17.2

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课本例题

例 1 设 S 为锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $x^2+y^2=2ax\ (a>0)$ 割下的部分, 求

$$I=\iint\limits_S (x^2y^2+y^2z^2+z^2x^2)dS.$$

解: 曲面 S 在 xy 平面上的投影为 $D=\{(x,y)\mid\ x^2+y^2\le 2ax\}$, 见图 17.2. 在直角坐标系中计算,

$$z_x=\frac{x}{\sqrt{x^2+y^2}}, z_y=\frac{y}{\sqrt{x^2+y^2}}, \sqrt{1+z_x^2+z_y^2}=\sqrt{2},$$

由公式(**??**)

$$I=\iint\limits_{x^2+y^2\le 2ax}[x^2y^2+(x^2+y^2)^2]\sqrt{2}\mathrm{d}x\mathrm{d}y.$$

作极坐标变换, 则

$$\begin{aligned}
I &= \sqrt{2}\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\int_0^{2a\cos\theta}(r^4\cos^2\theta\sin^2\theta+r^4)rdr\\
&= \sqrt{2}\int_{-\pi/2}^{\pi/2}(\cos^2\theta\sin^2\theta+1)\cdot\left(\frac{1}{6}r^6\Big|_0^{2a\cos\theta}\right)\mathrm{d}\theta\\
&= \frac{\sqrt{2}}{6}(2a)^6\int_{-\pi/2}^{\pi/2}\cos^6\theta(\cos^2\theta\sin^2\theta+1)d\theta\\
&= \frac{\sqrt{2}}{6}(2a)^6 2\int_0^{\pi/2}\cos^6\theta(\cos^2\theta-\cos^4\theta+1)d\theta\\
&= \frac{\sqrt{2}}{3}(2a)^6\Big(\frac{7!!}{8!!}\frac{\pi}{2}-\frac{9!!}{10!!}\frac{\pi}{2}+\frac{5!!}{6!!}\frac{\pi}{2}\Big)=\frac{29}{8}\sqrt{2}\pi a^6.
\end{aligned}$$

□

例 2 设 S 是立体 $\sqrt{x^2+y^2}\le z\le 1$ 的边界曲面, 求

$$\iint\limits_S (x^2+y^2)\mathrm{d}S.$$

解: 记 $S=S_1\cup S_2$, 其中

$$\begin{aligned}
&S_1: z=1, (x,y)\in D=\{(x,y)\mid x^2+y^2\le 1\},\\
&S_2: z=\sqrt{x^2+y^2}, (x,y)\in D.
\end{aligned}$$

则在 S_1 上, $\sqrt{1+z_x^2+z_y^2}=1$; 在 S_2 上, $\sqrt{1+z_x^2+z_y^2}=\sqrt{2}$, 因此

$$\begin{aligned}\iint\limits_S (x^2+y^2)\mathrm{d}S &= \iint\limits_{S_1}(x^2+y^2)\mathrm{d}S+\iint\limits_{S_2}(x^2+y^2)\mathrm{d}S \\ &= \iint\limits_{x^2+y^2\le 1}(x^2+y^2)\cdot 1\mathrm{d}x\mathrm{d}y+\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\cdot\sqrt{2}\mathrm{d}x\mathrm{d}y \\ &= (1+\sqrt{2})\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\mathrm{d}x\mathrm{d}y \\ &= (1+\sqrt{2})\int_0^{2\pi}\int_0^1 r^3\mathrm{d}r d\varphi \\ &= (1+\sqrt{2})2\pi\frac{1}{4}=\frac{\pi}{2}(1+\sqrt{2}). \qquad \square\end{aligned}$$

$\square$

例 3 利用公式 *(17.2.2)* 求例 *1* 中的积分.

解: $z=\sqrt{x^2+y^2}$, 球坐标系中的方程为 $\varphi=\dfrac{\pi}{4}$, 因此 S 的参数方程为

$$x=\frac{1}{\sqrt{2}}r\cos\theta, y=\frac{1}{\sqrt{2}}r\sin\theta, z=\frac{1}{\sqrt{2}}r, (r,\theta)\in D.$$

又 S 的边界曲线 $\begin{cases} z=\sqrt{x^2+y^2}, \\ x^2+y^2=2ax \end{cases}$ 的球坐标表示为

$$\varphi=\frac{\pi}{4},\quad r^2\sin^2\varphi=2ar\sin\varphi\cos\theta.$$

于是

$$D=\left\{(r,\theta)\middle| -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, 0\le r\le 2\sqrt{2}a\cos\theta\right\}.$$

计算得

$$E=\frac{r^2}{2}, F=0, G=1,$$

最后得到

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2}d\theta\int_0^{2\sqrt{2}a\cos\theta}\left(\frac{1}{4}r^4\cos^2\theta\sin^2\theta+\frac{1}{4}r^4\right)\frac{r}{\sqrt{2}}dr \\ &= \frac{29}{8}\sqrt{2}\pi a^6. \qquad \square\end{aligned}$$

$\square$

思考题

1. 写出第一型曲面积分的主要性质.

