THE FIELD OF DEFINITION OF A VARIETY.*

By André Weil.

Let V be a variety, defined over an overfield K of a groundfield k. Consider the following problems:

(P) Among the varieties, birationally equivalent to V over K, find one which is defined over k.

(P′) Among the varieties, birationally and biregularly equivalent to V over K, find one which is defined over k.

Problems of these types arise, for instance, in Chow's recent work on abelian varieties over function-fields ([**1**]), in my work on algebraic groups ([**4**]), and also in the unpublished work of Shimura and of Taniyama on complex multiplication. Criteria for those problems to have a solution are implicitly contained in Chow's paper ([**1**]) and in Lang's subsequent note on a related subject ([**2**]); the purpose of the present paper is to develop them more explicitly.

Without restricting the generality of the problem, we may assume that K is finitely generated over k; we shall make the restrictive assumption that it is separable over k. Then it is a regular extension of the algebraic closure k' of k in K; and k' is a separably algebraic extension of k of finite degree. Thus it will be enough for our purpose to discuss (P) and (P′), firstly when K is separably algebraic over k, and secondly when it is regular over k.

As usual, we do not distinguish between mappings and their graphs. In particular, we do not distinguish between a birational correspondence T between two varieties V and W (this being defined as a subvariety of $V \times W$ which satisfies certain conditions) and the mapping of V into W determined by T. The inverse mapping, T^{-1}, is then a mapping of W into V or also a birational correspondence between W and V. To prevent misunderstandings, I take this opportunity for pointing out that (by abuse of language) I call T *everywhere biregular* only when T is biregular at every point of V and T^{-1} is so at every point of W; when that is so, T might more suitably be

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called an *isomorphism* between V and W, and a *k-isomorphism* if k is a field of definition for V, W and T.

Section I. Separably Algebraic Extensions of the Groundfield.

1. Let k be a separably algebraic extension of a groundfield k_0, of finite degree n. Call $\mathfrak{J}$ the set of all distinct isomorphisms of k (over k_0, i.e., leaving all elements of k_0 invariant) into the algebraic closure $\bar{k}_0$ of k_0; $\mathfrak{J}$ consists of n distinct isomorphisms, including the identity automorphism ϵ of k. If $\sigma \varepsilon \mathfrak{J}$, and if ω is any isomorphism over k_0 of an overfield of k^σ, we denote by $\sigma\omega$ the isomorphism of k defined by putting $\xi^{\sigma\omega} = (\xi^\sigma)^\omega$ for every $\xi \varepsilon k$.

Let V be a variety, defined over k; assume that there is a variety V_0, defined over k_0, and a birational correspondence f, defined over k, between V_0 and V. Then, for $\sigma \varepsilon \mathfrak{J}$, $\tau \varepsilon \mathfrak{J}$, the mapping $f_{\tau,\sigma} = f^\tau \circ (f^\sigma)^{-1}$ is a birational correspondence between V^σ and V^τ. We now modify our problem (P) as follows:

(A) *Let k be a separably algebraic extension of k_0 of finite degree; let $\mathfrak{J}$ be the set of all distinct isomorphisms of k into $\bar{k}_0$. Let V be a variety, defined over k; for each pair (σ, τ) of elements of $\mathfrak{J}$, let $f_{\tau,\sigma}$ be a birational correspondence between V^σ and V^τ. Find a variety V_0, defined over k_0, and a birational correspondence f, defined over k, between V_0 and V, such that $f_{\tau,\sigma} = f^\tau \circ (f^\sigma)^{-1}$ for all $\sigma \varepsilon \mathfrak{J}$, $\tau \varepsilon \mathfrak{J}$.*

THEOREM 1. *Problem* (A) *has a solution if and only if the $f_{\tau,\sigma}$ are defined over a separably algebraic extension of k_0 and satisfy the following conditions:*

(i) $f_{\tau,\rho} = f_{\tau,\sigma} \circ f_{\sigma,\rho}$ *for all* ρ, σ, τ *in* $\mathfrak{J}$;

(ii) $f_{\tau\omega,\sigma\omega} = (f_{\tau,\sigma})^\omega$ *for all* σ, τ *in* $\mathfrak{J}$ *and all automorphisms* ω *of* $\bar{k}_0$ *over* k_0.

Moreover, when that is so, the solution is unique, up to a birational transformation on V_0, defined over k_0.

The conditions are obviously necessary. If the problem has two solutions (V_0, f) and (V_0', f'), put $F = f'^{-1} \circ f$; this is a birational correspondence between V_0 and V_0', defined over k. Writing that (V_0, f) and (V_0', f') are solutions of (A), we find $F^\sigma = F^\tau$ for all σ, τ; thus, F is invariant under all automorphisms of $\bar{k}_0$ over k_0; therefore it is defined over k_0; this proves the unicity assertion in Theorem 1.

Now assume that (i), (ii) are fulfilled; then, if σ, τ are in $\mathfrak{S}$ and ω is an automorphism of $\bar{k}_0$ over the compositum of k^σ and k^τ, (ii) shows that $f_{\tau,\sigma}$ is invariant under ω; if $f_{\tau,\sigma}$ is defined over a separably algebraic extension of k_0, this implies that it is defined over the compositum of k^σ and k^τ. Let x be a generic point of V over k; for each $\sigma \varepsilon \mathfrak{S}$, put $x_\sigma = f_{\sigma,\epsilon}(x)$, ϵ being the identity automorphism of k. If we put $\rho = \sigma$ in (i), we see that $f_{\sigma,\sigma}$ is the identity mapping of V^σ; therefore we have $x_\epsilon = x$.

