s_0, s)\gamma$. This proves our assertion when G is simply connected. Reasoning as in D^{I}, no. 11, one extends it at once to the general case.

Now let assumptions and notations be as in D^{I}, no. 12. From what we have proved above, it follows that, when Γ is made to act on $X \times G$ by

$$(x, g)\gamma = \big(x, g \cdot r_x(\gamma)\big) ,$$

it operates *properly* on $X \times G$. This means that, given any two compact subsets K, K' of $X \times G$, there are at most finitely many elements γ of Γ such that $K\gamma$ meets K'. Therefore the space $S = (X \times G)/\Gamma$ is separated (i.e., "Hausdorff"). While this fact had not expressly been stated in no. 12 of D^{I}, some of the assertions made there would not make sense unless this were so.

Once the above addition to the main theorem of D^{I} has been obtained, it is a trivial matter to strengthen it as follows: *one can choose W and $\mathfrak{U}'$ so that the sets $r^{-1}(W)$, for $r \in \mathfrak{U}'$, are all empty if $g \notin r_0(\Gamma)$ and all equal to $\{g\}$ if $g \in r_0(\Gamma)$.*

Appendix II

Let G be a connected Lie group of dimension n; in D^{I}, no. 6, we defined a G-structure on a manifold V of dimension n as being given by n everywhere linearly independent differential forms ω^{λ} on V satisfying the Maurer-Cartan equations for G. If the X_{λ}, at every point of V, are the vectors defined by $i(X_{\lambda})\omega^{\mu} = \delta^{\mu}_{\lambda}$, the X_{λ} make up n everywhere linearly independent vector-fields on V, satisfying the structural equations (1) for G; a G-structure may be considered as given by n such vector-fields, just as well as by structural forms ω^{λ}. The group G itself is always to be considered as carrying the G-structure determined by the right-invariant vector-fields X_{λ} (cf. no. 1); then, if Γ is a discrete subgroup of G, G/Γ (the space of right cosets $s\Gamma$ in G) carries a G-structure, determined in an obvious manner by that of G.

As observed in D^{I}, no. 6, the automorphisms of the G-structure of G are the right-translations; if Γ is a discrete subgroup of G, a right-translation $x \to xs$ determines an automorphism of G/Γ if and only if $s\Gamma s^{-1} = \Gamma$, i.e., if and only if s belongs to the normalizer $N(\Gamma)$ of Γ; and it is easily seen (using the elementary facts noted in D^{I}, no. 6, i.e., essentially nothing more than Frobenius's theorem) that all automorphisms of G/Γ can be obtained in this manner. The group of automorphisms of G/Γ may therefore be identified with $N(\Gamma)/\Gamma$.

Let $N_0(\Gamma)$ be the component of e in the closed subgroup $N(\Gamma)$ of G; it is a connected Lie group. For each $\gamma \in \Gamma$, the image of $N_0(\Gamma)$ by $x \to x\gamma x^{-1}$ must be a connected subset of Γ, containing γ, and is therefore $\{\gamma\}$. Thus $N_0(\Gamma)$ is also the component of e in the centralizer $Z(\Gamma)$ of Γ (consisting of the elements of G which commute with every $\gamma \in \Gamma$). In particular, the Lie algebra $\mathfrak{n}(\Gamma)$ of $N_0(\Gamma)$, which may be identified with those of $Z(\Gamma)$ and of $N(\Gamma)/\Gamma$, consists of the vectors X in the Lie algebra $\mathfrak{g}$ of G which are invariant under $\mathrm{Ad}(\gamma)$ for every $\gamma \in \Gamma$ (as usual, we denote by $\mathrm{Ad}(s)$ the automorphism of $\mathfrak{g}$ induced by the inner automorphism $x \to sxs^{-1}$ of G).

Now assume that G/Γ is compact, i.e., that there is a compact subset K of G such that $G = K\Gamma$. Then, if ρ is any representation of G in a finite-dimensional vector-space A over $\mathbf{R}$, any vector $a \in A$ which is invariant under $\rho(\gamma)$ for every $\gamma \in \Gamma$ has a compact orbit under $\rho(G)$. In this situation, we can apply the following lemma:

LEMMA. *Let ρ be a representation of a topological group G in a finite-dimensional vector-space A over* $\mathbf{R}$. *Let A' be the set of all the vectors in A whose orbit under $\rho(G)$ is relatively compact. Then A' is a subspace of A, invariant under $\rho(G)$; and ρ induces on A' a representation ρ' of G such that $\rho'(G)$ is contained in a compact group of automorphisms of A'.*

The first assertion is obvious. Now let $a_1, \cdots, a_n$ be a basis for A'; as the vectors $\rho(x)a_i$ belong to a bounded subset of A' for all $x \in G$, G induces on A' a relatively compact set of endomorphisms of A', hence also a relatively compact subset of the group of automorphisms of A'.

