数学分析习题课讲义

—分析十段天元手筋

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http://homepage.fudan.edu.cn/~yizhang/teaching/analysis-exercise.pdf

◊

Dedicate to Katie

创造的神秘,
有如夜间的黑暗,
是伟大的。

而知识的幻影,
不过如晨间之雾。

Contents

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♢

曾子曰：
大学之道，在明明德，
在亲民，在止于至善。
知止而后定，定而后能静，
静而后能安，安而后能虑，
虑而后能得。
物有本末，事有始终，
知所先后，则近道矣。

1. Problems on Sets,Sequences and Limits

1.1. Elementary Technique.

Exercise 1.1. Show that the set of all irrational number of $\mathbb{R}$ is uncountable.

Proof. Sufficient to show that the set of all irrational number in $[0,1]$ is uncountable. Otherwise, the set of real numbers in $[0,1]$ is countable, ie.,

$$[0,1] = \{x_1, x_2, \cdots\}.$$

Cover each $x_i, i = 1, \cdots$ with the corresponding interval

$$I_i := [x_i - \frac{1}{2}(\frac{1}{3})^i, x_i + \frac{1}{2}(\frac{1}{3})^i],$$

Then,there holds

$$[0,1] = \{x_1, x_2, \cdots\} \subset \bigcup_{i=1}^{\infty} I_i,$$

and so

$$1 = l([0,1]) \leq \sum_{i=1}^{\infty} l(I_i) = \sum_{i=1}^{\infty} l([x_i - \frac{1}{2}(\frac{1}{3})^i, x_i + \frac{1}{2}(\frac{1}{3})^i]) = \sum_{i=1}^{\infty} (\frac{1}{3})^i = \frac{1}{2}.$$

It is a contradiction.

$\square$

Example 1.2. Let $\alpha, \beta \in \mathbb{R}_+ \setminus \mathbb{Q}$ and $1/\alpha + 1/\beta = 1$. Let

$$A = \{[n\alpha] : n \in \mathbb{N}\}, \ \ B = \{[n\beta] : n \in \mathbb{N}\}$$

be two strictly increasing sequences of positive integers.

Show that

$$A \bigcup B = \mathbb{N}; \ \ A \bigcap B = \emptyset.$$

Proof. We have $\alpha > 1, \beta > 1$. W.L.O.G., we assume $\alpha < 2$. Then $1 \in A \cup B$.

1. If $a \cap B \neq \emptyset$, then there exist two integers m, n such that

$$[m\alpha] = [n\beta] = q \in \mathbb{N}.$$

Hence there holds that $q < m + n < q + 1$. A contradiction.

2. If there is a positive integer $p \in A \cup B$, then there exist two integers m, n such that

$$[m\alpha] < p < [(m+1)\alpha], \ \ \text{and} \ \ [n\beta] < p < [(n+1)\beta].$$

Therefore

$$m\alpha < p < p + 1 \leq [(m+1)\alpha] < (m+1)\alpha, \ \ \text{and} \ \ n\beta < p < +1 \leq [(n+1)\beta] < (n+1)\beta.$$

Since $1/\alpha + 1/\beta = 1$, we have

$$m + n < p < p + 1 < m + n + 2.$$

A contradiction.

$\square$

Example 1.3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a bounded function. Assume for any integers m, n there is a relation:

$$f(m+n) + f(m-n) = 2f(m)f(n).$$

To show all such function f.

Hint. a) $f(0) \in \{0,1\}$, and $f(0)=0 \Rightarrow f \equiv 0$.

b) Now, we assume $f(0)=1$. Then, $f(-n)=f(n)\forall n \in \mathbb{Z}$, and $f(2m)=2f(m)^2-1, \forall m \in \mathbb{Z}$. If there is an integer k such that $|f(k)|>1$, then

$$|f(2k)|=|2f(k)^2-1|=(f(k)^2-1)+f(k)^2>|f(k)|>1,$$

and so we have an increasing sequence of integer

$$|f(k)|<|f(2k)|<|f(4k)|<\cdots<|f(2^l k)|<\cdots.$$

But it contradict that f is a bounded function with all values in $\mathbb{Z}$. Therefore

$$f(\mathbb{Z}) \subset \{-1,0,1\}.$$

c) Let $f(1)=\cos\theta$ with $\theta \in \{\pi, \frac{\pi}{2}, 0\}$. We can show that

$$f(n)=\cos(n\theta), \forall n \in \mathbb{Z}$$

by induction and the formula

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta).$$

$\square$

Example 1.4. Define the function $G : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ by

$$\begin{cases} G(0)=0 & ; \\ G(n)=n-G(G(n-1)), & n \in \mathbb{N}. \end{cases}$$

To show that

$$G(n)=[\frac{\sqrt{5}-1}{2}(n+1)].$$

Proof. 1. Actually, we have

$$1 \le G(n) \le n \text{ and } G(n-1) \le G(n)\, \forall n \ge 1.$$

At first $G(1)=1, G(2)=1$.By induction, we assume

$$1 \le G(k) \le k \text{ and } G(k-1) \le G(k)\, \forall 1 \le k \le n-1.$$

Then, we have

$$1 \le G(n-1) \le n-1, \text{ and so } G(G(n-1)) \le G(n-1) \le n-1,$$

$$1=n-(n-1) \le G(n)=n-G(G(n)) \le n-1<n,$$

$$G(n)-G(n-1)=1-[G(G(n-1))-G(G(n-2))] \ge 0 \text{ since } 1 \le G(n-2) \le G(n-1) \le n-1.$$

2. By induction, we can show

$$G(n+1)-G(n)=1 \text{ or } 0\, \forall n.$$

3. Define $F(n)=[\alpha(n+1)]$ where $\alpha=\frac{\sqrt{5}-1}{2}$. Define

$$S(n)=F(n)+F(F(n-1)).$$

We are going to show that $S(n)=n$, and so $F(n)=G(n)$.

Let $K=[n\alpha]$, and so $n\alpha=K+\theta$ with $0<\theta<1$. Then

$$F(n)=[(n+1)\alpha]=[K+\alpha+\theta]=K+[\alpha+\theta],\ F(F(n-1))=F([n\alpha])=[(K+1)\alpha].$$

Since $\alpha^2=1-\alpha$, we have

$$(K+1)\alpha=\alpha+(n\alpha-\theta)\alpha=\alpha(1-\theta)+n\alpha^2=\alpha(1-\theta)+n-K-\theta,$$

$$S(n)=F(n)+F(F(n-1))=n+[\theta+\alpha]+[\alpha(1-\theta)-\theta].$$

Let $T = \alpha(1-\theta) - \theta$. Then

$$-1 < -\theta < T < \alpha < 1, \text{ and so } [T] = \begin{cases} 0, & T > 0; \\ -1, & T < 0. \end{cases}$$

On the other hand, we have

$$T = \alpha - \theta(1-\theta) = (1+\alpha)(\frac{\alpha}{1+\alpha} - \theta) = (1+\alpha)(\alpha^2 - \theta) = (1+\alpha)(1 - (\alpha+\theta)),$$

and so

$$[T] = \begin{cases} 0, & [\alpha+\theta] = 0; \\ -1, & [\alpha+\theta] = 1. \end{cases}$$

Therefore,

$$[\alpha+\theta] + [T] = 0, \text{i.e.}, S(n) = n.$$

□

Example 1.5. Let $\{a_n\}$ is strictly increasing sequence of positive integers, i.e.,$a_{n+1} > a_n$, and $\forall a_n \in \mathbb{N}$. Define

$$b_n = [a_1, a_2, \cdots, a_n] \text{ the least common multiple of } a_1, \cdots a_n.$$

To show that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} < \infty.$$

Proof. Let

$$d(n) = \#\{p : p \in \mathbb{N}\ p|n\}.$$

Then $d(n) < 2\sqrt{n}$, and so we have

$$n \le d(b_n) < 2\sqrt{b_n}.$$

By the previous exercise, we obtain the result.

□

Example 1.6. To show

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\sqrt[3]{1 + \frac{k}{n^2}} - 1) = \frac{1}{6}.$$

Proof. Let

$$\alpha_k = \sqrt[3]{1 + \frac{k}{n^2}} - 1.$$

Then $0 < \alpha_k < 1$ and $\alpha_k < \frac{k}{3n^2}$ by

$$1 + \frac{k}{n^2} = (1+\alpha_k)^3 > 1 + 3\alpha_k.$$

ON the other hand,

$$1 + \frac{k}{n^2} = (1+\alpha_k)^3 = 1 + 3\alpha_k^2 + 3\alpha_k + \alpha_k^3 \le 1 + 4\alpha_k^2 + 3\alpha_k \le 1 + 4(\frac{k}{3n^2})^2 + 3\alpha_k,$$

and so

$$\sum_{k=1}^{n} \frac{k}{3n^2} \ge \sum_{k=1}^{n} \alpha_k \ge \sum_{k=1}^{n} \frac{-4k^2}{27n^4} + \frac{k}{3n^2}.$$

Since

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

we have

$$\lim_{n\to\infty}(\frac{n(n+1)}{6n^2} - \frac{2n(n+1)(2n+1)}{3\times 27n^4}) \geq \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_k \geq \lim_{n\to\infty}\frac{n(n+1)}{6n^2}.$$

□

Remark. Let

$$g(x) = \sum_{i=1}^{n} i(1+x)^i.$$

Then,

$$g(x) - (1+x)g(x) = \sum_{i=1}^{n}(1+x)^i - n(1+x)^{n+1},$$

and so

$$g(x) = \frac{(1+x)-(1+x)^{n+1}}{x^2} + \frac{n(1+x)^{n+1}}{3x}.$$

Comparing the coefficient of the term x of the polynomial $g(x)$, we have:

$$\sum_{i=1}^{n} i^2 = -C_{n+1}^3 + nC_{n+1}^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example 1.7. Let $\{a_n\}, \{b_n\}$ be two sequence satisfying

(1)

$$(a_n+b_n)b_n \neq 0,\ n = 1, 2, \cdots,$$

(2)

$$|\sum_{n=1}^{\infty}\frac{a_n}{b_n}| = A < \infty,\ \sum_{n=1}^{\infty}(\frac{a_n}{b_n})^2 = B < \infty.$$

Then the sum

$$\sum_{n=1}^{\infty}\frac{a_n}{a_n+b_n} < \infty$$

is convergent.

Proof. By the condition (2), for any $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \frac{1}{2}$, there exist an integer $N > 0$, for any $n, l \geq N$, we have

$$|\frac{a_n}{b_n}| \leq \varepsilon,\ |\sum_{i=l}^{\infty}\frac{a_i}{b_i}| < \varepsilon,\ \text{and } \sum_{i=l}^{\infty}(\frac{a_i}{b_i})^2 < \varepsilon.$$

and so

$$\frac{a_n}{a_n+b_n} = \frac{\frac{a_n}{b_n}}{1+\frac{a_n}{b_n}} = \frac{\frac{a_n}{b_n}(1-\frac{a_n}{b_n})}{1-(\frac{a_n}{b_n})^2} = (\frac{a_n}{b_n} - (\frac{a_n}{b_n})^2)(1+\sum_{k=1}^{\infty}(\frac{a_n}{b_n})^{2k}),\ n \geq N \tag{1.7.1}$$

$$\delta_n = \sum_{k=1}^{\infty}(\frac{a_n}{b_n})^{2k} = \frac{(\frac{a_n}{b_n})^2}{1-(\frac{a_n}{b_n})^2} \leq 2(\frac{a_n}{b_n})^2 \leq 2\varepsilon^2\ (n \geq N), \tag{1.7.2}$$

$$(1.7.3)\qquad |\sum_{i=l}^{\infty}(\frac{a_i}{b_i})^3| \leq \sum_{i=l}^{\infty}|\frac{a_i}{b_i}|^3 \leq \sqrt{\sum_{i=l}^{\infty}(\frac{a_i}{b_i})^2}\sqrt{\sum_{i=l}^{\infty}(\frac{a_i}{b_i})^4} \leq \sum_{i=l}^{\infty}(\frac{a_i}{b_i})^2 \leq \varepsilon\ (l \geq N).$$

On the other hand, for $l \geq N$we have

$$|\sum_{n=l}^{\infty}\frac{a_n}{a_n+b_n}| = |\sum_{n=l}^{\infty}\{(\frac{a_n}{b_n}-(\frac{a_n}{b_n})^2)(1+\sum_{k=1}^{\infty}(\frac{a_n}{b_n})^{2k})\}| \leq 2\varepsilon + |\sum_{n=l}^{\infty}\{(\frac{a_n}{b_n}-(\frac{a_n}{b_n})^2)\sum_{k=1}^{\infty}(\frac{a_n}{b_n})^{2k}\}|.$$

But by 1.7.2 and 1.7.3 we have the following

$$|\sum_{n=N}^{\infty}\{(\frac{a_n}{b_n}-(\frac{a_n}{b_n})^2)\sum_{k=1}^{\infty}(\frac{a_n}{b_n})^{2k}\}| \leq |\sum_{n=l}^{\infty}\frac{a_n}{b_n}\delta_n| + |\sum_{n=l}^{\infty}(\frac{a_n}{b_n})^2\delta_n| \leq \sum_{n=l}^{\infty}\frac{|\frac{a_n}{b_n}|^3}{1-(\frac{a_n}{b_n})^2} + 2\varepsilon^3.$$

$$\sum_{n=l}^{\infty}\frac{|\frac{a_n}{b_n}|^3}{1-(\frac{a_n}{b_n})^2} \leq 2\sum_{n=l}^{\infty}|\frac{a_n}{b_n}|^3 \leq 2\varepsilon.$$

At all, we obtain

$$|\sum_{n=l}^{\infty}\frac{a_n}{a_n+b_n}| < 6\varepsilon,\ \ l \geq N,$$

and so by Cauchy principal, it is convergent.

□

Another clever proof by a student. It is sufficient to consider the convergence of

$$\sum_{n=1}^{\infty}(\frac{a_n}{a_n+b_n}-\frac{a_n}{b_n}).$$

We have

$$\sum_{n=1}^{\infty}(\frac{a_n}{b_n}-\frac{a_n}{a_n+b_n}) = \sum_{n=1}^{\infty}\frac{a_n^2}{(a_n+b_n)b_n} = \sum_{n=1}^{\infty}(\frac{a_n}{b_n})^2\frac{b_n}{a_n+b_n}.$$

Since

$$|\frac{a_n}{b_n}| \leq \frac{1}{2},\quad (n >> 0),$$

and $b_n \neq 0$, $\forall n \in \mathbb{N}$, we have

$$0 < \frac{b_n}{a_n+b_n} = \frac{1}{1+\frac{a_n}{b_n}} \leq 2,\quad (n >> 0),$$

and so $\frac{b_n}{a_n+b_n}$ has a positive upper bounded uniformly for $\forall n \in \mathbb{N}$. Therefore the sum

$$\sum_{n=1}^{\infty}(\frac{a_n}{a_n+b_n}-\frac{a_n}{b_n})$$

is convergent.

□

Example 1.8. Let $\{a_n\}$ be a sequence satisfying

$$a_{n+1}(2-a_n) = 1\ \forall n \in \mathbb{N}.$$

To shown that

$$\lim_{n\to\infty} a_n$$

exists, and

$$\lim_{n\to\infty} a_n = 1.$$

Hint. Let $b_n = 1 - a_n$, $n = 1, 2\cdots$. We have a new sequence $\{b_n\}$ with the relation

$$(1 + b_n)(1 - b_{n+1}) = 1, \text{ i.e., } b_{n+1} = \frac{b_n}{1 + b_n}.$$

We find $b_n = 0 \Longleftrightarrow b_{n+1} = 0$. If one $b_m = 0$ then $b_n = 0, \forall n \geq m$ and $\lim_{n\to\infty} a_n = 1$.

No we assume $b_n \neq 0$, $n = 1, 2, \cdots$. Then we have

$$\frac{1}{b_{n+1}} - \frac{1}{b_{n+1}} = 1,$$

and so

$$\frac{1}{b_{n+1}} = n + \frac{1}{b_1}, \text{ i.e., } b_{n+1} = \frac{b_1}{nb_1 + 1} \to 0 \ (n \to \infty).$$

□

Example 1.9. Let $\{a_n\}$ be a sequence satisfying

(1)

$$\sum_{n=1}^{\infty} a_n = 1;$$

(2) for all $n \in \mathbb{N}$,

$$0 < a_n \leq \sum_{k=n+1}^{\infty} a_k.$$

Then, for any $x \in (0, 1)$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\sum_{k=1}^{\infty} a_{n_k} = x.$$

Proof. Define the $n_1, n_2 \cdots$ by induction.

1. Let

$$n_1 = \min\{n \,|\, a_n < x\}.$$

It is well-defined since by the condition (1) $\lim_{n\to\infty} a_n = 0$. Then by the condition (2) we have

$$0 < a_{n_1 - 1} \leq \sum_{k=n_1}^{\infty} a_k,$$

and so

$$\sum_{k=n_1}^{\infty} a_k \geq x.$$

2. If

$$\sum_{k=n_1}^{\infty} a_k > x$$

then we define

$$n_2 = \min\{n > n_1 \,|\, a_{n_1} + a_n < x.\}.$$

We must show that

$$a_{n_1} + \sum_{k=n_2}^{\infty} a_k \geq x.$$

Otherwise, we have $n_2 > n_1 + 1$ since by step 1, and then by the condition (2) we have

$$a_{n_1} + a_{n_2-1} \leq a_{n_1} + \sum_{k=n_2}^{\infty} a_k < x.$$

A contradiction to the definition of n_2.

3. If

$$a_{n_1} + \sum_{k=n_2}^{\infty} a_k > x$$

then we define

$$n_3 = \min\{n > n_1 \,|\, a_{n_1} + a_{n_2} + a_n < x.\}.$$

and we have

$$a_{n_1} + a_{n_2} + \sum_{k=n_3}^{\infty} a_k \geq x.$$

4. By induction, assume we have $n_1, n_2, \cdots, n_k$ with

$$\sum_{j=1}^{k-1} a_{n_j} + \sum_{i=n_k}^{\infty} a_i \geq x. \tag{1.9.1}$$

If

$$\sum_{j=1}^{k-1} a_{n_j} + \sum_{i=n_k}^{\infty} a_i = x,$$

then the proof will be finished. Otherwise, we define

$$n_{k+1} = \min\{n > n_k \,|\, \sum_{i=1}^{k} a_{n_i} + a_n < x\}.$$

Only necessarily to show that

$$\sum_{j=1}^{k} a_{n_j} + \sum_{i=n_{k+1}}^{\infty} \geq x. \tag{1.9.2}$$

If 1.9.2 is false, then $n_{k+1} > n_k + 1$ since 1.9.1. By the condition (2) there must hold

$$a_{n_1} + a_{n_2} + \cdots + a_{n_k} + a_{n_{k+1}-1} \leq \sum_{j=1}^{k} a_{n_j} + \sum_{k=n_{k+1}}^{\infty} a_k < x.$$

It is a contradiction to the definition of n_{k+1}.

□

Example 1.10. 1. Let $f(x) = \sqrt{x^2 + c}$. Compute the iteration

$$f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

*Solute.*Let

$$\phi(x) = x^2, g(x) = x + c.$$

Then we have :

$$f = \phi^{-1} \circ g \circ \phi \ \text{ and } \ f^{(n)}(x) = \phi^{-1} \circ g^{(n)} \circ \phi,$$

where ϕ^{-1} is the inverse function of ϕ.

$$g^{(n)}(x) = x + nc, \ \text{ and so } \ f^{(n)}(x) = \sqrt{x^2 + nc}.$$

□

2. Let

$$f(x) = \frac{x}{\sqrt[3]{x^3+c}}.$$

Compute the iteration

$$f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

*Solute.*Let

$$\phi(x) = x^3, g(x) = \frac{x}{x+c}.$$

Then we have :

$$f = \phi^{-1} \circ g \circ \phi \text{ and } f^{(n)}(x) = \phi^{-1} \circ g^{(n)} \circ \phi,$$

where ϕ^{-1} is the inverse function of ϕ. By induction, we have:

$$g^{(n)}(x) = \frac{x}{(\sum_{i=o}^{n-1} c^i)x + c^n}.$$

Then,

$$f^{(n)}(x) = \frac{x}{\sqrt[3]{(\sum_{i=o}^{n-1} c^i)x^3 + c^n}}$$

□

Exercise 1.11. a) Let

$$f(x) = x^3 + 6x^2 + 12x + 6.$$

Compute the iteration

$$f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

b) Let

$$f(x) = \frac{x+4}{2x-1}.$$

Compute the iteration

$$f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

c) Let

$$f(x) = \frac{2x}{1-x^2}.$$

Compute the iteration

$$f^{(n)}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

Exercise 1.12. The sequence $\{a_0, a_1 \cdots, a_n\}$ is defined by

$$a_0 = \frac{1}{2}, a_k = a_{k-1} + \frac{1}{n}a_{k-1}^2.$$

To show that

$$1 - \frac{1}{n} < a_n < 1.$$

Hint. By induction to show for any $1 \leq k \leq n$, we have

$$\frac{n+1}{2n-k+2} < a_k < \frac{n}{2n-k},$$

and let $k = n$, we get

$$\frac{n+1}{n+2} < a_n < 1.$$

□

Exercise 1.13. Let $a - 1, a_2, \cdots, a_n$ be a sequence of different positive integers. To show that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

Hint. Let $S_k = a_1 + \cdots + a_k$. Then

$$S_k \geq \sum_{i=1}^{k} i \geq \frac{k(k+1)}{2},$$

and we have:

$$\sum_{k=1}^{n} \frac{a_k}{k^2} = \sum_{k=1}^{n} S_k[\frac{1}{k^2} - \frac{1}{(k+1)^2}] + \frac{S_n}{n^2} \geq \sum_{k=1}^{n} \frac{k(k+1)}{2}[\frac{1}{k^2} - \frac{1}{(k+1)^2}] + \frac{n+1}{2n}.$$

□

Exercise 1.14. Let $a_1 \geq a_2 \geq \cdots, \geq a_n$ and $b_1 \geq b_2 \geq \cdots, \geq b_n$. To show

$$(\sum_{k=1}^{n} a_k)(\sum_{k=1}^{n} b_k) \leq n \sum_{k=1}^{n} a_k b_k.$$

Proof. Let

$$D_\nu = \nu \sum_{k=1}^{\nu} a_k b_k - (\sum_{k=1}^{\nu} a_k)(\sum_{k=1}^{\nu} b_k).$$

We have $D_1 = 0$ and

$$D_{\nu+1} - D_\nu = \sum_{k=1}^{\nu} (a_k - a_{k+1})(b_k - b_{k+1}) \geq 0.$$

□

Exercise 1.15. 1. To show that

$$\log(n+1) < \sum_{k=1}^{n} \frac{1}{k} < \log n + 1.$$

2. Consider

$$\zeta_N(\alpha) = \sum_{k=1}^{N} \frac{1}{k^\alpha},\ \ \alpha \in \mathbb{R}.$$

To show that

$$\lim_{N\to\infty} \zeta_N(\alpha) < \infty, \alpha \in (1, \infty);$$

and

$$\lim_{N\to\infty} \zeta_N(\alpha) = \infty, \alpha \in (0, 1).$$

Exercise 1.16. Show the sum

$$S = C_n^2 x + C_n^5 x^2 + C_n^8 x^3 + \cdots + +C_n^{n-1} x^{\frac{n}{3}},$$

where n is a positive integer with $3|n$.

Hint. Let ω be the primitive root of $y^3 = 1$, i.e., $\omega^2 + \omega + 1 = 0$. Then, we have

$$\omega^{2(k+1)} + \omega^{k+1} + 1 = \begin{cases} 3, & 3|k+1; \\ 0, & \text{others.} \end{cases}$$

Let $a \in \{\sqrt[3]{x}, \omega\sqrt[3]{x}, \omega^2\sqrt[3]{x}\}$. Consider the equality

$$a(1+a)^n = \sum_{k=0}^{n} C_n^k a^{k+1},$$

Let a goes through the three values in the set $a \in \{\sqrt[3]{x}, \omega\sqrt[3]{x}, \omega^2\sqrt[3]{x}\}$, we then sum the three equalities and get

$$S = \frac{\sqrt[3]{x}\sum_{i=0}^{2} \omega^i (1 + \sqrt[3]{x}\omega^i)^n}{3}.$$

□

Exercise 1.17. Let the sequence $\{x_n\}$ satisfy the following condition:

$$\begin{cases} x_1 = 815, & ; \\ x_{n+1} = x^2 - 2, & n \in \mathbb{N}. \end{cases}$$

Show the limit

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_1 x_2 \cdots x_n}.$$

Hint. We have

$$x_{n+1}^2 - 4 = (x_1^2 - 4)(x_1 \cdots x_n)^2$$

and $x_n > 2^n$ for all $n \in \mathbb{N}$.

