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ON SOME ALGEBRAICAL PROPERTIES OF OPERATOR RINGS

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§1. The notations to be used in this paper agree with those of the papers quoted below, especially [2]. The results which we obtain will be used in [5], but they seem to have a certain interest of their own as well.

Bibliography

[1] J. von Neumann: "*Zur Algebra der Funktionaloperatoren . . .*," Math. Annalen, vol. 102, pp. 370–427 (1929).

[2] J. von Neumann: "*Über adjungierte Funktionaloperatoren*," Annals of Math., vol. 33, pp. 294–310 (1932).

[3] F. J. Murray: "*Linear transformations between Hilbert spaces*," Trans. Amer. Math. Soc., vol. 37, 301–338 (1935).

[4] F. J. Murray and J. von Neumann: "*On rings of operators*," Annals of Math., vol. 37, pp. 116–229 (1936).

[5] F. J. Murray and J. von Neumann: "*On rings of operators IV*," Annals of Math., vol. 44, (following immediately).

§2. We consider an operator ring $\mathbf{M}$ in a Hilbert space $\mathfrak{H}$ which contains 1. We say that a notion defined in $\mathbf{M}$ is *purely algebraical* if it can be expressed in terms of the entity 1 (the unit operator) and the operations αA (α any complex number), A^*, $A + B$, AB alone and referring only to operators belonging to $\mathbf{M}$. Thus if a mapping of $\mathbf{M}$ onto another ring is isomorphic, that is if the notions of 1, αA, A^*, $A + B$, AB are invariant, then every "purely algebraic" notion is invariant.

Notions which refer explicitly to elements of $\mathfrak{H}$ will not in general be purely algebraical.

Now the notions which follow do refer to elements of $\mathfrak{H}$.

(α) Definiteness of an $A \in \mathbf{M}$.

(β) The numerical value of the bound $|||\, A\, |||$ of an $A \in \mathbf{M}$.

(γ) The fact that $\lim_{n\to\infty}$ strong $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(δ) The fact that $\lim_{n\to\infty}$ weak $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(For the *strong* and *weak* notions of convergence, used above, and for the corresponding topologies, to be referred to below, cf. [1], pp. 381–388.) The object of this paper is to show that the notions (α)–(δ) are nevertheless purely algebraical.

We shall not prove the same thing concerning the strong and weak topology (for operators). It is probably not generally true. For an important special case where it is true, cf. [5], Theorem I.

(α) and (β) are easy to dispose of, cf. §3. (δ) follows from (γ) by a known argument which will be given in §6. So the main difficulty lies in establishing the character of (γ), which will be done in §5. The discussion of (γ) and (δ)

would be much easier if the notion of being purely algebraic did not exclude the use of operators not in $\mathbf{M}$, cf. §4.

§3. We begin by considering (α) and (β).

LEMMA 1. *$A \in \mathbf{M}$ is definite if and only if $A = B^*B$ for some $B \in \mathbf{M}$.*

PROOF. Sufficiency: Suppose $A = B^*B$ for $B \in \mathbf{M}$. Since $B^* \in \mathbf{M}$, $A \in \mathbf{M}$. Furthermore $(Af, f) = (B^*Bf, f) = (Bf, Bf) = \| Bf \|^2 \geqq 0$ and hence A is definite.

Necessity: If $A \in \mathbf{M}$ is definite, then there exists a definite $B \in \mathbf{M}$ with $B^2 = A$. (For the existence of B cf. [2], p. 307, Theorem 7, or [4], p. 142, §4.4. B is bounded along with A.[1] That $A \in \mathbf{M}$ implies $B \in \mathbf{M}$ is shown in [4], p. 143, Lemma 4.4.1. There exist also simple direct proofs, using functions of functional operators.) Now $B = B^*$ so $A = B^*B$.

