15.094J: Robust Modeling, Optimization, Computation

Lecture 5: Robust Mixed Integer Optimization

Outline

1. RMIO: Tractability
2. RMIO: Probabilistic Guarantees
3. Robust 0-1 Optimization
4. Robust Network Flows

Row-wise Polyhedral Uncertainty

- Primitives: Uncertainty sets U_i, $i = 1, \ldots, m$, $\boldsymbol{b}, \boldsymbol{c}$ (known, WLOG).
- RLO with row-wise uncertainty:

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \leq b_i. \quad \forall \boldsymbol{a}_i \in U_i, \ i = 1, \ldots, m, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \ x_i \in \mathcal{Z}, \ i = 1, \ldots, k. \end{array}$$

- $U_i = \{\boldsymbol{a}_i | \ \boldsymbol{D}_i\boldsymbol{a}_i \leq \boldsymbol{d}_i\}$, $\boldsymbol{D}_i : \ k_i \times n$.
- RC

$$\begin{array}{ll} \max_{\boldsymbol{x},\boldsymbol{p}_i} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{p}_i'\boldsymbol{d}_i \leq b_i, \quad i = 1, \ldots, m, \\ & \boldsymbol{p}_i'\boldsymbol{D}_i = \boldsymbol{x}', \quad i = 1, \ldots, m, \\ & \boldsymbol{p}_i \geq \boldsymbol{0}, \quad i = 1, \ldots, m, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \ x_i \in \mathcal{Z}, \ i = 1, \ldots, k. \end{array}$$

- RMIO reduces to MIO.
- Same even if uncertainty is not row-wise.

Row-wise Ellipsoidal uncertainty

- RO:

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \leq b_i. \\ & \boldsymbol{x} \geq \boldsymbol{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$

- $U_i = \{\boldsymbol{a}_i |\ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i'\boldsymbol{u}_i,\ ||\boldsymbol{u}_i||_2 \leq \rho\}$, $\boldsymbol{\Delta}_i :\ k_i \times n$, $\boldsymbol{u}_i : k_i \times 1$.
- RC:

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \overline{\boldsymbol{a}}_i'\boldsymbol{x} + \rho||\boldsymbol{\Delta}_i\boldsymbol{x}||_2 \leq b_i, \qquad i = 1, \ldots, m. \\ & \boldsymbol{x} \geq \boldsymbol{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$

- RMIO reduces to Mixed Integer Second order cone problem.
- Current versions of CPLEX and Gurobi support it, but more difficult than MIO.

Row-wise Budget of Uncertainty

- $$\begin{array}{rl} \text{minimize} & \tilde{\boldsymbol{c}}'\boldsymbol{x} \\ \text{subject to} & \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$
- **Uncertainty for matrix $\boldsymbol{A}$**: $a_{ij},\ j \in J_i$ is independent, symmetric and bounded random variable (but with unknown distribution) $\tilde{a}_{ij}, j \in J_i$ that takes values in $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.
- **Uncertainty for cost vector $\boldsymbol{c}$**: $c_j,\ j \in J_0$ takes values in $[c_j, c_j + d_j]$.

Budget of Uncertainty

- Consider an integer $\Gamma_i \in [0, |J_i|]$, $i = 0, 1, \ldots, m$.
- Γ_i adjusts the robustness of the proposed method against the level of conservativeness of the solution.
- Unlikely that all of the a_{ij}, $j \in J_i$ will change. We want to be protected against all cases that up to Γ_i of the a_{ij}'s are allowed to change.
- Nature will be restricted in its behavior, in that only a subset of the coefficients will change in order to adversely affect the solution.
- We will guarantee that if nature behaves like this then the robust solution will be feasible deterministically. Even if more than Γ_i change, then the robust solution will be feasible with very high probability.