解: 性质 1 (线性性质) 若 $\iint\limits_S f_i(x,y,z)\mathrm{d}S(i=1,2,\cdots,k)$ 存在, $c_i(i=1,2,\cdots,k)$ 为常数, 则 $\iint\limits_S\sum_{i=1}^k c_if_i(x,y,z)\mathrm{d}S$ 也存在, 且

$$\iint\limits_S\sum_{i=1}^k c_if_i(x,y,z)\mathrm{d}S=\sum_{i=1}^k c_i\iint\limits_S f_i(x,y,z)\mathrm{d}S.$$

性质 2 (积分路径可加性) 若分片曲线 S 由曲面 $S_1, S_2, \cdots, S_k$ 首尾相接而成, 且 $\iint\limits_{S_i} f(x,y,z)\mathrm{d}S$ $(i = 1, 2, \cdots, k)$ 都存在, 则 $\iint\limits_{S} f(x,y,z)\mathrm{d}S$ 也存在, 且

$$\iint\limits_{S} f(x,y,z)\mathrm{d}S = \sum_{i=1}^{k} \iint\limits_{S_i} f(x,y,z)\mathrm{d}S.$$

性质 3 (单调性) 若 $\iint\limits_{S} f(x,y,z)\mathrm{d}S$ 与 $\iint\limits_{S} g(x,y,z)\mathrm{d}S$ 都存在, 且在 S 上 $f(x,y,z) \le g(x,y,z)$, 则

$$\iint\limits_{S} f(x,y,z)\mathrm{d}S \le \iint\limits_{S} g(x,y,z)\mathrm{d}S.$$

性质 4 (绝对可积性) 若 $\iint\limits_{S} f(x,y,z)\mathrm{d}S$ 存在, 则 $\iint\limits_{S} |f(x,y,z)|\mathrm{d}S$ 也存在, 且

$$\Big|\iint\limits_{S} f(x,y,z)\mathrm{d}S\Big| \le \iint\limits_{S} |f(x,y,z)|\mathrm{d}S.$$

性质 5 (积分中值定理) 若 $\iint\limits_{S} f(x,y,z)\mathrm{d}S$ 存在, S 的弧长为 L, 则存在常数 c, 使得

$$\iint\limits_{S} f(x,y,z)\mathrm{d}S = cL,$$

其中 $\inf\limits_{S} f(x,y,z) \le c \le \sup\limits_{S} f(x,y,z)$.

□

2. 说明公式 (17.2.1) 是公式 (17.2.2) 的特殊情形.

解: 在公式 (17.2.1) 中的曲面 $z = z(x,y)$ 可视为用如下参数方程表示

$$S: \begin{cases} x = x, \\ y = y, \\ z = z(x,y), \end{cases} \qquad (x,y) \in D,$$

即在公式 (17.2.2) 中有 $u = x, v = y$, 从而可计算得到

$$\begin{aligned} E &= x_u^2 + y_u^2 + z_u^2 = 1 + z_x^2, \\ F &= x_u x_v + y_u y_v + z_u z_v = z_x z_y, \\ G &= x_v^2 + y_v^2 + z_v^2 = 1 + z_y^2. \end{aligned}$$

故有

$$EG - F^2 = 1 + z_x^2 + z_y^2,$$

从而结论得证. □

习题

1. 计算下列第一型曲面积分.

(1) $\iint\limits_S (x+y+z)^2 \mathrm{d}S$, 其中 S 为单位球面 $x^2+y^2+z^2=1$;

(2) $\iint\limits_S (x^4-y^4+y^2z^2-z^2x^2+1)\mathrm{d}S$, 其中 S 是锥面 $z^2=x^2+y^2$ 被柱面 $x^2+y^2=2x$ 割下部分;

(3) $\iint\limits_S |xyz|\mathrm{d}S$, 其中 S 是曲面 $|x|+|y|+|z|=1$;

(4) $\iint\limits_S z^2\mathrm{d}S$, 其中 S 是锥面 $z=\sqrt{x^2+y^2}$ 在球面 $x^2+y^2+z^2=R^2$ 内的部分.