Let K be the compositum of the fields k^σ, for all $\sigma \varepsilon \mathfrak{S}$; this is a Galois extension of k_0; call Γ its Galois group. Take any $\omega \varepsilon \Gamma$; as x and $x_{\epsilon\omega}$ are generic points of V and V^ω, respectively, over K, there is one and only one isomorphism ω^* of $K(x)$ onto $K(x_{\epsilon\omega})$ which induces ω on K and maps x onto $x_{\epsilon\omega}$. As $f_{\epsilon\omega,\epsilon}$ is a birational correspondence, defined over K, we have $K(x) = K(x_{\epsilon\omega})$, so that ω^* is an automorphism of $K(x)$. Applying (ii) to any extension of ω to an automorphism of $\bar{k}_0$, we get:

$$(x_\sigma)^{\omega^*} = f_{\sigma\omega,\epsilon\omega}(x_{\epsilon\omega})\,;$$

putting $\rho = \epsilon$ and $\sigma = \epsilon\omega$ in (i), we find that the right-hand side of this relation is $x_{\sigma\omega}$; therefore ω^* maps x_σ onto $x_{\sigma\omega}$ for all ω. From this it immediately follows that the mapping $\omega \to \omega^*$ is a homomorphism of Γ into the group of all automorphisms of $K(x)$, and more precisely an isomorphism of Γ onto a group Γ^* of automorphisms of $K(x)$. Call $k_0(y)$ the field consisting of those elements of $K(x)$ which are invariant under Γ^*; it is finitely generated over k_0 ([**4**], App., Prop. 3). As $K(x)$ is regular, hence separable, over K, and K is separable over k_0, $K(x)$ is separable over k_0 (Bourbaki, *Alg.*, Chap. V, § 7, no. 4, Prop. 7); hence $k_0(y)$ is separable over k_0. Any element of the algebraic closure of k_0 in $k_0(y)$ must be in the algebraic closure of K in $K(x)$, which is K since $K(x)$ is regular over K; as such an element is invariant under Γ^*, it must then be invariant under Γ, and so it must be in k_0. Thus we have proved that $k_0(y)$ is regular over k_0. Call V_0 the locus of y over k_0.

If an element ω of Γ induces the identity on k, ω^* leaves x invariant; as Γ^* is the Galois group of $K(x)$ over $k_0(y)$, this implies that $k(x) \subset k(y)$, so that we may write $x = f(y)$, where f is a mapping of V_0 into V, defined over k. We have $K(x) \subset K(y)$, hence $K(x) = K(y)$ since $K(y)$ is contained in $K(x)$ by definition. This shows that f is a birational correspondence between V_0 and V. Transforming the relation $x = f(y)$ by any ω^* in Γ^*, and calling σ the isomorphism of k induced on k by ω^*, we get $x_\sigma = f^\sigma(y)$, hence $f_{\sigma,\epsilon} = f^\sigma \circ f^{-1}$; by (i), this shows that (V_0, f) is a solution of our problem.

2. In the introduction, we formulated, in addition to the problem (P), a more precise problem (P′). We may modify the problem (A) similarly, by requiring f to be everywhere biregular; call (A′) this modified problem. For (A′) to have a solution, it is obviously necessary that the $f_{\tau,\sigma}$ should be everywhere biregular and satisfy the conditions in Theorem 1.

Assume that this is so; let (V_0, f) be a solution of (A). Then, if (V_0', f') is a solution of (A′), the unicity of the solution of (A) shows that we must have $f' = f \circ F^{-1}$, where F is a birational correspondence between V_0 and V_0', defined over k_0. Thus problem (A′) may be reformulated as follows:

(B) *Let k and k_0 be as in problem* (A); *let V and V_0 be varieties, respectively defined over k and over k_0; let f be a birational correspondence, defined over k, between V_0 and V. Find a variety V_0' and a birational correspondence F between V_0 and V_0', both defined over k_0, such that the birational correspondence $f \circ F^{-1}$ between V_0' and V is everywhere biregular.*

It is obvious that, if (B) has a solution, this is unique up to a k_0-isomorphism; therefore the same is true for (A′). If (B) has a solution, then ($\mathfrak{G}$ being defined as before) the birational correspondence $f^\tau \circ (f^\sigma)^{-1}$ between V^σ and V^τ must be everywhere biregular for all σ, τ in $\mathfrak{G}$. We will prove that this condition is also sufficient, at any rate if V is a k-open subset of a projective variety, defined over k. This will be an immediate consequence of the following result.

PROPOSITION 1. *Let k, k_0, $\mathfrak{G}$ be as in* (A). *Let V_0 be a variety, defined over k_0; let V be a projective (resp. affine) variety, defined over k; let f be a birational correspondence, defined over k, between V_0 and V. Then there is a projective (resp. affine) variety W and a birational correspondence F between V_0 and W, both defined over k_0, such that $F \circ f^{-1}$ is biregular at every point of V where the mappings $f^\sigma \circ f^{-1}$ are defined for all $\sigma \in \mathfrak{G}$.*

Let S be the ambient space of V, projective or affine; f may be regarded as a mapping of V_0 into S. Call $\sigma_1 = \epsilon, \sigma_2, \cdots, \sigma_n$ the elements of $\mathfrak{G}$, and put $F_1 = (f^{\sigma_1}, \cdots, f^{\sigma_n})$; this is a mapping of V_0 into the product $S \times \cdots \times S$ of n factors equal to S, and is defined over the compositum K of the fields k^σ. It is clear that $F_1 \circ f^{-1}$ is defined wherever all the $f^\sigma \circ f^{-1}$ are defined. Let x be a generic point of V_0 over k_0; let W_1 be the locus of $F_1(x)$ over K; put $u = F_1(x) = (x_1, \cdots, x_n)$. As $\sigma_1 = \epsilon$, we have $x_1 = f(x)$, so that the image of u by the mapping $f \circ F_1^{-1}$ is x_1; this shows that $f \circ F_1^{-1}$ is the mapping induced on W_1 by the projection of the product $S \times \cdots \times S$ onto its first factor, and is therefore everywhere defined. Thus the birational correspondence $F_1 \circ f^{-1}$ between V and W_1 is biregular wherever all the $f^\sigma \circ f^{-1}$ are defined.

