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ARITHMETIC ON ALGEBRAIC VARIETIES

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D. G. Northcott has recently contributed some interesting new theorems ([4a], [4b]) to a subject which I introduced in my thesis [1] under the above-given title, and which had been further developed by Siegel [2] and myself [3]. It is my purpose here, by making explicit some concepts which had remained implicit in these papers, to supply what seems to be the proper algebraic foundations for that theory, and to give a comprehensive account of its results, including some new ones, up to date.

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I. THE ALGEBRAIC FOUNDATIONS.

1. Specializations and valuations. Let K be a field; we denote by K_∞ the set consisting of K, with its structure as a field, and one additional element, denoted by ∞; the domain of definition of the algebraic operations in K is extended, in the usual manner, to K_∞ by putting $a/\infty = 0, a \pm \infty = \infty$, for $a \in K$; $a.\infty = a/0 = \infty$ for $a \in K_\infty$, $a \neq 0$. A function taking its values in K_∞ will frequently be said, incorrectly but more briefly, to take its values in K, when the context makes the meaning sufficiently clear.

By a *ring*, we shall always understand a commutative ring without zero-divisors, with a unit-element $1 \neq 0$. By a *specialization* of a ring A we understand a homomorphism f of A into some ring, such that $f(1) = 1$. Two specializations f, f' are called equivalent if $f' = g \circ f$, where g is an isomorphism of $f(A)$ onto $f'(A)$; thus a specialization of A is completely determined up to equivalence by its kernel $\mathfrak{p} = \overset{-1}{f}(0)$, which is a prime ideal of A. Let K be a field, A a subring of K, f a specialization of A with values in a field L; put $\mathfrak{p} = \overset{-1}{f}(0)$; by the *specialization-domain* of f we understand the subset of K_∞ consisting of all elements of the form u/v, with $u \in A, v \in A, uv \notin \mathfrak{p}$; f is canonically extended to that domain by putting $f(u/v) = f(u)/f(v)$. By the *specialization-ring* and the *specialization-ideal* of f we understand the subsets of the domain of f defined by $f(x) \neq \infty$, and by

$f(x) = 0$, respectively. The specialization-ring of f is A if and only if $\mathfrak{p}$ is the set of all non-units of A, i.e. of all non-invertible elements of A. Any ring A in which the set $\mathfrak{p}$ of all non-units is an ideal will be called a *specialization-ring*, and $\mathfrak{p}$ is called its *specialization-ideal*; this is of course a prime ideal; as $\mathfrak{p}$ is the maximal ideal in A, $A/\mathfrak{p}$ is a field called the *residue-field* of A; the canonical mapping of A onto $A/\mathfrak{p}$ is called the canonical specialization of A. A ring A is a specialization-ring if and only if x is a unit in A whenever $1 - x$ is a non-unit in A.

2. A subring R of a field K is called a *valuation-ring of* K if $K_\infty = R \cup R^{-1}$; such a ring is always a specialization-ring, and its specialization-ideal will be called its *valuation-ideal*; the extension to K_∞ of its canonical specialization, or of any specialization equivalent to it, will be called a *place* of K. If such a place maps R into a field L, it is said to be *L-valued*. If $R = K$, the place is an isomorphism of K onto a field K'; such a place is called *trivial*.

If A is a ring we denote by $U(A)$ the multiplicative group of units, i.e. of invertible elements of A; if K is a field, $U(K)$ is the group K^* of non-zero elements of K. If A is any subring of a field K, the relation $y \in xA$ between elements x, y of K^* is a preorder relation on the group K^* (i.e. $y \in xA$, $z \in yA$, imply $z \in xA$; and $y \in xA$ implies $yz \in xzA$), and it induces an ordering on the group $K^*/U(A)$; this is a total ordering if and only if A is a valuation-ring of K. Conversely, let ω be a homomorphism of K^* into a totally ordered group Γ (which we shall write additively); we extend ω to a mapping of K_∞ into the union Γ_∞ of Γ and of a set of two elements $\{-\infty, +\infty\}$ by putting $\omega(0) = +\infty$, $\omega(\infty) = -\infty$; in Γ_∞, we put $-\infty < \alpha < +\infty$, $-\infty + \alpha = -\infty$, $+\infty + \alpha = +\infty$ for all $\alpha \in \Gamma$. Such a homomorphism ω is called a *valuation* of K if its satisfies $\omega(x + y) \geqq \inf [\omega(x), \omega(y)]$ for all x, y in K, hence also for all x, y in K_∞; then the elements x of K such that $\omega(x) \geqq 0$ form a valuation-ring R of K, whose group of units and valuation-ideal are the subsets respectively determined by $\omega(x) = 0$ and by $\omega(x) > 0$. Two valuations ω, ω' are called *equivalent* if $\omega' = \lambda\circ\omega$, where λ is an isomorphism of the ordered group $\omega(K^*)$ onto the ordered group $\omega'(K^*)$. Let R be any valuation-ring of K; the canonical mapping ω of K^* onto $\Gamma = K^*/U(R)$ is a valuation of K if Γ is given the order determined by the preorder relation $y \in xR$ on K^*; this is called the canonical valuation of K, belonging to R; every valuation ω' of K, such that R is the ring of elements x for which $\omega'(x) \geqq 0$, is then equivalent to ω. Thus there is a one-to-one correspondence between (a) the valuation-rings of K, (b) the canonical valuations of K, (c) the classes of mutually equivalent valuations of K, (d) the classes of mutually equivalent places of K. To a trivial place of K there corresponds the trivial valuation of K, which maps K^* into 0.

3. Let A be a specialization-ring in a field K, and $\mathfrak{p}$ its maximal ideal. Let ω be a valuation of K, R its valuation-ring, and $\mathfrak{P}$ its valuation-ideal. Then we say that ω, or R, or any one of the places of K corresponding to ω and R, are *algebraic over* A if the following conditions are satisfied: (a) ω is $\geqq 0$ on A, i.e. $R \supset A$; (b) ω is > 0 on $\mathfrak{p}$, i.e. $\mathfrak{P} \supset \mathfrak{p}$; then we have $A \cap \mathfrak{P} = \mathfrak{p}$, so that we may identify the residue-field $A/\mathfrak{p}$ of A with a subfield of the residue-field $R/\mathfrak{P}$ of R; (c) the field $R/\mathfrak{P}$ is algebraic over $A/\mathfrak{p}$. If A is a subfield k of K, condition (b) is of course trivially satisfied. If A is a specialization-ring of K, $\mathfrak{p}$ its maximal ideal, and k a subfield of K, we shall also say that A is *algebraic over* k if $A \supset k$, so that the canonical mapping of A onto $A/\mathfrak{p}$ maps k isomorphically onto its image, and if the residue-field $A/\mathfrak{p}$ of A is algebraic over the image of k in it under that mapping.

Let the specialization-ring A in K be algebraic over the subfield k of K. Then a valuation ω of K is algebraic over A if and only if it is $\geqq 0$ on A and algebraic over k. In fact, these conditions are obviously necessary. Assume now that they are satisfied; call again $\mathfrak{p}$ the maximal ideal of A, R the valuation-ring of ω, $\mathfrak{P}$ its valuation-ideal. Identify k with its image in $R/\mathfrak{P}$; then $R/\mathfrak{P}$ is algebraic over k. The image of A in $R/\mathfrak{P}$ contains k and is a subring of $R/\mathfrak{P}$; such a subring must be a field, since every non-zero element z in it, being algebraic over k, has a reciprocal $1/z$ in $k[z]$. Therefore the kernel $A \cap \mathfrak{P}$ of the homomorphism of A onto its image in $R/\mathfrak{P}$ is the maximal ideal $\mathfrak{p}$ of A, so that the condition (b) in our definition is satisfied. The rest is obvious.

4. We shall now prove that, if A is any specialization-ring in a field K, there are valuations of K which are algebraic over A. This is in substance the same as the theorem on the extension of specializations in algebraic geometry, and can also be expressed as follows:

THEOREM 1. *Every specialization of a subring A of a field K, taking its values in an algebraically closed field Ω, can be extended to an Ω-valued place of K.*

In fact, by Zorn's lemma, among all the specializations of subrings of K with values in Ω which extend the given one, there must be a maximal specialization, i.e. one which cannot be further extended. Let therefore f be such a maximal specialization; let it be defined on a ring R; we have to show that $K_\infty = R \cup R^{-1}$. Put $\mathfrak{p} = \overset{-1}{f}(0)$; R is a specialization-ring and $\mathfrak{p}$ is its maximal ideal, for otherwise could be extended to the ring of elements u/v with $u \in R$, $v \in R$, $v \notin \mathfrak{p}$, and would not be maximal. Therefore $R/\mathfrak{p}$ is a field, which we may identify with $f(R)$; put $k = f(R) = R/\mathfrak{p}$. Let x be in K, and not in R; put $R' = R[x]$; we prove that $R'\mathfrak{p} = R'$. In fact, if $R'\mathfrak{p} \neq R'$, Zorn's lemma shows that there must be, among the R'-ideals containing $\mathfrak{p}$ but not 1, a maximal ideal $\mathfrak{p}'$. Then $R \cap \mathfrak{p}'$ is an R-ideal containing $\mathfrak{p}$ but not 1, hence $R \cap \mathfrak{p}' = \mathfrak{p}$. Let F be the canonical homomorphism of R' onto $R'/\mathfrak{p}'$; on R, F induces the canonical homomorphism of R onto $R/(R \cap \mathfrak{p}') = R/\mathfrak{p} = k$, which is f; hence it maps $R' = R[x]$ onto $k[\xi]$, with $\xi = F(x)$. If ξ is algebraic over k, there is an isomorphism σ of $k(\xi)$ into Ω which leaves the elements of k invariant. If ξ is transcendental over k, let σ be the homomorphism of $k[\xi]$ onto k defined by $\sigma(\xi) = 0$, $\sigma(\alpha) = \alpha$ for $\alpha \in k$. In both cases $\sigma \circ F$ is a specialization of R' with values in Ω and it coincides with f on R; as $R' \neq R$, this contradicts the assumption that f is maximal. Hence we have $R'\mathfrak{p} = R'$, i.e. $1 \in \mathfrak{p}R[x]$, and so there is a relation $1 - \sum_{\mu=0}^{m} \varpi_\mu x^\mu = 0$, with $\varpi_\mu \in \mathfrak{p}$ for $0 \leqq \mu \leqq m$; similarly, if $x^{-1} \notin R$, there must be a relation $1 - \sum_{\nu=0}^{n} \varpi'_\nu x^{-\nu} = 0$, with $\varpi'_\nu \in \mathfrak{p}$ for $0 \leqq \nu \leqq n$. We may assume that each one of the two relations we have just written is one of smallest degree among all the relations of the same form. Now, if $m \geqq n$, multiply the former relation by $1 - \varpi'_0$, the latter by $\varpi_m x^m$, and add them; if $n \geqq m$, multiply the former by $\varpi'_n x^{-n}$, the latter by $1 - \varpi_0$, and add them; in both cases we get a relation of the same form as one of those we have written, but of smaller degree. Hence the assumption $x \notin R$, $x^{-1} \notin R$ implies contradiction.

5. The following elementary theorem and its corollaries are also not new:

THEOREM 2. *Let A be a subring of a field K, f a specialization of A, and $x = (x_\alpha)$ a set of elements of K. Then the following assertions are equivalent:* (a) *there is a polynomial $P \in A[X] = A[(X_\alpha)]$, such that $P(x) = 0$ and $f[P(0)] \neq 0$;* (b) *there is no specialization g of $A[x]$, coinciding with f on A, and such that $g(x_\alpha) = 0$ for all α.*

Let $\mathfrak{A}$ be the ideal in $A[X]$ consisting of all those P for which $P(x) = 0$. If g has the properties stated in (b), then $g[P(x)] = f[P(0)]$ for all $P \in A[X]$. Hence $f[P(0)] = 0$ for all $P \in \mathfrak{A}$, and so (a) implies (b). Assume now that (a) is false, i.e. that $f[P(0)] = 0$ for all $P \in \mathfrak{A}$. Then the mapping $P \rightarrow f[P(0)]$, being a specialization of $A[X]$ which vanishes on $\mathfrak{A}$, determines a specialization of $A[X]/\mathfrak{A}$, which we may identify with $A[x]$; this specialization has the properties stated in (b).

COROLLARY 1. *Let A be a subring of a field K, and $x = (x_\alpha)$ a set of elements of K. Then the following assertions are equivalent:* (a_1) *there is a polynomial $P \in A[X]$ such that $P(x) = 0$, $P(0) = 1$;* (b_1) *there is no specialization g of $A[x]$ such that $g(x_\alpha) = 0$ for all α;* (c_1) *no valuation of K is $\geqq 0$ on A and > 0 at x_α for every α.*

Let $\mathfrak{A}$ be as in the proof of Theorem 2. If g is as in (b_1), then $g[P(x)] = g[P(0)]$ for all $P \in A[X]$, hence $g[P(0)] = 0$ for all $P \in \mathfrak{A}$; therefore (a_1) implies (b_1). Assume now that (a_1) is false. As the mapping $P \rightarrow P(0)$ is a homomorphism of $A[X]$ onto A, the image $\mathfrak{a}$ of $\mathfrak{A}$ by it is an ideal in A; (a_1) being false, we have $\mathfrak{a} \neq A$; hence $\mathfrak{a}$ is contained in some maximal ideal $\mathfrak{p}$ in A, other than A. As $\mathfrak{p}$ is prime, the canonical mapping f of A onto $A/\mathfrak{p}$ is a specialization of A. As $f(\mathfrak{a}) = 0$, (a) of Theorem 2 is false for f, and so, by Theorem 2, f can be extended to a specialization g of $A[x]$ such that $g(x_\alpha) = 0$ for all α; hence (b_1) is false. As to (c_1), it is equivalent to (b_1) by Theorem 1.

COROLLARY 2. *Let A be a specialization-ring in a field K, and $x = (x_\alpha)$ a set of elements of K. Then the following assertions are equivalent:* (a_2) *there is a polynomial $P \in A[X]$ such that $P(x) = 0$, $P(0) = 1$;* (b_2) *no valuation of K is $\geqq 0$ on A and >0 at x_α for every α;* (c_2) *no valuation of K is algebraic over A and >0 at x_α for every α.*

In fact, if f is the canonical specialization of A, (a_2) is equivalent to saying that there is no $P \in A[X]$ such that $P(x) = 0, f[P(0)] \neq 0$, since $f(A)$ is a field; by Theorem 2 and Theorem 1, this is equivalent to (c_2). As to (a_2), (b_2), they are equivalent by Corollary 1.

6. Valuation-functions. Let K be a field, and A a subring of K; by $\mathbf{V}(K/A)$ we shall denote the set of all non-trivial canonical valuations of K which are $\geqq 0$ on A; if A is a specialization-ring in K, and in particular if it is a subfield of K, we shall denote by $\mathbf{V}_0(K/A)$ the set of all non-trivial canonical valuations of K which are $\geqq 0$ on A and algebraic over A. If A is the prime ring of K (i.e. the ring of rational integers if the characteristic of K is 0, and the prime field k_0 of p elements if the characteristic of K is $p > 1$), $\mathbf{V}(K/A)$ is the set of all non-trivial canonical valuations of K; this will also be denoted by $\mathbf{V}(K)$.

If $\mathbf{V}$ is any subset of $\mathbf{V}(K)$, we shall denote by $\mathbf{F}'(\mathbf{V})$ the product $\prod_{\omega \in \mathbf{V}} \omega(K^*)$ of all the value-groups $\omega(K^*)$ for $\omega \in \mathbf{V}$, with its structure as an ordered group; as each one of its factors is a totally ordered group, hence a lattice, this product is a lattice, i.e. sup and inf are defined, as binary operations, hence also as operations on finite sets of elements, and they have the usual properties; in particular they are mutually distributive. If $\mathbf{V}' \subset \mathbf{V}$, $\mathbf{F}'(\mathbf{V}')$ is a partial product of $\mathbf{F}'(\mathbf{V})$, so that there is a natural homomorphism, viz. the projection, of $\mathbf{F}'(\mathbf{V})$ onto $\mathbf{F}'(\mathbf{V}')$.

