CHAPTER 2

SECTION 12

No exercise

SECTION 13

1.

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

For every x there is an open set U_x such that $x \in U_x \subset A$, therefore, $U = \cup_{x \in A} U_x$ is open, and $A = \cup_{x \in A} \{x\} \subset \cup_{x \in A} U_x \subset A$, i.e. $U = A$.

2.

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Let us enumerate the topologies by columns, as shown in Figure 1↓.

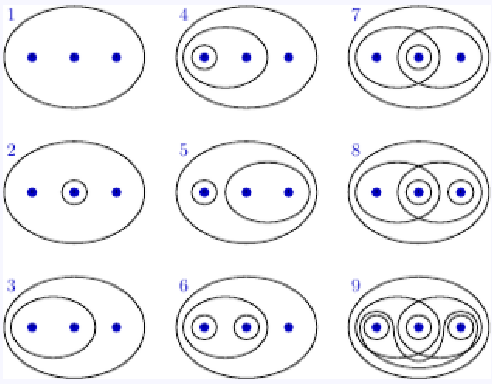


Figure 1 Three element topologies.

Let "$\prec$" be the partial relation between any two topologies that indicates that the topology on the left side is coarser (smaller) than the topology on the right side. Here we list all maximal ordered subsets of the set of topologies of Example 1 of §12 ordered by "$\prec$": $1 \prec 2 \prec 6 \prec 9$, $1 \prec 2 \prec 7 \prec 8 \prec 9$, $1 \prec 3 \prec 4 \prec 6 \prec 9$, $1 \prec 3 \prec 7 \prec 8 \prec 9$, and $1 \prec 5 \prec 9$.

> Note, that out of 29 possible topologies on the set of three elements, only 9 are shown here (because every other topology can be obtained from one of these by permutating its elements). However, this does not mean that such permutations preserve relations stated above. For example, if instead of the set 8 we had the set shown in Figure 2↓, we would additionally have $6 \prec 8'$, i.e. $1 \prec 3 \prec 4 \prec 6 \prec 8' \prec 9$.

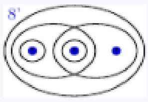


Figure 2 Another three element topology.

3.

Show that the collection $\mathcal{T}_c$ given in Example 4 of § 12 is a topology on the set X. Is the collection

$$T_\infty = \{U | X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

The proof is very similar to Example 3 of §12. The empty set and X are in the collection because their complements are X and the empty set, the complement of any union of open sets is the intersection of the countable complements of these sets, so it is countable as well, finally, the complement of the finite intersection of open sets is the union of the countable complements, so it is countable.

Now, $\mathcal{T}_\infty$ is the trivial topology if X is finite, but if X is infinite then it is not a topology, as, for example, we can partition X into three disjoint sets $X = X_1 \cup X_2 \cup \{x\}$ such that the first two sets are infinite, and then X_1 and X_2 are open but their union is not.

4.

(a) If $\{T_\alpha\}$ is a family of topologies on X, show that $\cap T_\alpha$ is a topology on X. Is $\cup T_\alpha$ a topology on X?

(b) Let $\{T_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections T_α, and a unique largest topology contained in all T_α.

(c) If $X = \{a, b, c\}$, let

$$T_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and } T_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing T_1 and T_2, and the largest topology contained in T_1 and T_2.

(a) The empty set, X, unions of sets from $\cap T_\alpha$ and their finite intersections all lie in every topology T_α (because every set from $\cap T_\alpha$ belongs to every T_α), and, therefore, they all are contained in $\cap T_\alpha$.

At the same time, the union $\cup T_\alpha$ does not have to be a topology. For example, the union of topologies $\{\emptyset, \{a, b, c\}, \{a\}\}$ and $\{\emptyset, \{a, b, c\}, \{b\}\}$ on $\{a, b, c\}$ is not a topology, as it is missing the union of the sets $\{a\}$ and $\{b\}$, i.e. $\{a, b\}$.

(b) The unique largest topology contained in all the topologies of the family $\{T_\alpha\}$ is clearly the intersection of the topologies $\cap T_\alpha$ (it is the largest collection of sets contained in all T_α, and it is a topology according to (a)).

The smallest topology containing all the topologies of the family $\{T_\alpha\}$ can be obtained in two different ways.

First, we can consider all the topologies containing each T_α, and take their intersection. The resulting collection of sets is the smallest topology (by (a)) containing every T_α.

Second, we will prove in Exercise 5, that the smallest topology containing every set from a collection of sets is the one generated by the collection as a subbasis (in general, we may need to include the whole space as a set into the collection, so that it would be indeed a subbasis, but in our case each T_α is already a topology). So, accordingly, we consider $\cup T_\alpha$ as a subbasis for a topology, and then it generates the smallest topology containing every T_α.

> The first approach is less constructive (as it operates with the family of all topologies containing every topology T_α without specifying how to obtain it), but it is more general, as whenever we need to find the smallest set satisfying property P that contains every set T_α, the first thing to do is to check whether the intersection of all sets satisfying P and containing every T_α, satisfies P. If it does (as in our case, where P is "being a topology"), then the intersection is automatically the smallest such set.
>
> The second approach, at the same time, is more constructive, as it prescribes exactly how to obtain a subbasis for the smallest topology, but it requires additional proof to show that the resulting topology is indeed the smallest (see Exercise 5).

(c) The following Figure 1↓ lists the topologies of Figure 12.1.

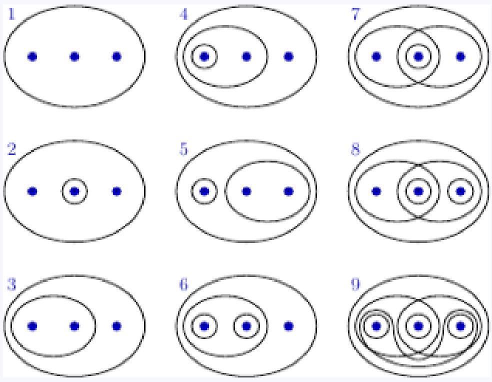


Figure 1 Three element topologies.

The topologies we are asked about are the topologies 4 and 5. Their intersection is the largest topology contained in both, it is the topology $\{\emptyset, X, \{a\}\}$ (the topology 2 where the central point is a). Their union $\{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}\}$ is a subbasis for the smallest topology containing both, all possible intersections of these sets gives as a basis for the topology, which is $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. In fact, the set is closed under unions, so it is already the smallest topology containing both topologies (the topology 8 where the left point is c).

> Note, that if we used the other method to obtain the smallest topology containing both given topologies, we would have to consider all 29 three-element topologies, including those that are not present in the picture above, check which ones contain both topologies given, and then take their intersection. With three elements this at least looks manageable, but the number of topologies grows as $e^{n^2/4}$ with the number of elements n, and the exact number of topologies is unknown for even small values of n. Imagine if our topologies were infinite, or if there were an infinite number of them. This is exactly what I meant when I said that the first method is less constructive.

5.

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Every topology containing the collection $\mathcal{A}$ must contain all unions of sets of $\mathcal{A}$, i.e. it must contain the topology generated by $\mathcal{A}$.

If $\mathcal{A}$ is a subbasis, then every topology containing $\mathcal{A}$ must contain all finite intersections of sets of $\mathcal{A}$, i.e. it must contain the basis generated by the subbasis $\mathcal{A}$.

In both cases, the topology generated by $\mathcal{A}$ contains $\mathcal{A}$, but at the same time is contained in every topology that contains $\mathcal{A}$, hence, it equals the intersection of such topologies (which is the smallest topology containing $\mathcal{A}$).

6.

Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology},\\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K,\\
\mathcal{T}_3 &= \text{the finite complement topology},\\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as basis},\\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a) = \{x | x < a\} \text{ as basis}.
\end{aligned}$$

Determine, for each of these topologies, which of the others it contains.

Let "$\prec$ " be the "being smaller" relation on the set of topologies (as in Exercise 2). Then $3 \prec 1 \prec 2 \prec 4$ and $5 \prec 1 \prec 2 \prec 4$, but 3 and 5 are not comparable. The relations are easy to see using Lemma 13.3. To see that 3 and 5 are not comparable consider point 1 and two open sets $\mathbb{R} - \{0\} \in \mathcal{T}_3$ and $(-\infty, 2) \in \mathcal{T}_5$ containing it. Neither of these open sets contains an open set from the other topology that contains 1 .

7.

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) | a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) | a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

(a) The standard topology is clearly finer than the topology generated by $\mathcal{B}$. To see that they are equivalent consider any set U open in the standard topology. Take any point $x \in U$. Since U is open, and the set of all open intervals is a basis for the standard topology, there is an interval (a, b) that contains x and lies in U. There are two rational points s and t such that $a < s < x < t < b$. The interval (s, t) contains x, lies in U and is a basis element in $\mathcal{B}$.

(b) The lower limit topology is finer than the topology generated by $\mathcal{C}$. Now, for point $\sqrt{2}$ having an open neighborhood $[\sqrt{2}, 2)$ in the lower limit topology, there is no basis element in $\mathcal{C}$ that would contain $\sqrt{2}$ being a subset of $[\sqrt{2}, 2)$.

> The topology is strictly finer than the standard topology, strictly coarser than the lower limit topology, and not comparable to either the K -topology or the countable complement topology or the upper limit topology.

SECTION 14

No exercise

SECTION 15

No exercise

SECTION 16

1.

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

B is open in $A \subset Y$ iff $B = A \cap C$ and C is open in $Y \subset X$ iff $B = A \cap C$, $C = Y \cap D$ and D is open in X iff $B = A \cap Y \cap D$ and D is open in X iff $B = A \cap D$ and D is open in X iff B is open in $A \subset X$.

2.

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Let Y' and Y be the subspaces of $(X, \mathcal{T}')$ and $(X, \mathcal{T})$, respectively. Then, Y' is finer but not necessarily strictly finer than Y. It is finer, because if we change the topology on X from $\mathcal{T}$ to $\mathcal{T}'$ then all subsets of X that were open are still open, and therefore their intersections with Y are still open in Y. It is not necessarily strictly finer as the new open sets from $\mathcal{T}' - \mathcal{T}$ may not produce new open sets in the subspace topology. For example, a one-point subset Y of any topological space X always have the same subspace topology regardless of the topology on X.

3.

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned}
A &= \{x | \tfrac{1}{2} < |x| < 1\}, \\
B &= \{x | \tfrac{1}{2} < |x| \le 1\}, \\
C &= \{x | \tfrac{1}{2} \le |x| < 1\}, \\
D &= \{x | \tfrac{1}{2} \le |x| \le 1\}, \\
E &= \{x | 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.
\end{aligned}$$

A and E are open in $\mathbb{R}$ (as unions of open intervals), and, therefore, in Y. B is the only set open in Y (as the intersection of a larger open set $\{x | \frac{1}{2} < |x| < 2\}$ with Y), but not in $\mathbb{R}$. C and D are not open in Y, as their points $\pm\frac{1}{2}$ belong to their boundaries.

4.

A map $f : X \to Y$ is said to be an ***open map*** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Let $U \subset X \times Y$ be an open set, and $x \in \pi_1(U)$. Then there exists y such that $x \times y \in U$. Since U is open, there is a basis set $A \times B$ in U that contains $x \times y$. Since it is a basis set, A is open in X. Moreover, $x \in A = \pi_1(A \times B) \subset \pi_1(U)$. Therefore, $\pi_1(U)$ is open. Similarly for $\pi_2(U)$.

5.

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

(a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) Does the converse of (a) hold? Justify your answer.

(a) Every basis set in $X \times Y$ is a basis set in $X' \times Y'$ (if one of the sets is empty, then, both topologies are trivial).

(b) Yes (assuming the sets are nonempty). If U is open in X, $x \in X$, V is open in Y, $y \in Y$, then $U \times V$ is open in $X \times Y$ and, therefore, open in $X' \times Y'$. Therefore, there exists a basis set $A \times B$ in $X' \times Y'$ such that it is a subset of $U \times V$ and it contains $x \times y$. Therefore, there are open sets $A \in \mathcal{T}'$ and $B \in \mathcal{U}'$ such that $x \in A \subset U$ and $y \in B \subset V$. So, U is open in X' and V is open in Y'.

> If $\mathcal{T}' \supsetneq \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then, assuming $Y \neq \emptyset$, the product topology on $X' \times Y'$ is strictly finer than the product topology on $X \times Y$. Indeed, suppose U is open in X' but not in X. Then, there is some $x \in U$ such that for every set W open in X, $x \in W$ implies $W \not\subset U$. Now, $U \times Y$ is open in $X' \times Y'$. If it were open in $X \times Y$, there would be a basis element $A \times B \subset X \times Y$ such that $x \times y \in A \times B \subset U \times Y$ for some $y \in Y$, and then we would have $x \in A \subset U$ where A is open in X, contradicting the assumption on x.
>
> In fact, all of these could have been easily proved using the following lemma.
>
> **Lemma** If A and B are nonempty, then $A \times B \subset X \times Y$ is open iff A and B are open in X and Y, respectively.
>
> This lemma should have been given in the text or exercises, unless I missed it.

6.

Show that the countable collection

$$\{(a, b) \times (c, d) | a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

The collection of sets (a, b) such that $a, b \in \mathbb{Q}$ is a basis for the standard topology on $\mathbb{R}$ (see Exercise 8(a) of §13).

7.

Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

No, for example, $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is a convex subset of $(\mathbb{Q}, <)$ which is not an interval in $\mathbb{Q}$.

8.

If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$. In each case it is a familiar topology.

A basis for $\mathbb{R}_l \times \mathbb{R}$ is the collection of sets $[a, b) \times (c, d)$. A basis for $\mathbb{R}_l \times \mathbb{R}_l$ is the collection of sets $[a, b) \times [c, d)$. In both cases the intersection of a basis set and the line is an interval on the line, either [) or () or (] or [] (we consider the line as a copy of $\mathbb{R}$ having a particular direction in the plane). And vice versa, that is if any such interval can be obtained as the intersection of the line with some basis set, then all same type intervals on the line can be obtained as the intersection of the line with some basis sets.

Now, not all combinations of these intervals are possible in all cases. If the line is vertical, then the respective intervals are () (produced by the intersection with $[a, b) \times (c, d)$), and [) or (] (produced by the intersection with $[a, b) \times (c, d)$, and depending on the direction of the line), and respective topologies are $\mathbb{R}$, and $\mathbb{R}_l$ or $\mathbb{R}_u$. If the line is horizontal, then in both cases the topology is either $\mathbb{R}_l$ or $\mathbb{R}_u$ depending on the direction of the line. If it has some slope then the first topology may generate either ([) and ()) or ((] and ()), depending on the direction of the line. That is, it is either the lower limit or the upper limit topology. For the second topology, if the line is downward sloping, we have intersections of the form [], [), (] and (), i.e. the discrete topology. Finally, if the line is upward sloping then the second topology generates a basis consisting of either [) or (], i.e. the topology is either $\mathbb{R}_l$ or $\mathbb{R}_u$.

Direction	$\mathbb{R}_l \times \mathbb{R}$	$\mathbb{R}_l \times \mathbb{R}_l$
↑	$\mathbb{R}$	$\mathbb{R}_l$
↗	$\mathbb{R}_l$	$\mathbb{R}_l$
→	$\mathbb{R}_l$	$\mathbb{R}_l$
↘	$\mathbb{R}_l$	$\mathbb{R}_d$
↓	$\mathbb{R}$	$\mathbb{R}_u$
↙	$\mathbb{R}_u$	$\mathbb{R}_u$
←	$\mathbb{R}_u$	$\mathbb{R}_u$
↖	$\mathbb{R}_u$	$\mathbb{R}_d$

9.

Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Every interval in the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the union of open sets in $\mathbb{R}_d \times \mathbb{R}$. At the same time, a basis for $\mathbb{R}_d \times \mathbb{R}$ is the collection of sets $\{x\} \times (a, b)$, where each is an interval in the dictionary order topology. Therefore, the topologies are the same.

These topologies are strictly finer then $\mathbb{R}^2$, see Exercise 5. For example, $\{0\} \times \mathbb{R}$ is open in the product topology $\mathbb{R}_d \times \mathbb{R}$ (and, hence, in the dictionary topology on $\mathbb{R} \times \mathbb{R}$), but not in $\mathbb{R}^2$.

10.

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

The first two topologies are not comparable. Indeed, $[0, 1] \times (0.5, 1]$ open in the first topology contains the point $(0, 1)$, but does not contain any its open neighborhood in the second topology, and $\{0\} \times (0, 1)$ open in the second topology contains the point $(0, 0.5)$, but does not contain any its open neighborhood in the first topology.

