# 15.094J: Robust Modeling, Optimization, Computation

Lectures 3: RLO: Tractability

September 12, 2012

# Outline

1. RLO with Row-wise uncertainty
2. RLO with Row-wise Polyhedral Uncertainty
3. RLO with Row-wise Ellipsoidal uncertainty
4. RLO with General Polyhedral Uncertainty

# Objectives Today

- Tractability of RLO
- Row-wise uncertainty
- General uncertainty

# Row-wise Uncertainty

- Primitives: Uncertainty sets $U_i$, $i = 1, \ldots, m$, $\boldsymbol{b}, \boldsymbol{c}$ (known, WLOG).
- RLO with row-wise uncertainty:

$$
\begin{array}{rll}
\max & \boldsymbol{c}'\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \le b_i. & \forall \boldsymbol{a}_i \in U_i, \ i = 1, \ldots, m, \\
 & \boldsymbol{x} \ge \boldsymbol{0}. &
\end{array}
$$

- Note that the problem has infinitely many constraints.
- Reformulation:

$$
\begin{array}{rl}
\max & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \le b_i \\
 & \boldsymbol{x} \ge \boldsymbol{0}.
\end{array}
$$

- Note that the uncertainty for different constraints is independent.

# Tractability

- Suppose that $U_i$, $i = 1, \ldots, m$ are convex sets.
- Given an $\boldsymbol{x}$, we can solve $i = 1, \ldots, m$:

$$\max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i' \boldsymbol{x},$$

  efficiently.
- How should we solve the RLO problem?

# Theoretical Tractability

- Solve the nominal problem; find $\boldsymbol{x}_0$.
- Separation problem: Given an $\boldsymbol{x}_0$, does there exist an $\boldsymbol{a}_i \in U_i$ that violates the constraint $\boldsymbol{a}_i'\boldsymbol{x} > b_i$?
- Solution: Solve $\max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x}$ and check whether

$$\max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \leq b_i.$$

- This shows that if $U_i$ are convex, we can solve the separation problem in polynomial time, thus we can solve the RLO with convex uncertainty sets in polynomial time using the Ellipsoid method (see Chapter 8 of Bertsimas and Tsitsiklis book).
- The key take away from this: Even though RLO has infinitely many constraints it is polynomially solvable.
- Question: How about practically solvable? The Ellipsoid method is not a practical algorithm.

# Practical Tractability

- Solve the nominal problem; find $\boldsymbol{x}_0$.
- Solve $\max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i' \boldsymbol{x}_0$, solution $\overline{\boldsymbol{a}}_i$.
- Solve (the dual Simplex method is the right choice)

$$\begin{array}{rl} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \overline{\boldsymbol{a}}_i'\boldsymbol{x} \le b_i \\ & \boldsymbol{x} \ge \mathbf{0}. \end{array}$$

- Find $\boldsymbol{x}_1$; iterate.

# Robust Counterpart-Polyhedral uncertainty

- $$\begin{array}{rl} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \leq b_i. \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$
- $U_i = \{\boldsymbol{a}_i |\ \boldsymbol{D}_i \boldsymbol{a}_i \leq \boldsymbol{d}_i\}$, $\boldsymbol{D}_i :\ k_i \times n$.
- Consider the problem and its dual:

$$\begin{array}{rl} \max & \boldsymbol{a}_i'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}_i \boldsymbol{a}_i \leq \boldsymbol{d}_i \end{array} \qquad\qquad \begin{array}{rl} \min & \boldsymbol{p}_i'\boldsymbol{d}_i \\ \text{s.t.} & \boldsymbol{p}_i'\boldsymbol{D}_i = \boldsymbol{x}' \\ & \boldsymbol{p}_i \geq \boldsymbol{0}. \end{array}$$

# Robust Counterpart continued

- RC becomes

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{p}_i} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{p}_i'\boldsymbol{d}_i \le b_i, \quad i = 1,\dots,m, \\ & \boldsymbol{p}_i'\boldsymbol{D}_i = \boldsymbol{x}', \quad i = 1,\dots,m, \\ & \boldsymbol{p}_i \ge \boldsymbol{0}, \quad i = 1,\dots,m, \\ & \boldsymbol{x} \ge \boldsymbol{0}. \end{aligned}$$

- Original nominal problem: $n$ variables, $m$ constraints.
- Uncertainty dimension: $k_i$.
- Size of Robust Counterpart: $n + \sum_{i=1}^{m} k_i$, variables; $m + m \cdot n$ constraints.