Let $z_1, \cdots, z_n$ be n points of S; if S is the projective m-space, put $z_i = (z_{i0}, \cdots, z_{im})$; and let z' be the point, in a projective space of suitable dimension, whose homogeneous coordinates are all the monomials $z_{1\mu_1} z_{2\mu_2} \cdots z_{n\mu_n}$, with $0 \leqq \mu_i \leqq m$ for every i. If S is the affine m-space, put $z_i = (z_{i1}, \cdots, z_{im})$, put $z_{i0} = 1$ for $1 \leqq i \leqq n$, and let z' be the point, in an affine space of suitable dimension, whose coordinates are the same monomials as before. In either case, put $z' = \Phi(z_1, \cdots, z_n)$; it is well-known that Φ is an everywhere biregular mapping of $S \times \cdots \times S$ onto its image in projective (resp. affine) space. Put now $F_2 = \Phi \circ F_1$; then F_2 is a birational correspondence between V_0 and $W_2 = \Phi(W_1)$, and $F_2 \circ f^{-1}$ is biregular wherever all the $f^\sigma \circ f^{-1}$ are defined.

If S is projective, let $(1, f_1(x), \cdots, f_m(x))$ be a set of homogeneous coordinates for $f(x)$; the f_μ are functions on V_0, defined over k. Put $f_0 = 1$. Then we have $F_2 = (g_0, \cdots, g_r)$, where the g_ρ are all the monomials

$$f_{\mu_1}{}^{\sigma_1} f_{\mu_2}{}^{\sigma_2} \cdots f_{\mu_n}{}^{\sigma_n}.$$

If ω is an automorphism of K over k, $g_\rho{}^\omega$ is again one of the g_ρ, which we may write as $g_{\omega(\rho)}$; the mapping $\rho \to \omega(\rho)$ determines a representation of Γ (the Galois group of K over k) as a group of permutations on the g_ρ. For a given ρ, let γ_ρ be the subgroup of Γ determined by $\omega(\rho) = \rho$; then, for $\omega \varepsilon \Gamma$, $\omega(\rho)$ takes a number of distinct values equal to the index d_ρ of γ_ρ in Γ. If K_ρ is the subfield of K consisting of the elements of K invariant under γ_ρ, g_ρ is defined over K_ρ; therefore, if $(\alpha_1, \cdots, \alpha_{d_\rho})$ is a basis of K_ρ over k_0, we may write $g_\rho = \sum_\nu \alpha_\nu h_{\rho\nu}$, where the $h_{\rho\nu}$, for $1 \leqq \nu \leqq d_\rho$, are functions on V_0, defined over k_0. Then we have, for all $\omega \varepsilon \Gamma$:

$$g_{\omega(\rho)} = \sum_\nu \alpha_\nu{}^\omega h_{\rho\nu}.$$

If, in this relation, we take for ω a set of representatives of the d_ρ cosets of γ_ρ in Γ, we get a linear substitution expressing the d_ρ distinct functions $g_{\omega(\rho)}$ in terms of the d_ρ functions $h_{\rho\nu}$; and, since K_ρ is separable over k_0, that substitution is invertible. From this it follows immediately that, if we call $F(x)$ the point whose homogeneous coordinates are all the functions $h_{\rho\nu}$ (where ρ runs through a set of representatives for the classes of equivalence determined by the permutation group Γ on the set $\{0, 1, \cdots, r\}$, and where, for each ρ, we take $1 \leqq \nu \leqq d_\rho$), F is of the form $\Psi \circ F_2$, where Ψ is an automorphism of the ambient projective space of W_2. If S is affine, we put

$f=(f_1,\cdots,f_m)$, $f_0=1$, and we define F by the same formulas as in the projective case, but regard it as a mapping of V_0 into an affine space; then we have again $F=\Psi\circ F_2$, Ψ being now an automorphism of the ambient affine space of W_2. In either case, the mapping F is defined over k_0; if W is the locus of $F(x)$ over k_0, W and F have the properties required by our proposition.

3. Before applying this to the problems (A′) and (B), we need a general lemma:

LEMMA 1. *Let f be a birational correspondence between two varieties U and V; let k be a field of definition for U, V, f. Then the sets of points where f and f^{-1} are respectively biregular are k-open, and f determines a k-isomorphism between them.*

Call U' the set of points of U where f is defined, U'' the set of points of U where it is biregular; call V' the set of points of V where f^{-1} is defined, V'' the set where it is biregular. By [**4**], App., Prop. 8, U' and V' are k-open. Call f' the restriction of f to $U'\times V'$ (i.e. the birational correspondence between U' and V' whose graph is the set-theoretic intersection of the graph of f with $U'\times V'$). If f is biregular at a point a of U, it is defined at a, so that $a\in U'$; and, if we put $b=f(a)$, f^{-1} is defined at b, so that $b\in V'$; therefore (a,b) is on the graph of f', and f' is defined at a. Conversely, let a be a point of U' where f' is defined; put $b=f'(a)$; then b is in V', so that f^{-1} is defined at b; as U' and V' are open, f is then defined at a, and we have $b=f(a)$; thus f is biregular at a. This shows that U'' is the set of points of U' where f' is defined; similarly V'' is the set of points of V' where f'^{-1} is defined; this implies that they are k-open. If f'' is the restriction of f to $U''\times V''$, it is everywhere biregular by definition (i.e., f'' is biregular at every point of U'', and f''^{-1} is so at every point of V'').

In order to formulate our results on problems (A′) and (B), we will say that a variety U, defined over a field k, is *projectively* (resp. *affinely*) *embeddable over k* if it is k-isomorphic to a k-open subset of a projective (resp. affine) variety, defined over k.

