



---

An Easy Proof of the Fundamental Theorem of Algebra

Author(s): Charles Fefferman

Source: *The American Mathematical Monthly*, Vol. 74, No. 7 (Aug. - Sep., 1967), pp. 854-855

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2315823>

Accessed: 25/07/2009 12:00

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).



Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

<http://www.jstor.org>

so that, for  $c$  as stated,  $w_k$  does converge to a root of (1).

Now let  $p$  be any nonconstant polynomial,  $z_0$  any complex number such that  $p'(z_0) \neq 0$ . With the substitutions  $z = z_0 + w$ ,  $a = p(z_0) + p'(z_0)c$ , the result just obtained shows that the equation  $p(z) = a$  has a root for all values of  $a$  close enough to  $p(z_0)$ . In other words, if  $z_0$  is not a zero of  $p'$ ,  $p(z_0)$  is an interior point of  $S$ . This proves the Lemma.

**THEOREM.** *With the notation of the lemma,  $S$  is the set of all complex numbers.*

*Proof.* The complement of  $S$  is open (B), and  $S - T$  is open by the lemma. But as  $T$  is finite (A) its complement cannot consist of two disjoint nonempty open sets (C). Hence the complement of  $S$  must be empty.

### AN EASY PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

CHARLES FEFFERMAN, University of Maryland

**THEOREM.** *Let  $P(z) = a_0 + a_1z + \cdots + a_nz^n$  be a complex polynomial. Then  $P$  has a zero.*

*Proof.* We shall show first that  $|P(z)|$  attains a minimum as  $z$  varies over the entire complex plane, and next that if  $|P(z_0)|$  is the minimum of  $|P(z)|$ , then  $P(z_0) = 0$ .

Since  $|P(z)| = |z|^n |a_n + a_{n-1}/z + \cdots + a_0/z^n|$  ( $z \neq 0$ ), we can find an  $M > 0$  so large that

$$(1) \quad |P(z)| \geq |a_0| \quad (|z| > M).$$

Now, the continuous function  $|P(z)|$  attains a minimum as  $z$  varies over the compact disc  $\{ |z| \leq M \}$ . Suppose, then, that

$$(2) \quad |P(z_0)| \leq |P(z)| \quad (|z| \leq M).$$

In particular,  $|P(z_0)| \leq P(0) = |a_0|$  so that, by (1),  $|P(z_0)| \leq |P(z)|$  ( $|z| > M$ ). Comparing with (2), we have

$$(3) \quad |P(z_0)| \leq |P(z)| \quad (\text{all complex } z).$$

Since  $P(z) = P((z - z_0) + z_0)$ , we can write  $P(z)$  as a sum of powers of  $z - z_0$ , so that for some complex polynomial  $Q$ ,

$$(4) \quad P(z) = Q(z - z_0).$$

By (3) and (4),

$$(5) \quad |Q(0)| \leq |Q(z)| \quad (\text{all complex } z).$$

We shall show that  $Q(0) = 0$ . This will establish the theorem since, by (4),  $P(z_0) = Q(0)$ .

Let  $j$  be the smallest nonzero exponent for which  $z^j$  has a nonzero coefficient in  $Q$ . Then we can write  $Q(z) = c_0 + c_j z^j + \cdots + c_n z^n$  ( $c_j \neq 0$ ). Factoring

$z^{j+1}$  from the higher terms of this expression, we have

$$(6) \quad Q(z) = c_0 + c_j z^j + z^{j+1} R(z),$$

where  $c_j \neq 0$  and  $R$  is a complex polynomial.

If we set  $-c_0/c_j = r e^{i\theta}$ , then the constant  $z_1 = r^{1/j} e^{i\theta/j}$  satisfies

$$(7) \quad c_j z_1^j = -c_0.$$

Let  $\epsilon > 0$  be arbitrary. Then, by (6),

$$(8) \quad Q(\epsilon z_1) = c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1).$$

Since polynomials are bounded on finite discs, we can find an  $N > 0$  so large that, for  $0 < \epsilon < 1$ ,  $|R(\epsilon z_1)| \leq N$ . Then, by (7) and (8) we have, for  $0 < \epsilon < 1$ ,

$$\begin{aligned} |Q(\epsilon z_1)| &\leq |c_0 + c_j \epsilon^j z_1^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &\leq |c_0 + \epsilon^j (c_j z_1^j)| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ (9) \quad &= |c_0 + \epsilon^j (-c_0)| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ &= (1 - \epsilon^j) |c_0| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ &= |c_0| - \epsilon^j |c_0| + \epsilon^{j+1} (|z_1|^{j+1} N). \end{aligned}$$

If  $|c_0| \neq 0$ , then we can take  $\epsilon$  so small that  $\epsilon^{j+1} (|z_1|^{j+1} N) < \epsilon^j |c_0|$ . In that case, by (9)

$$|Q(\epsilon z_1)| \leq |c_0| - \epsilon^j |c_0| + \epsilon^{j+1} (|z_1|^{j+1} N) < |c_0| = |Q(0)|,$$

contradicting (5). So  $|c_0| = 0$ , and therefore  $Q(0) = c_0 = 0$ .

## MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland  
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

*Material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue,  
N.W., Washington, D.C. 20005.*

### SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, III.

D. W. CROWE, University of Wisconsin

The Wisconsin High School Mathematical Talent Search operated for its third year in 1965-66, with continuing financial support from the National Science Foundation. Reports for preceding years are given in references [1], [2].