For each $\mathbf{V} \subset \mathbf{V}(K)$ and each $x \in K^*$, $\omega(x)$ is a function of ω, defined on $\mathbf{V}$, whose value at ω is an element of $\omega(K^*)$; this function determines an element of $\mathbf{F}'(\mathbf{V})$ which will be denoted by $[x]_{\mathbf{V}}$, or usually by $[x]$; we have $[1] = 0$, $[1/x] = -[x]$, $[xy] = [x] + [y]$, i.e. the mapping $x \rightarrow [x]$ is a homomorphism of the group K^* (with the trivial ordering, for which no two elements are comparable unless they are equal) into $\mathbf{F}'(\mathbf{V})$. We shall denote by $\mathbf{F}(\mathbf{V})$ the smallest subgroup of $\mathbf{F}'(\mathbf{V})$ containing $[x]$ for all $x \in K^*$, and closed with respect to the (binary) operations sup, inf in $\mathbf{F}'(\mathbf{V})$; the elements of this ordered group $\mathbf{F}(\mathbf{V})$ will be called the *valuation-functions* on $\mathbf{V}$. We shall write $\mathbf{F}(K)$, $\mathbf{F}(K/A)$, $\mathbf{F}_0(K/A)$, for $\mathbf{F}(\mathbf{V}(K))$, $\mathbf{F}(\mathbf{V}(K/A))$, $\mathbf{F}(\mathbf{V}_0(K/A))$, respectively. If we extend $\mathbf{F}(\mathbf{V})$ by the adjunction of one element $+\infty$, we write $[0] = +\infty$.

Because of the distributivity of sup, inf, every valuation-function can be written in the form $X = \inf_\mu \sup_i [x_{\mu,i}]$, where $(x_{\mu,i})_{1 \leqq \mu \leqq m; 1 \leqq i \leqq h_\mu}$ is a finite family of elements of K^*. Also because of the distributivity, the same element can be written as $X = \sup_{i(\mu)} \inf_\mu [x_{\mu,i(\mu)}]$, where μ runs over the set $\{1, \cdots, m\}$, and $i(\mu)$ over the set of all mappings of that set into the set of integers, satisfying $1 \leqq i(\mu) \leqq h_\mu$ for every μ. This gives $-X = \inf_{i(\mu)} \sup_\mu [1/x_{\mu,i(\mu)}]$. If $Y = \inf_\nu \sup_j [y_{\nu,j}]$ is another valuation-function, then $X + Y = \inf_{\mu,\nu} \sup_{i,j} [x_{\mu,i} y_{\nu,j}]$. The order relation will be completely determined by the knowledge of the valuation-functions which are $\succ 0$; and $X = Y$ is equivalent to $X - Y \succ 0$ and $Y - X \succ 0$. Since $\inf_\mu \sup_i [x_{\mu,i}] \succ 0$ if and only if $\inf_i [1/x_{\mu,i}] \prec 0$ for every μ, we see that the structure of $\mathbf{F}(\mathbf{V})$ is completely determined by the knowledge of those finite subsets (x_i) of K^* which are such that $\inf_i [x_i] \prec 0$ in $\mathbf{F}(\mathbf{V})$. In the cases in which we are interested, criteria for this are given by the corollaries of Theorem 2. In fact, if A is any subring of K, corollary 1 of Theorem 2 shows that $\inf_i [x_i] \prec 0$ in $\mathbf{F}(K/A)$ if and only if there is a polynomial $P \in A[X]$ such that $P(x) = 0$ and $P(0) = 1$; if A is a specialization-ring in K, the same criterion holds for $\mathbf{F}_0(K/A)$, by corollary 2 of Theorem 2; hence, in that case, the canonical homomorphism of $\mathbf{F}(K/A)$ onto $\mathbf{F}_0(K/A)$ is an isomorphism. From this it follows at once that a necessary and sufficient condition for $[y] \succ \inf_i [x_i]$ to hold in $\mathbf{F}(K/A)$, or in $\mathbf{F}_0(K/A)$, is that y

and the x_i should satisfy a relation $P(y, x_1, \cdots, x_n) = 0$, where P is a homogeneous polynomial with coefficients in A, such that $P(1, 0, \cdots, 0) = 1$. In particular, we have $[y] \succ 0$ if and only if y is integral over A, and $[y] = 0$ if and only if both y and $1/y$ are so. If k is a subfield of K, we have $[y] = 0$ in $\mathbf{F}(K/k)$ for the elements y of K^* which are algebraic over k, and only for those, and $[y] \succ 0$ only for those and for 0.

From what we have just proved it follows that, if A is a subring of K, the set A' of all elements of K which are integral over A is the set of all elements at which all the valuations in $\mathbf{V}(K/A)$ are $\geqq 0$, i.e. the intersection of their valuation-rings; hence A' is a ring, and $\mathbf{V}(K/A') = \mathbf{V}(K/A)$. In particular, if k is a subfield of K and k' its algebraic closure in K we have $\mathbf{V}(K/k') = \mathbf{V}(K/k)$, and $\mathbf{F}(K/k') = \mathbf{F}(K/k)$.

7. Valuation-functions and divisors. We shall now consider fields of functions on an algebraic variety V; the language will be that of F-VIII,[1] except that we do not include the constant ∞ among our functions. We shall say that a point P lies on a divisor X, or that X goes through P, if P lies on some component of X, and that some divisors have no point in common if there is no point lying on every one of them. If x is any non-zero function on V, (x), $(x)_0$ and $(x)_\infty$ denote the divisor of (x), and its divisors of zeros and of poles, respectively; the latter are positive divisors without common component.

Let $x_1, \cdots, x_n$ be functions on the variety V; let k be a field of definition for V and the x_i, M a generic point of V over k, and $x_i(M)$ the value of x_i at M; then the fields $k(x_1, \cdots, x_n)$, $k(x_1(M), \cdots, x_n(M))$ are isomorphic over k, and so every specialization $x' = (x_1', \cdots, x_n')$ of $(x_1(M), \cdots, x_n(M))$ over k is also one of $(x_1, \cdots, x_n)$ over k; if it is a specialization of $(x_1(M), \cdots, x_n(M))$ over $M \to P$ with respect to k, where P is a point of V, then we say that it is a specialization of $(x_1, \cdots, x_n)$ at P; this is the same as to say that $P \times x'$ is a point of the graph of the mapping $(x_1, \cdots, x_n)$ of V into a product of n projective straight lines; hence it does not depend upon the choice of M. A function x which does not admit the specialization ∞ at P is integral over the specialization-ring of P; if every such function is in that ring, i.e. is defined at P, we say that V is normal at P; V is called normal if it is so at all its points. If V is normal at P and its dimension is n, no multiple subvariety of V of dimension $n - 1$ can go through P.

LEMMA 1. *Let V be a variety of dimension n, and P a point of V not contained in any multiple subvariety of V of dimension $n - 1$. Let x be a function on V, other than 0. Then, if 0 is a specialization of x at P, P must lie on $(x)_0$; if ∞ is a specialization of x at P, P must lie on $(x)_\infty$. If P does not lie on $(x)_\infty$, and if V is normal at P, then x is defined and finite at P.*

If 0 is a specialization of x at P, $P \times 0$ lies on the graph Γ of x; hence, by F-VII$_4$, prop. 8, the intersection $\Gamma \cap (V \times 0)$ has a component of dimension at least $n - 1$ going through $P \times 0$; this must be of dimension $n - 1$, as other-

[1] By this, I mean my *Foundations*, Chap. VIII; by F-VII$_4$, the same, Chap. VII, §4; etc.

wise it would coincide both with Γ and with $V \times 0$, which are of dimension n; this would give $\Gamma = V \times 0$, i.e. $x = 0$. Thus that component must be of the form $X \times 0$, where X is a subvariety of V of dimension $n - 1$ and going through P, hence simple on V. As $X \times 0$ is contained in Γ, X is a component of $(x)_0$. The next assertion follows from this by replacing x by $1/x$, and the rest follows immediately.

THEOREM 3. *Let V be a complete abstract variety of dimension r without multiple subvarieties of dimension $r - 1$; let k be a field of definition for V. Let K be the field of functions on V which have k as a field of definition; let $x_1, \cdots, x_n$ be elements of K, other than 0. Then, for the relation $\inf_i[x_i] \prec 0$ to hold in $\mathbf{F}(K/k)$, it is necessary that the divisors $(x_i)_0$ should have no common component, and sufficient that they should have no common point.*

Let M be a generic point of V over k. Assume first that $\inf_i[x_i]$ is not $\prec 0$; then $(0, \cdots, 0)$ must be a specialization of $(x_1, \cdots, x_n)$, or, what is the same thing, of $(x_1(M), \cdots, x_n(M))$, over k; this can be extended so as to include a specialization P of M over k; as V is complete, P is a point of V. Then, for every i, 0 is a specialization of x_i at P; and so, by Lemma 1, P lies on all the $(x_i)_0$. Now assume that the $(x_i)_0$ have a common component X; this must be a subvariety of V of dimension $r - 1$, algebraic over k; let P be a generic point of X over $\bar{k}$. By F-VIII$_1$, prop. 5, the x_i are defined at P, and take the value 0 there; so $(0, \cdots, 0)$ is a specialization of $(x_1, \cdots, x_n)$ over k.

COROLLARY 1. *Let V, k and K be as in Theorem 3. Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_m)$ be two sets of non-zero elements of K such that $(x_i) = X_i - Z$, $(y_j) = Y_j - Z$, where Z is a divisor and the X_i, Y_j are positive divisors on V. Then, if the X_i have no point in common, we have $\inf_i[x_i] \prec \inf_j[y_j]$ in $\mathbf{F}(K/k)$; if at the same time the Y_j have no point in common, we have $\inf_i[x_i] = \inf_j[y_j]$ in $\mathbf{F}(K/k)$.*

We have $(x_i/y_j) = X_i - Y_j$, hence $(x_i/y_j)_0 \prec X_i$; if the X_i have no point in common, the same is true a fortiori of the positive divisors $(x_i/y_j)_0$ for each j, so that $\inf_i[x_i/y_j] \prec 0$, i.e. $\inf_i[x_i] \prec [y_j]$ for each j, by Theorem 3. The rest follows immediately.

COROLLARY 2. *Let V, k, K be as in Theorem 3. Let $(x_{\mu,i})$, (y_j) be two sets of non-zero elements of K such that $(x_{\mu,i}) = T + X_{\mu,i} - U_{\mu,i}$, $(y_j) = T + Y_j - V_j$, where T is a divisor and the $X_{\mu,i}$, $U_{\mu,i}$, Y_j, V_j are positive divisors. Then, if the divisors $\sum_i X_{\mu,i}$ have no point in common, and the divisors V_j also have no point in common, we have $\inf_\mu \sup_i [x_{\mu,i}] \prec \sup_j[y_j]$ in $\mathbf{F}(K/k)$.*

The conclusion is equivalent to $\inf_\mu[x_{\mu,i(\mu)}] \prec \sup_j[y_j]$, i.e. to $\inf_{\mu,j}[x_{\mu,i(\mu)}/y_j] \prec 0$, for all mappings $i(\mu)$, notations being as in No. 3. But

$$(x_{\mu,i(\mu)}/y_j) = X_{\mu,i(\mu)} + V_j - U_{\mu,i(\mu)} - Y_j.$$

Let P be any point on V; by our assumptions, there is μ such that P does not lie on any of the $X_{\mu,i}$, and so in particular not on $X_{\mu,i(\mu)}$; and there is a j such that P does not lie on V_j; so P does not lie on $X_{\mu,i(\mu)} + V_j$. By Theorem 3 this proves our assertion.

COROLLARY 3. *Let V, k, K be as in th. 3. Let $(x_{\mu,i})$, $(y_{\nu,j})$ be two sets of non-zero* *elements of K such that* $(x_{\mu,i}) = T + X_{\mu,i} - U_{\mu,i}$, $(y_{\nu,j}) = T + Y_{\nu,j} - V_{\nu,j}$, *where T is a divisor and the* $X_{\mu,i}$, $U_{\mu,i}$, $Y_{\nu,j}$, $V_{\nu,j}$ *are positive divisors, satisfying the following conditions:* a) *the divisors* $\sum_i X_{\mu,i}$ *have no point in common;* b) *for each* μ *the divisors* $U_{\mu,i}$ *have no point in common;* c) *the divisors* $\sum_j Y_{\nu,j}$ *have no point in common;* d) *for each* ν *the divisors* $V_{\nu,j}$ *have no point in common. Then we have* $\inf_\mu \sup_i [x_{\mu,i}] = \inf_\nu \sup_j [y_{\nu,j}]$ *in* $\mathbf{F}(K/k)$*; and this element of* $\mathbf{F}(K/k)$ *is* $\succ 0$ *if and only if* $T \succ 0$.

Put $X = \inf_\mu \sup_i [x_{\mu,i}]$, $Y = \inf_\nu \sup_j [y_{\nu,j}]$. By corollary 2 we have $X \prec \sup_j [y_{\nu,j}]$ for each ν, hence $X \prec Y$; we get $Y \prec X$ similarly. If T is positive, Theorem 3 shows that $\inf_i [1/x_{\mu,i}] \prec 0$ for each μ, hence $X \succ 0$. Assume now that $X \succ 0$, i.e. that $\inf_i [1/x_{\mu,i}] \prec 0$ for each μ; then, by Theorem 3, the divisors $(x_{\mu,i})_\infty$ are without common component for each μ. But if T has a component A with a coefficient <0 there must be a μ such that A is not a component of $\sum_i X_{\mu,i}$, since these divisors have no common point; for that value of μ, A must be a common component of all the $(x_{\mu,i})_\infty$. This shows that there cannot be a component such as A, i.e. that $T \succ 0$.

8. If T is a divisor, rational over k, and the $x_{\mu,i}$ are in K, we shall say that the element $X_T = \inf_\mu \sup_i [x_{\mu,i}]$ of $\mathbf{F}(K/k)$ is *attached to* T if the set $x_{\mu,i}$ satisfies the conditions a), b) of corollary 3 of Theorem 3, i.e. if

$$(x_{\mu,i}) = T + X_{\mu,i} - U_{\mu,i},$$

where the $X_{\mu,i}$, $U_{\mu,i}$ are positive divisors such that the divisors $\sum_i X_{\mu,i}$, and for each μ the divisors $U_{\mu,i}$ have no point in common. Corollary 3 of Theorem 3 shows that, if there is such an element X_T, it is uniquely determined. To the divisor (x) is attached the element $X_{(x)} = [x]$ of $\mathbf{F}(K/k)$.

It is easy to see that, if $X_T = \inf_\mu \sup_i [x_{\mu,i}]$ is attached to T, then $-X_T = \inf_{i(\mu)} \sup_\mu [1/x_{\mu,i(\mu)}]$ is attached to $-T$; hence, by corollary 3 of Theorem 3, $X_T \prec 0$ if and only if $T \prec 0$, and so $X_T = 0$ if and only if $T = 0$. If, moreover, $X_{T'} = \inf_\nu \sup_j [x'_{\nu,j}]$ is attached to T', then $X_T + X_{T'} = \inf_{\mu,\nu} \sup_{i,j} [x_{\mu,i} x'_{\nu,j}]$ is attached to $T + T'$. This shows that the divisors to which an element of $\mathbf{F}(K/k)$ is attached form a group, and that $T \to X_T$ is an isomorphic mapping of that ordered group into the ordered group $\mathbf{F}(K/k)$.

In particular, T being as above, assume that there are two sets of non-zero elements x_μ, u_ν of K such that $(x_\mu) = T + X_\mu - Z$, $(u_\nu) = U_\nu - Z$, where Z is a divisor, the X_μ are positive divisors with no point in common, and the U_ν are positive divisors with no point in common. Put

$$X_T = \inf_\mu \sup_\nu [x_\mu/u_\nu] = \inf_\mu [x_\mu] - \inf_\nu [u_\nu];$$

then, by our definitions, X_T is attached to T. One sees at once that the divisors T for which there exist sets x_μ, u_ν satisfying these conditions form a group.