THEOREM 2. *Problem* (B) *has a solution* (V_0',F') *provided V is projectively embeddable over k and the birational correspondence $f^\sigma\circ f^{-1}$ between V and V^σ is everywhere biregular for every isomorphism σ of k over k_0 into $\bar{k}_0$. When that is so, V_0' is projectively embeddable over k_0; it is affinely embeddable over k_0 if V is so over k.*

We may assume V to be a k-open subset of a projective (resp. affine) variety, defined over k. Take W and F as in Proposition 1; then $F \circ f^{-1}$ is biregular at every point of V; therefore, by Lemma 1, it is a k-isomorphism between V and the k-open subset V_0' of W where $f \circ F^{-1}$ is biregular. As the $f^\sigma \circ f^{-1}$ are everywhere biregular V_0' is also the subset of W where $f^\sigma \circ F^{-1}$ is biregular, for every σ; therefore it is invariant under all automorphisms of $\bar{k}_0$ over k_0, so that it is k_0-open, by [**4**], App., Prop. 9. Then, if F' is the restriction of F to $V_0 \times V_0'$, (V_0', F') is a solution of (B).

THEOREM 3. *Problem* (A′) *has a solution, i.e., problem* (A) *has a solution* (V_0, f) *for which* f *is everywhere biregular, provided* V *is projectively embeddable over* k *and the* $f_{\tau,\sigma}$ *are everywhere biregular and satisfy the conditions in Theorem* 1. *When that is so,* V_0 *is projectively embeddable over* k_0; *it is affinely embeddable over* k_0 *if* V *is so over* k. *The solution is unique up to a* k_0*-isomorphism.*

Section II. Regular Extensions of the Groundfield.

4. Let now k denote the groundfield. Let T be a variety, defined over k; let t be a generic point of T over k. When we denote by V_t a variety, defined over $k(t)$, we will agree, whenever t' is also a generic point of T over k, to denote by $V_{t'}$ the transform of V_t by the isomorphism of $k(t)$ onto $k(t')$ over k which maps t onto t'. Similarly, if a mapping, defined over $k(t)$, is denoted by f_t, $f_{t'}$ will denote its transform by the same isomorphism; if t, t', t'' are three independent generic points of T over k, and $f_{t',t}$ is a mapping, defined over $k(t, t')$, we denote by $f_{t'',t'}$ the transform of $f_{t',t}$ by the isomorphism of $k(t, t')$ onto $k(t', t'')$ over k which maps (t, t') onto (t', t''); etc.

Let V_t be a variety, defined over $k(t)$; assume that there is a variety V, defined over k, and a birational correspondence f_t, defined over $k(t)$, between V and V_t; then $f_{t'} \circ f_t^{-1}$ is a birational correspondence between V_t and $V_{t'}$. We therefore modify problem (P) of the introduction as follows:

(C) *Let* T *be a variety, defined over a field* k; *let* t, t' *be independent generic points of* T *over* k. *Let* V_t *be a variety, defined over* $k(t)$; *let* $f_{t',t}$ *be a birational correspondence, defined over* $k(t, t')$, *between* V_t *and* $V_{t'}$. *Find a variety* V, *defined over* k, *and a birational correspondence* f_t, *defined over* $k(t)$, *between* V *and* V_t, *such that* $f_{t',t} = f_{t'} \circ f_t^{-1}$.

THEOREM 4. *Problem* (C) *has a solution if and only if* $f_{t',t}$ *satisfies the condition:*

(i) $$f_{t'',t} = f_{t'',t'} \circ f_{t',t}$$

where t'' is a generic point of T over $k(t, t')$. When that is so, the solution is unique, up to a birational transformation on V, defined over k.

The condition is obviously necessary. The proof for the unicity of the solution, when one exists, is quite similar to the proof of the corresponding statement in Theorem 1. Now, assuming (i) to be fulfilled, we shall construct a solution of (C). We may replace T by any birational transform of T over k, and so we may assume that T is an affine variety. Similarly we may assume that V_t is an affine variety; and, taking x to be a generic point of V_t over $k(t)$, we may replace x by (x, t) and V_t by the locus of (x, t) over $k(t)$; after that is done, V_t is still an affine variety, and we have $k(t) \subset k(x)$; from now on, assume that this is so, and assume that x has been taken generic on V_t over $k(t, t', t'')$. By **[4]**, App., Prop. 1, $k(x)$ is a regular extension of k; call X the locus of x over k. Put $x' = f_{t',t}(x)$; this is a generic point of $V_{t'}$ over $k(t, t', t'')$; by the definition of $V_{t'}$, this implies that there is an isomorphism of $k(t, x)$ onto $k(t', x')$ over k, mapping t onto t' and x onto x'; therefore we have $k(t') \subset k(x')$, hence $k(x, t') \subset k(x, x')$. As the definition of x' shows $k(x, x')$ to be contained in $k(t', t, x)$, i.e. in $k(x, t')$, it follows that we have $k(x, x') = k(x, t')$; therefore x' has a locus W_x over $k(x)$. Let $k(v)$ be the smallest field of definition containing k for W_x; as $k(v) \subset k(x)$, $k(v)$ is a regular extension of k. Call V the locus of v over k; we may write $v = G(x)$, where G is a mapping of X into V, defined over k.

If we put $x'' = f_{t'',t}(x)$, W_x is also the locus of x'' over $k(x)$; as the fields $k(x, x')$ and $k(x, x'')$ are respectively the same as $k(x, t')$ and $k(x, t'')$ and are therefore algebraically independent over $k(x)$, W_x is also the locus of x'' over $k(x, x')$. But (i) may be written $x'' = f_{t'',t'}(x')$; therefore $W_{x'}$ is the same as W_x. This implies that the isomorphism of $k(x)$ onto $k(x')$ over k which maps x onto x' leaves invariant all the elements of the smallest field of definition of W_x, hence also all the elements of $k(v)$, so that we have $G(x) = G(x')$.