LEMMA 2. *$||| A |||$ is the smallest (real) number $\alpha \geqq 0$ such that $\alpha^2 \cdot 1 - A^*A$ is definite.*

$||| A |||$ is the smallest $\alpha \geqq 0$ such that for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$. The inequality may be rewritten $0 \leqq \alpha^2 \cdot \| f \|^2 - \| Af \|^2 = \alpha^2(f, f) - (Af, Af) = \alpha^2(f, f) - (A^*Af, f) = (\{\alpha^2 \cdot 1 - A^*A\}f, f)$. Thus the statement, "for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$" is equivalent to "$\alpha^2 \cdot 1 - A^*A$ is definite." Substituting in the first sentence yields the lemma.

Thus we have shown

THEOREM I. *The notions (α) and (β) (definiteness and bound) are purely algebraical.*

§4. Let us now interrupt our discussion in order to analyze (γ) and (δ) without the necessity of avoiding the use of operators not in $\mathbf{M}$. This is much easier than the discussion with the original observance, to be given in the two next sections.

$\lim_{n\to\infty}$ strong $A_n = 0$ means $\lim_{n\to\infty} \| A_n f \| = 0$ for all $f \in \mathfrak{H}$; $\lim_{n\to\infty}$ weak $A_n = 0$ means $\lim_{n\to\infty} | (A_n f, g) | = 0$ for every f and g in $\mathfrak{H}$. Clearly we may restrict ourselves in both cases to the f and g with $\| f \| = \| g \| = 1$.

Denote the closed linear set of all αf as usual by $[f]$, and its projection by $P_{[f]}$. Then one sees immediately that (for $\| f \| = \| g \| = 1$)

$$AP_{[f]}h = (f, h)Af, \qquad P_{[g]}AP_{[f]}h = (f, h)\cdot(Af, g)g.$$

Hence

$$||| AP_{[f]} ||| = \| Af \|, \qquad ||| P_{[g]}AP_{[f]} ||| = | (Af, g) | .$$

Now the $P_{[f]}$ are obviously the minimal projections of the ring $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$, i.e. those projections E for which the only projections $F \leqq E$

[1] We have $\| Bf \|^2 = (B^*Bf, f) = (Af, f) \leqq \| Af \| \cdot \| f \| \leqq ||| A ||| \cdot \| f \|^2$. Thus $\| Bf \| \leqq \sqrt{||| A |||} \cdot \| f \|$. Hence B is bounded along with A.

are $F = 0, E$. (Cf. [4], pp. 143, 144, Definition 5.1.2. The assertion concerning minimal projections is obvious.) So we see

$$\lim_{n\to\infty} \text{strong } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} \|| A_nE \||| = 0$$

$$\lim_{n\to\infty} \text{weak } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} \||| GA_nE \||| = 0.$$

Here E, G run over all minimal projections of the ring **B** of all bounded operators in $\mathfrak{H}$.

The drawback in all this is that we had to refer to the ring **B** instead of **M**.

§5. We now proceed to the more difficult analysis of (γ) in the original sense.

DEFINITION. *A sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is a* Σ *sequence if it possesses the properties*

(i) *The (numerical) sequence* $\||| A_1 \|||, \||| A_2 \|||, \cdots$ *is bounded.*

(ii) *There exists an operator* $X \in \mathbf{M}$ *such that* 1.) *for all* $C \in \mathbf{M}$ $CX = 0$ *implies* $C = 0$, *and* 2.) *all the operators,*

$$1 - \sum_{m=1}^{n} (A_m X)^*(A_m X), \qquad (n = 1, 2, \cdots) \quad \textit{are definite.}$$

LEMMA 3. *For every* Σ *sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *we have* $\lim_{n\to\infty} \text{strong } A_n = 0$.

PROOF. For every $f \in \mathfrak{H}$ we have

$$(\{1 - \sum_{m=1}^{n} (A_m X)^*(A_m X)\}f, f) \geqq 0,$$

i.e.

$$\sum_{m=1}^{n} ((A_m X)^*(A_m X)f, f) \leqq (f, f),$$

i.e.

$$\sum_{m=1}^{n} \| A_m Xf \|^2 \leqq \| f \|^2.$$

Consequently $\sum_{m=1}^{\infty} \| A_m Xf \|^2 \leqq \| f \|^2$ and therefore $\lim_{m\to\infty} \| A_m Xf \| = 0$.