Now let again G be a Lie group with the Lie algebra $\mathfrak{g}$; call $\mathfrak{c}$ the set of the vectors in $\mathfrak{g}$ whose orbits under the adjoint group are relatively compact. It is clear that this is not only a vector-subspace of $\mathfrak{g}$ but a Lie subalgebra of $\mathfrak{g}$, invariant under $\mathrm{Ad}(G)$ and even under all automorphisms of $\mathfrak{g}$. The adjoint representation $x \to \mathrm{Ad}(x)$ of G induces on $\mathfrak{c}$ a representation whose kernel is the centralizer C of $\mathfrak{c}$ in G; in view of the lemma, this implies that G/C has an injective representation into a compact group. In the case with which we are mainly concerned in this paper, G is connected and semisimple without compact components, so that it has no non-trivial representation into a compact group; therefore, in that case, we have $C = G$, hence $\mathfrak{c} = \{0\}$. If now Γ is a discrete subgroup of G such that G/Γ is compact, it follows from what has been said above that every vector in $\mathfrak{n}(\Gamma)$ has a compact orbit under the adjoint group, so that $\mathfrak{n}(\Gamma) \subset \mathfrak{c}$; therefore:

THEOREM. *Let G be a connected semi-simple Lie group without compact components; let Γ be a discrete subgroup of G such that G/Γ is compact; let $\mathfrak{n}(\Gamma)$ be the set of all the vectors in the Lie algebra of G which are invariant under* $\mathrm{Ad}(\gamma)$ *for every $\gamma \in \Gamma$. Then $\mathfrak{n}(\Gamma) = \{0\}$.*

This is of course contained in a deeper result proved by Borel for the case when G/Γ is merely assumed to have finite measure [2, Corollary 4.4].

COROLLARY 1. *Let G and Γ be as in the theorem; then the normalizer $N(\Gamma)$ of Γ in G is discrete, $G/N(\Gamma)$ is compact, and Γ is of finite index in $N(\Gamma)$.*

In fact, the first assertion amounts to saying that $N_0(\Gamma) = \{e\}$, and this is equivalent to our theorem. The other assertions follow from this at once.

COROLLARY 2. *Let G and Γ be as in the theorem, and let Δ be a subgroup of finite index of Γ. Then the space of the representations of Γ into G which induce the identity on Δ is discrete.*

The subgroup Δ' of Γ which induces the identity mapping on the homogeneous space Γ/Δ is of finite index in Γ (at most equal to $d!$ if d is the index of Δ in Γ) and is a normal subgroup of Γ; replacing Δ by Δ', we see that it is enough to consider the case when Δ itself is normal in Γ. As Δ is of finite index in Γ, G/Δ is compact, so that $N(\Delta)$ is discrete by corollary 1. Now, if a representation r of Γ into G induces the identity on Γ, we have $r(\Gamma) \subset N(\Delta)$. Therefore, if r, r' are two such representations, and if (γ_i) is a finite set of generators for Γ, we must have $r(\gamma_i) = r'(\gamma_i)$ for all i, and therefore $r = r'$, as soon as r' is close enough to r.

COROLLARY 3. *Let G', G'' be two connected semisimple Lie groups without compact components; let Γ be a discrete subgroup of $G = G' \times G''$ with compact quotient. Let Δ', Δ'' be the intersections of Γ with G' and with G'' considered as subgroups of G, and let Γ', Γ'' be its projections on G' and on G''. Assume that Γ'' is discrete in G''. Then Γ' is discrete in G'; G'/Γ' and G''/Γ'' are compact; Γ has a finite index in $\Gamma' \times \Gamma''$, and $\Delta' \times \Delta''$ has a finite index in Γ.*

