15.094, Problem Set 3

Due: 18 March 2015 at 9am

Problem 1 - The Price of Robustness (40 points)

We consider an instance of the facility location problem which consists of F candidate facilities (potential sites where a facility can be opened) and C demand points that must be serviced (possibly from a combination of facilities). Opening facility $f \in \mathcal{F} := \{1, \ldots, F\}$ incurs a cost c_f, while servicing all the demand of customer $c \in \mathcal{C} := \{1, \ldots, C\}$ from facility f incurs a cost d_{fc} (delivery cost). We assume that the cost of opening a facility is precisely known (perfect information). The servicing costs are uncertain, but it is known that d_{fc}, $f \in \mathcal{F}$, $c \in \mathcal{C}$ are independent, symmetric and bounded random variables with support $\mathcal{S}_{fc} := \left[\overline{d}_{fc}(1-\rho), \overline{d}_{fc}(1+\rho)\right]$, where $\rho > 0$ and $\overline{d}_{fc}$ denotes the nominal value of d_{fc}.

We wish to be immunized against variations in the servicing costs when at most $\Gamma \in \{0, \ldots, FC\}$ of these costs can deviate from their nominal values.

(a) (10 points) Formulate the robust facility location problem that minimizes costs in the worst-case realization of the uncertain parameters.

(b) (10 points) Reformulate the robust facility location problem as a deterministic optimization problem.

(c) (20 points) Let $\rho = 5\%$. Using the data in the companion Excel spreadsheet, investigate the price of robustness in this problem. That is, calculate the worst-case cost incurred and the associated bound on the violation probability in dependence of Γ. Plot the trade-off curves.

Problem 2 - Probabilistic Guarantees (T/F) (35 points)

For each of the following statements, indicate if the statement is true or false. Provide also a brief justification, sketch of a proof, or counterexample.

(a) Assume throughout that $\tilde{\mathbf{u}} \sim \mathbb{P}^*$ is a random variable coming from some distribution $\mathbb{P}^*$ which we may not know.

i. (5 points) Suppose $\mathcal{U}_1$ implies a probabilistic guarantee at level ϵ_1 and $\mathcal{U}_2$ implies a probabilistic guarantee at level ϵ_2. Furthermore, suppose $\epsilon_1 < \epsilon_2$. Then for any set $\mathcal{X}$, we have that

$$\min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{u} \in \mathcal{U}_1} \mathbf{u}^T \mathbf{x} \geq \min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{u} \in \mathcal{U}_2} \mathbf{u}^T \mathbf{x}$$

ii. (5 points) Recall the data-driven uncertainty set described in Lecture 7 using a hypothesis test for the mean:

$$\mathcal{U} = \left\{ \mathbf{u} \in \mathbb{R}^d : (\mathbf{u} - \hat{\boldsymbol{\mu}})^T \boldsymbol{\Sigma}^{-1} (\mathbf{u} - \hat{\boldsymbol{\mu}}) \le \left(\Gamma + \sqrt{1/\epsilon - 1} \right)^2 \right\}$$

Here $\hat{\boldsymbol{\mu}}$ is the sample mean and $\Gamma \equiv \frac{R^2}{N} \left(2 + \sqrt{2 \log(1/\delta)} \right)$. Let $\mathcal{U}_1$ be the result of applying this construction to this data with parameter $\delta = \delta_1$, and let $\mathcal{U}_2$ be the result from applying this construction with parameter $\delta = \delta_2$. Assume $\delta_1 < \delta_2$. Then for any set $\mathcal{X}$, we have that

$$\min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{u} \in \mathcal{U}_1} \mathbf{u}^T \mathbf{x} \ \ge \ \min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{u} \in \mathcal{U}_2} \mathbf{u}^T \mathbf{x}$$

(b) For the remaining parts, consider the following two stage adaptive linear optimization problem. Again assume that $\tilde{\mathbf{b}} \sim \mathbb{P}^*$ is a random variable coming from some distribution we may not know.

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{y}(\cdot)} \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{b}) \ge \mathbf{b} \quad \forall \mathbf{b} \in \mathcal{U} \end{aligned} \tag{1}$$

Assume that $\mathcal{U} \subseteq \mathbb{R}^m$.

i. (5 points) Consider the robust approximation to this problem, i.e. where we impose that $\mathbf{y}(\cdot) \equiv \mathbf{y}_0 \in \mathbb{R}^{n_2}$. Suppose that $\mathcal{U}$ implies a probabilistic guarantee at level ϵ. Then, if $(\mathbf{x}, \mathbf{y}_0)$ are robust feasible, they are feasible with probability at least $1 - m\epsilon$ to problem (1).