One says that a divisor T is "everywhere locally equivalent to 0" (or "everywhere locally a complete intersection") if, for each point P on V, there is a function x such that P does not lie on $T - (x)$. If $X_T = \inf_\mu \sup_i [x_{\mu,i}]$ is attached to T, then corresponding to each P there are μ, i such that P does not lie on

$T - (x_{\mu,i})$, so that T is everywhere locally equivalent to 0; on a projective variety V the converse of this is also true. Here we shall only deal with the non-singular case:

THEOREM 4. *Let V be a non-singular projective variety defined over a field k; let T be a divisor on V rational over k. Then there are functions x_μ, u_ν on V, defined over k, such that $(x_\mu) = T + X_\mu - Z$, $(u_\nu) = U_\nu - Z$, where Z is a divisor, the X_μ are positive divisors with no point in common, and the U_ν are positive divisors with no point in common.*

As mentioned above, the divisors T with that property form a group, so that it is enough to prove the theorem under the assumption that T is a prime rational divisor over k. Let r, n be the dimensions of V and of the ambient projective space P^n. We shall use homogeneous coordinates; this amounts to considering, instead of V, the corresponding cone of dimension $r + 1$ in the affine space S^{n+1}. By a form we understand a homogeneous polynomial in the homogeneous coordinates $X_0, \cdots, X_n$; we define, in an obvious manner, the divisor (F) on V of a form F which is not 0 on V. In the ring $k[x] = k[X_0, \cdots, X_n]$ let $\mathfrak{A}$ be the homogeneous prime ideal generated by the forms which are 0 on the components of T. If F is any form in $\mathfrak{A}$, not 0 on V, we have $(F) \succ T$. Take $(r + 1)(n + 1)$ independent variables $z_{\rho i}$ over k; put $Z_\rho = \sum z_{\rho i} X_i$ $(0 \leqq \rho \leqq r)$, and $k_z = k(z) = k(z_{00}, \cdots, z_{rn})$. Consider the ideal $\mathfrak{A}_z = k_z[X]\mathfrak{A}$ generated by $\mathfrak{A}$ in the ring $k_z[X]$, and the ideal $\mathfrak{A}' = \mathfrak{A}_z \cap k_z[Z]$ in $k_z[Z] = k_z[Z_0, \cdots, Z_r]$. By well-known elementary theorems, $\mathfrak{A}'$ is a homogeneous prime ideal of dimension $r - 1$ in $k_z[Z]$, hence a principal ideal, which can be generated by an element $P(z, Z)$ of that ring, where P is a polynomial in the $z_{\rho i}$ and the Z_ρ, homogeneous in the Z_ρ, with coefficients in k. Geometrically speaking, $P = 0$ is the equation of the "monoid" determined by T and by the linear variety $Z_0 = \cdots = Z_r = 0$ in the projective space P^n, or of the projecting cylinder of T in the direction $Z_0 = \cdots = Z_r = 0$ in the associated affine space S^{n+1}. Let d be the degree of P in the Z_ρ; we can write $P(z, Z) = \sum_\mu p_\mu(z) F_\mu(X)$, where the p_μ are in $k[z]$, and the F_μ are forms of degree d in $k[X]$, linearly independent over k. As P is 0 on T, the F_μ must be in $\mathfrak{A}$. Moreover, if M is any point on V, and the $z_{\rho i}$ have been chosen as independent variables, not merely over k, but over $k(M)$, it is known (since M is simple on V) that M does not lie on the positive divisor $(P) - T$ on V; geometrically speaking, the intersection of $P = 0$ with V is reduced to T in a "neighborhood" of M. Hence, corresponding to every point M on V, there is a μ such that M does not lie on $(F_\mu) - T$. Now let $M_\nu(X)$ be all the monomials of degree d in $X_0, \cdots, X_n$, in any order; take for $M_0(X)$ one such monomial which is not 0 on V; put $x_\mu = F_\mu/M_0$ for all the μ such that F_μ is not 0 on V, and $u_\nu = M_\nu/M_0$ for all the ν such that M_ν is not 0 on V. Put $X_\mu = (F_\mu) - T$, $U_\nu = (M_\nu)$, $Z = U_0 = (M_0)$. These functions and divisors on V satisfy the requirements of our theorem.

II. ABSOLUTE VALUES, DISTRIBUTIONS, HEIGHTS.

9. Absolute values. By an *absolute value* on a field K, we understand a function v on K, with values in the closed interval $[0, +\infty]$, such that $v(0) = 0$, $v(1) = 1$,

$v(xy) = v(x)v(y)$ and $v(x + y) \leqq v(x) + v(y)$ for all x, y in K; this is called *trivial* if $v(x) = 1$ for all $x \neq 0$. We put $v(\infty) = +\infty$.

If v is as above, the subset of K defined by $v(x) < +\infty$ is a subring R of K. Since $v(x) = +\infty$ implies $v(1/x) = 0$, we have $K_\infty = R \cup R^{-1}$, i.e. R is a valuation-ring of K; the valuation-ideal $\mathfrak{P}$ of R is the set defined by $v(x) = 0$. Since $v(-x)^2 = v(x)^2 = v(x^2)$ we have $v(-x) = v(x)$; hence, if $v(y) = 0$, we have both $v(x + y) \leqq v(x)$ and $v[(x + y) + (-y)] \leqq v(x + y)$, hence $v(x + y) = v(x)$; therefore v induces an absolute value v', which is nowhere $+\infty$, on the residue-field $K' = R/\mathfrak{P}$.

In a finite field every element $\neq 0$ is a root of unity; hence there is no non-trivial absolute value. On the other hand, if K is of characteristic 0 it contains the ring Z of the rational integers, and the field Q of rational numbers; v being an absolute value on K, and $\mathfrak{P}$ being defined as above, $Z \cap \mathfrak{P}$ must be either $\{0\}$ or a prime ideal $(p) = pZ$ in Z; in the latter case v induces an absolute value, which must be the trivial one, on the finite field $Z/(p)$, i.e. we have $v(a) = 0$ or 1 for $a \in Z$ according as $a \equiv 0$ or $a \not\equiv 0$ mod. p.

If x, y are in K, and such that $v(y) \leqq v(x)$, then we have, for every integer $n > 0$, $v(x + y)^n \leqq \sum_{r=0}^{n} v(C_r^n)v(x)^n$, where the C_r^n are binomial coefficients. Hence, if K is of characteristic $p > 1$, or if it is of characteristic 0 and v is $\leqq 1$ on Z, we have $v(x + y)^n \leqq (n + 1)v(x)^n$ for all n, whence $v(x + y) \leqq v(x)$. In other words, in those two cases, we have $v(x + y) \leqq \sup[v(x), v(y)]$ for all x, y in K. When that is so, the subset of K defined by $v(x) \leqq 1$ is a valuation-ring R_1 of K, contained in R; its valuation-ideal $\mathfrak{P}_1$ is the set defined by $v(x) < 1$; and the mapping $x \rightarrow v(x)$ of the group of units $U(R)$ of R into the real line is a homomorphism of $U(R)$ into the multiplicative group of real numbers >0, with the kernel $U(R_1)$. Let $\Gamma = K^*/U(R_1)$ be the value-group of the canonical valuation ω of K associated with R_1; the image γ of $U(R)$ in Γ is then a subgroup of Γ; and what we have said shows that there is an order-reversing isomorphism λ of γ into the multiplicative group of real numbers >0, with the following properties: a) $v(x) = \lambda[\omega(x)]$ for $\omega(x) \in \gamma$, i.e. for $x \in U(R)$; b) $v(x) = 0$ for $\omega(x) > \gamma$, i.e. $x \in \mathfrak{P}$; c) $v(x) = +\infty$ for $\omega(x) < \gamma$, i.e. $x \notin R$.

In particular, assume that the characteristic of K is 0, that $v \leqq 1$ on Z, and that $Z \cap \mathfrak{P} = \{0\}$; then $Z \cap \mathfrak{P}_1$ is either $\{0\}$, in which case v is trivial on Z, hence on Q, or it is a prime ideal $(p) = pZ$ in Z; then we have $v(a) = 1$ for $a \in Z$, $a \notin \mathfrak{P}_1$, i.e. $a \not\equiv 0$ mod. p, and $0 < v(p) < 1$; and if $r = p^n a/b$, where a, b, n are integers, and a, b are prime to p, we have $v(r) = v(p)^n$. When that is so, v will be called *p-adic*, and *normed* if $v(p) = 1/p$. It is clear that every p-adic absolute value is of the form $v(x)^\rho$, where v is p-adic and normed, and $\rho > 0$.

Finally, assume that K is of characteristic 0, and that the absolute value v is not everywhere $\leqq 1$ on Z; as we have seen, this implies $Z \cap \mathfrak{P} = \{0\}$, and hence $v \neq +\infty$ on Z. It is known that in this case v must induce on Q an absolute value of the form $v(r) = |r|^\rho$, where ρ is a constant satisfying $0 < \rho \leqq 1$; v is then called *archimedean*, and *normed* if $\rho = 1$. As $Q \cap \mathfrak{P} = \{0\}$, Q is mapped isomorphically into the residue-field $K' = R/\mathfrak{P}$; that being so, it is known that

K' may be identified with a subfield of the field C of complex numbers in such a way that $v'(x') = |x'|^\rho$. In other words, every archimedean absolute value is of the form $v(x) = |f(x)|^\rho$, where f is a C-valued place of K, with the valuation-ring R, and $0 < \rho \leqq 1$.

An absolute value will henceforth be called *proper* if it is either p-adic or archimedean, and normed. The above discussion shows that every absolute value which takes other values than 0 and 1 on the prime ring (i.e. Z if K is of characteristic 0, and the prime field otherwise) is of the form $v(x)^\rho$, where v is proper and $\rho > 0$; if v is archimedean ρ must also be $\leqq 1$.

As the set of elements of a field K where a non-archimedean absolute value is $\leqq 1$ is a valuation-ring, and hence (by No. 6) integrally closed, such an absolute value is $\leqq 1$ on the ring of algebraic integers in K; for a similar reason, every proper absolute value on K is $< +\infty$ on the algebraic closure of Q in K.

By an *algebraic number-field* we shall understand a finite algebraic extension of Q. On such a field k it is known that the proper non-archimedean absolute values are in a one-to-one correspondence with the prime ideals in k; the proper p-adic absolute values correspond to the prime divisors of p in k; for each p they are finite in number. The proper archimedean absolute values of k are also finite in number, and are in a one-to-one correspondence with the isomorphisms of k into the real number-field and with the pairs of mutually conjugate isomorphisms of k into the complex number-field C.

If k is an algebraic number-field we define a *k-divisor* as a real-valued function $\delta(v)$ of the proper absolute values on k, with the following properties: (a) $\delta(v) > 0$ for all v's, and $\delta(v) = 1$ for all but a finite number of the v's; (b) for each non-archimedean v, there is an $\alpha \in k^*$ such that $\delta(v) = v(\alpha)$. If $\alpha \in k^*$, $v(\alpha)$ is a k-divisor; so is every function of v obtained from functions of this form by the operations of sup and inf. If K is any field containing k we may extend any k-divisor δ to a function of the proper absolute values on K by putting $\delta(w) = \delta(v)$ when w is a proper absolute value on K which induces v on k.

10. Let A be any subring of a field K; let the x_i be in K, and such that $\inf_i[x_i] \prec 0$ in $\mathbf{F}(K/A)$; then, by corollary 1 of Theorem 2, there is a polynomial $P \in A[X]$ such that $P(x) = 0$ and $P(0) = 1$; in other words there is a relation $1 = \sum_{\nu=1}^{n} a_\nu M_\nu(x)$, where the a_ν are in A, and the M_ν are monomials of degree $\geqq 1$. Let v be any absolute value on K; either $\sup_i v(x_i) \geqq 1$, or $v(x_i) < 1$ for all i; assume that we are in the latter case. Then, putting $\alpha = \sup_\nu v(a_\nu)$, we have $1 \leqq n\alpha \sup_i v(x_i)$ always, and $1 \leqq \alpha \sup_i v(x_i)$ if v is non-archimedean. This shows that our assumption on the x_i implies

$$\sup_i v(x_i) \geqq \gamma_v = \inf_\nu(1, v(a_\nu^{-1})/n_v),$$

with $n_v = n$ or 1 according as v is archimedean or not. Therefore:

THEOREM 5. *Let K be a field, and A a subring of K; let v_0 be an absolute value on K which is $< +\infty$ on A. Let the x_i be in K, and such that* $\inf_i[x_i] \prec 0$ *in* $\mathbf{F}(K/A)$. *Then there is a constant $\gamma > 0$ such that $\sup_i v(x_i) \geqq \gamma$ for all the absolute values v on K which coincide with v_0 on A.*

Now, in addition to the assumptions made above on K, A and the x_i, assume that K is of characteristic 0, that A is a specialization-ring in K, and that A is absolutely algebraic, i.e. algebraic over the prime field Q; according to No. 3, this means that $A \supset Q$, and that $A/\mathfrak{p}$ is algebraic over Q, where $\mathfrak{p}$ is the maximal ideal of A. Let v be any absolute value on K which is 0 on $\mathfrak{p}$; then, for $a \in A$, $v(a)$ depends only upon the class of a mod. $\mathfrak{p}$, i.e. upon the image a' of a in $A/\mathfrak{p}$, and we can write $v(a) = v'(a')$, where v' is an absolute value on $A/\mathfrak{p}$. The a_ν being defined as above, we have $\sup_i v(x_i) \geqq \gamma_v = \inf_\nu(1, v'(a_\nu'^{-1})/n_\nu)$. Let k be the subfield of $A/\mathfrak{p}$ generated by the a_ν' over Q; this is an algebraic number-field. As the normed archimedean absolute values v' on k are finite in number, there is a real number $\rho > 0$ such that $v'(a_\nu'^{-1})/n \geqq \rho$ for all such v' and all ν; then, if $r \in Q$ is such that $0 < r \leqq \inf(1, \rho)$, we have $\sup_i v(x_i) \geqq v(r)$ for all the archimedean absolute values v on K which are 0 on $\mathfrak{p}$. Let m be an integer >0 such that all the ma_ν' are algebraic integers; then we have $v'(a_\nu'^{-1}) \geqq v'(m)$ for all non-archimedean absolute values v' on k and all ν. Therefore:

THEOREM 6. *Let K be a field of characteristic* 0, *A an absolutely algebraic specialization-ring in K, and $\mathfrak{p}$ the maximal ideal in A. Let the x_i be in K, and such that* $\inf_i[x_i] \prec 0$ *in* $\mathbf{F}(K/A)$. *Then there are a rational number $r > 0$ and an integer $m > 0$ such that, if v is any absolute value on K which is 0 on $\mathfrak{p}$,* $\sup_i v(x_i) \geqq \inf(v(r), v(m))$.

If A and $\mathfrak{p}$ are as in Theorem 6, we shall say that an absolute value v on K *belongs properly to A* if it is proper and if it is 0 on $\mathfrak{p}$; v' being defined as before, v' is then a proper absolute value on $A/\mathfrak{p}$, and so is $< +\infty$ on $A/\mathfrak{p}$ since $A/\mathfrak{p}$ is algebraic over Q; therefore v must be $< +\infty$ on A. We can now rephrase the most important case of Theorem 6 as follows:

THEOREM 6'. *Let K be a field of characteristic* 0, *and A an absolutely algebraic specialization-ring in K. Let the x_i be in K, and such that* $\inf_i[x_i] \prec 0$ *in* $\mathbf{F}(K/A)$. *Then there is a Q-divisor δ such that* $\sup_i v(x_i) \geqq \delta(v)$ *for all the absolute values v on K which belong properly to A.*

11. Distributions. As in No. 6, let the $x_{\mu,i}$ be a finite set of elements of a field K; in No. 6, we have attached to it the element $X = \inf_\mu \sup_i [x_{\mu,i}]$ of $\mathbf{F}(K)$; now we also attach to it the function $\Delta_x(v) = \sup_\mu \inf_i v(x_{\mu,i})$ of the absolute values v on K, with values in $[0, +\infty]$. Such a function, or its restriction to various subsets of the set of all absolute values on K, will be called a *distribution*; Δ_x will be called the distribution *belonging to the expression* $\inf_\mu \sup_i[x_{\mu,i}]$; we also say that it is a distribution belonging to the element X of $\mathbf{F}(K)$, or, as the case may be, of one of the groups $\mathbf{F}(K/A)$, defined by that same expression. The distribution belonging to the expression $\inf_{i(\mu)}\sup_\mu[1/x_{\mu,i(\mu)}]$ of $-X$ is Δ_x^{-1}; if $y_{\nu,j}$ is another set of elements of K, and if we put $Y = \inf_\nu \sup_j [y_{\nu,j}]$, the distribution $\Delta_{xy}(v)$ belonging to the expression $\inf_{\mu,\nu}\sup_{i,j}[x_{\mu,i}y_{\nu,j}]$ of $X + Y$ is equal to $\Delta_x(v)\Delta_y(v)$ whenever this product is defined, that is except when one of its factors is 0 and the other $+\infty$.