The third topology is strictly finer than the first and second one. Indeed, according to Exercise 9, the third topology is generated by the basis consisting of the sets $\{x\} \times ((a, b) \cap [0, 1])$. So, every basis element $((a, b) \cap [0, 1]) \times ((c, d) \cap [0, 1])$ of the first topology is the union of some basis sets of the third topology, and every basis set $(a, b) < (x, y) < (c, d)$ of the second topology is the union of some basis sets of the third topology as well. The fact that the third topology is strictly finer than the first and second topologies follows from the fact that the first and second topologies are not comparable.

SECTION 17

1.

Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\emptyset$ and X are in $\mathcal{C}$, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection

$$\mathcal{T} = \{X - C | C \in \mathcal{C}\}$$

is a topology on X.

The proof is similar to Theorem 17.1, just the other direction. The empty set and X are in $\mathcal{T}$ because they are complements of X and $\emptyset$, respectively, which are in $\mathcal{C}$. And the complement $X - B$, where B is an arbitrary union (a finite intersection) of some elements of $\mathcal{T}$, is the intersection (the finite union) of the complements of these elements, which belong to $\mathcal{C}$, therefore, $B \in \mathcal{T}$.

2.

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

If Y is closed in X, then we have A is closed in Y iff $A = Y \cap B$ for some B closed in X iff $A \subset Y$ and A is closed in X.

3.

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

$A \times B = X \times Y - (((X - A) \times Y) \cup (X \times (Y - B)))$ is closed.

4.

Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

$U - A = U \cap (X - A)$ is open in X. $A - U = A \cap (X - U)$ is closed in X.

5.

Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subset [a, b]$. Under what conditions does equality hold?

$[a, b] = X - ((-\infty, a) \cup (b, +\infty))$ is closed and contains (a, b), so it contains the closure of (a, b). It equals the closure iff both endpoints are limit points of the interval, i.e. if (a, b) is not empty and for every $x \in (a, b)$ there are $s, t \in (a, b)$ such that $a < s < x < t < b$. This is equivalent to the requirement that a has no immediate successor, and b has no immediate predecessor. Otherwise, if a has an immediate successor c then $(-\infty, c)$ is an open set containing a that does not intersect (a, b), and, similarly, if b has an immediate predecessor c then $(c, +\infty)$ is an open set containing b that does not intersect (a, b).

6.

Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{\cup A_\alpha} \supset \cup \overline{A}_\alpha$; give an example where equality fails.

(a) $\overline{B}$ is closed and contains B, and, hence, A, therefore, it contains the closure of A.

(b) (c) Every closed set containing the union of A_α contains each set A_α, and, hence, its closure $\overline{A}_\alpha$. Therefore, the closure of the union (the left hand side), which is the intersection of such closed sets, contains all $\overline{A}_\alpha$ and their union (the right hand side).

Now, the other direction. For any finite number of sets $\{A_i\}_{i=1}^n$: $\overline{\cup_{i=1}^n A_i} = \cap_{E \supset (\cup_{i=1}^n A_i), E \text{ is closed}} E = \cap_{\{C_i \supset A_i, C_i \text{ is closed}\}_{i=1}^n} (\cup_{i=1}^n C_i) = \cup_i \overline{A}_i$ (in fact, this is a proof for both directions in the case of a finite number of sets). For infinite number of sets, some collections of closed sets C_i have non-closed unions so that the intersection over all closed sets E is the intersection over a subcollection of sets $\{C_i\}_{i=1}^n$ (which makes the right hand side smaller, as we intersect over a larger collection of sets). This suggests that the right hand side is a proper subset of the left hand side if there is a collection of closed sets C_α containing A_α, respectively, such that their union is not closed, and there is no closed set contained in the union that would contain all A_α. For example, if we take $A_n = \{\frac{1}{n}\}$ and $A = \cup_{n \in \mathbb{Z}_+} A_n$ then $\overline{A} = A \cup \{0\} \supsetneq \cup_n \overline{A}_n = A$ (here, for example, $C_n = [\frac{1}{2n}, 2]$ is closed and contains A_n, but the union $\cup_{n \in \mathbb{Z}_+} C_n = (0, 2]$ is not closed, and there is no closed E such that $A \subset E \subset (0, 2]$, so the right hand side has an "extra" intersection with $(0, 2]$ which the left hand side does not have).

7.

Criticize the following "proof" that $\overline{\cup A_\alpha} \subset \cup \overline{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\cup A_\alpha}$, then every neighborhood U of x intersects $\cup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \cup \overline{A}_\alpha$.

The problem of the proof is that for different U ' s the set A_α may be different, so that we actually did not show that every neighborhood U of x intersects some particular A_α, we showed that every neighborhood U of x intersects some A_α where α may depend on the choice of U, and we cannot conclude from this that x lies in the closure of some particular A_α. As an example, we use the one from the previous exercise, $A_n = \{\frac{1}{n}\}$, $A = \cup_{n \in \mathbb{Z}_+} A_n$. $0 \in A$, and $I_n = (-\frac{1}{n}, \frac{1}{n})$ contains 0 and intersects A at points $\frac{1}{m}$ for $m > n$, so I_n does, indeed, intersect sets A_m for $m > n$, but no set A_m intersects every I_n.

8.

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(b) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.

(c) $\overline{A - B} = \overline{A} - \overline{B}$.

(a) (b) The inclusion $\subset$ holds in both cases. Using 6(a), $\overline{\cap A_\alpha} \subset \overline{A_\alpha}$ for every α, and, therefore, $\overline{\cap A_\alpha} \subset \cap \overline{A_\alpha}$. The equality does not hold in general, as A_α ' s might have some common limit points which are not in their intersection and not limit points of the intersection. For example, the intersection can be empty, while there is a common limit point: $(0,1)$ and $(1,2)$ is an example. Another example is $A = \mathbb{Q} \subset \mathbb{R}$ and $B = \mathbb{R} - \mathbb{Q}$, where $\overline{A \cap B} = \emptyset$ and $\overline{A} \cap \overline{B} = \mathbb{R}$.

(c) The inclusion $\supset$ holds, and, in fact, $\overline{A - B} - \overline{B} = \overline{A} - \overline{B}$. Therefore, the inclusion is proper iff the closures of $A - B$ and B share some points. The same examples illustrate the fact: if $A = (0,1)$ and $B = (1,2)$, then $\overline{A} - \overline{B} = \overline{A - B} - \overline{B} = [0,1) \subset \overline{A - B} = [0,1]$, and if $A = \mathbb{Q}$ and $B = \mathbb{R} - \mathbb{Q}$, then $\overline{A} - \overline{B} = \overline{A - B} - \overline{B} = \mathbb{R} - \mathbb{R} = \emptyset \subset \overline{A - B} = \mathbb{R}$.

Proof 1. Using 6(a,b), $\overline{A} - \overline{B} = (\overline{A - B} \cup \overline{A \cap B}) - \overline{B} = \overline{A - B} - \overline{B}$.

Proof 2. Assume that x is in $\overline{A} - \overline{B}$, i.e. every neighborhood of x intersects A but there is some neighborhood V of x that does not intersect B. Suppose there is some neighborhood U of x that does not intersects $A - B$, then the neighborhood $U \cap V$ of x does not intersects $A - B$ and it does not intersects B, i.e. it does not intersects A at all, contradiction. So, $x \in \overline{A - B}$ but $x \notin \overline{B}$. The other direction is immediate using 6(a).

9.

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

$(x, y) \in \overline{A \times B}$ iff for every basis element $U \times V$ of $X \times Y$, where U is open in X and V is open in Y, such that $(x, y) \in U \times V$ the intersection $U \times V \cap A \times B \neq \emptyset$ iff for every neighborhood U of x and every neighborhood V of y, $U \cap A \neq \emptyset \neq V \cap B$ iff $x \in \overline{A}$ and $y \in \overline{B}$ iff $(x, y) \in \overline{A} \times \overline{B}$.

10.

Show that every order topology is Hausdorff.

If $x < y$ then either there is some c such that $x < c < y$ and $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighborhoods of x and y, respectively, or there is no element between x and y, so that $(-\infty, y)$ and $(x, +\infty)$ are disjoint neighborhoods of x and y, respectively.

11.

Show that the product of two Hausdorff spaces is Hausdorff.

Take any two points (x, y) and (x', y') in the product space. If $x = x'$, let $U = U'$ be any neighborhood of $x = x'$, otherwise let U and U' be disjoint neighborhoods of x and x'. Similarly, if $y = y'$, let $V = V'$ be any neighborhood of $y = y'$, otherwise let V and V' be disjoint neighborhoods of y and y'. If $(x, y) \neq (x', y')$ then U and U' are disjoint and/or V and V' are disjoint. Hence, $U \times V$ and $U' \times V'$ are disjoint neighborhoods of (x, y) and (x', y'), respectively.

12.

Show that a subspace of a Hausdorff space is Hausdorff.

If X is Hausdorff, $x, y \in Y \subset X$, $x \neq y$, and U and V are disjoint neighborhoods of x and y in X, respectively, then $Y \cap U$ and $Y \cap V$ are their disjoint neighborhoods in Y.

13.

Show that is Hausdorff if and only if the ***diagonal*** is closed in .

Cool! is closed in iff for every there are two open sets containing and containing such that for no point iff any pair of different points have disjoint neighborhoods.

14.

In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

For any point and any its neighborhood (which in the finite complement topology is $\mathbb{R}$ minus a finite number of points) there is only finite number of points in the sequence that may be not in the neighborhood (all points of the sequence are different), so the sequence converges to every point in $\mathbb{R}_{fc}$.

15.

Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

X is a T_1-space iff $\forall x \in X$, $\{x\}$ is closed iff $\forall x \in X$, $X - \{x\}$ is open iff $\forall x \in X$ and $y \in X - \{x\}$, there is an open set U such that $y \in U \subset X - \{x\}$.

16.

Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of § 13.

(a) Determine the closure of the set $K = \{1/n | n \in \mathbb{Z}_+\}$ under each of these topologies.

(b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

(a) and (b) The closure is the union of the set and its limit points. So we describe only limit points.

Topology:	$\mathcal{T}_1 = \mathbb{R}$	$\mathcal{T}_2 = \mathbb{R}_K$	$\mathcal{T}_3 = \mathbb{R}_{fc}$	$\mathcal{T}_4 = \mathbb{R}_u$	$\mathcal{T}_5 = \mathbb{R}_{-\infty}$
K'	$\{0\}$	$\emptyset$	$\mathbb{R}$	$\emptyset$	$\overline{\mathbb{R}}_+$
Hausdorff	+	+	-	+	-
T_1	+	+	+	+	-

17.

Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

The topology $\mathcal{C}$ is given by $[a, b)$ where $a, b \in \mathbb{Q}$. We noted in the solution for Exercise 8 of §13 that this topology is strictly finer than the standard topology and strictly coarser than the lower limit topology. The finer is the topology, the (weakly) smaller is the closure of any set, as there are more neighborhoods of points not in the set. So, we expect the closures of the two sets in both topologies to be subsets of their closures in $\mathbb{R}$, i.e. $[0, \sqrt{2}]$ and $[\sqrt{2}, 3]$.

In the lower limit topology, $\overline{A} = [0, \sqrt{2})$ and $\overline{B} = [\sqrt{2}, 3)$, as the closure of any interval (a, b) is $[a, b)$ (point b has the neighborhood $[b, +\infty)$ not intersecting the interval). So, additionally, in $\mathcal{C}$ we would expect the closures to be also supersets of these half-open intervals.

In $\mathcal{C}$, $\overline{A} = [0, \sqrt{2}]$ and $\overline{B} = [\sqrt{2}, 3)$. Indeed, for the set A the argument is close to the one described in the solution of Exercise 8 of §13: the set $[\sqrt{2}, 2)$ is open in the lower limit topology, but not in $\mathcal{C}$, where every open interval containing $\sqrt{2}$ has a rational lower bound and, hence, a point below $\sqrt{2}$ from the interval A.

18.

Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 | n \in \mathbb{Z}_+\},$$
$$B = \{(1 - 1/n) \times \tfrac{1}{2} | n \in \mathbb{Z}_+\},$$
$$C = \{x \times 0 | 0 < x < 1\},$$
$$D = \{x \times \tfrac{1}{2} | 0 < x < 1\},$$
$$E = \{\tfrac{1}{2} \times y | 0 < y < 1\}.$$

The ordered square is $[0,1] \times [0,1]$ in the dictionary order topology. The closures are the unions of the sets and their limit points, so we describe limit points only. $A' = \{(0,1)\}$ as every neighborhood of $(0,1)$ is $((0,a),(b,c))$ for some $a < 1$ and $b > 0$, which contains a point $(1/n, 0)$ for some sufficiently large n. Similarly, $B' = \{(1,0)\}$, $C' = D' = (0,1] \times \{0\} \cup [0,1) \times \{1\}$, and $E' = \{\frac{1}{2}\} \times [0,1]$.

19.

If $A \subset X$, we define the ***boundary*** of A by the equation

$$\mathrm{Bd} A = \overline{A} \cap \overline{(X - A)}.$$

(a) Show that $\mathrm{Int} A$ and $\mathrm{Bd} A$ are disjoint, and $\overline{A} = \mathrm{Int} A \cup \mathrm{Bd} A$.

(b) Show that $\mathrm{Bd} A = \emptyset \Leftrightarrow A$ is both open and closed.

(c) Show that U is open $\Leftrightarrow \mathrm{Bd} U = \overline{U} - U$.

(d) If U is open, is it true that $U = \mathrm{Int}(\overline{U})$? Justify your answer.

(a) $x \in \overline{A}$ iff for every open U, $x \in U$ implies $U \cap A \neq \emptyset$ iff there is an open V such that $x \in V \subset A$ or for every open U, $x \in U$ implies $U \cap A \neq \emptyset \neq U \cap (X - A)$ iff $x \in Int(A)$ or $x \in Bd(A)$. Also, the two cases are disjoint (either there is V that does not intersect $X - A$ or every U intersects $X - A$), so that $Int(A) \cap Bd(A) = \emptyset$.

(b) $Bd(A) = \emptyset$ iff $\forall x$ there is an open set U s.t. $x \in U \subset A$ or $x \in U \subset (X - A)$ iff $A, (X - A) \in \mathcal{T}$.

(c) U is open iff (according to (a)) $Bd(U) = \overline{U} - Int(U) = \overline{U} - U$.

(d) No, $U \subset \overline{U}$ is open, therefore, $U \subset Int(\overline{U})$, but, for example, $\mathbb{R} - \{0\} \subsetneq Int(\overline{\mathbb{R} - \{0\}}) = \mathbb{R}$.

20.

Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y | y = 0\}$

(b) $B = \{x \times y | x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y | x \text{ is rational}\}$

(e) $E = \{x \times y | 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y | x \neq 0 \text{ and } y \leq 1/x\}$

Set	Bd	+	Int	=	Cl
A	A		$\emptyset$		A
B	$\{0\} \times \mathbb{R} \cup$ $\mathbb{R}_+ \times \{0\}$		B		$\overline{\mathbb{R}}_+ \times \mathbb{R}$
C	$\mathbb{R}_- \times \{0\} \cup$ $\{0\} \times \mathbb{R}$		$\mathbb{R}_+ \times \mathbb{R}$		$\mathbb{R}_- \times \{0\} \cup \overline{\mathbb{R}}_+ \times \mathbb{R}$
D	$\mathbb{R} \times \mathbb{R}$		$\emptyset$		$\mathbb{R} \times \mathbb{R}$
E	$\{(x,y) \| \|x\| = \|y\|$ or $x^2 - y^2 = 1\}$		$\{(x,y) \| 0 < x^2 - y^2 < 1\}$		$\{(x,y) \| 0 \leq x^2 - y^2 \leq 1\}$
F	$\{(x,y) \| (x \neq 0$ and $y = 1/x)$ or $x = 0\}$		$\{(x,y) \| x \neq 0$ and $y < 1/x\}$		$\{(x,y) \| (x \neq 0$ and $y \leq 1/x)$ or $x = 0\}$

21.

* (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \overline{A}$ and complementation $A \to X - A$ are functions from this collection to itself.

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

(a) We use the following notation here: we define four operations as follows, $iA = Int(A)$, $cA = \overline{A}$, $bA = Bd(A)$, $xA = X - A$. So that, for example, $xicA = X - Int(\overline{A})$ and $cA = iA \cup bA$ (Exercise 19(a)). First, several statements.

