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Part II. Topology and Abstract Algebra as Two Roads of Mathematical Comprehension*

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Hermann Weyl

Note: The first part of this article appeared in 1995, in the May issue of the *Monthly* (pp. 453). What follows is a short summary of the first part and the concluding part of the article.

SUMMARY OF PART I. Weyl begins by saying that

> We are not very pleased when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and of the road; we want to understand the *idea* of the proof, the deeper context.

and goes beyond this familiar notion of understanding to what he calls modes of understanding. These are ways of looking at mathematics as well as the branches of mathematics associated with them. Two such modes of understanding "have proved, in our time, to be especially penetrating and fruitful. The two are topology and abstract algebra."

Now the discussion begins to involve some technical matters. Weyl explains what is meant by purely topological investigations of continua, discusses the motives that have led to the development of abstract algebra, and uses "a simple example to show how the same issue can be looked at from a topological and from an abstract-algebraic viewpoint. The (not so simple) example which he considers from the two viewpoints is the theory of algebraic functions of a single variable. ■

After all these general remarks I want to use two simple examples that illustrate the different kinds of concept building in algebra and in topology. The classical example of the fruitfulness of the topological method is Riemann's theory of

*The original German version of this article is found in vol. 3, pp. 348–358, of the four-volume edition of Hermann Weyl's collected works published by Springer-Verlag in 1968. The translation is by Abe Shenitzer.

algebraic functions and their integrals. Viewed as a topological surface, a Riemann surface has just one characteristic, namely its connectivity number or genus p. For the sphere $p = 0$ and for the torus $p = 1$. How sensible it is to place topology ahead of function theory follows from the decisive role of the topological number p in function theory on a Riemann surface. I quote a few dazzling theorems: The number of linearly independent everywhere regular differentials on the surface is p. The total order (that is, the difference between the number of zeros and the number of poles) of a differential on the surface is $2p - 2$. If we prescribe more than p arbitrary points on the surface, then there exists just one single-valued function on it that may have simple poles at these points but is otherwise regular; if the number of prescribed poles is exactly p, then, if the points are in general position, this is no longer true. The precise answer to this question is given by the Riemann-Roch theorem in which the Riemann surface enters only through the number p. If we consider all functions on the surface that are everywhere regular except for a single place $\mathfrak{p}$ at which they have a pole, then its possible orders are all numbers $1, 2, 3, \ldots$ except for certain powers of p (the Weierstrass gap theorem). It is easy to give many more such examples. The genus p permeates the whole theory of functions on a Riemann surface. We encounter it at every step, and its role is direct, without complicated computations, understandable from its topological meaning (provided that we include, once and for all, the Thomson-Dirichlet principle as a fundamental function-theoretic principle).

The Cauchy integral theorem gives topology the first opportunity to enter function theory. The integral of an analytic function over a closed path is 0 only if the domain that contains the path and is also the domain of definition of the analytic function is simply connected. Let me use this example to show how one "topologizes" a function-theoretic state of affairs. If $f(z)$ is analytic, then the integral $\int_\gamma f(z)\,dz$ associates with every curve a number $F(\gamma)$ such that

$$(\dagger) \qquad F(\gamma_1 + \gamma_2) = F(\gamma_1) + F(\gamma_2).$$

$\gamma_1 + \gamma_2$ stands for the curve such that the beginning of γ_2 coincides with the end of γ_1. The functional equation (†) marks the integral $F(\gamma)$ as an additive path function. Also, each point has a neighborhood such that $F(\gamma) = 0$ for each closed path γ in that neighborhood. I will call a path function with these properties a topological integral, or briefly, an integral. In fact, all this concept assumes is that there is given a continuous manifold on which one can draw curves; it is the topological essence of the analytic notion of an integral. Integrals can be added and multiplied by numbers. The topological part of the Cauchy integral theorem states that on a simply connected manifold every integral is homologous to 0 (not only in the small but in the large), that is, $F(\gamma) = 0$ for every closed curve γ on the manifold. In this we can spot the definition of "simply connected." The function-theoretic part states that the integral of an analytic function is a topological integral in our sense of the term. The definition of the order of connectivity [that we are about to state] fits in here quite readily. Integrals $F_1, F_2, \ldots, F_n$ on a closed surface are said to be linearly independent if they are not connected by a homology relation