First let A be a subring of K, and v_0 an absolute value on K which is $< +\infty$ on A, and apply Theorem 5. Let X, Y, Δ_x, Δ_y be defined as above; then, if

$X \prec Y$ in $\mathbf{F}(K/A)$, there is a constant $\gamma > 0$ such that $\Delta_x(v) \geqq \gamma\Delta_y(v)$ for all absolute values v on K coinciding with v_0 on A. In fact, the distribution belonging to the expression $\sup_{i(\mu),\nu}\inf_{\mu,j}[x_{\mu,i(\mu)}/y_{\nu,j}]$ of $X - Y$ is equal to $\Delta_x(v)\Delta_y(v)^{-1}$ except when $\Delta_x(v)$, $\Delta_y(v)$ are both 0 or both $+\infty$; by Theorem 5, for each $i(\mu)$ and each ν there must be a $\gamma > 0$ such that $\sup_{\mu,j}v(x_{\mu,i(\mu)}/y_{\nu,j}) \geqq \gamma$ for all v's coinciding with v_0 on A; replacing all these γ's by the smallest one among them, we get $\Delta_x(v)\Delta_y(v)^{-1} \geqq \gamma$ except when $\Delta_x(v)$, $\Delta_y(v)$ are both 0 or both $+\infty$, which is equivalent to what we had to prove. From this it follows that, if $X = Y$ in $\mathbf{F}(K/A)$, there are constants γ, γ', both >0, such that $\gamma\Delta_y(v) \leqq \Delta_x(v) \leqq \gamma'\Delta_y(v)$ for all v's coinciding with v_0 on A. In particular, if $X \succ 0$ we have $X = \sup(X, 0)$, so that this last inequality holds if we take for Δ_y the distribution $\Delta_y(v) = \sup_\mu\inf_\iota(v(x_{\mu,i}), 1) = \inf(\Delta_x(v), 1)$ belonging to the expression

$$Y = \sup(X, 0) = \inf_\mu\sup_i([x_{\mu,i}], [1]).$$

Quite similarly, the application of Theorem 6′ gives the following results. Let K be of characteristic 0, and let A be an absolutely algebraic specialization-ring in K. Let X, Y, Δ_x, Δ_y be defined as before. Then, *if $X \prec Y$ in $\mathbf{F}(K/A)$, there is a Q-divisor δ such that $\Delta_x(v) \geqq \delta(v)\Delta_y(v)$ for all absolute values v on K belonging properly to A*. From this it follows that, *if $X = Y$ in $\mathbf{F}(K/A)$, there are Q-divisors δ, δ' such that $\delta(v)\Delta_y(v) \leqq \Delta_x(v) \leqq \delta'(v)\Delta_y(v)$ for all absolute values v on K belonging properly to A*; we express this by saying that Δ_x, Δ_y are then *quasi-equal*. As above it follows that, if $X \succ 0$, $\inf(\Delta_x(v), 1)$ is a distribution and is quasi-equal to $\Delta_x(v)$.

12. The size of a distribution. By the *multiplicity* of a proper absolute value v of an algebraic number-field k we understand the degree of the completion of k over the completion of Q, both with respect to v. By $\prod_{v/k}$ we denote a product taken over all the proper absolute values v on k, each one of these being repeated a number of times equal to its multiplicity. Then if α is an element of k^* we have the "product-formula" $\prod_{v/k} v(\alpha) = 1$. If v is a proper absolute value of multiplicity μ on k, and if k' is an extension of k of finite degree d, the sum of the multiplicities of the proper absolute values on k' that induce v on k is equal to μd.

If δ is a k-divisor, we put $S_k(\delta) = \prod_{v/k} \delta(v)$; then, if k' is an extension of k of degree d, we have $\prod_{v'/k'} \delta(v') = S_k(\delta)^d$. By the *size* of a k-divisor δ we shall understand the number $s(\delta) = S_k(\delta)^{1/n}$, where $n = [k:Q]$ is the degree of k over Q; then the size of the k'-divisor determined by δ is again equal to $s(\delta)$. If δ is a k-divisor, we have $0 < s(\delta) < +\infty$. If δ is a function which, for all proper absolute values v on k, is identically 0 or identically $+\infty$, then we put $s(\delta) = 0$ or $s(\delta) = +\infty$, respectively.

Now if Δ is a distribution on a field K of characteristic 0, we consider the restriction of Δ to those absolute values of K which are of the form $v \circ f$, where f is a $\bar{Q}$-valued[2] place of K, and v a proper absolute value on $\bar{Q}$; then if Δ belongs

[2] As usual, $\bar{k}$ denotes the algebraic closure of k. In particular, $\bar{Q}$ is the field of all algebraic numbers, as we denote by Q the field of rational numbers.

to the expression $\inf_\mu \sup_i [x_{\mu,i}]$, we have $\Delta(v \circ f) = \sup_\mu \inf_i v[f(x_{\mu,i})]$. The $x_{\mu,i}$ being given, denote by k_f, for each f, the extension of Q generated by the $f(x_{\mu,i})$ (or, more correctly, by those of the $f(x_{\mu,i})$ which are not ∞); then for a given f, either $\Delta(v \circ f)$ is 0 for all v's (viz., if, for each μ, there is an i such that $f(x_{\mu,i}) = 0$), or it is $+\infty$ for all v's (viz., if there is a μ such that $f(x_{\mu,i}) = \infty$ for all i's), or else it is a k_f-divisor. So we may put $\Sigma(f) = s[\Delta(v \circ f)]$; this is a function of the $\bar{Q}$-valued places of K, with values in $[0, +\infty]$, which we call the *size* of Δ.

From our results of No. 11 about distributions we immediately get corresponding results about their sizes. If two distributions Δ, Δ' are quasi-equal, then there are constants γ, γ', both >0, such that their sizes Σ, Σ' satisfy the inequality $\gamma\Sigma'(f) \leqq \Sigma(f) \leqq \gamma'\Sigma'(f)$ for all f; this will be the case, therefore, if Δ and Δ' belong to one and the same element of $\mathbf{F}(K/Q)$. Similarly, if Σ_x, Σ_y are the sizes of distributions Δ_x, Δ_y, belonging respectively to two elements X, Y of $\mathbf{F}(K/Q)$, and if $X \prec Y$ in $\mathbf{F}(K/Q)$, there is a constant $\gamma > 0$ such that $\Sigma_x(f) \geqq \gamma\Sigma_y(f)$ for all f.

If x is any element of K^*, the size $\Sigma_{[x]}(f)$ of the distribution $\Delta_{[x]}(v) = v(x)$ belonging to the expression $[x]$ is given by $\Sigma_{[x]}(f) = s(v[f(x)])$; this is equal to 0, $+\infty$, or 1, according as $f(x)$ is 0, ∞, or an algebraic number other than 0. From this it follows that, if Σ is the size of a distribution belonging to an element X of $\mathbf{F}(K/Q)$, and Σ' the size of a distribution belonging to $X + [x]$, there are constants γ, γ', both >0, such that $\gamma\Sigma'(f) \leqq \Sigma(f) \leqq \gamma'\Sigma'(f)$ except possibly for those f for which $f(x) = 0$ or ∞.

13. The height of a point. Let $(\alpha_0, \cdots, \alpha_n)$ be a set of $n + 1$ elements of $\bar{Q}_\infty = \bar{Q} \cup \{\infty\}$; let k be any algebraic number-field containing all the α_i which are not ∞; for every proper absolute value v on k put $\delta(v) = \sup_i v(\alpha_i)$; this is either 0 for all v (viz., if all the α_i are 0), or $+\infty$ for all v (viz., if some α_i is ∞), or a k-divisor. Put $h(\alpha) = s(\delta)$; this does not depend upon the choice of k. If ξ is any non-zero algebraic number we have $h(\xi\alpha) = h(\alpha)$, so that in particular, if the α_i are all $\neq \infty$ and not all 0, $h(\alpha)$ depends only upon the point α in the projective space P^n with the homogeneous coordinates $(\alpha_0, \cdots, \alpha_n)$; then, if e.g. $\alpha_0 \neq 0$, we have $\sup_i v(\alpha_i) \geqq v(\alpha_0)$, hence $h(\alpha) \geqq s[v(\alpha_0)] = 1$. So we have $1 \leqq h(\alpha) < +\infty$ whenever α is a point in P^n, i.e. when the α_i are all $\neq \infty$ and not all 0. Also, if σ is any automorphism of $\bar{Q}$, it merely induces a permutation of the proper absolute values of any normal extension of Q, so that we have $h(\alpha^\sigma) = h(\alpha)$ for every α.

If α is a point of P^n with algebraic coordinates, $h(\alpha)$ will be called the *height* of α; this is a slight modification of a concept introduced by Northcott [4a][3]. Let us also denote by $d(\alpha)$ the degree over Q of the extension of Q generated by the α_i/α_j; then *the number of points* α in P^n *for which* $d(\alpha) \leqq d_0$, $h(\alpha) \leqq h_0$,

[3] Siegel ([2]) first attached to a point α, with coordinates in k, the number equal, in our notation, to $S_k(\delta)$, with δ as above; cf. H. Hasse, *Monatshefte f. Math.* 48 (1939), p. 205, where that number is called the height of α. Northcott saw the advantage of using a number independent of k; the one he uses (called by him the "complexity" of α) differs from our $h(\alpha)$ by a factor $\geqq 1$ and $\leqq n + 1$.

is finite when n, d_0 and h_0 are given. Also this important theorem is due to Northcott, and his proof for it is as follows. If $d(\alpha) = d$ we may take the α_i themselves in a field k of degree d over Q; then by definition, we have $h(\alpha)^d = \prod_{v/k} \sup_i v(\alpha_i)$. In this product, the partial product corresponding to the archimedean absolute values can be written as $\prod_\sigma \sup_i |\alpha_i^\sigma|$, the product being taken over all the distinct isomorphisms σ of k into the field C of complex numbers. On the other hand, if v is the proper p-adic absolute value on k corresponding to the prime ideal $\mathfrak{p}$ of k, and if μ is its multiplicity, then $v(\alpha_i)^\mu$ is equal to $N(\mathfrak{p})^{-\rho_i}$, where $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$, taken in k over Q, and ρ_i is the exponent of $\mathfrak{p}$ in the expression of the fractional ideal (α_i) in k as a product of powers of prime ideals. Therefore, if $\mathfrak{m}$ is the G.C.D. of the principal ideals (α_i) in k, we have

$$h(\alpha)^d = N(\mathfrak{m})^{-1}\prod_\sigma \sup_i |\alpha_i^\sigma| .$$

Now let $U_0, \cdots U_n$ be indeterminates; consider the polynomial

$$F(U) = \prod_\sigma \left(\sum_{i=0}^{n} \alpha_i^\sigma U_i\right);$$

it is homogeneous of degree d with rational coefficients; if m is the (possibly fractional) G.C.D. of its coefficients, then $F_0 = F/m$ has mutually prime integral coefficients. Obviously m is an integral multiple of $N(\mathfrak{m})$; as a matter of fact, Gauss' lemma shows that $m = N(\mathfrak{m})$, but this is not needed here. Therefore $F_0(U)$ is majorized by $\Phi(U) = \left[h(\alpha)\sum_{i=0}^{n} U_i\right]^d$, in the sense that the (ordinary) absolute value of each coefficient in F_0 is at most equal to the corresponding coefficient in Φ. Hence, for given d and n, there can be only a finite number of polynomials F_0 corresponding to points α for which $h(\alpha) \leqq h_0$. But such an F_0, if it can be factored into linear forms, can be so factored essentially in only one way, and so can correspond to no more than d points α in P^n.

Let $\alpha = (\alpha_i)$, $\beta = (\beta_j)$ be two absolutely algebraic points in two projective spaces P^n, P^m, respectively; let γ be the point in P^{nm+n+m} with the homogeneous coordinates $(\alpha_i\beta_j)$; then we have $h(\gamma) = h(\alpha)h(\beta)$; and the corresponding result holds for points in any number. In particular, in the projective space of dimension $(n+1)^m - 1$, call $T_m(\alpha)$ the point whose homogeneous coordinates are all the products $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m}$, where the i_μ run independently over the set $\{0, 1, \cdots, n\}$; then we have $h[T_m(\alpha)] = h(\alpha)^m$. It is well-known, and easy to verify, that T_m is a one-to-one biregular mapping of P^n onto a non-singular subvariety of the projective space of dimension $(n+1)^m - 1$.

As to the effect on the height of a change of coordinates in P^n, it is easy to see (reasoning as in No. 10) that, if L is any matrix with $m+1$ rows and $n+1$ columns and with coefficients in $\bar{Q}$, there is a Q-divisor δ such that for $\alpha' = L(\alpha)$, $\sup_j v(\alpha'_j) \leqq \delta(v)\sup_i v(\alpha_i)$; hence there is a constant γ such that $h(\alpha') \leqq \gamma h(\alpha)$. In particular, if L is an invertible square matrix, or, what amounts to the same thing, if it defines a change of coordinates in P^n, there are two constants γ, γ', both >0, such that $\gamma h(\alpha) \leqq h(\alpha') \leqq \gamma' h(\alpha)$. This, however, is only a special case of the results which will be proved presently.

14. Let $x_0, \cdots, x_n$ be elements, not all 0, of a field K of characteristic 0. Let Δ_x be the distribution on K which belongs to the expression $X = \inf_i[x_i]$, and let Σ_x be its size; thus, for every $\bar{Q}$-valued place f of K, we have $\Sigma_x(f) = s[\Delta(v \circ f)] = s(\sup_i v[f(x_i)])$. In other words, if we put $\xi_f = (f(x_0), \cdots, f(x_n))$, we have $\Sigma_x(f) = h(\xi_f)$. Thus every theorem on the size of distributions gives a theorem on the height of the points of a "projective model" of K. In particular:

THEOREM 7. *Let $(x_0, \cdots, x_n)$, $(y_0, \cdots, y_m)$ be two sets of non-zero elements in a field K of characteristic 0; let t be a transcendental quantity over K. For every $\bar{Q}$-valued place f of $K(t)$, put $\xi_f = (f(tx_0), \cdots, f(tx_n))$, $\eta_f = (f(ty_0), \cdots, f(ty_m))$. Put $X = \inf_i[x_i]$, $Y = \inf_j[y_j]$. Then, if $X \prec Y$ in $\mathbf{F}(K/Q)$, there is a constant $\gamma > 0$ such that $h(\xi_f) \geqq \gamma h(\eta_f)$ for all f; if $X = Y$ in $\mathbf{F}(K/Q)$, there are constants γ, γ', both >0, such that $\gamma h(\eta_f) \leqq h(\xi_f) \leqq \gamma' h(\eta_f)$.*

Put $X' = \inf_i[tx_i] = [t] + X$, $Y' = \inf_j[ty_j] = [t] + Y$; if we have $X \prec Y$ in $\mathbf{F}(K/Q)$, we also have $X \prec Y$ in $\mathbf{F}(K(t)/Q)$, hence $X' \prec Y'$ in $\mathbf{F}(K(t)/Q)$; if $X = Y$ in $\mathbf{F}(K/Q)$, $X' = Y'$ in $\mathbf{F}(K(t)/Q)$. As we have seen above that $h(\xi_f)$, $h(\eta_f)$ are nothing else than the sizes of the distributions on $K(t)$ belonging respectively to X' and to Y', our assertions are contained in the results of No. 12.

We shall now combine this with the results of No. 7, but first give some preliminary definitions and results. If V is a variety defined over a field k, and $x_0, \cdots, x_n$ are functions, not all 0, defined over k on V, there is a mapping φ of V into the projective space P^n, such that, if P is any point of V at which the x_i are all defined and $\neq \infty$ and not all 0, $\varphi(P)$ is the point with the homogeneous coordinates $(x_0(P), \cdots, x_n(P))$; φ is defined over k, and, as such, is determined by the fact that, if M is generic on V over k, $\varphi(M)$ is the point with the homogeneous coordinates $(x_0(M), \cdots, x_n(M))$. If some of the x_i are 0, φ maps V into a linear subvariety of P^n, which can be identified with a projective space of lower dimension; so there is no real loss of generality in assuming that none of the x_i is 0. If V is complete the points of the image $\varphi(V)$ of V by φ are the points x' of P^n whose homogeneous coordinates $(x'_0, \cdots, x'_n)$ are specializations of $(tx_0, \cdots, tx_n)$ over k, t being any transcendental quantity over $k(x_0, \cdots, x_n)$. We say that φ is the mapping of V into P^n *determined by* the set $(x_0, \cdots, x_n)$. If z is any non-zero function on V, φ is also the mapping of V into P^n determined by the set of functions $(zx_0, \cdots, zx_n)$. In particular, if P is a point of V, and if a function z on V can be found such that the functions $x'_i = zx_i$ are all defined and $\neq \infty$ and not all 0 at P, then φ is defined at P, and $\varphi(P)$ is the point $(x'_0(P), \cdots, x'_n(P))$ in P^n. In particular, assume that $(x_i) = X_i - Z$, where the X_i are positive divisors, and Z is any divisor on V; let P be a point not on X_0 at which V is normal; then, if we take $z = 1/x_0$, and put again $x'_i = zx_i = x_i/x_0$, we have $(x'_i) = X_i - X_0$, and so, by Lemma 1 of No. 7, the x'_i are all defined and finite at P, and not all 0 since $x'_0 = 1$; so φ is defined at P, hence in general at every point not lying on all the X_i at which V is normal.