1. i and c are monotone. If $A \subset B$ then $i(A) \subset i(B)$ and $c(A) \subset c(B)$. Indeed, every open set contained in A is contained in B as well, and every closed set containing B contains A as well.

2. $ccA = cA$, $xxA = A$, $iiA = iA$, but $bbA \neq bA$ in general. Indeed, cA is closed, $xxA = X - (X - A) = A$, and iA is open, but $bb(\mathbb{Q}) = b(\mathbb{R}) = \emptyset$.

3. $bA = bxA$, and $cA = iA \cup bA$ and $X = iA \cup bA \cup ixA$ are partitions (the sets on the right are disjoint). This follows from the definition of the boundary of a set, Exercise 19(a), and the following facts: $X = A \cup xA \subset cA \cup cxA$, $iA \subset A$, and $ixA \subset xA$.

4. $xcA = ixA$, $xiA = cxA$, and $iA = xcxA$. These follow from property 3 immediately, indeed, $ixA = x(iA \cup bA) = xcA$. Hence, using property 2, $cxA = xxcxA = xixxA = xiA$, and $iA = xxiA = xcxA$.

5. $iA \subset icA$, $ciA \subset cA$. Indeed, $iA \subset cA$ where iA is open, and cA is closed. Both inclusions can be proper: consider $A = \{0\} \cup [1, 2) \cup (2, +\infty)$.

6. $icicA = icA$, $ciciA = ciA$. Using properties 5 and 2, $cicA \subset ccA = cA$, but $icA \subset cicA$, therefore, using property 1, $icA = iicA \subset icicA \subset icA$. Hence, $icicA = icA$. Now, using this and properties 2 and 4, $ciciA = cicixxA = xicicxA = xicxA = cixxA = ciA$.

7. icA, ciA and A are, in general, not comparable by inclusion. Consider the following set: $A = [0, 1] \cap \mathbb{Q} \cup (2, +\infty)$. Then $icA = (0, 1) \cup (2, +\infty)$ and $ciA = [2, +\infty)$, and each of the following points belongs to one set only: 0, $1/\sqrt{2}$ and 2.

Property 4 shows that even if we take Int as an additional operation we are not going to get more sets.

Property 2 shows that in a sequence of operations of closure and complementation we can substitute any two operations of the same type by one or zero operations of the same type. Therefore, we may construct only two sequences of sets: starting with two initial sets A and xA we apply the sequence of operations c, then x, then c etc.

Further, we may use properties 4 and 2 to move all operations x to the right (by switching it with another operation on the right: $xc = ix$ and $xi = cx$) and to cancel two x ' s in a row ($xxA = A$).

Accordingly, the first few members of the sequence starting with A are A, cA, $xcA = ixA$, $cxcA = cixA$, $xcxcA = icA$, $cxcxcA = cicA$, $xcxcxcA = icixA$, $cxcxcxcA = cicixA$, and the second sequence starts with xA, cxA, $xcxA = iA$, $cxcxA = ciA$, $xcxcxA = icxA$, $cxcxcxA = cicxA$, $xcxcxcxA = iciA$, $cxcxcxcxA = ciciA$. Note that, according to property 6, in each sequence the 8th member equals the 4th member, therefore, there can be at most 7 distinct sets in each sequence, and 14 sets overall.

(b) Every set of each sequence presented above starting from the second one is open or closed. Let A be neither open nor closed. This ensures that no other set in the sequence equals A or xA. The following table divides the sets in the sequences into 4 different groups according to whether a set is a sequence of operations i and c applied to A or xA, and whether it is closed or open. The table also shows inclusion relations among the sets within one group, using properties 1, 2 and 5 (property 7 tells us that at least some sets from different groups can be made incomparable).

	A	xA
open	$iA \subset iciA \subset icA$	$ixA \subset icixA \subset icxA$
closed	$ciA \subset cicA \subset cA$	$cixA \subset cicxA \subset cxA$

The sets in different groups can easily be made different. For example, closed sets and open sets are different in the standard topology on the real line (with the exception for the empty set and whole space). Also, we can easily have some different points in sequences starting from A and xA (both operations i and c preserve interior points). So, the main challenge is to try to make all sets within one group different.

For example, to make ciA and $cicA$ different, we need

a) make iA and icA different, so A must have a limit point such that it is an interior point of cA but not an interior point of A, which can be easily done by removing a point from an interval;

b) make iA and icA so different, that their closures would differ as well, so we might want to remove a whole dense set of points from an interval, but such that the closure of the remaining points is still the closed interval, and the set we used to illustrate property 7 is an example of how this can be done.

But that example will not work for $cicA$ and cA. We also need to add an isolated point to the set, which will disappear in icA.

Having a set like this, we already have all sets in the group of closed sets derived from A different. This also implies that icA is different from iA and $iciA$. So, additionally, we only need to ensure that iA and $iciA$ are different, but this can be easily done if we include a punctured interval, that we would need for xA anyway.

So, A consists of an isolated point, an interval with a dense subset removed, and an interval with a point removed.

Now, the same should work for the complement xA as well, but the isolated point of A becomes an interval with a point removed in xA, and vice versa, and the interval with a dense set of points removed becomes the dense set. So, basically, we don' t need any additional constructions for xA.

Here is an example:

	A	xA
	$\{-2\}\cup([-1,1]\cap\mathbb{Q})$ $\cup(1,2)\cup(2,+\infty)$	$(-\infty,-2)\cup(-2,-1)$ $\cup([-1,1]-\mathbb{Q})\cup\{2\}$
c	$\{-2\}\cup[-1,+\infty)$	$(-\infty,1]\cup\{2\}$
$xc=ix$	$(-\infty,-2)\cup(-2,-1)$	$(1,2)\cup(2,+\infty)$
$cxc=cix$	$(-\infty,-1]$	$[1,+\infty)$
$xcxc=ic$	$(-1,+\infty)$	$(-\infty,1)$
$cxcxc=cic$	$[-1,+\infty)$	$(-\infty,1]$
$xcxcxc=icix$	$(-\infty,-1)$	$(1,+\infty)$

In both cases the next closure equals the 4th member of the sequence. So, there are maximum of 14 sets that can be obtained from a given set by taking closure and complement (and interior, as the latter can be expressed in terms of the former two operations), and above is an example of a set for which all 14 sets are different.

SECTION 18

1.

Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ definition of continuity implies the open set definition.

Let $V \subset \mathbb{R}$ be open and f be continuous according to the $\epsilon - \delta$ definition. For every $x \in f^{-1}(V)$, $f(x) = y \in V$ and $\exists\, \delta > 0$ such that $(y - \delta, y + \delta) \subset V$. Further, there is $\epsilon > 0$ such that $f((x - \epsilon, x + \epsilon)) \subset (y - \delta, y + \delta)$ implying $(x - \epsilon, x + \epsilon) \subset f^{-1}(V)$. Therefore, $f^{-1}(V)$ is open.

2.

Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X , is it necessarily true that $f(x)$ is a limit point of $f(A)$?

No, for example, f can be a constant function.

To elaborate a bit. Suppose we wanted to prove this statement true. Having an open neighborhood V of $f(x)$ we would want to show that it intersects $f(A)$ in a point different from $f(x)$. $U = f^{-1}(V)$ is an open neighborhood of x , hence, there is some $z \in f^{-1}(V) \cap A - \{x\}$, and $f(z) \in V \cap f(A)$. By this, we only showed that $f(x) \in \overline{f(A)}$, but it can be the case that for every $z \in f^{-1}(V) \cap A$, $f(z) = f(x)$. So, if $f(x)$ has a neighborhood V such that $f(f^{-1}(V) \cap A) = \{f(x)\}$, then $f(x)$ is not a limit point of $f(A)$.

3.

Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.

(a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

(a) Both mean that "every set open in $\mathcal{T}$ is open in $\mathcal{T}'$ ".

(b) Follows from (a), as i^{-1} is the same identity function.

4.

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \text{ and } g(y) = x_0 \times y$$

are imbeddings.

Let $V = f(U)$. Then V is open in $X \times \{y_0\}$ iff $U \times \{y_0\}$ is open in $X \times \{y_0\}$ iff (using Exercise 4 of §16 one way, and by definition the other way) U is open in X . Similarly for g .

5.

Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

In both cases $f(x) = (x - a)/(b - a)$ works as a homeomorphism (I believe Munkres means $a < b$ in both cases, especially in the second case). The inverse function is $g(y) = (b - a)y + a$. Both are continuous (as Munkres says, "these are familiar facts from calculus").

6.

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

For example, $f(x) = x$ if $x \in \mathbb{Q}$, or 0 otherwise. For every x, the image of the open interval $I_\epsilon = (x - \epsilon, x + \epsilon)$, $\epsilon > 0$, is $(I_\epsilon \cap \mathbb{Q}) \cup \{0\}$. So, if $x \neq 0$, then, regardless of the value of $f(x)$, for no $\epsilon > 0$ the image of I_ϵ is a subset of, say, $(f(x) - |x|, f(x) + |x|)$. But if $x = 0$, then for every $\epsilon < \delta$, the image of I_ϵ is a subset of I_δ.

7.

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_l$ to $\mathbb{R}$.

(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_l$? As maps from $\mathbb{R}_l$ to $\mathbb{R}_l$? We shall return to this question in Chapter 3.

> When we make the topology of the domain finer, there are, in general, more continuous functions, as some preimages of open sets that were not open can now become open. Similarly, we may have more continuous functions if we make the topology of the range coarser. The opposite is true if we make the topology of the domain coarser or the topology of the range finer, i.e. in these cases some continuous functions may become not continuous. However, when we make both domain and range finer (or coarser) at the same time, the set of continuous functions may change unpredictably. After all, if both domain and range have indiscrete topology, every function is continuous, but the same is true if both have the discrete topology. So, while changing the topology from one to the other, we may first loose some continuous functions but then gain them again.
>
> In our case, this means that we expect all standard (i.e. as a function from $\mathbb{R}$ to $\mathbb{R}$) continuous functions to be continuous when considered as functions from $\mathbb{R}_l$ to $\mathbb{R}$ (the domain is strictly finer), and we can have more continuous functions (according to (a) all continuous from the right functions become continuous in these topologies). However, not every standard continuous function is going to be continuous as a function from $\mathbb{R}$ to $\mathbb{R}_l$. Of course, all constant functions are still going to be continuous. But are there any others? And in the case of functions from $\mathbb{R}_l$ to $\mathbb{R}_l$, we cannot even say (without further investigation) whether any standard continuous function is still going to be continuous, or whether there going to be any new continuous functions.

(a) If V is open in $\mathbb{R}$ and $x \in f^{-1}(V)$ then $y = f(x) \in V$, and for some $\delta > 0$, $(y - \delta, y + \delta) \subset V$. Therefore, $\exists \epsilon > 0$ such that $f([x, x + \epsilon)) \subset (y - \delta, y + \delta) \subset V$ and $[x, x + \epsilon) \subset f^{-1}(V)$. Thus, $f^{-1}(V)$ is open in $\mathbb{R}_l$.

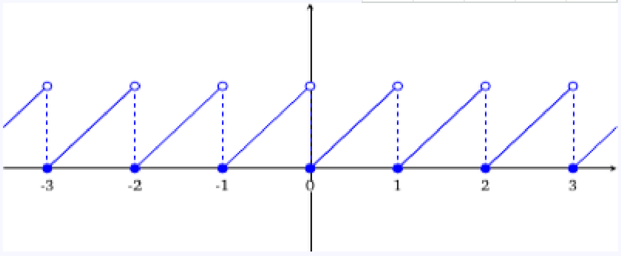
In fact, the opposite is also true. Indeed, if $f : \mathbb{R}_l \to \mathbb{R}$ is continuous, then for every $\delta > 0$, $I = (f(x) - \delta, f(x) + \delta)$ is open, $f^{-1}(I)$ is open and contains x, there is a basis element $[x, x + \epsilon)$ such that $[x, x + \epsilon) \subset f^{-1}(I)$, hence, $f([x, x + \epsilon)) \subset I$. So, we can say that a function is continuous from the right iff it is continuous as a function from $\mathbb{R}_l$ to $\mathbb{R}$.

(b) From $\mathbb{R}$ to $\mathbb{R}_l$ only constant functions are continuous. Indeed, the inverse image of any $[a, b)$ has to be both open and closed, and there are only two such sets in $\mathbb{R}$, namely, the empty set and $\mathbb{R}$.

Now, from $\mathbb{R}_l$ to $\mathbb{R}_l$. $f : \mathbb{R}_l \to \mathbb{R}_l$ is continuous iff for every $x \in \mathbb{R}$ and $\delta > 0$, there exists $\epsilon > 0$ such that $f([x, x+\epsilon)) \subset [f(x), f(x)+\delta)$. This can be proved exactly the same way as we did in (a). Now, what this $\epsilon - \delta$ definition of continuity means... f must be continuous from the right and locally non-decreasing from the right.

> As we noted in the beginning, indeed, in this case some standard continuous functions are not continuous anymore (for example, the strictly decreasing $f(x) = -x$ where $f^{-1}([0, 1)) = (-1, 0]$ is not open), but we also have some new continuous functions, such as the one shown in Figure 1↓. The function also illustrates that a "locally non-decreasing from the right" function does not have to be non-decreasing (from both sides) even locally.

Figure 1 A continuous funct



8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x | f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

(a) $\{x | f(x) > g(x)\} = \cup_{y \in Y} \{x | f(x) > y > g(x)\} \cup \cup_{y < y' : (y, y') = \emptyset} \{x | f(x) > y, g(x) < y'\}$ is open as all sets on the right are intersections of two open sets (preimages of open rays).

(b) $h(x) = f(x)$ on $\{x | f(x) \leq g(x)\}$, and $h(x) = g(x)$ on $\{x | f(x) \geq g(x)\}$, therefore, h restricted to these sets is continuous, and both sets are closed by (a). Using the pasting lemma, h is continuous.

9.

Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \cup_\alpha A_\alpha$. Let $f: X \to Y$; suppose that $f|A_\alpha$, is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

(c) An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

(a) The pasting lemma applied several times. Or, we can just say that for a closed $B \subset Y$, $f^{-1}(B) = \cup_\alpha (f|A_\alpha)^{-1}(B)$ (this follows from the fact that $\cup_\alpha A_\alpha = X$) is a finite union of closed sets (this follows from the fact that every A_α is closed).

(b) $f(x) = 0$ on $[1/(n+1), 1/n]$, $f(x) = 1$ on $(-\infty, 0]$ are continuous, but f on $(-\infty, 1]$ is not. The reason is that if the collection is not finite, the union of preimages as in (a) may be not closed, as is the case for $f^{-1}(\{0\})$, for example.

(c) Let B be a closed subset of Y. Then $A = f^{-1}(B) = \cup_\alpha (f|A_\alpha)^{-1}(B)$ (this follows from the fact that $\cup_\alpha A_\alpha = X$). Suppose $x \notin A$. There is a neighborhood U of x such that it intersects only a finite number of sets in the collection: $A_1, \ldots, A_n$. For each $i = 1, \ldots, n$: $(f|A_i)^{-1}(B) = S_i$ is closed in A_i and, therefore, closed in X (as A_i is closed). Moreover, $x \notin S_i$. Hence, there is a neighborhood U_i of x such that $U_i \cap S_i = \emptyset$. The intersection $U' = U \cap \cap_i U_i$ is a neighborhood of x such that $U' \cap A = \emptyset$. We conclude that A is closed.

10.

Let $f: A \to B$ and $g: C \to D$ be continuous functions. Let us define a map $f \times g: A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

Let $U \times V$ be a basis element for the product topology of $B \times D$, then $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$. Since f and g are continuous, the inverse images on the right are open, and their product is open in $A \times C$.

11.

Let $F: X \times Y \to Z$. We say that F is ***continuous in each variable separately*** if for each y_0 in Y, me map $h: X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k: Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

Let $x \in h^{-1}(A)$ where A is open in Z, then $F(x, y_0) \in A$ and there is a basis neighborhood $U \times V$ of $x \times y_0$ in $X \times Y$ such that $U \times V \subset F^{-1}(A)$. It follows that $h(U) \subset A$, $x \in U \subset h^{-1}(A)$, and, therefore, $h^{-1}(A)$ is open.

12.

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2+y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous.

(a) If $y_0 = 0$ then $F(x, y_0) = 0$ is continuous, otherwise $F(x, y_0) = xy_0/(x^2 + y_0^2)$ is continuous as well.

(b) $g(x) = \begin{cases} \frac{1}{2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(c) $F(x \times y) = \frac{1}{2}$ implies $x \times y \neq 0 \times 0$ and $(x-y)^2 = 0$. Therefore, $F^{-1}(\{\frac{1}{2}\}) = \{(x,x) | x \neq 0\}$ is not closed.