On the other hand, let K be an overfield of k, algebraically independent from $k(x, x')$ over k; if ϕ is any function on X, defined over K, it will induce on W_x a function which is defined over $K(v)$; if $\phi(x) = \phi(x')$, that function is a constant, so that its constant value must be in $K(v)$. This shows that $K(v)$ is the subfield of $K(x)$ consisting of the elements of $K(x)$ which are invariant under the isomorphism of $K(x)$ onto $K(x')$ over K mapping x onto x'.

Now the relation $x'' = f_{t'',t}(x)$ shows that x'' is rational over $k(t'', t, x)$,

i. e. over $k(t'', x)$, so that we may write $x'' = \phi_{t''}(x)$, where $\phi_{t''}$ is a mapping of X into $V_{t''}$, defined over $k(t'')$. The relation $x'' = f_{t'',t'}(x')$ may then be written as $x'' = \phi_{t''}(x')$. Applying to the field $K = k(t'')$ and to the function $\phi_{t''}$ what we have proved above, we conclude from this that $k(x'') \subset k(t'', v)$. As we have $G(x) = G(x')$, hence also $G(x) = G(x'')$, the isomorphism of $k(x)$ onto $k(x'')$ over k which maps x onto x'' leaves $v = G(x)$ invariant; applying the inverse of that isomorphism to the relation $k(x'') \subset k(t'', v)$, we get $k(x) \subset k(t, v)$, hence $k(x) = k(t, v)$ since $k(t)$ and $k(v)$ are both contained in $k(x)$. Also, since $k(x)$ and $k(t')$ are algebraically independent over k, the same is true of $k(v)$ and $k(t')$; as the isomorphism of $k(x')$ onto $k(x)$ over k which maps x' onto x maps t' onto t and v onto itself, this implies that $k(v)$ and $k(t)$ are algebraically independent over k. As the relation $k(x) = k(t, v)$ can also be written $k(t, x) = k(t, v)$, we conclude that V_t and V are birationally equivalent over $k(t)$, so that we may write $x = f_t(v)$, where f_t is a birational correspondence between V and V_t, defined over $k(t)$. Then we have $x' = f_{t'}(v)$. Therefore (V, f_t) is a solution of our problem. We also see that X is birationally equivalent to $T \times V$ over k.

5. Just as in Section I, we consider the problem (C′) which consists in finding a solution (V, f_t) of (C) such that f_t is everywhere biregular. For such a solution to exist, it is necessary that $f_{t',t}$ should be everywhere biregular; it will be shown that this is sufficient.

As in Section I, if we make use of Theorem 4, we see that (C′) may be reformulated as follows:

(D) *Let k, T and t be as in* (C) ; *let V and V_t be varieties, respectively defined over k and over $k(t)$; let f_t be a birational correspondence, defined over $k(t)$, between V and V_t. Find a variety V' and a birational correspondence F between V and V', both defined over k, such that the birational correspondence $f_t \circ F^{-1}$ between V' and V_t is everywhere biregular.*

In order to solve (D), we need some preliminary results.

LEMMA 2. *Let F and H be mappings of a variety X into two varieties W, T, all these being defined over a field k; x being a generic point of X over k, assume that $t = H(x)$ is generic over k on T and that x has a locus V_t over $k(t)$. Let F_t be the mapping of V_t into W induced by F on V_t. Then F is defined at every point of V_t where F_t and H are both defined.*

It is clearly enough to treat the case in which X is an affine variety and W is the affine line. Then F_t is the function on V_t, defined over $k(t)$, such that $F_t(x) = F(x)$. If F_t is defined at a point a of V_t, we can write it as $F_t(x) = P(x)/Q(x)$, where P, Q are polynomials with coefficients in $k(t)$, such that $Q(a) \neq 0$. More explicitly, we have

$$P(X) = \sum_i \lambda_i(t) M_i(X), \qquad Q(X) = \sum_j \mu_j(t) N_j(X),$$

where the λ_i, μ_j are functions on T, defined over k, and the M_i, N_j are monomials in the indeterminates (X); and we have

$$\sum_j \mu_j(t) N_j(a) \neq 0. \tag{1}$$

Then we have $F(x) = \Phi(x)/\Psi(x)$, with

$$\Phi(x) = \sum_i \lambda_i(H(x)) M_i(x), \qquad \Psi(x) = \sum_j \mu_j(H(x)) N_j(x). \tag{2}$$

As a is on V_t, (t, a) is a specialization of (t, x) over k; if H is defined at a, we must have $H(a) = t$. As t is generic on T over k, the functions λ_i, μ_j are defined at t; therefore the functions $\lambda_i \circ H$, $\mu_j \circ H$ are defined at a on X, with the values $\lambda_i(t)$, $\mu_j(t)$. That being so, the relations (1), (2) show that F is defined at a on V.

Proposition 2. *Let k, T, t, t' be as in* (C); *let V be a variety, defined over k; let V_t be a variety, defined and projectively* (*resp. affinely*) *embeddable over $k(t)$; let f_t be a birational correspondence, defined over $k(t)$, between V and V_t. Then:*

(i) *if a is a point of V_t where $f_{t'} \circ f_t^{-1}$ is biregular, there is an affine variety W and a birational correspondence F between V and W, both defined over k, such that $F \circ f_t^{-1}$ is biregular at a;*

(ii) *if $f_{t'} \circ f_t^{-1}$ is everywhere biregular, there is a variety W, defined and projectively* (*resp. affinely*) *embeddable over k, and a birational correspondence F between V and W, defined over k, such that $F \circ f_t^{-1}$ is everywhere biregular.*

We may assume that V_t is a $k(t)$-open subset of a variety, defined over $k(t)$, in a projective (resp. affine) space S. We may also assume that T is a projective (resp. affine) variety; let S' be its ambient space. If S, S' are affine, $S \times S'$ is an affine space; if they are projective, call Φ the well-known biregular embedding of $S \times S'$ into a projective space S'' of suitable dimension. Let v be generic on V over $k(t, t')$, and put $x = f_t(v)$. We may replace V_t by a suitable $k(t)$-open subset of the locus of (x, t) over $k(t)$ in the affine case, of $\Phi(x, t)$ over $k(t)$ in the projective case; after that is done, we have $k(t) \subset k(x)$, and therefore $k(x) = k(t, x) = k(t, v)$, so that x has a locus X over k, birationally equivalent to $T \times V$, and that we may write $t = H(x)$, where H is a mapping of X into T, defined over k; moreover, the mapping H is everywhere defined on X.