15. THEOREM 8. *Let V be an abstract variety, complete and normal, defined over* *an algebraic number-field k. Let $(x_0, \cdots, x_n)$, $(y_0, \cdots, y_m)$ be two sets of functions other than* 0, *defined over k on V; let φ, ψ be the mappings of V into the projective spaces P^n, P^m, respectively determined by these two sets. Assume that $(x_i) = X_i - Z$, $(y_j) = Y_j - Z$, where the X_i, Y_j are positive divisors. Then, if the X_i have no common point, φ is everywhere defined; ψ is defined at every point P which does not lie on all the Y_j; and there is a constant $\gamma > 0$ such that $h[\varphi(P)] \geqq \gamma h[\psi(P)]$ at all the absolutely algebraic points P of V which do not lie on all the Y_j. If at the same time the Y_j have no common point, then also ψ is everywhere defined, and there are constants γ, γ', both > 0, such that $\gamma h[\psi(P)] \leqq h[\varphi(P)] \leqq \gamma' h[\psi(P)]$ at all absolutely algebraic points P of V.*

We have already seen above that ψ is defined at each point which does not lie on all the Y_j, and φ at each point which does not lie on all the X_i; so φ is everywhere defined. Let K be the field of functions defined over k on V. By corollary 1 of Theorem 3, No. 7, we have $\inf_i[x_i] \prec \inf_j[y_j]$ in $\mathbf{F}(K/k)$, or, what is the same thing by No. 6, in $\mathbf{F}(K/Q)$; by the same corollary, we have $\inf_i[x_i] = \inf_j[y_j]$ in $\mathbf{F}(K/Q)$ if the Y_j have no common point. Now let P be a point on V, not lying on Y_0, say; put $z = 1/y_0$, $x_i' = zx_i = x_i/y_0$, $y_j' = zy_j = y_j/y_0$; if t is a transcendental quantity over K, $(x_0', \cdots, x_n', y_0', \cdots, y_m')$ is a specialization of $(tx_0, \cdots, tx_n, ty_0, \cdots, ty_m)$ over k. We have $(x_i') = X_i - Y_0$, $(y_j') = Y_j - Y_0$, and so, by Lemma 1 of n° 7, the x_i', y_j' are all defined and finite at P; as $y_0' = 1$, and as P does not lie on all the X_i, neither all the x_i' nor all the y_j' are 0 at P, and so $\varphi(P)$, $\psi(P)$ are respectively the points with the homogeneous coordinates $(x_0'(P), \cdots, x_n'(P))$, $(y_0'(P), \cdots, y_m'(P))$. These $n + m + 2$ quantities form a specialization of $(x_0', \cdots, x_n', y_0', \cdots, y_m')$, hence also one of $(tx_0, \cdots, tx_n, ty_0, \cdots, ty_m)$ over k; this is absolutely algebraic if P is absolutely algebraic; by Theorem 1, it can then be extended to a $\bar{Q}$-valued place f of the field $K(t)$. Our theorem now appears as a special case of Theorem 7.

Corollary. *Let V and k be as in Theorem* 8; *let the x_μ, u_ν and y_j be three sets of functions, defined over k on V, such that $(x_\mu) = T + X_\mu - Z$, $(u_\nu) = U_\nu - Z$, $(y_j) = Y_j - T$, where the X_μ, the U_ν, and the Y_j, are positive divisors, the X_μ have no common point, and the U_ν have no common point. Let φ, ω, ψ be the mappings of V into projective spaces, respectively determined by the sets (x_μ), (u_ν), (y_j). Then φ, ω are everywhere defined; ψ is defined at all the points which do not lie on all the Y_j; and there is a constant γ such that $h[\psi(P)] \leqq \gamma h[\omega(P)]/h[\varphi(P)]$ at all absolutely algebraic points P of V, not lying on all the Y_j. If, moreover, the Y_j have no common point, then ψ is everywhere defined, and there are constants γ, γ', both >0, such that $\gamma' h[\omega(P)]/h[\varphi(P)] \leqq h[\psi(P)] \leqq \gamma h[\omega(P)]/h[\varphi(P)]$ at all absolutely algebraic points P of V.*

In fact, we obtain these results by applying Theorem 8 to the two sets (u_ν), $(x_\mu y_j)$.

Northcott's main theorems ([4a], Theorem 2 = [4b], Theorem 2; [4b], Theorem 1) are special cases of the corollary of Theorem 8. He takes for V a projective variety, and for φ the mapping T_m defined above in No. 13. His assumption that V be non-singular is necessary for his method of proof (using the "decomposition theorem"; cf. *infra*, §III), but not for the validity of his results.

Theorem 8 can also be conveniently expressed in the language of "linear series". Every mapping φ of a variety V into a projective space P^n determines uniquely a "linear series" without fixed component, consisting of the positive divisors $\bar{\varphi}^1(L) = pr_V[\Gamma \cdot (V \times L)]$ on V, where Γ is the graph of the mapping φ in $V \times P^n$, and where one takes for L all hyperplanes, i.e. all linear varieties of dimension $n - 1$, in P^n. Conversely, it is well-known that every linear series without fixed component can be so obtained, and essentially in only one way. All the divisors in a linear series are linearly equivalent to each other, i.e. each linear series is contained in a class of divisors (for linear equivalence). Now we can rephrase Theorem 8, or rather (for the sake of brevity) its latter half, as follows:

THEOREM 8′. *Let V be an abstract variety, complete and normal, defined over an algebraic number-field k. Let φ, ψ be two mappings of V into projective spaces, both defined over k; assume that the linear series without fixed components, respectively defined by φ and by ψ, are both without fixed points, and that the divisors in one series are linearly equivalent to the divisors in the other. Then φ, ψ are everywhere defined, and there are constants γ, γ', both >0, such that*

$$\gamma h[\psi(P)] \leqq h[\varphi(P)] \leqq \gamma' h[\psi(P)]$$

at all absolutely algebraic points P of V.

16. With the notations of the corollary of Theorem 8, suppose now that we have two other sets of functions x'_ρ, u'_σ such that $(x'_\rho) = T + X'_\rho - Z'$, $(u'_\sigma) = U'_\sigma - Z'$, where the X'_ρ are positive divisors without common point and so are the U'_σ, and T is the same as before; call φ', ω' the mappings of V into projective spaces respectively defined by the sets (x'_ρ), (u'_σ). Applying Theorem 8 to the two sets $(x_\mu u'_\sigma)$, $(x'_\rho u_\nu)$, we see that the two functions $h[\omega(P)]/h[\varphi(P)]$, $h[\omega'(P)]/h[\varphi'(P)]$ are equivalent, in the sense that each is less than a constant multiple of the other; in this sense, the "order of magnitude" of such a function depends only upon T. Moreover, it does not change if T is replaced by a linearly equivalent divisor T'; for, if $(z) = T' - T$, put $x'_\mu = zx_\mu$; the two sets (x'_μ), (u_ν) have the same properties with respect to T' as (x_μ), (u_ν) with respect to T; and they define the same mappings φ, ω.

Now assume that V is a non-singular projective variety defined over an algebraic number-field k. Consider all the classes of divisors on V (with respect to linear equivalence) which contain absolutely algebraic divisors, i.e. divisors which are rational over $\bar{k} = \bar{Q}$. For every such class C, choose an absolutely algebraic divisor T in C, and choose two sets x_μ , u_ν of functions on V defined over $\bar{k}$, and having, with respect to T, the properties described in the corollary of Theorem 8. Theorem 4 of No. 8 tells us that such choices are always possible. Call φ, ω the mappings of V into projective spaces defined by the sets (x_μ), (u_ν) respectively and put $h(C, P) = h[\omega(P)]/h[\varphi(P)]$ for every absolutely algebraic point P on V; for every such P we have $0 < h(C, P) < +\infty$. When C is given, any other choice of T and of the x_μ , u_ν leads to a function $h'(C, P)$ equivalent to $h(C, P)$, i.e., there are constants γ, γ', both >0, such that

$$\gamma h(C, P) \leqq h'(C, P) \leqq \gamma' h(C, P).$$

In particular, if T, (x_μ), (u_ν) is a permissible choice for C, so is $-T$, (u_ν), (x_μ) for $-C$; therefore *$h(-C, P)$ is equivalent to $h(C, P)^{-1}$*, in the sense defined above. If at the same time T', (x'_ρ), (u'_σ) is a permissible choice for a class C', then $T + T'$, $(x_\mu x'_\rho)$, $(u_\nu u'_\sigma)$ is one for $C + C'$; and therefore *$h(C + C', P)$ is equivalent to $h(C, P)h(C', P)$.*

Let ψ be a mapping of V, defined over $\bar{k}$, into a projective space; ψ determines a linear series without fixed components; let $\mathfrak{B}(\psi)$ be the set of points lying on all the divisors of this series (the "base-points" of the series); let C be the class to which the divisors in that linear series belong. Then, by the corollary of Theorem 8, ψ is defined everywhere outside $\mathfrak{B}(\psi)$; and *there is a constant γ such that $h[\psi(P)] \leqq \gamma h(C, P)$ at all the absolutely algebraic points P of V which do not lie in $\mathfrak{B}(\psi)$.* If, moreover, $\mathfrak{B}(\psi) = \emptyset$, i.e. if the linear series is without "base-points", then ψ is everywhere defined and *$h[\psi(P)]$ is equivalent to $h(C, P)$.*

More generally, C being given as above, let $\mathfrak{B}(C)$ be the set of those points which lie on all the positive divisors belonging to the class C. Let T, (x_μ), (u_ν) be the divisor and the sets of functions which have been chosen for the determination of $h(C, P)$. As T is rational over $\bar{k}$, it is easy to show by means of F-VIII$_3$, Theorem 10, that there is a finite set of functions y_j, defined over $\bar{k}$, such that $(y_j) = Y_j - T$, where the Y_j are positive divisors (obviously belonging to the class C), and that every function y satisfying $(y) \succ -T$ is a linear combination of the y_j; $\mathfrak{B}(C)$ is then the set of points lying on all the Y_j; if the class C contains no positive divisor, then the set (y_j) is empty and $\mathfrak{B}(C) = V$; otherwise, let ψ be the mapping of V into a projective space, determined by the set (y_j). By the corollary of Theorem 8, there is a constant $\gamma > 0$ such that

$$h(C, P) \geqq \gamma h[\psi(P)].$$

If there is only one function y_j, then ψ maps V into P^0, i.e. onto a point, and $h[\psi(P)] = 1$; in any case, we have $h[\psi(P)] \geqq 1$ when $\psi(P)$ is defined. Therefore, *for every C there is a constant $\gamma > 0$ such that $h(C, P) \geqq \gamma$ for all absolutely algebraic points P, not in $\mathfrak{B}(C)$.* Of course, if $\mathfrak{B}(C) = V$, i.e. if there is no positive divisor in the class C, this statement gives no information about $h(C, P)$.

III. The theorem of decomposition.

17. We shall now apply the results of No. 8 not merely to heights, i.e. to the sizes of distributions, as in No. 16, but to the distributions themselves. More precisely, V being a variety defined over a field k, and K being the field of functions defined over k on V, we wish to discuss distributions on K in their relation with the divisors on V.

First let V be an abstract variety, complete and normal, defined over a field k. Until the end of No. 18 we shall restrict the use of the words "function on V", "point of V", and "place", as follows: by a point on V we shall understand one that is rational over k; by the functions on V we understand those that have k as a field of definition; we call K the field of those functions; by a place we understand a k-valued place of K/k, i.e. a k-valued place of K that induces on k the identical automorphism. Once for all (until the end of No. 18), we choose an absolute value v on k, everywhere $< +\infty$ on k, hence everywhere >0 on k^*. On K, we consider only such absolute values as are of the form $v \circ f$, where f is a place (in the present sense of the word). Consequently, if Δ is a distribution on K, we write $\Delta(f)$ instead of $\Delta(v \circ f)$.

Every place f, being a simultaneous specialization of all elements of K, determines a specialization P of a generic point of V over k; as V is complete, this must be a point of V (in our present sense of the word), and is called the *center* of f; then, if x is a function on V defined at P we have $f(x) = x(P)$.

We shall say that a distribution Δ is *defined at a point* P if it takes the same value for all the places f with their center at P; then we write $\Delta(P)$ for that value. Consider a distribution $\sup_\mu \inf_i v[f(x_{\mu,i})]$; it will be defined at P if every one of the distributions $\inf_i v[f(x_{\mu,i})]$ is defined at P. Now consider a distribution of the form $\Delta(f) = \inf_i v[f(x_i)]$; this will be defined at P if at least one of the x_i is defined and is equal to 0 at P, in which case $\Delta(P) = 0$, or also if all the x_i are defined at P, in which case $\Delta(P) = \inf_i v[x_i(P)]$. In particular, assume that $(x_i) = X - U_i$, where X and the U_i are positive divisors; if e.g. P does not lie on U_0, then either P lies on X, in which case $x_0(P) = 0$ and $\Delta(P) = 0$, or it does not, in which case all the x_i are defined at P, and $\Delta(P) = \inf_i v[x_i(P)]$; as $x_0(P) \neq \infty$, $\Delta(P) \neq +\infty$; so Δ is defined and $<+\infty$ at every point P not lying on all the U_i, and it is 0 at such a point P if and only if P lies on X.

Now, as in No. 8 and No. 16, consider functions x_μ, u_ν such that

$$(x_\mu) = T + X_\mu - Z, \qquad (u_\nu) = U_\nu - Z,$$

where T, the X_μ, and the U_ν, are positive divisors, the X_μ, and likewise the U_ν, being without common points. By what we have just seen, the distribution Δ_T, belonging to the expression $\inf_\mu \sup_\nu [x_\mu/u_\nu]$, is everywhere defined and $<+\infty$ on V, and is 0 only on T; this distribution will be said to be *attached to* the positive divisor T. The results of No. 11 show that any other distribution Δ'_T attached in this sense to the divisor T (i.e., by means of another similar choice of the x_μ, u_ν) is equivalent to Δ_T, by which we mean that for all P $\gamma\Delta_T(P) \leqq \Delta'_T(P) \leqq \gamma'\Delta_T(P)$, with constant $\gamma > 0$, $\gamma' > 0$. Similarly we see that $\inf(\Delta_T(P), 1)$ is a distribution equivalent to $\Delta_T(P)$, and so can be substituted everywhere for Δ_T in our final results, in stating which we may therefore take our distributions Δ_T to be $\leqq 1$. Also, if Δ_T, $\Delta_{T'}$ are respectively attached to the positive divisors T, T', any distribution attached to $T + T'$ is equivalent to $\Delta_T(P)\Delta_{T'}(P)$. Finally, if T, T' are equivalent divisors such that $(z) = T' - T$, and if Δ_T has been defined by means of functions x_μ, u_ν, then a distribution attached to T' can be defined by means of the functions zx_μ, u_ν; this will be $\sup_\mu \inf_\nu v[f(zx_\mu/u_\nu)]$, hence equal to $v[z(P)]\Delta_T(P)$ at every point P not lying on T nor on T'; and so there are constants γ, γ', both >0, such that $\gamma\Delta_{T'}(P)/\Delta_T(P) \leqq v[z(P)] \leqq \gamma'\Delta_{T'}(P)/\Delta_T(P)$ unless P lies both on T and on T'.

18. The field k can be topologized by taking, as distance of two elements x, y, the real number $v(x-y)$; this defines a topology on every finite-dimensional vector-space over k, i.e. on the set of points with coordinates in k in the affine space S^n, for every n. For that topology on k, polynomials with coefficients in k are continuous functions; hence the zeros of any ideal in $k[X_1, \cdots, X_n]$, with coordinates in k, form a closed set. Every birational correspondence defined over k between varieties, also defined over k, in affine spaces, is bicontinuous at every pair of points with coordinates in k at which it is biregular. Thus a topology is defined on the set of points (in our present sense, i.e. rational over k) on our abstract variety V; for that topology, the set of points lying on a divisor T, rational over k, is closed; and every function x is continuous at every point at which it is defined and finite; hence also $v(x)$ is continuous at such points. Now let $x_0, \cdots, x_n$ be functions such that $(x_i) = X - U_i$, where X and the U_i are positive divisors, rational over k. Let P be a point of V, and assume for the sake of definiteness that P lies on $U_{r+1}, \cdots, U_n$ and not on $U_0, \cdots, U_r$, where $r \geqq 0$. Put $\Delta(Q) = \inf_i v[x_i(Q)]$; we have $\Delta(Q) = 0$ if Q lies on X and not on U_0, and $\Delta(Q) = v[x_0(Q)] \cdot \sup_i v[u_i(Q)]^{-1}$, with $u_i = x_0/x_i$, if Q does not lie on X nor on U_0. As $(u_i) = U_i - U_0$, we have $u_i(P) = 0$ for $r + 1 \leqq i \leqq n$, and so there is a neighborhood $\mathfrak{B}$ of P having no point in common with any of the divisors $U_0, \cdots, U_r$, such that $v[u_i(Q)] \leqq 1$ for $Q \in \mathfrak{B}$, $r + 1 \leqq i \leqq n$. As $u_0 = 1$, we have, therefore, $\sup_i v[u_i(Q)] = \sup_{0 \leqq i \leqq r} v[u_i(Q)]$ for $Q \in \mathfrak{B}$, hence $\Delta(Q) = \inf_{0 \leqq i \leqq r} v[x_i(Q)]$ for $Q \in \mathfrak{B}$; as $x_0, \cdots, x_r$ are defined and finite in $\mathfrak{B}$, this shows that Δ is continuous in $\mathfrak{B}$ and so in particular at P. Therefore every distribution Δ_T, attached to a positive divisor T, is everywhere continuous on V.