13.

Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Let g and h be two continuous functions on $\overline{A}$ such that they both agree with f on A. Let $C = \{x \in \overline{A} | g(x) = h(x)\}$. In particular, $A \subset C \subset \overline{A}$. According to Theorem 18.4, $(g, h) : \overline{A} \to Y \times Y$ is continuous, and $C = (g,h)^{-1}(\triangle)$ where $\triangle = \{(y,y) | y \in Y\}$. According to Exercise 13 of §17, $\triangle$ is closed in $Y \times Y$ as Y is Hausdorff, and, hence, C is closed. Therefore, $\overline{A} \subset C \subset \overline{A}$.

> The part "if f may be extended" is needed, because not every continuous function can be extended onto the closure of its domain: for example, $1/x$ defined on $\mathbb{R}_+$ cannot be continuously extended onto $\overline{\mathbb{R}}_+$. Another example is $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$ defined on $A = \mathbb{R} - \{0\}$ which cannot be continuously extended onto $\overline{A} = \mathbb{R}$.
>
> The part "let Y be Hausdorff" is needed for the uniqueness of a possible extension. Indeed, if Δ is not closed, then there are some points $(a, b) \in \overline{\Delta}$ such that $a \neq b$. But then $C' = \{x \in \overline{A} | (g(x), h(x)) \in \overline{\Delta}\}$ is closed and contains A, and, hence, $C' = \overline{A}$. In particular, $\overline{A}$ may contain some point x such that $g(x) = a \neq h(x) = b$. The easiest example is the two element set $Y = \{a, b\}$ with the topology $\{\emptyset, \{a\}, Y\}$. $f : X \to Y$ is continuous iff $f^{-1}(\{a\})$ is open in X. Let $f : \mathbb{R} - \{0\} \to Y$ be such that $f(x) = a$ for $x \neq 0$. Then regardless of whether we define $g(0) = a$ or $g(0) = b$, the resulting function is going to be a continuous function from $\mathbb{R}$ to Y.
>
> As an alternative proof, one can use Exercises 5, 6 and 7 of Chapter 3 Supplementary Exercises: Nets. Namely, take $x \in \overline{A}$, use Exercise 6 to argue that there are nets in A converging to x, use Exercise 7 to argue that then the value at x is determined by the values at the points of the net, and finally use Exercise 5 to conclude the argument.

SECTION 19

1.

Prove Theorem 19.2.

The product of any basis elements is open in the box topology, and the product of finitely many basis elements and all other spaces is open in the product topology. Hence, the topologies defined in the theorem are coarser than the box and product topologies, respectively. Further, $\prod_\alpha U_\alpha = \cup_{B_\alpha \subset U_\alpha} \prod_\alpha B_\alpha$ where each B_α is a basis element if U_α is a proper subset of X_α or X_α if $U_\alpha = X_\alpha$. Hence, every open basis element for either topology can be represented as a union of some basis elements as defined in the theorem, and the topologies defined in the theorem are finer than the box and product topologies, respectively.

2.

Prove Theorem 19.3.

$\sqcap_\alpha U_\alpha \cap \sqcap_\alpha A_\alpha = \sqcap_\alpha (U_\alpha \cap A_\alpha)$. On the left hand side we have a basis element for the subspace topology (where if $\prod_\alpha X_\alpha$ is in the product topology, for all but finitely many α , $U_\alpha = X_\alpha$), and on the right hand side we have a basis element for the topology of the product where each A_α is in its subspace topology (where, again, if the product is in the product topology, for all but finitely many α , we may assume $U_\alpha = X_\alpha$ so that $U_\alpha \cap A_\alpha = A_\alpha$). This shows that the bases for the two topologies (the subspace topology of the product and the topology of the product of subspace topologies) are the same, regardless of whether both products are given the box or product topology.

3.

Prove Theorem 19.4.

If two points $\mathbf{x}, \mathbf{y} \in \prod_\alpha X_\alpha$ are different, then they have at least one different coordinate $x_\alpha \neq y_\alpha$. Then let U and V be disjoint open subsets of X_α containing x_α and y_α , respectively. We have, $\prod_\beta U_\beta$ and $\prod_\beta V_\beta$, where $U_\beta = V_\beta = X_\beta$ for $\beta \neq \alpha$ and $U_\alpha = U$ and $V_\alpha = V$, are two disjoint open neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively, in either topology.

4.

Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.

The homeomorphism is $f(((x_1, \ldots, x_{n-1}), x_n)) = (x_1, \ldots, x_n)$. Indeed, U is open in $(X_1 \times \cdots \times X_{n-1}) \times X_n$ iff every point $(x_1 \times \cdots \times x_{n-1}) \times x_n$ of U has a basis neighborhood $V \times U_n \subset U$ where V is open in $X_1 \times \cdots \times X_{n-1}$ iff every point $(x_1 \times \cdots \times x_{n-1}) \times x_n$ of U has a basis neighborhood $(U_1 \times \cdots \times U_{n-1}) \times U_n \subset U$ iff every point $x_1 \times \cdots \times x_n$ of $f(U)$ has a basis neighborhood $U_1 \times \cdots \times U_n \subset f(U)$.

5.

One of the implications stated in Theorem 19.6 holds for the box topology. Which one?

If $f : A \to \prod_{\alpha\in J} X_\alpha$ is continuous, and $\prod_{\alpha\in J} X_\alpha$ is in the box topology, then, since the product topology is coarser than the box topology, f is still continuous if $\prod_{\alpha\in J} X_\alpha$ is given the product topology, and, by Theorem 19.6, each f_α , $\alpha \in J$, is continuous.

6.

Let $\mathbf{x}_1, \mathbf{x}_2, \dots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \dots$ converges to $\pi_\alpha(\mathbf{x})$ for each α . Is this fact true if one uses the box topology instead of the product topology?

Suppose that the sequence converges in the product topology. Let U be an open neighborhood of $\pi_\alpha(\mathbf{x})$ in X_α . Then the product of U and all other spaces is an open neighborhood of $\mathbf{x}$ in the product topology and all members of the sequence starting from some N lie in the neighborhood. Hence, $\pi_\alpha(\mathbf{x}_n) \in U$ for $n \geq N$. This works for the box topology as well.

The other direction. Suppose, that the projections of the sequence converge. Take any neighborhood of $\mathbf{x}$, it contains a basis set U containing $\mathbf{x}$. U is the product of open sets U_α in each coordinate space, and the space is in the product topology, for all but a finite number of α ' s: $U_\alpha = X_\alpha$. For every U_α there is a number N_α such that for $n \geq N_\alpha$, $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$. If $U_\alpha = X_\alpha$, we choose $N_\alpha = 1$. Then, for the product topology, there is only a finite number of N_α ' s that are greater than 1 , and we can take the maximum of all N_α ' s, $N = \max_\alpha N_\alpha$. All elements of the sequence starting from N are contained in U . For the box topology this direction may fail: there can be infinite number of $N_\alpha > 1$ and no greatest N_α . Some examples are in Exercise 4(b) of §20 (note that in all examples every projection converges to 0).

7.

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

> The finer is the topology on a set, the smaller (at least, not larger) is the closure of any its subset. Indeed, a finer topology has more closed sets, so the intersection of all closed sets containing a given subset is, in general, smaller in a finer topology than in a coarser topology. Another way to see this is that if a point is not in the closure in a coarser topology, then it has a neighborhood that does not intersect the subset, and the same neighborhood will work in a finer topology, so the point is still not in the closure. Yet another way to remember this is that in the finest topology, the discrete topology, the closure of any subset equals the subset itself, while in the coarsest topology, the indiscrete topology, the closure of any nonempty subset is the whole space.

So, in our case, we would expect the closure of $\mathbb{R}^\infty$ to be larger (or at least not smaller) in the product topology than in the box topology.

If $\mathbb{R}^\omega$ is given the product topology, for every point $\mathbf{x} \in \mathbb{R}^\omega$ and every its neighborhood $U = \prod_{n \in \mathbb{Z}_+} U_n$, where for all but finitely many values of n, $U_n = \mathbb{R}$, choose a point $\mathbf{y}$ such that $y_n \in U_n$ and $y_n = 0$ if $U_n = \mathbb{R}$. Since for only finitely many values of n, U_n is a proper subset of $\mathbb{R}$, $\mathbf{y} \in \mathbb{R}^\infty$, hence, $\mathbf{y} \in U \cap \mathbb{R}^\infty$. We conclude that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology (see also Example 7 of §23).

An alternative way to see that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ is as follows. According to Exercise 6, a sequence in the product space converges to a point iff all coordinate projections converge to the corresponding projections of the point. For $\mathbf{x} = (x_1, x_2, \ldots)$ in $\mathbb{R}^\omega$ define a sequence of points $\mathbf{y}_k \in \mathbb{R}^\infty$ such that the first k coordinates of $\mathbf{y}_k$ are equal to the first k coordinates of $\mathbf{x}$, and all others are zero. Then, clearly, the projections of $\mathbf{y}_k$ ' s onto any coordinate n converge to x_n, and, therefore, the sequence $\mathbf{y}_1, \mathbf{y}_2, \ldots$ converges to $\mathbf{x}$, i.e. every $\mathbf{x}$ is in the closure of $\mathbb{R}^\infty$ in the product topology.

Now, if $\mathbb{R}^\omega$ is given the box topology, every point not in $\mathbb{R}^\infty$ has a disjoint open neighborhood. Indeed, in the box topology, if a point $(x_1, x_2, \ldots) \notin \mathbb{R}^\infty$ then there is an increasing infinite sequence $\{n_k\}$ of indexes such that $x_{n_k} \neq 0$ for all k, and for each k there is a neighborhood U_{n_k} of x_{n_k} that does not contain 0. For all other coordinates m (not in $\{n_k\}$) let us take $U_m = \mathbb{R}$. Then, $\prod_n U_n$ is open in the box topology, contains $\mathbf{x}$, but does not intersect $\mathbb{R}^\infty$. Therefore, $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open, and $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$ in the box topology.

8.

Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots).$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Clearly, h is bijective (as a product of bijective functions). Also note, that the inverse function has the same form with different coefficients: $g(\mathbf{y}) = (\frac{1}{a_1}y_1 - \frac{b_1}{a_1}, \frac{1}{a_2}y_2 - \frac{b_2}{a_2}, \ldots)$. So, all we need to prove is that h is continuous. Take a point $y = h(x) \in U$ where U is open in the product, then y has an open basis neighborhood $B \subset U$, which is the product of some open sets $B_n \subset \mathbb{R}$ (and in the product topology only finitely many of B_n ' s are proper subsets of $\mathbb{R}$). The preimage $A_n = \{x_n | a_nx_n + b_n \in B_n\}$ of each such set is open in $\mathbb{R}$ as well, and if $B_n = \mathbb{R}$ then $A_n = \mathbb{R}$. Therefore, every point x in $h^{-1}(U)$ has an open neighborhood $\prod_{n\in\mathbb{Z}_+} A_n \subset h^{-1}(U)$ (in the case of the product topology only finitely many of A_n ' s are proper subsets of $\mathbb{R}$), and, therefore, $h^{-1}(U)$ is open, h is continuous. This works for both product and box topologies.

9.

Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha\in J}$ of nonempty sets, with $J \neq 0$, the cartesian product

$$\prod_{\alpha\in J} A_\alpha$$

is not empty.

We use the fact that the choice axiom is equivalent to the existence of a choice function (see page 59).

Given the definition of the cartesian product on page 113, the product is not empty if there is at least one function $\mathbf{x} : J \to \cup_{\alpha\in J} A_\alpha$ such that for every $\alpha \in J$, $x_\alpha \in A_\alpha$, but this just says that there is a choice function (see page 59). More formally, we would have to consider a bijective function $h : J \to \{A_\alpha\}$ (see page 36), a set $\mathcal{B} = \{A_\alpha\} = \{B | B = h(\alpha) \text{ for some } \alpha \in J\}$ and $\mathbf{x}' : \mathcal{B} \to \cup_{B\in\mathcal{B}} B$, and then define $\mathbf{x} = \mathbf{x}' \circ h$, where the existence of $\mathbf{x}'$ is ensured by the choice axiom.

Vice versa, for every collection $\mathcal{B}$ of nonempty sets, let $J = \mathcal{B}$ and define the indexing function $h : \mathcal{B} \to \mathcal{B}$ by $h(B) = B$. Then, the product $\prod_{B\in\mathcal{B}} B$ is not empty, and there is at least one function $\mathbf{x} : \mathcal{B} \to \cup_{B\in\mathcal{B}} B$ such that for every $B \in \mathcal{B}$, $\mathbf{x}(B) \in B$, but then $\mathbf{x}$ is a choice function.

10.

Let A be a set; let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha\in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

(a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.

(b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta\},$$

and let $\mathcal{S} = \cup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.

(d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha\in J};$$

let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z .

(a) Let $\mathfrak{T}$ be the collection of topologies on A such that relative to each one each f_α is continuous. $\mathfrak{T}$ is nonempty, as there exists at least one topology on A such that every function having A as its domain is continuous, namely, the discrete topology. The intersection of an arbitrary subcollection of $\mathfrak{T}$ is a topology (Exercise 4(a) of §13) contained in $\mathfrak{T}$ (for every open set $U \subset X_\alpha$, $f_\alpha^{-1}(U)$ is open in every topology in the subcollection, hence, it is open in their intersection). Therefore, $\mathcal{T} = \cap_{\mathcal{U}\in\mathfrak{T}}\mathcal{U}$ is the coarsest topology such that relative to it each of the functions f_α is continuous. The uniqueness follows immediately, as the constructed topology must be a subset of any other topology in $\mathfrak{T}$.

(b) For every topology $\mathcal{U} \in \mathfrak{T}$, every set in $\mathcal{S}$ must be open in $\mathcal{U}$, as every f_β is continuous relative to $\mathcal{U}$. So, every topology in $\mathfrak{T}$ contains $\mathcal{S}$. The coarsest such topology $\mathcal{T}$ is generated by $\mathcal{S}$ as a subbasis (Exercise 5 of §13).

> Note, that (b) provides another way to construct $\mathcal{T}$. Indeed, every topology in $\mathfrak{T}$ must contain all sets from each $\mathcal{S}_\beta$. Otherwise, f_β would not be continuous relative to this topology. Moreover, a topology is in $\mathfrak{T}$ iff it contains $\mathcal{S}$ (the other direction is also clear). Hence, according to my solution of Exercise 4(b) of §13, there are two ways to construct the coarsest topology containing each $\mathcal{S}_\beta$, namely, we can either consider the intersection of all topologies containing each $\mathcal{S}_\beta$ as in (a), or generate the topology from $\mathcal{S} = \cup_\beta \mathcal{S}_\beta$ as its subbasis. See also the discussion of the difference between the two methods from the point of view of their theoretical and practical values.

(c) Let $h_\alpha = f_\alpha \circ g$. Then if g is continuous, every h_α is continuous. If each h_α is continuous, then for every U open in A , U is the union of finite intersections of some subbasis elements $f_\beta^{-1}(U_\beta)$, where each U_β is open in X_β . Therefore, $g^{-1}(U)$ is the union of finite intersections of $g^{-1}(f_\beta^{-1}(U_\beta)) = h_\beta^{-1}(U_\beta)$, and, hence, open in Y .

(d) Let $U \in \mathcal{T}$ and $\mathbf{x} \in f(U)$. Take $a \in U$ such that $\mathbf{x} = f(a)$. Since U is open there is a basis element V in $\mathcal{T}$ such that $a \in V \subset U$ and $V = \cap_{i=1,\dots,n} f_{\alpha_i}^{-1}(U_{\alpha_i})$ where U_{α_i} is open in X_{α_i} (without loss of generality all α ' s are assumed to be distinct). For all other α ' s assume $U_\alpha = X_\alpha$. Then, $V = f^{-1}(\sqcap_\alpha U_\alpha)$ and $f(V) = f(A) \cap \sqcap_\alpha U_\alpha$. This set is open in the subspace topology (as only finitely many of U ' s are distinct from their spaces) and $\mathbf{x} \in f(V) \subset f(U)$. Therefore, $f(U)$ is open in the subspace topology.

SECTION 20

1.