Now, since X is birationally equivalent to $T \times V$ over k, and V is birationally equivalent to V_t over $k(t)$, X is birationally equivalent to $T \times V_t$ over $k(t)$. More explicitly, if we put $x' = f_{t'}(v)$, x' is generic over $k(t)$ on X, and we have $k(x') = k(t', v)$, hence $k(t, x') = k(t, t', x)$, so that we may write $x' = g_t(t', x)$, where g_t is a birational correspondence, defined over $k(t)$, between $T \times V_t$ and X. We have $t' = H(x')$, and we may write $x = \phi_t(x')$, where ϕ_t is a mapping of X into V_t, defined over $k(t)$; then (H, ϕ_t) is the mapping of X into $T \times V_t$, inverse to g_t. The mapping g_t induces on the subvariety $t' \times V_t$ of $T \times V_t$ the mapping $(t', x) \rightarrow x' = f_{t'}(f_t^{-1}(x))$; and ϕ_t induces on $V_{t'}$ the mapping $x' \rightarrow x$, i. e. the mapping $f_t \circ f_{t'}^{-1}$. Applying Lemma 2, we see that g_t is defined at (t', a) whenever a is a point of V_t where $f_{t'} \circ f_t^{-1}$ is defined, and that ϕ_t is defined at every point of $V_{t'}$ where $f_t \circ f_{t'}^{-1}$ is defined. Therefore g_t is biregular at (t', a) whenever a is a point of V_t where $f_{t'} \circ f_t^{-1}$ is biregular.

Now let A_0 be the $k(t)$-closed subset of $T \times V_t$ where g_t is not biregular; and assume first that a is a point of V_t with the property stated in (i). Then (t', a) is not in A_0, so that $T \times a$ is not contained in A_0; let A_1 be the (non-dense) $k(t, a)$-closed subset of T consisting of those points t_1 such that $(t_1, a) \varepsilon A_0$. By [**4**], App., Prop. 12, there is a $\bar{k}$-closed subset A_2 of T containing all $\bar{k}$-closed subsets of T contained in A_1; in particular, every point of A_1 which is algebraic over k must be in A_2. Let A_3 be the union of the components of A_2 and of their conjugates over k; put $T' = T - A_3$; this is a k-open subset of T such that, if t_1 is any algebraic point over k in T', g_t is biregular at (t_1, a).

On the other hand, assume, as in (ii), that $f_{t'} \circ f_t^{-1}$ is everywhere biregular. Then g_t is biregular at every point of $t' \times V_t$, so that A_0 has no point in common with $t' \times V_t$. This implies that the projection of A_0 on T is non-dense in T, so that, if we call A_1' the closure of that projection, it is a (non-dense) $k(t)$-closed subset of T. Let A_2' be the maximal $\bar{k}$-closed subset of T contained in A_1'; let A_3' be the union of the components of A_2' and of their conjugates over k; put $T'' = T - A_3'$. Then T'' is k-open on T; and, if t_1 is any algebraic point over k in T'', g_t is biregular at every point of $t_1 \times V_t$.

Now let t_1 be a separably algebraic point over k in T' (resp. T''); if k is finite, we may take for t_1 any algebraic point over k in T' (resp. T''), since in that case every algebraic extension of k is separable; if k is infinite, we apply [**4**], App., Prop. 13. Let $t_1, \cdots, t_n$ be the distinct conjugates of t_1 over k. As they are in T' (resp. T''), g_t is biregular at (t_i, a) (resp. at every point of $t_i \times V_t$), and a fortiori at (t_i, x), for $1 \leqq i \leqq n$; therefore it induces on $t_i \times V_t$ a birational correspondence g_i between V_t and the locus V_i of the point $g_i(x) = g_t(t_i, x)$ over $k(t, t_i)$ in the projective (resp. affine) ambient space of X; and g_i is biregular at a (resp. at every point of V_t). But, as we have already observed, the relation $k(x) = k(t, v)$ shows that X is birationally equivalent to $T \times V$; we may write $x = f(t, v)$, where f is a birational correspondence between $T \times V$ and X, defined over k; then we have $x' = f(t', v)$; and f is the product of g_t and of the birational correspondence $(t', v) \rightarrow (t', x)$ between $T \times V$ and $T \times V_t$. As the latter correspondence is biregular at (t_i, v), and g_t is biregular at (t_i, x), for $1 \leqq i \leqq n$, we see that f is biregular at (t_i, v), and that we have

$$g_i(x) = g_t(t_i, x) = f(t_i, v).$$

As the point $f(t_i, v)$ has the same locus over $k(t_i)$ as over $k(t, t_i)$, this shows that V_i is defined over $k(t_i)$. As every automorphism of $\bar{k}$ over k can be extended to an automorphism of $\bar{k}(v)$ over $k(v)$, this also shows that V_i is the transform of V_1 by the isomorphism of $k(t_1)$ onto $k(t_i)$ over k which maps t_1 onto t_i. Also, if f_i is the mapping of V into V_i, defined over $k(t_i)$, which is such that $f_i(v) = f(t_i, v)$, we have $f_i = g_i \circ f_t$; and f_i is the transform of f_1 by the isomorphism of $k(t_1)$ onto $k(t_i)$ over k which maps t_1 onto t_i.