□

Exercise 1.18. If the following is true for all $n \in \mathbb{N}$,

$$[Bn] = [A[An]] + 1,$$

show that

$$B = A^2.$$

Hint.

$$\lim_{n\to\infty}\frac{[\alpha n]}{n}=\alpha.$$

□

1.2. **Applications of the Stolz theorem.**

Theorem (Stolz theorem). *Let $\{y_n\}$ is strictly increasing sequence, i.e.,$y_{n+1}>y_n$, and $y_n\to\infty$. If*

$$\lim_{n\to\infty}\frac{x_n-x_{n+1}}{y_n-y_{n+1}}=a,\ a\in[-\infty,+\infty],$$

Then

$$\lim_{n\to\infty}\frac{x_n}{y_n}=a.$$

Example 1.19. Let $\{a_n\},\{b_n\}$ be two sequences satisfying that

$$b_{n+1}=a_n+918a_{n+1}\ \forall n\in\mathbb{N}.$$

Then, we have:

$$\lim_{n\to\infty}b_n \text{ exists} \iff \lim_{n\to\infty}a_n \text{ exists}.$$

Proof. We only prove the "$\Rightarrow$" part. Let $b=\lim_{n\to\infty}b_n$. We define two new sequences

$$\alpha_n=\frac{b}{919}-a_n,\beta_n=\frac{b-b_n}{918},\ n=1,2,\cdots.$$

Let $\lambda=-\frac{1}{918}$. then the sequences $\{\alpha_n\},\{\beta_n\}$ satisfy

$$\beta_{n+1}+\lambda\alpha_n=\alpha_{n+1}.$$

By induction, we have

$$\alpha_{n+1}=\sum_{i=1}^{n+1}\beta_i\lambda^{n+1-i}+\lambda^{n+1}\alpha_0=\frac{\sum_{i=1}^{n+1}\beta_i(\frac{1}{\lambda})^i+\alpha_0}{(\frac{1}{\lambda})^{n+1}},$$

and so

$$|\alpha_{n+1}|\le\frac{\sum_{i=1}^{n+1}|\beta_i||\frac{1}{\lambda}|^i+|\alpha_1|}{|\frac{1}{\lambda}|^{n+1}}.$$

By the Stolz theorem, we have

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n+1}|\beta_i||\frac{1}{\lambda}|^i+|\alpha_1|}{|\frac{1}{\lambda}|^n}=\lim_{n\to\infty}\frac{|\beta_{n+1}||\frac{1}{\lambda}|^{n+1}}{|\frac{1}{\lambda}|^n(|\frac{1}{\lambda}|-1)}=0.$$

□

1.20. Exercises.

1. Let $p_1,p_2,\cdots,p_n,\cdots$ be positive numbers. If

$$\lim_{n\to\infty}p_n=p\in(0,\infty),$$

then

$$\lim_{n\to\infty}\sqrt[n]{p_1p_2\cdots p_n}=p.$$

2. Let $a_0, a_1, a_2, \cdots, a_n, \cdots$ be positive numbers. If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p < \infty,$$

then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = p.$$

3. To show for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} = \frac{1}{k+1}.$$

*Proof.*By Stolz theorem, we compute

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n+1} i^k - \sum_{i=1}^{n} i^k}{(n+1)^{k+1} - n^{k+1}} = \lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} = \lim_{n\to\infty} \frac{\frac{(n+1)^k}{n^k}}{\frac{(n+1)^{k+1}-n^{k+1}}{n^k}}.$$

On the other hand,

$$\frac{(n+1)^k}{n^k} = 1 + O(\frac{1}{n}),$$

$$\frac{(n+1)^{k+1} - n^{k+1}}{n^k} = k + 1 + O(\frac{1}{n}).$$

□

4. To show for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} n(\frac{\sum_{i=1}^{n} i^k}{n^{k+1}} - \frac{1}{k+1}) = \frac{1}{2}.$$

Proof.

$$n(\frac{\sum_{i=1}^{n} i^k}{n^{k+1}} - \frac{1}{k+1}) = \frac{1}{2} = \frac{(k+1)(\sum_{i=1}^{n} i^k) - n^{k+1}}{(k+1)n^k}.$$

By Stolz theorem, it is sufficient to compute that

$$\lim_{n\to\infty} \frac{(k+1)(n+1)^k - (n+1)^{k+1} + n^{k+1}}{(k+1)((n+1)^k - n^k)}$$

On the other hand

$$(k+1)(n+1)^k - (n+1)^{k+1} + n^{k+1} = (k+1)\sum_{i=0}^{k} C_k^i n^i - \sum_{i=0}^{k} C_{k+1}^i n^i = \sum_{i=0}^{k-1}((k+1)C_k^i - C_{k+1}^i)n^i$$

So

$$\frac{(k+1)(n+1)^k - (n+1)^{k+1} + n^{k+1}}{(k+1)((n+1)^k - n^k)} = \frac{\frac{1}{2}k(k+1) + O(\frac{1}{n})}{k(k+1) + O(\frac{1}{n})}$$

□

5. Let $x_1 \in (0,1)$ and $\{x_n\}$ be a sequence with $x_{n+1} = x_n(1-x_n) \forall n \in \mathbb{N}$. Then we have:

$$\lim_{n\to\infty} n x_n = 1.$$

*Proof.*Since $x_2 = x_1(1 - x_1) \le (1/2)^2 < 1$, by induction we have $x_n \in (0, 1) \forall n \in \mathbb{N}$. On the other hand,

$$\frac{x_{n+1}}{x_n} = 1 - x_n < 1,$$

so $\{x_n\}$ is a strictly decreasing sequence with a finite bound, and then

$$\lim_{n\to\infty} x_n = A \in [0, 1).$$

Moveover, by $A = A(1 - A)$, we have $A = 0$. By Stolz theorem,

$$\lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} \frac{n - (n-1)}{\frac{1}{x_n} - \frac{1}{x_{n-1}}} = \lim_{n\to\infty} (1 - x_n) = 1.$$

□

6. let $\{a_k\}$ be a sequence satisfying that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = A < \infty.$$

Let $\{p_k\}$ be a strictly increasing sequence of positive numbers with $p_k \to +\infty$ $(k \to \infty)$. Show that

$$\lim_{n\to\infty} \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_n} = 0.$$

*Proof.*Assume $A_0 = 0$, we have:

$$\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_n} = \frac{\sum_{k=1}^{n} p_k (A_k - A_{k-1})}{p_k} = \frac{\sum_{k=1}^{n-1} A_k (p_k - p_{k+1}) + A_n p_n}{p_n}$$

By STolz theorem, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} A_k (p_k - p_{k+1})}{p_n} = \lim_{n\to\infty} \frac{A_n (p_n - p_{n+1})}{p_{n+1} - p_n} = -A$$

□

2. Inequality

Theorem 2.1 (Hölder Inequality)**.** *Let* $a_k, b_k \geq 0(k = 1, \cdots, n)$*,* $q, p > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then, we have*

$$\sum_{k=1}^{n} a_k b_k \leq (\sum_{k=1}^{n} a_k^p)^{\frac{1}{p}} (\sum_{k=1}^{n} b_k^q)^{\frac{1}{q}}.$$

Moreover, the inequality becomes to be an equality if and only if there exists a $t \in \mathbb{R}_+$ *such that*

$$a_k^p = t b_k^q \ (k = 1, \cdots, n).$$

The proof of the Hölder inequality depends heavily on the following lemma.

Lemma 2.2. *Let* $A, B \geq 0$*. then for any* $\alpha \in [0, 1]$*, we have:*

$$A^{\alpha} B^{1-\alpha} \leq \alpha A + (1 - \alpha) B,$$

and the inequality becomes to be an equality if and only if $A = B$*.*

Proof. W.L.O.G. we assume $A \geq B > 0, \alpha \neq 0$. Then we have

$$(\frac{A}{B})^{\alpha} = \alpha \int_1^{\frac{A}{B}} x^{\alpha - 1} d\,x + 1 < \alpha \int_1^{\frac{A}{B}} 1 \cdot d\,x + 1 = \alpha(\frac{A}{B} - 1) + 1 = \alpha(\frac{A}{B}) + (1 - \alpha).$$

□

Exercise 2.3 (Young Inequality)**.** Let $A, B \geq 0$. and $q, p > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$AB < \varepsilon A^p + \varepsilon^{-\frac{q}{p}} B^q \ \forall \varepsilon > 0.$$

Exercise 2.4 (Minkowski Inequality)**.** Let $a_k, b_k \geq 0(k = 1, \cdots, n), p > 1$. Then, we have

$$\{\sum_{k=1}^{n} (a_k + b_k)^p\}^{\frac{1}{p}} \leq \{\sum_{k=1}^{n} a_k^p\}^{\frac{1}{p}} + \{\sum_{k=1}^{n} b_k^p\}^{\frac{1}{p}}.$$

Moreover, the inequality becomes to be an equality if and only if there exists a $t \in \mathbb{R}$ such that

$$a_k = t b_k \ (k = 1, \cdots, n).$$

Theorem 2.5 (**Arithmetic Mean-Geometry Mean Inequality**)**.** *Let* $a_1, \cdots, a_n$ *be positive real numbers and* $a_1, \cdots, a_n$*be positive real numbers with* $p_1 + \cdots + p_n = 1$*. Then we have:*

$$G_n = \prod_{i=1}^{n} a_i^{p_i} \leq \sum_{i=1}^{n} p_i a_i = A_n,$$

moreover the inequality becomes to be an equality if and only if $a_1 = a_2 = \cdots = a_n$*.*

Proof. W.L.O.G. we assume $a_1 \leq a_2 \leq \cdots \leq a_n$. Then there exists an integer $k \in [1, n-1]$ such that

$$a_k \leq G_n \leq a_{k+1}.$$

Consider the following formula

$$\begin{aligned}
\frac{A_n}{G_n}-1 &= \sum_{i=1}^{n} p_i(\frac{a_i-G_n}{G_n})-\sum_{i=1}^{n} p_i(\frac{\log a_i-\log G_n}{G_n}) \\
&= \sum_{i=1}^{k} -p_i(\frac{G_n-a_i}{G_n}+(\log G_n--\log a_i))+\sum_{i=k+1}^{n} p_i(\frac{a_i-G_n}{G_n}-(\log a_i-\log G_n)) \\
&= \sum_{i=1}^{k} -p_i\int_{a_i}^{G_n}(\frac{1}{G_n}-\frac{1}{t})dt+\sum_{i=k+1}^{n} p_i\int_{G_n}^{a_i}(\frac{1}{G_n}-\frac{1}{t})dt \\
&= \sum_{i=1}^{k} +p_i\int_{a_i}^{G_n}(\frac{1}{t}-\frac{1}{G_n})+\sum_{i=k+1}^{n} p_i\int_{G_n}^{a_i}(\frac{1}{G_n}-\frac{1}{t}) \\
&\geq 0.
\end{aligned}$$

Moreover the inequality becomes to be an equality if and only if

$$a_1=a_2=\cdots=a_n=G_n.$$

□

2.6. Exercises.

1. Let $a_k, b_k \geq 0(k=1,\cdots,n)$, with $\frac{1}{p}+\frac{1}{q}=1$ with $0<p<1$ Assume that

$$(a_k^p)_{k=1}^n \neq t(b_k^q)_{k=1}^n,\ \forall t\in\mathbb{R}.$$

To show that

$$\sum_{k=1}^{n} a_k b_k > (\sum_{k=1}^{n} a_k^p)^{\frac{1}{p}}(\sum_{k=1}^{n} b_k^q)^{\frac{1}{q}}.$$

*Hint.*We have

$$0<\frac{1}{(\frac{1}{p})}<1,\ 0<\frac{1}{(-\frac{q}{p})}=1-k<1,$$

and

$$\frac{1}{(\frac{1}{p})}+\frac{1}{(-\frac{q}{p})}=1.$$

Then, we use the Hölder inequality to the following

$$\sum_{i=1}^{n} a_i^p=\sum_{i=1}^{n}(a_ib_i)^p b_i^{-p}=\sum_{i=1}^{n}(a_ib_i)^{\frac{1}{1/p}} b_i^{\frac{1}{-q/p}}.$$

□

2. Let $a_k\geq 0(k=1,\cdots,n), p>0$. We define

$$M_p(a)=\{\frac{\sum_{k=1}^{n} a_k^p}{n}\}^{\frac{1}{p}}.$$

To show that for $0<r<s$, we always have

$$M_r(a)<M_s(a)$$

except for $a_1=a_2=\cdots=a_n$.

3. To show

$$\sum_{k=1}^{n}\sqrt{\frac{1}{k}\sin(\frac{k\pi}{n})}\le\sqrt{\frac{1+\log n}{\sin\frac{\pi}{2n}}}.$$

4. To show

$$\sum_{k=1}^{n}\frac{1}{k^2}\sqrt{\arctan\frac{1}{k^2+k+1}}\le\sqrt{\frac{\pi}{3}}.$$

Example 2.7 (Hilbert Inequality). Assume $\sum_{n=1}^{\infty}a_n^2\le\infty$. Then we have:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\le\pi\sum_{n=1}^{\infty}a_n^2.$$

Proof. Consider

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n}=\sum_{m=1}^{N}\sum_{n=1}^{N}(\frac{m}{n})^{\frac14}\frac{a_m}{\sqrt{m+n}}\cdot(\frac{n}{m})^{\frac14}\frac{a_n}{\sqrt{m+n}}.$$

Thus, by the Hölder inequality, we have:

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n}\le\{\sum_{m=1}^{N}\sum_{n=1}^{N}(\frac{m}{n})^{\frac12}\frac{a_m^2}{m+n}\}^{\frac12}\cdot\{\sum_{m=1}^{N}\sum_{n=1}^{N}(\frac{n}{m})^{\frac12}\frac{a_n^2}{m+n}\}^{\frac12}=\sum_{m=1}^{N}\sum_{n=1}^{N}(\frac{m}{n})^{\frac12}\frac{a_m^2}{m+n},$$

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n}\le\sum_{m=1}^{N}a_m^2\sum_{n=1}^{N}(\frac{m}{n})^{\frac12}\frac{1}{m+n}.$$

On the other hand, we have:

$$(\frac{m}{n})^{\frac12}\frac{1}{m+n}=\frac{\frac{1}{m}}{(\frac{n}{m})^{\frac12}(1+\frac{n}{m})}\le\int_{\frac{n-1}{m}}^{\frac{n}{m}}\frac{dx}{(1+x)\sqrt{x}}.$$

$$\sum_{n=1}^{N}(\frac{m}{n})^{\frac12}\frac{1}{m+n}\le\sum_{n=1}^{N}\int_{\frac{n-1}{m}}^{\frac{n}{m}}\frac{dx}{(1+x)\sqrt{x}}\le\int_0^{\frac{N}{m}}\frac{dx}{(1+x)\sqrt{x}}\le\int_0^{\infty}\frac{dx}{(1+x)\sqrt{x}}=\pi.$$

At all, taking limit $n,m\to\infty$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}\le\sum_{m=1}^{\infty}a_m^2\pi=\pi\sum_{m=1}^{\infty}a_m^2.$$

□

Example 2.8 (Hardy Inequality). Let $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume $\sum_{n=1}^{\infty}a_n^p\le\infty$ and $\sum_{m=1}^{\infty}b_n^q\le\infty$ Then we have:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n b_m}{m+n}\le\frac{\pi}{\sin(\frac{\pi}{p})}(\sum_{n=1}^{\infty}a_n^p)^{\frac1p}(\sum_{m=1}^{\infty}b_m^q)^{\frac1q}.$$

Hint. Consider the following arguments.

a)

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n}=\sum_{m=1}^{N}\sum_{n=1}^{N}(\frac{m}{n})^{\frac{1}{pq}}\frac{a_m}{\sqrt{m+n}}\cdot(\frac{n}{m})^{\frac{1}{pq}}\frac{a_n}{\sqrt{m+n}}.$$

b)

$$(\frac{m}{n})^{\frac{1}{q}}\frac{1}{m+n}=\frac{\frac{1}{m}}{(\frac{n}{m})^{\frac{1}{q}}(1+\frac{n}{m})}\leq\int_{\frac{n-1}{m}}^{\frac{n}{m}}\frac{dx}{(1+x)x^{\frac{1}{q}}}.$$

□

Example 2.9 (Hardy-Landan inequality). Let p and a_k, $k=1,\cdots,n$ be positive numbers. To show for $\forall p>1$, we have:

$$\sum_{k=1}^{n}(\frac{a_1+\cdots+a_k}{k})^p<(\frac{p}{p-1})^p\sum_{k=1}^{n}a_k^p.$$

Hint. Let

$$A_k=\frac{a_1+\cdots+a_k}{k}.$$

Then, we have:

a)

$$A_k^p-\frac{p}{p-1}A_k^{p-1}a_k=A_k^p-\frac{p}{p-1}A_k^{p-1}(A_k-A_{k-1})=A_k^p(1-\frac{kp}{p-1})+\frac{(k-1)p}{p-1}A_k^{p-1}A_{k-1}$$

$$A_k^p-\frac{p}{p-1}A_k^{p-1}a_k\leq A_k^p(1-\frac{kp}{p-1})+\frac{k-1}{p-1}[(p-1)A_k^p+A_{k-1}^p]=\frac{1}{p-1}[(k-1)A_{k-1}^p-kA_k^p].$$

b) Thus, we have

$$\sum_{k=1}^{n}A_k^p-\frac{p}{p-1}A_k^{p-1}a_k<\frac{p}{p-1}\sum_{k=1}^{n}A_k^{p-1}a_k,$$

again use Hölder inequality.

□

Example 2.10 (Carleman inequality). Let a_k, $k=\in\mathbb{N}$ be positive numbers. If

$$\sum_{k=1}^{\infty}a_k<\infty,$$

then

$$\sum_{k=1}^{\infty}(a_1a_2\cdots a_k)^{1/k}\leq e\sum_{k=1}^{\infty}a_k.$$

Another proof directly by elementary technique. We only show that

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{1/k}\leq e\sum_{k=1}^{n}a_k.$$

Let $b_j=j(1+\frac{1}{j})^j$. Then, we have

$$\frac{b_j}{j}\leq e\ \text{ and }\ b_1b_2\cdots b_k=(k+1)^k,$$

and so

$$(a_1a_2\cdots a_k)^{1/k}=\frac{1}{k+1}\sqrt[k]{(a_1b_1)\cdots(a_kb_k)}\leq\frac{\sum_{i=1}^{k}a_ib_i}{k(k+1)}=\sum_{i=1}^{k}[\frac{1}{k}-\frac{1}{k+1}]a_ib_i.$$

Therefore,

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{1/k} \le \sum_{k=1}^{n}\sum_{i=1}^{k}[\frac{1}{k}-\frac{1}{k+1}]a_ib_i = \sum_{i=1}^{n}\sum_{k=i}^{n}[\frac{1}{k}-\frac{1}{k+1}]a_ib_i = \sum_{i=1}^{n}a_ib_i\sum_{k=i}^{n}[\frac{1}{k}-\frac{1}{k+1}],$$

and so

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{1/k} \le \sum_{i=1}^{n}a_ib_i[\frac{1}{i}-\frac{1}{n}] \le \sum_{i=1}^{n}a_ib_i\frac{1}{i} = \sum_{i=1}^{n}a_i\frac{b_i}{i} \le e\sum_{i=1}^{n}a_i.$$

□

2.11. Homework.

1. Let $A_k \ge 0, t_k > 0,\ k = 1, \cdots, n$. Denote $T_l = \sum_{k=1}^{l} t_k$. Show that

$$(\prod_{k=1}^{n}A_k^{t_k})^{\frac{1}{T_n}} \le \frac{1}{T_n}\sum_{k=1}^{n}t_kA_k.$$

*Hint.*Consider the following argument. Let

$$L_m = \sum_{k=1}^{m}A_k^{t_k}.$$

Then,we have:

$$(L_m)^{\frac{1}{T_m}} = (A_m^{t_m})^{\frac{1}{T_m}} = (L_{m-1}^{\frac{1}{T_{m-1}}})^{\frac{T_{m-1}}{T_m}} \cdot A_m^{\frac{t_m}{T_m}} \le \frac{T_{m-1}}{T_m}L_{m-1}^{\frac{1}{T_{m-1}}} + \frac{t_m}{T_m}A_m.$$

By induction, we prove the statement.

□

2. To show the Carleman inequality by using the Hardy-Landan inequality.

*Proof.*By the Hardy-Landan inequality, for all $p > 1$, we have

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{1/k} \le \sum_{k=1}^{n}(\frac{a_1^{\frac{1}{p}} + a_2^{\frac{1}{p}} + \cdots + a_k^{\frac{1}{p}}}{k})^p \le (\frac{p}{p-1})^p\sum_{k=1}^{n}a_k.$$

Since

$$\lim_{p\to+\infty}(\frac{p}{p-1})^p = e,$$

we get

$$\sum_{k=1}^{n}(a_1a_2\cdots a_k)^{1/k} \le e\sum_{k=1}^{n}a_k.$$

□

2.12 (Appendix)**.** Computing the integral

$$\int_0^\infty \frac{dx}{(1+x)x^{\frac{1}{q}}}$$

with residue theory.

Since $\frac{1}{q}+\frac{1}{p} = 1$. Then

$$\int_0^\infty \frac{dx}{(1+x)x^{\frac{1}{q}}} = \int_0^\infty \frac{x^{\frac{1}{p}-1}}{1+x}dx.$$

Let $f(z) = \frac{1}{(1+z)}$. $f(z)$ is a holomorphic at $\mathbb{C} - \{0\}$. We have

$$\begin{aligned} 2\pi\sqrt{-1}\mathrm{Res}_{z=-1}(z^{\frac{1}{p}-1}f(z)) &= \int_\varepsilon^R x^{\frac{1}{p}-1}f(x)dx + \int_R^\varepsilon e^{(\frac{1}{p}-1)2\pi\sqrt{-1}}x^{\frac{1}{p}-1}f(x)dx \\ &\quad + \int_{|z|=\varepsilon} z^{\frac{1}{p}-1}f(z)dz - \int_{|z|=R} z^{\frac{1}{p}-1}f(z)dz \end{aligned}$$

$$\begin{aligned} \mathrm{Res}_{z=-1}(z^{\frac{1}{p}-1}f(z)) &= e^{(\frac{1}{p}-1)\log z}|_{z=-1} \\ &= e^{(\frac{1}{p}-1)(\log|-1|+\sqrt{-1}\arg(-1))} \\ &= e^{(\frac{1}{p}-1)\pi\sqrt{-1}} = -e^{\frac{1}{p}\pi\sqrt{-1}} \end{aligned}$$

On the other hand,

$$|f(z)| \leq 1(|z| << 1), |f(z)| \geq \frac{1}{|z|}(|z| >> 1).$$

Therefore, we have

$$\begin{aligned} |\int_{|z|=\varepsilon} z^{\frac{1}{p}-1}f(z)dz| &\leq \varepsilon^{\frac{1}{p}-1}2\pi\varepsilon \to 0(\varepsilon \to 0), \\ |\int_{|z|=R} z^{\frac{1}{p}-1}f(z)dz| &\leq \frac{|R|^{\frac{1}{p}-1}}{R}2\pi R \to 0(R \to \infty), \\ \int_\varepsilon^R x^{\frac{1}{p}-1}f(x)dx + \int_R^\varepsilon e^{(\frac{1}{p}-1)2\pi\sqrt{-1}}x^{\frac{1}{p}-1}f(x)dx &= (1 - e^{(\frac{1}{p}-1)2\pi\sqrt{-1}})\int_\varepsilon^R x^{\frac{1}{p}-1}f(x)dx \end{aligned}$$

At all, we have

$$(1 - e^{(\frac{1}{p}-1)2\pi\sqrt{-1}})\int_0^\infty x^{\frac{1}{p}-1}f(x)dx = -2\pi\sqrt{-1}e^{\frac{1}{p}\pi\sqrt{-1}},$$

and so

$$\int_0^\infty \frac{dx}{(1+x)x^{\frac{1}{q}}} = \frac{-2\pi\sqrt{-1}e^{\frac{1}{p}\pi\sqrt{-1}}}{(1 - e^{\frac{2}{p}\pi\sqrt{-1}})} = \frac{2\pi\sqrt{-1}}{e^{\frac{1}{p}\pi\sqrt{-1}} - e^{-\frac{1}{p}\pi\sqrt{-1}}} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

3. Orders Estimate of Infinitesimal

3.1. **Notations and Examples.**

3.1. Notations.

1. If
$$\lim_{x\to x_o}\frac{f(x)}{g(x)}=0,$$
we denote $f(x)=o(g(x))$.It is obvious that $o(h)+o(g)=o(|h|+|g|)$.
In particular, let $\{a_n\}$ be a sequence, if
$$\lim_{n\to\infty}a_n=0$$
we denote $a_n=o(1),\ n\to\infty$.

2. If
$$\lim_{x\to x_o}\frac{f(x)}{g(x)}=1,$$
we say $f(x)$ and $g(x)$ is equivalent for $x\to x_0$, and we denote
$$f(x)\sim g(x),\ x\to x_0$$

3. Let $g(x)>0$, if there is a constant $A>0$, such that
$$|f(x)|\le Ag(x),\ x\in(a,b),$$
then we denote $f(x)=O(g(x)),\ x\in(a,b)$.

4. Here we have some well-know elementary functions: For n even, we denote $n!!=n(n-2)(n-4)\cdots4\cdot2$; For n odd, we denote $n!!=n(n-2)(n-4)\cdots3\cdot1$.
a)
$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\ (x\in\mathbb{R}),\ \text{and}\ \cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\ (x\in\mathbb{R}),$$
b)
$$\arctan x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\ (|x|<1),$$
c)
$$\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\ (|x|<1),$$
d)
$$(1+x)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k\ (|x|<1),$$
e)
$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}\ (x\in\mathbb{R}).$$

f)

$$\arctan x = \sum_{k=}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \ (x \in \mathbb{R}).$$

g)

$$\arcsin x = x + \frac{1}{3}\frac{1}{2!}x^3 + \frac{1}{5}\frac{3!!}{4!!}x^5 + \cdots \frac{1}{2n+1}\frac{(2n-1)!!}{(2n)!!}x^{2n+1} + \cdots \ (x \in \mathbb{R}).$$

Theorem. *In the neighborhood of x_0, if $f^{(n)}(x)$ exists and $|f^{(n)}(x_0)| \le M$ then*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + O(|x-x_0|^n).$$

For examples, we have:
a) $\sin x = x - \frac{x^3}{3!} + O(|x|^5) \ (x \in \mathbb{R}),$ and $\cos x = 1 - \frac{x^2}{2} + O(|x|^4) \ (x \in \mathbb{R}),$
b) $\log(1+x) = x - \frac{x^2}{2} + O(|x|^3) \ (|x| < 1),$
c) $(1+x)^\alpha = 1 + \alpha x + O(x^2) \ \forall \alpha \in \mathbb{R}, \quad (|x| < 1),$
d) $e^x = 1 + x + \sum_{k=1}^n \frac{x^k}{k!} + O(|x|^{n+1}) \ (x \in \mathbb{R}),$
e) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7)(x \in \mathbb{R}),$
f) $\arcsin x = x + \frac{1}{3}\frac{1}{2!}x^3 + \frac{1}{5}\frac{3!!}{4!!}x^5 + +O(x^7) \ (x \in \mathbb{R}).$

Example 3.2. Let $\varepsilon > 0$ and A be any two constants. Then for any $\alpha > 0$ we have :
1. $x^A = o((1+\alpha)^{\varepsilon x}), \ x \to \infty;$

2. $\log x^A = o(x^\varepsilon), \ x \to \infty;$

3. $(f(x))^A = o(e^{\varepsilon f(x)})$ for any increasing function $f(x)$ with

$$\lim_{x\to\infty} f(x) = \infty.$$

Proof. Let $n = [x]$ be the integer part of x and $m = [A]+1$. If $x \to \infty$ then $n \to \infty$. We have :

$$(1+\alpha)^x \ge (1+\alpha)^n \ge C_n^{m+1}\alpha^{m+1} \ge \frac{\alpha^{m+1}}{(m+1)!} \cdot (\frac{n}{2})^{m+1},$$

if $n \ge 2m+1$. Thus, if $x \to \infty$, we have

$$\frac{(1+\alpha)^x}{x^A} \ge \frac{\alpha^{m+1}}{2^{m+1}(m+1)!}\frac{n^{m+1}}{(1+n)^m}.$$

Let $x = \varepsilon y$, we get (1).
Let $\alpha = e-1, x = \varepsilon \log y$, we get (2).
(3) is obvious.

