Geometry and Physics
Author(s): Michael Atiyah
Source: *The Mathematical Gazette*, Vol. 80, No. 487, Centenary Issue (Mar., 1996), pp. 78-82
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3620334
Accessed: 24/07/2009 20:06

Geometry and Physics

MICHAEL ATIYAH

1 Introduction

In my 1982 Presidential Address to the Mathematical Association I tried to explain the general role of Geometry in Mathematics so it seems appropriate, on this centenary occasion, that I should move beyond the confines of Mathematics and discuss the interrelation of Geometry and Physics. There are two very good reasons for doing this. One is historical and arises from the close ties between the two subjects in their early evolution. A second and more topical reason is that, over the past two decades, there has been a remarkable burst of interaction of a quite unexpected kind between Geometry and Physics.

I will begin therefore by a brief look at the historical development before moving to describe the exciting recent events.

2 The classical period

From the earliest times, the geometry of space has served as the arena for physics. Euclid worked hard to abstract geometry out of space, but it was always realized that this was essentially a way of imposing a formal mathematical structure onto the physical world. The task of the geometer was to describe and study the possible shapes and mutual relations of idealised spatial objects such as lines, triangles or circles. These would then provide the language for the physicist.

By the time of Isaac Newton the requirements of dynamics, bringing in the notion of time, opened up a new area. The crucial new idea was that of the gravitational force, acting at a distance. According to Newton a force manifested itself through a deviation from uniform straight line motion. Forces produced curved paths and the strength of the force was reflected in the amount of curvature.

This linkage between force and curvature has proved to be one of the most long-lasting and fruitful ideas in the whole of Physics. Its greatest triumph was in Einstein's Theory of General Relativity where gravitational force is interpreted as the curvature of 4-dimensional space-time. Moreover the other great classical force, electromagnetism, can also be interpreted as a suitable curvature. However, this interpretation requires the addition of a fifth angular or phase variable.

This use of 5 or higher dimensions (known generically as Kaluza-Klein theories) has been increasingly adopted as Physics has developed. On the one hand, other forces operating at the nuclear level appear to be most easily interpreted in these terms. Moreover quantum mechanics, with its emphasis on vibrating modes, tells us that, if the additional dimensions (above 4) have small overall scale (eg if the radius of the phase circle is small) then they are not detected except at very high energies. This explains why, in ordinary life, 4 dimensions are all that we notice.

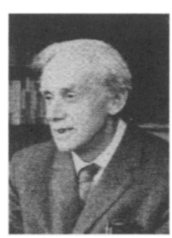
In all theories of the Einstein-Maxwell type (now called gauge theories) the underlying physical entity is the geometry of space-time with all its additional dimensions. Geometry here, as the etymology implies, represents measurement of distances. The resulting force is the curvature or distortion, representing the deviation of measurements from what they would be in a vacuum.

3 Topology and quantum theory

This classical picture, identifying force with curvature, acquires totally new features when we consider quantum physics in a serious way. Ever since the 1920s our picture of classical physics has been profoundly changed. At the fundamental level of small distances or high energies quantum features become significant. In particular there are two characteristic aspects of quantum theory which appear at first sight very non-geometric. First, there is the fact that certain physical quantities such as angular momentum or electric charge only appear in discrete integer multiples of some basic unit. The familiar continuum of Geometry appears to be discarded. Secondly, the notion of precise localisation in space cannot always be maintained, undermining the fundamentals of Geometry.

Although quantum theory appears in these ways to be ungeometrical it has slowly been realised that there is a resolution of these difficulties. This involves Topology, the branch of Mathematics which has arisen out of Geometry and which concerns itself with global and qualitative aspects.

The simplest illustration of a topological concept is to consider a closed path in the plane which does not go through the origin. Such a path 'winds round' the origin a certain number of times. This 'winding number' is a discrete integer (which can be negative if we go backwards or anti-clockwise) and is a topological quantity. It does not depend on the precise shape or length of the closed path. It is also a global notion, because we cannot ascribe a winding number to a small portion of the path, only to the whole path.



The first connection between Topology and quantum theory was the argument produced by Dirac to explain the 'quantisation' of electric charge – the fact that all particles in nature have an electric charge which is an integral multiple of the charge of an electron. Dirac's argument was based on the assumption of a hypothetical 'magnetic monopole', a particle which would radiate a magnetic field like a charged particle radiates an electric field. Such magnetic monopoles are allowed by Maxwell's equations of electromagnetism and Dirac studied the behaviour of an electron wandering around in the field of such a monopole. He showed that its wave-function would not be single-valued but would have integer jumps related to a '2-dimensional winding number' (analogous to the 1-dimensional winding number of a closed planar path).

4 Knots

A more interesting topological problem concerns knots in 3-dimension space. By a knot we mean here a closed path in space that does not cross itself. It can, of course, be visualized as a closed knotted piece of string.

The study of knots is a branch of topology, because what is of interest is not the exact shape or length of the string but its 'knottedness'. The fundamental question we can ask is whether two given knots are equivalent, that is whether we can move one around until it looks like the other.

Although ideas related to winding numbers are relevant to knots there is no simple integer which measures the 'knottedness' of a knot. More subtle ways of distinguishing knots are needed. A very useful tool for this purpose was discovered in 1928 by the American mathematician J. W. Alexander. He showed how, for each knot k, one could define a polynomial invariant $A_k(t)$. Here t is an indeterminate and $A_k(t)$ has integer coefficients (and negative powers of t are also allowed). This Alexander polynomial can be easily calculated from any plane projection of the knot in terms of the usual over/under crossings. The important point is that the result is independent of the particular projection and is the same for equivalent knots. For example, for the trefoil knot, the Alexander polynomial turns out to be

$$A(t) = t - 1 + t^{-1}$$

On the other hand the standard circle (which is 'unknotted') has Alexander polynomial equal to 1. This shows formally that the trefoil knot is not equivalent to a circle.