解: (1) 解法一:

在 S 上有 $x^2+y^2+z^2=1$, 于是

$$\begin{aligned}\iint\limits_S (x+y+z)^2\mathrm{d}S &= \iint\limits_S \left(x^2+y^2+z^2+2xz+2xy+2yz\right)\mathrm{d}S \\ &= \iint\limits_S (1+2xz+2xy+2yz)\mathrm{d}S. \qquad (1)\end{aligned}$$

记 $S=S_1\cup S_2$, 其中

$$\begin{aligned} S_1 &: \ z=\sqrt{1-x^2-y^2}, \quad (x,y)\in D=\{(x,y)\mid x^2+y^2\le 1\}, \\ S_2 &: \ z=-\sqrt{1-x^2-y^2}, \quad (x,y)\in D. \end{aligned}$$

在曲面 S_1 上, 有

$$z_x=-\frac{x}{\sqrt{1-x^2-y^2}}, \quad z_y=-\frac{y}{\sqrt{1-x^2-y^2}}, \quad \sqrt{1+z_x^2+z_y^2}=\frac{1}{\sqrt{1-x^2-y^2}},$$

从而有

$$\begin{aligned}&\iint\limits_{S_1} 1+2xz+2xy+2yz\mathrm{d}S \\ &= \iint\limits_D \left(1+2xy+2x\cdot\sqrt{1-x^2-y^2}+2y\sqrt{1-x^2-y^2}\right)\cdot\frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_D \left(\frac{1}{\sqrt{1-x^2-y^2}}+2xy+2x+2y\right)\mathrm{d}x\mathrm{d}y. \qquad (2)\end{aligned}$$

在曲面 S_2 上, 有

$$z_x=\frac{x}{\sqrt{1-x^2-y^2}}, \quad z_y=\frac{y}{\sqrt{1-x^2-y^2}}, \quad \sqrt{1+z_x^2+z_y^2}=\frac{1}{\sqrt{1-x^2-y^2}},$$

从而有

$$
\begin{aligned}
&\iint\limits_{S_2} 1+2xz+2xy+2yz\mathrm{d}S \\
&\quad=\iint\limits_{D}\left(1+2xy-2x\cdot\sqrt{1-x^2-y^2}-2y\sqrt{1-x^2-y^2}\right)\cdot\frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}x\mathrm{d}y \\
&\quad\iint\limits_{D}\left(\frac{1}{\sqrt{1-x^2-y^2}}+2xy-2x-2y\right)\mathrm{d}x\mathrm{d}y. \qquad (3)
\end{aligned}
$$

把 (2) 和 (3) 代入 (1) 中, 有

$$
\iint\limits_{S}(1+2xz+2xy+2yz)\mathrm{d}S=2\iint\limits_{D}\left(\frac{1}{\sqrt{1-x^2-y^2}}+2xy\right)\mathrm{d}x\mathrm{d}y.
$$

作极坐标变换 $\left\{\begin{array}{l} x=r\cos\theta, \\ y=r\sin\theta, \end{array}\right.$ 则 D 与 $\triangle=\{(r,\theta)|0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant 2\pi\}$ 一一对应, 于是有

$$
\begin{aligned}
&\iint\limits_{D}\left(\frac{1}{\sqrt{1-x^2-y^2}}+\frac{2xy}{\sqrt{1-x^2-y^2}}\right)\mathrm{d}x\mathrm{d}y \\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(\frac{1}{\sqrt{1-r^2}}+2r^2\sin\theta\cos\theta\right)\cdot r\mathrm{d}r \\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(\frac{r}{\sqrt{1-r^2}}+2r^3\sin\theta\cos\theta\right)\mathrm{d}r \\
&=\int_0^{2\pi}\mathrm{d}\theta\left(-\frac{1}{2}\int_0^1\frac{1}{\sqrt{1-r^2}}\mathrm{d}(1-r^2)+\int_0^1(\sin 2\theta)\cdot(r^3)\mathrm{d}\right) \\
&=\int_0^{2\pi}\left(-\sqrt{1-r^2}\Big|_0^1+\sin 2\theta\frac{r^4}{4}\Big|_0^1\right)\mathrm{d}\theta \\
&=\int_0^{2\pi}\left(1+\sin 2\theta\frac{1}{4}+\frac{2}{3}\cos\theta+\frac{2}{3}\sin\theta\right)\mathrm{d}\theta \\
&=\theta|_0^{2\pi}-\frac{1}{4}\cdot\cos 2\theta|_0^{2\pi} \\
&=2\pi.
\end{aligned}
$$

因此, 有

$$
\iint\limits_{S}(1+2xz+2xy+2yz)\mathrm{d}S=4\pi.
$$

解法二:

因为 S 的参量方程为 $x=\sin\phi\cos\theta, y=\sin\phi\sin\theta, z=\cos\phi$, 其中 $(\phi,\theta)\in D=[0,\pi]\times[0,2\pi]$, 且有

$$
E=x_\phi^2+y_\phi^2+z_\phi^2=1,\quad F=x_\phi x_\theta+y_\phi y_\theta+z_\phi z_\theta=0, G=x_\theta^2+y_\theta^2+z_\theta^2=\sin^2\phi,
$$