Let K be a compact subset of G such that $G = K\Gamma$; call K', K'' its projections on G' and on G''. For any x' in G', we can write $(x', e'') = k\gamma$

with $k \in K$ and $\gamma = (\gamma', \gamma'') \in \Gamma$; then γ'' belongs to $K''^{-1} \cap \Gamma''$, which is a finite set since Γ'' is discrete and K'' is compact. Choose a finite number of elements $\gamma_i = (\gamma'_i, \gamma''_i)$ of Γ so that $K''^{-1} \cap \Gamma'' = \{\gamma''_1, \cdots, \gamma''_m\}$. Then, if x', k, γ are as above, there is an i such that $\gamma = \gamma_i\delta'$ with $\delta' \in \Delta'$. This gives

$$x' \in (\textstyle\bigcup_i K'\gamma'_i) \cdot \Delta' ,$$

which shows that G'/Δ' is compact. Therefore, by corollary 1, $N(\Delta')$ is discrete in G'. On the other hand, one sees at once that Γ' is contained in $N(\Delta')$, so that it is discrete. Exchanging now G' and G'' in the above proof, we see that G''/Δ'' is compact. Our other assertions are now obvious.

Appendix III

(*This appendix reproduces, with minor verbal changes, a communication of L. Hörmander to the author, and is published here with his permission*).

Let notations be as in nos. 2 and 3; take any vector-field $Y_0 \in \mathfrak{Y}$ on S, and consider, as in no. 11, the solutions of the differential system determined on S by that vector-field. For every point M of S, call $\Phi(M)$ the point where the solution of that system which goes through M cuts the fibre $\pi^{-1}(0)$. Then, for each t, Φ induces on the fibre $\pi^{-1}(t)$ a homeomorphism of class C^∞ of that fibre onto $\pi^{-1}(0)$, and the mapping $M \rightarrow (\pi(M), \Phi(M))$ is a homeomorphism of class C^∞ of S onto $I \times \pi^{-1}(0)$. Put $V = \pi^{-1}(0)$; then the mapping of $\pi^{-1}(t)$ onto V induced by Φ will map the vector-fields X_λ onto n vector-fields $X_\lambda(t)$ on V; similarly, if Y and the f^μ_λ are as in no. 3, it will map the functions induced by the f^μ_λ on $\pi^{-1}(t)$ onto functions $f^\mu_\lambda(t)$ on V; it maps the volume element $d\Omega$ on $\pi^{-1}(t)$ onto a volume element $d\Omega_t = \delta(t)^2 d\Omega$ on V, with a density $\delta(t)^2$ which is nowhere 0; and the $X_\lambda(t)$, the $f^\mu_\lambda(t)$ and $\delta(t)$ are all of class C^∞ on $I \times V$. After replacing the $X_\lambda(t)$ by $\delta(t)X_\lambda(t)$, writing $f_{\mu\lambda}(t)$ instead of $\delta(t)f^\mu_\lambda(t)$, $c_{\mu\lambda\rho}(t)$ instead of $\delta(t)c^\mu_{\lambda\rho}$ and φ_μ instead of φ^μ, we can now state the variational problem of no. 3 as follows: *for each value of t, the functions φ_μ are to be chosen so as to minimize the integral*

$$\int_V \sum_{\mu\lambda} \left(f_{\mu\lambda}(t) + X_\lambda(t)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(t)\varphi_\rho\right)^2 d\Omega . \tag{23}$$

Here all the data are assumed to be of class C^∞ on $I \times V$; for each t, the $X_\lambda(t)$ are everywhere linearly independent vector-fields; and one wishes to show that the problem has a unique solution, of class C^∞ on $I \times V$, under the assumption that, for each value of t, the system

$$X_\lambda(t)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(t)\varphi_\rho = 0 \tag{24}$$

has no solution of class C^∞ on V, other than 0.

On the space L^2 of functions of integrable square on V, we use the norm $\| g \| = \left(\int g^2 d\Omega\right)^{1/2}$; similarly, for a system $g = (g_{\mu\lambda})$ of such functions, we use the norm given by

$$\| g \|^2 = \sum_{\lambda,\mu} \| g_{\mu\lambda} \|^2 = \int \sum_{\lambda,\mu} (g_{\mu\lambda})^2 d\Omega \ ;$$

with this norm, the space of such systems will also be denoted by L^2 (this will cause no confusion). On the other hand, for $\varphi = (\varphi_\mu)$, we introduce the norm given by

$$||| \varphi |||^2 = \sum_{\lambda,\mu} \| X_\lambda(0)\varphi_\mu \|^2 + \sum_\mu \| \varphi_\mu \|^2 \ ,$$

and we call H the space of the $\varphi = (\varphi_\mu)$ for which this is finite, i.e., for which all the φ_μ are in L^2 and their first derivatives, in the distribution sense, are also in L^2.

