15.094, Problem Set 1

Due: 18 February 2014 at 5pm EST (via Stellar)

Problem 1 (20 points)

Let d_t be the demand for a product at time $t \in \mathcal{T} := \{1, \ldots, T\}$. From historical data, we have estimated the mean and covariance matrix for $\mathbf{d} = (d_t)_{t \in \mathcal{T}} \in \mathbb{R}^T$, which we denote by $\boldsymbol{\mu} \in \mathbb{R}^T$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{T \times T}$, respectively.

(a) (10 points) Formulate an uncertainty set for $\mathbf{d}$ that takes into account this historical information.

(b) (10 points) Suppose that the rank of $\boldsymbol{\Sigma}$ is $\kappa \ll T$. Formulate an uncertainty set for $\mathbf{d}$ that incorporates this new information.

Problem 2 (25 points)

Every instance of robustification we have focused on thus far has considered constraints of the form

$$\mathbf{a}'\mathbf{x} \leq b \;\forall \mathbf{a} \in \mathcal{U}.$$

In this exercise we will focus on what happens with equality constraints.

(a) (10 points) Consider the robust equality constraint

$$\mathbf{a}'\mathbf{x} = b \;\forall \mathbf{a} \in \mathcal{U},$$

where $\mathcal{U} = \{\mathbf{a} : \mathbf{A}\mathbf{a} \leq \mathbf{d}\}$. Using the fact that

$$\mathbf{a}'\mathbf{x} = b \;\forall \mathbf{a} \in \mathcal{U} \iff \mathbf{a}'\mathbf{x} \leq b \;\forall \mathbf{a} \in \mathcal{U} \quad \text{and} \quad \mathbf{a}'\mathbf{x} \geq b \;\forall \mathbf{a} \in \mathcal{U}$$

rewrite the uncertain constraint $\mathbf{a}'\mathbf{x} = b \;\forall \mathbf{a} \in \mathcal{U}$ as a deterministic, finite number of linear inequality constraints (using the usual duality methods).

(b) (10 points) Despite the fact that part (a) suggests that we can in fact robustify equality constraints, this is essentially never done. The primary reason is that robust equality constraints often lead to infeasibility. In this part we consider the homogenous case when $b = 0$. Prove that $\{\mathbf{x} : \mathbf{a}'\mathbf{x} = b \;\forall \mathbf{a} \in \mathcal{U}\} = \{\mathbf{0}\}$ if and only if $\mathcal{U}$ contains a basis (here, basis has the usual linear algebra definition: a linear independent spanning set).

(c) (5 points) Now consider the non-homogenous case where $b \neq 0$. State and prove an analogous claim as to that given in part (b).

Hint: affine independence.

N.B. Note that in particular, this implies that if $\mathcal{U}$ is full-dimensional, then the robustified equality constraint is infeasible (if $b \neq 0$).

N.B. Is there hope for equality constraints? Yes. It is often the case that often one really cares about only one of the constraints $\mathbf{a}'\mathbf{x} \leq b$ or $\mathbf{a}'\mathbf{x} \geq b$, and so you can robustify the one that matters. Further, based on problem structure, it is also the case that you may only need to include one of these, and you can guarantee that it is binding (at equality) in all optimal solutions. This is a very standard technique throughout optimization theory.

Problem 3 (30 points)

The typical mantra of robust optimization is that "the robust analogue of (Optimization Problem of Type X) is an (Optimization Problem of Type X) of comparable size." We will explore the validity of such a statement for robust LPs. Modern LP solvers can solve massive scale LPs (with millions of variables and constraints). It is possible to solve even larger problems when the systems of inequalities are well-structured or sparse. Sparse LPs are ones for which many of the coefficients in the constraints are zeros. The focus of the question is, *what happens to sparsity properties of an LP under robustification?*

(a) (10 points) Consider the constraint $\mathbf{a}'\mathbf{x} \leq b \ \forall \mathbf{a} \in \mathcal{U}$, where $\mathcal{U} = \{\mathbf{a} : \|\mathbf{a} - \widehat{\mathbf{a}}\|_\infty \leq \epsilon\}$ (here $\widehat{\mathbf{a}} \in \mathbb{R}^n$ and $\epsilon > 0$ are fixed; $\|\cdot\|_\infty$ denotes the ℓ_∞ norm, with $\|\mathbf{a}\|_\infty = \max_i |a_i|$). Rewrite the semi-infinite constraint as a finite number of linear inequality constraints. *Hint: you will likely need to use auxiliary variables.*

(b) (5 points) Comment on the sparsity of the new representation in part (a). If $\widehat{\mathbf{a}}$ has mostly zero entries (in which case we would call the constraint $\widehat{\mathbf{a}}'\mathbf{x} \leq b$ "sparse"), are the constraints in the new representation of the uncertain constraint still sparse? Please be as specific as possible.

(c) (10 points) Repeat parts (a) and (b) with a different uncertainty set:

$$\mathcal{U} = \{\mathbf{a} : \|\mathbf{a} - \widehat{\mathbf{a}}\|_1 \leq \epsilon\}.$$

(Here $\|\mathbf{a}\|_1 = \sum_i |a_i|$.)

(d) (5 points) What differences, if any, did you observe between parts (b) and (c)? Offer an explanation for why there are (or are not) differences. Does this say anything about how to choose uncertainty sets?

Problem 4 (25 points)

Consider the following robust optimization problem

$$\begin{aligned}
\min_{x_1, x_2} \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1, x_2 \in \mathbb{R} \\
& \left.\begin{array}{l} x_1 \geq \xi_1 \\ x_2 \geq \xi_2 \end{array}\right\} \quad \forall\, (\xi_1, \xi_2) \in \Xi,
\end{aligned} \tag{1}$$

where $\Xi = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| + |\xi_2| \leq 1\}$.

(a) (20 points) Formulate and solve the robust counterpart of (1). Provide us with your commented code and the optimal solution obtained.

(b) (5 points) Do you find the solution intuitive? Is it what you expected?

N.B. For this exercise, you may use the programming language and solver of your choice. For LPs and SOCPs, the solvers most commonly used by academics are CPLEX and Gurobi (both freely available for students).

Problem 5 (20 points, **OPTIONAL EXTRA CREDIT**)

Fix $\mathbf{a}_i \in \mathbb{R}^n$ and $\epsilon_i > 0$ for $i = 1, \ldots, m$. Fix $p \in [1, \infty]$ and let $\|\cdot\|_p$ denote the usual ℓ_p norm. Let $\mathcal{U}_i = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{a}_i - \mathbf{x}\|_p \le \epsilon_i\}$. Write the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1' \\ \vdots \\ \mathbf{a}_m' \end{pmatrix} \in \mathbb{R}^{m \times n},$$

and suppose that $\mathbf{b} \in \mathbb{R}^m$ so that $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \le b\}$ is a full-dimensional, bounded polyhedron. Let $\mathrm{vol}(S)$ denote the n-dimensional volume for a set $S \subseteq \mathbb{R}^n$. Compute explicitly (or provide estimates of) the ratio

$$\frac{\mathrm{vol}\left(\{\mathbf{x} : \mathbf{a}'\mathbf{x} \le b_i \ \forall \mathbf{a} \in \mathcal{U}_i \ \forall i\}\right)}{\mathrm{vol}(P)}.$$

Can you provide an estimate in the case when only one ϵ_i is nonzero? How does your answer depend on p?

Hint: it is instructive to focus on different examples, such as hypercubes, simplices, etc.