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ON HODGE'S THEORY OF HARMONIC INTEGRALS

BY HERMANN WEYL

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The attempt which W. V. D. Hodge made in Chapter III of his beautiful book[1] to establish *the existence of harmonic integrals with preassigned periods* has not been entirely successful because the proof is partly based on a false statement (p. 136) concerning the behavior of the solution of a non-homogeneous integral equation when the spectrum parameter approaches an eigen value. In a Princeton seminar on the subject, H. F. Bohnenblust pointed out that counter examples are readily available even for linear equations with a finite number of unknowns. For instance the equation $\lambda x + Ax = c$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is solvable for $\lambda = 0$ (x_1 arbitrary, $x_2 = 1$) and yet the solution for $\lambda \neq 0$,

$$x_1 = 1/\lambda, \qquad x_2 = 0$$

does not tend to a limit with $\lambda \to 0$.

In his book Hodge uses the *parametrix method* first developed for a single elliptic differential equation by E. E. Levi and D. Hilbert.[2] Building on the formal foundations laid by Hodge, I will show here how the argument can be made conclusive. Hilbert's procedure served me as a model.

Let n be the dimensionality of our Riemannian manifold. I denote by $^*u$, Du the dual form and the derivative of any (linear differential) form u and use the abbreviation Δ for the operator D^*D.[3] For two forms u, v of rank p, $n - p$ respectively (v, u) designates the integral of the product $v \cdot u$ over the whole manifold. $(^*u, u)$ is positive unless $u = 0$. An immediate consequence is

LEMMA I. *$\Delta u = 0$ implies $Du = 0$.*

Indeed $(D^*Du, u) = 0$ leads to $(^*Du, Du) = 0$, hence $Du = 0$.

In the following, f, u, φ, η are forms of rank p and g, v, ψ, ϑ forms of rank $n - p$.

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[1] *The Theory and Applications of Harmonic Integrals*, Cambridge, 1941. See also Proc. London Math. Soc. (2) **41,** 1936, pp. 483–496 where Hodge ascribes the idea of using Hilbert's parametrix method to H. Kneser. I find it hard to judge whether a previous proof along different lines (Proc. London Math. Soc. (2) **38,** 1933, p. 72) is complete, or rather how much effort is needed to make it complete. For the Euclidean case, see W. V. D. Hodge, Proc. London Math. Soc. (2) **36,** 1932, p. 257, and H. Weyl, Duke Math. Jour. **7,** 1940, pp. 411–444.

[2] E. E. Levi, Memorie della Società italiana delle Scienze, Ser. 3ª, Tom. 16, 1909; D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, 1912, pp. 223–231.

[3] The author intended $*u$, the printer evidently disliked it and replaced it by $^*u$. If a standardized notation in the theory of linear differential forms is adopted the author would recommend to follow him and not the printer.

The rank p is fixed; no induction with respect to p takes place. The goal is to prove the following

THEOREM I. *For any given null form g, $g \sim 0$, the equation*

$$\Delta u = g \tag{1}$$

has a solution u.

I copy Hodge's two basic formulas (3) and (4) on pp. 132, 133 of his book, replacing $p - 1$ by p and using the abbreviation $1/\gamma = (-1)^{np}(n - 2)\alpha_n$. Let K be the operator with the kernel $\gamma \cdot K_p(x, y)$ which carries any form $u(x)$ into $\gamma \int K_p(x, y) \cdot u(y)$, and K' its transpose. The "parametrix" operators Q, P with the kernels $\gamma \cdot \omega_p(x, y)$ and $\gamma \cdot \omega_{p-1}(x, y)$ are symmetric,

$$(Qv, g) = (Qg, v).$$

Finally, I set $DPD = \Pi$. Hodge's formulas read

$$Ku - u = Q\Delta u + (-1)^n \Pi^* u, \tag{2}$$

$$K'v - v = \Delta Qv + (-1)^{n*}\Pi v. \tag{2'}$$

The solutions of the equations

$$Ku - u = 0, \qquad K'v - v = 0$$

will be called the eigen forms of the kernels K and K' (*scilicet* "for the eigen value 1"). We try to solve our problem by means of the non-homogeneous integral equation suggested by (2),