于是有

$$
\begin{aligned}
&\iint\limits_S (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\mathrm{d}S \\
= &\iint\limits_S x^2 + y^2 + z^2 + 2xz + 2xy + 2yz\mathrm{d}S \\
= &\iint\limits_S 1 + 2xz + 2xy + 2yz\mathrm{d}S \\
= &\iint\limits_D \left(1 + 2\sin^2\phi\cos\theta\sin\theta + 2\sin\phi\cos\theta\cos\phi + 2\sin\phi\sin\theta\cos\phi\right)\sqrt{EG - F^2}\mathrm{d}\phi\mathrm{d}\theta \\
= &\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \left(1 + 2\sin^2\phi\cos\theta\sin\theta + 2\sin\phi\cos\theta\cos\phi + 2\sin\phi\sin\theta\cos\phi\right)\sin\phi\mathrm{d}\phi \\
= &\int_0^{2\pi} \mathrm{d}\theta \left(\int_0^{\pi}\mathrm{d}\phi + \int_0^{\pi} 2\sin^3\phi\cos\theta\sin\theta\mathrm{d}\phi + \int_0^{\pi} 2\sin^2\phi\cos\theta\cos\phi\mathrm{d}\phi + \int_0^{\pi} 2\sin^2\phi\sin\theta\cos\phi\mathrm{d}\phi\right) \\
= &\int_0^{2\pi} \left(-\cos\phi|_0^{\pi} + \sin 2\theta\cdot 2\cdot\frac{2}{3}\bigg|_0^{\pi} + 2\cos\theta\frac{\sin^3\phi}{3}\bigg|_0^{\pi} + 2\sin\theta\frac{\sin^3\phi}{3}\bigg|_0^{\pi}\right)\mathrm{d}\theta \\
= &\int_0^{2\pi} 2 + \frac{3}{4}\sin 2\theta\mathrm{d}\theta \\
= &\ 2\theta + \frac{3}{4}\left(-\frac{\cos 2\theta}{2}\right)\bigg|_0^{2\pi} \\
= &\ 4\pi.
\end{aligned}
$$

(2) 记 $S = S_1 \cup S_2$, 其中

$$
\begin{aligned}
&S_1 : z = \sqrt{x^2 + y^2}, (x, y) \in D = \{(x, y) \mid (x-1)^2 + y^2 \le 1\}, \\
&S_2 : z = -\sqrt{x^2 + y^2}, (x, y) \in D.
\end{aligned}
$$

在曲面 S_1 上, 有

$$
z_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad z_y = \frac{y}{\sqrt{x^2 + y^2}}, \quad \sqrt{1 + z_x^2 + z_y^2} = \sqrt{2},
$$

于是有

$$
\begin{aligned}
&\iint\limits_{S_1} (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\mathrm{d}S \\
&= \iint\limits_D (x^4 - y^4 + y^2(x^2 + y^2) - (x^2 + y^2)x^2 + 1)\cdot\sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_D 1\cdot\sqrt{2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\iint\limits_D \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

在曲面 S_2 上, 有

$$
z_x = -\frac{x}{\sqrt{x^2 + y^2}}, \quad z_y = -\frac{y}{\sqrt{x^2 + y^2}}, \quad \sqrt{1 + z_x^2 + z_y^2} = \sqrt{2},
$$

于是有

$$
\begin{aligned}
&\iint\limits_{S_2}(x^4-y^4+y^2z^2-z^2x^2+1)\mathrm{d}S\\
&\quad=\iint\limits_{D}(x^4-y^4+y^2(x^2+y^2)-(x^2+y^2)x^2+1)\cdot\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y\\
&\quad=\iint\limits_{D}1\cdot\sqrt{2}\mathrm{d}x\mathrm{d}y=\sqrt{2}\iint\limits_{D}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

注意到 D 是半径为 1 的圆, 所以 $\iint\limits_{D}\mathrm{d}x\mathrm{d}y=\pi\cdot 1^2=\pi$, 所以

$$
\begin{aligned}
&\iint\limits_{S}(x^4-y^4+y^2z^2-z^2x^2+1)\mathrm{d}S\\
&\quad=\left(\iint\limits_{S_1}+\iint\limits_{S_2}\right)(x^4-y^4+y^2z^2-z^2x^2+1)\mathrm{d}S\\
&\quad=2\sqrt{2}\iint\limits_{D}\mathrm{d}x\mathrm{d}y=2\sqrt{2}\pi.
\end{aligned}
$$

(3) S 图形如 (1) 记 $S=S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6\cup S_7\cup S_8$, 其中