(a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x},\mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for x, y $\in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

(a) The basis elements when $n = 2$ are squares turned by 45 degrees (right angle "rhombuses"). The fact that it is a metric follows from the inequality on page 122. Further, based on Theorem 20.3, to show that d' induces the usual topology on $\mathbb{R}^n$ we can show that d' induces the same topology as ρ . Indeed, using Lemma 20.2, every such "rhombus" of size ϵ contains a "square" of size ϵ/n , because $d'(\mathbf{x},\mathbf{y}) \leq n\rho(\mathbf{x},\mathbf{y})$, and every "square" of size ϵ contains a "rhombus" of size ϵ , because $\rho(\mathbf{x},\mathbf{y}) \leq d'(\mathbf{x},\mathbf{y})$.

(b) Similarly, it follows from $\rho \leq d' \leq n^{1/p}\rho$.

2.

Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Let $d_d((x,y),(a,b)) = 1$ if $x \neq a$ and $\overline{d}(y,b) = \min\{|y-b|,1\}$ otherwise (i.e. if $x = a$). This is a metric, as the only nontrivial property to check is the triangle inequality, but it is easy to check using the facts that the distance between any two points is at most 1 , and $\overline{d}$ is a metric on $\mathbb{R}$. Now, using Lemma 20.2, every ball of size ϵ under this metric is either a vertical interval open in the dictionary order topology or the whole space (if $\epsilon > 1$). In either case it is open in the dictionary order topology on $\mathbb{R} \times \mathbb{R}$. Vice versa, for every open set in the dictionary order topology and any its point there is an open vertical interval centered at the point and contained in the set. Therefore, there is an open ball contained in the set.

Alternatively, the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ can be generated by the set of vertical intervals $\{x\} \times (y,z)$ as its basis, and even by the set of "small" vertical intervals $\{x\} \times (y,z)$ where $y < z \leq y+2$. But each such interval can be rewritten as $\{x\} \times (\frac{y+z}{2} - \frac{z-y}{2}, \frac{y+z}{2} + \frac{z-y}{2}) = B_{d_d}((x,\frac{y+z}{2}),\frac{z-y}{2})$ where $0 < \frac{z-y}{2} \leq 1$. And vice versa, for every $\epsilon \in (0,1]$, $B_{d_d}((x,y),\epsilon) = \{x\} \times (y-\epsilon, y+\epsilon)$ where $y - \epsilon < y + \epsilon \leq y - \epsilon + 2$. Hence, the dictionary order topology is generated by the basis consisting of all ϵ -balls, $\epsilon \leq 1$, in the d_d metric. But the same basis generates the topology induced by d_d , because to generate a metric topology it is enough to consider only "small" balls in its basis (the first paragraph on page 122).

3.

Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

(a) Let U be an open subset of $\mathbb{R}$, and $x \times y \in d^{-1}(U)$. Then, $a = d(x \times y) \in U$, and there is an open interval $(a-\epsilon, a+\epsilon) \subset U$. Take any point $x' \times y' \in B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$. Then, $|d(x' \times y') - d(x \times y)| \leq d(x' \times x) + d(y \times y') < \epsilon$, $d(x' \times y') \in (a-\epsilon, a+\epsilon) \subset U$, and $x' \times y'$ is contained in $d^{-1}(U)$. Therefore, the set $B = B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$ open in $X \times X$ is such that $x \times y \in B \subset d^{-1}(U)$, and $d^{-1}(U)$ is open.

(b) If $d : X' \times X' \to \mathbb{R}$ is continuous, then for every fixed $x \in X'$, $d_x(y) : X' \to \mathbb{R}$, $d_x(y) = d(x, y)$, is continuous (Exercise 11 of §18). Therefore, every d-ball $B_d(x, r) = \{y | d_x(y) < r\} = d_x^{-1}((-\infty, r))$ must be open in X'.

> We could have solved both (a) and (b) at once. Let $\mathfrak{T}$ be the family of all topologies on X such that $d : X \times X \to \mathbb{R}$ is continuous. Then, $\mathfrak{T}$ is nonempty, as, for example, the discrete topology on X is in $\mathfrak{T}$. Further, for every subfamily of topologies $\{\mathcal{T}_\alpha\} \subset \mathfrak{T}$, $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on X (Exercise 4(a) of §13) such that d is continuous (the preimage of every open set is open relative to every topology in $\{\mathcal{T}_\alpha\}$, and, hence, relative to their intersection). Therefore, $\mathcal{T} = \bigcap_{\mathcal{U} \in \mathfrak{T}} \mathcal{U}$ is the coarsest topology relative to which d is continuous. It remains to note, that d is continuous relative to $\mathcal{U}$ iff $\mathcal{U}$ contains every ball $B_d(x, \epsilon)$. Indeed, if d is continuous, then it must be continuous in each variable, and, in particular, $B_d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$ must be open for every x. And if every ball $B_d(x, \epsilon)$ is open, then $(x, y) \in A = d^{-1}((a-\epsilon, a+\epsilon))$ implies $(x, y) \in B_d(x, \delta) \times B_d(y, \delta) \subset A$ where $\delta = \min\{\frac{a-d(x,y)+\epsilon}{2}, \frac{d(x,y)-a+\epsilon}{2}\}$, hence, A is open, and d is continuous. Therefore, $\mathcal{T}$ is the coarsest topology containing all balls $B_d(x, \epsilon)$, which is the one generated by the balls as its basis (Exercise 5 of §13).

4.

Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots), \\ g(t) &= (t, t, t, \ldots), \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots). \end{aligned}$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned} \mathbf{w}_1 &= (1,1,1,1,\ldots), & \mathbf{x}_1 &= (1,1,1,1,\ldots), \\ \mathbf{w}_2 &= (0,2,2,2,\ldots), & \mathbf{x}_2 &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots), \\ \mathbf{w}_3 &= (0,0,3,3,\ldots), & \mathbf{x}_3 &= (0,0,\tfrac{1}{3},\tfrac{1}{3}\ldots), \\ &\ldots & &\ldots \\ \mathbf{y}_1 &= (1,0,0,0,\ldots), & \mathbf{z}_1 &= (1,1,0,0,\ldots), \\ \mathbf{y}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,\ldots), & \mathbf{z}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,\ldots), \\ \mathbf{y}_3 &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,\ldots), & \mathbf{z}_3 &= (\tfrac{1}{3},\tfrac{1}{3},0,0,\ldots), \\ &\ldots & &\ldots \end{aligned}$$

(a) If a function is continuous, and we change the topology of the range to a coarser one, then the function remains continuous. And if a function is not continuous, and we make the topology of the range finer, the function remains noncontinuous. So, for each function we may specify the finest topology out of the three given topologies such that the function is continuous relative to it. For f it is the product topology, so that f is continuous relative to the product topology only, and for g and h it is the uniform topology, so that they are not continuous in the box topology only.

The box topology. $((-1,1),(-\frac{1}{4},\frac{1}{4}),\ldots,(-\frac{1}{k^2},\frac{1}{k^2}),\ldots)$ is open in the box topology, but all three functions have its inverse image equal to $\{0\}$. So, neither function is continuous relative to the box topology.

The uniform topology. $f^{-1}(B_{\bar{\rho}}(\mathbf{0},1)) \subset f^{-1}(\prod_{n\in\mathbb{Z}_+}(-1,1)) = \{0\}$. At the same time, if the function $k(t) = (a_1 t, a_2 t, \ldots)$ equals g ($a_n = 1$) or h ($a_n = \frac{1}{n}$), and $k(t) \in B_{\bar{\rho}}(\mathbf{x},\epsilon)$, then for every $n \in \mathbb{Z}_+$, $|x_n - a_n t| \leq \sup_{n\in\mathbb{Z}_+}|x_n - a_n t| \overset{def}{=} \delta < \epsilon$, and for $|z| < \frac{\epsilon-\delta}{2}$, $|x_n - a_n(t+z)| \leq |x_n - a_n t| + a_n|z| < \delta + \frac{\epsilon-\delta}{2} = \frac{\epsilon+\delta}{2} < \epsilon$. Hence, $k((t-\frac{\epsilon-\delta}{2}, t+\frac{\epsilon-\delta}{2})) \subset B_{\bar{\rho}}(\mathbf{x},\epsilon)$, and $k^{-1}(B_{\bar{\rho}}(\mathbf{x},\epsilon))$ is open. So, f is the only function that is not continuous relative to the uniform topology.

The product topology. It is easy to see that all three functions are continuous relative to the product topology. For f it follows from the fact that if $f(t) \in U$ open in the product topology, then $f(t) \in \prod_{n\in\mathbb{Z}_+} U_n \subset U$ where for finitely many $n \in \mathbb{Z}_+$, i.e. for $n \in \{n_1, \ldots, n_k\}$, $U_n = (nt - n\epsilon_n, nt + n\epsilon_n)$, and for every other $n \in \mathbb{Z}_+$, $U_n = \mathbb{R}$. Therefore, if $\epsilon = \min_{i=1,\ldots,k} \epsilon_{n_k}$, $f((t-\epsilon, t+\epsilon)) \subset U$, and $f^{-1}(U)$ is open. Therefore, f is continuous relative to the product topology. For g and h we can either use a similar argument, or use the fact that they are already continuous relative to the finer uniform topology.

(b) If a sequence converges to a point, and we change the topology to a coarser one, then the sequence still converges to the point (it may converge to new points as well but not in a Hausdorff space). And if a sequence does not converge to a point (or at all), and we change the topology to a finer one, then the sequence still does not converge to the point (or at all). Therefore, for each sequence we may specify the finest topology out of the three given topologies in which it converges to some point (which is unique because all three topologies are Hausdorff). For $\{\mathbf{w}_n\}$ it is the product topology, for $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ it is the uniform topology, and for $\{\mathbf{z}_n\}$ it is the box topology (i.e. it converges in all three topologies).

In the product topology (metricized) $\{\mathbf{w}_n\}$ converges to $\mathbf{0}$ as $\overline{d}(\mathbf{w}_k, 0)/k = 1/k$, and, therefore, any ball centered at $\mathbf{0}$ has all elements of the sequence starting from some index. Therefore, if it did converge in the uniform topology, it would converge to $\mathbf{0}$, but in the uniform topology the distance between any $\mathbf{w}_n$ and $\mathbf{0}$ is always 1 ($B_{\overline{\rho}}(\mathbf{0}, 1)$ does not contain any members of the sequence). So, $\{\mathbf{w}_n\}$ converges in the product topology only.

$\{\mathbf{x}_n\}$ converges to $\mathbf{0}$ in the uniform topology (and, hence, in the product topology), as for $n > \frac{1}{\epsilon}$, $\mathbf{x}_n \in B_{\overline{\rho}}(\mathbf{0}, \epsilon)$, but it does not converge in the box topology, as there are no elements of the sequence contained in the open neighborhood $A = (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots$ of $\mathbf{0}$ (and $\mathbf{0}$ is the only point it may converge to in the box topology).

$\{\mathbf{y}_n\}$ converges to $\mathbf{0}$ in the uniform topology (and, hence, in the product topology), as for $n > \frac{1}{\epsilon}$, $\mathbf{y}_n \in B_{\overline{\rho}}(\mathbf{0}, \epsilon)$, but not in the box topology (the same set A shows this).

$\{\mathbf{z}_n\}$ converges to $\mathbf{0}$ in all three topologies, as it clearly converges to $\mathbf{0}$ in the finest one — the box topology (for $n > \frac{1}{\min\{\epsilon_1, \epsilon_2\}}$, $\mathbf{z}_n \in (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \times \ldots$).

> Another way to see that each sequence converges to $\mathbf{0}$ in the product topology, and, hence, may converge to $\mathbf{0}$ only in any finer topology, is to use Exercise 6 of §19, according to which a sequence in the product space converges to a point iff all coordinate projections converge to the corresponding projections of the point. And in our case all coordinate projections for each sequence converge to $\mathbf{0}$.

5.

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

> The finer is the topology on a set, the smaller (at least, not larger) is the closure of any its subset. Indeed, a finer topology has more closed sets, so the intersection of all closed sets containing a given subset is, in general, smaller in a finer topology than in a coarser topology. Another way to see this is that if a point is not in the closure in a coarser topology, then it has a neighborhood that does not intersect the subset, and the same neighborhood will work in a finer topology, so the point is still not in the closure. Yet another way to remember this is that in the finest topology, the discrete topology, the closure of any subset equals the subset itself, while in the coarsest topology, the indiscrete topology, the closure of any nonempty subset is the whole space.

So, in our case, we would expect the closure of $\mathbb{R}^\infty$ to be larger (or at least not smaller) than in the box topology, and smaller (or at least not larger) than in the product topology. Exercise 7 of §19 shows that the closure of $\mathbb{R}^\infty$ in the box topology is the set itself, and in the product topology is the whole space. So here the answer can be either one of those or anything in between (which does not, actually, help :)).

Let $X \in \mathbb{R}^\omega$ be the set of all sequences of real numbers that converge to 0 in $\mathbb{R}$ ($\mathbf{x} \in X$ iff for every $\epsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for $n \geq N$, $|x_n| < \epsilon$). Note, that $\mathbb{R}^\infty \subset X$. If $\mathbf{y} \notin X$, then there is $\epsilon > 0$ such that for every $k \in \mathbb{Z}_+$ there is $n_k \geq k$ such that $|y_{n_k}| \geq \epsilon$. Hence, if $\mathbf{z} \in B_{\overline{\rho}}(\mathbf{y}, \frac{\epsilon}{2})$, for every $k \in \mathbb{Z}_+$, $|z_{n_k}| > |y_{n_k}| - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}$, and $B_{\overline{\rho}}(\mathbf{y}, \frac{\epsilon}{2})$ does not contain any points of X. Therefore, X is closed and contains the closure of $\mathbb{R}^\infty$. At the same time, for every $\mathbf{x} \in X$ and $\epsilon > 0$, there is $N \in \mathbb{Z}_+$ such that for $n \geq N$, $|x_n| < \frac{\epsilon}{2}$, and $\mathbf{y} = (x_1, \ldots, x_N, 0, 0, \ldots) \in B_{\overline{\rho}}(\mathbf{x}, \epsilon) \cap \mathbb{R}^\infty$. So, the closure of $\mathbb{R}^\infty$ in the uniform topology is the set of all sequences of real numbers converging to zero in $\mathbb{R}$.

5.

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

> The finer is the topology on a set, the smaller (at least, not larger) is the closure of any its subset. Indeed, a finer topology has more closed sets, so the intersection of all closed sets containing a given subset is, in general, smaller in a finer topology than in a coarser topology. Another way to see this is that if a point is not in the closure in a coarser topology, then it has a neighborhood that does not intersect the subset, and the same neighborhood will work in a finer topology, so the point is still not in the closure. Yet another way to remember this is that in the finest topology, the discrete topology, the closure of any subset equals the subset itself, while in the coarsest topology, the indiscrete topology, the closure of any nonempty subset is the whole space.

So, in our case, we would expect the closure of $\mathbb{R}^\infty$ to be larger (or at least not smaller) than in the box topology, and smaller (or at least not larger) than in the product topology. Exercise 7 of §19 shows that the closure of $\mathbb{R}^\infty$ in the box topology is the set itself, and in the product topology is the whole space. So here the answer can be either one of those or anything in between (which does not, actually, help :)).

Let $X \in \mathbb{R}^\omega$ be the set of all sequences of real numbers that converge to 0 in $\mathbb{R}$ ($\mathbf{x} \in X$ iff for every $\epsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for $n \geq N$, $|x_n| < \epsilon$). Note, that $\mathbb{R}^\infty \subset X$. If $\mathbf{y} \notin X$, then there is $\epsilon > 0$ such that for every $k \in \mathbb{Z}_+$ there is $n_k \geq k$ such that $|y_{n_k}| \geq \epsilon$. Hence, if $\mathbf{z} \in B_{\bar\rho}(\mathbf{y}, \frac{\epsilon}{2})$, for every $k \in \mathbb{Z}_+$, $|z_{n_k}| > |y_{n_k}| - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}$, and $B_{\bar\rho}(\mathbf{y}, \frac{\epsilon}{2})$ does not contain any points of X . Therefore, X is closed and contains the closure of $\mathbb{R}^\infty$. At the same time, for every $\mathbf{x} \in X$ and $\epsilon > 0$, there is $N \in \mathbb{Z}_+$ such that for $n \geq N$, $|x_n| < \frac{\epsilon}{2}$, and $\mathbf{y} = (x_1, \dots, x_N, 0, 0, \dots) \in B_{\bar\rho}(\mathbf{x}, \epsilon) \cap \mathbb{R}^\infty$. So, the closure of $\mathbb{R}^\infty$ in the uniform topology is the set of all sequences of real numbers converging to zero in $\mathbb{R}$.

6.

Let $\bar\rho$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

(a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ -ball $B_{\bar\rho}(\mathbf{x}, \epsilon)$.