$\square$

Example 3.3. Let $\{a_n\}, \{b_n\}$ be two sequences with $a_n = o(b_n) \ n \to \infty$ and $b_n > 0$. If $\sum_{n=1}^\infty b_n = \infty$, then we have:

$$\sum_{n=1}^N a_n = o(\sum_{n=1}^N b_n), N \to \infty.$$

Proof. For any $\varepsilon > 0$ there exists $M > 0$ such that for any $n > M$ we always have $|a_n| < \varepsilon b_n$.
Thus, for any $N > M$, we have:

$$|\sum_{n=1}^{N} a_n| \leq |\sum_{n=1}^{M} a_n| + \varepsilon \sum_{n=M+1}^{N} b_n \leq \sum_{n=1}^{M} |a_n| + \varepsilon \sum_{n=M+1}^{N} b_n.$$

Since $\sum_{n=1}^{\infty} b_n = \infty$, there exists $N_0 > M$ such that if $N > N_0$

$$\varepsilon \sum_{n=1}^{N} b_n > \sum_{n=1}^{M} |a_n|.$$

So

$$|\sum_{n=1}^{N} a_n| \leq 2\varepsilon \sum_{n=1}^{N} b_n.$$

□

Example 3.4. Let $\{a_k\}$ be a sequences with $a_k > 0$ and $\sum_{k=1}^{\infty} a_k = L < \infty$, if

$$\sum_{k=n}^{\infty} a_n = O(a_n),$$

then we have:

$$\sum_{k=1}^{n} \frac{1}{a_k} = O(\frac{1}{a_n}).$$

Proof. Let

$$A_n = \sum_{k=n}^{\infty} a_n,$$

then $a_n = A_n - A_{n+1} < A_n$ and $A_n = O(a_n)$. Thus, there exists a finite constant $A > 1$ such that $A_n \leq A a_n$. We therefore have:

$$\frac{A_n}{A_n - A_{n+1}} \leq A,$$

$$A_{n+1} \leq \frac{A-1}{A} A_n \leq \cdots \leq (\frac{A-1}{A})^{l+1} A_{n-l}.$$

Thus,

$$\sum_{k=1}^{n} \frac{1}{a_k} \leq \frac{A_n}{a_n} \sum_{k=1}^{n} \frac{A}{A_k} \leq \frac{A}{a_n} \sum_{k=1}^{n} \frac{A_n}{A_k} \leq \frac{A}{a_n} \sum_{l=0}^{n-1} (\frac{A-1}{A})^l < \frac{A}{a_n} \sum_{l=0}^{\infty} (\frac{A-1}{A})^l = \frac{A^2}{a_n}.$$

□

Example 3.5 (Stolz theorem)**.** Let $\{y_n\}$ is strictly increasing sequence, i.e.,$y_{n+1} > y_n$, and $y_n \to \infty$. If

$$\lim_{n\to\infty} \frac{x_n - x_{n+1}}{y_n - y_{n+1}} = a, \ a \in [-\infty, +\infty],$$

Then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = a.$$

Proof. In case of $a = +\infty$, there is a $N \in \mathbb{N}$ such that $x_{n+1} - x_n > y_{n+1} - y_n$ for $\forall n > N$ and $x_n \to +\infty$, and

$$\lim_{n\to\infty} \frac{y_n - y_{n+1}}{x_n - x_{n+1}} = 0.$$

In case of $a = -\infty$, let $t_n = -x_n$. Thus these two cases are reduced to the case finite constant a.

Now we assume that a is a finite constant. Then we have

$$x_k - x_{k-1} = a(y_k - y_{k-1}) + o(y_k - y_{k-1}),$$

sum with k, we get

$$\sum_{k=2}^{n}(x_k - x_{k-1}) = \sum_{k=2}^{n} a(y_k - y_{k-1}) + \sum_{k=2}^{n} o(y_k - y_{k-1}).$$

Since $y_{n+1} > y_n \forall n \in \mathbb{N}$ and $y_n \to \infty$ we have

$$\sum_{k=2}^{n} o(y_k - y_{k-1}) = o(\sum_{k=2}^{n} y_k - y_{k-1}) = o(y_n - y_1) = o(y_n).$$

Thus,

$$x_n - x_1 = a(y_n - y_1) + o(y_n).$$

□

3.2. **Exercises and homework.**

3.6. Exercises.

1. To show

$$\lim_{n\to\infty} \frac{n(\sqrt[n]{n} - 1)}{\log n} = 1.$$

Proof.

$$\sqrt[n]{n} = \exp(\frac{\log n}{n}) = 1 + \frac{\log n}{n} + O(\frac{\log^2 n}{n^2}),$$

$$n(\sqrt[n]{n} - 1) = \log n + O(\frac{\log^2 n}{n}).$$

□

*Another proof given by a student.*Let $t = \sqrt[n]{n} - 1$. Then

$$t \to 0^+ \,(n \to \infty) \text{ and } n = (1+t)^n,$$

and so

$$\frac{n(\sqrt[n]{n} - 1)}{\log n} = \frac{nt}{n\log(1+t)} = \frac{t}{\log(1+t)} = \frac{1}{\log(1+\frac{1}{1/t})^{1/t}} \sim \log e = 1\,(n \to \infty).$$

□

2. To prove for $x > 0$,

$$\lim_{n\to\infty} n^2(\sqrt[n]{x} - \sqrt[n+1]{x}) = \log x.$$

Proof.

$$n^2(\sqrt[n]{x} - \sqrt[n+1]{x}) = n^2 x^{1/n}(1 - x^{-1/(n(n+1))}) = n^2 x^{1/n}(1 - \exp(\frac{-\log x}{n(n+1)})).$$

$$\exp(\frac{-\log x}{n(n+1)}) = 1 - \frac{\log x}{n(n+1)} + O(\frac{1}{n^4}),$$

so we have:

$$n^2(\sqrt[n]{x} - \sqrt[n+1]{x}) = n^2 x^{1/n}(\frac{\log x}{n(n+1)} + O(\frac{1}{n^4})) = x^{1/n}\log x + O(\frac{1}{n^2}).$$

□

3. To prove

$$\lim_{n\to\infty} \cos^n(\frac{x}{\sqrt{n}}) = e^{-\frac{x^2}{2}}.$$

Proof.

$$(\cos(\frac{x}{\sqrt{n}}))^n = (1 - \frac{x^2}{2n} + O(\frac{1}{n^2}))^n = \exp(n\log(1 - \frac{x^2}{2n} + O(\frac{1}{n^2}))),$$

Since

$$\log(1 - \frac{x^2}{2n} + O(\frac{1}{n^2})) = -\frac{x^2}{2n} + O(\frac{1}{n^2}),$$

we have:

$$(\cos(\frac{x}{\sqrt{n}}))^n = e^{-\frac{x^2}{2}}(1 + O(\frac{1}{n})).$$

□

Another proof given by a student.

$$(\cos(\frac{x}{\sqrt{n}}))^n = (1 - \sin^2(\frac{x}{\sqrt{n}}))^{\frac{n}{2}} \sim ((1 - \frac{x^2}{n})^{\frac{-n}{x^2}})^{-\frac{x^2}{2}} \sim e^{-\frac{x^2}{2}},\ (n \to \infty).$$

□

4. To prove for $0 < \alpha < 2$, we have

$$\lim_{x\to\infty} \sqrt{x + \sqrt{x + \sqrt{x^\alpha}}} - \sqrt{x} = \frac{1}{2}.$$

Proof.

$$\sqrt{x + \sqrt{x + \sqrt{x^\alpha}}} = \sqrt{x}(1 + \sqrt{\frac{1}{x} + x^{\frac{\alpha}{2}-2}})^{1/2} = \sqrt{x}(1 + \frac{1}{2\sqrt{x}}\sqrt{1 + x^{\frac{\alpha}{2}-1}}) + O(\frac{1}{\sqrt{x}}),$$

thus

$$\sqrt{x + \sqrt{x + \sqrt{x^\alpha}}} - \sqrt{x} = \frac{1}{2}(1 + \frac{1}{2}x^{\frac{\alpha}{2}-1} + O(x^{\alpha-2})) + O(\frac{1}{\sqrt{x}}).$$

□

5. To show

$$\lim_{x\to 0}(\frac{x}{\tan x})^{\frac{1}{x^2}} = e^{\frac{1}{3}}.$$

Proof.

$$\frac{x}{\tan x}=\frac{x\cos x}{\sin x}=x\frac{1-\frac{x^2}{2}+O(x^4)}{x-\frac{x^3}{6}+O(x^5)}=(1-\frac{x^2}{2}+O(x^4))(1+\frac{x^2}{6}+O(x^4))=1-\frac{x^2}{3}+O(x^4),$$

$$(\frac{x}{\tan x})^{\frac{1}{x^2}}=\exp(\frac{\log 1-\frac{x^2}{3}+O(x^4)}{x^2})=\exp(\frac{\frac{x^2}{3}+O(x^4)}{x^2})=e^{\frac{1}{3}+O(x^2)}.$$

□

3.7. Homework.

1. **Euler constant.**To show that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\log n=\gamma<\infty.$$

*Proof.*Let

$$a_n=\sum_{k=1}^{n-1}\frac{1}{k}-\log n,$$
$$b_n=\sum_{k=1}^{n}\frac{1}{k}-\log n.$$

Then $b_n-a_n=\frac{1}{n}$ and

$$a_{n+1}-a_n=\frac{1}{n}-(\log(n+1)-\log n)=\int_n^{n+1}(\frac{1}{n}-\frac{1}{t})dt>0,$$
$$b_{n+1}-b_n=\frac{1}{n+1}-(\log(n+1)-\log n)=\int_n^{n+1}(\frac{1}{n+1}-\frac{1}{t})dt<0.$$

Thus

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=\gamma<\infty.$$

□

2. In the textbook, e is defined as

$$e:=\lim_{n\to\infty}(1+\frac{1}{n})^n.$$

Show that

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}.$$

*Proof.*Let

$$a_n=\sum_{k=0}^{n}\frac{1}{k!},$$
$$b_n=(1+\frac{1}{n})^n.$$

then

$$b_n=1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})\le a_n$$

On the other hand, given any $k\in\mathbb{N}$, for $n>k$ we have

$$b_n>1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k}{n}).$$

Thus,

$$e \geq \lim_{n\to\infty} 1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k}{n}) = a_k.$$

Therefore,

$$b_k \leq a_k \leq e,\ \forall k \in \mathbb{N},\ \text{ and so } \lim_{k\to\infty} a_k = e.$$

□

3. To show that for $\alpha \in (1,\infty)$

$$\sum_{k=n}^{\infty}\frac{1}{k^\alpha} = O(\frac{1}{n^{\alpha-1}}),$$

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} = C + O(\frac{1}{n^{\alpha-1}}),$$

where C is a constant.

*Proof.*Since

$$\int_k^{k+1}\frac{dx}{x^\alpha} < \frac{1}{k^\alpha} < \int_{k-1}^{k}\frac{dx}{x^\alpha},$$

So

$$\sum_{k=n}^{\infty}\frac{1}{k^\alpha} < \int_{n-1}^{\infty}\frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\int_{n-1}^{\infty} dx^{1-\alpha} = O(\frac{1}{n^{\alpha-1}}).$$

□

4. Application of Infinitesimal: Wallis Formula and Stirling Formula

4.1. (Wallis Formula).

1. Let $n \in \mathbb{N}$. If n is even, we denote
$$n!! = n(n-2)(n-4)\cdots 4\cdot 2.$$
If n is odd, we denote
$$n!! = n(n-2)(n-4)\cdots 3\cdot 1.$$
Show the integral
$$J_m = \int_0^{\frac{\pi}{2}} \sin^m x d\, x = \begin{cases} \frac{\pi}{2}\frac{(m-1)!!}{m!!} & m \text{ is even };\\ \frac{(m-1)!!}{m!!}, & m \text{ is odd.}\end{cases}$$
*Proof.*By partial integral, we have
$$\begin{aligned} J_m &= -\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos x d\, x \\ &= sin^{m-1} x \cos x|_0^{\frac{\pi}{2}} - (m-1)\int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^2 x d\, x \\ &= (m-1)J_{m-2} - (m-1)J_m, \end{aligned}$$
and so,
$$J_m = \frac{m-1}{m} J_{m-1}.$$
□

2. To show the Wallis Formula
$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{1}{2n+1}[\frac{(2n)!!}{(2n-1)!!}]^2.$$
*Proof.*From
$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x d\, x \le \int_0^{\frac{\pi}{2}} \sin^{2n} x d\, x \int_0^{\frac{\pi}{2}} \sin^{2n-1} x d\, x,$$
we have
$$\frac{(2n)!!}{(2n+1)!!} \le \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2} \le \frac{(2n-2)!!}{(2n-1)!!}$$
$$\frac{1}{2n+1}[\frac{(2n)!!}{(2n-1)!!}]^2 \le \frac{\pi}{2} \le \frac{1}{2n}[\frac{(2n)!!}{(2n-1)!!}]^2 \tag{4.1.1}$$
But by the above inequality, we have
$$\frac{1}{2n}[\frac{(2n)!!}{(2n-1)!!}]^2 - \frac{1}{2n+1}[\frac{(2n)!!}{(2n-1)!!}]^2 = \frac{1}{2n(2n+1)}[\frac{(2n)!!}{(2n-1)!!}]^2 \le \frac{\pi}{4n} \to 0\ (n\to\infty).$$
Therefore,
$$\lim_{n\to\infty} \frac{1}{2n+1}[\frac{(2n)!!}{(2n-1)!!}]^2 = \lim_{n\to\infty} \frac{1}{2n}[\frac{(2n)!!}{(2n-1)!!}]^2 = \frac{\pi}{2}.$$
□

4.2. (Stirling Formula).

1. To show

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+\gamma+O(\frac{1}{n}),$$

where γ is the EuLer constant.

*Proof.*Since

$$\log(1+\frac{1}{k})=\frac{1}{k}+C_k$$

where $C_k=O(\frac{1}{k^2})$. we have

$$\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=1}^{n}\log(1+\frac{1}{k})+\sum_{k=1}^{n}C_k=\sum_{k=1}^{n}\log(1+\frac{1}{k})+\sum_{k=1}^{\infty}C_k-\sum_{k=n+1}^{\infty}C_k=\log n+\gamma+O(\frac{1}{n}).$$

□

2. To show that

$$\log n!=n\log n-n+\frac{1}{2}\log n+C+O(\frac{1}{n}),\tag{4.2.1}$$

where C is a constant, i.e.,

$$n!\sim Ae^{-n}n^{n+\frac{1}{2}}.$$

Proof.

$$\log n!=\log n\sum_{k=1}^{n-1}=\log n+\int_1^n\log t dt-\sum_{k=1}^{n-1}\int_k^{k+1}\log\frac{t}{k}dt=(n+1)\log n-n+1-\sum_{k=1}^{n-1}\int_0^1\log\frac{t+k}{k}dt$$

$$\log n!=(n+1)\log n-n+1-\sum_{k=1}^{n-1}\int_0^1\log(1+\frac{t}{k})dt=(n+1)\log n-n+1-\sum_{k=1}^{n-1}\int_0^1(\frac{t}{k}+O(\frac{t^2}{k^2}))dt$$

$$\log n!=(n+1)\log n-n+1-\frac{1}{2}(\sum_{k=1}^{n-1}(\frac{1}{k}+O(\frac{1}{k^2})))$$

Since

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+\gamma+O(\frac{1}{n}),$$

where γ is the Euler constant, we have:

$$\log n!=n\log n-n+\frac{1}{2}\log n+C+O(\frac{1}{n}),$$

i.e.,

$$n!=Ae^{-n}n^{n+\frac{1}{2}}(1+O(\frac{1}{n})).$$

□

3. To show

$$\frac{1}{n+\frac{1}{2}}<\log(1+\frac{1}{n})<\frac{1}{2}(\frac{1}{n+1}+\frac{1}{n}).$$

*Proof.*The right inequality is clear by the picture of area and the convex of picture of

$$y = f(x) = 1/x.$$

The left inequality is equivalent to the following inequality :

$$(n+\frac{1}{2})\log(1+\frac{1}{n})-1>0.$$

If $n=1$, it is obvious. If $n\geq 2$, we consider the following formula:

$$\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\ (|x|<1).$$

Thus,for $n \gg 0$ we have

$$(n+\frac{1}{2})\log(1+\frac{1}{n})-1=(n+\frac{1}{2})(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+O(\frac{1}{n^4}))-1=\frac{1}{12n^2}+O(\frac{1}{n^3})>0.$$

The equality is true for all n, but we only need the inequality for $n \gg 0$ to prove the Stirling formula. □

4. Use the above inequality to give another proof of the following formula:

$$n! \sim Ae^{-n}n^{n+\frac{1}{2}}\ (n\to\infty).$$

*Proof.*Let

$$a_n=\frac{n!}{e^{-n}n^{n+\frac{1}{2}}}.$$

Then we have:

$$\log\frac{a_n}{a_{n+1}}=(n+\frac{1}{2})\log(1+\frac{1}{n})-1.$$

From the the above inequality, we have

$$0<\log\frac{a_n}{a_{n+1}}<\frac{1}{4}(\frac{1}{n}-\frac{1}{n+1}),$$

thus $\{a_n\}$ is a strictly decreasing sequence and $\{b_n\}$ is a strictly decreasing sequence where $b_n = a_n e^{-\frac{1}{4n}}$. Moreover,

$$0<a_n-b_n=a_n(1-e^{-\frac{1}{4n}})\leq a_1(1-e^{-\frac{1}{4n}})\to 0\ (n\to\infty).$$

Therefore, there exists a finite constant A such that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=A.$$

□

5. Use the Wallis Formula to show that the constant A is exact $\sqrt{2\pi}$.

*Proof.*Let $n\in\mathbb{N}$. If n is even, we denote

$$n!!=n(n-2)(n-4)\cdots 4\cdot 2.$$

If n is odd, we denote

$$n!!=n(n-2)(n-4)\cdots 3\cdot 1.$$

The Wallis Formula says that

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{1}{2n+1}[\frac{(2n)!!}{(2n-1)!!}]^2.$$

Thus

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{1}{2n+1}\{\frac{[(2n)!!]^2}{(2n)!}\}^2 = \lim_{n\to\infty} \frac{1}{2n+1}\{\frac{(2^n n!)^2}{(2n)!}\}^2 = \lim_{n\to\infty} \frac{1}{2n+1}\{\frac{2^{2n}(n!)^2}{(2n)!}\}^2.$$

on the other hand, we have

$$n! \sim Ae^{-n}n^{n+\frac{1}{2}} \text{ and } (2n)! \sim Ae^{-2n}(2n)^{2n+\frac{1}{2}} \ (n\to\infty)$$

Thus

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{1}{2n+1}\{\frac{2^{2n}A^2 n^{2n+1}e^{-2n}}{A(2n)^{2n+1/2}e^{-2n}}\}^2 = \lim_{n\to\infty} \frac{1}{2n+1}(\sqrt{\frac{n}{2}}A)^2 = \frac{A^2}{4}.$$

At all, we obtain

$$A = \sqrt{2\pi},$$

and so

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n(1+O(\frac{1}{n})) \tag{4.2.2}$$

□

5. Topic on the Gamma functions

5.1. Γ functions.

5.1. Definition and basic properties. Refer to the textbook. The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ x > 0. \tag{5.1.1}$$

The Gamma function has the following properties:

1. $\Gamma(x) \in C^\infty((0,\infty))$, *moreover*

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}t^{z-1}(\log t)^n dt.$$

2.

$$\Gamma(x+1) = x\Gamma(x),$$

so that

$$\lim_{x\to 0+} \Gamma(s) = +\infty$$

and for any $n \in \mathbb{N}$,

$$\Gamma(n) = n!.$$

3. *For any* $\alpha, \beta > 0$,

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^\infty \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}dt = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy. \tag{5.1.2}$$

so that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \ 0 < x < 1 \tag{5.1.3}$$

and

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

*Proof.*Put $y = xt$ in the formula 5.1.1, there is

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty x^{\alpha-1}e^{-x}dx\int_0^\infty y^{\beta-1}e^{-y}dy \\
&= \int_0^\infty x^{\alpha-1}e^{-x}dx\int_0^\infty x^{\beta}t^{\beta-1}e^{-xt}dt \\
&= \int_0^\infty x^{\alpha+\beta-1}e^{-x}dx\int_0^\infty t^{\beta-1}e^{-xt}dt \\
&= \int_0^\infty t^{\beta-1}dt\int_0^\infty x^{\alpha+\beta-1}e^{-x(1+t)}dx.
\end{aligned}$$

Put back $y = x(1+t)$ in the above formula, there holds

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty t^{\beta-1}dt\int_0^\infty x^{\alpha+\beta-1}e^{-x(1+t)}dx \\
&= \int_0^\infty t^{\beta-1}dt\int_0^\infty \frac{y^{\alpha+\beta-1}e^{-y}}{(1+t)^{\alpha+\beta}}dy \\
&= \Gamma(\alpha+\beta)\int_0^\infty \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}dt.
\end{aligned}$$

Put $t = \tan^2\theta$, we obtain

$$\int_0^\infty \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}dt = 2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\beta-1}(\cos\theta)^{2\alpha-1}d\theta = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy.$$

□

4. **Legendre formula**

$$2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \Gamma(\frac{1}{2})\Gamma(2x). \tag{5.1.4}$$

*Proof.*Put $\alpha = \beta = x$ into the formula 5.1.2, there holds

$$\frac{(\Gamma(x))^2}{\Gamma(2x)} = \int_0^1 y^{x-1}(1-y)^{x-1}dy = 2\int_0^{1/2} y^{x-1}(1-y)^{x-1}dy.$$

Let $y = (1-\sqrt{t})/2$, we the obtain

$$\begin{aligned}
2\int_0^{1/2} y^{x-1}(1-y)^{x-1}dy &= \frac{1}{2}\int_0^1 (\frac{1-t}{4})^{x-1}t^{\frac{1}{2}}dt \\
&= 2^{1-2x}\int_0^1 (1-t)^{x-1}t^{\frac{1}{2}}dt \\
&= 2^{1-2x}\frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}.
\end{aligned}$$

□

Lemma 5.2. *Let a be any constant, then there holds*

$$\frac{\Gamma(x)}{\Gamma(x+a)} = x^{-a} + O(x^{-a-1}). \tag{5.2.1}$$

Proof. If $a < 1$ then there is a $k \in \mathbb{N}$ with $a + k > 1$, and so

$$\frac{\Gamma(x)}{\Gamma(x+a)} = \frac{\Gamma(x)}{\Gamma(x+a+k)} \prod_{i=1}^{k} (x+a+i).$$

Thus we reduce the question to the case $a > 1$. Now we suppose that $a > 1$. From 5.1.2, there is

$$\frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} = \int_0^1 y^{x-1}(1-y)^{a-1} dy = \int_0^\infty (1-e^{-t})^{a-1} e^{-xt} dt = I_1 + I_2.$$

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{\sqrt{x}}} (1-e^{-t})^{a-1} e^{-xt} dt \\
&= \int_0^{\frac{1}{\sqrt{x}}} (t+O(t^2))^{a-1} e^{-xt} dt \\
&= \int_0^{\frac{1}{\sqrt{x}}} (t)^{a-1}(1+O(t)) e^{-xt} dt \\
&= \int_0^{\frac{1}{\sqrt{x}}} (t)^{a-1} e^{-xt} dt + O(\int_0^{\frac{1}{\sqrt{x}}} (t)^{a} e^{-xt} dt) \\
&= x^{-a} \int_0^{\sqrt{x}} (t)^{a-1} e^{-t} dt + O(x^{-a-1} \int_0^{\sqrt{x}} (t)^{a-1} e^{-t} dt) \\
&= x^{-a}\Gamma(a) + O(x^{-a-1}).
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{\frac{1}{\sqrt{x}}}^\infty (1-e^{-t})^{a-1} e^{-xt} dt \\
&= O(\int_{\frac{1}{\sqrt{x}}}^\infty e^{-xt} dt) \\
&= O(x^{-a-1})
\end{aligned}$$

At all , we obtain

$$\frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} = x^{-a}\Gamma(a) + O(x^{-a-1}),$$

i.e.,

$$\frac{\Gamma(x)}{\Gamma(x+a)} = x^{-a} + O(x^{-a-1}).$$

□

Theorem 5.3 (Stirling's Formula)**.**

(5.3.1) $$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}(1 + O(\frac{1}{x})), \ x > 0.$$

In particular, if $x = n \in \mathbb{N}$, there is the Stirling formula:

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n (1 + O(\frac{1}{n})).$$

Proof. The formula 9.16.3 says:

$$\log \Gamma(n) = n \log n - n + \frac{1}{2} \log n + C + O(\frac{1}{n}),$$

where C is a constant. Let $x = n + a$ where $n \in \mathbb{N}$ and $0 < a < 1$. The formula 5.2.1 says that

$$\log\Gamma(x) = \log\Gamma(n+a) = \log\Gamma(n) + a\log n + O(\frac{1}{n}).$$

The formula 9.16.3 says:

$$\log\Gamma(n) = n\log n - n + \frac{1}{2}\log n + C + O(\frac{1}{n}),$$

where C is a constant, thus we obtain

$$\begin{aligned}
\log\Gamma(x) &= (n-\frac{1}{2})\log n - n + C + a\log n + O(\frac{1}{n}) \\
&= (x-a-\frac{1}{2})\log(x-a)n - x + a + C + a\log(x-a) + O(\frac{1}{x}) \\
&= (x-\frac{1}{2})\log x - x + C + O(\frac{1}{x})
\end{aligned}$$

On the other hand, the Legendre formula says

$$(2x-1)\log 2 + \log\Gamma(x) + \log\Gamma(x+\frac{1}{2}) = \log\Gamma(\frac{1}{2}) + \log\Gamma(2x).$$

We then have
(5.3.2)

$$(2x-\frac{1}{2})\log 2x - 2x + C + \log\Gamma(\frac{1}{2}) + O(\frac{1}{x}) = (2x-1)\log 2 + (x-\frac{1}{2})\log x + x\log(x+\frac{1}{2}) - 2x - \frac{1}{2} + 2C + O(\frac{1}{x})$$

Therefore,

$$C = \log\sqrt{2\pi}.$$

□

Theorem 5.4 (Euler Product Formula)**.**

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}(1+\frac{1}{n})^s(1+\frac{s}{n})^{-1},\ s>0 \tag{5.4.1}$$

Proof. 1. First, we show that

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt = \lim_{n\to\infty}\int_0^n t^{s-1}(1-\frac{t}{n})^n dt \tag{5.4.2}$$

Let

$$\int_0^n t^{s-1}(1-\frac{t}{n})^n dt = \int_0^{n^{\frac{1}{3}}} t^{s-1}(1-\frac{t}{n})^n dt + \int_{n^{\frac{1}{3}}}^n t^{s-1}(1-\frac{t}{n})^n dt = I_1 + I_2.$$

$$\begin{aligned} I_1 &= \int_0^{n^{\frac{1}{3}}} t^{s-1}(1-\frac{t}{n})^n dt \\ &= \int_0^{n^{\frac{1}{3}}} t^{s-1}e^{n\log n(1-\frac{t}{n})}dt \\ &= \int_0^{n^{\frac{1}{3}}} t^{s-1}e^{-t+o(n^{\frac{1}{3}})}dt \\ &= \int_0^{n^{\frac{1}{3}}} t^{s-1}e^{-t}dt + O(n^{\frac{1}{3}}) \\ &= \Gamma(s)+O(n^{\frac{1}{3}}). \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{n^{\frac{1}{3}}}^n t^{s-1}(1-\frac{t}{n})^n dt = I_1+I_2. \\ &= O(\int_{n^{\frac{1}{3}}}^n t^{s-1}(1-\frac{n^{\frac{1}{3}}}{n})^n dt) \\ &= O(\frac{n^s}{s}e^{-n^{\frac{1}{3}}}). \end{aligned}$$

2. We then show that

$$\int_0^n t^{s-1}(1-\frac{t}{n})^n dt == \frac{(1+\frac{s}{n})^{-1}}{s}\prod_{k=1}^{n-1}(1+\frac{1}{k})^s(1+\frac{s}{k})^{-1}. \tag{5.4.3}$$

$$\int_0^n t^{s-1}(1-\frac{t}{n})^n dt = n^s\int_0^1(1-t)^n t^{s-1}dt = \frac{n^s}{s}\int_0^1(1-t)^n dt^s = n^s\frac{n}{s}\int_0^1(1-t)^{n-1}t^s dt.$$

By induction, we obtain:

$$\int_0^n t^{s-1}(1-\frac{t}{n})^n dt = n^s\frac{n(n-1)\cdots 1}{s(s+1)\cdots(s+n-1)}\int_0^1 t^{s+n-1}dt = n^s\frac{n(n-1)\cdots 1}{s(s+1)\cdots(s+n)},$$

so

$$\int_0^n t^{s-1}(1-\frac{t}{n})^n dt = \frac{(1+\frac{s}{n})^{-1}}{s}\prod_{k=1}^{n-1}(1+\frac{1}{k})^s(1+\frac{s}{k})^{-1}.$$

□

To be Continuous .....

5.2. **Exercises.**

Exercise 5.5. To show that for $n = 1, 2, 3, \cdots$, there holds

$$(xe^{2(x-n)})^n = O(e^{(x^2+x)}), x > 0$$

Proof. Consider the function

$$f_n(x) = xe^{2(x-n)})^n - e^{(x^2+x)}.$$

Then

$$f_n'(x) = e^{2n(x-n)-x^2-x}(nx^{n-1}+x^n(-2x+2n-1)),$$

and $f_n'(n) = 0$,$f_n(x)$ gets the maximal value at $x = n$,

$$f_n(n) = e^{n\log n-n^2-n} = O(1).$$

□

Exercise 5.6. For any $\delta > 0$, to show that

$$\log x = o(x^\delta).$$

Exercise 5.7. Define

$$\mathrm{Li}x = \lim_{\eta \to 0} \left(\int_0^{1-\eta} + \int_{1-\eta}^{x} \right) \frac{dt}{\log t}.$$

To show that

$$\mathrm{Li}x \sim \frac{x}{\log x}.$$

Exercise 5.8. To show that

$$\int_0^1 (\log \frac{1}{t})^n dt = n!.$$

Exercise 5.9. Denote γ be the Euler constant. For any $x > 0$, show that

$$\sum_{n \in \mathbb{N}, n \leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}).$$

Exercise 5.10. To show that

$$-\Gamma'(1) = \lim_{n \to \infty} \left(\int_0^1 \frac{1 - y^n}{1 - y} dy - \log n \right) = \lim_{n \to \infty} \{ \sum_{i=1}^{n} \frac{1}{i} - \log n \} = \gamma.$$

6. Working Technique in function theory

6.1. Iteration technique.

Example 6.1 (De Giorgi-Nash iteration)**.** Let $\varphi(t)$ be a nonnegative function decreasing on $[k_0, \infty]$. Assume that φ satisfies

$$\varphi(h) \leq (\frac{M}{h-k})^{\alpha}[\varphi(k)]^{\beta},\ \forall h > k \geq k_0, \tag{6.1.1}$$

where $M, \alpha > 0, \beta > 1$. Then

$$\varphi(k_0 + d) = 0$$

where

$$d = M[\varphi(k_0)]^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}.$$

Proof. Define sequence $\{k_n\}$ by the following rule

$$k_n = k_0 + d - \frac{d}{2^n},\ n = 0, 1, 2, \cdots .$$

Then $\{\varphi(k_n)\}$ is a decreasing sequence. From the rule 6.2.1, we have

$$\varphi(k_{n+1}) \leq \frac{M^{\alpha}2^{(n+1)\alpha}}{d^{\alpha}}[\varphi(k_n)]^{\beta}, n = 0, 1, 2, \cdots .$$

If we can show that

$$\varphi(k_n) \leq \frac{\varphi(k_0)}{r^n} \text{ for some } r > 1,\ n = 0, 1, 2, \cdots , \tag{6.1.2}$$

where r is under-determined. Then, we have

$$0 \leq \varphi(k_0 + d) \leq \varphi(k_n) \leq \frac{\varphi(k_0)}{r^n},\ n = 0, 1, 2, \cdots .$$

Take the limit, we then obtain the result.

Now, we choose the number r an prove the inequality 6.1.2 by using induction : if the inequality is true for n, then, we have

$$\varphi(k_{n+1}) \leq \frac{M^{\alpha}2^{(n+1)\alpha}}{d^{\alpha}}[\varphi(k_n)]^{\beta} \leq \frac{\varphi(k_0)}{r^{n+1}} \frac{M^{\alpha}2^{(n+1)\alpha}}{d^{\alpha}r^{n(\beta-1)-1}}[\varphi(k_0)]^{\beta-1}.$$

Choose $r \geq 2^{\alpha/(\beta-1)}$, we have

$$\frac{M^{\alpha}2^{(n+1)\alpha}}{d^{\alpha}r^{n(\beta-1)-1}}[\varphi(k_0)]^{\beta-1} \leq \frac{M^{\alpha}2^{\alpha\beta/(\beta-1)}}{d^{\alpha}}[\varphi(k_0)]^{\beta-1} \leq 1.$$

□

Example 6.2. Let $\varphi(t)$ be a nonnegative function increasing on $[0, R_0]$. Assume that for some $\theta, \eta \in [0, 1)$ and $\gamma \in (0, 1], K \geq 0$, φ satisfies

$$\varphi(\theta R) \leq \eta\varphi(R) + KR^{\gamma},\ R \in (0, R_0]. \tag{6.2.1}$$

Then, we have

$$\varphi(R) \leq C(\frac{R}{R_0})^{\alpha}[\varphi(R_0) + KR_0^{\gamma}],\ \ R \in (0, R_0], \tag{6.2.2}$$

where $\alpha = \alpha(\theta, \eta, \gamma) \in (0, \gamma)$ and $C = C(\theta, \eta, \gamma) > 0$ are constant.

Proof. Assume that $\theta^{-\alpha}\eta > 1$. Let $\widetilde{R}_i = \theta^i \widetilde{R}_0,\ i = 0, 1, 2, \cdots$. From 6.2.1, we have

$$\varphi(\widetilde{R}_{i+1}) \le \eta^{\varphi}(\widetilde{R}_i) + K\widetilde{R}_i^{\gamma},$$

and by iteration, we have

$$\begin{aligned}
\varphi(\widetilde{R}_i) &\le \eta^i\varphi(\widetilde{R}_0) + \sum_{m=0}^{i-1} K\eta^m \widetilde{R}_{i-m-1}^{\gamma}, \\
&\le \eta^i\varphi(\widetilde{R}_0) + K\widetilde{R}_0^{\gamma}\theta^{\gamma(i-1)}\sum_{m=0}^{i-1}(\theta^{-\gamma}\eta)^m \\
&= \eta^i\varphi(\widetilde{R}_0) + K\widetilde{R}_0^{\gamma}\theta^{\gamma(i-1)}\frac{(\theta^{-\gamma}\eta)^i - 1}{\theta^{-\gamma}\eta - 1} \\
&\le \eta^i\varphi(\widetilde{R}_0) + K\widetilde{R}_0^{\gamma}\theta^{\gamma(i-1)}\frac{(\theta^{-\gamma}\eta)^i}{\theta^{-\gamma}\eta - 1} \\
&= \eta^i[\varphi(\widetilde{R}_0) + C_1K\widetilde{R}_0^{\gamma}],
\end{aligned}$$

where

$$C_1 = \frac{\theta^{-\gamma}}{\theta^{-\gamma}\eta - 1}.$$

Since

$$i = \frac{\log(\frac{\widetilde{R}_i}{\widetilde{R}_0})}{\log\theta},$$

let $\alpha = \log\eta/\log\theta \in (o, \gamma)$, we have

$$\varphi(\widetilde{R}_i) \le (\frac{\widetilde{R}_i}{\widetilde{R}_0})^{\alpha}[\varphi(\widetilde{R}_0) + C_1K\widetilde{R}_0^{\gamma}],\ i = 0, 1, 2\cdots.,$$

When $\widetilde{R}_0 \in (\theta R_0, R_0]$, let $C_2 = \max(C_1, 1)$, we have

$$\varphi(\widetilde{R}_i) \le \frac{C_1}{\theta^{\alpha}}(\frac{\widetilde{R}_i}{R_0})^{\alpha}[\varphi(R_0) + KR_0^{\gamma}],\ i = 0, 1, 2\cdots.,$$

On the other hand, for any $R \in [0, R_0]$, there are $i \in \mathbb{N}$, $\widetilde{R}_0 \in (\theta R_0, R_0]$ such that $R = \widetilde{R}_i$. Let $C = \frac{C_1}{\theta^{\alpha}}$, we then have the inequality 6.2.2.

□

Example 6.3. Let $0 \le T_0 < T_1$, $\varphi(t)$ be a nonnegative bounded function on $[T_0, T_1]$. Assume that for some t, s with $0 \le T_0 \le t < s \le T_1$, φ satisfies

$$\varphi(t) \le \theta\varphi(s) + \frac{A}{(s-t)^{\alpha}} + B, \tag{6.3.1}$$

where θ, A, B, α are non-negative constant and $\theta < 1$. Then, we have

$$\varphi(\rho) \le C(\frac{A}{(R-\rho)^{\alpha}} + B),\ \forall T_0 \le \rho < R \le T_1, \tag{6.3.2}$$

where $C = C(\alpha, \theta) > 0$ is a constant.

Proof. Let $T_0 \le \rho < R \le T_1$. Define

$$t_0 = \rho,\ t_{i+1} = t_i + (1-\tau)\tau^i(R-\rho)\ (i = 0, 1, 2, \cdots),$$

where $\tau \in (0,1)$ is under-determined. By the relation 6.3.2, we have

$$\varphi(t_i) \le \theta\varphi(t_{i+1}) + \frac{A}{((1-\tau)\tau^i(R-\rho))^\alpha} + B,\ i = 0, 1, \cdots.$$

by induction, for any $k \ge 1$ we have

$$\phi(t_0) \le \theta^k\varphi(t_k) + (\frac{A}{(1-\tau)^\alpha(R-\rho)^\alpha} + B)\sum_{i=0}^{k-1}(\theta\tau^{-\alpha})^i.$$

Choose τ with $\theta\tau^{-\alpha} < 1$, and let $k \to \infty$, we then obtain the inequality 6.3.2.

□

Example 6.4. Let $0 \le T_0 < T_1$, $\varphi(t)$ be a nonnegative function increasing on $[0, R_0]$. Assume that for some constants β, α with $0 < \beta < \alpha$, φ satisfies

(6.4.1) $$\varphi(\rho) \le C(\frac{\rho}{R})^\alpha[\varphi(R) + BR^\beta],\ 0 < \rho < R \le R_0.$$

where C is a constant. Then, we have

(6.4.2) $$\varphi(\rho) \le C(\frac{\rho}{R})^\beta[\varphi(R) + BR^\beta],\ 0 < \rho < R \le R_0,$$

where $C = C(A, \alpha, \beta) > 0$ is a constant.

Proof. Let $\nu = \frac{1}{2}(\alpha + \beta)$. Choose $\tau \in (0,1)$ such that $A\tau^{\alpha-\nu} < 1$. Then, we have

$$\varphi(\tau R) \le A\tau^\alpha\varphi(R) + BR^\beta = A\tau^{\alpha-\nu}\tau^\nu\varphi(R) + BR^\beta \le \tau^\nu\varphi(R) + BR^\beta,$$

and

$$\begin{aligned}
\varphi(\tau^{k+1}R) &= \tau^{(k+1)\nu}\varphi(R) + B(\tau^{k\nu} + \tau^{(k-1)\nu+\beta} + \cdots + \tau^{k\beta})R^\beta \\
&= \tau^{(k+1)\nu}\varphi(R) + B\tau^{k\beta}(\tau^{k(\nu-\beta)} + \tau^{(k-1)(\nu-\beta)} + \cdots + 1)R^\beta \\
&= \tau^{(k+1)\nu}\varphi(R) + B\frac{\tau^{k\beta}(1 - \tau^{(k+1)(\nu-\beta)}}{1 - \tau^{(\nu-\beta)}}R^\beta \\
&\le C_1\tau^{k\beta}[\varphi(R) + BR^\beta],
\end{aligned}$$

Where $C_1 \ge 1$ is a constant independent on k. Thus,

(6.4.3) $$\varphi(\tau^{k+1}R) \le C_1\tau^{(k-1)\beta}[\varphi(R) + BR^\beta],\ k \ge 0.$$

for any ρ with $0 < \rho < R \le R_0$, we choose any positive integer k such that

$$\tau^{k+1}R < \rho \le \tau^k R,$$

then we have

$$\begin{aligned}
\varphi(\rho) &\le \varphi(\tau^k R) \le C_1\tau^{(k-1)\beta}[\varphi(R) + BR^\beta] \\
&\le C_1\tau^{-2\beta}(\frac{\rho}{R})^\beta[\varphi(R) + BR^\beta] \\
&= C(\frac{\rho}{R})^\beta[\varphi(R) + BR^\beta],
\end{aligned}$$

where $C = C_1\tau^{-2\beta}$.

□

6.2. **Exercises and Homework.**

Exercise 6.5. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. define

$$f^{(n)}(x)=\underbrace{f\circ f\circ\cdots\circ f}_{n}.$$

Assume that there exist $a, C\in\mathbb{R}$ with $C>o$ such that for all $n\in\mathbb{N}$,

$$|f^{(n)}(a)|<C.$$

Then, the function f has a fixed point, i.e., $\exists x_0\in\mathbb{R}, f(x_0)=x_0$.

Proof. Let $g(x)=f(x)-x$. We can assume

$$g(x)>0, \forall x\in\mathbb{R} \text{ or } g(x)>0, \forall x\in\mathbb{R}.$$

Otherwise, by the Cauchy mean value theorem we will have ar least a zero point of g.

a) If

$$g(x)>0, \forall x\in\mathbb{R}$$

we will get a bounded increasing $\{x_n\}$ defined by

$$x_0=a,\ x_{n+1}=f(x_n),\ n=1,2\cdots.$$

Since $A=\lim_{n\to\infty}x_n$ exists, by f is continuous we then have

$$A=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}f(x_n)=f(\lim_{n\to\infty}x_n)=f(A).$$

b) If

$$g(x)<0, \forall x\in\mathbb{R}$$

we will get a bounded decreasing $\{y_n\}$ defined by

$$y_0=a,\ y_{n+1}=f(y_n),\ n=1,2\cdots.$$

Similarly, $B=\lim_{n\to\infty}y_n$ exists and is a fixed point of f.

□

Exercise 6.6. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying

$$f(2x^2-1)=2xf(x),\ \forall x ı\mathbb{R}.$$

To show that

$$f(x)\equiv 0, \forall x\in[-1,1].$$

Proof. We have $f(x)=-f(-x), \forall x\neq 0$. By the continuity of f, $f(0)=-f(0)$, and so $f(0)=0$. Now, for any θ, we have

$$f(\cos 2\theta)=2\cos\theta f(\cos\theta),$$

and so

$$f(\cos\theta)=0\Longleftrightarrow f(\cos 2\theta)=0.$$

Again the continuity of f, for any θ, we have

$$\frac{f(\cos 2\theta)}{\sin 2\theta}=\frac{f(\cos\theta)}{\sin\theta}.$$

On the other hand,

$$f(-\cos 2\theta)=f(2\sin^2\theta-1)=2\sin\theta f(\sin\theta),$$

then

$$\cos\theta f(\cos\theta)=\sin\theta f(\sin\theta),$$

and so

$$\frac{f(\sin\theta)}{\cos\theta} = \frac{f(\cos\theta)}{\sin\theta}.$$

Therefore, for any $t = \sin\xi \in [-1,1]$, we have

$$\frac{f(\sin\xi)}{\cos\xi} = \frac{f(\sin\frac{\xi}{2})}{\cos\frac{\xi}{2}} = \cdots = \frac{f(\sin\frac{\xi}{2^n})}{\cos\frac{\xi}{2^n}} \to \frac{f(0)}{1} = 0 \ (n\to\infty).$$

□

Exercise 6.7. Let $f(x), g(x)$ be functions defined on $(a,+\infty)$. Suppose that $f(x), g(x)$ satisfy the following conditions :

a) $f(x), g(x)$ are bounded on any finite interval (a,b);

b)

$$g(x+1) > g(x),\ \forall x \in (a,+\infty);$$

c)

$$\lim_{x\to+\infty} g(x) = +\infty.$$

If

$$\lim_{x\to+\infty} \frac{f(x+1)-f(x)}{g(x+1)-g(x)} = l,$$

where $l \in [-\infty,+\infty]$. Then, we have

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = l.$$

Hint. Refer to the proof of the Stolz theorem.

□

Exercise 6.8. (1) Show that the set of all irrational number of $\mathbb{R}$ is uncountable.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(\mathbb{Q}) \subset \mathbb{R}\backslash\mathbb{Q}, \text{ and } f(\mathbb{R}\backslash\mathbb{Q}) \subset \mathbb{Q}.$$

To show that f is not a continuous function, and show an example.

7. Applications of Differential

Example 7.1 (Liouville)**.** Suppose $\alpha \in \mathbb{R}$ is an algebraic integer of degree $d > 0$, i.e, α is a root of a polynomial with integer coefficients

$$f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d, \ \forall a_i \in \mathbb{Z},$$

which is irreducible in $\mathbb{Q}[x]$. Then there exists a constant $c = C(\alpha) > 0$ such that for each pair of integers (p, q), there holds

$$|\alpha - \frac{p}{q}| \geq \frac{c}{|q|^d}.$$

Proof. For a given pair (p, q), if $|\alpha - \frac{p}{q}| > 1$ then let $c = 1$. Thus,We suppose $|\alpha - \frac{p}{q}| \leq 1$. We have:

$$f(\frac{p}{q}) = f(\frac{p}{q}) - f(\alpha) = f'(\zeta)(\frac{p}{q} - \alpha), \text{ for some } \zeta \in [-|\alpha| - 1, |\alpha| + 1].$$

Let M be the maximal value of $|f'(x)|$ on $[|\alpha| - 1, |\alpha| + 1]$, and let $c = 1/M$.

Since $f(\frac{p}{q}) \neq 0$ for any national number $\frac{p}{q}$ (we omit the proof here), so

$$|\frac{p}{q} - \alpha| \geq \frac{|f'(\zeta)||\frac{p}{q} - \alpha|}{M} = c|f(\frac{p}{q})| = c\frac{|\sum_{i=0}^{d} a_i p^{d-i} q^i|}{|q|^d} \geq c \cdot \frac{1}{|q|^d}.$$

□

Exercise 7.2. For any $n \in \mathbb{N}$, define

$$P_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

and denote $F_n = P_{n+1}(x)P_n(x)$. To show that $F_n(x)$ has a unique root in $\mathbb{R}$.

Hint. Sufficiently to prove that if n is even then $P_n(x)$ has no root in $\mathbb{R}$, and if n is odd then $P_n(x)$ has a unique root in $\mathbb{R}$. □

To be continuous......

8. Treasures in Calculus

8.1. **Euler's identification** $\zeta(2) = 1 + \frac{1}{2^2} \cdots + \frac{1}{n^2} + \cdots \equiv \frac{\pi^2}{6}$**.** At first, there holds

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy &= \int_0^1 \int_0^1 \sum_{i \geq 0} x^i y^i dxdy \\
&= \sum_{i \geq 0} \int_0^1 \int_0^1 x^i y^i dxdy \\
&= \sum_{i \geq 0} \int_0^1 x^i dx \int_0^1 y^i dy \\
&= \sum_{i \geq 0} \frac{1}{(i+1)^2} \\
&= \zeta(2).
\end{aligned}$$

On the other hand, replacing x^2 with X and y^2 with Y, there holds

$$\begin{aligned}
\int_0^1 \int_0^1 (\frac{1}{1-xy} - \frac{1}{1+xy}) dxdy &= \int_0^1 \int_0^1 (\frac{2xy}{1-x^2y^2}) dxdy \\
&= \frac{1}{2} \int_0^1 \int_0^1 \frac{1}{1-XY} dXdY \\
&= \frac{\zeta(2)}{2}
\end{aligned}$$

and let $x = \sin\theta / \cos\phi, y = \sin\phi / \cos\theta$, we obtain

$$\begin{aligned}
\int_0^1 \int_0^1 (\frac{1}{1-xy} + \frac{1}{1+xy}) dxdy &= 2 \int_0^1 \int_0^1 (\frac{1}{1-x^2y^2}) dxdy \\
&= 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}-\theta} \frac{1}{1-(\frac{\sin\theta}{\cos\phi})^2 (\frac{\sin\phi}{\cos\theta})^2} (1 - \tan^2\theta \tan^2\phi) d\phi \\
&= \frac{\pi^2}{4}
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
2\zeta(2) &= \int_0^1 \int_0^1 (\frac{1}{1-xy} - \frac{1}{1+xy}) dxdy + int_0^1 \int_0^1 (\frac{1}{1-xy} + \frac{1}{1+xy}) dxdy \\
&= \frac{\zeta(2)}{2} + \frac{\pi^2}{4},
\end{aligned}$$

and so

$$\zeta(2) = 1 + \frac{1}{2^2} \cdots + \frac{1}{n^2} + \cdots \equiv \frac{\pi^2}{6} \tag{8.0.1}$$

8.2. **Irrationality of** $\pi, \log 2, \zeta(2), \zeta(3)$**.**

8.1. Legendre polynomial

1. For any $n \in \mathbb{N}$, the Legendre polynomial is define by

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n). \tag{8.1.1}$$

It is obvious all coefficients of P_n are integers, i.e.,

$$P_n = \sum_{j=0}^{n} p_{n,j}x^j, \ \forall p_{n,j} \in \mathbb{Z}.$$

For examples, $P_0(x) = 0, P_1(x) = 1 - 2x, P_2(x) = 2 - 12x - 12x^2$.

2. The following is an important property of Legendre polynomials:

$$\int_0^1 P_n(x)f(x)dx = (-1)^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n f(x)}{dx^n}dx. \tag{8.1.2}$$

3. **A technique to show a number** ξ **is irrational by using Legendre polynomials.**
   - Suppose there is a family of integrals such that ($j \in \mathbb{N}$)

   $$\int_0^1 x^j f(x) = R_j + S_j\xi,$$

   where $R_j, S_j \in \mathbb{Q}$ and $f(x)$ is an under-determined function.
   - Assume ξ is rational, then

   $$\int_0^1 P_n(x)f(x) = \frac{A_n}{B_n} \text{ with } A_n, B_n \in \mathbb{Z}, \forall n \in \mathbb{N}.$$

   - On the other hand, if $A_n \neq 0 n >> 0$ and

   $$|B_n(-1)^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n f(x)}{dx^n}dx| \longrightarrow 0 \ (n \to \infty)$$

   then there is a confliction:

   $$1 \leq |A_n| \longrightarrow 0 \ (n \to \infty),$$

   and so ξ is irrational.