RMIO

$$
\begin{aligned}
RMIO: \quad & \text{minimize} \quad \boldsymbol{c}'\boldsymbol{x} + \max_{\{S_0 \mid \; S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j |x_j| \right\} \\
& \text{subject to} \quad \sum_j a_{ij} x_j + \max_{\{S_i \mid \; S_i \subseteq J_i, |S_i| \leq \Gamma_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} |x_j| \right\} \leq b_i, \quad \forall i \\
& \qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}, \;\; x_i \in \mathcal{Z}, \;\; i = 1, \ldots, k.
\end{aligned}
$$

$$
\begin{aligned}
RC: \quad & \text{minimize} && \boldsymbol{c}'\boldsymbol{x} + z_0 \Gamma_0 + \textstyle\sum_{j \in J_0} p_{0j} \\
& \text{subject to} && \sum_j a_{ij} x_j + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \leq b_i && \forall i \\
& && z_0 + p_{0j} \geq d_j y_j && \forall j \in J_0 \\
& && z_i + p_{ij} \geq \hat{a}_{ij} y_j && \forall i \neq 0, j \in J_i \\
& && p_{ij}, y_j, z_i \geq 0 && \forall i, j \in J_i \\
& && -y_j \leq x_j \leq y_j && \forall j \\
& && \boldsymbol{x} \geq \boldsymbol{0}, x_i \in \mathcal{Z}, \;\; i = 1, \ldots, k.
\end{aligned}
$$

Proof

- Given a vector $\boldsymbol{x}^*$, we define:

$$\beta_i(\boldsymbol{x}^*) = \max_{\{S_i|\ S_i \subseteq J_i, |S_i| = \Gamma_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} |x_j^*| \right\}.$$

- This equals to:

$$\begin{aligned} \beta_i(\boldsymbol{x}^*) = \max \quad & \sum_{j \in J_i} \hat{a}_{ij} |x_j^*| z_{ij} \\ \text{s.t.} \quad & \sum_{j \in J_i} z_{ij} \leq \Gamma_i \\ & 0 \leq z_{ij} \leq 1 \qquad \forall i, j \in J_i. \end{aligned}$$

- Dual:

$$\begin{aligned} \beta_i(\boldsymbol{x}^*) = \min \quad & \sum_{j \in J_i} p_{ij} + \Gamma_i z_i \\ \text{s.t.} \quad & z_i + p_{ij} \geq \hat{a}_{ij} |x_j^*| && \forall j \in J_i \\ & p_{ij} \geq 0 && \forall j \in J_i \\ & z_i \geq 0 && \forall i. \end{aligned}$$

Size

- Original Problem has n variables and m constraints
- RC has $2n + m + l$ variables, where $l = \sum_{i=0}^{m} |J_i|$ is the number of uncertain coefficients, and $2n + m + l$ constraints.
- Sparsity is preserved, attractive!

Probabilistic Guarantees

- $\boldsymbol{x}^*$ an optimal solution of RMIO.
- $\tilde{a}_{ij}, j \in J_i$ independent, symmetric and bounded random variables, support $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.
- $$\Pr\left(\sum_j \tilde{a}_{ij} x_j^* > b_i\right) \leq \frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor\nu\rfloor}^{n}\binom{n}{l} + \mu\sum_{l=\lfloor\nu\rfloor+1}^{n}\binom{n}{l}\right\},$$
  $n = |J_i|$, $\nu = \frac{\Gamma_i + n}{2}$ and $\mu = \nu - \lfloor\nu\rfloor$; bound is tight.
- As $n \to \infty$
  $$\frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor\nu\rfloor}^{n}\binom{n}{l} + \mu\sum_{l=\lfloor\nu\rfloor+1}^{n}\binom{n}{l}\right\} \sim 1 - \Phi\left(\frac{\Gamma_i - 1}{\sqrt{n}}\right).$$

$\lvert J_i\rvert$	Γ_i
5	5
10	8.3565
100	24.263
200	33.899

Experimental Results

- Knapsack Problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in N} c_i x_i \\ \text{subject to} \quad & \sum_{i \in N} w_i x_i \leq b \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned}$$

- $\tilde{w}_i$ independently distributed and follow symmetric distributions in $[w_i - \delta_i, w_i + \delta_i]$;
- $\boldsymbol{c}$ is not subject to data uncertainty.
- $|N| = 200,\ b = 4000,$
- w_i randomly chosen from $\{20, 21, \ldots, 29\}$.
- c_i randomly chosen from $\{16, 17, \ldots, 77\}$.
- $\delta_i = 0.1 w_i$.