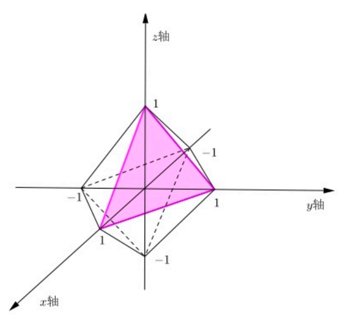

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图 1: 曲面 $|x|+|y|+|z|=1$

$$
\begin{aligned}
&S_1: z=1-x-y,(x,y)\in D_1=\{(x,y)\mid 0<y<1-x,0<x<1\},\\
&S_2: z=1+x-y,(x,y)\in D_2=\{(x,y)\mid 0<y<1+x,-1<x<0\},\\
&S_3: z=1-x+y,(x,y)\in D_3=\{(x,y)\mid x-1<y<0,0<x<1\},\\
&S_4: z=1+x+y,(x,y)\in D_4=\{(x,y)\mid -x-1<y<0,-1<x<0\},\\
&S_5: z=x+y-1,(x,y)\in D_5=D_1\\
&S_6: z=-x+y-1,(x,y)\in D_6=D_2\\
&S_7: z=x-y-1,(x,y)\in D_7=D_3\\
&S_8: z=-x-y-1,(x,y)\in D_8=D_4.
\end{aligned}
$$

利用 S 的对称性及被积函数的对称性有

$$\iint\limits_{S} |xyz|\mathrm{d}S = 8\iint\limits_{S_1} |xyz|\mathrm{d}S,$$

在 S_1 上,

$$z_x = -1, \quad z_y = -1, \quad \sqrt{1+z_x^2+z_y^2} = \sqrt{3},$$

$$\begin{aligned}
\iint\limits_{S_1} |xyz|\mathrm{d}S &= \iint\limits_{D} xy(1-x-y)\cdot\sqrt{3}\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 x\mathrm{d}x\int_0^{1-x} \sqrt{3}y(1-x-y)\mathrm{d}y \\
&= \int_0^1 \sqrt{3}x\left(\frac{y^2(1-x)}{2}-\frac{y^3}{3}\right)\Bigg|_0^{1-x}\mathrm{d}x \\
&= \int_0^1 \frac{\sqrt{3}}{6}x(1-x)^3\mathrm{d}x \\
&= \frac{\sqrt{3}}{120}.
\end{aligned}$$

于是

$$\iint\limits_{S} |xyz|\mathrm{d}S = 8\frac{\sqrt{3}}{120} = \frac{\sqrt{3}}{15}.$$

(4) 曲面 S 在 xy 平面上的投影为 $D=\left\{(x,y)\ \middle|\ x^2+y^2\le\frac{R^2}{2}\right\}$, 注意到锥面方程 $z=\sqrt{x^2+y^2}$ 满足

$$z_x = \frac{x}{\sqrt{x^2+y^2}}, \quad z_y = \frac{y}{\sqrt{x^2+y^2}}, \quad \sqrt{1+z_x^2+z_y^2} = \sqrt{2},$$

于是

$$\iint\limits_{S} z^2\mathrm{d}S = \iint\limits_{D} \sqrt{2}(x^2+y^2)\mathrm{d}x\mathrm{d}y$$

作极坐标变换 $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$ 则 D 与 $\triangle=\left\{(r,\theta)\ \middle|\ 0\leqslant r\leqslant\frac{\sqrt{2}R}{2}, 0\leqslant\theta\leqslant 2\pi\right\}$ 一一对应, 于是有

$$\begin{aligned}
\iint\limits_{D} \sqrt{2}(x^2+y^2)\mathrm{d}x\mathrm{d}y &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\sqrt{2}}{2R}} r\cdot r^2\mathrm{d}r \\
&= \sqrt{2}\cdot 2\pi\cdot\frac{r^4}{4}\Bigg|_0^{\frac{\sqrt{2}}{2R}} \\
&= \frac{\sqrt{2}}{8}\pi R^4.
\end{aligned}$$

□

2. 求 $f(t) = \iint\limits_{x^2+y^2+z^2=t^2} f(x,y,z)\mathrm{d}S$, 其中

$$f(x,y,z) = \begin{cases} x^2+y^2, & z \ge \sqrt{x^2+y^2}, \\ 0, & z < \sqrt{x^2+y^2}. \end{cases}$$

解: 球面 $S := \{(x,y,x) \mid x^2+y^2+z^2=t^2\}$ 如图 2所示.