Now apply Proposition 1 to the variety V, defined over the groundfield k, to the variety V_1, defined over $k(t_1)$, and to the birational correspondence f_1; this gives a projective (resp. affine) variety W and a birational correspondence F between V and W, both defined over k, such that $F \circ f_1^{-1}$ is biregular wherever all the $f_i \circ f_1^{-1}$ are defined, i. e. wherever all the $g_i \circ g_1^{-1}$ are defined. Now, in case (i), all the g_i are biregular at a, so that all the $g_i \circ g_1^{-1}$ are biregular at the point $g_1(a)$; therefore $F \circ f_t^{-1}$, which is the same as $(F \circ f_1^{-1}) \circ g_1$, is biregular at a; as this involves merely a local property of W at the image of a by that mapping, we may replace W, in the projective case, by one of its affine representatives. Thus we have solved our problem in case (i). In case (ii), g_i is biregular at every point of V_t; as we have just shown, this implies that $F \circ f_t^{-1}$ is biregular at every point of V_t, so that it determines an isomorphism of V_t onto a $k(t)$-open subset W' of W. The assumption in (ii) implies that W' is invariant under the isomorphism of $k(t)$ onto $k(t')$ over k which maps t onto t'. From this and from [**4**], App., Prop. 9, it follows easily that W' is k-open; thus (W', F) is a solution of our problem.

COROLLARY. *Let k, T, t and t' be as in* (C); *let V be a variety, defined over k; let V_t be a variety, defined over $k(t)$; let f_t be a birational correspondence between V and V_t, defined over $k(t)$ and such that $f_{t'} \circ f_t^{-1}$ is everywhere biregular. Then, if a is any point of V_t, there is an affine variety W and a birational correspondence F between V and W, both defined over k, such that $F \circ f_t^{-1}$ is biregular at a.*

We may assume that t' has been taken generic on T over $k(t, a)$; take t'' generic on T over $k(t, t', a)$. Call a', a'' the images of a by $f_{t'} \circ f_t^{-1}$ and by $f_{t''} \circ f_t^{-1}$, respectively. The isomorphism of $k(t, a, t')$ onto $k(t, a, t'')$ over $k(t, a)$ which maps t' onto t'' maps a' onto a''; therefore, if $V_{t'\alpha}$ is a representative of the (abstract) variety $V_{t'}$ on which a' has a representative a_α', the point a'' of $V_{t''}$ has a representative a_α'' on $V_{t''\alpha}$. Let $f_{t'\alpha}$ be the birational correspondence between V and $V_{t'\alpha}$ which is determined by $f_{t'}$. As $f_{t''} \circ f_{t'}^{-1}$ is everywhere biregular and maps a' onto a'', $f_{t''\alpha} \circ f_{t'\alpha}^{-1}$ is biregular at a_α'. Applying Proposition 2(i) to V, $V_{t'\alpha}$ and $f_{t'\alpha}$, we get a solution (W, F) of our problem.

6. Now we can deal with problems (D) and (C′).

THEOREM 5. *Problem* (D) *has a solution if and only if $f_{t'} \circ f_t^{-1}$ is everywhere biregular for t' generic over $k(t)$ on T.*

The condition being obviously necessary, assume that it is fulfilled. By the corollary of Proposition 2, there is, to every point a of V_t, an affine variety W_a and a birational correspondence F_a between V and W_a, both defined over k, such that $F_a \circ f_t^{-1}$ is biregular at a; call Ω_a the $k(t)$-open subset of V_t where $F_a \circ f_t^{-1}$ is biregular, and call W_a' its image on W_a by $F_a \circ f_t^{-1}$, which is a $k(t)$-open subset of W_a. Then W_a' is the subset of W_a where $f_t \circ F_a^{-1}$ is biregular; as in the proof of Proposition 2, this implies that W_a' is invariant under the isomorphism of $k(t)$ onto $k(t')$ over k which maps t onto t', and we again conclude from this that W_a' is k-open. As we have $a \in \Omega_a$ for every $a \in V_t$, the open sets Ω_a form a covering of V_t; by the well-known "compactoid" property of open sets in the Zariski topology, there must be finitely many points a_α on V such that the sets Ω_{a_α} cover V_t. Then the k-open subsets W_{a_α}' of the affine varieties W_{a_α}, together with the birational correspondences $F_{a_\beta} \circ F_{a_\alpha}^{-1}$ between them, define an abstract variety, which, together with the obvious birational correspondence between it and V, solves our problem.

THEOREM 6. *Problem* (C′) *has a solution, i.e., problem* (C) *has a* *solution* (V, f_t) *for which* f_t *is everywhere biregular, if and only if* $f_{t',t}$ *is everywhere biregular and satisfies condition* (i) *in Theorem* 4. *The solution is unique up to a* k*-isomorphism.*

This is an immediate consequence of Theorems 4 and 5.

7. As to the projective or affine embeddability of the solution of problems (D) and (C′), we have the following result.

THEOREM 7. *Let* V *be a variety, defined over a field* k*, and projectively* (*resp. affinely*) *embeddable over an overfield* K *of* k*. Then* V *is projectively* (*resp. affinely*) *embeddable over* k *provided* (i) K *is separable over* k *or* (ii) V *is everywhere normal with reference to* k.

The assumption means that there is a birational correspondence f, defined over K and biregular at every point of V, between V and a subvariety of a projective (resp. affine) space; if we regard f as a mapping of V into that space, it has a smallest field of definition k' containing k; we may replace K by k'; after that is done, K is finitely generated over k. If K is separable over k, it is a regular extension $k_1(t)$ of the algebraic closure k_1 of k in K, and k_1 is a separably algebraic extension of k of finite degree. Proposition 2(ii) shows that V is then projectively (resp. affinely) embeddable over k_1; by Theorem 2, this implies that the same is true over k; this completes the proof in case (i). If K is not separable over k, let k^* be the union of the fields $k^{p^{-n}}$, for $n = 1, 2, \cdots$; then the compositum K^* of K and k^* is separable over k^*, so that, by what we have just proved, V is projectively (resp. affinely) embeddable over k^*. In order to deal with case (ii), it is therefore enough to prove our theorem in the case in which V is everywhere normal with reference to k, and K is purely inseparable over k; I owe the proof for this to T. Matsusaka; it is as follows.