$$Ku - u = Qg. \tag{E}$$

It is essential to study this equation not only for null forms but in a wider set $\mathcal{G}$; the success of the method depends on the proper choice of that linear space $\mathcal{G}$. Here is my definition:

g belongs to $\mathcal{G}$ whenever PDg is closed,

$$DPDg = \Pi g = 0.$$

Every form of the type

$$f = \Pi v \qquad (v \text{ arbitrary})$$

is said to belong to $\mathcal{F}$. Evidently $\mathcal{G}$ contains all *closed* forms g whereas all elements f of $\mathcal{F}$ are *null* forms. $\mathcal{F}$ and $\mathcal{G}$ are orthogonal:

LEMMA II. $(g, f) = 0$ *for* $g \in \mathcal{G}$, $f \in \mathcal{F}$.

Indeed, if PDg is closed, then

$$(PDg, Dv) = 0 = (Dg, PDv),$$

an equation which may at once be changed into

$$(g, DPDv) = 0.$$

I take over Hodge's Lemma I on p. 142:

LEMMA III. *If ψ is any eigen form of K' then $Q\psi$ is closed.*

For the sake of completeness I repeat the simple proof. Equation (2′) yields for $\xi = Q\psi$:

$$\Delta\xi = (-1)^{n-1}{}^*\Pi\psi, \tag{3}$$

hence $D^*\Delta\xi = D^*D^*D\xi = 0$ and then by double application of Lemma I,[4]

$$D^*D\xi = 0, \qquad D\xi = 0.$$

Incidentally we learn from (3) and the intermediate equation $\Delta\xi = 0$ that $\Pi\psi = 0$, or that *the eigen forms ψ of K' lie in $\mathfrak{G}$.*

We analyze the eigen forms of K and K' as follows. Within the linear space of all eigen forms $\bar{\varphi}$ of K we consider the subspace $\mathfrak{f}$ of the *closed* eigen forms φ and choose our basis

$$\varphi_1, \cdots, \varphi_l, \qquad \bar{\varphi}_1, \cdots, \bar{\varphi}_m$$

for all eigen forms accordingly, i.e. $\varphi_1, \cdots, \varphi_l$ span $\mathfrak{f}$. Equation (2) yields

$$Q\Delta\bar{\varphi} = (-1)^{n-1}\Pi^*\bar{\varphi}. \tag{4}$$

This proves on the one hand that each closed eigen form φ of K satisfies the condition $\Pi^*\varphi = 0$,

LEMMA IV. ${}^*\varphi \in \mathfrak{G}$ *for every* $\varphi \in \mathfrak{f}$.

It shows on the other hand that $\bar{\psi} = \Delta\bar{\varphi}$ satisfies the conditions

$$\Delta Q\bar{\psi} = 0, \qquad \Pi\bar{\psi} = 0$$

because the operators $\Delta\Pi$ and $\Pi\Delta$ annihilate. It then follows from (2′) that $\bar{\psi}$ is an eigen form of K'. The m forms $D\bar{\varphi}_1, \cdots, D\bar{\varphi}_m$ are linearly independent by construction, and hence by Lemma I the same is true for the forms

$$\bar{\psi}_1 = \Delta\bar{\varphi}_1, \cdots, \qquad \bar{\psi}_m = \Delta\bar{\varphi}_m.$$

The transposed kernel K' has the same number $l + m$ of linearly independent eigen forms as K. We determine a basis

$$\bar{\psi}_1, \cdots, \bar{\psi}_m; \qquad \psi_1, \cdots, \psi_l \tag{5}$$

of which the $\bar{\psi}$'s are a part.