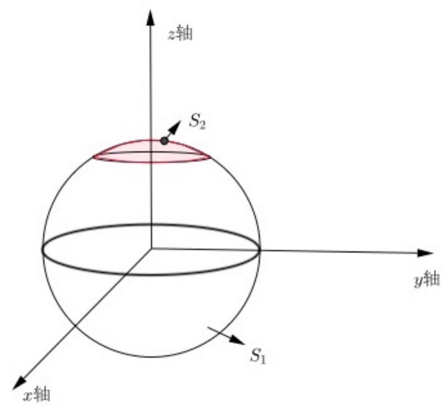

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图 2: 曲面 S_1 和 S_2

记 $S = S_1 \cup S_2$, 其中

$$\begin{aligned} S_1 &: \{(x,y,z) \in S \mid z < \sqrt{x^2+y^2}\}, \\ S_2 &: \{(x,y,z) \in S \mid z \geqslant \sqrt{x^2+y^2}\}. \end{aligned}$$

注意在 S_1 上, 有 $f(x,y,z)=0$, 从而有

$$\iint\limits_{S_1} f(x,y,z)\mathrm{d}S = 0;$$

在曲面 S_2 上, 联立方程组

$$\begin{cases} x^2+y^2+z^2=t^2 \\ z=\sqrt{x^2+y^2} \end{cases}$$

可求得交线在 xy 平面上的投影为 $\left\{(x,y) \;\middle|\; x^2+y^2=\dfrac{t^2}{2}\right\}$, 即得到 S_2 在 xy 平面上的投影为 $D = \{(x,y) \mid x^2+y^2 \leqslant t^2/2\}$, 且有

$$z_x = -\frac{x}{\sqrt{t^2-x^2-y^2}}, \quad z_y = -\frac{y}{\sqrt{t^2-x^2-y^2}}, \quad \sqrt{1+z_x^2+z_y^2} = \frac{t}{\sqrt{t^2-x^2-y^2}},$$

从而有

$$\iint\limits_{S_2} f(x,y,z)\mathrm{d}S = \iint\limits_{D} (x^2+y^2)\cdot\frac{t}{\sqrt{t^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y.$$

作极坐标变换 $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$ 则 D 与 $\triangle = \left\{(r,\theta)\,\middle|\, 0 \leqslant r \leqslant \dfrac{\sqrt{2}t}{2}, 0 \leqslant \theta \leqslant 2\pi\right\}$ 一一对应, 于是有

$$
\begin{aligned}
&\iint\limits_D (x^2+y^2)\cdot\frac{t}{\sqrt{t^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y \\
&= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\sqrt{2}t}{2}} r\cdot r^2\cdot\frac{t}{\sqrt{t^2-r^2}}\mathrm{d}r \\
&= \int_0^{2\pi}\mathrm{d}\theta\frac{1}{2}\int_0^{\frac{\sqrt{2}t}{2}} r^2\cdot\frac{t}{\sqrt{t^2-r^2}}\mathrm{d}r^2 \\
&= \int_0^{2\pi}\mathrm{d}\theta\frac{1}{2}\left(\int_0^{\frac{\sqrt{2}t}{2}}\sqrt{t^2-r^2}\mathrm{d}(t^2-r^2)+\int_0^{\frac{\sqrt{2}t}{2}} t^2\cdot\frac{t}{\sqrt{t^2-r^2}}\mathrm{d}r^2\right) \\
&= \int_0^{2\pi}\frac{1}{2}\cdot\frac{2}{3}\left((t^2-r^2)^{\frac{3}{2}}\Big|_0^{\frac{\sqrt{2}t}{2}}+t^2(\sqrt{t^2-r^2})\Big|_0^{\frac{\sqrt{2}t}{2}}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{2}{3}t^3-\frac{5}{12}t^3\right)\mathrm{d}\theta \\
&= \left(\frac{4}{3}-\frac{5\sqrt{2}}{6}\right)\pi t.
\end{aligned}
$$

综上所述, 有

$$
f(t) = \iint\limits_{x^2+y^2+z^2=t^2} f(x,y,z)\mathrm{d}S = \iint\limits_{S_1\cup S_2} f(x,y,z)\mathrm{d}S = \iint\limits_{S_2} f(x,y,z)\mathrm{d}S = \left(\frac{4}{3}-\frac{5\sqrt{2}}{6}\right)\pi t.
$$

□

3. 求上半球面 $z=\sqrt{a^2-x^2-y^2}$ 被 $x^2+y^2=ax$ 截取部分的面积和重心坐标 (x_0,y_0,z_0), 其中 $a>0$, 球面的面密度为 1.

解: 首先求上半球面 $z=\sqrt{a^2-x^2-y^2}$ 被 $x^2+y^2=ax$ 截取部分的面积.