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with $\lim_{m\to\infty} \| A_m g \| = 0$ contains the range of X.

$\mathfrak{S}$ is clearly a linear set, and since the $A_1, A_2, \cdots$ are uniformly bounded (by (i)), $\mathfrak{S}$ is closed. Let E be the projection on $\mathfrak{S}$. We next show $\mathfrak{S} \,\eta\, \mathbf{M}$. (For this notation, cf. [4], p. 141, Def. 4.2.1.) For suppose $U' \in \mathbf{M}'$ is unitary and $f \in \mathfrak{S}$. Then $\lim_{m\to\infty} \| A_m U'f \| = \lim_{m\to\infty} \| U'A_m f \| = \lim_{m\to\infty} \| A_m f \| = 0$. Thus $f \in \mathfrak{S}$ implies $U'f \in \mathfrak{S}$ and $\mathfrak{S}$ is invariant under every $U' \in \mathbf{M}'$ or $\mathfrak{S} \,\eta\, \mathbf{M}$. This implies $E \in \mathbf{M}$.

Since the range of X is contained in $\mathfrak{S}$, $EX = X$ or $(1 - E)X = 0$. Hence by (ii), $1 - E = 0$ and $1 = E$. Thus $\mathfrak{S} = \mathfrak{H}$ that is $\lim_{m\to\infty} \text{strong } A_m = 0$.

LEMMA 4. *For every sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *with* $\lim_{m\to\infty} \text{strong } A_n = 0$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *which is a* Σ *sequence.*

PROOF. $\||| A_1 \|||, \||| A_2 \|||, \cdots$ is bounded by [1], p. 382, footnote 35), hence $\||| A_{1'} \|||, \||| A_{2'} \|||, \cdots$ is a *fortiori* bounded for every subsequence.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

For every $i = 1, 2, \cdots$, $\lim_{n\to\infty} \| A_n f_i^0 \| = 0$. Choose accordingly $k(i)$ such that for $n \geqq k(i)$, $\| A_n f_j^0 \| \leqq 1/2^i$ for all $j = 1, \cdots, i$. Choose a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ with $1' < 2' < \cdots$ and $i' \geqq k(i)$. Then $\| A_{n'} f_j^0 \| \leqq 1/2^n$ if $n \geqq j$. Consequently $\sum_{n=1}^{\infty} \| A_{n'} f_j^0 \|^2$ is finite for $j = 1, 2, \cdots$.

Thus: The set $\mathfrak{F}$ of all $g \in \mathfrak{H}$ for which $\sum_{n=1}^{\infty} \| A_{n'} g \|^2$ is finite, contains all $f_1^0, f_2^0, \cdots$ and therefore it is everywhere dense in $\mathfrak{H}$.

Form the space $\infty \otimes \mathfrak{H}$ of all sequences $\langle f_1, f_2, \cdots \rangle$ of elements of $\mathfrak{H}$ with $\sum_{i=1}^{\infty} \| f_i \|^2 < \infty$. $\infty \otimes \mathfrak{H}$ is a Hilbert space. (Cf. [4], pp. 136–137, §2.4.) Consider the following operator T from $\mathfrak{H}$ to $\infty \otimes \mathfrak{H}$.

$$Tf = \langle f, A_{1'} f, A_{2'} f, \cdots \rangle$$

(Cf. [3], Def. 1.2, p. 303.) The domain of this T is obviously the above defined set $\mathfrak{F}$. T is linear and it is also easy to verify that T is closed.[2] Therefore we may apply an argument for operators between Hilbert spaces similar to that of [2], p. 309, Satz 10, according to which there exists one and only one self-adjoint (i.e. hypermaximal) definite operator B in $\mathfrak{H}$ which is metrically equivalent to T, i.e. such that it has the same domain as T and always $\| Bf \| = \| Tf \|$. (Cf. [3], §4, pp. 311, 312.)