8.2. π is irrational.

1. It is obvious for any $j \in \mathbb{N}$,

$$\int_0^1 x^j \sin(\pi x)dx = h(\frac{1}{\pi}),$$

where $h(x)$ is a polynomial in $\mathbb{Z}[x]$ with degree at most j.

2. Thus, if $\pi = a/b$ is rational then for any Legendre polynomial $P_n(x)$, there holds

$$\int_0^1 P_n(x)\sin(\pi x)dx = \frac{A_n}{a^n} \text{ with } A_n\mathbb{Z}.$$

3. On the other hand,by stirling formula, there is

$$\begin{aligned} |A_n| &= |a^n \int_0^1 P_n(x)\sin(\pi x)dx| \\ &= |a^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n}{dx^n}(\sin(\pi x))dx| \\ &= |a^n \int_0^1 \frac{1}{n!}x^n(1-x)^n\pi^n dx| \\ &\le |\frac{1}{n!}(\frac{a\pi}{4})^n| \longrightarrow 0(n\to\infty). \end{aligned}$$

8.3. $\log 2$ **is irrational.** In this case, we choose $f(x) = 1/(1+x)$.

1. Since $x^j = x^{j-1}(x+1) - x^{j-2}(x+1) + \cdots \mp (x+1) \pm 1$ for $j \in \mathbb{N}$, it is obvious that

$$\int_0^1 \frac{x^j}{1+x}dx = \frac{1}{j} - \frac{1}{j-1} + \cdots \mp 1 \pm \log 2 = C_j \pm \log 2,$$

where $C_j \in \mathbb{Q}$.

2. Thus, if $\log 2 = a/b$ is rational then for any Legendre polynomial $P_n(x)$, there holds

$$\int_0^1 P_n(x)\frac{1}{x+1}dx = \frac{A_n}{bd_n} \text{ with } A_n\mathbb{Z},$$

and $d_n = LCM(1,2,\cdots,n)$.

3. On the other hand,

$$\begin{aligned} |A_n| &= |bd_n \int_0^1 P_n(x)\frac{1}{(1+x)}dx| \\ &= |bd_n \int_0^1 \frac{1}{n!}x^n(1-x)^n[\frac{d^n}{dx^n}\frac{1}{x+1}]dx| \\ &= |bd_n \int_0^1 (\frac{x(1-x)}{1+x})^n \frac{1}{1+x}dx|. \end{aligned}$$

Since

$$\max_{x\in[0,1]}(\frac{x(1-x)}{1+x}) = 3-2\sqrt{2},$$

and $d_n \le 3^n$,

$$|A_n| \le |b|(3(3-2\sqrt{2}))^n \longrightarrow 0.$$

8.4. $\zeta(2)$ **is irrational.** In this case, we choose $f(x) = \int_0^1 \frac{(1-y)^n}{1-xy}dy$.

1. Consider the integral family($j \in \mathbb{N}$)

$$\int_0^1 x^j[\int_0^1 \frac{(1-y)^n}{1-xy}dy]dx = \sum_{r,s\in\mathbb{N}} p_{j;r,s}\int_0^1\int_0^1 \frac{x^ry^s}{1-xy}dydx, p_{j;r,s}\in\mathbb{Z},$$

By Euclean's division method, for any pair $(r,s) \in \mathbb{N}^2$, we have:

$$\begin{aligned} \int_0^1\int_0^1 \frac{x^ry^s}{1-xy}dydx &= \sum_{p,q\in\mathbb{N}} A_{r,s;p,q}\int_0^1\int_0^1 x^py^q dydx + \sum_{p\in\mathbb{N}} B_{r,s;p}\int_0^1\int_0^1 \frac{x^p}{1-xy}dydx \\ &+ \sum_{q\in\mathbb{N}} C_{r,s;q}\int_0^1\int_0^1 \frac{y^q}{1-xy}dydx + D_{r,s}\int_0^1\int_0^1 \frac{1}{1-xy}dydx. \end{aligned}$$

Therefore, we obtain

$$\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}dydx = \begin{cases} \frac{k_{r,s}}{d_{n+1}^2}, & r \neq s \\ \zeta(2) - \sum_{i=1}^r \frac{1}{i^2}, & r = s, \end{cases}$$

where each $k_{r,s}$ is an integer and $d_n = LCM(1, \cdots, n)$.

2. Suppose $\zeta(2) = a/b$ is rational, then or any Legendre polynomial $P_n(x)$,

$$|\int_0^1 P_n(x)f(x)dx| = \frac{|A_n|}{bd_{n+1}^2},\ A_n \in \mathbb{Z}.$$

3. On the other hand,

$$\begin{aligned} |A_n| &= |bd_{n+1}^2 \int_0^1 P_n(x)f(x)dx| \\ &= |bd_{n+1}^2 \int_0^1 \frac{1}{n!}x^n(1-x)^n[\frac{d^n}{dx^n}f(x)]dx| \\ &= |bd_{n+1}^2 \int_0^1\int_0^1 (\frac{x(1-x)y(1-y)}{1+xy})^n \frac{1}{1-xy}dydx|. \end{aligned}$$

Since

$$d_{n+1} \leq 3^{n+1},$$

and

$$\max_{(x,y)\in[0,1]\times[0,1]}(\frac{x(1-x)y(1-y)}{1-xy}) = 3-2\sqrt{2} = (\frac{-1+\sqrt{5}}{2})^5,$$

then

$$|A_n| \leq |9b\xi(2)|(9(\frac{-1+\sqrt{5}}{2})^5)^n \longrightarrow 0,$$

since

$$9(\frac{-1+\sqrt{5}}{2})^5 = \frac{45\sqrt{5}-99}{2} < 1.$$

Exercise 8.5. To show that

$$\zeta(3) = \sum_{i=1}^{\infty}\frac{1}{i^3}$$

is irrational.

Hint. Consider the function

$$f(x) = \int_0^1 \frac{P_n(y)}{1-xy}\log xydy,$$

and the family of integrals($j \in \mathbb{N}$)

$$\int_0^1 x^j f(x)dx.$$

Show that if $\zeta(3) = a/b$ is rational then

$$|\int_0^1 P_n(x)f(x)| = \frac{|A_n|}{|bd_{n+1}^3|}.$$

□

8.3. **The properties of Tchebychev's functions.** Denote $\mathbb{P}$ be the set of all prime numbers.

8.6. Tchebychev's functions

1. Define

$$\pi(x) = \sum_{p\in\mathbb{P}, p\le x} 1,$$

and

$$\theta(x) = \sum_{p\in\mathbb{P}, p\le x} \log p,\ \ x > 0.$$

Example. For any $n \in \mathbb{N}$, there holds

$$\sum_{p\le n, p\in\mathbb{P}} [\frac{n}{p}] \log p = n\log n + O(n). \tag{8.6.1}$$

*Proof.*At first,

$$n! = \prod_{p\le n, p\in\mathbb{P}} p^{\alpha_p},$$

where

$$\alpha_p = \sum_{i=1}^{\infty} [\frac{n}{p^i}].$$

Then,we obtain

$$\log n! = \sum_{p\le n, p\in\mathbb{P}} \alpha_p \log p = \sum_{p\le n, p\in\mathbb{P}} [\frac{n}{p}] \log p + \sum_{p\le n, p\in\mathbb{P}} \sum_{i=2}^{\infty} [\frac{n}{p^i}] \log p.$$

But

$$\begin{aligned}
\sum_{p\le n, p\in\mathbb{P}} \sum_{i=2}^{\infty} [\frac{n}{p^i}] \log p &\le n \sum_{p\le n, p\in\mathbb{P}} \sum_{i=2}^{\infty} \frac{1}{p^i} \log p \\
&\le n \sum_{p\le n, p\in\mathbb{P}} \frac{\log p}{p^2} \cdot \frac{p}{p-1} \\
&\le n \sum_{k=2}^{\infty} \frac{\log k}{k^2} \cdot \frac{k}{k-1} \\
&\le 2n \sum_{k=2}^{\infty} \frac{\log k}{k^2} \\
&= O(n).
\end{aligned}$$

On the other hand, the Stirling formula says that

$$\log n! = n\log n + O(n).$$

Comparing the above two formula, we obtain 8.6.1. □

Theorem. *Let $x \ge 2$, there is important relations between the functions $\theta(x)$ and $\pi(x)$:*

$$\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt \tag{8.6.2}$$

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} dt \tag{8.6.3}$$

The proof is dependent of the following lemma heavily :

Lemma (Abel identity)**.** *Let $a(n):\mathbb{N}\to\mathbb{C}$ be a arithmetic function, its sum function is*

$$A(x)=\sum_{n\le x}a(n),\ \ x\ge 1.$$

Let $f(t)$ is a differential function on $[y,x]$ where $y\in[1,x]$. Then, there holds

$$\sum_{y<n\le x}a(n)f(n)=A(x)f(x)-A(y)f(y)-\int_y^x A(t)f'(t)dt \tag{8.6.4}$$

*Proof.*We leave it as an exercise.

□

2. Define **Mangoldt** function on $\mathbb{N}$

$$\Lambda(n)=\begin{cases}\log p, & n=p^m, m\in\mathbb{N} \text{ for some } p\in\mathbb{P};\\ 0, & \text{others.}\end{cases}$$

Define

$$\psi(x)=\sum_{n\in\mathbb{N},n\le x}\Lambda(n),\ \ x>0.$$

Example. There holds $\psi(x)\sim\theta(x)$, moreover

$$\psi(x)=\theta(x)+O(\sqrt{x}\log x),\ (x\to\infty). \tag{8.6.5}$$

*Proof.*It is obvious that there holds

$$\psi(x)=\theta(x)+\sum_{i=2}^{\infty}\theta(x^{\frac{1}{i}}). \tag{8.6.6}$$

We note

$$0\le\theta(x^{\frac{1}{i}})\le x^{\frac{1}{i}}\log(x^{\frac{1}{i}}),$$

if $i\ge 2$; and

$$\theta(x^{\frac{1}{i}})=0$$

if $i>[\frac{\log x}{\log 2}]=M$. Thus

$$\theta(x)\le\psi(x)\le\theta(x)+\sum_{i=2}^{M}\theta(x^{\frac{1}{i}})\le\theta(x)+\sum_{i=2}^{M}x^{\frac{1}{i}}\log(x^{\frac{1}{i}})\le\theta(x)+Mx^{\frac{1}{2}}\log(x^{\frac{1}{2}}).$$

□

3. Moreover, we have

$$\overline{\lim_{x\to\infty}}\frac{\pi(x)}{x(\log x)^{-1}}=\overline{\lim_{x\to\infty}}\frac{\theta(x)}{x}=\overline{\lim_{x\to\infty}}\frac{\psi(x)}{x}. \tag{8.6.7}$$

*Proof.*i. At first, from 8.6.6we obtain

$$\psi(x)=\sum_{p\in\mathbb{P},p\le x}[\frac{\log x}{\log p}]\log p,$$

and so

$$\theta(x)\le\psi(x)\le\sum_{p\in\mathbb{P},p\le x}\frac{\log x}{\log p}\log p=\pi(x)\log x$$

Thus, we obtain

$$\overline{\lim_{x\to\infty}}\frac{\theta(x)}{x}\le\overline{\lim_{x\to\infty}}\frac{\psi(x)}{x}=\overline{\lim_{x\to\infty}}\frac{\pi(x)}{x(\log x)^{-1}}.$$

ii. On the other hand, for any α with $0 < \alpha < 1$, and $x > 1$, there holds

$$\theta(x) \geq \sum_{p \in \mathbb{P}, x^\alpha < p \leq x} \log x \geq (\pi(x) - \pi(x^\alpha)) \log x^\alpha \geq \alpha(\pi(x) - x^\alpha) \log x.$$

Thus

$$\overline{\lim_{x\to\infty}} \frac{\theta(x)}{x} \geq \alpha \overline{\lim_{x\to\infty}} \frac{\pi(x)}{x(\log x)^{-1}}.$$

Since it holds for any $0 < \alpha < 1$, there is

$$\overline{\lim_{x\to\infty}} \frac{\theta(x)}{x} \geq \overline{\lim_{x\to\infty}} \frac{\pi(x)}{x(\log x)^{-1}}.$$

□

Similarly, we obtain

(8.6.8) $$\underline{\lim}_{x\to\infty} \frac{\pi(x)}{x(\log x)^{-1}} = \underline{\lim}_{x\to\infty} \frac{\theta(x)}{x} = \underline{\lim}_{x\to\infty} \frac{\psi(x)}{x}.$$

Corollary. *There holds*

(8.6.9) $$\pi(x) \sim \frac{\theta(x)}{\log x} \sim \frac{\psi(x)}{\log x}, \ (x \to \infty).$$

4. Define **Möblius** function $\mu : \mathbb{N} \to \{= 1, 0, 1\}$ by

$$\mu(n) = \begin{cases} 1, & n = 1; \\ (-1)^r, & n = q_1 q_2 \cdots q_r, q_1 < q_2 < \cdots < q_r; \\ 0, & \text{others.} \end{cases}$$

Thus, if $(n_1, n_2) = 1$, then $\mu(n_1 n_2) = \mu(n_1)\mu(n_2)$, and for any $n \in \mathbb{N}$, there holds

(8.6.10) $$\sum_{k|n} \mu(k) = [\frac{1}{n}].$$

Example. For any $x \geq 1$, there holds

(8.6.11) $$|\sum_{d \leq x} \frac{\mu(d)}{d}| \leq 1.$$

Proof.

$$1 = \sum_{n \leq x} [\frac{1}{n}] = \sum_{n \leq x} \sum_{d|n} \mu(d) = \sum_{dl \leq x; d \leq 1, l \leq 1} \mu(d) = \sum_{d \leq x} \mu(d) \sum_{1 \leq l \leq x/d} 1 = \sum_{d \leq x} \mu(d)[\frac{x}{d}],$$

thus we obtain

$$x \sum_{d \leq x} \frac{\mu(d)}{d} = 1 - \sum_{d \leq x} \mu(d) \{\frac{x}{d}\}.$$

Since

$$\sum_{2 \leq d \leq x} \mu(d)\{\frac{x}{d}\} < \sum_{2 \leq d \leq x} 1 \leq [x] - 1 = x - \{x\} - 1,$$

we have

$$|1 - \sum_{d \leq x} \mu(d)\{\frac{x}{d}\}| = |1 - \{x\} - \sum_{2 \leq d \leq x} \mu(d)\{\frac{x}{d}\}| \leq x.$$

□

Lemma (Möbius Transform). *Let $f(n), g(n)$ be arithmetic functions, i.e., from $\mathbb{N}$ to $\mathbb{C}$. Then the follow two formulas are equivalent:*

$$g(n) = \sum_{d|n} f(d); \tag{8.6.12}$$

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}). \tag{8.6.13}$$

Corollary. *For any $n \in \mathbb{N}$, there holds*

$$\sum_{d|n} \Lambda(d) = \log n.$$

*Proof.*Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$. Then, by induction on the number r, there is

$$\begin{aligned}
\sum_{d|n} \Lambda(d) &= \sum_{\lambda_1=0}^{\alpha_1} \sum_{\lambda_2=0}^{\alpha_2} \cdots \sum_{\lambda_r=0}^{\alpha_r} \Lambda(q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_r^{\lambda_r}) \\
&= \sum_{\lambda_1=1}^{\alpha_1} \sum_{\lambda_2=0}^{\alpha_2} \cdots \sum_{\lambda_r=0}^{\alpha_r} \Lambda(q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_r^{\lambda_r}) + \sum_{\lambda_2=0}^{\alpha_2} \cdots \sum_{\lambda_r=0}^{\alpha_r} \Lambda(q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_r^{\lambda_r}) \\
&= \sum_{\lambda_1=1}^{\alpha_1} \Lambda(q_1^{\lambda_1}) + \sum_{d|q_2^{\alpha_2} \cdots q_r^{\alpha_r}} \Lambda(d) \\
&= \alpha_1 \log q_1 + \log q_2^{\alpha_2} \cdots q_r^{\alpha_r} \\
&= \log n.
\end{aligned}$$

□

Example. [Tchebychev's identity]

$$\sum_{n \leq x} \psi(\frac{x}{n}) = \log[x]!. \tag{8.6.14}$$

Proof.

$$\begin{aligned}
\log([x]!) &= \sum_{p \in \mathbb{P}, p \leq [x]} (\sum_{i=1}^{\infty} [\frac{[x]}{p^i}]) \log p = \sum_{p \in \mathbb{P}, p \leq [x]} (\sum_{i=1}^{\infty} [\frac{x}{p^i}]) \log p \\
&= \sum_{p \in \mathbb{P}, p \leq [x]} \sum_{i=1}^{\infty} \Lambda(p^i) [\frac{x}{p^i}] = \sum_{p \in \mathbb{P}, p^i \leq [x]} \Lambda(p^i) [\frac{x}{p^i}] \\
&= \sum_{d \leq x} \Lambda(d) [\frac{x}{d}] = \sum_{d \leq x} \psi(\frac{x}{d}).
\end{aligned}$$

□

By the exercise 8.6.16, we obtain

Corollary.

$$\sum_{n \leq x} \psi(\frac{x}{n}) = x \log x - x + O(\log x), \ x \geq 1. \tag{8.6.15}$$

Example (Tchebychev's inequality). For $x \geq 2$, there holds

$$(\frac{\log 2}{4})x \leq \psi(x) \leq (4\log 2)x. \tag{8.6.16}$$

*Proof.*Let $n \in \mathbb{N}$. we obtain

$$\begin{aligned}\log(2n)! - 2\log n! &= \sum_{k\leq 2n} \psi(\frac{2n}{k}) - 2\sum_{k\leq n}\psi(\frac{n}{k}) \\ &= \sum_{k\leq 2n} \psi(\frac{2n}{k}) - 2\sum_{k\leq n}\psi(\frac{2n}{2k}) \\ &= \sum_{k\leq 2n} (-1)^{k-1}\psi(\frac{2n}{k}),\end{aligned}$$

then

$$\psi(2n) - \psi(n) \leq \log\frac{(2n)!}{(n!)^2} \leq \psi(2n). \tag{8.6.17}$$

It is obvious that

$$2^n \leq \frac{(2n)!}{(n!)^2} \leq 4^n, n \geq 2,$$

thus

$$\psi(2n) \geq n\log 2,\ \ \psi(2n) - \psi(n) \leq 2n\log 2.$$

Then, for $x \geq 2$, there is $m \in \mathbb{N}$ with $2^{m-1} \leq x \leq 2^m$we obtain

$$\psi(x) \geq \psi(2[\frac{x}{2}]) \geq [\frac{x}{2}]\log 2 > (\frac{\log 2}{4})x,$$

and

$$\psi(x) \leq \psi(2^m) = \sum_{i=0}^{m-1}\psi(2^{i+1}) - \psi(2^i) \leq (\sum_{i=0}^{m-1} 2^{i+1})\log 2 < 2^{m+1}\log 2 < (4\log 2)x.$$

□

Exercise 8.7. 1. To show that for any $n \in \mathbb{N}$, there holds

$$\log n\sum_{d|n}\mu(d) = 0,$$

and so

$$\Lambda(n) = \sum_{d|n}\mu(d)\log\frac{n}{d} = -\sum_{d|n}\mu(d)\log d,$$

i.e., The Möbius transform of $-\mu(n)\log n$ is $\Lambda(n)$. Therefore, there is

$$-\mu(n)\log n = \sum_{d|n}\Lambda(\frac{n}{d}),\ \ n \in N. \tag{8.7.1}$$

2. **Generalized Möbius Transform** Let $F(x), G(x)$ be two functions defined over $x \geq 1$. Then the follow two formulas are equivalent:

$$G(x) = \sum_{n\leq x} F(\frac{x}{n}); \tag{8.7.2}$$

$$F(x) = \sum_{n\leq x}\mu(n)G(\frac{x}{n}). \tag{8.7.3}$$

3. To show the Tchebychev's identity 4
$$\sum_{n\leq x}\psi(\frac{x}{n}) = \log[x]!.$$
by the generalized Möbius transform 8.7.2.

4. To show that for any $x \geq 1$, there holds
$$\log[x]! = x\log x - x + O(\log(x)). \tag{8.7.4}$$
*Proof.*Using
$$\int_1^x \log t dt = x\log x - x + 1,$$
we obtain
$$\begin{aligned}\log[x]! &= \sum_{n\leq[x]-1}\log n + \log x\\ &\leq \sum_{n\leq[x]-1}\int_n^{n+1}\log t dt + \log x\\ &\leq \int_1^x \log t dt + \log x\\ &= x\log x - x + 1 + \log x,\end{aligned}$$
and
$$\begin{aligned}\log[x]! &= \sum_{2\leq n\leq[x]}\log n\\ &\geq \sum_{2\leq n\leq[x]}\int_{n-1}^{n}\log t dt\\ &= \int_1^x \log t dt - \int_{[x]}^x \log t dt\\ &\geq \int_1^x \log t dt - \log x\\ &= x\log x - x + 1 - \log x,\end{aligned}$$
□

5. To show that
$$\theta(x) = O(x),\ \ x \geq 1.$$
*Proof.*At first, by 8.6.1 we have
$$\sum_{p\leq n, p\in\mathbb{P}}[\frac{n}{p}]\log p = n\log n + O(n),$$
thus
$$\sum_{p\in\mathbb{P}, p\leq 2n}[\frac{2n}{p}]\log p - 2[\frac{n}{p}]\log p = O(n).$$

Since

$$[\frac{2n}{p}] - 2[\frac{n}{p}] = \begin{cases} 1, & n < p \leq 2n \\ \geq 0, & p \leq n, \end{cases}$$

there is

$$\theta(2n) - \theta(n) = \sum_{n<p\leq 2n} \log p = O(n),$$

and so

$$\begin{aligned} \theta(2x) - \theta(x) &= \theta([2x]) - \theta([x]) \\ &= \theta([2x]) - \theta(2[x]) + \theta(2[x]) - \theta([x]) \\ &= O(\log[2x]) + O([x]) \\ &= O([x]). \end{aligned}$$

Define $k(x) = [\log x/log2] + 1$, then for $x \geq 1$, we obtain

$$\theta(x) = \sum_{i=1^{k(x)}} (\theta(\frac{x}{2^{i-1}}) - \theta(\frac{x}{2^i})) = O(1) \sum_{i=1}^{\infty} \frac{x}{2^i} = O(x).$$

$\square$

8.4. **Mertens' Theorem and Selberg's Inequality.**

8.8. Mertens' Theorems

1. By 8.7.2, the generalized möbius transform of $\psi(x)$ is

(8.8.1) $$T(x) = \sum_{n\leq x} \psi(\frac{x}{n}) = \sum_{m,n\in\mathbb{N}, mn\leq x} \Lambda(m) = \sum_{m\leq x} \Lambda(m)[\frac{x}{m}].$$

2. For $x \geq 1$, there holds

(8.8.2) $$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

*Proof.*By 8.8.1, we obtain

$$\sum_{m\leq x} \Lambda(m)[\frac{x}{m}] = T(x) = x\log x + O(x),$$

and so the Tchebychev inequality say

$$0 \leq \sum_{n\leq x} \Lambda(n)(\frac{x}{n} - [\frac{x}{n}]) \leq \sum_{n\leq x} \Lambda(n) = \psi(x) < 4x.$$

□

3. Also, we have

(8.8.3) $$\sum_{p\leq x, p\in\mathbb{P}} \frac{\log p}{n} = \log x + O(1),\ x \geq 1.$$

*Proof.*At first, we have

$$\begin{aligned}\sum_{n\leq x} \frac{\Lambda(n)}{n} &- \sum_{p\leq x, p\in\mathbb{P}} \frac{\log p}{p} \\ &= \sum_{m\geq 2} \sum_{p\in\mathbb{P}, p\leq x^{1/m}} \frac{\log p}{p^m} \\ &< \sum_{p\in\mathbb{P}} \log p(\sum_{i=2}^{\infty} \frac{1}{p^i}) \\ &= \sum_{p\in\mathbb{P}} \frac{\log p}{p(p-1)} \\ &< \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)} = O(1).\end{aligned}$$

Then,by 8.8.3we obtain the 8.8.3.

□

4.

Lemma. *Let $\{\lambda_n\}$ be an increasing sequence with $\lambda_n \to \infty (n \to \infty)$ and $\{a_n\}$ be any sequence. Let $b(x)$ be a function on $\mathbb{R}$ such that it is continuous on any finite interval $[\lambda_1, \xi]$. Then, there holds*

(8.8.4) $$\sum_{\lambda_1\leq\lambda_n\leq x} a_n b(\lambda_n) = S(x)b(x) - \int_{\lambda_1}^{x} S(x)b'(x)dt,$$

where

$$S(x) = \sum_{\lambda_1 \leq \lambda_n \leq x} a_n.$$

Proof.

$$\begin{aligned} S(x)b(x) & - \sum_{\lambda_1 \leq \lambda_n \leq x} a_n b(\lambda_n) \\ &= \sum_{\lambda_1 \leq \lambda_n \leq x} a_n (b(x) - b(\lambda_n)) \\ &= \sum_{\lambda_1 \leq \lambda_n \leq x} \int_{\lambda_n}^{x} a_n b'(t) dt \\ &= \int_{\lambda_1}^{x} (\sum_{\lambda_1 \leq \lambda_n \leq t} a_n) b'(t) dt \\ &= - \int_{\lambda_1}^{x} S(x) b'(x) dt. \end{aligned}$$

□

Theorem. *For $x \geq 2$, there is*

(8.8.5) $$\sum_{p \in \mathbb{P}, p \leq x} \frac{1}{p} = \log\log x + A_1 + O(\frac{1}{\log x}),\ \ x \leq 2$$

where A_1 is a constant.