ii. (5 points) Consider the affine approximation to this problem, i.e. where we impose that $\mathbf{y}(\cdot) \equiv \mathbf{F}\mathbf{b} + \mathbf{y}_0$ for some matrix $\mathbf{F} \in \mathbb{R}^{n_2 \times m}$ and vector $\mathbf{y}_0 \in \mathbb{R}^m$. Then, if $(\mathbf{x}, \mathbf{y}(\cdot))$ are robust feasible, they are feasible with probability at least $1 - m\epsilon$ to problem (1).

iii. (5 points) Now consider the fully adaptive version of this problem where $\mathbf{y}(\cdot)$ is permitted to be any function of the data. Then, if $(\mathbf{x}, \mathbf{y}(\cdot))$ are robust feasible, they may not be feasible with probability at least $1 - m\epsilon$ to problem (1).

iv. (5 points) Finally, suppose $\mathbb{P}(\tilde{\mathbf{b}} \in \mathcal{U}) \ge 1 - \epsilon$. (Recall from the lecture this is a *stronger* requirement than implying a probabilistic guarantee). Then, if $(\mathbf{x}, \mathbf{y}(\cdot))$ are robust feasible, they will be feasible with probability at least $1 - \epsilon$ to problem (1).

v. (5 points) **Important:** What does this problem tell you about constructing uncertainty sets for multistage optimization problems?

Problem 3 - A two stage adaptive RO problem (25 points)

Consider the two-stage adaptive robust problem

$$\begin{aligned} \min_{u_1 \in \mathbb{R}} \quad & c u_1 + \max_{\substack{w_1 \in \mathbb{R}: \\ -1 \le w_1 \le 1 \\ L \le u_1 + w_1 \le U}} h(x_1 + u_1 + w_1) \\ \text{s.t.} \quad & L \le u_1 \le U, \end{aligned} \tag{2}$$

where $c, x_1 \in \mathbb{R}$ are fixed.

(a) (10 points) Suppose $c > 0$, $0 < L < 1 < L + 1 < U$, and $h(y) = y$. Derive an expression for the optimal cost of (2). *Hint:* it should an affine function of c.

(b) (5 points) What would change if you repeated (a) for $c \leq 0$? (No need to do the full analysis, mention which cases (if any) you would consider.)

(c) (10 points) Outline your approach for an arbitrary convex function $h(.)$. (Write down the optimal w_1 for the inner problem, the final minimization problem, and the optimal u_1 for the outer problem. Is this outer problem still convex?)

Problem 4 (20 points, **OPTIONAL EXTRA CREDIT**)

As we have seen, the primary constraint encountered in two-stage adaptive optimization is something of the form

$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{d} \; \forall \boldsymbol{\xi} \in \mathcal{U},$$

where $\mathbf{A}, \mathbf{B}$, and $\mathbf{d}$ are all known. In reality, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{d}$ are often estimated. If there are uncertainty in $\mathbf{A}$ and $\mathbf{d}$, we can handle this in the usual robust optimization framework. The focus of this question is what if $\mathbf{B}$ is uncertain? For simplicity we will only consider a basic model of uncertainty and only look at a single constraint. Fix $\mathbf{a}, d$ of appropriate dimensions, and consider the constraint

$$\mathbf{a}'\mathbf{x} + \mathbf{b}'\mathbf{y}(\boldsymbol{\xi}) \leq d \; \forall \boldsymbol{\xi} \in \mathcal{U}, \mathbf{b} \in \mathcal{V}, \tag{3}$$

where $\mathbf{y}(\boldsymbol{\xi}) = \bar{\mathbf{y}} + \mathbf{E}\boldsymbol{\xi}$, and $\bar{\mathbf{y}}, \mathbf{E}$ are decision variables in the outer problem.

(a) Propose a general solution technique for *tractably* solving such a problem with a constraint as given in (3) under general (convex) sets $\mathcal{U}$ and $\mathcal{V}$. You should formally prove any tractability claims you make. Can you even make such a claim for well-structured (e.g. polyhedral) sets $\mathcal{U}$ and $\mathcal{V}$?

(b) The type of uncertainty assumed aboved requires (essentially) independence in uncertainty between $\mathbf{b}$ and $\mathbf{y}$. This is likely unrealistic. Let us consider instead the constraint

$$\mathbf{a}'\mathbf{x} + \mathbf{b}(\boldsymbol{\xi})'\mathbf{y}(\boldsymbol{\xi}) \leq d \; \forall \boldsymbol{\xi} \in \mathcal{U},$$

where now we take for example $\mathbf{b}(\boldsymbol{\xi}) = \bar{\mathbf{b}} + \mathbf{D}\boldsymbol{\xi}$, where $\bar{\mathbf{b}}$ and $\mathbf{D}$ are fixed. How do you solve such a problem?