Thus the domain of B is $\mathfrak{F}$ and always

$$(1) \qquad \| Bf \|^2 = \| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2.$$

Consider a unitary $U' \in \mathbf{M}'$. Then $U' A_{i'} = A_{i'} U'$ and hence (1) implies $\| BU'f \| = \| Bf \|$ and $\| U'^{-1} BU'f \| = \| Bf \|$. Thus $U'^{-1} BU'$ possesses the above properties which characterize B uniquely. So $U'^{-1} BU' = B$, i.e. $U'B = BU'$. Thus $B \, \eta \, \mathbf{M}$. (Cf. again [4], p. 141, Def. 4.2.1.)

By (1), $Bf = 0$ implies $f = 0$. This and the self-adjointness of B imply that B^{-1} is also self-adjoint. By (1), $\| Bf \| \geqq \| f \|$, hence $\| B^{-1} g \| \leqq \| g \|$ and so B^{-1} is bounded. Now $B \, \eta \, \mathbf{M}$ yields $B^{-1} \, \eta \, \mathbf{M}$ and since B^{-1} is bounded, $B^{-1} \in \mathbf{M}$. (Cf. [4], p. 141, Lemma 4.2.1.) Put $X = B^{-1}$. Thus $X \in \mathbf{M}$. Since $X = B^{-1}$, $CX = 0$ implies $Cf = CXBf = (CX) \cdot Bf = 0$ for f in the domain of B which is dense. For C bounded, this implies $C = 0$ and thus we have (ii) 1 of the definition of this section.

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[2] We must prove $\lim_{n \to \infty} \| f_n - f \| = 0$, $\lim_{n \to \infty} \| Tf_n - f^* \| = 0$ imply the existence of Tf and $Tf = f^*$.

Put $f^* = \langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle$. Then $\lim_{n \to \infty} \| Tf_n - f^* \|^2 = 0$ means that

$$\lim_{n \to \infty} (\| f_n - f^{(0)} \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f_n - f^{(i)} \|^2) = 0.$$

Hence, *a fortiori*

$$\lim_{n \to \infty} \| f_n - f^{(0)} \| = 0, \qquad \lim_{n \to \infty} \| A_{i'} f_n - f^{(i)} \| = 0.$$

Thus $f = f^{(0)}$ and $A_{i'} f = f^{(i)}$. Consequently

$$\| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2 = \| f^0 \|^2 + \sum_{i=1}^{\infty} \| f^{(i)} \|^2 = \| f^* \|^2$$

i.e. $\sum_{i=1}^{\infty} \| A_{i'} f \|^2$ is finite.

So Tf is defined and $Tf = \langle f, A_{1'} f, A_{2'} f, \cdots \rangle = \underline{\langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle} = f^*$ as desired.

By equation (1), $\sum_{m=1}^{n} \| A_{m'} f \|^2 \leqq \| Bf \|^2$, hence $\sum_{m=1}^{n} \| A_{m'} Xg \|^2 \leqq \| g \|^2$. This may be written

$$\sum_{m=1}^{n} ((A_{m'} X)^*(A_{m'} X)g, g) \leqq (g, g)$$

or

$$(\{1 - \sum_{m=1}^{n} (A_{m'} X)^*(A_{m'} X)\}g, g) \geqq 0.$$

Thus the operator $1 - \sum_{m=1}^{n} (A_{m'}X)^*(A_{m'}X)$ is definite, i.e. we have (ii)2. This completes the proof.

LEMMA 5. *For a sequence* $A_1, A_2, \cdots, \epsilon \mathbf{M}$ *we have* $\lim_{n\to\infty}$ strong $A_n = 0$ *if and only if every subsequence* $A_{\bar{1}}, A_{\bar{2}}, \cdots$ *of* $A_1, A_2, \cdots$ *possesses a subsequence which is a* Σ*-sequence.*

PROOF: Necessity: $\lim_{n\to\infty}$ strong $A_n = 0$ implies for every subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ that $\lim_{n\to\infty}$ strong $A_{\bar{n}} = 0$. Hence Lemma 4 applies to $A_{\bar{1}}, A_{\bar{2}}, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ which is a Σ-sequence.