*Proof.*Define a sequence $\{\lambda_n\}$ by

$$\lambda_1 = 2, \lambda_n = p_n \in \mathbb{P}(n \geq 2);$$

and define sequence $\{a_n\}$ by

$$a_n = \frac{\log p_n}{\log p_n}, p_n \in \mathbb{P}.$$

From 8.8.3, we obtain

$$S(x) = \sum_{\lambda_1 \leq \lambda_n \leq x} a_n = \log x + O(1).$$

Let $b(x) = \frac{1}{\log x}$ and $r(x) = S(x) - logx$ From8.8.4, we have

$$\begin{aligned} \sum_{p \in \mathbb{P}, p \leq x} \frac{1}{p} &= \frac{S(x)}{\log x} + \int_2^x \frac{S(t)}{t(\log t)^2} dt \\ &= \frac{\log x + O(1)}{\log x} + \int_2^x \frac{\log t + O(1)}{t(\log t)^2} dt \\ &= \log\log x + O(\frac{1}{\log x}) + A + O(1) \int_x^{\infty} \frac{dt}{t(\log t)^2} \\ &= \log\log x + O(\frac{1}{\log x}) + A, \end{aligned}$$

where

$$A = 1 - \log\log 2 + \int_2^{\infty} \frac{r(t) dt}{t(\log t)^2}$$

is a constant. □

8.9. Selberg's Asymptotic Formula

1.

Lemma 8.10. *Let $F(x), G(x)$ be two functions defined over $x \geq 1$ with $F(1) = G(1)$. The the following two relations are equivalent :*

$$G(x) = \sum_{1 \leq n \leq x} F(\frac{x}{n}) \tag{8.10.1}$$

and

$$\sum_{1 \leq n \leq x} \mu(n) G(\frac{x}{n}) \log \frac{x}{n} = F(x) \log x + \sum_{1 \leq n \leq x} F(\frac{x}{n}) \Lambda(n) \tag{8.10.2}$$

*Proof.*If the relation

$$G(x) = \sum_{1 \leq n \leq x} F(\frac{x}{n})$$

holds then we obtain

$$\begin{aligned}
\sum_{1 \leq n \leq x} \mu(n) G(\frac{x}{n}) \log \frac{x}{n} &= \sum_{n \leq x} \mu(n) \log \frac{x}{n} \sum_{m \leq x/n} F(\frac{x}{mn}) \\
&= \sum_{k \leq x} F(\frac{x}{k}) \sum_{n|k} \mu(n) \log \frac{x}{n} \\
&= \log x \sum_{n \leq x} F(\frac{x}{n}) \sum_{k|n} \mu(k) - \sum_{n \leq x} F(\frac{x}{n}) \sum_{k|n} \mu(k) \log k \\
&= F(x) \log x + \sum_{1 \leq n \leq x} F(\frac{x}{n}) \Lambda(n).
\end{aligned}$$

Assume the relation

$$\sum_{1 \leq n \leq x} \mu(n) G(\frac{x}{n}) \log \frac{x}{n} = F(x) \log x + \sum_{1 \leq n \leq x} F(\frac{x}{n}) \Lambda(n)$$

holds.Let

$$k(n) = \sum_{1 \leq n \leq x} \mu(n) G(\frac{x}{n}) \log \frac{x}{n},$$

then by the generalized Möbius transform 8.7.2 we obtain

$$G(x) \log x = \sum_{1 \leq n \leq x} k(\frac{x}{n}),$$

thus,

$$\begin{aligned}
G(x) \log x &= \sum_{1 \leq n \leq x} \{F(\frac{x}{n}) \log \frac{x}{n} + \sum_{1 \leq m \leq \frac{x}{n}} F(\frac{x}{nm}) \Lambda(m)\} \\
&= \log x \sum_{1 \leq n \leq x} F(\frac{x}{n}) - \sum_{1 \leq n \leq x} \{F(\frac{x}{n}) \log n + \sum_{1 \leq k \leq x} F(\frac{x}{k}) \sum_{m|k} \Lambda(m) \\
&= \log x \sum_{1 \leq n \leq x} F(\frac{x}{n}) - \sum_{1 \leq n \leq x} \{F(\frac{x}{n}) \log n + \sum_{1 \leq k \leq x} F(\frac{x}{k}) \log k \\
&= \log x \sum_{1 \leq n \leq x} F(\frac{x}{n})
\end{aligned}$$

$\square$

2. Denote $\psi_1(x) = x - \gamma - 1$, where γ is the Euler constant. Let

$$T(x) = \sum_{n \leq x} \psi(\frac{x}{n}),\ T_1(x) = \sum_{n \leq x} \psi_1(\frac{x}{n}).$$

Since

$$\sum_{n \in \mathbb{N}, n \leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}),\ \ x \geq 1,$$

Then

$$T_1(x) = \sum_{n \leq x} (\frac{x}{n} - \gamma - 1) = x \log x - x + O(1).$$

Moreover, since there is

$$T(x) = [n]! = x \log x - x + O(\log(x))\ \ x \geq 1,$$

we obtain

$$T(x) - T_1(x) = \sum_{n \leq x} \{\psi(\frac{x}{n}) - \psi_1(\frac{x}{n})\} = O(\log x), \tag{8.10.3}$$

and for any $l \in \mathbb{Z}_{\geq 0}$, we obtain

$$\sum_{n \leq x} \mu(n) \{T(\frac{x}{n}) - T_1(\frac{x}{n})\} (\log \frac{x}{n})^l << \sum_{n \leq x} (\log \frac{x}{n})^{l+1} << \sum_{n \leq x} (\frac{x}{n})^{\frac{1}{2}} << x. \tag{8.10.4}$$

Actually we again obtain the Tchebychev inequality while $l = 0$.

Theorem 1 (Selberg's Inequality)**.**

$$(\psi(x) - x) \log x + \sum_{n \leq x} \Lambda(n) (\psi(\frac{x}{n}) - \frac{x}{n}) = O(x). \tag{8.10.5}$$

*Proof.*Denote

$$F(x) = \psi(x) - \psi_1(x) = \psi(x) - (x - \gamma - 1),\ \ \text{and } G(x) = T(x) - T_1(x).$$

From 8.10.2 and 8.10.4, we obtain

$$\begin{aligned} (\psi(x) - (x - \gamma - 1)) \log x \quad &+ \quad \sum_{1 \leq n \leq x} \Lambda(n) (\frac{x}{n} - \gamma - 1)) \\ &= \quad \sum_{1 \leq n \leq x} \mu(n) (T(\frac{x}{n}) - T_(\frac{x}{n})) \log \frac{x}{n} \\ &<< \quad \sum_{n \leq x} (\log \frac{x}{n})^2 << x. \end{aligned}$$

□

3. Selberg's inequality says that

$$\psi(x) \log x + \sum_{n \leq x} \Lambda(n) \psi(\frac{x}{n}) = \log x + \sum_{n \leq x} \Lambda(n) \frac{x}{n} + O(x),$$

then withith Merterns' theorem,we immediately obtain

Corollary (Selberg's Asymptotic Formula)**.**

(8.10.6) $$\psi(x)\log x + \sum_{n\le x}\Lambda(n)\psi(\frac{x}{n}) = 2x\log x + O(x).$$

(8.10.7) $$\sum_{p\le x, p\in\mathbb{P}}\log^2 p + \sum_{p,q\in\mathbb{P}, pq\le x}\log p\log q = 2x\log x + O(x).$$

*Proof.*Only 8.10.7 is left to prove. Since

$$\psi(x) = \theta(x) + O(\sqrt{x}\log x),\ (x\to\infty),$$

the formula 8.10.6 can be formulated into

(8.10.8) $$\theta(x)\log x + \sum_{p\le x, p\in\mathbb{P}}\theta(\frac{x}{n})\log p = 2x\log x + O(x).$$

On the other hand,

$$\begin{aligned}\theta(x) - \sum_{p\le x, p\in\mathbb{P}}\log^2 p &= \sum_{p\le x, p\in\mathbb{P}}\log p\log\frac{x}{p}\\ &= \sum_{p\le x, p\in\mathbb{P}}\log p(\sum_{n\le\frac{x}{p}}\frac{1}{n} + O(1))\\ &= \sum_{n\le x}\frac{1}{n}\sum_{n\le\frac{x}{p}}\log p + O(\theta(x))\\ &= O(x\sum_{n\le x}\frac{1}{n^2}) + O(x)\\ &= O(x).\end{aligned}$$

□

8.5. An Elementary Proof of The Prime Number Theorem by Selberg and Erdös.

8.6. **Gauss's Proof of The Fundamental Theorem of Algebra.**

Theorem. *Let*

$$f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

be a polynomial with $a_i \in \mathbb{C}\,\forall i$. *Then, there exists a root* $\alpha \in \mathbb{C}$ *of* $f(x)$, *i.e.,* $f(\alpha) = 0$.

Proof. Let $x = re^{\sqrt{-1}\theta} = r(\cos\theta + \sin\theta)$. Then $f(re^{\sqrt{-1}\theta}) = P(r,\theta) + \sqrt{-1}Q(r,\theta)$ with

$$P(r,\theta) = r^n\cos(n\theta) + \cdots, \;\; Q(r,\theta) = r^n\sin(n\theta) + \cdots .$$

We know that

$$F(r,\theta) = (P^2+Q^2)(r,\theta) = r^{2n} + \cdots$$

is a function of (r,θ). If there is a pair (r_0,θ_0) satisfying $F(r_0,\theta_0) = 0$ then $r_0 e^{\sqrt{-1}\theta_0}$ is a root of the polynomial $f(x)$. Suppose that $F(r,\theta)$ has no zero point, we show there must be a confliction: We can define a differentiable function of (r,θ) by $U = \arctan\frac{P}{Q}$ since $F(r,\theta)$ has no zero point. Thus, there is

$$\frac{\partial U}{\partial r} = \frac{1}{P^2+Q^2}(Q\frac{\partial P}{\partial r} - P\frac{\partial Q}{\partial r}), \;\; \frac{\partial U}{\partial \theta} = \frac{1}{P^2+Q^2}(Q\frac{\partial P}{\partial \theta} - P\frac{\partial Q}{\partial \theta}),$$

and so

$$\frac{\partial^2 U}{\partial r\partial\theta} = \frac{H}{P^2+Q^2},$$

where H is a continuous function of (r,θ).

Denote

$$I_1 = \int_0^R (\int_0^{2\pi} \frac{\partial^2 U}{\partial r\partial\theta} d\theta)dr, \;\text{ and }\; I_2 = \int_0^{2\pi}(\int_0^R \frac{\partial^2 U}{\partial r\partial\theta} dr)d\theta.$$

Since $\frac{\partial^2 U}{\partial r\partial\theta}$ is a continuous function of (r,θ), there holds $I_1 = I_2$.

On the other hand, we have the following calculations

i. $I_1 = 0$ is given by

$$\int_0^{2\pi} \frac{\partial^2 U}{\partial r\partial\theta} d\theta = \frac{\partial U}{\partial r}|_0^{2\pi} = 0.$$

ii.

$$\frac{\partial P}{\partial \theta} = -nr^n\sin(n\theta) + \cdots, \text{ and } \frac{\partial Q}{\partial\theta} = nr^n\cos(n\theta) + \cdots,$$

and so,

$$Q\frac{\partial P}{\partial\theta} - P\frac{\partial Q}{\partial\theta} = -nr^{2n} + \cdots,$$

$$\frac{\partial U}{\partial\theta} = \frac{1}{P^2+Q^2}(Q\frac{\partial P}{\partial\theta} - P\frac{\partial Q}{\partial}) = \frac{-nr^{2n}+\cdots}{r^{2n}+\cdots}.$$

Regarding $Q\frac{\partial P}{\partial\theta} - P\frac{\partial Q}{\partial\theta}$ and P^2+Q^2 as polynomials of r, then all coefficients in both polynomials are bounded function of θ. Therefore

$$\lim_{r\to\infty}\frac{\partial U}{\partial\theta} = -n \text{ (uniformly for all } \theta),$$

$$\int_0^R \frac{\partial^2 U}{\partial r\partial\theta}dr = \frac{\partial U}{\partial\theta}|_0^R \longrightarrow -n \text{ (uniformly for all } \theta),$$

and so

$$\lim_{R\to\infty} I_2 = -2n\pi.$$

It is a confliction to that $I_1 = I_2$. Thus P^2+Q^2 has at least one zero point.

□

9. Advanced Techniques in Analysis

9.1. **Preliminary Results in Analysis.**

9.1. Some useful techniques.

1. **Unit decomposition.** Let $K \subset \mathbb{R}^n$ is a compact set, $U_1, \cdots, U_N$ is open covering of K. Then there exist functions $\eta_1 \in C_0^\infty(U_1), \cdots, \eta_N \in C_0^\infty(U_N)$, such that
   i.
   $$0 \leq \eta_i(x) \leq 1, \forall x \in U_i \; (i = 1, \cdots, N);$$
   ii.
   $$\sum_i^N \eta_i(x) \equiv 1, \; \forall x \in K.$$

2. **Local C^k-flattening** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We call the boundary $\partial\Omega$ is C^k, if for any $x^0 \in \partial\Omega$ there is a neighborhood $U \subset \mathbb{R}^n$ of x^0 and a bijective function $\Psi : U \to B_1(0)$ with $\Psi(x^0) = 0$ such that
   i. Ψ, Ψ^{-1} are C^k.
   ii.
   $$\Psi(U \cap \Omega) = B_1^+(0) = \{y \in B_1(0) \, : \, y_n > 0\},$$
   iii.
   $$\Psi(U \cap \partial\Omega) = \partial B_1^+(0) = \{y \in B_1(0) \, : \, y_n = 0\}.$$

3. **Lipschitz flattening** An open subset $\Omega \subset \mathbb{R}^n$ is said to be **Lipschitz** if for every the boundary $\partial\Omega$ is C^k, if if for any $x^0 \in \partial\Omega$ there exist a neighborhood $U \subset \mathbb{R}^n$ of x^0 and a bijective function $\Psi_{x^0} : U \to B_1(0)$ with $\Psi(x^0) = 0$ such that
   i. Ψ, Ψ^{-1} are Lipschitz.
   ii.
   $$\Psi(U \cap \Omega) = B_1^+(0) = \{y \in B_1(0) \, : \, y_n > 0\},$$
   iii.
   $$\Psi(U \cap \partial\Omega) = \partial B_1^+(0) = \{y \in B_1(0) \, : \, y_n = 0\}.$$

   Corollary. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain.There exists a continuous linear extension operator*
   $$E : W^{1,2}(\Omega) \longrightarrow W^{1,2}(\mathbb{R}^n)$$
   such that
   $$||E(u)||_{W^{1,2}(\mathbb{R}^n)} \leq C(\Omega)||u||_{W^{1,2}(\Omega)}, \; \forall u \in W^{1,2}(\Omega).$$
   Moreover, it is possible to arrange such an extension E such that
   $$\{x \in \mathbb{R}^n \, : \; E(u) \neq 0\} \subset \{x \in \mathbb{R}^n \, : \, dist(x, \overline{\Omega}) \leq 1\}.$$

9.2. Friedrichs smoothing mollifiers.

1. Consider a function $j \in C_0^\infty(\mathbb{R}^n)$ having support in $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ with the properties
   - $j(x) \geq 0$,
   - $\int_{B_1(0)} j(x)dx = 1$.

Example. Denote

$$j(x) = \begin{cases} \frac{1}{A}\exp(\frac{1}{|x|^2-1}), & |x| < 1; \\ 0, & |x| \geq 1; \end{cases}$$

where

$$A = \int_{B_1(0)} \exp(\frac{1}{|x|^2-1})dx.$$

2. For any $\epsilon > 0$, we define the mollifiers

$$j_\epsilon(x) = \frac{1}{\epsilon^n} j(\frac{x}{\epsilon}),$$

then j_ϵ has compact support in $B_\epsilon(0) = \{x \in \mathbb{R}^n : |x| < \epsilon\}$ with the properties

- $j_\epsilon(x) \geq 0$,
- $\int_{B_\epsilon(0)} j_\epsilon(x)dx = 1$.

Remark. The following are useful remarks:

a) For any $\varepsilon > 0$, there is a δ with $0 < \delta < 1$ such that

$$0 < \int_{\mathbb{R}^n \setminus B_R(0)} j(x) \leq \varepsilon, \ \forall \delta \leq R < 1,$$

and

$$0 < \int_{\mathbb{R}^n \setminus B_{\epsilon R}(0)} j_\epsilon(x) \leq \varepsilon, \ \forall \delta \leq R < 1.$$

b) If $j(x)$ has additional property $j(x) = j(|x|)$, then we have

$$\frac{n\omega_n}{\epsilon^n} \int_0^\epsilon j(\frac{r}{\epsilon}) r^{n-1} dr = 1,$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$, actually

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.$$

3. We can use the mollifiers j_ϵ to smooth a function $u \in L^1_{loc}(\Omega)(\Omega \subset \mathbb{R}^n)$. Let

$$\Omega_\epsilon := \{x \in \Omega \ : \ dist(x, \partial\Omega) > \epsilon\}$$

and for any $x \in \Omega_\epsilon$, denote

$$J_\epsilon u(x) = (u * j_\epsilon)(x) := \int_\Omega u(y) j_\epsilon(x-y)dy,$$

$J_\epsilon u$ is evidently smooth on Ω_ϵ. In fact, by virtue of the usual *differential under the integral* lemma, for any $x \in \Omega_\epsilon$,

$$D^\alpha_x J_\epsilon u(x) = \int_\Omega u(y) D^\alpha_x j_\epsilon(x-y)dy.$$

We then call that $J_\epsilon u$ is the **smoothing (mollifying)of** u, and J_ϵ the **smoothing operator**, j_ϵ the **smoothing (mollifying) kernel**.

Remark. Mollifying kernels were indroduced into PDE theory by K.O.Friedrichs.Therefore, they are often called "**Friedrichs mollifiers**".

Proposition 9.3. *Let u be a function on $\mathbb{R}^n$ with*

$$\mathrm{Supp} u \subset \Omega \subset \mathbb{R}^n.$$

We will put $u(y) = 0$ for $y \in \mathbb{R}^n \setminus \Omega$.(We shall always use that convention in the sequel.) Then, we have the following properties:

a) *If* $u \in L^1_{loc}(\Omega)$, *then* $J_\epsilon u \in C^\infty(\Omega_\epsilon)$.

b) *If* $\text{dist}(\text{Supp}u, \partial\Omega) > \epsilon$, *then* $J_\epsilon u \in C_0^\infty(\Omega)$.

c)

$$J_\epsilon u \xrightarrow{a.e.} u \text{ on } \Omega,$$

and moreover for all compact sets $K \subset \Omega$,

i. $J_\epsilon u \longrightarrow u$ *on* K *(on* $\overline{\Omega}$ *for* $u \in C^0(\overline{\Omega})$*) uniformly if* u *is continuous.*

ii.

$$J_\epsilon u \xrightarrow{L^1(K)} u.$$

Proof. Now let $u \in C^0(\Omega)$. At first, Let

$$\rho = \frac{\text{dist}(K, \partial\Omega)}{2}.$$

Then, for any given $\varepsilon > 0$, there exists δ with $\rho > \delta > 0$ such that

- $B_\delta(x) \subset K_\rho$, $\forall x \in K$, where K_ρ is a compact set in Ω denoted by

$$K_\rho = \{x \in \Omega \mid \text{dist}(x, K) \le \rho\};$$

- if $0 < \epsilon < \delta$ then

$$\sup_{z \in B_1(0)} |u(x + \epsilon z) - u(x)| < \varepsilon, \ \forall x \in K,$$

since u is uniformly continuous on the compact set K_ρ.

If $\epsilon < \rho$ then

$$J_\epsilon u(x) = (u * j_\epsilon)(x) = \int_\Omega u(y) \frac{1}{\epsilon^n} j(\frac{x-y}{\epsilon}) dy = \int_{B_1(0)} j(z) u(x + \epsilon z) dz.$$

Therefore, for all $x \in K$, if $\epsilon < \delta$ then

$$|J_\epsilon u(x) - u(x)| \le \int_{B_1(0)} j(z)|u(x+\epsilon z) - u(x)| dz \le \sup_{z \in B_1(0)} |u(x+\epsilon z) - u(x)| \int_{B_1(0)} j(z) dz \le \varepsilon.$$

□

d) *If* $u \in L^p(\Omega) (1 \le p < \infty)$, *then* $J_\epsilon u \in L^p(\Omega)$, *and*

$$\lim_{\epsilon \to 0^+} ||J_\epsilon u - u||_{L^p(\Omega)} = 0.$$

Moreover, if $u \in L^P_{loc}(\Omega)$ *then for all compact sets* $K \subset \Omega$, *we have*

$$J_\epsilon u \xrightarrow{L^p(K)} u.$$

Proof. Denote

$$\Omega_\rho = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) \le \rho\}.$$

At first,

$$J_\epsilon u(x) = (u * j_\epsilon)(x) = \int_\Omega u(y) \frac{1}{\epsilon^n} j(\frac{x-y}{\epsilon}) dy = \int_{B_1(0)} j(z) u(x + \epsilon z) dz,$$

then for $q > 0$ with $1/q + 1/q = 1$, we obtain

$$|J_\epsilon u(x)|^p \le (\int_{B_1(0)} j(z) dz)^{\frac{p}{q}} (\int_{B_1(0)} j(z)|u(x+\epsilon z)|^p dz).$$

If $\epsilon < \rho$, it follows that

(9.3.1)
$$\int_\Omega |J_\epsilon u(x)|^p dx \le \int_\Omega \int_{B_1(0)} j(z)|u(x+\epsilon z)|^p dz dx = \int_{B_1(0)} j(z)(\int_\Omega |u(x+\epsilon z)|^p dx) dz \le \int_{\Omega_\rho} |u(x)|^p dx.$$

For a given $\varepsilon > 0$, we choose $w \in C^0(\Omega_\rho)$ with

$$||u - w||_{L^p(\Omega_\rho)} < \frac{\varepsilon}{3}.$$

On the other hand, for sufficiently small ϵ,

$$||w - J_\epsilon w||_{L^p(\Omega_\rho)} < \frac{\varepsilon}{3}.$$

Applying 9.3.1 to $u - w$, we obtain

$$||J_\epsilon u - J_\epsilon w||_{L^p(\Omega)} \le ||u - w||_{L^p(\Omega_\rho)} < \frac{\epsilon}{3},$$

and hence

$$||J_\epsilon u - u||_{L^p(\Omega)} \le ||J_\epsilon u - J_\epsilon w||_{L^p(\Omega)} + ||J_\epsilon w - w||_{L^p(\Omega)} + ||u - w||_{L^p(\Omega)} \le \varepsilon.$$

□

Remark 9.4. In the proof, we did not use the smoothness of the kernel j at all. Thus, some results holds for other kernels, and in particular for

$$\sigma(x) = \begin{cases} \frac{1}{\omega_n}, & |x| \le 1; \\ 0, & |x| \ge 1; \end{cases}$$

The corresponding convolution is

$$u * \sigma_\epsilon = \frac{1}{\omega_n \epsilon^n} \int_\Omega \sigma(\frac{x-y}{\epsilon}) u(y) dy = \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) dy.$$

We obtain the the following result:

Corollary. *If $u \in L^p(\Omega)(1 \le p < \infty)$, then $u * \sigma_\epsilon \in L^p(\Omega)$, and*

$$\lim_{\epsilon \to 0^+} ||\frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) dy - u||_{L^p(\Omega)} = 0.$$

Moreover,

$$\frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) dy \xrightarrow{a.e.} u \text{ on } \Omega,$$

and for all compact sets $K \subset \Omega$,

i. $\frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) dy \longrightarrow u$ *on K (on $\overline{\Omega}$ for $u \in C^0(\overline{\Omega})$) uniformly if u is continuous.*

ii.
$$\frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} u(y) dy \xrightarrow{L^1(K)} u.$$

9.5. Cut-off functions.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $\Omega' \subset\subset$(i.e, $\overline{\Omega'} \subset \Omega$). Let $d := \frac{1}{4} dist(\Omega', \partial\Omega,)$ then $d > 0$. Let

$$\Omega^{''} := \{x \in \Omega \, : \, dist(x, \Omega') < d\},$$

then $dist(\Omega^{''}, \partial\Omega) = 3d$.

1. Let $\chi_{\Omega''}$ be characteristic function of Ω'', we consider the smoothing function of $\chi_{\Omega''}$
$$\eta(x) := J_d(\chi_{\Omega''}).$$
Then, we have
   - $\eta \in C_0^\infty(\Omega)$; and
   - $$0 \le \eta \le 1 \text{ with } \eta(x) \equiv 1 \text{ on } \Omega';$$
   - $$|\nabla \eta| \le \frac{C(\Omega)}{d},$$

   where $C(\Omega)$ is a constant only dependent on Ω. We then call η the **cut-off function**.

2. In application, we always consider the case
$$\Omega = B_R(x^0) := \{x \in \mathbb{R}^n : \ |x - x^0| < R\}.$$
Let $0 < \rho < R$ and $\eta(x)$ be the cut-off function related to $B_\rho(x^0)$. We have moreover property of $\eta(x)$:
$$|D^k \eta(x)| \le \frac{C}{|R-\rho|^k},$$
$$[D^k \eta]_\alpha \le \frac{C}{|R-\rho|^{k+\alpha}},$$
where the constant $C(\Omega)$ absolutely does not depend on R and ρ.

9.6. Applications of smoothing kernels and cut-off functions.

1. **Mean-value formula.** Let u be a continuous function on a domain $\Omega \subset \mathbb{R}^n$ such that
$$u(x) = |\partial B(x,r)|^{-1} \int_{\partial B(x,r)} u(y) dS(y) \tag{9.6.1}$$
for any ball $B(x,r) \subset \Omega$. Then, u is a harmonic function.