Although the Alexander polynomial is very useful it does not distinguish all knots. For example, it cannot distinguish a knot from its mirror image. The trefoil in particular comes in two forms (left-handed and right-handed) which are inequivalent but have the same Alexander polynomial.

It was therefore a great surprise when in 1984 the New Zealand mathematician Vaughan Jones discovered another polynomial invariant of knots which could distinguish knots from their mirror images. Unlike the Alexander polynomial the Jones polynomial has no simple interpretation in terms of standard topological ideas (like winding numbers). Instead it turns out that it can best be understood in terms of quantum theory.

The link between knots and quantum theory involves 2-dimensional physics, in which we ignore the third spatial dimension. Instead, by adding time, we consider space-time as 3-dimension and the knot represents a graph of a moving collection of point-particles. If these particles are of an appropriate type, governed by suitable forces, then the Jones polynomial is related to various quantities computed from the quantum fields associated with the space-time graph. The invariance of the Jones polynomial, in particular its independence of the choice of space-time axes, is then a reflection of the relativistic invariance of the corresponding physical theory. The fact that the Jones polynomial can distinguish mirror images rests on the fact that the physical theory involved is 'chiral', that is it distinguishes left from right. Such chiral theories are an important part of real physics and correspond to the lack of 'parity conservation' in certain physical processes.

5 Other applications

There are many other applications of quantum ideas to Geometry and Topology. Knots are just the easiest to describe. In particular, very exciting new results have been obtained in 4-dimensional topology by using quantum physics in conventional 3 space and 1 time dimensions. Again quantum theory can be used to produce invariants which help in classification. This time, instead of knots, one is interested in classifying closed 4-dimensional manifolds. These are the analogue of closed 2-dimensional surfaces, where the topological classification is rather simple. Every oriented closed surface is topologically a sphere with a certain number of handles attached. This number characterises the surface.

The idea of considering closed surfaces (or manifolds) of higher-dimension is not as mysterious or bizarre as it sounds. In fact this was a theme of my presidential address.

The great surprise was that the theory in 4-dimensions is quite unlike that in any other dimension. It is much subtler and the subtlety can be detected by ideas coming from quantum theory. This was the great discovery of Simon Donaldson from Oxford and the development of this theory has been a major feature of the past decade.

Within the past few months, remarkable developments and simplifications have taken place in this 4-dimensional theory and these again have arisen from physics. The key idea has been to exploit the duality between electricity and magnetism that is latent in Maxwell's equations and that lay behind Dirac's use of magnetic monopoles. Extending this duality to the theories of the more complicated nuclear forces has been a great challenge, but a recent breakthrough by N. Seiberg and E. Witten has opened up great possibilities. In particular, new light is shed on Donaldson's invariants.

6 Gravity

The physical theories involved in the examples above are all generalizations of Maxwell's theory of electromagnetism and the curvature is that of some higher dimensional spaces superimposed on space-time. By contrast Einstein's theory of General Relativity deals with 4-dimensional space-time alone. Its curvature represents the Gravitational Field.

As yet there is no satisfactory way of combining General Relativity and Quantum Theory. In other words we do not yet have a theory of Quantum

Gravity. The search for such a theory is the driving force in current research and there are many partial results and insights.

The fundamental conceptual difficulty is that in a quantum theory we have fluctuations and uncertainty. It is one thing to have a field fluctuating in a fixed space-time background. It is quite another thing to contemplate a space-time that is itself fluctuating, particularly when the fluctuations may be topological in nature such as attaching handles or boring worm-holes.

Because of these difficulties considerable attention has been paid to simple situations with lower dimensional space-times. In particular 1-dimensional space leading to 2-dimensional space-times have been much studied. As a mathematical by-product many surprising new facts have been discovered about the geometry of 2-dimensional surfaces.

7 Conclusion

Over the past two decades ideas from quantum physics have led directly to remarkable new mathematical discoveries across a very wide range of problems in Geometry. Usually the physical theories are 'formal' in the sense that they are not yet in rigorous mathematical form. Mathematicians have therefore to produce proofs based on alternative ideas and techniques. However, without the physical intuition and background the results in question would probably not have been discovered. The physics also provides an overall unifying conceptual framework, whereas the mathematics frequently degenerates into uninformative and varied techniques.

The reverse benefits in which the physicists benefit from the mathematics are also present though more difficult to assess. In physics the ultimate test is whether the theory explains all the experimental data and that stage has not yet been reached. But what is certainly true is that a new dialogue has been set up between mathematicians and physicists and ideas are constantly flowing both ways. The future looks exciting.

SIR MICHAEL ATIYAH, President 1981-82
Master's Lodge, Trinity College, Cambridge CB2 1TQ

Graham Hoare graduated in mathematics at Imperial College, London, in 1958. He became a teacher, first in Canterbury then at Dr Challoner's Grammar School, Amersham where he was Head of Mathematics and Deputy Head. He is co-author of *Mathematics with a Microcomputer* (1985) and has written several articles and notes for the *Gazette*, starting in 1966, and for other journals. He joined the Association in 1962 and is chair of the Organising Committee of the Royal Institution Mathematics Masterclasses, Editor of *Problem Corner* and a member of Council. He enjoys philosophy, logic, the music of Bach and 'exercise in high places'.