# Row-wise Ellipsoidal uncertainty

- RO:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \le b_i. \\ & \boldsymbol{x} \ge \boldsymbol{0}. \end{aligned}$$

- $U_i = \{\boldsymbol{a}_i |\ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i'\boldsymbol{u}_i,\ ||\boldsymbol{u}_i||_2 \le \rho\}$, $\boldsymbol{\Delta}_i :\ k_i \times n,\ \boldsymbol{u}_i : k_i \times 1.$
- RC:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \overline{\boldsymbol{a}}_i'\boldsymbol{x} + \rho||\boldsymbol{\Delta}_i\boldsymbol{x}||_2 \le b_i, \qquad i = 1, \ldots, m. \\ & \boldsymbol{x} \ge \boldsymbol{0}. \end{aligned}$$

- Second order cone problem, nearly as tractable as linear optimization.

# Proof

- $Z^* = \max_{\boldsymbol{a} \in U} \boldsymbol{a}'\boldsymbol{x} = \overline{\boldsymbol{a}}'\boldsymbol{x} + \max_{||\boldsymbol{u}|| \leq \rho} \boldsymbol{u}'(\boldsymbol{\Delta x})$
- Lagrangean dual:

$$Z(\lambda) = \overline{\boldsymbol{a}}'\boldsymbol{x} + \max \ \boldsymbol{u}'(\boldsymbol{\Delta x}) - \lambda(\boldsymbol{u}'\boldsymbol{u}/2 - \rho^2/2).$$

- $\boldsymbol{u}^* = \boldsymbol{\Delta x}/\lambda.$
- $$Z(\lambda) = \overline{\boldsymbol{a}}'\boldsymbol{x} + \frac{1}{2}\left(\frac{||\boldsymbol{\Delta x}||^2}{\lambda} + \lambda\rho^2\right).$$
- For $\lambda \geq 0$, $Z^* \leq Z(\lambda)$ and strong duality: $Z^* = \min_{\lambda \geq 0} Z(\lambda)$.
- $\lambda^* = ||\boldsymbol{\Delta x}||/\rho.$
- $Z^* = \overline{\boldsymbol{a}}'\boldsymbol{x} + \rho||\boldsymbol{\Delta x}||.$

# Robust Counterpart-General Norm uncertainty

- RO:

$$\begin{array}{rl} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \leq b_i. \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

- $U_i = \{\boldsymbol{a}_i | \ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i'\boldsymbol{u}_i, \ ||\boldsymbol{u}_i|| \leq \rho\}$, $\boldsymbol{\Delta}_i : \ k_i \times n, \ \boldsymbol{u}_i : k_i \times 1.$
- Dual norm:

$$||\boldsymbol{s}||^* = \max_{\{||\boldsymbol{x}|| \leq 1\}} |\boldsymbol{s}'\boldsymbol{x}|.$$

- The dual of the $L_p$-norm $||\boldsymbol{x}||_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$:
- $||\boldsymbol{s}||^* = ||\boldsymbol{s}||_q$ with $q = 1 + \dfrac{1}{p-1}$.
- The dual norm of the $L_2$ norm is $L_2$.
- The dual norm of the $L_1$ norm is the $L_\infty$ norm.
- RC:

$$\begin{array}{rll} \max & \boldsymbol{c}'\boldsymbol{x} & \\ \text{s.t.} & \overline{\boldsymbol{a}}_i'\boldsymbol{x} + \rho||\boldsymbol{\Delta}_i\boldsymbol{x}||^* \leq b_i, & i = 1, \ldots, m. \\ & \boldsymbol{x} \geq \boldsymbol{0}. & \end{array}$$

# General Polyhedral Uncertainty

- $$\begin{array}{lll} \max & \boldsymbol{c}'\boldsymbol{x} & \\ \text{s.t.} & \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \boldsymbol{b}, & \forall \tilde{\boldsymbol{A}} \in \mathcal{U} \\ & \boldsymbol{x} \in P. & \end{array}$$
- $\mathcal{U} = \{\text{vec}(\tilde{\boldsymbol{A}}) \mid \boldsymbol{G} \cdot \text{vec}(\tilde{\boldsymbol{A}}) \leq \boldsymbol{d}\}$,
- $\boldsymbol{G} \in \Re^{l \times (m \cdot n)}$, $\boldsymbol{d} \in \Re^{l \times 1}$, and $\text{vec}(\tilde{\boldsymbol{A}}) \in \Re^{(m \cdot n) \times 1}$.