We may again assume that K is finitely generated over k; as it is purely inseparable, it is contained in some field $k' = k^{1/q}$, where q is a power of the characteristic. Then there is a mapping f' of V into a projective (resp. affine) space, defined over k', such that f' determines a birational correspondence, biregular at every point of V, between V and the closure W' of its image by f'; then W' is a projective (resp. affine) variety, defined over k', and f' determines a k'-isomorphism between V and a k'-open subset of W'. Call π the automorphism $\xi \rightarrow \xi^q$ of the universal domain; put $W = W'^{\pi}$; W is then a projective (resp. affine) variety, defined over k. Let x be a generic point of V over k; then W' is the locus of the point $y' = f'(x)$ over k', and W is the locus of the point $y = y'^{\pi}$ over k. As y' is rational over $k'(x)$, y is so over $k(x^{\pi})$; we may write $y = g(x)$, where g is a mapping of V into W, defined over k; as we have $k'(y') = k'(x)$, we have $k(y) = k(x^{\pi})$, which implies that $k(x)$ is purely inseparable over $k(y)$. In the projective case, let U be the projective variety derived from W by normalization in the field $k(x)$[1]; U is birationally equivalent to V over k; let z be the point of U which corresponds to x on V. In the affine case, we take for z a point in a suitable affine space such that $k[z]$ is the integral closure of the ring $k[y]$ in the field $k(x)$, and for U the locus of z over k. In either case we may write $z = f(x)$, where f is a birational correspondence between V and U, defined over k. By definition, U is everywhere normal with reference to k, and the mapping $h = g \circ f^{-1}$ of U into W is everywhere defined and such that the (set-theoretic) inverse image of every point of W for that mapping consists of finitely many points of U. Let a be any point of V; let (a, b) be a specialization of (x, z) over $x \to a$ with reference to k; then, as h is defined at b, $(a, b, h(b))$ is a specialization of (x, z, y) over k. As f' is defined at a, g is also defined there, so that we must have $h(b) = g(a)$; therefore b is one of the finitely many points of U whose image by h is $g(a)$. As V is normal at a by assumption, with reference to k, this implies that f is defined at a, and that we have $b = f(a)$. We have $g(a) = f'(a)^{\pi}$, hence $f'(a) = g(a)^{\pi^{-1}}$; as $g(a) = h(b)$, this shows that $f'(a)$ is the unique specialization of y' over $z \to b$ with reference to k'; as f' is biregular at a, f'^{-1} is defined at $f'(a)$, and therefore x has no other specialization than a over $z \to b$ with reference to k', hence also with reference to k by F-II_1, Prop. 3. As U is normal at b, with reference to k, this implies that f^{-1} is defined at $b = f(a)$. We have thus shown that f is biregular at every point of V, so that it is a k-isomorphism between V and a k-open subset of U.

As a special case (already contained in Proposition 2), we see that, in problem (D), V' is projectively (resp. affinely) embeddable over k if V_t is so over $k(t)$; similarly, in problem (C′), V is projectively (resp. affinely) embeddable over k if V_t is so over $k(t)$.

8. In [**4**], the construction carried out in Nos. 7-9 can be advantageously replaced by the application of our Theorem 6 to the situation described in No. 6 of that paper. The application is entirely straightforward, so that no further details need be given; this shows that the recourse to the Lang-Weil

[1] U is the "derived normal model of W in the field $k(x)$" according to Zariski's definition ([**5**], pp. 69-70); cf. also [**3**].

Theorem, i.e., in substance, to the so-called "Riemann hypothesis" in the case of a finite groundfield (loc. cit., p. 374) was unnecessary; so is the assumption of normality in the final result (loc. cit., p. 375); normality had to be assumed there merely because of the use made of the Chow point in the construction on p. 370, whereas in the present paper a different device was adopted (in the proof of Proposition 1). Of course, in the main theorem of **[4]** (p. 375), parts (i) and (ii) remain unchanged. For the sake of completeness, we give here the improved result by which part (iii) of that theorem may now be replaced:

PROPOSITION 3. *Let G be a group and W a chunk of transformation-space with respect to G, both defined over k. Then there is a transformation-space S with respect to G, and a birational correspondence f between W and S, both defined over k, with the following properties:* (a) *f is biregular at every point of W;* (b) *for every $s \varepsilon G$ and $a \varepsilon W$ such that sa is defined, we have $f(sa) = sf(a)$;* (c) *every point of S can be written in the form $sf(a)$, with $s \varepsilon G$ and $a \varepsilon W$. Moreover, S is uniquely determined by these properties up to a k-isomorphism compatible with the operations of G.*

UNIVERSITY OF CHICAGO.

REFERENCES.

[1] W. L. Chow, "Abelian varieties over function-fields," *Transactions of the American Mathematical Society*, vol. 78 (1955), pp. 253-275.

[2] S. Lang, "Abelian varieties over finite fields," *Proceedings of the National Academy of Sciences*, vol. 41 (1955), pp. 174-176.

[3] T. Matsusaka, "A note on my paper 'Some theorems on abelian varieties'," *Nat. Sc. Rep. Ochanomizu Univ.*, vol. 5 (1954), pp. 21-23.

[4] A. Weil, "On algebraic groups of transformations," *American Journal of Mathematics.*, vol. 77 (1955), pp. 355-391.

[5] O. Zariski, "Theory and applications of holomorphic functions on algebraic varieties over arbitrary groundfields," *Memoirs of the American Mathematical Society*, no. 5 (1951), pp. 1-90.