so that, for c as stated, w_k does converge to a root of (1).

Now let p be any nonconstant polynomial, z_0 any complex number such that $p'(z_0)\neq 0$. With the substitutions $z=z_0+w$, $a=p(z_0)+p'(z_0)c$, the result just obtained shows that the equation $p(z)=a$ has a root for all values of a close enough to $p(z_0)$. In other words, if z_0 is not a zero of p', $p(z_0)$ is an interior point of S. This proves the Lemma.

THEOREM. *With the notation of the lemma, S is the set of all complex numbers.*

Proof. The complement of S is open (B), and $S-T$ is open by the lemma. But as T is finite (A) its complement cannot consist of two disjoint nonempty open sets (C). Hence the complement of S must be empty.

AN EASY PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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THEOREM. *Let $P(z)=a_0+a_1z+\cdots+a_nz^n$ be a complex polynomial. Then P has a zero.*

Proof. We shall show first that $|P(z)|$ attains a minimum as z varies over the entire complex plane, and next that if $|P(z_0)|$ is the minimum of $|P(z)|$, then $P(z_0)=0$.

Since $|P(z)|=|z|^n|a_n+a_{n-1}/z+\cdots+a_0/z^n|$ $(z\neq 0)$, we can find an $M>0$ so large that

$$|P(z)| \geqq |a_0| \qquad (|z|>M). \tag{1}$$

Now, the continuous function $|P(z)|$ attains a minimum as z varies over the compact disc $\{|z| \,|z|\leqq M\}$. Suppose, then, that

$$|P(z_0)| \leqq |P(z)| \qquad (|z|\leqq M). \tag{2}$$

In particular, $|P(z_0)|\leqq P(0)=|a_0|$ so that, by (1), $|P(z_0)|\leqq|P(z)|$ $(|z|>M)$. Comparing with (2), we have

$$|P(z_0)| \leqq |P(z)| \qquad \text{(all complex } z). \tag{3}$$

Since $P(z)=P((z-z_0)+z_0)$, we can write $P(z)$ as a sum of powers of $z-z_0$, so that for some complex polynomial Q,

$$P(z) = Q(z-z_0). \tag{4}$$

By (3) and (4),

$$|Q(0)| \leqq |Q(z)| \qquad \text{(all complex } z). \tag{5}$$

We shall show that $Q(0)=0$. This will establish the theorem since, by (4), $P(z_0)=Q(0)$.

Let j be the smallest nonzero exponent for which z^j has a nonzero coefficient in Q. Then we can write $Q(z)=c_0+c_jz^j+\cdots+c_nz^n$ $(c_j\neq 0)$. Factoring

z^{j+1} from the higher terms of this expression, we have

$$Q(z) = c_0 + c_j z^j + z^{j+1}R(z), \tag{6}$$

where $c_j \neq 0$ and R is a complex polynomial.

If we set $-c_0/c_j = re^{i\theta}$, then the constant $z_1 = r^{1/j}e^{i\theta/j}$ satisfies

$$c_j z_1^j = -c_0. \tag{7}$$

Let $\epsilon > 0$ be arbitrary. Then, by (6),

$$Q(\epsilon z_1) = c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1). \tag{8}$$

Since polynomials are bounded on finite discs, we can find an $N > 0$ so large that, for $0 < \epsilon < 1$, $|R(\epsilon z_1)| \leqq N$. Then, by (7) and (8) we have, for $0 < \epsilon < 1$,

$$\begin{aligned} |Q(\epsilon z_1)| &\leqq |c_0 + c_j \epsilon^j z_1^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &\leqq |c_0 + \epsilon^j (c_j z_1^j)| + \epsilon^{j+1}(|z_1|^{j+1} N) \\ &= |c_0 + \epsilon^j(-c_0)| + \epsilon^{j+1}(|z_1|^{j+1}N) \\ &= (1 - \epsilon^j)|c_0| + \epsilon^{j+1}(|z_1|^{j+1}N) \\ &= |c_0| - \epsilon^j |c_0| + \epsilon^{j+1}(|z_1|^{j+1}N). \end{aligned} \tag{9}$$

If $|c_0| \neq 0$, then we can take ϵ so small that $\epsilon^{j+1}(|z_1|^{j+1}N) < \epsilon^j |c_0|$. In that case, by (9)

$$|Q(\epsilon z_1)| \leqq |c_0| - \epsilon^j |c_0| + \epsilon^{j+1}(|z_1|^{j+1}N) < |c_0| = |Q(0)|,$$

contradicting (5). So $|c_0| = 0$, and therefore $Q(0) = c_0 = 0$.

MATHEMATICAL EDUCATION NOTES

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SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, III.

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