If now we take for V a non-singular projective variety, and apply Theorem 4 of No. 8, we get the following result:

THEOREM 9. *Let V be a non-singular projective variety defined over a field k. Let v be an absolute value, everywhere $< +\infty$, on k. Then to every prime rational divisor W over k on V one can attach a function $\Delta_W(P)$, defined at all the points P of V which are rational over k, taking its values in* [0, 1], *in such a way that the following properties hold*:

(a) *$\Delta_W(P)$ is 0 if and only if P lies on W, and it is continuous everywhere for the topology defined by v*;

(b) *if z is any function, defined over k on V, with the divisor $(z) = \sum_i m_i W_i$, where the W_i are prime rational divisors over k, then there are constants γ, γ', both >0, such that*

$$\gamma \prod_i \Delta_{W_i}(P)^{m_i} \leqq v[z(P)] \leqq \gamma' \prod_i \Delta_{W_i}(P)^{m_i}$$

at all points P, rational over k on V, at which z is defined.

If k is the complex number-field C, and v is the ordinary absolute value on C, it is easy to verify Theorem 9 by elementary topological methods, and to extend it to all compact complex-analytic manifolds.

19. Now we consider an abstract variety V, complete and normal, over an algebraic number-field k. We restrict the words "function on V", "point of V", "place", just as in n° 17, taking, however, as "ground-field", not k, but its algebraic closure $\bar{k} = \bar{Q}$, i.e. the field of all algebraic numbers. So "a function on V" will be one that has $\bar{k}$ as a field of definition, and K will be the field of such functions; a "point of V", or simply "a point", will be one that is rational over $\bar{k}$, i.e. absolutely algebraic; "a place" will be an absolutely algebraic, i.e. $\bar{k}$-valued, place of K, that induces on $\bar{k}$ the identical automorphism. We consider only such absolute values on K as are of the form $v \circ f$, where f is a place (in our present sense), and v a proper absolute value on $\bar{k} = \bar{Q}$. Accordingly, distributions are written $\Delta(v \circ f)$.

Let Δ_x be the distribution belonging to the expression $\inf_\mu \sup_i [x_{\mu,i}]$; if a finite algebraic extension k' of k is a field of definition for all the $x_{\mu,i}$, we say that Δ_x is *defined over* k'. If σ is any automorphism of $\bar{k}$ over k, it can be extended in one and only one way to an automorphism of K, leaving invariant every function on V which has k as a field of definition; the latter automorphism being also denoted by σ, we denote by Δ_x^σ the distribution belonging to the expression $\inf_\mu \sup_i [x^\sigma_{\mu,i}]$; this will be the same as Δ_x whenever σ induces the identical automorphism on k'. If f is a place of K, we denote by f^σ the place defined by

$$f^\sigma(x) = [f(x^{\sigma^{-1}})]^\sigma;$$

then if P is the center of f, P^σ is the center of f^σ. If, at the same time, we denote by v^σ the absolute value on $\bar{k}$ defined by $v^\sigma(\xi) = v(\xi^{\sigma^{-1}})$ for all $\xi \in \bar{k}$, we have $\Delta_x(v \circ f) = \Delta_x^\sigma(v^\sigma \circ f^\sigma)$.

We shall say that the distribution Δ, defined over k', is *defined at a point* P of V if it satisfies the following conditions: (a) for each v, $\Delta(v \circ f)$ has the same value $\Delta(P, v)$ for all places f having their center at P; (b) either $\Delta(P, v) = 0$ for all v, or $\Delta(P, v) = +\infty$ for all v, or $\Delta(P, v)$ depends only upon the absolute value induced by v on $k'(P)$, and, as such, is a $k'(P)$-divisor. Just as in No. 17, one sees that the distribution $\Delta(v \circ f) = \inf_i v[f(x_i)]$ is defined at the point P if $(x_i) = X - U_i$, where X and the U_i are positive divisors, and P does not lie on all the U_i.

Now we can proceed just as in Nos. 17–18, and obtain the following theorem:

THEOREM 10. *Let V be a non-singular projective variety of dimension r, defined over an algebraic number-field k. Then to every absolutely algebraic $(r-1)$-dimensional subvariety W of V one can attach a function $\Delta_W(P, v)$, defined for all absolutely algebraic points P of V and all proper absolute values v of $\bar{k} = \bar{Q}$, taking its values in* $[0, 1]$, *in such a way that the following properties hold:*

(a) *$\Delta_W(P, v)$ is 0 if and only if P lies on W; and, for given v, W, $\Delta_W(P, v)$ is a continuous function of P for the topology defined on V by v;*

(b) *if k_W is the smallest extension of k over which W is defined, then, for each P not on W, $\Delta_W(P, v)$ depends only upon the absolute value induced by v on $k_W(P)$, and, as such, is a $k_W(P)$-divisor;*

(c) *if σ is any automorphism of $\bar{k}$ over k, $\Delta_{W^\sigma} = \Delta_W^\sigma$ for all W;*

(d) *if z is a function, defined over $\bar{k}$ on V, with the divisor $(z) = \sum_i m_i W_i$, then there are Q-divisors δ, δ', such that*

$$\delta(v) \prod_i \Delta_{W_i}(P, v)^{m_i} \leqq v[z(P)] \leqq \delta'(v) \prod_i \Delta_{W_i}(P, v)^{m_i}$$

for all v, and for all absolutely algebraic P at which z is defined.

If one pays attention only to the non-archimedean absolute values, then $\Delta_W(P, v)$ determines, for each W and P, an (integral) ideal $\mathfrak{a}_W(P)$ in the field $k_W(P)$; this ideal is 0 if and only if P lies on W; and (d) asserts that for a given z there are non-zero rational numbers r, r' such that, if k' is a common field of definition for z and the W_i, the principal ideal $(z(P))$ in $k'(P)$ is a multiple of $r \prod_i \mathfrak{a}_{W_i}(P)^{m_i}$ and divides $r' \prod_i \mathfrak{a}_{W_i}(P)^{m_i}$ for all P at which z is defined.

Some further properties of the distributions Δ_W can easily be deduced from our general theory. Consider for instance *a set of varieties* W_λ *without common point*; then (as observed by Northcott, [4a], Lemma 3) *there is a Q-divisor* δ *such that* $\sup_\lambda \Delta_{W_\lambda}(P, v) \geqq \delta(v)$ *for all* v *and* P. For each λ, in fact, Δ_{W_λ} is quasi-equal to a distribution belonging to an expression $\inf_\mu \sup_\nu [x_{\lambda,\mu}/u_{\lambda,\nu}]$, with

$$(x_{\lambda,\mu}) = W_\lambda + X_{\lambda,\mu} - Z_\lambda, \qquad (u_{\lambda,\nu}) = U_{\lambda,\nu} - Z_\lambda,$$

where, for each λ, the $X_{\lambda,\mu}$ are positive divisors without common point, and so are the $U_{\lambda,\nu}$. The distribution $\sup_\lambda \Delta_{W_\lambda}$ is then quasi-equal to the distribution belonging to $\inf_{\lambda,\mu}\sup_\nu [x_{\lambda,\mu}/u_{\lambda,\nu}]$, and so, by No. 11, all we need do is to show that this last expression is $\prec 0$ in $\mathbf{F}(K/Q) = \mathbf{F}(K/\bar{k})$, i.e. that $\inf_{\lambda,\mu}[x_{\lambda,\mu}/u_{\lambda,\nu(\lambda,\mu)}] \prec 0$ for each choice of the function $\nu(\lambda, \mu)$; this is an immediate consequence of Theorem 3 of No. 7, and of our assumptions.

Finally, we have the following relation between our distributions Δ_W and the functions $h(C, P)$ defined in No. 16. Let W be an absolutely algebraic $(r - 1)$-dimensional subvariety of V, and let $C(W)$ be the class of the divisor W. Let the x_μ, u_ν be a "permissible choice" of functions for W, i.e. one such that $(x_\mu) = W + X_\mu - Z$, $(u_\nu) = U_\nu - Z$, where the X_μ are positive divisors without common point, and so are the U_ν. Let φ, ω be the mappings of V into projective spaces determined by the sets of functions (x_μ), (u_ν); then the function $h'(P) = h[\omega(P)]/h[\varphi(P)]$ is equivalent to the function $h(C(W), P)$ defined (by means of a similar choice) in No. 16; and the distribution Δ' belonging to the expression $\inf_\mu \sup_\nu [x_\mu/u_\nu]$ is quasi-equal to Δ_W. Let f be a place, and P its center; assume for the sake of definiteness that P is not in U_0; put $x'_\mu = x_\mu/u_0$, $u'_\nu = u_\nu/u_0$. We have

$$h[\omega(P)] = s(\sup_\nu v[u'_\nu(P)]);$$

also, if P is not in W, the x'_μ are not all 0 at P, and then $h[\varphi(P)] = s(\sup_\mu v[x'_\mu(P)])$. In all cases, we have

$$\Delta'(P, v) = \sup_\mu \inf_\nu v[f(x'_\mu/u'_\nu)] = \sup_\mu v[x'_\mu(P)]/\sup_\nu v[u'_\nu(P)].$$

Therefore we have $s[\Delta'(P, v)] = h'(P)$ if P is not on W. Hence:

THEOREM 11. *Notations being as in Theorem* 10, *call* $C(W)$ *the class of the divisor* W; *let* $h(C(W), P)$ *be the function belonging to this class, as defined in No.* 16. *Then there are two constants* γ, γ', *both* >0, *such that*

$$\gamma h(C(W), P)^{-1} \leqq s[\Delta_W(P, v)] \leqq \gamma' h(C(W), P)^{-1}$$

for all absolutely algebraic points P *on* V *which do not lie on* W.

IV. The case of curves.

20. Heights. When applied to curves, or algebraically speaking to algebraic function-fields of degree of transcendency 1, our theory undergoes far-reaching simplifications due to the fact that, if K is such a function-field over a ground-field k, the set $\mathbf{V}(K/k)$ of non-trivial valuations of K, trivial on k, is nothing else than the set of all prime rational divisors over k on a complete non-singular model of K; our group $\mathbf{F}(K/k)$ of "valuation-functions" can therefore be identified in that case with the group of rational divisors over k on such a model. In other words, there is then no distinction to be made between valuation-functions and divisors; and our whole algebraic theory reduces to the usual theory of divisors on a curve. All we need do is therefore to summarize the main results of our arithmetical theory in that case.

Let therefore Γ be a curve, which we may assume to be given as a non-singular curve in a projective space, defined over an algebraic number-field k. Consider the group of all the divisor-classes C on Γ which contain absolutely algebraic divisors. To each such class C, we have learned (in No. 16) to attach a function $h(C, P)$, defined at all absolutely algebraic points P of Γ, with values in the open interval $]0, +\infty[$, and with the following properties:

(a) *$h(-C, P)$ is equivalent to $h(C, P)^{-1}$; and, if C' is another class, $h(C + C', P)$ is equivalent to $h(C, P)h(C', P)$;*

(b) *If the class C contains a positive divisor, there is a constant $\gamma > 0$ such that $h(C, P) \geqq \gamma$ for all P;*

(c) *if φ is a mapping, defined over $\bar{k}$, of Γ into a projective space, and C is the class of the divisors in the linear series* (without fixed point) *determined on Γ by φ, then $h[\varphi(P)]$ is equivalent to $h(C, P)$.*

We recall that two functions $h(P)$, $h'(P)$ are said to be equivalent if each is less than a constant multiple of the other.

One more important property of the functions $h(C, P)$ can be deduced from the theorem of Riemann-Roch (Siegel [2]). Let A, B be two absolutely algebraic points on Γ; let g be the genus of Γ; by that theorem, the class of the divisor $(m + g)A - mB$ contains a positive divisor for every value of the integer m. But we may take m as large as we please, and so there is, for each $\varepsilon > 0$, a constant $\gamma > 0$ such that $h(C(A), P) \geqq \gamma h(C(B), P)^{1-\varepsilon}$, where $C(A)$, $C(B)$ denote the classes of the divisors A, B. If for a fixed A we put $h_0(P) = \sup(1, h(C(A), P))$, we get the following result:

(d) *there is a function $h_0(P)$, taking its values in $[1, +\infty[$, such that, to every class C of divisors of degree d, and to every $\varepsilon > 0$, there are constants γ, γ', both >0, for which*

$$\gamma h_0(P)^{d-\varepsilon} \leqq h(C, P) \leqq \gamma' h_0(P)^{d+\varepsilon}$$

for all P.

From this, and from Northcott's theorem (No. 13), it follows in particular that $\log h(C, P)/\log h_0(P)$ tends to the degree d of the class C when P runs through any infinite sequence of distinct points of bounded degree over k on Γ. It is also easy now to obtain Siegel's second fundamental inequality ([2], p. 52), in a slightly more general form. Let $x_1, \cdots, x_n$ be functions defined over k on Γ; put $x_0 = 1$; let φ be the mapping of Γ into P^n determined by $(x_0, \cdots, x_n) = (1, x_1, \cdots, x_n)$. Let P be a point of Γ such that all the $x_i(P)$ are algebraic integers; then, for each non-archimedean absolute value v, we have $v[x_i(P)] \leqq 1$. From this it follows that $\sup_{0 \leqq j \leqq n} v[x_i(P)] = 1$, and hence that

$$h[\varphi(P)] = \Big(\prod_\sigma \sup_i | x_i(P)^\sigma |\Big)^{1/m},$$

where m is the degree of $k(P)$ over Q, and σ runs over all the m isomorphisms of $k(P)$ into the complex number-field. Also, if C is the class of the divisors in the linear series (without fixed point) determined by φ on Γ, $h[\varphi(P)]$ is equivalent to $h(C, P)$; so, if d is the degree of that class, i.e. the degree of the divisor $\sup_i(x_i)_\infty$, there is, to every $\varepsilon > 0$, a $\gamma > 0$ such that $h[\varphi(P)] \geqq \gamma h_0(P)^{d-\varepsilon}$. With these notations we have, therefore, $\sup_{i,\sigma} | x_i(P)^\sigma | \geqq \gamma h_0(P)^{d-\varepsilon}$, for all points P such that all the $x_i(P)$ are algebraic integers. For $n = 1$ and $k(P) = k$, this is Siegel's result.

21. Distributions. Here the subvarieties W of V in the general theory of No. 19 become points on Γ, so that the distributions $\Delta_W(P, v)$ of No. 19 become functions $\Delta_A(P, v) = \Delta(A, P, v)$ of pairs of points A, P, and of absolute values v on $\bar{k}$. This suggests going over to the product $\Gamma \times \Gamma$ of Γ by itself; this can of course be represented as a non-singular projective variety; as usual, we denote the diagonal on it by Δ.