(b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.

(c) Show that

$$B_{\bar\rho}(\mathbf{x}, \epsilon) = \bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta).$$

(a) $\mathbf{y} = (x_1 + \frac{\epsilon}{2}, x_2 + \frac{2\epsilon}{3}, x_3 + \frac{3\epsilon}{4}, \dots)$ is in $U(\mathbf{x}, \epsilon)$ but not in $B_{\bar\rho}(\mathbf{x}, \epsilon)$. But, of course, $B_{\bar\rho}(\mathbf{x}, \epsilon) \subset U(\mathbf{x}, \epsilon)$. So that $B_{\bar\rho}(\mathbf{x}, \epsilon) \subsetneq U(\mathbf{x}, \epsilon)$.

(b) For the point $\mathbf{y}$ from (a) there is no ball centered at it and contained in $U(\mathbf{x}, \epsilon)$.

(c) Clearly, for $\delta < \epsilon$, $U(\mathbf{x}, \delta) \subset B_{\bar\rho}(\mathbf{x}, \epsilon)$: the distance between any point in $U(\mathbf{x}, \delta)$ and $\mathbf{x}$ is less or equal to $\delta < \epsilon$. Hence, $\bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta) \subset B_{\bar\rho}(\mathbf{x}, \epsilon)$. Further, for $\mathbf{y} \in B_{\bar\rho}(\mathbf{x}, \epsilon)$, $\bar\rho(\mathbf{x}, \mathbf{y}) < \epsilon$, and, therefore, $\mathbf{y} \in U(\mathbf{x}, \frac{\epsilon + \bar\rho(\mathbf{x},\mathbf{y})}{2})$ where $\frac{\epsilon + \bar\rho(\mathbf{x},\mathbf{y})}{2} < \epsilon$. Hence, $B_{\bar\rho}(\mathbf{x}, \epsilon) \subset \bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta)$.

7.

Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i, and b_i is h continuous? a homeomorphism?

Recall that $h((x_1, x_2, \ldots)) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots)$, $a_i > 0$, h is bijective, and $h^{-1}(\mathbf{y}) = (\frac{1}{a_1}y_1 - \frac{b_1}{a_1}, \frac{1}{a_2}y_2 - \frac{b_2}{a_2}, \ldots)$. So, if we find the conditions on a_i ' s and b_i ' s such that h is continuous, the same conditions on $\frac{1}{a_i}$ ' s and $\frac{-b_i}{a_i}$ ' s will ensure that h^{-1} is continuous. The combination of these two sets of conditions will give us the conditions under which h is a homeomorphism.

According to the solution of Exercise 6(a), $A = h^{-1}(B_{\bar{\rho}}(\mathbf{y}, \epsilon)) \subset h^{-1}(U(\mathbf{y}, \epsilon)) = \prod_{n\in\mathbb{Z}_+}(\frac{1}{a_n}y_n - \frac{b_n}{a_n} - \frac{\epsilon}{a_n}, \frac{1}{a_n}y_n - \frac{b_n}{a_n} + \frac{\epsilon}{a_n}) = \prod_{n\in\mathbb{Z}_+} A_n$. Note, that if $\{a_n\}$ is unbounded, then for any $\delta > 0$, there is some n such that $\frac{\epsilon}{a_n} < \delta$, and the diameter of the interval A_n is less than 2δ, so that A cannot contain any ball of size δ. Hence, one condition on a_i ' s is that $\{a_i\}$ must be bounded. Now, if for all $n \in \mathbb{Z}_+$, $a_n \le M$, then for every $\mathbf{x} \in A$, according to Exercise 6(c), there is $\delta < \epsilon$ such that $\mathbf{x} \in h^{-1}(U(\mathbf{y}, \delta))$, and for every $n \in \mathbb{Z}_+$, $x_n \in (\frac{1}{a_n}y_n - \frac{b_n}{a_n} - \frac{\delta}{a_n}, \frac{1}{a_n}y_n - \frac{b_n}{a_n} + \frac{\delta}{a_n})$. Hence, if $\mathbf{z} \in B_{\bar{\rho}}(\mathbf{x}, \frac{\epsilon-\delta}{2M}) \subset U(\mathbf{x}, \frac{\epsilon-\delta}{2M})$, then $z_n \in (x_n - \frac{\epsilon-\delta}{2M}, x_n + \frac{\epsilon-\delta}{2M}) \subset (\frac{1}{a_n}y_n - \frac{b_n}{a_n} - (\frac{\epsilon-\delta}{2M} + \frac{\delta}{a_n}), \frac{1}{a_n}y_n - \frac{b_n}{a_n} + (\frac{\epsilon-\delta}{2M} + \frac{\delta}{a_n}))$ where $a_n(\frac{\epsilon-\delta}{2M} + \frac{\delta}{a_n}) = a_n\frac{\epsilon-\delta}{2M} + \delta \le \frac{\epsilon-\delta}{2} + \delta = \frac{\epsilon+\delta}{2}$, and $h(\mathbf{z}) \in U(\mathbf{y}, \frac{\epsilon+\delta}{2}) \subset B_{\bar{\rho}}(\mathbf{y}, \epsilon)$.

To summarize, h is continuous in the uniform topology iff $\{a_i\}$ is bounded, i.e. there is some M such that $a_i \le M$ for all $i \in \mathbb{Z}_+$. Similarly, h^{-1} is continuous iff $\frac{1}{a_i}$ is bounded, or, in other words, iff there is some $m > 0$ such that $a_i \ge m$ for all $i \in \mathbb{Z}_+$. Hence, h is a homeomorphism iff there are $m, M \in \mathbb{R}_+$ such that for every $i \in \mathbb{R}_+$, $m \le a_i \le M$.

8.

Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the l^2 *-topology*. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset l^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n\in\mathbb{Z}_+}[0, 1/n]$$

is contained in X; it is called the ***Hilbert cube***. Compare the four topologies that H inherits as a subspace of X.

(a) To show that a topology $\mathcal{T}$ is finer than a metric topology d, it is enough to show that for any $B_d(x,\epsilon)$ there is $U \in \mathcal{T}$ such that $x \in U \subset B_d(x,\epsilon)$ (i.e. we need to show there is a neighborhood contained in the ball of the center point only, as for any other point in the ball we can first find another ball centered at it; the argument is similar to the second part of Lemma 20.2). Let $M(\mathbf{x}) = \{\mathbf{y} | |x_i - y_i|^2 < \epsilon^2/2^i\}$. Then it is open in the box topology, and for every $\mathbf{y} \in M(\mathbf{x})$, $\sum_{i=1}^{\infty} y_i^2 < \sum_{i=1}^{\infty}(|x_i| + \frac{\epsilon^2}{2^i})^2$ converges (because $|x_i|$ ' s are bounded) and $\sum_{i=1}^{\infty} |x_i - y_i|^2 < \epsilon^2$, so that $M(\mathbf{x}) \subset B_{l^2}(\mathbf{x},\epsilon) \subset B_{\overline{\rho}}(\mathbf{x},\epsilon)$ (the second inclusion follows from $\overline{\rho}(\mathbf{x},\mathbf{y}) \le d(\mathbf{x},\mathbf{y})$).

(b) The finer the topology of a space, the finer the topology of any its subspace. So, given (a) and the fact that the uniform topology is finer then the product topology, to show they are all different as subspace topologies, we only need to find subsets of $\mathbb{R}^\infty$ open in a finer topology but not in a coarser one. The set A we used in 4(b) is open in the box topology, and its intersection A' with $\mathbb{R}^\infty$ is open in the subspace. $\mathbf{0} \in A'$, but for every $\epsilon > 0$, $B_{l^2}(\mathbf{0},\epsilon) \cap \mathbb{R}^\infty$ has a point not in A' (just make a sufficiently large coordinate $\epsilon/2$ and all others 0). Further, for every $\epsilon > 0$ and $\delta > 0$, for a sufficiently large n , $(\underbrace{\delta/2,\dots,\delta/2}_{n},0,0,\dots) \in (B_{\overline{\rho}}(\mathbf{0},\delta) - B_{l^2}(\mathbf{0},\epsilon)) \cap \mathbb{R}^\infty$. Finally, for every $0 < \epsilon < 1$ and $\delta > 0$, for a sufficiently large k , $(\frac{\delta}{2},\delta,\frac{3\delta}{2},\dots,\frac{k\delta}{2},0,0,\dots) \in (B_D(\mathbf{0},\delta) - B_{\overline{\rho}}(\mathbf{0},\epsilon)) \cap \mathbb{R}^\infty$.

(c) The box topology is strictly finer than the l^2 -topology on H : $A \cap H = \prod_{n=1}^{\infty}[0,\frac{1}{n+1})$ is an open neighborhood of $\mathbf{0}$ in H_b , but the intersection of any $B_{l^2}(\mathbf{0},\epsilon)$ with H has a point $(\underbrace{0,\dots,0}_{n-1},\frac{1}{n+1},0,0,\dots)$ for a sufficiently large n which is not in $\prod_{n=1}^{\infty}[0,\frac{1}{n+1})$.

The other three topologies are the same on H . To show this we show that the product topology is finer than the l^2 -topology on H . Intuitively, this is true because unbounded spaces in the product of open sets (a basis element for the product topology) becomes bounded by the sequence $\{\frac{1}{n}\}$ which converges in squares. For every $\epsilon > 0$, consider the ball $B = B_{l^2}(\mathbf{x},\epsilon)$. We only need to show that $\mathbf{x}$ has an open neighborhood in H_p that lies within the ball (see the comment for (a)). Consider the following set $B' = \prod_{i \le n}((x_i - \frac{\epsilon^2}{2n}, x_i + \frac{\epsilon^2}{2n}) \cap [0,\frac{1}{i}]) \times \prod_{i>n}[0,\frac{1}{i}]$ where n is sufficiently large such that $\sum_{i>n}\frac{1}{i^2} < \frac{\epsilon^2}{2}$. B' is open in H_p . Further, for $\mathbf{y} \in B'$, $d^2(\mathbf{x},\mathbf{y}) < \sum_{i \le n}\frac{\epsilon^2}{2n} + \sum_{i>n}\frac{1}{i^2} < \epsilon^2$, hence, $\mathbf{x} \in B' \subset B$.

9.

Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, \ldots, x_n+y_n),$$
$$c\mathbf{x} = (cx_1, \ldots, cx_n),$$
$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n.$$

(a) Show that $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{z})$.

(b) Show that $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$. [*Hint:* If $\mathbf{x},\mathbf{y} \neq \mathbf{0}$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \ge 0$.]

(c) Show that $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$. [*Hint:* Compute $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

(a) Just use the distributive law (and, of course, commutative and associative laws) for the expression on the left hand side.

(b) If $\mathbf{x}$ or $\mathbf{y}$ is zero, then the equality holds, otherwise, using the hint, as well as $\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x}$, $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$, $(a\mathbf{x})\cdot(b\mathbf{y}) = ab(\mathbf{x}\cdot\mathbf{y})$ and (a), $0 \le \|a\mathbf{x} \pm b\mathbf{y}\|^2 = a^2\|\mathbf{x}\|^2 + b^2\|\mathbf{y}\|^2 \pm 2ab(\mathbf{x}\cdot\mathbf{y}) = 2 \pm 2ab(\mathbf{x}\cdot\mathbf{y})$. Therefore, $|\mathbf{x}\cdot\mathbf{y}| \le \frac{1}{ab} = \|\mathbf{x}\|\|\mathbf{y}\|$.

(c) Using the hint and (b), $0 \le \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2|\mathbf{x}\cdot\mathbf{y}| \le (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$ and both are positive.

(d) $d(\mathbf{x},\mathbf{y}) \ge \mathbf{0}$ and is $\mathbf{0}$ iff $\mathbf{x} = \mathbf{y}$. The triangle inequality holds due to (c) because $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$.

10.

Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x},\mathbf{y} \in X$, then $\sum |x_iy_i|$ converges. [*Hint:* Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x},\mathbf{y} \in X$, then so are $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i-1}^{\infty}(x_i - y_i)^2\right]^{1/2}$$

is a well-defined metric on X.

(a) Using the hint and Exercise 9(b), for any $\mathbf{z}$, let $\mathbf{z}' = (|z_1|, |z_2|, \ldots)$ and $\mathbf{z}_n = (z_1, \ldots, z_n)$, then $\|\mathbf{z}'\| = \|\mathbf{z}\|$, and $\sum_{i-1}^{n} |x_iy_i| = |\mathbf{x}'_n \cdot \mathbf{y}'_n| \le \|\mathbf{x}'_n\|\|\mathbf{y}'_n\| = \|\mathbf{x}_n\|\|\mathbf{y}_n\|$, and the expression on the right converges.

(b) $\sum (x_i+y_i)^2 = \sum |x_i+y_i|^2 \le \sum (|x_i| + |y_i|)^2 = \sum x_i^2 + \sum y_i^2 + 2\sum |x_iy_i|$ which converges by (a). Alternatively, it follows immediately from Exercise 9(c) (the partial sums are bounded). $\sum c^2x_i^2 = c^2 \sum x_i^2$ also converges.

(c) It is. The sum converges by (b), it is nonnegative, and equals zero iff $x_i = y_i$ for every $i \in \mathbb{Z}_+$. The symmetry is clear as well. Further, the triangle inequality holds by Exercise 9(c) (used for partial sums).

11.

* Show that if d is a metric for X, then

$$d'(x,y) = d(x,y)/(1+d(x,y))$$

is a bounded metric that gives the topology of X. [*Hint:* If $f(x) = x/(1+x)$ for $x > 0$, use the mean-value theorem to show that $f(a+b) - f(b) \leq f(a)$.]

First, we show that d' is a bounded metric on X. Using the hint, on $\overline{\mathbb{R}}_+$, $f(x) = 1 - \frac{1}{1+x} : \overline{\mathbb{R}}_+ \to [0,1)$ is well-defined, nonnegative, zero at $x=0$ only, strictly increasing and bounded. Hence, $d' = f \circ d : X \times X \to [0,1)$ is well-defined, nonnegative, zero iff d is zero iff $x = y$, symmetric and bounded. So, we only need to show the triangle inequality. For $a, b \geq 0$, $f(a+b) - f(b) = \frac{a}{(1+b)(1+a+b)} \leq \frac{a}{1+a} = f(a)$. Using this, and the facts that $d' = f \circ d$ and f is increasing, $d'(x,y) + d'(y,z) \geq f(d(x,y) + d(y,z)) \geq d'(x,z)$. So, d' is a metric.

Now, we show that d' induces the same topology as d. Since f and $f^{-1}(y) = \frac{y}{1-y} : [0,1) \to \overline{\mathbb{R}}_+$ are continuous, $d' = f \circ d$ and $d = f^{-1} \circ d'$, d' is continuous in the d-topology (using Exercise 3(a)), and d is continuous in the d'-topology, implying that the topologies are the same (Exercise 3(b)).

SECTION 21

1.

Let $A \subset X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology on A.

The collection of balls $B_{d'}(a,r) \subset A$ centered at points $a \in A$ in the metric $d' = d|A \times A$ is a basis for the metric topology on A. A as a subspace has a basis element $B'(x,r) = A \cap B_d(x,r) = \{y \in A | d(x,y) < r\}$ for all $x \in X$. Clearly, $B_{d'}(a,r) = A \cap B_d(a,r) = B'(a,r)$ so that the subspace topology on A is finer than the metric topology on A. Further, consider any basis element $B'(x,r)$ for the subspace topology. If $a \in B'(x,r) \subset A$, let $r' = r - d(x,a)$. Then $a \in B_{d'}(a,r') = B_d(a,r') \cap A \subset B_d(x,r) \cap A = B'(x,r)$. Therefore, $B'(x,r)$ is open in the metric topology on A, and the metric topology on A is finer than the subspace topology on A.

2.

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an ***isometric imbedding*** of X in Y.

By the properties of metric it is injective, therefore, f from X onto $f(X)$ is bijective. The image of any open ball in X is open in $f(X)$ and the inverse image of any open ball in $f(X)$ (which, by Exercise 1, is a basis element for the subspace topology of $f(X)$) is an open ball in X, as $y \in B_{d_Y}(f(x), r) \cap f(X)$ iff $y = f(x')$ for some $x' \in B_{d_X}(x, r)$.

3.

Let X_n be a metric space with metric d_n , for $n \in \mathbb{Z}_+$.

(a) Show that

$$\rho(x,y) = \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\}$$

is a metric for the product space $X_1 \times \cdots \times X_n$.