$$c_1 F_1 + c_2 F_2 + \cdots + c_n F_n \sim 0$$

with constant coefficients c_i other than the trivial one, when all the c_i vanish. The order of connectivity of a surface is the maximal number of linearly independent integrals. For a closed two-sided surface the order of connectivity h is always an even number $2p$, where p is the genus. From a homology between integrals we can go over to a homology between closed paths. The path homology

$$n_1\gamma_1 + n_2\gamma_2 + \cdots + n_r\gamma_r \sim 0$$

states that for every integral F we have the equality

$$n_1F(\gamma_1) + n_2F(\gamma_2) + \cdots + n_rF(\gamma_r) = 0.$$

If we go back to the topological skeleton that decomposes the surface into elementary pieces and replace the continuous point-chains of paths by the discrete chains constructed out of elementary pieces, then we obtain an expression for the order of connectivity h in terms of the numbers s, k and e of pieces, edges and vertices. The expression in question is the well-known Euler polyhedral formula $h = k - (e + s) + 2$. Conversely, if we start with the topological skeleton, then our reasoning yields the result that this combination h of the number of pieces, edges and vertices is a topological invariant, namely it has the same value for "equivalent" skeletons which represent the same manifold in different subdivisions.

When it comes to application to function theory, it is possible, using the Thompson-Dirichlet principle, to "realize" the topological integrals as actual integrals of everywhere regular-analytic differentials on a Riemann surface. One can say that all of the constructive work is done on the topological side, and that the topological results are realized in a function-theoretic manner with the help of a universal transfer principle, namely the Dirichlet principle. This is, in a sense, analogous to analytic geometry, where all the constructive work is carried out in the realm of numbers, and then the results are geometrically "realized" with the help of the transfer principle lodged in the coordinate concept.

All this is seen more perfectly in uniformization theory, which plays a central role in all of function theory. But at this point, I prefer to point to another application which is probably close to many of you. I have in mind enumerative geometry, which deals with the determination of the number of points of intersection, singularities, and so on, of algebraic relational structures, which was made into a general, but very poorly justified, system by Schubert and Zeuthen. Here, in the hands of Lefschetz and v.d. Waerden, topology achieved a decisive success in that it led to definitions of multiplicity valid without exception, as well as to laws likewise valid without exception. Of two curves on a two-sided surface one can cross the other at a point of intersection from left to right or from right to left. These points of intersection must enter every setup with opposite weights $+1$ and -1. Then the total of the weights of the intersections (which can be positive or negative) is invariant under arbitrary continuous deformations of the curves; in fact, it remains unchanged if the curves are replaced by homologous curves. Hence it is possible to master this number through finite combinatorial means of topology and obtain transparent general formulas. Two algebraic curves are, actually, two closed Riemann surfaces embedded in a space of four real dimensions by means of an analytic mapping. But in algebraic geometry a point of intersection is counted with positive multiplicity, whereas in topology one takes into consideration the sense of the crossing. This being so, it is surprising that one can resolve the algebraic question by topological means. The explanation is that in the case of an analytic manifold, crossing always takes place with the same sense. If the two curves are represented in the x_1, x_2-plane in the vicinity of their point of intersection by the functions $x_1 = x_1(s), x_2 = x_2(s)$, and $x_1 = x_1^*(t), x_2 = x_2^*(t)$, then the sense ± 1 with which the first curve intersects the second is given by the sign of the