Sufficiency: Assume that we do not have $\lim_{n\to\infty}$ strong $A_n = 0$. Then there exists an $f \epsilon \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} \| A_n f \| = 0$. Consequently there exists a subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} \| A_{\bar{n}} f \| = \alpha$ for an $\alpha \neq 0$ (but possibly $\alpha = \infty$). Thus for every subsequence, $A_{1'}, A_{2'}, \cdots$ of $A_{\bar{1}}, A_{\bar{2}}, \cdots$ equally $\lim_{n\to\infty} \| A_{n'} f \| = \alpha$ thus excluding $\lim_{n\to\infty}$ strong $A_{n'} = 0$. Hence Lemma 3 excludes that $A_{\bar{1}}, A_{\bar{2}}, \cdots$ be a Σ sequence.

Therefore, if the condition of our lemma is satisfied, we must have $\lim_{n\to\infty}$ strong $A_n = 0$.

Replacing $A_1, A_2, \cdots$ by $A_1 - A, A_2 - A, \cdots$ we can conclude from Lemma 5,

THEOREM II. *The notion* (γ) *(strong convergence) is purely algebraical.*

§6. (δ) can be deduced from (γ) by a known argument, which, nevertheless, we will give in full for the sake of completeness.

LEMMA 6. *If* $\lim_{n\to\infty}$ weak $A_n = A$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *of* $A_1, A_2, \cdots$ *such that* $\lim_{n\to\infty}$ strong $\frac{1}{n} \sum_{m=1}^{n} A_{m'} = A$.

PROOF. Since we may replace $A, A_1, A_2, \cdots$ by $0, A_1 - A, A_2 - A, \cdots$ there is no loss of generality if we assume that $A = 0$, i.e. $\lim_{n\to\infty}$ weak $A_n = 0$.

By [1], p. 382, footnote 35, $||| A_1 |||, ||| A_2 |||, \cdots$ is bounded. Let α be a bound, i.e. $||| A_n ||| \leqq \alpha < \infty$.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

We shall define a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ by induction. Assume therefore that $1', 2', \cdots, (m-1)'$ are already defined. We shall now define m'.

Since $\lim_{n\to\infty}$ weak $A_n = 0$, $\lim_{n\to\infty} (A_n f, g) = 0$ for any two f and $g \epsilon \mathfrak{H}$. Hence, in particular $\lim_{n\to\infty} (A_n f_i^0, A_l f_i^0) = 0$ for all $l = 1', 2', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Consequently there exists a $k(m)$ such that $n \geqq k(m)$ implies $| (A_n f_i^0, A_{l'} f_i^0) | \leqq 1/2^m$ for all $l' = 1', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Now choose $m' \geqq k(m)$ and $> (m-1)'$.

Thus $1' < 2' < \cdots$ and $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^m$ for $m > l$ and $m \geqq i$. Interchanging m, l [3] gives $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^l$, for $m < l$ and $l \geqq i$. Summing up:

$$\{|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^{\mathrm{Max}(m,l)} \quad \text{if } m \neq l \text{ and } \mathrm{Max}\,(m, l) \geqq i. \tag{2}$$

Now $n \geqq i$ yields

$$\begin{aligned} \|(\textstyle\sum_{m=1}^n A_{m'})f_i^0\|^2 &= (\textstyle\sum_{m=1}^n A_{m'}f_i^0, \sum_{m=1}^n A_{m'}f_i^0) \\ &= \textstyle\sum_{m,l=1}^n (A_{m'}f_i^0, A_{l'}f_i^0) \\ &= \textstyle\sum_{m,l=1}^{i-1} (A_{m'}f_i^0, A_{l'}f_i^0) + \sum_{m=i}^n (A_{m'}f_i^0, A_{m'}f_i^0) \\ &\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^n (A_{m'}f_i^0, A_{l'}f_i^0) \\ &\leqq (i-1)^2\alpha^2\|f_i^0\|^2 + (n-i+1)\alpha^2\|f_i^0\|^2 \\ &\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^n 1/2^{\mathrm{Max}(m,l)} \\ &= \alpha^2\cdot\|f_i^0\|^2(n + (i-1)(i-2)) + \textstyle\sum_{k=i}^n \dfrac{2k-2}{2^k}\,{}^{4} \\ &\leqq \alpha^2\|f_i^0\|^2(n + (i-1)(i-2)) + 3\,{}^{5} \end{aligned}$$

and so

$$\left\|\left(\frac{1}{n}\sum_{m=1}^n A_{m'}\right)f_i^0\right\| \leqq \frac{1}{n}\sqrt{\|f_i^0\|^2\alpha^2[n + (i-1)(i-2)] + 3}$$