By Theorem 4 of No. 8, we can find on $\Gamma \times \Gamma$ two sets of functions x_μ, u_ν, defined over k, such that $(x_\mu) = \Delta + X_\mu - Z$, $(u_\nu) = U_\nu - Z$, where the X_μ are positive divisors without common point, and so are the U_ν; let $\Delta(P, Q, v)$ be the distribution belonging to the expression $\inf_\mu \sup_\nu [x_\mu/u_\nu]$ on $\Gamma \times \Gamma$; this is defined at all points (P, Q) of that surface. Let A be any (absolutely algebraic) point on Γ; if $A \times \Gamma$ is a component of Z with the coefficient m, take any function z, defined over k on $\Gamma \times \Gamma$, such that $A \times \Gamma$ is a component of the divisor (z), with the coefficient $-m$; then $A \times \Gamma$ is not a component of $Z' = Z + (z)$. Put $x'_\mu = zx_\mu$, $u'_\nu = zu_\nu$; the x'_μ, u'_ν will then induce functions x''_μ, u''_ν on $A \times \Gamma$; x''_μ is 0 if $A \times \Gamma$ is a component of X_μ, whereas otherwise its divisor is $(x''_\mu) = (A \times A) + X''_\mu - Z''$, with $X''_\mu = X_\mu \cdot (A \times \Gamma)$, $Z'' = Z' \cdot (A \times \Gamma)$; and the divisors X''_μ, corresponding to non-zero x''_μ have no common component, since such a point would lie on all the X_μ. Similarly the divisors of the non-zero u''_ν are given by $(u''_\nu) = U''_\nu - Z''$, where the U''_ν have no common component. From this, it follows that $\Delta_A(P, v) = \Delta(A, P, v)$ is a distribution attached to the divisor A on Γ. If now we replace $\Delta(P, Q, v)$ by $\inf(1, \Delta(P, Q, v), \Delta(Q, P, v))$, which is a distribution quasi-equal to it by n° 11, we can state our results as follows:

There is a function $\Delta(P, Q, v)$ of pairs of absolutely algebraic points P, Q on Γ, and of proper absolute values v on $\bar{k}$, with values in $[0, 1]$, *and with the following properties:*

(a) $\Delta(P, Q, v) = \Delta(Q, P, v)$ *for all* P, Q; $\Delta(P, Q, v) = 0$ *if and only if* $P = Q$;

for a given v, $\Delta(P, Q, v)$ is a continuous function of (P, Q) for the topology defined by v on $\Gamma \times \Gamma$;

(b) *for given P, Q, with $P \neq Q$, $\Delta(P, Q, v)$, as a function of v, is a $k(P, Q)$-divisor;*

(c) *if σ is any automorphism of $\bar{k}$ over k, $\Delta(P, Q, v) = \Delta(P^{\sigma}, Q^{\sigma}, v^{\sigma})$;*

(d) *if $P \neq P'$, there is a Q-divisor δ* (depending upon P, P') *such that* $\sup(\Delta(P, Q, v), \Delta(P', Q, v)) \geqq \delta(v)$ *for all v and Q;*

(e) *if z is a function defined over $\bar{k}$ on Γ, with the divisor $(z) = \sum_i m_i A_i$, there are Q-divisors δ, δ', such that for all P and v,*

$$\delta(v) \prod_i \Delta(A_i, P, v)^{m_i} \leqq v[z(P)] \leqq \delta'(v) \prod_i \Delta(A_i, P, v)^{m_i};$$

(f) *for each A on Γ, $s[\Delta(A, P, v)]$, as a function of P, is equivalent to $h(C(A), P)^{-1}$, where $C(A)$ is the class of the divisor A, and h is as in No.* 20. In particular, to each A and each $\varepsilon > 0$, there are constants γ, γ', both >0, such that

$$\gamma h_0(P)^{-1-\varepsilon} \leqq s[\Delta(A, P, v)] \leqq \gamma' h_0(P)^{-1+\varepsilon},$$

where $h_0(P)$ is as in No. 20.

If we pay attention only to the non-archimedean absolute values v, $\Delta(P, Q, v)$ determines an integral ideal $\mathfrak{a}(P, Q)$ of the field $k(P, Q)$, the properties of which are implicit in those given above. In particular, if z is as in (e), and k' is a finite extension of k over which z and the A_i are defined, then there are non-zero rational numbers r, r' such that the principal ideal $(z(P))$ in $k'(P)$ is a multiple of $r\prod_i \mathfrak{a}(A_i, P)^{m_i}$, and divides $r'\prod_i \mathfrak{a}(A_i, P)^{m_i}$, for all P. Also, in the last inequality in (f), if we restrict the product implicit in the symbol s to the non-archimedean v, then, since the factors thus left out are all $\leqq 1$, the first half of the inequality remains true. If $d(P)$ is the degree of $k(A, P)$ over Q, and N denotes the norm (over Q) of ideals in $k(A, P)$, we thus find that, if A and $\varepsilon > 0$ are given, there is a constant γ such that $N(\mathfrak{a}(A, P))^{1/d(P)} \leqq \gamma h_0(P)^{1+\varepsilon}$ for all P. This, for $k(P) = k$, is Siegel's first fundamental inequality ([2], p. 50).

V. Valuation-functions and local ideals.

22. We now abandon the arithmetical considerations of §§II, III, IV, and go back to the algebraic theory of §I; our purpose is in part to provide some substantial motivation for our concept of valuation-functions "attached" to divisors (No. 8), which may seem to have been introduced solely on the ground of expediency. This will require some further general concepts.

If A is a subring of a field K, let us write K_A for K when it is taken, not with its structure as a field, but merely with its structure as an A-module. Let the x_i be a finite set of elements of K; they generate a submodule $M = \sum_i x_i A$ of K_A; if ω is a valuation of K which is $\geqq 0$ on A, we have $\inf_i \omega(x_i) = \inf_{x \in M} \omega(x)$, and so the element $\inf_i [x_i]$ of $\mathbf{F}(K/A)$ is completely determined by the module M.

If K is the field of fractions of the ring A, then a fractional A-ideal in K is defined to be any submodule of K_A which, as an A-module, is isomorphic to an ideal in the ring A, or, what is the same thing, which is contained in xA for some $x \in K$. Any finitely generated submodule $I = \sum_i x_i A$ of K_A is a fractional A-ideal; as above, this determines the element $\inf_i [x_i]$ of $\mathbf{F}(K/A)$. If A' is the integral closure of A, the set of the $x \in K$ such that $[x] \succ \inf_i [x_i]$ in $\mathbf{F}(K/A)$ is a fractional A'-ideal I'; if $A = A'$, and $I = I'$, I is sometimes called "complete"[4] or "integrally closed".[5]

Let K be the field of fractions of the ring A; let $I = \sum_i x_i A$, $J = \sum_j y_j A$ be two finitely generated fractional A-ideals in K, determining respectively the elements $X = \inf_i [x_i]$, $Y = \inf_j [y_j]$ of $\mathbf{F}(K/A)$; then the A-ideals $I + J$ and $I \cdot J$ determine respectively the elements $\inf (X, Y)$ and $X + Y$ of $\mathbf{F}(K/A)$, while $I \cap J$, if it is finitely generated, determines an element Z of $\mathbf{F}(K/A)$ satisfying $Z \succ \sup (X, Y)$; if $I \supset J$ we have $X \prec Y$, but the converse need not be true. Thus there is only an incomplete parallelism between valuation-functions and ideals. The two A-ideals I, J will be called *equivalent* if $X = Y$, i.e. if they determine the same element of $\mathbf{F}(K/A)$.

23. From now on, the "ground-field" k will remain fixed once for all; V will be a variety (mostly abstract) defined over k; the word "function" will be restricted to functions on V having k as a field of definition; K will be the field of such functions. The word "valuation" will be restricted to valuations of K which vanish on k, that is to elements of $\mathbf{V}(K/k)$. By a "point" we understand any point on V, not necessarily rational nor even algebraic over k.

Each point P on V determines a subring A_P of K, the "specialization-ring of P in K", consisting of the functions on V which are defined and finite at P; to say that A_P is integrally closed is the same as to say that V is normal at P relatively to k. If P, Q are two points on V, we have $A_P \supset A_Q$ if and only if Q is a specialization of P over k, and $A_P = A_Q$ if and only if P and Q are generic specializations of each other over k.

From now on, let V be a complete abstract variety. Let ω be any valuation in $\mathbf{V}(K/k)$; if the place f of K belongs to ω, then it is trivial on k, i.e. it induces on k an isomorphism σ of k onto some field k'; the isomorphism σ^{-1} of k' onto k can then be extended to an isomorphism τ of $f(K)$ into some field L (more correctly, into $L_\infty = L \cup \{\infty\}$), and then $f' = \tau \circ f$ is a place of K, equivalent to f, which induces on k the identical automorphism. Such a place f' determines a specialization P of a generic point of V over k, which, since V is complete, is a point of V; then P is such that ω is $\geqq 0$ on A_P, and 0 on the maximal ideal of A_P; if Q is any other point with these two properties, then $A_Q = A_P$, and Q is a generic specialization of P over k; if Q is merely such that ω is $\geqq 0$ on A_Q, then $A_Q \subset A_P$, and Q is a specialization of P over k. Thus the points Q such that $\omega \geqq 0$ on A_Q are those which lie on the union of the components of the locus of P over k; this union is called the *center* of ω.

[4] Cf. O. Zariski, Am. J. of Math. 60 (1938), p. 151.

[5] Cf. W. Krull, *Idealtheorie*, Erg. d. math. Wiss. IV-3, Berlin 1935, pp. 128–129. Krull's "fundamental theorem" (p. 129), identifying "integrally closed" ideals with "valuation-ideals", is substantially equivalent to our corollary 1 of Theorem 2 (No. 5), and to the consequences derived from it in No. 6.

From this, it follows that $\mathbf{V}(K/k) = \bigcup_{P \in V} \mathbf{V}(K/A_P)$; and so an element of $\mathbf{F}(K/k)$ is completely determined by the elements it induces in the groups $\mathbf{F}(K/A_P)$ for all P. These are not independent; in fact, it follows from what we have seen that $\mathbf{V}(K/A_P) \cap \mathbf{V}(K/A_Q)$ is the set of all valuations whose center contains both P and Q.

24. Consider an element X of $\mathbf{F}(K/k)$; this is a function $X(\omega)$ of the valuations $\omega \in \mathbf{V}(K/k)$, and, for any P on V, its canonical image X_P in $\mathbf{F}(K/A_P)$ is the restriction of the function $X(\omega)$ to the valuations $\omega \in \mathbf{V}(K/A_P)$. Let y_j be a set of elements of K, and put $Y = \inf_j[y_j]$; if $X_P = Y_P$, i.e. if $X(\omega) = Y(\omega)$ for $\omega \in \mathbf{V}(K/A_P)$, we say that X_P is *defined by* the A_P-ideal $I_P = \sum_j y_j A_P$, and also that X is *defined at* P by that ideal. If X and P are given there need not exist an ideal I_P defining X at P; and, if there is one, it is determined only up to equivalence. If X is such that it is defined at every point P of V by an A_P-ideal I_P, we say that it is *definable by local ideals*. If that is so, and if the I_P are given, they determine X_P for every P, and so X is uniquely determined. We shall now obtain some necessary and sufficient conditions for a set of local ideals I_P to determine an element X of $\mathbf{F}(K/k)$.

It will be convenient to use some topological terms. On $\mathbf{V}(K/k)$ Zariski has introduced a (non-separated) topology which can be defined as follows. For each $x \in K$, let Ω_x be the set of those $\omega \in \mathbf{V}(K/k)$ for which $\omega(x) \geqq 0$. The open sets in the Zariski topology are those which can be obtained from the Ω_x by the operations of union and of finite intersection. More important for us, however, are those sets which can be obtained from the Ω_x by the operations of finite union and finite intersection; these will be called *finitely open*; every such set can be defined by a relation $X(\omega) \geqq 0$, with $X \in \mathbf{F}(K/k)$. As shown by Zariski, and as one can easily verify, using corollary 1 of Theorem 2, his open sets, hence a fortiori the finitely open sets, satisfy the "compactness axiom": every covering of $\mathbf{V}(K/k)$ by such sets contains a finite covering.

On the other hand, we define as closed sets on V itself (relatively to the ground-field k) all bunches of subvarieties of V which are normally algebraic over k; in other words, the union of all conjugates over k of a subvariety of V which is algebraic over k will be a closed subset of V; and all finite unions of such sets will be closed sets. The complements of these will be the open sets on V; clearly they satisfy the compactness axiom. The locus over k of a point P of V is the smallest closed subset containing P, i.e. it is the closure of P; hence, if it is contained in the union of two closed sets, it must be contained in one of them. This applies in particular to the center of a valuation. If W, W' are two open subsets of V, and if a valuation ω is in $\mathbf{V}(K/A_P) \cap \mathbf{V}(K/A_{P'})$, with $P \in W$ and $P' \in W'$, then the center of ω cannot be contained in the complement $C(W)$ of W, nor in $C(W')$, and so is not contained in $C(W) \cup C(W') = C(W \cap W')$; hence there is a $Q \in W \cap W'$ such that $\omega \in \mathbf{V}(K/A_Q)$.

Let Ω be a finitely open set in $\mathbf{V}(K/k)$; then the set W of the points P of V such that $\mathbf{V}(K/A_P) \subset \Omega$ is open. In fact, let Ω be defined by $X(\omega) \geqq 0$, with $X = \inf_\mu \sup_i[x_{\mu,i}]$; then it is the intersection of the sets Ω_μ respectively defined by $\inf_i \omega(1/x_{\mu,i}) \leqq 0$, and W is the intersection of the corresponding subsets of V, so that it is enough to prove our statement for each Ω_μ. Taking complements, what we have to prove amounts to this: $y_1, \cdots, y_m$ being given elements of K, the set Y of points P, such that there exists an $\omega \in \mathbf{V}(K/A_P)$ for which $\omega(y_j) > 0$ for all j, is closed. In fact, Y is the set of points P such that $(0, \cdots, 0)$ is a specialization of $(y_1, \cdots, y_m)$ at P. So, if Γ is the graph in $V \times S^m$ of the mapping $(y_1, \cdots, y_m)$ of V into S^m, we have $Y \times 0 = \Gamma \cap (V \times 0)$, which proves our assertion.

In particular, if X, Y are two elements of $\mathbf{F}(K/k)$, the set of points P such that $X_P = Y_P$, i.e. such that $X(\omega) = Y(\omega)$ for all $\omega \in \mathbf{V}(K/A_P)$, is open; of course, it may be empty. Now, for each element Y of $\mathbf{F}(K/k)$ of the form $Y = \inf_j[y_j]$, consider the set $W(Y)$ of the points P of V such that $X_P = Y_P$, i.e. such that X is defined at P by the ideal $\sum_j y_j A_P$; if we assume that X is definable by local ideals, the open sets $W(Y)$ form a covering of V, from which one can therefore extract a finite covering. In other words, there exists a finite covering of V by open sets W_λ, and, for each λ, an element $Y_\lambda = \inf_j[y_{\lambda,j}]$ of $\mathbf{F}(K/k)$, such that $X_P = (Y_\lambda)_P$ for each $P \in W_\lambda$, that is, such that X is defined by the ideal $\sum_j y_{\lambda,j} A_P$ at every $P \in W_\lambda$, for each λ.

25. Assume that to every P on V we have assigned a fractional A_P-ideal I_P in K; the I_P will be said to form a *coherent system of local ideals* if there is a finite covering of V by open sets W_λ, and, for each λ, a set $y_{\lambda,j}$ of elements of K, such that, whenever $P \in W_\lambda$, the ideals I_P and $\sum_j y_{\lambda,j} A_P$ are equivalent. Then:

THEOREM 12. *Let a fractional A_P-ideal I_P in K be given for every point P on V; the I_P will define an element X of $\mathbf{F}(K/k)$ if and only if they form a coherent system.*

We have just proved that this condition is necessary. In order to prove that it is sufficient we need various lemmas, and in the first place the following one on abstract varieties.

LEMMA. *Let V be an abstract variety defined over k; let a finite covering of V be given by subsets W_λ of V, open on V relatively to k. Then there is an abstract variety $V' = [V'_\rho; \emptyset; T'_{\sigma\rho}]$, defined over k by affine varieties V'_ρ with empty frontiers, and by the birational correspondences $T'_{\sigma\rho}$ between the V'_ρ, such that:* (a) *there is an everywhere biregular birational correspondence T between V' and V over k;* (b) *for each ρ, the image $T(V'_\rho)$ of V'_ρ on V by T is contained in some W_λ.*

In stating (b), we have identified each V'_ρ with the corresponding subset of V', i.e. with the set of those points of V' which have a representative in V'_ρ. In proving the lemma, we may, if necessary, replace the covering W_λ by a finite subcovering, and so assume that the W_λ are finite in number. Each W_λ is then an abstract variety defined by a finite number of affine representatives $W_{\lambda\alpha}$ with frontiers $F_{\lambda\alpha}$. Consider one of them, say $W_{\lambda\alpha}$; it is a variety in an affine space S^N; let $(x_1, \cdots, x_N)$ be a generic point of it over k. Take a finite "basis," i.e. a finite set of generators, for the ideal in $k[X_1, \cdots, X_N]$, consisting of the polynomials which are 0 on $F_{\lambda\alpha}$; let the $P_h(X)$ be those elements in that set which are not 0 on $W_{\lambda\alpha}$. For each h let $W'_{\lambda\alpha h}$ be the locus of $(x_1, \cdots, x_N, 1/P_h(x))$ over k in S^{N+1}. As our varieties V'_ρ we take all the $W'_{\lambda\alpha h}$, with the obvious birational correspondences between them; these have all the properties stated in our lemma.