Jacobian

$$\begin{vmatrix} \dfrac{dx_1}{ds} & \dfrac{dx_2}{ds} \\ \dfrac{dx_1^*}{dt} & \dfrac{dx_2^*}{dt} \end{vmatrix} = \frac{\partial(x_1, x_2)}{\partial(x, t)},$$

evaluated at the point of intersection. In the case of complex-algebraic "curves" this criterion always yields the value $+1$. Indeed, let z_1, z_2 be complex coordinates in the plane and let s and t be the respective complex parameters on the two "curves." The real and imaginary parts of z_1 and z_2 play the role of real coordinates in the plane. In their place we can take $z_1, \bar{z}_1, z_2, \bar{z}_2$. But then the determinant whose sign determines the sense of the crossing is

$$\frac{\partial(z_1, \bar{z}_1, z_2, \bar{z}_2)}{\partial(s, \bar{s}, t, \bar{t})} = \frac{\partial(z_1, z_2)}{\partial(s, t)} \cdot \frac{\partial(\bar{z}_1, \bar{z}_2)}{\partial(\bar{s}, \bar{t})} = \left| \frac{\partial(z_1, z_2)}{\partial(s, t)} \right|^2,$$

and thus invariably positive. Note that the Hurwitz theory of correspondence between algebraic curves can likewise be reduced to a purely topological core.

On the side of abstract algebra, I will emphasize just one fundamental concept, namely the concept of an ideal. If we use the algebraic method, then an algebraic manifold is given in 3-dimensional space with complex cartesian coordinates x, y, z by means of a number of simultaneous equations

$$f_1(x, y, z) = 0, \ldots, f_n(x, y, z) = 0.$$

The f_i are polynomials. In the case of a curve it is not at all true that two equations suffice. Not only do the polynomials f_i vanish at points of the manifold but also every polynomial f of the form

$$(**) \qquad f = A_1 f_1 + \cdots + A_n f_n \ (A_i \text{ are polynomials}).$$

Such polynomials f form an "ideal" in the ring of polynomials. Dedekind defined an ideal in a given ring as a system of ring elements closed under addition and subtraction as well as under multiplication by ring elements. This concept is not too broad for our purposes. The reason is that, according to the Hilbert basis theorem, every ideal in the polynomial ring has a finite basis; there are finitely many polynomials $f_1, \ldots, f_n$ in the ideal such that every polynomial in the ideal can be written in the form $(**)$. Hence the study of algebraic manifolds reduces to the study of ideals. On an algebraic surface there are points and algebraic curves. The latter are represented by ideals that are divisors of the ideal under consideration. The fundamental theorem of M. Noether deals with ideals whose manifold of zeros consists of finitely many points, and makes membership of a polynomial in such an ideal dependent on its behavior at these points. This theorem follows readily from the decomposition of an ideal into prime ideals. The investigations of E. Noether show that the concept of an ideal, first introduced by Dedekind in the theory of algebraic number fields, runs through all of algebra and arithmetic like Ariadne's thread. v.d. Waerden was able to justify the enumerative calculus by means of the algebraic resources of ideal theory.

If one operates in an arbitrary abstract number field rather than in the continuum of complex numbers, then the fundamental theorem of algebra, which asserts that every complex polynomial in one variable can be [uniquely] decomposed into linear factors, need not hold. Hence the general prescription in algebraic work: See if a proof makes use of the fundamental theorem or not. In every algebraic theory there is a more elementary part that is independent of the fundamental theorem, and therefore valid in every field, and a more advanced part for which the fundamental theorem is indispensable. The latter part calls for the algebraic closure of the field. In most cases the fundamental theorem marks a crucial split; its use should be avoided as long as possible. To establish theorems that hold in an arbitrary field it is often useful to embed the given field in a larger field. In particular, it is possible to embed any field in an algebraically closed field. A well-known example is the proof of the fact that a real polynomial can be decomposed over the reals into linear and quadratic factors. To prove this, we adjoin i to the reals and thus embed the latter in the algebraically closed field of complex numbers. This procedure has an analogue in topology which is used in the study and characterization of manifolds; in the case of a surface, this analogue consists in the use of its covering surfaces.