S 在 xy 平面上的投影为 $D=\{(x,y)|\ x^2+y^2\leqslant ax\}$, 所以

$$
z_x = -\frac{x}{\sqrt{a^2-x^2-y^2}}, z_y = -\frac{y}{\sqrt{a^2-x^2-y^2}}, \sqrt{1+z_x^2+z_y^2} = \frac{a}{\sqrt{a^2-x^2-y^2}},
$$

,

作极坐标变换 $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$ 则 D 与

$$
\triangle = \left\{(r,\theta)\,\middle|\, 0 \le r \le a\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}
$$

一一对应, 因为球面的面密度为 1, 于是有

$$\begin{aligned}
\iint\limits_S 1\mathrm{d}S &= \iint\limits_D \frac{a}{\sqrt{a^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\cdot r\mathrm{d}r \\
&= -\frac{a}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\mathrm{d}(a^2-r^2) \\
&= -\frac{a}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sqrt{a^2-r^2})\Big|_0^{a\cos\theta} \mathrm{d}\theta \\
&= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a-a|\sin\theta|)\mathrm{d}\theta \\
&= a\int_{-\frac{\pi}{2}}^{0} (a+a\sin\theta)\mathrm{d}\theta + a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a-a\sin\theta)\mathrm{d}\theta \\
&= a^2\ (\theta-\cos\theta)|_{-\frac{\pi}{2}}^{0} + a^2\ (\theta+\cos\theta)|_{-\frac{\pi}{2}}^{0} \\
&= a^2(\pi-2).
\end{aligned}$$

其次求上半球面 $z=\sqrt{a^2-x^2-y^2}$ 被 $x^2+y^2=ax$ 截取部分的重心坐标.

(1) 由对称性可知, $y_0=0$.

(2) 由重心坐标公式可得

$$x_0=\frac{\iint\limits_S x\mathrm{d}S}{\iint\limits_S 1\mathrm{d}S}=\iint\limits_D \frac{ax}{\sqrt{a^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y\cdot\frac{1}{a^2(\pi-2)}$$

作极坐标变换 $\begin{cases} x=r\cos\theta, \\ y=r\sin\theta \end{cases}$ 则 D 与 $\triangle=\left\{(r,\theta)\ \middle|\ 0\leqslant r\leqslant a\cos\theta, -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}\right\}$ 一一对应, 且有

$$\begin{aligned}
&\iint\limits_D \frac{ax}{\sqrt{a^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y\cdot\frac{1}{a^2(\pi-2)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} \frac{ar\cos\theta}{\sqrt{a^2-r^2}}\cdot r\mathrm{d}r \\
&= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\mathrm{d}\theta \int_0^{a} \frac{r^2}{\sqrt{a^2-r^2}}\mathrm{d}r \\
&= \frac{a}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\left(-r\sqrt{a^2-r^2}+a^2\arcsin(\cos\theta)\right)\Big|_0^{a\cos\theta}\mathrm{d}\theta \\
&= \frac{a^3}{2}\left[\int_{-\frac{\pi}{2}}^{0}\left(\cos^2\theta\sin\theta+(\frac{\pi}{2}+\theta)\right)\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\left(-\cos^2\theta\sin\theta+(\frac{\pi}{2}-\theta)\right)\mathrm{d}\theta\right] \\
&= \frac{a^3}{2}\left[-\frac{\cos^3\theta}{3}+\frac{\pi}{2}\sin\theta+\theta\sin\theta+\cos\theta\Big|_{-\frac{\pi}{2}}^{0}\right]+\frac{a^3}{2}\left[\frac{\cos^3\theta}{3}+\frac{\pi}{2}\sin\theta-\theta\sin\theta-\cos\theta\Big|_0^{\frac{\pi}{2}}\right] \\
&= \frac{a^3}{2}\cdot\frac{4}{3} \\
&= \frac{2a^3}{3}.
\end{aligned}$$

于是

$$x_0 = \frac{2a^3}{3} \cdot \frac{1}{a^2(\pi-2)} = \frac{2z}{3(\pi-2)};$$

(3) 由重心坐标公式可得

$$z_0 = \frac{\iint\limits_S z\mathrm{d}S}{\iint\limits_S 1\mathrm{d}S} = \iint\limits_D \frac{a\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y \cdot \frac{1}{a^2(\pi-2)} = \iint\limits_D \mathrm{d}x\mathrm{d}y \cdot \frac{1}{a(\pi-2)}$$

作极坐标变换 $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$ 则 D 与 $\triangle = \left\{(r,\theta) \;\middle|\; 0 \leqslant r \leqslant a\cos\theta, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}\right\}$ 一一对应, 且有

$$\begin{aligned}
\iint\limits_D \mathrm{d}x\mathrm{d}y &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} r\mathrm{d}r \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left.\frac{r^2}{2}\right|_0^{a\cos\theta} \mathrm{d}\theta \\
&= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta \\
&= a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta \\
&= a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{\pi a^2}{4}.
\end{aligned}$$

于是

$$z_0 = \frac{\pi a^2}{4} \cdot \frac{1}{a(\pi-2)} = \frac{\pi a}{4(\pi-2)}.$$

综上所述, 重心坐标为 $\left(\frac{2z}{3(\pi-2)}, 0, \frac{\pi a}{4(\pi-2)}\right)$. □

---

4. 设 $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, 利用公式 (17.2.1) 推出公式 (17.2.2).