*Proof.*Let $j(x)$ be the smoothing kernel having the additional property $j(x) = j(|x|)$, then we have
$$\frac{n\omega_n}{\epsilon^n} \int_0^\epsilon j(\frac{r}{\epsilon}) r^{n-1} dr = 1,$$
Let $j_\epsilon(x) := \frac{1}{\epsilon^n} j(\frac{x}{\epsilon}), \forall \epsilon > 0$. For any sufficient small $\epsilon > 0$, denote
$$\Omega_\epsilon := \{x \in \Omega \ : \ dist(x, \partial\Omega) > \epsilon\}$$
and then for any $x \in \Omega_\epsilon$, denote
$$J_\epsilon u(x) = (u * j_\epsilon)(x) := \int_\Omega u(y) j_\epsilon(x-y) dy.$$

Then $J_\epsilon u(x) \in C^\infty(\Omega_\epsilon)$, and on Ω_ϵ, we have

$$\begin{aligned}
J_\epsilon u(x) &= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n} u(y)j_\epsilon(x-y)dy \\
&= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n} j(\frac{x-y}{\epsilon})u(y)dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon j(\frac{r}{\epsilon})\int_{\partial B(x,r)} u(y)dS(y)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon j(\frac{r}{\epsilon})|\partial B(x,r)|u(x)dr \\
&= u(x)\frac{n\omega_n}{\epsilon^n}\int_0^\epsilon j(\frac{r}{\epsilon})r^{n-1}dr = u(x).
\end{aligned}$$

Hence $u \in C^\infty(\Omega)$. On the other hand,for any $x \in \Omega$, denote

$$\phi(r) := |\partial B(0,1)|^{-1}\int_{\partial B(0,1)} u(x+rz)dS(z).$$

Then

$$\phi(r) = |\partial B(x,r)|^{-1}\int_{\partial B(x,r)} u(y)dS(y) \equiv u(x),$$

and so

$$\begin{aligned}
0 &\equiv \frac{\partial \phi(r)}{\partial r} \\
&= |\partial B(0,1)|^{-1}\int_{\partial B(0,1)} Du(x+rz)\cdot z dS(z) \\
&= |\partial B(x,r)|^{-1}\int_{\partial B(x,r)} Du(y)\cdot \frac{y-x}{r}dS(y) \\
&= |\partial B(x,r)|^{-1}\int_{\partial B(x,r)} \frac{\partial u(y)}{\partial \mathbf{n}}dS(y) \\
&= \frac{r}{n}|B(x,r)|^{-1}\int_{B(x,r)} \Delta u(y)dy.
\end{aligned}$$

Since u is smooth, let $r \to 0$, we then obtain

$$\Delta u(x) = 0, \ \forall x \in \Omega.$$

□

Remark. A continuous function on Ω is harmonic if and only if u satisfies 9.6.1.

2. Let u be a harmonic function on a domain $\Omega \subset \mathbb{R}^n$ and $\widetilde{\Omega} \subset\subset \Omega$, $d = dist(\widetilde{\Omega}, \partial\Omega)$. Then $u \in C^\infty(\Omega)$ and there holds for every multi-index α

$$\sup_{\widetilde{\Omega}} |D^\alpha u| \leq C(\alpha, d)||u||_{L^1(\Omega)}.$$

*Proof.*Let $j_d(x)$ be the mollifier smooth function related to Ω having additional property $j_d(x) = j_d(|x|)$. Then, for every fixed $y \in \widetilde{\Omega}$, we have

$$u(y) = \int_{\mathbb{R}^n} j_d(x-y)u(x)dx,$$

and hence

$$|D_y^\alpha u(y)| \le \int_{B_d(y)} |D_y^\alpha j_d(x-y)||u(x)|dx \le \sup_{\widetilde{\Omega}} |D^\alpha j_d(x)| \int_\Omega |u(x)|dx.$$

Since

$$|D^\alpha j_d(x)| = C_{n,\alpha} d^{-\alpha},$$

we complete the proof. □

3. **Weyl Lemma.**

Definition. Let $u \in W^{1,2}(\Omega)$, where Ω is an open set in $\mathbb{R}^n$, we say that u is **weakly harmonic** on Ω if

$$\int_\Omega Du \cdot D\phi = 0, \ \forall \phi \in C_0^\infty(\Omega).$$

Theorem 9.7 (Weyl Lemma). *Suppose that $u \in W^{1,2}(\Omega)$ is weakly harmonic. Then the L^2 class of u has a C^∞ representative which is harmonic.*

*Proof.*For any $\epsilon > 0$, let $j_\epsilon(x)$ be the mollifier smooth function. then $J_\epsilon u$ is the smoothing of u defined on

$$\Omega_\epsilon := \{x \in \Omega \ : \ dist(x, \partial\Omega) > \epsilon\},$$

and $\Omega_\epsilon \subset\subset \Omega$.

i. Since

$$\Delta_x j_\epsilon(x-y) = (-1)^2 \Delta_y j_\epsilon(x-y) = \Delta_y j_\epsilon(x-y),$$

for any $x \in \Omega_\epsilon$, we have

$$\begin{aligned} \Delta_x J_\epsilon u(x) &= \int_\Omega u(y)\Delta_x j_\epsilon(x-y)dy \\ &= \int_\Omega u(y)\Delta_y j_\epsilon(x-y)dy \\ &= -\sum_{i=1}^n \int_\Omega D_{y_i}u(y) D_{y_i}[j_\epsilon(x-y)]dy \\ &= 0, \end{aligned}$$

i.e., $J_\epsilon u$ is C^∞ and harmonic on Ω_ϵ.

ii. For any $K \subset\subset \Omega$, we can find a ϵ_0 such that if $0 < \epsilon < \epsilon_0$, then $\overline{K} \subset \Omega_\epsilon$. Let $d = dist(K, \partial\Omega)$. We have for any $x \in K$, any $0 < \epsilon_1, \epsilon_2 < \epsilon_0$

$$|D_x^\alpha(J_{\epsilon_1}u(x) - J_{\epsilon_2}u(x))| \le C_{n,\alpha} d^{-\alpha} \int_\Omega |J_{\epsilon_1}u(y) - J_{\epsilon_2}u(y)|dy = C_{n,\alpha} d^{-\alpha} ||J_{\epsilon_1}u - J_{\epsilon_2}u||_{L^1(\Omega)}.$$

Denote

$$\widetilde{u} = \lim_{\epsilon \to 0^+} J_\epsilon u,$$

then by Arzela-Ascoli lemma $\widetilde{u}$ is in $C^\infty(\Omega)$ and is harmonic.

□

4. **Application.**

Lemma 9.8 (Rellich Compactness Lemma). *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and u_k is a sequence of $W^{1,2}(\Omega)$ with $\sup_k ||u_k||_{W^{1,2}(\Omega)} < \infty$. Then there is a subsequence u_{k_i} and $u \in W^{1,2}(\Omega)$ such that*

i.

$$u_{k_i} \xrightarrow{weakly} u \ in \ W^{1,2}(\Omega);$$

ii.

$$u_{k_i} \xrightarrow{strongly} u \text{ in } L^2(\Omega);$$

(The same in fact is true for $L^p(\Omega)$ with $p < 2^* := \frac{2n}{n-2}$)

iii.

$$\int_\Omega |Du|^2 \le \liminf_{i\to\infty} \int_\Omega |Du_{k_i}|^2$$

Theorem 9.9 (Poincaré inequality)**.** *Suppose that Ω is a bounded and connected Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant C_Ω dependent only on the domain Ω such that for every $u \in W^{1,2}(\Omega)$ there holds*

$$\int_\Omega |u-\lambda|^2 \le C_\Omega \int_\Omega |Du|^2, \tag{9.9.1}$$

where $\lambda = |\Omega|^{-1}\int_\Omega u$.

*Proof.*a) Suppose the assertion is false: Then for each $k \in \mathbb{N}$, there exist function $u_k \in W^{1,2}(\Omega)$ such that 9.9.1 fails for $C_\Omega = k$:

$$\int_\Omega |Du_k|^2 \le \frac{1}{k}\int_\Omega |u_k-\lambda_k|^2 \equiv \frac{1}{k}\inf_{\lambda\in\mathbb{R}}\int_\Omega |u_k-\lambda|^2,$$

where $\lambda_k = |\Omega|^{-1}\int_\Omega u_k$. Define

$$v_k := \frac{u_k-\lambda_k}{||u_k-\lambda_k||_{L^2(\Omega)}},$$

we find

$$||v_k||_{L^2(\Omega)} = 1 \text{ and } ||Dv_k|| \le \frac{1}{\sqrt{k}},$$

and hence v_k is a bounded sequence in $W^{1,2}(\Omega)$.

b) By Rellich Compactness Lemma, there exists $v \in W^{1,2}(\Omega)$ and a subsequence v_{k_i} of v_k such that

- $$v_{k_i} \xrightarrow{strongly} v \text{ in } L^2(\Omega),$$

  and so

  $$\int_\Omega v = 0;$$

- $$v_{k_i} \xrightarrow{weakly} v \text{ in } W^{1,2}(\Omega),$$

  and so

  $$Dv_{k_i} \xrightarrow{weakly} Dv \text{ in } L^2(\Omega);$$

- $$\int_\Omega |Dv|^2 \le \liminf_{i\to\infty}\int_\Omega |Dv_{k_i}|^2$$

  and so

  $$\int_\Omega |Dv|^2 \le \liminf_{i\to\infty}\frac{1}{\sqrt{k}} = 0.$$

c) Hence $Dv = 0$ a.e on Ω. Since Ω is connected and

$$D(J_\epsilon v) = J_\epsilon(Dv) = 0,$$

$J_\epsilon v$ is a constant on any connected component $\widetilde{\Omega} \subset \Omega$. Hence $v = 0$ a.e. on Ω by $\int_\Omega v = 0$, but it contradicts to that $||v||_{L^2(\Omega)} = 1$.

□

Remark. By using the special case for $\Omega = B_1(0)$ and by changing scale $x \mapsto Rx$, the Poincaré inequality is : For any $u \in W^{1,2}(B_R(x_0))$,

$$R^{-n} \int_{B_R(x_0)} |u - \lambda|^2 \le CR^{2-n} \int_{B_R(x_0)} |Du|^2, \tag{9.9.2}$$

where $\lambda = |B_R(x_0)|^{-1} \int_{B_R(x_0)} u$ and C is a constant depending only on n.

9.10. More properties on harmonic functions. Let u be a harmonic function on an open set $\Omega \subset \mathbb{R}^n$.

1. **Harnark inequality.**

Proposition. *Let $B_R(x) \subset \Omega$ be a ball.For any ball $B_r(x) \subset B_R(x)$, there holds*

$$\sup_{y \in B_r(x)} u(y) \le (\frac{R+r}{R-r})^n \inf_{y \in B_r(x)} u(y). \tag{9.10.1}$$

Moreover, for any connected compact subset $V subset \Omega$, we have

$$\sup_{y \in V} u(y) \le C \inf_{y \in V} u(y), \tag{9.10.2}$$

where the constant C depends only on n and $dist(V, \partial\Omega)$.

2.

Proposition. *For any ball $B_r(x) \subset \Omega$ and any multi-index α with $k = |\alpha|$, there holds*

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}} \int_{B_r(x)} |u(y)| dy, \tag{9.10.3}$$

where

$$C_0 = \frac{1}{\omega_n},\ C_k = \frac{(n+k)^{n+k}(n+1)^k}{\omega_n (n+1)^{n+1}}\ (k = 1, 2, \cdots).$$

Proof. At first, we have

$$|u_{x_i}(x)| \le \frac{n+1}{\omega_n r^{n+1}} \int_{B_r(x)} |u(y)| dy.$$

Since $D^\alpha u$ is still a harmonic function, we have

$$|D^\alpha u(x)| \le \frac{n+1}{\omega_n (tr)^{n+1}} \int_{B_{tr}(x)} |D^\beta u(y)| dy,$$

where β is a multi-index with $|\beta| = k-1$ such that there is $i \in \{1, 2, \cdots, n\}$ with $D^\alpha u = (D^\beta u)_{x_i}$.

Since $y \in B_{tr}(x)$ implies that $B_{(1-t)r}(y) \subset B_r(x) \subset$, by induction on k there holds

$$|D^\beta u(y)| \le \frac{C_{k-1}}{((1-t)r)^{n+k-1}} \int_{B_{(1-t)r}(y)} |u(z)| dz \le \frac{C_{k-1}}{((1-t)r)^{n+k-1}} \int_{B_r(x)} |u(z)| dz,$$

Hence

$$|D^\alpha u(x)| \le \frac{n+1}{\omega_n (tr)^{n+1}} \omega_n (tr)^n \max_{y \in B_{tr}(x)} |D^\beta u(y)| \le \frac{(n+1) C_{k-1}}{r^{n+k} t (1-k)^{n+k-1}} \int_{B_r(x)} |u(z)| dz.$$

Denote

$$C_k := \max_{t\in[0,1]} \frac{(n+1)C_{k-1}}{t(1-k)^{n+k-1}},$$

then

$$C_k = \frac{(n+1)C_{k-1}(n+k)^{n+k}}{(n+k-1)^{n+k-1}} = \frac{(n+k)^{n+k}(n+1)^k}{\omega_n(n+1)^{n+1}}.$$

□

3. As a corollary, by Harnark inequality,we have:

   Corollary. *Let u be a harmonic function on the total $\mathbb{R}^n$. Suppose u has upper bound (or lower bound), then u is a constant.*

4.

   Proposition. *A u harmonic function on Ω is an analytic function.*

   Proof. Fix any point $x_0 \in \Omega$, we only need to check the convergent radios of the following the Taylor series of u at x_0. First the Taylor-Maclaurin formula says

$$u(x) = \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha + R_N(x),$$

where

$$R_N(x) = \sum_{k=N}^{\infty} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0+\xi(x-x_0))}{\alpha!}(x-x_0)^\alpha, \xi \in (0,1).$$

Let $r = \frac{1}{4}dist(x_0, \partial\Omega)$, and denote

$$A = \frac{1}{\omega_n(n+1)^{n+1}r^n} \int_{B_{2r}(x_0)} |u(y)|dy.$$

Then, for any $x \in B_{2r}(x_0)$ (so that $B_r(x) \subset B_{2r}(x_0) \subset \Omega$), the formula 9.10.3 says that for $N >> o$,

$$\begin{aligned} |R_N(x)| &\leq \sum_{|\alpha|=N} \frac{|D^\alpha u(x_0+\xi(x-x_0))|}{\alpha!}|x-x_0|^\alpha \\ &\leq A \sum_{|\alpha|=N} \frac{1}{\alpha!}(n+N)^{n+N}[\frac{(n+1)|x-x_0|}{r}]^N \\ &= 2^n e^n A(\sum_{|\alpha|=N})(\frac{N!}{\alpha!})e^N N^{n-1/2}[\frac{(n+1)|x-x_0|}{r}]^N \\ &\leq 2^n e^n A[\frac{en(n+1)|x-x_0|}{r}]^N N^{n-1/2}. \end{aligned}$$

Here we use the combination formula

$$\sum_{|\alpha|=N} \frac{1}{\alpha!} = n^N,$$

and the Stirling formula (refer to the formula 4.2.2):

$$\lim_{N\to\infty} \frac{N^{N+1/2}}{N!e^N} = \frac{1}{\sqrt{2\pi}} < 1,$$

and so if $N >> 0$

$$(n+N)^{n+N} = (1+n/N)^N (N+n)^n N^N \le e^n (2N)^n N^N \le 2^n e^n N! e^N N^{n-1/2}.$$

Hence if

$$x \in B_{\frac{r}{2n(n+1)e}}(x_0)$$

then

$$|R_N(x)| \le 2^n e^n A \frac{N^{n-1/2}}{2^N},$$

and so

$$|R_N(x)| \xrightarrow{uniformly} 0 \ (N \to \infty).$$

□

5. The following harmonic approximation (or "blow up") lemma will be of fundamental importance.

Proposition (Harmonic Approximation Lemma)**.** *Let $f \in W^{1,2}(B_1(0))$. If for each $\varepsilon > 0$ there is $\delta = \delta(n, \varepsilon) > 0$ such that f satisfies*
i.

$$\int_{B_1(0)} |Df|^2 < 1,$$

and
ii.

$$|\int_{B_1(0)} Df \cdot D\varphi| < \delta \sup_{B_1(0)} |D\varphi|, \ \forall \varphi \in \mathbb{C}_0^\infty(B_1(0)).$$

Then there is a harmonic function u on $B_1(0)$ such that $\int_{B_1(0)} |Du|^2 < 1$,and

$$\int_{B_1(0)} |u-f|^2 \le \varepsilon^2.$$

*Proof.*If this fails for some ε, then there is a sequence $\{f_k\} \subset W^{1,2}(B_1(0))$ with
- $\int_{B_1(0)} |Df_k|^2 < 1$,
- $|\int_{B_1(0)} Df_k \cdot D\varphi| < \frac{1}{k} \sup_{B_1(0)} |D\varphi|, \ \forall \varphi \in \mathbb{C}_0^\infty(B_1(0))$,

and such that

(9.10.4) $$\int_{B_1(0)} |u - f_k|^2 > \varepsilon^2,$$

for every harmonic u on $B_1(0)$ with $\int_{B_1(0)} |Du|^2 < 1$.

Notice that since the same holds with $\widetilde{f_k} = f_k - \lambda_k$ for any choice of constants λ_k, we can assume without loss of generality that $\int_{B_1(0)} f_k = 0$ for each k. But then by the Poincaré inequality(c.f.9.9.1) we conclude

$$\sup \lim_{k\to\infty} \int_{B_1(0)} (|f_k|^2 + |Df_k|^2) < \infty,$$

and hence by the Rellich Compactness Theorem we have a subsequence $f_{k'}$ and an $f \in W^{1,2}(B_1(0))$ such that

$$f_{k'} \xrightarrow{strongly} f \text{ in } L^2(B_1(0)); \text{ and } Df_{k'} \xrightarrow{weakly} Df \text{ in } L^2(B_1(0)).$$

But we then conclude

$$\int_{B_1(0)} Df \cdot D\varphi = 0 \ \forall \varphi \in \mathbb{C}_0^\infty(B_1(0)),$$

so that f is a weakly harmonic function on $B_1(0)$, and the Weyl's lemma guarantees that f is a smooth harmonic function on $B_1(0)$. Moreover, since by Lebsegue lemma, we have

$$\int_{B_1(0)} |Df|^2 \leq \inf \lim_{k'} |Df_{k'}|^2 \leq 1,$$

hence that $f_{k'} \xrightarrow{strongly} f$ in $L^2(B_1(0))$ contradicts to the relation 9.10.4. $\square$

9.2. **Scaling Technique and Schauder Estimate.**

9.11. Notions. Let $\Omega \subset \mathbb{R}^n$ be an open domain and $\alpha \in (0,1]$. Let u be a function defined on Ω.

1. u is said to be **Hölder continuous with exponent** α on Ω if there is a constant C such that

(9.11.1) $$|u(x) - u(y)| \leq C|x-y|^{\alpha}, \ \forall\, x, y \in \Omega$$

2. Let u be a Hölder continuous function with exponent α on Ω. The **Hölder coefficient** of u is defined by
$$[u]_{\alpha;\Omega} := \sup_{x \neq y, x, y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}}.$$
we define the set of all bounded Hölder continuous functions with exponent α on Ω by
$$C^{0,\alpha}(\overline{\Omega}) = \{u |\ u \text{ is a function defined on } \Omega \text{ with } [u]_{\alpha;\Omega} < \infty\}.$$
We call u **Lipschitz continuous** if u is a Hölder continuous function with exponent 1.

Remark. $[\cdot]_{\alpha;\Omega}$ is only a semi-norm in the space $C^{0,\alpha}(\overline{\Omega})$. It is not a norm since it is zero for any constant function.But by the Arzela-Ascoli theorem, the space $C^{0,\alpha}(\overline{\Omega})$ becames a Banach space in the norm
$$|u|_{0,\alpha;\Omega} := |u|_{0;\Omega} + [u]_{\alpha;\Omega}.$$
where $|u|_{0;\Omega} = ||u||_{L^{\infty}(\Omega)}$.

3. u is called **locally Hölder continuous with exponent** α on Ω if it is Hölder continuous on each $\Omega' \subset\subset \Omega$, i.e., $u \in C^{0,\alpha}(\overline{\Omega'})$, where $\Omega' \subset\subset \Omega$ means the closure of Ω' in Ω is a compact set of Ω. We denote the space of all locally Hölder continuous functions with exponent α on Ω be $C^{0,\alpha}(\Omega)$.

4. **Scaling.** Let $u \in C^{0,\alpha}(\overline{\Omega})$. For any $R > 0$, we define the **scaled function**
$$\widetilde{u}(x) = R^{-\alpha} u(Rx) \ x \in \widetilde{\Omega}$$
where the domain $\widetilde{\Omega} \subset \mathbb{R}^n$ is defined by
$$\widetilde{\Omega} := \{R^{-1}y \mid y \in \Omega\}.$$
Then, $\widetilde{u} \in C^{0,\alpha}(\overline{\widetilde{\Omega}})$, and
$$[u]_{\alpha;\Omega} = [\widetilde{u}]_{\alpha;\widetilde{\Omega}}.$$

5. **Oscillation.** For any real function $u : \Omega \to \mathbb{R}$. The **oscillation** is defined as
$$\mathrm{osc}_{\Omega} u := \sup_{x \in \Omega} u(x) - \inf_{x \in \Omega} u(x).$$

Corollary 9.12. *Let $u : B_R(x_0) \to \mathbb{R}$ be a real value function. If for every $y \in B_{R/2}(x_0)$, and every $\rho \leq \frac{R}{2}$, $\mathrm{osc}_{B_{R(y)}} u < \infty$ and there is a fixed $\theta \in (0,1)$ such that*

(9.12.1) $$\mathrm{osc}_{B_{\theta\rho}(y)} u < \frac{1}{2} \mathrm{osc}_{B_{\rho}(y)} u,$$

then $u \in C^{0,\alpha}(\overline{B}_{R/2}(x_0))$ *with* $\alpha = -\frac{\log 2}{\log \theta}$*, and moreover*

$$[u]_{\alpha;B_{R/2}(x_0)} \leq C_\theta R^{-\alpha} \mathrm{osc}_{B_R(x_0)} u.$$

Proof. By induction we get from 9.12.1 that for every $y \in B_{R/2}(x_0)$, and every $\rho \leq \frac{R}{2}$, the following estimates

$$\mathrm{osc}_{B_{\theta^k \rho}(y)} u < \frac{1}{2^k} \mathrm{osc}_{B_\rho(y)} u,\ k = 1, 2. \cdots \tag{9.12.2}$$

are valid. Let $x, y \in B_{R/2}(x_0)$.

a) If $r := |x - y| \geq \frac{R}{2}$, then we have

$$|u(x) - u(y)| \leq (\frac{2|x-y|}{R})^\alpha \mathrm{osc}_{B_R(x_0)} u,$$

and so

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq 2^\alpha R^{-\alpha} \mathrm{osc}_{B_R(x_0)} u.$$

b) If $r := |x - y| \leq \frac{R}{2}$, we can choose an integer k with

$$\theta^{k+1} \leq \frac{2r}{R} < \theta^k.$$

Using 9.12.2 with $\rho = R/2$, we have

$$|u(x) - u(y)| \leq \mathrm{osc}_{B_{\theta^k \frac{R}{2}}} u \leq 2^{-k} \mathrm{osc}_{B_{\frac{R}{2}}(y)} u \leq 2^{-k} \mathrm{osc}_{B_R(x_0)} u,$$

and so

$$|u(x) - u(y)| \leq \theta^{k\alpha} \mathrm{osc}_{B_R(x_0)} u \leq \theta^{-\alpha} (\frac{2|x-y|}{R})^\alpha \mathrm{osc}_{B_R(x_0)} u.$$

Since $\theta^\alpha = \frac{1}{2}$, we have

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq 2^{\alpha+1} R^{-\alpha} \mathrm{osc}_{B_R(x_0)} u.$$

Let $C_\theta = 2^{\alpha+1}$. The claim is then established.

□

9.13. Notions.

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $\alpha \in (0, 1]$.. Let $\beta = (\beta_1, \cdots, \beta_n)$ be the multiple index with $\beta_1, \cdots, \beta_n$ non-negative integers.
Denote $|\beta| := \beta_1 + \cdots + \beta_n$.

Let u be a function defined on Ω.

1. If $u \in C^k(\overline{\Omega})$, denote

$$D^\beta u = \frac{\partial^{|\beta| u}}{\partial^{\beta_1} x_1 \cdots \partial^{\beta_n} x_n}.$$

The space $C^k(\overline{\Omega})$ is a Banach Space with the norm defined by

$$|u|_{k;\Omega} = \sum_{|\beta| \leq k} \sup_\Omega |D^\beta u|.$$

2. Recall
$$C^{0,\alpha}(\overline{\Omega}) = \{u|\ u \text{ is a function defined on } \Omega \text{ with } [u]_{\beta;\Omega} < \infty\},$$
and it is a Banach space with the norm
$$|u|_{0,\alpha;\Omega} := |u|_{0;\Omega} + [u]_{\alpha;\Omega}.$$
We define
$$C^{k,\alpha}(\overline{\Omega}) = \{u|\ u \text{ is a function defined on } \Omega \text{ with } D^\beta u \in C^{0,\alpha}(\overline{\Omega}) \text{ for any mult-index } \beta \text{ with } |\beta| \le k.\}$$
The space $C^{k,\alpha}(\overline{\Omega})$ will be a Banach space if it is denoted the following two norms:
$$|u|_{k,\alpha;\Omega} = \sum_{|\beta|\le k} |D^\beta u|_{\alpha;\Omega},$$
and
$$|u|_{k,0;\Omega} = |u|_{k;\Omega} = \sum_{|\beta|\le k} |D^\beta u|_{0;\Omega}.$$
Remark. The following two are not norms but semi-norms:
$$[u]_{k,\alpha;\Omega} = \sum_{|\beta|=k} [D^\beta u]_{\alpha;\Omega},$$
and
$$[u]_{k,0;\Omega} = [u]_{k;\Omega} = \sum_{|\beta|=k} |D^\beta u|_{0;\Omega}.$$

Lemma 9.14. *Let u be a harmonic function over $\mathbb{R}^n$. If there are constants $C, q > 0$ such that*
$$|u|_{0;B_R} \le CR^q\ \forall R \ge 1.$$
Then u is a polynomial.