# RC

- The RC is

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{p}_i'\boldsymbol{G} = \boldsymbol{x}_i', \quad && i = 1, \dots, m \\ & \boldsymbol{p}_i'\boldsymbol{d} \leq b_i, \quad && i = 1, \dots, m \\ & \boldsymbol{p}_i \geq \boldsymbol{0}, \quad && i = 1, \dots, m \\ & \boldsymbol{x} \in P, \end{aligned}$$

- $\boldsymbol{p}_i \in \Re^{l \times 1}$.
- $\boldsymbol{x}_i \in \Re^{(m \cdot n) \times 1}$, $i = 1, \dots, m$; $\boldsymbol{x}_i$ contains $\boldsymbol{x}$ in entries $(i-1) \cdot n + 1$ through $i \cdot n$, and zero everywhere else.

# Proposition

- Suppose $\mathcal{U} \neq \emptyset$.
- A given $\hat{\boldsymbol{x}}$ satisfies $\tilde{\boldsymbol{a}}_i'\hat{\boldsymbol{x}} \leq b_i$ for all $\tilde{\boldsymbol{A}} \in \mathcal{U}$ if and only if there exists a vector $\boldsymbol{p}_i \in \Re^{l\times 1}$ such that

$$
\begin{aligned}
\boldsymbol{p}_i'\boldsymbol{d} &\leq b_i \\
\boldsymbol{p}_i'\boldsymbol{G} &= \hat{\boldsymbol{x}}_i' \\
\boldsymbol{p}_i &\geq \boldsymbol{0}
\end{aligned}
$$

- $\hat{\boldsymbol{x}}_i \in \Re^{(m\cdot n)\times 1}$ contains $\hat{\boldsymbol{x}}$ in entries $(i-1)\cdot n+1$ through $i\cdot n$, and zero everywhere else.

# Proof

- Consider the primal-dual pair

$$\begin{aligned} \max_{\boldsymbol{A}} \quad & \boldsymbol{a}_i'\hat{\boldsymbol{x}} \\ \text{s.t.} \quad & \boldsymbol{G}\cdot \mathrm{vec}(\boldsymbol{A}) \leq \boldsymbol{d} \end{aligned}$$

$$\begin{aligned} \min_{\boldsymbol{p}_i} \quad & \boldsymbol{p}_i'\boldsymbol{d} \\ \text{s.t.} \quad & \boldsymbol{p}_i'\boldsymbol{G} = \hat{\boldsymbol{x}}_i' \\ & \boldsymbol{p}_i \geq \boldsymbol{0}. \end{aligned}$$

- Suppose that $\hat{\boldsymbol{x}}$ satisfies $\tilde{\boldsymbol{a}}_i'\hat{\boldsymbol{x}} \leq b_i$ for all $\tilde{\boldsymbol{A}} \in \mathcal{U}$.
- Then, $\max_{\boldsymbol{A}} \boldsymbol{a}_i'\hat{\boldsymbol{x}} \leq b_i$.
- Then primal is feasible and bounded, and so is its dual.
- Thus, there exists a vector $\boldsymbol{p}_i \in \Re^{(m\cdot n)\times 1}$ satisfying the dual constraints.
- By strong duality, the optimal objective function value of the dual equals $\max_{\boldsymbol{A}} \boldsymbol{a}_i'\hat{\boldsymbol{x}}$ and is less than $b_i$.