26. If a coherent system of local ideals is given on V, there is a covering of V with the properties stated in the definition of such systems in No. 25; to this covering we apply the lemma above, and then identify V with V' by means of the correspondence T. Then, after an obvious change of notations, the situation can be described as follows. We have an abstract variety $V = [V_\alpha ; T_{\beta\alpha}]$ given by the affine representatives V_α (with empty frontiers), and the birational correspondences $T_{\beta\alpha}$ between the V_α, all these being defined over k; put $\Omega_\alpha = \bigcup_{P \epsilon V_\alpha} \mathbf{V}(K/A_P)$; by No. 24, we have $\Omega_\alpha \cap \Omega_\beta = \bigcup_{P \epsilon V_{\alpha\beta}} \mathbf{V}(K/A_P)$, with $V_{\alpha\beta} = V_\alpha \cap V_\beta$. Also, for each α we have a set $y_{\alpha,j}$ of elements of K; and, for $Y_\alpha = \inf_j [y_{\alpha,j}]$, we have $Y_\alpha(\omega) = Y_\beta(\omega)$ whenever $\omega \epsilon \Omega_\alpha \cap \Omega_\beta$, for all α, β. In order to prove Theorem 12 we have to construct an $X \epsilon \mathbf{F}(K/k)$ such that $X(\omega) = Y_\alpha(\omega)$ for $\omega \epsilon \Omega_\alpha$, for all α.

Assume that we have constructed elements $X_{\alpha\beta}$ of $\mathbf{F}(K/k)$ for all $\alpha \neq \beta$, such that $X_{\alpha\beta} \geqq Y_\alpha$ on Ω_α, and $X_{\alpha\beta} \leqq Y_\beta$ on $\Omega_\beta \cap C(\Omega_\alpha)$. Then, for each α, put $X'_\alpha = \inf_{\beta \neq \alpha} X_{\alpha\beta}$, $X''_\alpha = \inf(Y_\alpha, X'_\alpha)$, and $X = \sup_\alpha X''_\alpha$. Take any $\omega \epsilon \Omega_\alpha$; we have $X_{\alpha\beta}(\omega) \geqq Y_\alpha(\omega)$ for all $\beta \neq \alpha$, hence $X'_\alpha(\omega) \geqq Y_\alpha(\omega)$, $X''_\alpha(\omega) = Y_\alpha(\omega)$. Also, for each $\beta \neq \alpha$ we have either $\omega \epsilon \Omega_\beta$ or $\omega \epsilon C(\Omega_\beta)$; in the first case we have $Y_\beta(\omega) = Y_\alpha(\omega)$, and so $X''_\beta(\omega) \leqq Y_\alpha(\omega)$, while in the second case

$$X_{\beta\alpha}(\omega) \leqq Y_\alpha(\omega), \qquad X'_\beta(\omega) \leqq Y_\alpha(\omega), \qquad X''_\beta(\omega) \leqq Y_\alpha(\omega).$$

As we have $X''_\beta(\omega) = Y_\alpha(\omega)$, and, for all $\beta \neq \alpha$, $X''_\beta(\omega) \leqq Y_\alpha(\omega)$, we have

$$X(\omega) = Y_\alpha(\omega);$$

hence X is as required. So all we need do is to construct $X_{\alpha\beta}$ for each pair $\alpha \neq \beta$. Again, for a given pair $\alpha \neq \beta$, assume that we have constructed, for each j, an element Z_j of $\mathbf{F}(K/k)$ such that $Z_j \geqq Y_\alpha$ on Ω_α, and $Z_j(\omega) \leqq \omega(y_{\beta,j})$ for $\omega \epsilon \Omega_\beta \cap C(\Omega_\alpha)$; then one sees at once that $X_{\alpha\beta} = \inf_j Z_j$ will be as required. As $Y_\alpha = Y_\beta$ on $\Omega_\alpha \cap \Omega_\beta$, we have, for each j, $\omega(y_{\beta,j}) \geqq Y_\alpha(\omega)$ on $\Omega_\alpha \cap \Omega_\beta$. So our problem will be solved if we prove the following lemma:

27. LEMMA. *Let V, V' be two affine varieties, defined and birationally equivalent over k; let x, x' be corresponding generic points of V, V' over k; put $K = k(x) = k(x')$. Put $\Omega = \bigcup_{P \epsilon V} \mathbf{V}(K/A_P)$, $\Omega' = \bigcup_{Q \epsilon V'} \mathbf{V}(K/A_Q)$. Let the y_j and y' be non-zero elements of K such that $\omega(y') \geqq \inf_j \omega(y_j)$ for all $\omega \epsilon \Omega \cap \Omega'$. Then there is a finite set of monomials $M_\nu(x)$ in the coordinates of x such that, if we put $Z = \inf_{\nu,j}[M_\nu(x)y_j]$, we have $Z(\omega) \geqq \inf_j \omega(y_j)$ for all $\omega \epsilon \Omega$, and $Z(\omega) \leqq \omega(y')$ for all $\omega \epsilon \Omega'$.*

In fact, Ω is no other than the set $\mathbf{V}(K/k[x])$ of the valuations which are $\geqq 0$ on the ring $k[x]$ generated over k by the coordinates of x; for those coordinates are in A_P for every P on V, and so, for $P \epsilon V$, we have $A_P \supset k[x]$, and therefore $\mathbf{V}(K/A_P) \subset \mathbf{V}(K/k[x])$; and if ω is $\geqq 0$ on $k[x]$, a place belonging to ω and inducing on k the identical automorphism will induce a finite specialization P of x over k, i.e. a point of V, and then we have $\omega \epsilon \mathbf{V}(K/A_P)$. Similarly we have $\Omega' = \mathbf{V}(K/k[x'])$, whence $\Omega \cap \Omega' = \mathbf{V}(K/k[x, x'])$. As $\omega \epsilon \Omega$ implies that ω is $\geqq 0$ on $k[x]$, we have then $\omega(M_\nu(x)) \geqq 0$ for every monomial $M_\nu(x)$, so that the first inequality in our lemma is satisfied for every choice of the monomials $M_\nu(x)$; it remains for us to show that the last inequality is satisfied for a suitable choice of the $M_\nu(x)$.

We do not change either our assumptions or our conclusions if we replace everywhere y' by 1, and each y_j by y_j/y'. So we may assume that $y' = 1$. Then our assumption is that $\inf_j\omega(y_j) \leqq 0$ for all $\omega \in \mathbf{V}(K/k[x, x'])$; and we have to prove that, for some suitable choice of the $M_\nu(x)$, $\inf_{\nu,j}\omega(M_\nu(x)y_j) \leqq 0$ for all $\omega \in \mathbf{V}\ (K/k[x'])$. By corollary 1 of Theorem 2 (No. 5), our assumption implies that there is a polynomial P in the indeterminates Y_j, with coefficients in $k[x, x']$, such that $P(y) = 0$, $P(0) = 1$. So we have a relation $1 + \sum_\lambda p_\lambda N_\lambda(y) = 0$, with $p_\lambda \in k[x, x']$, the N_λ being monomials of degree $\geqq 1$ in the y_j. This can also be written as $1 + \sum_{\nu=1}^{n} q_\nu(x')M_\nu(x)N'_\nu(y) = 0$, with $q_\nu \in k[x']$, the M_ν being monomials in the coordinates of x, and the N' monomials of degree $\geqq 1$ in the y_j. Now put $M_0(x) = 1$, and $w_{\nu,j} = M_\nu(x)y_j$, for all j, and $0 \leqq \nu \leqq n$; then we have $w_{0,j} = y_j$; so, for each ν, if we choose j_ν so that y_{j_ν} is a factor of $N'_\nu(y)$, $M_\nu(x)N'_\nu(y)$ can be written as the product of w_{ν,j_ν} and of some of the $w_{0,j}$, i.e. as a monomial $R_\nu(w)$ of degree $\geqq 1$ in the $w_{\nu,j}$. Therefore we have the relation $1 + \sum_\nu q_\nu(x')R_\nu(w) = 0$; by corollary 1 of Theorem 2, this proves our conclusion, so that the proof of our lemma, and with it that of Theorem 12, are now complete.

28. As an example, we shall apply our results to systems of principal local ideals. Assume again that V is a complete abstract variety defined over a field k; let X be an element of $\mathbf{F}(K/k)$; and assume that X can be defined by a coherent system of principal local ideals. For each $y \neq 0$ in K, let W_y be the set of points P of V for which $X_P = [y]_P$, i.e. such that $X(\omega) = \omega(y)$ for all $\omega \in \mathbf{V}(K/A_P)$; the W_y are open subsets of V, and our assumption means that they form a covering of V. Therefore there is a finite covering of V by open subsets W_λ, and, for each λ, a $y_\lambda \in K^*$, such that, for all $P \in W_\lambda$, we have $X(\omega) = \omega(y_\lambda)$ for $\omega \in \mathbf{V}(K/A_P)$. Then we must have $\omega(y_\lambda y_\mu^{-1}) = 0$ for $\omega \in \mathbf{V}(K/A_P)$ and $P \in W_\lambda \cap W_\mu$; this means that neither 0 nor ∞ can be a specialization of $y_\lambda y_\mu^{-1}$ at any point of $W_\lambda \cap W_\mu$. Conversely, if a finite covering of V by open sets W_λ and a set of functions y_λ on V have this last property, then we can choose, for each P on V, a λ such that $P \in W_\lambda$; for that λ, put $I_P = y_\lambda A_P$; these will form a coherent system of principal local ideals. When such sets W_λ and such functions y_λ are given, we shall say, briefly, that they form a *(W, y)-system.*

Now, r being the dimension of V, assume that V has no multiple subvarieties of dimension $r - 1$. Then, by Lemma 1 of No. 7, a (W, y)-system can be defined as consisting of a finite covering of V by open sets W_λ, and of functions $y_\lambda \neq 0$ on V, such that, for all λ, μ, all the components of $(y_\lambda/y_\mu) = (y_\lambda) - (y_\mu)$ are contained in the closed set $C(W_\lambda \cap W_\mu) = C(W_\lambda) \cup C(W_\mu)$, i.e. either in $C(W_\lambda)$ or in $C(W_\mu)$. Such a system being given, there is one and only one divisor T such that, for each λ, all the components of $T - (y_\lambda)$ are contained in $C(W_\lambda)$. In fact, we can construct such a divisor T by taking as its components all the $(r - 1)$-dimensional subvarieties A of V such that there is a λ for which A is a component of (y_λ) and is not contained in $C(W_\lambda)$, and by taking as the coefficient of such a variety A in T its coefficient in (y_λ); this is unambiguous, for, if at the same time A is a component of (y_μ) and is not contained in $C(W_\mu)$, then it cannot be a component of $(y_\lambda) - (y_\mu)$, and so it has the same coefficient in (y_μ) as in (y_λ). Since to every point P on V there is a λ such that P is not in $C(W_\lambda)$, the divisor T is uniquely determined, and is everywhere locally equivalent to 0, in the sense defined in No. 8; it is also clear that it is rational over k.

Conversely, let T be a divisor, rational over k, and everywhere locally equivalent to 0; then to every point P of V there is a function y_P such that P is contained in the complement W_P of the union of the components of the divisor $T - (y_P)$; so the open sets W_P form a covering of V, from which one can extract a finite covering W_{P_λ}; then the W_{P_λ} and y_{P_λ} form a (W, y)-system which determines T as above. As this system also determines a coherent system of local ideals, hence an element X of $\mathbf{F}(K/k)$, we can summarize part of our results as follows:

THEOREM 13. *Let V be a complete abstract variety of dimension r defined over k without multiple subvarieties of dimension $r - 1$; let K be the field of functions defined over k on V. Let T be a divisor on V, rational over k, everywhere locally equivalent to* 0. *Then there is an element X_T of $\mathbf{F}(K/k)$ with the following property: if P is any point on V, and if $y \in K$ is such that $T - (y)$ has no component going through P, then $X(\omega) = \omega(y)$ for every $\omega \in \mathbf{V}(K/A_P)$. The mapping $T \rightarrow X_T$ is an isomorphism of the ordered group of divisors T onto the subgroup of the ordered group $\mathbf{F}(K/k)$, consisting of all the elements definable by principal local ideals.*

It may be seen at once that, if there exists an element of $\mathbf{F}(K/k)$ "attached" to the divisor T in the sense defined in No. 8, this must be the same as the divisor X_T defined in Theorem 13. The converse of this is also true under fairly general assumptions, and in particular if V is a non-singular projective variety; for, in that case, the element X_T, defined in Theorem 13, can be written as $X_T = \inf_\mu \sup_\nu [x_\mu/u_\nu]$, where the x_μ, u_ν are the functions described in Theorem 4 of No. 8. This is the true justification for the concepts introduced in No. 8.

Finally, if the variety V is normal, a (W, y)-system can also be defined as one consisting of a finite covering of V by open sets W_λ, and of functions $y_\lambda \neq 0$ on V such that, for all λ, μ, the function $y_{\lambda\mu} = y_\lambda/y_\mu$ is everywhere defined, finite and $\neq 0$ in $W_\lambda \cap W_\mu$; these are exactly the conditions under which a covering W_λ of V and a system of functions $y_{\lambda\mu}$ can be used to define a "fibre-space" over V, with the multiplicative group in one variable. In fact, the fibre-space which is so defined is precisely the one whose invariant is the class (for linear equivalence) containing the divisor T defined by the given (W, y)-system.

APPENDIX

Divisorial valuations

As in §V we consider a variety V of dimension r defined over a field k, and the field K of functions defined over k on V. A valuation in $\mathbf{V}(K/k)$ will be called *divisorial* if its residue-field has the degree of transcendency $r - 1$ over k;

we shall write $\mathbf{V}_d(K/k)$ for the set of all such valuations, and $\mathbf{F}_d(K/k)$ for $\mathbf{F}[\mathbf{V}_d(K/k)]$, i.e. for the ordered group of the restrictions to $\mathbf{V}_d(K/k)$ of the valuation-functions in $\mathbf{F}(K/k)$. It is well-known that the value-group $\omega(K^*)$ of every divisorial valuation ω is isomorphic to the additive group of integers, with which it may be canonically identified. We shall prove here that $\mathbf{F}_d(K/k)$ is isomorphic to $\mathbf{F}(K/k)$, or, more precisely, that the canonical homomorphism of $\mathbf{F}(K/k)$ onto $\mathbf{F}_d(K/k)$ is an isomorphism; from this it will follow that all our results hold true if $\mathbf{F}(K/k)$ is replaced everywhere by $\mathbf{F}_d(K/k)$.

In view of our results in No. 3, what we have to prove is that, if $x_1, \cdots, x_n$ is any set of elements of K, $\inf_i[x_i] \prec 0$ in $\mathbf{F}_d(K/k)$ implies the same in $\mathbf{F}(K/k)$, or, in other words, that, if $0 = (0, \cdots, 0)$ is a specialization of $x = (x_1, \cdots, x_n)$ over k, there is a valuation in $\mathbf{V}_d(K/k)$ which is >0 at the x_i; this is well-known, and may be proved as follows. By Theorem 1, it will be enough to show that, for every $s < r$, there are s independent variables $t_1, \cdots, t_s$ over k in K such that 0 is still a specialization of the set x over $k(t_1, \cdots, t_s)$; using induction on s, one sees that one need only do this for $s = 1 < r$. If there exists a transcendental element t_1 in K over $K' = k(x)$, this will be as required; so we may suppose that K' has the dimension $r \geqq 2$ over k, and proceed to select t_1 out of K'. Let U be the locus of x over k in the affine space S^n; by F-App. II, prop. 5 and 6, there are quantities $y = (y_1, \cdots, y_m)$ in K', finite at every point of U, and such that the locus U' of (x, y) over k in S^{n+m} is relatively normal with respect to k at all its points. Let y' be one of the specializations of y over $x \to 0$ with respect to k; these are all finite, hence algebraic over k; so the point $P = (0, y')$ is on U' and is algebraic over k. Every element of $K' = k(x) = k(x, y)$ which is finite at P is defined at P. More generally, every element u of K' which has only a finite number of specializations at P is defined at P; for, if k is infinite, this assumption on u implies that there is a $c \in k$ such that $1/u - c$ is finite at P, hence defined there; and if k is finite it is perfect, so that U' is (absolutely) normal at P, and we may reason in the same manner, taking for c a constant which is transcendental over k. Hence all we need do is to find an element t_1 of K' which is not defined at P. To do this, take a hyperplane $L(X, Y) = 0$ in S^{n+m}, going through P, algebraic over k, and not containing U'. As each component of its intersection with U' has the dimension $r - 1 \geqq 1$, the hyperplanes which contain such a component will make up a linear variety of dimension $\leqq n + m - 2$ in the $(n + m - 1)$-dimensional projective space of hyperplanes through P. Hence there is a hyperplane $L'(X, Y) = 0$, going through P, algebraic over k, such that neither this nor any of its conjugates over k contains any component of the intersection of U' with $L = 0$. Put

$$F(X, Y) = \prod_\sigma L^\sigma(X, Y), \quad F'(X, Y) = \prod_\sigma L'^\sigma(X, Y),$$

where σ runs over all automorphisms of some finite normal extension of k containing the coefficients in L and L'; and put $t_1 = F(x, y)/F'(x, y)$. As $(t_1)_0$ and $(t_1)_\infty$ both have components going through P, both 0 and ∞ are specializations of t_1 at P, so that t_1 is not defined at P, and solves our problem.

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