We shall now prove that the inequality

$$||| \varphi |||^2 \leqq C^2 \sum_{\lambda,\mu} \| X_\lambda(0)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(0)\varphi_\rho \|^2 \tag{25}$$

holds, for a suitable choice of the constant C, for all $\varphi \in H$. In fact, if this were not so, there would be a sequence $\varphi^{(i)}$ with $||| \varphi^{(i)} ||| = 1$ such that the right-hand side of (25) tends to 0. By a well-known principle based on Poincaré's inequality (cf. e.g., Courant-Hilbert, Vol. 2, pp. 488–490), every sequence for which $||| \varphi^{(i)} |||$ is bounded has a subsequence which converges in L^2; therefore we may assume that $\varphi^{(i)}$ converges in L^2 towards a limit φ. As the right-hand side of (25) tends to 0 for this sequence, this implies that the $X_\lambda(0)\varphi_\mu$ converge in L^2. Therefore φ is in H, we have $||| \varphi ||| = 1$, and φ is a solution of the system (24) with $t = 0$. As (24) for $t = 0$ implies

$$\sum_\lambda X_\lambda(0)\big(X_\lambda(0)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(0)\varphi_\rho\big) = 0 \ ,$$

and as this is an elliptic system, the theory of elliptic systems (cf. e.g., [5] or [6]) shows that φ can be modified on a null-set so as to become C^∞. We have thus obtained a non-zero solution of (24) for $t = 0$, of class C^∞, against our assumption. This proves (25).

From the continuity of the data in t, it follows now at once that we have the inequality

$$||| \varphi |||^2 \leqq 4C^2 \sum_{\lambda,\mu} \| X_\lambda(t)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(t)\varphi_\rho \|^2 \tag{26}$$

for all $\varphi \in H$ and all t in some neighborhood I' of 0 in I. From now on, we assume that t is in I'.

From (26), it follows that the mapping G_t of H into L^2 given by

$$\varphi \to G_t(\varphi) = \left(X_\lambda(t)\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}(t)\varphi_\rho\right)$$

has a closed range. Let $P(t)$ be the projection on this range in L^2. Then the problem of minimizing (23) has the unique solution

$$\varphi = -G(t)^{-1}P(t)f$$

in H, and it follows from (26) that $|||\varphi||| \leqq 2C\,||f||$.

The solution φ of our variational problem must also satisfy the Euler equations of that problem

$$\int \sum_{\mu,\lambda} (f_{\mu\lambda} + X_\lambda\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}\varphi_\rho)(X_\lambda\psi_\mu + \sum_\rho c_{\mu\lambda\rho}\psi_\rho)d\Omega = 0$$

for all $\psi = (\psi_\mu)$ of class C^∞ on V. This is a weak form of a system

$$L_\mu\varphi = F_\mu\ , \tag{27}$$

where the leading term in L_μ is $-\sum_\lambda X_\lambda^2\varphi_\mu$, so that this is an elliptic system; F_μ is the effect of an operator of the first order acting on f, so that F_μ is of class C^∞ on $I' \times V$. Therefore, for each $t \in I'$, φ can be modified on a null-set so that it will be of class C^∞ on V (cf. [5], [6]).

The system (27) has no other solution than φ in H. In fact, assume now that φ is any solution of (27) in H. In view of the definition of L_μ, we have

$$\int \sum_{\lambda,\mu} (X_\lambda\varphi_\mu + \sum_\rho c_{\mu\lambda\rho}\varphi_\rho)^2 d\Omega = \int \sum_\mu (L_\mu\varphi)\varphi_\mu d\Omega = \int \sum_\mu F_\mu\varphi_\mu d\Omega\ .$$

Combining this with (26), we get

$$|||\varphi|||^2 \leqq 4C^2\,||F||\cdot||\varphi|| \leqq 4C^2\,||F||\cdot|||\varphi|||\ ,$$

and therefore

$$|||\varphi||| \leqq 4C^2\,||F||\ . \tag{28}$$

In particular, $F = 0$ implies $\varphi = 0$, as we asserted.

If $\alpha = (\alpha_1, \cdots, \alpha_n)$, where the α_i are integers $\geqq 0$, we shall write, as usual, $|\alpha| = \sum_\lambda \alpha_\lambda$; and we write X_0^α for the differential operator of order $|\alpha|$ given by

$$X_0^\alpha = X_1(0)^{\alpha_1} \cdots X_n(0)^{\alpha_n}\ .$$

With this notation, the proofs for the regularity of solutions of elliptic systems (cf. again [5], [6]) give estimates

$$\sup_V |X_0^\alpha\varphi_\mu| \leqq C_\alpha\left(|||\varphi||| + \sum_{|\beta|\leqq|\alpha|+n+1} \sup_{V,\mu} |X_0^\beta F_\mu|\right)\ ,$$

with C_α independent of t. In view of (28), we get now

$$\text{(29)} \qquad \sup_V |X_0^\alpha \varphi_\mu| \leqq C'_\alpha \sum\nolimits_{|\beta| \leqq |\alpha|+n+1} \sup_{V,\mu} |X_0^\beta F_\mu| \;,$$

which is valid whenever (27) has the solution $\varphi \in H$.