(b) Let $\overline{d}_i = \min\{d_i, 1\}$. Show that

$$D(x,y) = \sup\{\overline{d}_i(x_i,y_i)/i\}$$

is a metric for the product space $\prod X_i$.

(a) It is well-defined as the product is finite. All properties including the triangle inequality are clearly satisfied (similar to the proof on page 122). Finally, $B_\rho(\mathbf{x},r) = \prod_{i=1}^n B_{d_i}(x_i,r)$ is open in the product topology, and for a basis element $\prod_{i=1}^n B_{d_i}(x_i,r_i)$ for the product topology and any its point $\mathbf{y}$, for each $i = 1,\ldots,n$, we can find s_i such that $B_{d_i}(y_i,s_i) \subset B_{d_i}(x_i,r_i)$, and, by letting $s = \min\{s_1,\ldots,s_n\}$, $\mathbf{y} \in B_\rho(\mathbf{y},s) = \prod_{i=1}^n B_{d_i}(y_i,s) \subset \prod_{i=1}^n B_{d_i}(y_i,s_i) \subset \prod_{i=1}^n B_{d_i}(x_i,r_i)$.

(b) $\overline{d}_i$ is a metric on X_i inducing the same topology as d_i (Theorem 20.1), and, therefore, so is $\overline{d}_i/i$. D is well-defined as the set $\{\overline{d}_i(x_i,y_i)/i\}$ is bounded from above, and all the metric properties hold for D (similar to Theorem 20.5). Further, again similar to Theorem 20.5, for $B_D(\mathbf{x},r)$, $\prod_{i=1}^N B_{d_i}(x_i,r) \times \mathbb{R} \times \mathbb{R} \times \cdots \subset B_D(\mathbf{x},r)$ if $\frac{1}{N} \leq r$, and for $U = \prod_{i\in\mathbb{Z}_+} U_i$ where $U_i = \mathbb{R}$ if $i > N$, and any point $\mathbf{y} \in U$, we can find $B_{d_i}(y_i,s_i) \subset U_i$, $s_i \leq 1$, for $i = 1,\ldots,N$, take $s = \min\{\frac{s_1}{1},\ldots,\frac{s_N}{N}\}$, and then $\mathbf{y} \in B_D(\mathbf{y},s) \subset B_{d_1}(y_1,s_1) \times \cdots \times B_{d_N}(y_N,s_N) \times \mathbb{R} \times \mathbb{R} \times \cdots \subset U$.

4.

Show that $\mathbb{R}_l$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)

For $x \in \mathbb{R}_l$, an example of a countable basis at x is $\{[x, x+\frac{1}{n})\}_{n\in\mathbb{Z}_+}$.

For $a \times b \in I_o^2$, an example of a countable basis at $a \times b$ is

- if $b \in (0,1)$, $\{((a, b-\frac{b}{n}), (a, b+\frac{1-b}{n}))\}_{n\in\mathbb{Z}_+}$,
- if $b = 0$ and $a > 0$, $\{((a-\frac{a}{n}, 1), (a, \frac{1}{n}))\}_{n\in\mathbb{Z}_+}$,
- if $b = 0$ and $a = 0$, $\{[(0,0), (0, \frac{1}{n}))\}_{n\in\mathbb{Z}_+}$,
- if $b = 1$ and $a < 1$, $\{((a, 1-\frac{1}{n}), (a+\frac{1-a}{n}, 0))\}_{n\in\mathbb{Z}_+}$,
- if $b = 1$ and $a = 1$, $\{((1, 1-\frac{1}{n}), (1,1)]\}_{n\in\mathbb{Z}_+}$.

5.

Theorem. Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then

$$x_n + y_n \to x + y,$$
$$x_n - y_n \to x - y,$$
$$x_n y_n \to xy,$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n / y_n \to x/y.$$

[*Hint:* Apply Lemma 21.4; recall from the exercises of §19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$.]

Using the hint, Exercise 6 of §19 shows that $x_n \times y_n \to x \times y$, and we may now apply Lemma 21.4 and Theorem 21.3.

6.

Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) does not converge uniformly.

For every $x < 1$, $f_n(x) \to f(x) = 0$, and $f_n(1) \to f(1) = 1$, but for every $\epsilon > 0$ and points $x < 1$ sufficiently close to 1, $f_n(x) > \epsilon$ (something like $(1-\delta)^n \geq 1 - n\delta$, which can be proved by induction, might help). Another way to argue that $(f_n(x))$ does not converge uniformly is by using Theorem 21.6, as f is not continuous.

7.

Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\overline{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \overline{\rho})$.

Once the notion of the uniform metric on $\mathbb{R}^X$ is clarified, the rest is an easy implication. By definition, $\bar{\rho}(f, g) = \sup_{x \in X}\{\min\{|f(x) - g(x)|, 1\}\}$ so that for every $\epsilon \leq 1$, $g \in B_{\overline{\rho}}(f, \epsilon)$ iff there is some $0 < \delta < \epsilon$ such that for every $x \in X$, $|f(x) - g(x)| < \delta$. Now, it is clear that if for every $0 < \epsilon \leq 1$ starting from some N, $f_n \in B_{\bar{\rho}}(f, \epsilon)$, then for $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for every $x \in X$, which is just the definition of uniform convergence. And vice versa, if f_n converges uniformly to f then for every $\epsilon > 0$ starting from some N, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for every $x \in X$, then for $n \geq N$, $\bar{\rho}(f_n, f) = \sup_{x \in X}\{\min\{|f_n(x) - f(x)|, 1\}\} \leq \frac{\epsilon}{2} < \epsilon$ and $f_n \to f$. Generally speaking, the definition of uniform convergence is (almost) the same as the definition of convergence in the uniform metric.

8.

Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x . Show that if the sequence (f_n) converges uniformly to f , then $(f_n(x_n))$ converges to $f(x)$.

f is continuous, by Theorem 21.6, therefore, for every $\epsilon > 0$, there is an open neighborhood U of x such that $f(U) \subset B_{d_Y}(f(x), \frac{\epsilon}{2})$. Let N be such that for $n > N$, $x_n \in U$ and $d_Y(f_n(x), f(x)) < \frac{\epsilon}{2}$ for all $x \in X$. Then, by the triangle inequality, $d_Y(f_n(x_n), f(x)) \leq d_Y(f_n(x_n), f(x_n)) + d_Y(f(x_n), f(x)) < \epsilon$ for $n > N$.

> This is of course not true if the convergence is not uniform. For example, $x^n \to \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$ on $[0, 1]$, but $(1 - \frac{1}{n})^n \to \frac{1}{e} \neq 1$. But this is also not always true even if all f_n and f are continuous while the convergence is not uniform. For example, $\min\{|nx - 1|, 1\} \to 1$, but $\min\{|n\frac{1}{n} - 1|, 1\} \to 0 \neq 1$. See also the function f of Exercise 9, which converges to 0 but $f(\frac{1}{n}) \to 1$.

9.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(b) Show that f_n does not converge uniformly to f . (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform.)

(a) f_n is positive and unimodal as the denominator is a positive parabola, the mode is at $x = \frac{1}{n}$, hence, for $x \leq 0$, $f_n(x) \leq f_n(0) \to 0$, and for $x > 0$, $n \geq \frac{2}{x}$, $f_n(x) \leq f_n(\frac{2}{n}) = f_n(0) \to 0$ as well.

(b) $|f_n(\frac{1}{n}) - f(\frac{1}{n})| = 1$.

> See an example of a similar function in the solution of Exercise 8.

10.

Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:

$$A = \{x \times y | xy = 1\},$$
$$S^1 = \{x \times y | x^2 + y^2 = 1\},$$
$$B^2 = \{x \times y | x^2 + y^2 \leq 1\}.$$

The set B^2 is called the (closed) ***unit ball*** in $\mathbb{R}^2$.

Using Lemma 21.4, the sets are continuous preimages of closed subsets of $\mathbb{R}$, and we may apply Theorem 18.1.

11.

Prove the following standard facts about infinite series:

(a) Show that if (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then (s_n) converges.

(b) Let (a_n) be a sequence of real numbers; define

$$s_n = \sum_{i=1}^{n} a_i.$$

If $s_n \to s$, we say that the ***infinite series***

$$\sum_{i=1}^{\infty} a_i$$

converges to s also. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum(ca_i + b_i)$ converges to $cs + t$.

(c) Prove the ***comparison test*** for infinite series: If $|a_i| \leq b_i$ for each i, and if the series $\sum b_i$ converges, then the series $\sum a_i$ converges. [*Hint:* Show that the series $\sum |a_i|$ and $\sum c_i$ converge, where $c_i = |a_i| + a_i$.]

(d) Given a sequence of functions $f_n : X \to \mathbb{R}$, let

$$s_n(x) = \sum_{i=1}^{n} f_i(x).$$

Prove the ***Weierstrass M-test*** for uniform convergence: If $|f_i(x)| < M_i$, for all $x \in X$ and all i, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s. [*Hint:* Let $r_n = \sum_{i=n+1}^{\infty} M_i$. Show that if $k > n$, then $|s_k(x) - s_n(x)| \leq r_n$; conclude that $|s(x) - s_n(x)| \leq r_n$.]

(a) It converges to $s = \sup_{n\in\mathbb{Z}_+}\{s_n\}$, as for every $\epsilon > 0$, there is N such that $s_N \in (s-\epsilon, s]$, and, hence, for $n \geq N$, $s_N \leq s_n \leq s$ implies $s_n \in (s-\epsilon, s]$.

(b) If $c = 0$ then $\sum(ca_i + b_i) = \sum b_i$ converges to $cs + t = t$. If $c \neq 0$, then for every $\epsilon > 0$, there is a sufficiently large N such that for $n \geq N$, $|\sum_{i=1}^{n} a_i - s| < \frac{\epsilon}{2|c|}$ and $|\sum_{i=1}^{n} b_i - s| < \frac{\epsilon}{2}$, therefore, $|\sum_{i=1}^{n}(ca_i + b_i) - (cs + t)| < \epsilon$.

(c) Using the hint, $s_n = \sum_{i=1}^{n} |a_i|$ is a sequence such that $s_n \leq s_{n+1}$, and $s_n \leq \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{\infty} b_i = b$, hence, by (a), (s_n) converges. Similarly, $t_n = \sum_{i=1}^{n} c_i$ form a non-decreasing sequence of real numbers that is bounded by $2b$. Therefore, by (b), $\sum a_i = \sum(-1 \cdot |a_i| + c_i)$ converges.

(d) By (c), s_n converges point-wise to some function s. Now, using the hint, for $k > n$, $|s_k(x) - s_n(x)| \leq \sum_{i=n+1,\ldots,k} |f_i(x)| \leq r_n$. Therefore, $|s(x) - s_n(x)| \leq r_n$ uniformly for all x (if that was not true than for some $x \in X$, $|s(x) - s_n(x)| - r_n = \delta > 0$, and for some sufficiently large k such that $|s(x) - s_k(x)| < \frac{\delta}{2}$, $|s_k(x) - s_n(x)| \geq |s(x) - s_n(x)| - |s(x) - s_k(x)| > r_n + \delta - \frac{\delta}{2} > r_n$). Now, note that for any $\epsilon > 0$ we can find N such that for $n \geq N$, $r_n = \sum_{i\in\mathbb{Z}_+} M_i - \sum_{i=1}^{n} M_i < \epsilon$.

12.

Prove continuity of the algebraic operations on $\mathbb{R}$, as follows: Use the metric $d(a,b)=|a-b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ given by the equation

$$\rho((x,y),(x_0,y_0))=\max\{|x-x_0|,|y-y_0|\}.$$

(a) Show that addition is continuous. [*Hint:* Given ϵ, let $\delta=\epsilon/2$ and note that

$$d(x+y,x_0+y_0)\le|x-x_0|+|y-y_0|.]$$

(b) Show that multiplication is continuous. [*Hint:* Given (x_0,y_0) and $0<\epsilon<1$, let

$$3\delta=\epsilon/(|x_0|+|y_0|+1)$$

and note that

$$d(xy,x_0y_0)\le|x_0||y-y_0|+|y_0||x-x_0|+|x-x_0||y-y_0|.]$$

(c) Show that the operation of taking reciprocals is a continuous map from $\mathbb{R}-\{0\}$ to $\mathbb{R}$. [*Hint:* Show the inverse image of the interval (a,b) is open. Consider five cases, according as a and b are positive, negative, or zero.]

(d) Show that the subtraction and quotient operations are continuous.

(a) Using the hint, if $(x,y)\in B_\rho((x_0,y_0),\delta)$ then $d(x+y,x_0+y_0)\le|x-x_0|+|y-y_0|<2\delta=\epsilon$, i.e. $x+y\in B_d(x_0+y_0,\epsilon)$.

(b) I am not sure why there is 3 in the expression for δ. Also, instead of assuming $\epsilon<1$, we can just take $\delta=\min\{\epsilon/(|x_0|+|y_0|+1),\frac{1}{2}\}$. Using the hint, if $(x,y)\in B_\rho((x_0,y_0),\delta)$ then $d(xy,x_0y_0)\le|x_0||y-y_0|+|y_0||x-x_0|+|x-x_0||y-y_0|\le|x_0|\delta+|y_0|\delta+\delta^2<|x_0|\delta+|y_0|\delta+\delta\le\epsilon$, i.e. $xy\in B_d(x_0y_0,\epsilon)$.

(c) The inverse image of (a,b) is (we may consider finite intervals only): $(1/b,1/a)$ if they have the same sign, $(-\infty,1/a)\cup(1/b,+\infty)$ if they have different signs, $(1/b,+\infty)$ if $a=0$, and $(-\infty,1/a)$ if $b=0$.

(d) Since $f(x)=-x$ is continuous (the preimage of any interval (a,b), $(-b,-a)$, is open), by Exercise 10 of §18 as well as (a), (b) and (c), $x-y=x+(-y)$ and $\frac{x}{y}=x\cdot\frac{1}{y}$ are continuous as composites of continuous functions.

SETCTION 22

1.

Check the details of Example 3.

Considering all 8 subsets of A and their preimages, $\emptyset$, $\{a,b,c\}$, $\{a\}$, $\{b\}$ and their union $\{a,b\}$ should be open in A, while $\{c\}$, $\{c,a\}$ and $\{c,b\}$ should not be open (but they are closed).

2.

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

(b) If $A \subset X$, a ***retraction*** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

(a) We can reformulate as follows: if a continuous function has a continuous right inverse then it is a quotient map. Suppose p has a right inverse, then, by Exercise 5(a) of §2, it is surjective. Since it is also continuous, all we need to show is that it maps open saturated sets to open sets. Let $A = p^{-1}(B)$ be open in X. Then, $f^{-1}(A) = f^{-1}(p^{-1}(B)) = (p \circ f)^{-1}(B) = B$ is open in Y as f is continuous.

(b) The inclusion map is a continuous right inverse of the retraction, apply (a).

3.

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

Consider $f : A \to \mathbb{R} \times \{0\}$ defined as $f(x, y) = (x, 0)$. Then, f is a retraction (as a continuous function on a restricted domain), hence, it is a quotient map (Exercise 2(b)). There is an obvious homeomorphism h of $\mathbb{R} \times \{0\}$ with $\mathbb{R}$ defined by $h(x, 0) = x$ (see also Exercise 4 of §18). Moreover, $q = h \circ f$. Therefore, q is a quotient map as well (Theorem 22.2). But $q(\mathbb{R} \times \mathbb{R}_+ \cap A) = \overline{\mathbb{R}}_+$ is not open in $\mathbb{R}$, and $q(\{(x, 1/x) | x > 0\}) = \mathbb{R}_+$ is not closed in $\mathbb{R}$.

4.

(a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0 + y_0^2 = x_1 + y_1^2.$$

Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [*Hint:* Set $g(x \times y) = x + y^2$.]

(b) Repeat (a) for the equivalence relation

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

(a) $\mathbb{R}$. Using the hint and Theorem 22.2, or rather Corollary 22.3, g is surjective and continuous (the preimage of an interval is the set of all points between two parabolas), and, by Corollary 22.3, it induces a bijective continuous map $f : X^* \to \mathbb{R}$. Moreover, $g \circ h = i_{\mathbb{R}}$ where $h(x) = x \times 0$ is a continuous (the preimage of $U \times V$ is either $\emptyset$ or U) right inverse of g. So, by Exercise 2(a), g is a quotient map, and, by Corollary 22.3, f is a homeomorphism.