Experimental Results. continued

Γ	Violation Probability	Optimal Value	Reduction
0	0.5	5592	0%
2.8	0.449	5585	0.13%
36.8	5.71×10^{-3}	5506	1.54%
82.0	5.04×10^{-9}	5408	3.29%
200	0	5283	5.50%

Robust 0-1 Optimization

- Nominal 0-1 optimization:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \in X \subset \{0,1\}^n. \end{aligned}$$

- Reformulation:

$$\begin{aligned} Z^* = \quad \text{minimize} \quad & \boldsymbol{c}'\boldsymbol{x} + \max_{\{S \mid \; S \subseteq J, |S| \leq \Gamma\}} \sum_{j \in S} d_j x_j \\ \text{subject to} \quad & \boldsymbol{x} \in X, \end{aligned}$$

Contrast

- Other approaches to robustness are hard. Scenario based uncertainty:

$$\begin{aligned} \text{minimize} \quad & \max(\boldsymbol{c}_1'\boldsymbol{x}, \boldsymbol{c}_2'\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in X. \end{aligned}$$

  is NP-hard for the shortest path problem.
- $d_1 \geq d_2 \geq \ldots \geq d_n$. Optimal robust solution is

$$Z^* = \min_{l=1,\ldots,n+1} d_l\Gamma + \min_{\boldsymbol{x}\in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l}(d_j - d_l)x_j.$$

- Thus, if nominal problem is polynomially solvable the robust problem is also.

Proof

$$
\begin{aligned}
\text{Primal}: Z^* = \min_{\boldsymbol{x}\in X} \boldsymbol{c}'\boldsymbol{x} + \max \quad & \sum_j d_j x_j u_j \\
\text{s.t.} \quad & 0 \le u_j \le 1, \qquad \forall\, j \\
& \sum_j u_j \le \Gamma
\end{aligned}
$$

$$
\begin{aligned}
\text{Dual}: Z^* = \min_{\boldsymbol{x}\in X} \boldsymbol{c}'\boldsymbol{x} + \min \quad & \theta\Gamma + \sum_j y_j \\
\text{s.t.} \quad & y_j + \theta \ge d_j x_j, \qquad \forall\, j \\
& y_j, \theta \ge 0
\end{aligned}
$$

Proof, continued

- Solution: $y_j = \max(d_j x_j - \theta, 0)$
- $$Z^* = \min_{\boldsymbol{x} \in X, \theta \geq 0} \theta\Gamma + \sum_j (c_j x_j + \max(d_j x_j - \theta, 0))$$
- Since $X \subset \{0,1\}^n$,

$$\max(d_j x_j - \theta, 0) = \max(d_j - \theta, 0)\, x_j$$

- $$Z^* = \min_{\boldsymbol{x} \in X, \theta \geq 0} \theta\Gamma + \sum_j (c_j + \max(d_j - \theta, 0))\, x_j$$

Proof, continued

- $d_1 \geq d_2 \geq \ldots \geq d_n \geq d_{n+1} = 0.$
- For $d_l \geq \theta \geq d_{l+1}$,

$$\min_{\boldsymbol{x} \in X, d_l \geq \theta \geq d_{l+1}} \theta\Gamma + \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - \theta) x_j =$$

$$d_l\Gamma + \min_{\boldsymbol{x} \in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j = Z_l$$

- $$Z^* = \min_{l=1,\ldots,n+1} d_l\Gamma + \min_{\boldsymbol{x} \in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j.$$

Algorithm A

- **Input:** Vectors $\boldsymbol{c}, \boldsymbol{d} \in \Re^n_+$, an integer Γ, and a polynomial time algorithm that solves the problem $Z = \min \boldsymbol{c'x}$ subject to $\boldsymbol{x} \in X \subseteq \{0,1\}^n$ for all $\boldsymbol{c} \geq \boldsymbol{0}$.
- **Output:** A solution $\boldsymbol{x}^* \in X$ such that $\boldsymbol{x}^* = \mathrm{argmin} \left(\boldsymbol{c'x} + \max_{\{S|\ S \subseteq J, |S| = \Gamma\}} \sum_{j \in S} d_j x_j \right)$.