This implies $\lim_{n\to\infty}\left\|\left(\frac{1}{n}\sum_{m1}^n A_{m'}\right)f_i^0\right\| = 0.$

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with

$$\lim_{n\to\infty}\left\|\left(\frac{1}{n}\sum_{m=1}^n A_{m'}\right)g\right\| = 0 \quad \text{contains all } f_1^0, f_2^0, \cdots.$$

Hence $\mathfrak{S}$ is everywhere dense in $\mathfrak{H}$. Since all $|||A_{m'}||| \leqq \alpha$ $\left|\left|\left|\frac{1}{n}\sum_{m=1}^n A_{m'}\right|\right|\right| \leqq \alpha$ for $n = 1, 2, \cdots$. Thus the $\frac{1}{n}\sum_{m=1}^n A_{m'}$ are uniformly bounded and consequently $\mathfrak{S}$ is a closed set. So $\mathfrak{S} = \mathfrak{H}$ and $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^n A_{m'} = 0$.

This completes the proof.

LEMMA 7. *$\lim_{n\to\infty} A_n = A$ if and only if every subsequence $A_{\bar 1}, A_{\bar 2}, \cdots$ of $A_1, A_2, \cdots$ possesses a subsequence $A_{1'}, A_{2'}, \cdots$ such that*

$$\lim_{n\to\infty} \text{strong } \frac{1}{n}\sum_{m=1}^n A_{m'} = A.$$

[3] Remember that $(A_{m'}f_i^{(0)}, A_{l'}f_i^{(0)}) = (A_{l'}f_i^{(0)}, A_{m'}f_i^{(0)})$.

[4] We introduce a new summation variable, $k = \mathrm{Max}\,(i, j)$. Given k the number of paris i, j which belong to it is clearly $2k - 2$.

[5] We have $\sum_{k=1}^\infty \frac{2k-2}{2^k} \leqq \sum_{k=1}^\infty \frac{2k-1}{2^k} = 3.$

PROOF. Necessity: Assume $\lim_{n\to\infty}$ weak $A_n = A$. For every subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ of $A_1, A_2, \cdots$, $\lim_{n\to\infty}$ weak $A_{\bar{n}} = A$. Hence Lemma 6 applies to $A_{\bar{1}}, A_{\bar{2}}, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$. Thus the necessity of the hypothesis is established.

Sufficiency: Assume that the hypothesis is true but $\lim_{n\to\infty}$ weak $A_n = A$ is not true. Then there exists an f and $g \in \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} (A_n f, g) = (Af, g)$. Consequently there exists a subsequence $A_{1^*}, A_{2^*}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} (A_{n^*}f, g) = \alpha$ for an $\alpha \neq (Af, g)$ (but possibly $\alpha = \infty$). Hence if $A_{1'}, A_{2'}, \cdots$ is any subsequence of $A_{1^*}, A_{2^*}, \cdots$

$$\lim_{n\to\infty} (A_{n'}f, g) = \alpha \quad \text{and} \quad \alpha = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} (A_{m'}f, g)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'}f, g\right).$$

On the other hand, by hypothesis there is a subsequence $A_{1'}, A_{2'}, \cdots$ of $A_{1^*}, A_{2^*}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$ and consequently $\lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'}f, g\right) = (Af, g) \neq \alpha$. This contradicts the last result of the preceding paragraph, which states that for every such sequence, the limit is α.

This contradiction proves the sufficiency of the hypothesis.

With Lemma 7, we have shown

THEOREM III. *The notion* (δ) (*weak convergence*) *is purely algebraical.*

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