Proof. At first, it is well known that
$$\frac{|\partial B_R|}{|B_R|} = \frac{n}{R}.$$
Denote ω_n be the volume of B_1.

For any $x \in \mathbb{R}^n$, any harmonic function u, we have
$$u(x) = \frac{1}{\rho^n \omega_n} \int_{B_\rho} u(x+y)dy, \forall \rho > 0.$$
and
$$u(x) = \frac{1}{\rho^{n-1} n\omega_n} \int_{\partial B_\rho} u(x+y)dS(y), \forall \rho > 0.$$
Since $D_i u$ is still a harmonic function on $\mathbb{R}^n$, we have
$$\begin{aligned} D_i u(x) &= \frac{1}{R^n \omega_n} \int_{B_R} D_i u(x+y)dy \\ &= \frac{1}{R^n \omega_n} \int_{\partial B_R} u(x+y) \cos(\overrightarrow{n}, y) dS(y) \\ &= \frac{1}{R\omega_n} \int_{|w|=1} u(x+Rw) w^i dw, \end{aligned}$$
where $\sum_{i=1}^n (w^i)^2 = 1$. By induction, we have

$$\begin{aligned} D_{i_1}D_{i_2}\cdots D_{i_k}u(x)) &= \frac{1}{R\omega_n}\int_{|w_1|=1} D_{i_2}\cdots D_{i_k}u(x+Rw_1)w_1^{i_1}dw_1 \\ &= \frac{1}{R^k\omega_n^k}\int_{|w_1|=1} dw_1\cdots\int_{w_k=1} u(x+Rw_1+\cdots Rw_k)w_1^{i_1}\cdots w_1^{i_k}dw_k, \end{aligned}$$

and so for all $x\in\mathbb{R}^n, k\geq 1$, we have

$$|D_{i_1}D_{i_2}\cdots D_{i_k}u(x)|\leq \frac{C}{R^k\omega_n^k}(|x|+kR)^q(n\omega_n)^k.$$

Fix R_0 such that $|x|\leq R_0$, Let $R\geq R_0$, if $k>q$ we then have

$$|D_{i_1}D_{i_2}\cdots D_{i_k}u(x)|\leq \frac{Cn^k}{R^{k-q}}(\frac{R_0}{R}+k)^q\to 0(R\to\infty).$$

Therefore, $D^ku(x)\equiv 0\ \forall x\in B_{R_0}$ if $k>q$, and then u has to be a polynomial of degree less than $q+1$.

□

Corollary 9.15. *Let $u\in C^2(\overline{\mathbb{R}^n_+})$ with $u(x',0)=0$, and u is a harmonic function over $\mathbb{R}^n_+$. If there are constants $C,q>0$ such that*

$$|u|_{0;B_R\cap\mathbb{R}^n_+}\leq CR^q\ \forall R\geq 1.$$

Then u is a polynomial.

Proof. Define

$$\widetilde{u}(x',x_n)=\begin{cases} u(x',x_n), & x_n\geq 0;\\ -u(x',-x_n), & x_n<0.\end{cases}$$

Then, $\widetilde{u}\in C^0(\mathbb{R}^n)$ and

$$D^2_{ij}\widetilde{u}\in C^0(\mathbb{R}^n),\ i+j\leq 2n-1.$$

On the other hand, we have

$$\begin{cases} \Delta\widetilde{u}=0, & x_n>0;\\ \Delta\widetilde{u}=-\Delta u(x'-x_n)=0, & x_n<0.\end{cases}$$

Thus, we have

$$\widetilde{u}_{x_nx_n}(x',+0)=u_{x_nx_n}(x',+0)=-\sum_{i=1}^{n-1}u_{x_ix_i}(x',0),$$

$$\widetilde{u}_{x_nx_n}(x',-0)=\sum_{i=1}^{n-1}u_{x_ix_i}(x',0)=\widetilde{u}_{x_nx_n}(x',+0);$$

and so $\widetilde{u}\in C^2(\mathbb{R}^n)$ and it is harmonic on $\mathbb{R}^n$. Since

$$|\widetilde{u}|_{0,B_R}=|u|_{0;B_R\cap\mathbb{R}^n_+}\leq CR^q\ \forall R\geq 1,$$

by the above lemma $\widetilde{u}$ is a polynomial, and then so is u.

□

Theorem 9.16. *Let $\alpha\in(0,1)$. Then there is a constant $C=C(n,\alpha)$ such that for any $u\in C^{2,\alpha}(\mathbb{R}^n)$, we have*

(9.16.1) $$[u]_{2,\alpha}\leq C[\Delta u]_\alpha.$$

Proof. 1. Assume the estimate 9.16.1 is not true for all $u \in C^{2,\alpha}(\mathbb{R}^n)$, then we have a series $u_k,\ k = 1, 2, \cdots$, such that

$$1 = [u_k]_{2,\alpha} \geq k[\Delta u_k]_\alpha,\ k = 1, 2, \cdots,$$

i.e., we have

$$[\Delta u_k]_\alpha < \frac{1}{k},\ k = 1, 2, \cdots. \tag{9.16.2}$$

2. By the definition, we have

$$[u]_{2,\alpha} = \sum_{|\beta|=2} [D^\beta u]_\alpha = \sum_{|\beta|=2} \sup_{x\in\mathbb{R}^n, e\in S^{n-1}, h>0} \frac{|D^\beta u_k(x + he) - D^\beta u_k(x)|}{h^\alpha},$$

it implies that for any $k \in \mathbb{N}$, We can choose $x_k \in \mathbb{R}^n, h_k > 0$ and $\beta^*(|\beta| = 2), e^* \in S^{n-1}$ with β^*, e^* independent of k, satisfying

$$\frac{|D^{\beta^*} u_k(x_k + h_k e^*) - D^{\beta^*} u_k(x_k)|}{h_k^\alpha} \geq \frac{1}{2n^3}.$$

(actually, we can choose $e^* = \mathbf{e}_{i_k}$ such that $i_k \neq \gamma, \delta$ where $|\beta^*| = \beta^*_\gamma + \beta^*_\delta$.)

3. Using scaling technique: Let $x = x_k + h_k y$ and denote $v_k(y) = h_k^{-\alpha-2} u_k(x_k + h_k y)$. Then, we have

$$D_y^2 v_k(y) == h_k^{-\alpha} D_x^2 u_k(x_k + h_k y),$$

and so we have: for $\forall k \in \mathbb{N}$, there is

$$[v_k]_{2,\alpha} = \sum_{|\beta|=2} [D^\beta v_k]_\alpha = \sum_{|\beta|=2} \sup_{y,y'\in\mathbb{R}^n} \frac{|D^\beta u_k(x_k + h_k y) - D^\beta u_k(x_k + h_k y')|}{h^\alpha |y - y'|^\alpha} = \sum_{|\beta|=2} [D^\beta u_k]_\alpha = [u_k]_{2,\alpha} = 1; \tag{9.16.3}$$

$$[\Delta v_k]_\alpha = [\Delta u_k]_\alpha \leq \frac{1}{k},\ \forall k \in \mathbb{N}; \tag{9.16.4}$$

$$|D^{\beta^*} v_k(e^*) - D^{\beta^*} v_k(0)| \geq \frac{1}{2n^3},\ \forall k \in \mathbb{N}. \tag{9.16.5}$$

4. Denote

$$\widetilde{v}_k(y) = v_k(y) - v_k(0) - \sum_{1\leq i\leq n} D_{y_i} v_k(0) y_i - \frac{1}{2} \sum_{1\leq i,j\leq n} D_{y_i y_j} v_k(0) y_i y_j.$$

Then, we have

$$\widetilde{v}_k(0) = 0,\ D\widetilde{v}_k(0) = 0,\ D^2\widetilde{v}_k(0) = 0,\ \forall k \in \mathbb{N}; \tag{9.16.6}$$

$$[\Delta \widetilde{v}_k]_\alpha \leq \frac{1}{k},\ \forall k \in \mathbb{N}; \tag{9.16.7}$$

$$|D^2\widetilde{v}_k(y)| \leq [\widetilde{v}_k]_{2,\alpha} |y|^\alpha = |y|^\alpha,\ |D\widetilde{v}_k(y)| \leq |y|^{\alpha+1},\ |\widetilde{v}_k(y)| \leq |y|^{\alpha+2},\ \forall k \in \mathbb{N}; \tag{9.16.8}$$

$$|D^{\beta^*} \widetilde{v}_k(e^*)| \geq \frac{1}{2n^3},\ \forall k \in \mathbb{N}. \tag{9.16.9}$$

Then, we obtain that for $\forall R > 0, \forall k \in \mathbb{N}$, $\widetilde{v}_k$ are bounded uniformly and is equai-continuous on $\overline{B_R}$, and so on $\overline{B_R}$ with any $R > 0$, we have

$$\widetilde{v}_k \Longrightarrow v^* \in C^2(\overline{B_R}), v^* \in C^{2,\alpha}(\overline{B_R});$$

and

$$[v^*]_{\alpha;B_R} \leq 2, \Delta v_k \Longrightarrow \Delta v^*$$

5. From 9.16.7, we have $[\Delta v^*]_\alpha \equiv 0$, and so Δv^* is a constant.Moreove, by 9.16.8 and let $y \to 0$, we have $\Delta v^* \equiv 0$, i.e., v^* is a harmonic function.Using the formula 9.16.8 and the above lemma 9.14,v^* is polynomial of degree at least 4. Thus,by 9.16.7,$D^2 v^*(0) = 0$ and so

$$D^{\beta^*} v^*(te^*) = a_i t^i + \cdots + a_1 t, \text{ with } i \le 2 \text{ and } a_i \ne 0.$$

   i. If $i \ge 1$, we have a contradiction from

$$\frac{|D^{\beta^*} v^*(te^*)|}{t^\alpha} \le [v^*]_{2,\alpha} \le 2 \ (\forall t > 0).$$

   ii. If $i = 0$, then $D^{\beta^*} v^*(te^*) \equiv 0$. But, by the formula 9.16.9 we have

$$|D^{\beta^*} v^*(e^*)| \ge \frac{1}{2n^3}.$$

   It is a contradiction.

□

Theorem 9.17. *Let $\alpha \in (0,1)$. Then there is a constant $C = C(n,\alpha)$ such that for any $u \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$ with $u|_{\partial \mathbb{R}^n_+} = 0$, we have*

(9.17.1) $$[u]_{2,\alpha;\mathbb{R}^n_+} \le C[\Delta u]_{\alpha;\mathbb{R}^n_+}.$$

Proof. 1. Assume the estimate 9.17.1 is not true for all $u \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$, then we have a series $u_k,\ k = 1, 2, \cdots$, such that

$$1 = [u_k]_{2,\alpha;\mathbb{R}^n_+} \ge k[\Delta u_k]_{\alpha;\mathbb{R}^n_+},\ k = 1, 2, \cdots,$$

i.e., we have

(9.17.2) $$[\Delta u_k]_{\alpha;\mathbb{R}^n_+} < \frac{1}{k},\ k = 1, 2, \cdots.$$

By the definition, we have

$$[u]_{2,\alpha;\mathbb{R}^n_+} = \sum_{|\beta|=2} [D^\beta u]_{\alpha;\mathbb{R}^n_+} = \sum_{|\beta|=2} \sup_{x \in \mathbb{R}^n_+, e \in S^{n-1}, h>0} \frac{|D^\beta u_k(x+he) - D^\beta u_k(x)|}{h^\alpha},$$

it implies that for any $k \in \mathbb{N}$, Similarly as in the proof of the theorem 9.16,we can choose $x_k \in \mathbb{R}^n_+, h_k > 0$ and $\beta^*(|\beta| = 2), e^* \in S^{n-1}$ with β^*, e^* independent of k, satisfying

$$\frac{|D^{\beta^*} u_k(x_k + h_k e^*) - D^{\beta^*} u_k(x_k)|}{h_k^\alpha} \ge \frac{1}{2n^3}.$$

2. Using scaling technique: Let $t_k = x_{k;n} h_k^{-1}$ where $x_k = (x_{k;1}, \cdots, x_{k;n})$. We consider the following two cases:
   a) In case of $t_k \to +\infty\,(h_k \to +0)$. Denote $x = x_k + y h_k$, then $y = h_k^{-1} x - x_k h_k^{-1}$ and $y_n > -t_k$ with $t_k \to -\infty$. Denote $v_k(y) = h_k^{-\alpha-2} u_k(x_k + h_k y)$. Then,

$$D_y^2 v_k(y) == h_k^{-\alpha} D_x^2 u_k(x_k + h_k y),$$

   also, we have: for $\forall k \in \mathbb{N}$,

(9.17.3)
$$[v_k]_{2,\alpha} = \sum_{|\beta|=2} [D^\beta v_k]_\alpha = \sum_{|\beta|=2} \sup_{y,y' \in \mathbb{R}^n} \frac{|D^\beta u_k(x_k + h_k y) - D^\beta u_k(x_k + h_k y')|}{h^\alpha |y - y'|^\alpha} = \sum_{|\beta|=2} [D^\beta u_k]_\alpha = [u_k]_{2,\alpha} = 1,$$

   and so

(9.17.4) $$[\Delta v_k]_\alpha = [\Delta u_k]_\alpha \le \frac{1}{k},\ \forall k \in \mathbb{N},$$

$$|D^{\beta^*}v_k(e^*) - D^{\beta^*}v_k(0)| \geq \frac{1}{2n^3},\ \forall k \in \mathbb{N}. \tag{9.17.5}$$

Similar as in the proof of the theorem 9.16, we get a harmonic function $v^*(y)$ defined on whole $\mathbb{R}^n$ with

$$[v^*]_{2,\alpha} \leq 2,\ |v^*(y)| \leq |y|^{2+\alpha}, y \in \mathbb{R}^n \ \text{ and } \ |D^{\beta^*}v^*(e^*) - D^{\beta^*}v^*(0)| \geq \frac{1}{2n^3}.$$

We then have a contradiction.

b) In case of $t_k \to t^* < +\infty$ ($h_k \to +0$). Writing $x_k = (x_k', x_{k;n})$, we denote $v_k(y) = h_k^{-\alpha-2}u_k((x_k', 0)+h_k y)$. Then,

$$v_k(y)|_{y_n=0} = 0,\ [v_k]_{2,\alpha} = 1,\ [\Delta v_k]_\alpha \leq \frac{1}{k},\ \forall k \in \mathbb{N}.$$

and

$$|D^{\beta^*}v_k((0,t_k)+e^*) - D^{\beta^*}v_k(0,t_k)| \geq \frac{1}{2n^3}.$$

Denote

$$\widetilde{v}_k(y) = v_k(y) - v_k(0) - \sum_{1\leq i\leq n} D_{y_i}v_k(0)y_i - \frac{1}{2}\sum_{1\leq i,j\leq n} D_{y_iy_j}v_k(0)y_iy_j.$$

Then, we have

$$\widetilde{v}_k(0) = 0,\ D\widetilde{v}_k(0) = 0,\ D^2\widetilde{v}_k(0) = 0,\ \forall k \in \mathbb{N}; \tag{9.17.6}$$

$$[\Delta\widetilde{v}_k]_\alpha \leq \frac{1}{k},\ \forall k \in \mathbb{N}; \tag{9.17.7}$$

$$|D^2\widetilde{v}_k(y)| \leq [\widetilde{v}_k]_{2,\alpha}|y|^\alpha = |y|^\alpha,\ |D\widetilde{v}_k(y)| \leq |y|^{\alpha+1},\ |\widetilde{v}_k(y)| \leq |y|^{\alpha+2},\ \forall k \in \mathbb{N}; \tag{9.17.8}$$

$$|D^{\beta^*}\widetilde{v}_k((0,t_k)+e^*) - D^{\beta^*}\widetilde{v}_k((0,t_k))| \geq \frac{1}{2n^3},\ \forall k \in \mathbb{N}. \tag{9.17.9}$$

Similar as in the proof of the theorem 9.16, we obtain $v^* \in C^{2,\alpha}(\overline{\mathbb{R}}^n_+)$ satisfying

$$[v^*]_{2,\alpha;\overline{\mathbb{R}}^n_+} \leq 2,\ |v^*(y)| \leq |y|^{2+\alpha}, \forall y \in \overline{\mathbb{R}}^n_+ \ \text{ and } \ |D^{\beta^*}v^*((0,t^*)+e^*) - D^{\beta^*}v^*(0,t^*)| \geq \frac{1}{2n^3}, \tag{9.17.10}$$

$$\begin{cases} \Delta v^* = 0, \text{in } \overline{\mathbb{R}}^n_+ \\ v^*|_{\partial\overline{\mathbb{R}}^n_+} = 0. \end{cases} \tag{9.17.11}$$

Thus, from 9.17.10,9.17.11 and the corollary 9.15, v^* is a polynomial on $\overline{\mathbb{R}}^n_+$. Then it contradicts to that

$$|D^{\beta^*}v^*((0,t^*)+e^*) - D^{\beta^*}v^*(0,t^*)| \geq \frac{1}{2n^3}.$$

□

9.3. **Campanato's characterization of L^2 functions to be Hölder continuous.**

Theorem 9.18 (Campanato Lemma). *Suppose $u \in L^2(B_{2R}(x_0)), \alpha \in (0,1], \beta > 0$ are constants, and*

$$\inf_{\lambda\in\mathbb{R}} \rho^{-n}\int_{B_\rho(y)} |u-\lambda|^2 \le \beta^2(\frac{\rho}{R})^{2\alpha} \tag{9.18.1}$$

for every ball $B_\rho(y)$ such that $y \in B_R(x_0)$ and $\rho \le R$. Then there is a Hölder continuous representation $\widetilde{u}$ for the L^2-class of u with

$$|\widetilde{u}(x) - \widetilde{u}(y)| \le C_{n,\alpha}\beta(\frac{|x-y|}{R})^\alpha.\ \forall\, x, y \in B_R(x_0),$$

where $C_{n,\alpha}$ depends only on n and α.

Remark. Actually,

$$\inf_{\lambda\in\mathbb{R}}\int_{B_\rho(y)} |u-\lambda|^2 = \int_{B_\rho(y)} |u-\lambda_{y,\rho}|^2,$$

where

$$\lambda_{y,\rho} = \frac{1}{\omega_n\rho^n}\int_{B_\rho(y)} u.$$

Proof. a) First by the inequality 9.18.1, we note that

$$(\frac{\rho}{2})^{-n}\int_{B_{\rho/2}(y)} |u-\lambda_{y,\rho}|^2 \le 2^n\rho^{-n}\int_{B_\rho(y)} |u-\lambda_{y,\rho}|^2 \le 2^n\beta^2(\frac{\rho}{R})^{2\alpha},$$

where

$$\lambda_{y,\rho} = \frac{1}{\omega_n\rho^n}\int_{B_\rho(y)} u.$$

On the other hand,we have

$$(\frac{\rho}{2})^{-n}\int_{B_{\rho/2}(y)} |u-\lambda_{y,\rho/2}|^2 = \inf_{\lambda\in\mathbb{R}}(\frac{\rho}{2})^{-n}\int_{B_{\rho/2}(y)} |u-\lambda|^2 \le (\frac{\rho}{2})^{-n}\int_{B_{\rho/2}(y)} |u-\lambda_{y,\rho}|^2 \le 2^n\beta^2(\frac{\rho}{R})^{2\alpha}.$$

Since

$$|\lambda_{y,\rho} - \lambda_{y,\rho/2}|^2 \le 2(|u-\lambda_{y,\rho}|^2 + |u-\lambda_{y,\rho/2}|^2),$$

we conclude that

$$|\lambda_{y,\rho/2} - \lambda_{y,\rho}| \le 2^N\omega_n^{-\frac{1}{2}}\beta(\frac{\rho}{R})^\alpha$$

provided that $\rho < R$ and $y \in B_R(x_0)$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

b) For any $\nu = 0, 1, 2, \cdots$, we can choose $\rho = 2^{-\nu}R$ and obtain

$$|\lambda_{y,2^{-\nu-1}\rho} - \lambda_{y,2^{-\nu}\rho}| \le 2^N\omega_n^{-\frac{1}{2}}\beta 2^{-\nu\alpha}. \tag{9.18.2}$$

Denote

$$s_k = \sum_{\nu=0}^{k}(\lambda_{y,2^{-\nu-1}\rho} - \lambda_{y,2^{-\nu}\rho}).$$

We find

$$\sum_{\nu=0}^{\infty}|\lambda_{y,2^{-\nu-1}\rho} - \lambda_{y,2^{-\nu}\rho}| < \infty,$$

hence $\lim_{k\to\infty} s_k$ exists and so does $\lim_{k\to\infty}\lambda_{y,2^{-\nu}\rho}$:

$$\lambda_y := \lim_{k\to\infty}\lambda_{y,2^{-\nu}\rho} = \lim_{k\to\infty} s_k + \lambda_{y,\rho} < \infty.$$

Moreover, denote $\widetilde{u} := \lambda_y$,using the Lebesgue Lemma, we have

$$\widetilde{u}(y) = u(y), \text{ a.e. on } B_{2R}(x_0),$$

and

$$|\lambda_{y,2^{-\nu}\rho} - \widetilde{u}y| \leq \sum_{j=\nu}^{\infty} |\lambda_{y,2^{-j-1}\rho} - \lambda_{y,2^{-j}\rho}| \leq C\beta 2^{-\nu\alpha},$$

where the constant $C = C(n, \alpha)$.

c) Using

$$|u - \widetilde{u}(y)|^2 \leq 2(|u - \lambda_{y,2^{-\nu}\rho}|^2 + |\widetilde{u}(y) - \lambda_{y,2^{-\nu}\rho}|^2),$$

we obtain

$$\rho^{-n} \int_{B_\rho(y)} |u - \widetilde{u}(y)|^2 dx \leq C\beta^2 (\frac{\rho}{R})^{2\alpha}, \forall \rho = 2^{-\nu} R \ \nu = 0, 1, 2, \cdots .$$

On the other hand for any $\rho \in (0, R/2]$ there is an integer $\nu_0 \leq 1$ such that $2^{-nu-1}R < \rho \leq 2^{-\nu}R$. Replacing $2^{n+2\alpha}C$ with C, we conclude that

$$\rho^{-n} \int_{B_\rho(y)} |u - \widetilde{u}(y)|^2 dx \leq C\beta^2 (\frac{\rho}{R})^{2\alpha}, \forall \rho \in (0, R/2] \tag{9.18.3}$$

d) Now take any pair $y, z \in B_R(x_0)$ with $|y - z| \leq R/4$ and let $\rho = |y - z|$. Since

$$B_{\rho/2}(\frac{y+z}{2}) \subset B_\rho(y) \cap B_\rho(z),$$

and

$$|\widetilde{u}(y) - \widetilde{u}(z)|^2 \leq 2(|u - \widetilde{u}(y)|^2 + |u - \widetilde{u}(z)|^2),$$

the inequality 9.18.3 gives

$$\begin{aligned}
|\widetilde{u}(y) - \widetilde{u}(z)|^2 &\leq |B_{\rho/2}(\frac{y+z}{2})|^{-1} \int_{B_{\rho/2}(\frac{y+z}{2})} 2(|u - \widetilde{u}(y)|^2 + |u - \widetilde{u}(z)|^2) dx \\
&\leq 2C\rho^{-n} \int_{B_\rho(y)} |u - \widetilde{u}(y)|^2 dx + 2C\rho^{-n} \int_{B_\rho(z)} |u - \widetilde{u}()|^2 dx \\
&\leq 4C\beta^2 (\frac{\rho}{R})^{2\alpha}.
\end{aligned}$$

Hence,

$$|\widetilde{u}(y) - \widetilde{u}(z)| \leq 2C\beta(\frac{\rho}{R})^\alpha = 2^{1+\alpha} C\beta(\frac{|y-z|}{R})^\alpha, \ \forall y, z \in B_R(x_0) \text{ with } |y - z| \leq R/4. \tag{9.18.4}$$

e) If $y, z \in B_R(x_0)$, we can pick points $z_1 = y, \cdots, z_8 = z$ on the segment jointing y, z such that $|z_i - z_{i+1}| \leq R/4$. Applying 9.18.4, we obtain

$$|\widetilde{u}(y) - \widetilde{u}(z)| \leq C(n, \alpha)\beta \frac{|y-z|^\alpha}{R^\alpha}, \ \forall y, z \in B_R(x_0).$$

We complete the proof.

□

Theorem 9.19 (Morrey's Lemma)**.** *Let $B_R(x_0) \subset \mathbb{R}^n$ be an open ball. Suppose $u \in W^{1,2}(B_R(x_0)), \alpha \in (0, 1], \beta > 0$ are constants, and*

$$\rho^{2-n} \int_{B_| Rho(y)} |Du|^2 dx \leq \beta^2 (\frac{\rho}{R})^{2\alpha}, \ \forall y \in B_R(x_0), \ \rho \in (0, \frac{R}{2}].$$

Then $u \in C^{0,\alpha}(\overline{B}_R(x_0))$ *(there is a Hölder continuous representation for the* L^2*-class of* u *), and in fact*

$$|u(x) - u(y)| \leq C\beta(\frac{|x-y|}{R})^{\alpha}, \forall x, y \in B_R(x_0),$$

where C *depends only on* n*.*

Proof. The Poincaré inequality (c.f. the remark 9.9.2) gives

$$\int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 dx \leq C\rho^{2-n} \int_{B_\rho(y)} |Du|^2 dx \leq \beta^2 (\frac{\rho}{R})^{2\alpha},$$

for each $y \in B_{\frac{R}{2}}(x_0)$ and each $\rho \in (0, \frac{R}{2}]$, where

$$\lambda_{y,\rho} = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u.$$

Using the Campanato Lemma (c.f. the theorem 9.18.1), we then have the required result. □

10. Advanced Topics in Analysis

10.1. Topic on Riemann Zeta Function(To be Continuous). To be Continuous .....

10.2. **Elementary on Nevanlinna Theory(To be Continuous). To be Continuous .....**

10.3. **Elementary on p-adic Series(To be Continuous). To be Continuous .....**