# Proof continued

- For the reverse, since $\mathcal{U} \neq \emptyset$, primal is feasible. Suppose there exists a vector $\boldsymbol{p}_i \in \Re^{l \times 1}$ that satisfies the dual constraints.
- Since both problems are feasible, they must be bounded and their optimal objective function values must be equal.
- Then $\min_{\boldsymbol{p}_i} \boldsymbol{p}_i'\boldsymbol{d} \leq \boldsymbol{p}_i'\boldsymbol{d} \leq b_i$.
- By strong duality, $\max_{\boldsymbol{A}} \boldsymbol{a}_i'\hat{\boldsymbol{x}} = \min_{\boldsymbol{p}_i} \boldsymbol{p}_i'\boldsymbol{d} \leq b_i$, and hence $\hat{\boldsymbol{x}}$ satisfies $\boldsymbol{a}_i'\hat{\boldsymbol{x}} \leq b_i$ for all $\tilde{\boldsymbol{A}} \in \mathcal{U}$.

# RC

- RO:

$$
\begin{array}{ll}
\max & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} & \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \boldsymbol{b}, \quad \forall \tilde{\boldsymbol{A}} \in \mathcal{U} \\
& \boldsymbol{x} \in P.
\end{array}
$$

- $\mathcal{U} = \{\text{vec}(\tilde{\boldsymbol{A}}) \mid \boldsymbol{G} \cdot \text{vec}(\tilde{\boldsymbol{A}}) \leq \boldsymbol{d}\}$.
- The RC is

$$
\begin{array}{lll}
\max & \boldsymbol{c}'\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{p}_i'\boldsymbol{G} = \boldsymbol{x}_i', & i = 1, \ldots, m \\
& \boldsymbol{p}_i'\boldsymbol{d} \leq b_i, & i = 1, \ldots, m \\
& \boldsymbol{p}_i \geq \boldsymbol{0}, & i = 1, \ldots, m \\
& \boldsymbol{x} \in P. &
\end{array}
$$

# General uncertainty sets under a general norm

- RO:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \tilde{\boldsymbol{A}}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \in P \\ & \forall \tilde{\boldsymbol{A}} \in \mathcal{U} = \left\{ \tilde{\boldsymbol{A}} \mid ||\boldsymbol{M}(\text{vec}(\tilde{\boldsymbol{A}}) - \text{vec}(\overline{\boldsymbol{A}}))|| \le \Delta \right\}. \end{aligned}$$

- RC:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \overline{\boldsymbol{a}}_i\boldsymbol{x} + \Delta||\boldsymbol{M}^{-1}\boldsymbol{x}_i||^* \le \boldsymbol{b}_i, \quad i = 1, \ldots, m \\ & \boldsymbol{x} \in P, \end{aligned}$$

- $\boldsymbol{x}_i \in \Re^{(m\cdot n)\times 1}$ contains $\boldsymbol{x} \in \Re^{n\times 1}$ in entries $(i-1)\cdot n + 1$ through $i \cdot n$, and 0 everywhere else.

# Proof

- $\boldsymbol{y} = \dfrac{\boldsymbol{M}(\mathrm{vec}(\tilde{\boldsymbol{A}}) - \mathrm{vec}(\overline{\boldsymbol{A}}))}{\Delta}$.
- Then, $\mathcal{U} = \{\boldsymbol{y} : \ ||\boldsymbol{y}|| \leq 1\}$.

$$
\begin{aligned}
\max_{\{\mathrm{vec}(\tilde{\boldsymbol{A}}) \in \mathcal{U}\}} \left\{\tilde{\boldsymbol{a}}_i{}' \boldsymbol{x}\right\} &= \max_{\{\mathrm{vec}(\tilde{\boldsymbol{A}}) \in \mathcal{U}\}} \left\{(\mathrm{vec}(\tilde{\boldsymbol{A}}))' \boldsymbol{x}_i\right\} \\
&= \max_{\{\boldsymbol{y}:\ ||\boldsymbol{y}|| \leq 1\}} \left\{(\mathrm{vec}(\overline{\boldsymbol{A}}))' \boldsymbol{x}_i + \Delta(\boldsymbol{M}^{-1}\boldsymbol{y})' \boldsymbol{x}_i\right\} \\
&= \overline{\boldsymbol{a}}_i' \boldsymbol{x} + \Delta \max_{\{\boldsymbol{y} | ||\boldsymbol{y}|| \leq 1\}} \left\{\boldsymbol{y}'(\boldsymbol{M}^{-1}\boldsymbol{x}_i)\right\} \\
&= \overline{\boldsymbol{a}}_i \boldsymbol{x} + \Delta ||\boldsymbol{M}^{-1}\boldsymbol{x}_i||^*
\end{aligned}
$$