解: 因为 $\begin{cases} x = x(u,v), \\ y = y(u,v) \end{cases}$ 且 $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, 则由反函数存在定理可得, 存在 $\begin{cases} u = u(x,y), \\ v = v(x,y) \end{cases}$ 且有

$$u_x = -x_u, u_y = -y_u, v_x = -x_v, v_y = -y_v,$$

于是

$$z(u,v) = z(u(x,y), v(x,y)) = z(x,y), \quad z_x = z_u u_x + z_v v_x, \quad z_y = z_u u_y + z_v v_y,$$

于是

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+(z_u u_x + z_v v_x)^2 + (z_u u_y + z_v v_y)^2}, \tag{4}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = x_u y_v - x_v y_u, \tag{5}$$

于是利用坐标变换 $\begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases}$

则利用 (4), (5)及式子 (17.2, 1) 可得,

$$
\begin{aligned}
& \iint\limits_{S} f(x,y,z)\mathrm{d}S \\
= & \iint\limits_{D_{xy}} f(x,y,z(x,y))\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y \\
= & \iint\limits_{D_{uv}} f(x(u,v),y(u,v),z(x,y)(x,y)) \cdot \sqrt{1+z_x^2+z_y^2} \cdot J\mathrm{d}x\mathrm{d}y \\
= & \iint\limits_{D_{uv}} f(x(u,v),y(u,v),z(x,y)(x,y)) \cdot \sqrt{1+(z_u u_x + z_v v_x)^2 + (z_u u_y + z_v v_y)^2} \cdot (x_u y_v - x_v y_u)\mathrm{d}u\mathrm{d}v \\
= & \iint\limits_{D_{uv}} f(x(u,v),y(u,v),z(x,y)(x,y)) \cdot \left((x_u^2+y_u^2+z_u^2)\cdot(x_v^2+y_v^2+z_v^2) - (x_u x_v + y_u y_v + z_u z_v)^2\right)\mathrm{d}u\mathrm{d}v \\
= & \iint\limits_{D_{uv}} f(x(u,v),y(u,v),z(x,y)(x,y)) \cdot \left|EG-F^2\right|\mathrm{d}u\mathrm{d}v.
\end{aligned}
$$

其中

$$
\begin{aligned}
E &= x_u^2+y_u^2+z_u^2, \\
F &= x_u x_v + y_u y_v + z_u z_v, \\
G &= x_v^2+y_v^2+z_v^2.
\end{aligned}
$$

于是 (17.2.4) 得证. □

5. 计算 $\iint\limits_{S} x^2\mathrm{d}S$, 其中 S 为圆锥表面的一部分:

$$
S: \begin{cases} x = r\cos\phi\sin\theta, \\ y = r\sin\phi\sin\theta, \\ z = r\cos\theta, \end{cases} \qquad D: \begin{cases} 0 \le r \le a, \\ 0 \le \phi \le 2\pi, \end{cases}
$$

这里 θ 为常数 $(0 < \theta < \frac{\pi}{2})$.

解: 由公式可得

$$
\begin{aligned}
E &= x_r^2+y_r^2+z_r^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \\
F &= x_r x_\phi + y_r y_\phi + z_r z_\phi = -r\sin\phi\cos\phi\sin^2\theta + r\sin\phi\cos\phi\sin^2\theta + 0 = 0, \\
G &= x_\phi^2+y_\phi^2+z_\phi^2 = r^2\sin^2\phi\sin^2\theta + r^2\cos\phi\sin^2\theta + 0 = r^2\sin^2\theta,
\end{aligned}
$$

所以

$$
\begin{aligned}
\iint\limits_{S} x^2 \mathrm{d}S &= \iint\limits_{D} r^2 \sin^2\theta \cos^2\phi |EG - F^2| \mathrm{d}S \\
&= \int_0^{2\pi} \cos^2\phi \mathrm{d}\phi \int_0^a r^3 \sin^3\theta \mathrm{d}r \\
&= \frac{a^4}{4} \sin^3\theta \int_0^{2\pi} \frac{\cos 2\phi + 1}{2} \mathrm{d}\phi \\
&= \frac{a^4}{8} \sin^3\theta \ (\sin 2\phi + \phi)|_0^{2\pi} \\
&= \frac{\pi a^4}{4} \sin^3\theta.
\end{aligned}
$$

$\square$