The integral equation (E) is solvable if and only if

$$(Qg, \psi) = 0 = (g, Q\psi)$$

for every eigen form ψ of K', or with the notation $\xi = Q\psi$, if

$$(g, \xi) = 0. \tag{6}$$

[4] One differentiation may be saved here by applying the formula $(Ds, Dt) = 0$ holding for any two forms s, t with continuous first derivatives of rank $p - 1$ and $n - p - 1$ (see Weyl, l.c.[1], p. 426) to $s = PD\psi$ and $t = {}^*D\xi$ with the result $(\Pi\psi, \Delta\xi) = 0 = ({}^*\Delta\xi, \Delta\xi)$ whence $\Delta\xi = 0 = \Pi\psi$.

Let us say that ψ is *of the first kind* when $\xi = Q\psi \in \mathfrak{F}$. The forms $\bar{\psi}_1, \cdots, \bar{\psi}_m$ are of the first kind, on account of the equation (4). We choose our basis (5) so that

$$\bar{\psi}_1, \cdots, \bar{\psi}_m; \qquad \psi_1, \cdots, \psi_\nu$$

span the linear manifold of all eigen forms of K' of the first kind. By Lemma II the relation (6) holds good for *any* $g \in \mathfrak{G}$ in case ψ is of the first kind, and thus the $m + l$ conditions (6) reduce to the last $l - \nu$ of them,

$$(7) \qquad (g, Q\psi_{\nu+1}) = 0, \cdots, \qquad (g, Q\psi_l) = 0.$$

Let $\mathfrak{G}_1$ denote the set of those forms $g \in \mathfrak{G}$ which satisfy the conditions (7). *We have found that under the assumption $g \in \mathfrak{G}_1$ the integral equation* (E) *has a solution u.*

For this solution u we obtain from (2):

$$(8) \qquad Q(g - \Delta u) = (-1)^n \cdot \Pi^* u,$$

hence $\Delta Q(g - \Delta u) = 0$. Combining this with $\Pi(g - \Delta u) = 0$ and applying (2′) to $v = g - \Delta u$ one finds

$$g - \Delta u = \psi = \bar{c}_1\bar{\psi}_1 + \cdots + \bar{c}_m\bar{\psi}_m + c_1\psi_1 + \cdots + c_l\psi_l$$

to be an eigen form of K'. More precisely, because of (8), $Q\psi \in \mathfrak{F}$, ψ is an eigen form of the first kind, which forces $c_{\nu+1}, \cdots, c_l$ to vanish. Writing u for $u + \bar{c}_1\bar{\varphi}_1 + \cdots + \bar{c}_m\bar{\varphi}_m$ we arrive at the following

INTERMEDIARY PROPOSITION: *For any $g \in \mathfrak{G}_1$ there exists a form u and ν constants $c_1, \cdots, c_\nu$ such that*

$$(9) \qquad g - \Delta u = c_1\psi_1 + \cdots + c_\nu\psi_\nu.$$

We know from Lemma IV that the dual form $*\varphi$ of any element φ of $\mathfrak{f}$ lies in $\mathfrak{G}$. That subspace of $\mathfrak{f}$ the elements φ of which satisfy the conditions

$$(*\varphi, Q\psi_{\nu+1}) = 0, \cdots, (*\varphi, Q\psi_l) = 0$$

is of a dimensionality $\mu \geqq \nu$. Let the basis $\varphi_1, \cdots, \varphi_l$ of $\mathfrak{f}$ be so chosen that $\varphi_1, \cdots, \varphi_\mu$ span this subspace. From (9) we obtain for the ν unknowns c_β the μ linear equations

$$(10) \qquad \sum\nolimits_\beta H_{\alpha\beta} \cdot c_\beta = (g, \varphi_\alpha) \qquad \begin{pmatrix} \alpha = 1, \cdots, \mu; \\ \beta = 1, \cdots, \nu \end{pmatrix}$$

where

$$H_{\alpha\beta} = (\psi_\beta, \varphi_\alpha).$$

I maintain:

LEMMA V. *$\| H_{\alpha\beta} \|$ is a non-singular square matrix.*

Once this is established we have reached the goal. For then the ν conditions

$$(g, \varphi_\alpha) = 0 \qquad (\alpha = 1, \cdots, \nu)$$

imply $c_\alpha = 0$ whereby (9) reduces to $g - \Delta u = 0$. In other words, if $g \in \mathfrak{G}$ satisfies the relations

$$(11) \quad (g, Q\psi_{\nu+1}) = 0, \ \cdots, \ (g, Q\psi_l) = 0; \qquad (g, \varphi_1) = 0, \ \cdots, \ (g, \varphi_\nu) = 0$$

then the equation (1) is solvable. A null form g fulfills all our requirements, because the φ_i and $Q\psi_i$ are closed, the first by construction, the others by Lemma III.