(b) $\overline{\mathbb{R}}_+$. Similar to (a), using $g(x \times y) = x^2 + y^2$. g is surjective and continuous (the preimage of (a, b) for $0 \leq a < b$ is the set of all points between two circles, or punctured open disc if $a = 0$, and the preimage of $[0, c)$ is an open disc). Moreover, $g \circ h = i_{\overline{R}+}$ where $h : \overline{\mathbb{R}}_+ \to \mathbb{R}^2$ defined as $h(x) = \sqrt{x} \times 0$ is continuous as the composite of two continuous functions $\sqrt{x} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ (the preimage of (a, b) for $0 \leq a < b$ is (a^2, b^2), and the preimage of $[0, c)$ is $[0, c^2)$) and $x \times 0 : \overline{\mathbb{R}}_+ \to \mathbb{R}^2$ (the preimage of $U \times V$ is either $\emptyset$ or $U \cap \overline{R}_+$). Hence, g induces a homeomorphism of $X^* = \mathbb{R}^2 / \sim$ with $\overline{\mathbb{R}}_+$.

5.

Let $p: X \to Y$ be an open map. Show that if A is open in X, then the map $q: A \to p(A)$ obtained by restricting p is an open map.

If $U \subset A$ is open in A, then, since A is open in X, U is open in X, hence, $q(U) = p(U) \subset p(A)$ is open in X, and in $p(A)$.

This exercise, I believe, shows, in particular, that, in Theorem 22.1, if p is an open quotient map, and A is open, then even if A is not saturated, the restriction q is an open quotient map. In other words, in Theorem 22.1, instead of assuming that A is saturated, we can assume that both (1) A is open and (2) p is open.

6.

Recall that $\mathbb{R}_K$ denotes the real line in the K-topology. (See §13.) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point; let $p: \mathbb{R}_K \to Y$ be the quotient map.

(a) Show that Y satisfies the T_1 axiom, but is not Hausdorff.

(b) Show that $p \times p: \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. [*Hint:* The diagonal is not closed in $Y \times Y$, but its inverse image is closed in $\mathbb{R}_K \times \mathbb{R}_K$.]

(a) Recall that the set K is closed in $\mathbb{R}_K$ (but not in $\mathbb{R}$). Therefore, every one-point set $\{[1]\}$ and $\{x\}$, $x \notin K$, in Y is closed, and it is a T_1-topological space. At the same time, there is no neighborhood of 0 in $\mathbb{R}$ that does not contain points in K. Further, every neighborhood of 0 in $\mathbb{R}_K$ is of the form U or $U - K$ where U is open in $\mathbb{R}$, and, hence, contains a point $x \in K$. Then, U intersects every neighborhood V of x in $\mathbb{R}_K$. Overall, every neighborhood of 0 in Y that does not contain $[1]$ is of the form $U - K$, and every neighborhood of $[1]$ is the union of some neighborhoods of all points of K in $\mathbb{R}$, and the two must intersect as we said.

(b) Y is not Hausdorff, therefore, the diagonal Δ of $Y \times Y$ is not closed (by (a), every neighborhood of $0 \times [1]$ contains some $x \times x$: see §17, Hausdorff Spaces). At the same time,

$$\begin{aligned}(p \times p)^{-1}(\Delta) &= \{x \times y \in \mathbb{R}_K \times \mathbb{R}_K | x = y \text{ or } x, y \in K\} \\ &= \Delta_K \cup (K \times K)\end{aligned}$$

is closed as $\mathbb{R}_K$ is Hausdorff (the diagonal Δ_K is closed) and K is closed in $\mathbb{R}_K$.

Note that p is not an open quotient map, in particular, $B = p((\frac{1}{2}, 2)) = (\frac{1}{2}, 1) \cup (1, 2) \cup \{[1]\}$ is not open as $[1]$ has no neighborhood contained in B.

Supplementary Exercises

1.

Let H denote a group that is also a topological space satisfying the T_1 axiom. Show that H is a topological group if and only if the map of $H \times H$ into H sending $x \times y$ into $x \cdot y^{-1}$ is continuous.

If H is a topological group, then $f(x, y) = x \cdot y^{-1}$ is continuous as the composite of the following two continuous functions: $x \times y \to x \times y^{-1}$ (by Exercise 10 of §18) and $x \times y \to x \cdot y$. If f is continuous then it is continuous in both variables (Exercise 11 of §18), and, in particular, $f(1, y)$ and $f(x, f(1, y))$ ($x \times y \to x \times f(1, y) \to f(x, f(1, y))$) are continuous.

2.

Show that the following are topological groups:

(a) $(\mathbb{Z}, +)$

(b) $(\mathbb{R}, +)$

(c) $(\mathbb{R}_+, \cdot)$

(d) $(S^1, \cdot)$, where we take S^1 to be the space of all complex numbers z for which $|z| = 1$.

(e) The *general linear group* $GL(n)$, under the operation of matrix multiplication. ($GL(n)$ is the set of all nonsingular n by n matrices, topologized by considering it as a subset of euclidean space of dimension n^2 in the obvious way.)

(a) (b) (c) The spaces are Hausdorff, the operations are continuous (Exercise 12 of §21, and for $\mathbb{Z}$ we don' t even need that).

(d) The space is Hausdorff (as a subspace of a Hausdorff space). Let $f(x, y) = x \cdot y^{-1}$. If U is open in S^1, and $x \times y \in f^{-1}(U)$, then $x \cdot y^{-1} \in U \subset S^1$, and if $x = e^{i\phi}$ and $y = e^{i\theta}$, then $z = x \cdot y^{-1} = e^{i(\phi-\theta)}$. There is some $\epsilon > 0$ such that $z \in A_{\phi-\theta,\epsilon} \subset U$ where $A_{\alpha,\delta} = \{e^{i\gamma} | \alpha - \delta < \gamma < \alpha + \delta\}$, and for $x' \in A_{\phi,\epsilon/2}$ and $y' \in A_{\theta,\epsilon/2}$, we have $f(x', y') \in A_{\phi-\theta,\epsilon} \subset U$. Since $x \times y \in A_{\phi,\epsilon/2} \times A_{\theta,\epsilon/2} \subset f^{-1}(U)$, and $A_{\phi,\epsilon/2} \times A_{\theta,\epsilon/2}$ is open in $S^1 \times S^1$, $f^{-1}(U)$ is open in $S^1 \times S^1$, and f is continuous. By Exercise 1, $(S^1, \cdot)$ is a topological group.

(e) It is Hausdorff (as a subspace of a Hausdorff space). The operation of multiplication can be represented as a product of compositions of addition and multiplication (using Exercise 10 of §18 and Exercise 12 of §21). This should help.

3.

Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\overline{H}$ are topological groups.

Let $f : G \times G \to G$ be given by $f(x, y) = x \cdot y^{-1}$. $\overline{H}$ is a subgroup iff $f(\overline{H} \times \overline{H}) \subset \overline{H}$, which is true as $f(\overline{H} \times \overline{H}) = f(\overline{H \times H}) \subset \overline{f(H \times H)} \subset \overline{H}$ (f is continuous, see also Theorem 19.5). Now, if H (or $\overline{H}$) is a subgroup, then the restriction $f|_H : H \times H \to H$ (or $f|_{\overline{H}}$) is continuous (see Exercise 1), moreover, both H and $\overline{H}$ are T_1 -spaces.

4.

Let α be an element of G. Show that the maps $f_\alpha, g_\alpha : G \to G$ defined by

$$f_\alpha(x) = \alpha \cdot x \text{ and } g_\alpha(x) = x \cdot \alpha$$

are homeomorphisms of G. Conclude that G is a *homogeneous space*. (This means that for every pair x, y of points of G, there exists a homeomorphism of G onto itself that carries x to y.)

Note that $f_\alpha \circ f_{\alpha^{-1}}$, $f_{\alpha^{-1}} \circ f_\alpha$, $g_\alpha \circ g_{\alpha^{-1}}$ and $g_{\alpha^{-1}} \circ g_\alpha$ are the identity maps, so, f_α and g_α (and their inverse functions) are bijective and continuous (Exercise 11 of §18). For every x and y, $f_{y \cdot x^{-1}}(x) = y$, or $g_{x^{-1}y}(x) = y$.

5.

Let H be a subgroup of G. If $x \in G$, define $xH = \{x \cdot h | h \in H\}$; this set is called a ***left coset*** of H in G. Let G/H denote the collection of left cosets of H in G; it is a partition of G. Give G/H the quotient topology.

(a) Show that if $a \in G$, the map f_a of the preceding exercise induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$. Conclude that G/H is a homogeneous space.

(b) Show that if H is a closed set in the topology of G, then one-point sets are closed in G/H.

(c) Show that the quotient map $p : G \to G/H$ is open.

(d) Show that if H is closed in the topology of G and is a normal subgroup of G, then G/H is a topological group.

Note that for each x, $x = x \cdot e \in xH$, and $xH = yH$ iff $y = x \cdot z$ for some $z \in H$.

(a) Let $p : G \to G/H$ be the quotient map. Then, $p_\alpha = p \circ f_\alpha : G \to G/H$ is a quotient map as the composite of a homeomorphism with a quotient map. It maps every $x \in G$ to $\alpha x H \in G/H$. Now, $\alpha x H = \alpha y H$ iff $\alpha \cdot x \cdot z = \alpha \cdot y$ for some $z \in H$ iff $y = x \cdot z$ for some $z \in H$ iff $xH = yH$, therefore, p_α is constant on sets $p^{-1}(xH)$. By Corollary 22.3, p_α induces a homeomorphism $h_\alpha : G/H \to G/H$ carrying each xH to $\alpha x H$. Moreover, $h_{y \cdot x^{-1}}(xH) = yH$.

(b) H is closed, f_α is a homeomorphism, therefore, $xH = f_x(H)$ is closed in G, and $\{xH\}$ is closed in G/H.

(c) Let U be open in G. Since g_α is a homeomorphism, $p(U) = \cup_{x \in U} xH = \cup_{x \in U} \cup_{h \in H} x \cdot h = \cup_{h \in H} g_h(U)$ is open in G, and $p(U)$ is open in G/H.

(d) The operation is defined on G/H as follows: $xH \cdot yH = (x \cdot y)H$ which we denote simply as xyH. This operation is well-defined as if $xH = x'H$ and $yH = y'H$, then for some $z, w \in H$, $x \cdot y = x' \cdot z \cdot y' \cdot w$, and, since H is normal, $z \cdot y' = y' \cdot z'$ for some $z' \in H$, therefore, $x \cdot y = x' \cdot y' \cdot (z' \cdot w)$ and $xyH = x'y'H$. Note, that eH is the identity element of G/H, $\cdot$ is associative, and $(xH)^{-1} = x^{-1}H$, hence, $(G/H, \cdot)$ is a group. Using (b), G/H is also a T_1-space. Using Exercise 1, it is sufficient to show that $h(xH, yH) = (xH) \cdot (yH)^{-1}$ is continuous. Now, let $f(x, y) = x \cdot y^{-1}$, which is continuous by Exercise 1 again. $h(xH, yH) = f(x, y)H$, therefore, $h \circ (p \times p) = p \circ f$. $p \circ f$ is continuous as a composite of continuous maps, moreover, by (c), p is an open quotient map so that $p \times p$ is an open quotient map as well, therefore, using Theorem 22.2, $p \circ f$ induces the continuous function h via the quotient $p \times p$.

6.

The integers $\mathbb{Z}$ are a normal subgroup of $(\mathbb{R}, +)$. The quotient $\mathbb{R}/\mathbb{Z}$ is a familiar topological group; what is it?

$[0, 1)$ with addition by modulus, which is homeomorphic and isomorphic to $(S^1, \cdot)$. Indeed, $x\mathbb{Z} = y\mathbb{Z}$ iff $y = x + z$ for some $z \in \mathbb{Z}$, so let $f(x) = e^{i2\pi x}$, then f is a quotient map and $f(x) = f(y)$ iff $y = x + z$ for some $z \in \mathbb{Z}$ iff $x\mathbb{Z} = y\mathbb{Z}$, i.e., by Corollary 22.3, f induces the homeomorphism $h : \mathbb{R}/\mathbb{Z} \to S^1$ (via the quotient $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$) such that $h(x\mathbb{Z}) = f(p^{-1}(x\mathbb{Z})) = f(\{x + z | z \in \mathbb{Z}\}) = e^{i2\pi x}$, in particular, $h(x\mathbb{Z} + y\mathbb{Z}) = h((x + y)\mathbb{Z}) = e^{i2\pi(x+y)} = h(x\mathbb{Z}) \cdot h(y\mathbb{Z})$.

7.

If A and B are subsets of G, let $A \cdot B$ denote the set of all points $a \cdot b$ for $a \in A$ and $b \in B$. Let A^{-1} denote the set of all points a^{-1}, for $a \in A$.

(a) A neighborhood V of the identity element e is said to be ***symmetric*** if $V = V^{-1}$. If U is a neighborhood of e, show there is a symmetric neighborhood V of e such that $V \cdot V \subset U$. [*Hint:* If W is a neighborhood of e, then $W \cdot W^{-1}$ is symmetric.]

(b) Show that G is Hausdorff. In fact, show that if $x \neq y$, there is a neighborhood V of e such that $V \cdot x$ and $V \cdot y$ are disjoint.

(c) Show that G satisfies the following separation axiom, which is called the ***regularity axiom***. Given a closed set A and a point x not in A, there exist disjoint open sets containing A and x, respectively. [*Hint:* There is a neighborhood V of e such that $V \cdot x$ and $V \cdot A$ are disjoint.]

(d) Let H be a subgroup of G that is closed in the topology of G; let $p : G \to G/H$ be the quotient map. Show that G/H satisfies the regularity axiom. [*Hint:* Examine the proof of (c) when A is saturated.]

(a) $x \cdot y$ and $x \cdot y^{-1}$ are continuous, therefore, there is a neighborhood V' of e such that $V' \cdot V' \subset U$ and W such that $V = W \cdot W^{-1} \subset V'$. Therefore, $V \cdot V \subset U$, and $x \in V^{-1}$ iff $x = (y \cdot y^{-1})^{-1} = y \cdot y^{-1}$ for some $y \in W$ iff $x \in V$.

(b) We need a symmetric neighborhood of e such that for no $u^{-1}, v \in V : u^{-1} \cdot x = v \cdot y$ or $x \cdot y^{-1} = u \cdot v$. In other words, we need a symmetric neighborhood V of e such that $V \cdot V$ does not contain $x \cdot y^{-1}$. G is a T_1 -space, therefore, $\{x \cdot y^{-1}\}$ is closed. Using (a), we construct V for $U = G - \{x \cdot y^{-1}\}$. It remains to note that since g_x and g_y are homeomorphisms (Exercise 4), $V \cdot x$ and $V \cdot y$ are disjoint open neighborhoods of x and y.

(c) We need a symmetric neighborhood of e such that for no $u^{-1}, v \in V$ and $y \in A$: $u^{-1} \cdot y = v \cdot x$ or $y \cdot x^{-1} = u \cdot v$. In other words, we need a symmetric neighborhood V of e such that $V \cdot V$ does not intersect $A \cdot x^{-1}$. $A \cdot x^{-1}$ is closed ($g_{x^{-1}}$ is a homeomorphism by Exercise 4). Using (a), we construct V for $U = G - A \cdot x^{-1}$. It remains to note that since g_x and g_y for $y \in A$ are homeomorphisms (Exercise 4), $V \cdot x$ and $V \cdot A = \cup_{y \in A} V \cdot y$ are disjoint open neighborhoods of x and A.

(d) H is closed, therefore, G/H is a T_1 -space (Exercise 5(b)), and xH is closed (both as a subset of G and as an element of G/H). A closed subset of G/H is an image of a closed saturated subset of G. Let A be a saturated closed subset of G that does not intersect xH. Then, $A = A \cdot H$ ($A = p^{-1}(p(A)) = p^{-1}(\cup_{a \in A} aH) = \cup_{a \in A} aH = A \cdot H$). Using (c), let V be a symmetric neighborhood of e such that $V \cdot A$ does not intersect $V \cdot x$. Using Exercise 5(c), $p(V \cdot A) = \cup_{v \in V, a \in A} vaH$ is an open neighborhood of $p(A)$ disjoint from $p(V \cdot x) = \cup_{v' \in V} v' xH$, an open neighborhood of xH. Indeed, we they are not disjoint, then for some elements $v, v' \in V$ and $h, h' \in H$, $v \cdot a \cdot h = v' \cdot x \cdot h'$, and $v \cdot a \cdot h \cdot h'^{-1} = v' \cdot x$ where $a \cdot h \cdot h'^{-1} \in A \cdot H = A$, contradicting $V \cdot A \cap V \cdot x = \emptyset$.