Until now we have only used the fact that the data, and in particular the F_μ, are of class C^∞ on V for each t. Now, in order to discuss the differentiability in t, we exhibit the dependence upon t in (27) by writing it as $L_\mu(t)\varphi(t) = F_\mu(t)$. Put

$$\psi(t, h) = h^{-1}[\varphi(t + h) - \varphi(t)] \;;$$

this is a solution of the system

$$\text{(30)} \qquad L_\mu(t)\psi(t, h) = G_\mu(t, h) \;,$$

where we have put

$$G_\mu(t, h) = h^{-1}[F_\mu(t + h) - F_\mu(t)] - h^{-1}[L_\mu(t + h) - L_\mu(t)]\varphi(t + h) \;.$$

As $\psi(t, h)$ is in H, we can apply (29), substituting $\psi(t, h), G_\mu(t, h)$ for φ, F_μ. In view of (29) and of our assumptions on the differentiability of the data on $I \times V$, this gives at once a uniform bound for $|X_0^\alpha \psi_\mu(t, h)|$ for each α (and for all $t, t + h$ in I'); this shows that the $X_0^\alpha \varphi_\mu(t)$ are Lipschitz-continuous in t.

For $h \to 0$, $G_\mu(t, h)$ tends to a limit $G_\mu(t)$; for each t, this is of class C^∞ on V; moreover, from the differentiability of the data on $I \times V$, it follows that, for every α, $X_0^\alpha G_\mu(t, h)$ tends uniformly to $X_0^\alpha G_\mu(t)$ for $h \to 0$. Consider the system $L_\mu(t)\psi = G_\mu(t)$, which is just (27) differentiated formally with respect to t; as (30) has the solution $\psi(t, h)$, and as the range of an elliptic system is closed, this has a solution ψ in H. Then we have

$$L_\mu(t)[\psi(t, h) - \psi] = G_\mu(t, h) - G_\mu(t) \;.$$

Applying the estimates (29) to this system, we see now that $X_0^\alpha \psi_\mu(t, h)$ tends uniformly to $X_0^\alpha \psi_\mu$, for every α, for $h \to 0$. Therefore we have $\psi = \partial\varphi/\partial t$, and this is of class C^∞ on V for each t. Repeating the same argument for ψ and G_μ, we find that $\partial^2\varphi/\partial t^2$ is of class C^∞ on V, etc. This proves that φ is of class C^∞ on $I' \times V$. Since the same conclusion must hold true for some neighborhood I'' of any point t of I, we see that φ is of class C^∞ on $I \times V$, as was to be proved.

REMARK. The assumption that (24) has no solution can be replaced by the assumption that the dimension of the space of solutions of (24) is the same for all $t \in I$; this dimension is necessarily finite by Ascoli's theorem. One merely has to replace H by a supplement in H of that space of solutions for $t = 0$. If no such assumption is made, the result is false, as shown by the following example. Take $V = \mathbf{R}/\mathbf{Z}$ (the real line modulo 1);

let f be any function of class C^∞ on V such that $\int f dx \neq 0$, and consider the problem of minimizing the integral

$$\int\left(f + \frac{d\varphi}{dx} + t\varphi\right)^2 dx \,.$$

This has a unique solution $\varphi(t)$ for $t \neq 0$, but $\int \varphi(t) dx$ tends to ∞ for $t \to 0$.

This concludes Hörmander's communication and completes the proof of the facts which were needed in no. 4 of this paper. It is natural to conjecture that, if the data of the variational problem in no. 4 are real-analytic, the solution is real-analytic; but Hörmander informs me that a proof for this would presumably require more delicate arguments; anyway, it would not be true unless one assumes that $\mathfrak{n}_t = \{0\}$, or at least that the dimension of $\mathfrak{n}_t$ is constant. As the argument I had in mind in writing the footnote on page 383 of D^{I} depended upon this, I must now withdraw that footnote; I see no other way at present of proving the real-analytic equivalence of the fibres considered there than by using Grauert's theorem.

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