PROOF OF LEMMA V. We have found the equations (10) to be solvable if $g \in \mathfrak{G}_1$. For

$$(12) \qquad \varphi = a_1\varphi_1 + \cdots + a_\mu\varphi_\mu$$

the integral $({}^*\varphi, \varphi)$ is a positive definite quadratic form of $a_1, \cdots, a_\mu$. Hence we can determine the coefficients a_i in (12) so as to assign arbitrary values b_α to the integrals

$$({}^*\varphi, \varphi_\alpha) \qquad (\alpha = 1, \cdots, \mu).$$

But $g = {}^*\varphi \in \mathfrak{G}_1$. Hence we see that the equations

$$\sum_\beta H_{\alpha\beta} c_\beta = b_\alpha \qquad \begin{pmatrix} \alpha = 1, \cdots, \mu; \\ \beta = 1, \cdots, \nu \end{pmatrix}$$

have a solution c_β for arbitrary b_α. In view of $\mu \geqq \nu$ this statement is equivalent to our lemma.

In proving Theorem I we actually showed that the equation $\Delta u = g$ is solvable if $g \in \mathfrak{G}$ satisfies the conditions (11). Hence each such g is a null form, and the linear space $\mathfrak{G}$ is of finite dimensionality $\leqq l$ modulo the space of null forms. As $\mathfrak{G}$ contains all closed forms of rank $n - p$, we find *a fortiori* that the number R'_{n-p} of linearly independent closed forms of rank $n - p$ modulo null is finite and $\leqq l$. The conditions (11) are of the type $(g, f) = 0$ where f runs over certain specified closed forms of rank p. Consider the "inner product" (g, f) of any two closed forms g, f of rank $n - p$ and p respectively; the factors matter only modulo null. Our proof implies this further fact:

THEOREM II. *If the inner product (g, f) vanishes for a given closed g and all closed f, then $g \sim 0$.*

It is of course also true that the product cannot vanish for a given closed f and all closed g unless $f \sim 0$. Both facts together give the duality law

$$(13) \qquad R'_{n-p} = R'_p .$$

Theorem II has nothing to do with any Riemannian metric. de Rham's second theorem follows at once from it by means of the expression of the product (g, f) in terms of the periods of g and f (Hodge, p. 85, last line), but it is essentially simpler since it deals with closed forms only, and not with forms and cycles. Its proof on an arbitrary manifold should be correspondingly easier.

The following proposition is equivalent to Theorem I for the rank $p - 1$ instead of p:

THEOREM III. *For any form f there exists a uniquely determined $\eta \sim f$ such that $^*\eta$ is closed. If f be closed, then η is harmonic.*

Indeed, set $f = Dt + \eta$, t being of rank $p - 1$. The requirement $D^*\eta = 0$ leads to the equation $D^*Dt = D^*f$ which is solvable by Theorem I.

The new proposition shows at once that for any rank p the space of closed forms modulo null may be identified with the space of harmonic forms. This makes the equation (13) particularly lucid because $^*u$ is harmonic if u is and vice versa. The same proposition provides another proof for Theorem II, because one has merely to substitute $^*\eta$ for ϑ in order to see that the vanishing of the inner product (η, ϑ) of a fixed harmonic form η with every harmonic ϑ implies $\eta = 0$. The observation (Hodge, p. 139) that on account of (2) the harmonic p-forms are eigen forms of K again proves the inequality $R'_p \leqq l$.

The link with the homology theory of cycles is established by de Rham's first theorem stating that a p-cycle C is homologous zero if the integral of every closed p-form f over C vanishes.

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.