At the center of today's interest is noncommutative algebra in which one does not insist on the commutativity of multiplication. Its rise is dictated by concrete needs of mathematics. Composition of operations is a kind of noncommutative operation. Here is a specific example. We consider the symmetry properties of functions $f(x_1, x_2, \ldots, x_n)$ of a number of arguments. The latter can be subjected to an arbitrary permutation s. A symmetry property is expressed in one or more equations of the form

$$\sum_s a(s) \cdot sf = 0.$$

Here $a(s)$ stands for the numerical coefficients associated with the permutation. These coefficients belong to a given field K. $\Sigma_s a(s) \cdot s$ is a "symmetry operator." These operators can be multiplied by numbers, added and multiplied, that is, applied in succession. The result of the latter operation depends on the order of the "factors." Since all formal rules of computation hold for addition and multiplication of symmetry operators, they form a "noncommutative ring" (hypercomplex number system). The dominant role of the concept of an ideal persists in the noncommutative realm. In recent years, the study of groups and their representations by linear substitutions has been almost completely absorbed by the theory of noncommutative rings. Our example shows how the multiplicative group of $n!$ permutations s is extended to the associated ring of magnitudes $\Sigma_s a(s) \cdot s$ that admit, in addition to multiplication, addition and multiplication by numbers. Quantum physics has given noncommutative algebra a powerful boost.

Unfortunately, I cannot here produce an example of the art of building an abstract-algebraic theory. It consists in setting up the right general concepts, such as fields, ideals, and so on, in decomposing an assertion to be proved into steps (for example, and assertion "A implies B," or $A \rightarrow B$, may be decomposed into steps $A \rightarrow C, C \rightarrow D, D \rightarrow B$), and in the appropriate generalization of these partial assertions in terms of general concepts. Once the main assertion has been subdivided in this way and the inessential elements have been set aside, the proofs of the individual steps do not, as a rule, present serious difficulties.

Whenever applicable, the topological method appears, thus far, to be more effective than the algebraic one. Abstract algebra has not yet produced successes comparable to the successes of the topological method in the hands of Riemann. Nor has anyone reached by an algebraic route the peak of uniformization scaled topologically by Klein, Poincaré and Koebe. Here are questions to be answered in the future. But I do not want to conceal from you the growing feeling among mathematicians that the fruitfulness of the abstracting method is close to exhaustion. It is a fact that beautiful general concepts do not drop out of the sky. The truth is that, to begin with, there are definite concrete problems, with all their undivided complexity, and these must be conquered by individuals relying on brute force. Only then come the axiomatizers and conclude that instead of straining to break in the door and bloodying one's hands one should have first constructed a magic key of such and such shape and then the door would have opened quietly, as if by itself. But they can construct the key only because the successful breakthrough enables them to study the lock front and back, from the outside and from the inside. Before we can generalize, formalize and axiomatize there must be mathematical substance. I think that the mathematical substance on which we have practiced formalization in the last few decades is near exhaustion and I predict that the next generation will face in mathematics a tough time.

[The sole purpose of this lecture was to give the audience a feeling for the intellectual atmosphere in which a substantial part of modern mathematical research is carried out. For those who wish to penetrate more deeply I give a few bibliographical suggestions. The true pioneers of abstract axiomatic algebra are Dedekind and Kronecker. In our own time, this orientation has been decisively advanced by Steinitz, by E. Noether and her school, and by E. Artin. The first great advance in topology came in the middle of the 19th century and was due to Riemann's function theory. The more recent developments are linked primarily to a few works of H. Poincaré devoted to analysis situs (1895–1904). I mention the following books:

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Gauss oncc said, "Mathematics is the queen of the sciences and number theory thc queen of mathematics." If this is true we may add that the disquisitions is the Magna Charter of number theory.

—*M. Cantor*
Allgemeine Deutsche Biographie, Bd. 8. 1878 p. 435.