Algorithm A, continued

$\boldsymbol{x}^1 \leftarrow \arg\min\{\boldsymbol{c}'\boldsymbol{x} : x \in X\}$

FOR $l \in 2, \ldots, r$

  IF $d_l < d_{l-1}$

    $$\boldsymbol{x}^l \leftarrow \arg\min\{\boldsymbol{c}'\boldsymbol{x} + \sum_{j=1}^{l}(d_j - d_l)x_j : x \in X\}$$

    $$Z_l \leftarrow \boldsymbol{c}'\boldsymbol{x}^l + \max_{\{S| \ S\subseteq J, |S|=\Gamma\}} \sum_{j\in S} d_j x_j^l$$

  ELSE

    $\boldsymbol{x}^l \leftarrow \boldsymbol{x}^{l-1}$

    $Z_l \leftarrow Z_{l-1}$

  END IF

END FOR

Algorithm A, continued

$$
\begin{aligned}
11 \quad &: \quad \boldsymbol{x}^{\boldsymbol{r+1}} \leftarrow \arg\min\{\boldsymbol{c}'\boldsymbol{x} + \sum_{j \in J} d_j x_j : x \in X\} \\
12 \quad &: \quad Z_{r+1} \leftarrow \boldsymbol{c}'\boldsymbol{x}^{\boldsymbol{r+1}} + \max_{\{S| \; S \subseteq J, |S| = \Gamma\}} \sum_{j \in S} d_j x_j^{r+1} \\
13 \quad &: \quad \pi \leftarrow \arg\min\{Z_j : j \in J \cup \{r+1\}\} \\
14 \quad &: \quad Z^* = Z_\pi; \; \boldsymbol{x}^* = \boldsymbol{x}^{\boldsymbol{\pi}}.
\end{aligned}
$$

Theorem

- Algorithm A correctly solves the robust 0-1 optimization problem.
- It requires at most $|J|+1$ solutions of nominal problems. Thus, If the nominal problem is polynomially time solvable, then the robust 0-1 counterpart is also polynomially solvable.
- Robust minimum spanning tree, minimum assignment, minimum matching, shortest path and matroid intersection, are polynomially solvable.

Robust Approximation Algorithms

- If the nominal problem is α-approximable, is the robust counterpart also α-approximable?
- Use an α-approximate solution to

$$\min_{\boldsymbol{x}\in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j.$$

- Theorem: Overall algorithm is α-approximate.

Ellipsoidal Uncertainty

- $$\min_{\boldsymbol{x} \in X} \boldsymbol{c}'\boldsymbol{x} + \max_{\tilde{\boldsymbol{s}} \in D} \tilde{\boldsymbol{s}}'\boldsymbol{x}$$
- $D = \{\boldsymbol{s} : \|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{s}\|_2 \leq \Omega\}$
- Equivalent to:
  $$\min_{\boldsymbol{x} \in X} \boldsymbol{c}'\boldsymbol{x} + \Omega\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}$$
  $\boldsymbol{\Sigma}$ is the covariance matrix of the random cost coefficients: NP-hard
- D a polyhedron: NP-hard.

Uncorrelated uncertainty

- For $\boldsymbol{\Sigma} = diag(d_1^2, \ldots, d_n^2)$,

$$Z^* = \min_{\boldsymbol{x} \in X} \boldsymbol{c}'\boldsymbol{x} + \Omega\sqrt{\boldsymbol{d}'\boldsymbol{x}}$$

  Complexity Open.

- Theorem: For $d_1 = \ldots = d_n = \sigma$,

$$Z^* = \min_{w=0,1,\ldots,n} Z(w),$$

$$Z(w) = \begin{cases} \min_{\boldsymbol{x} \in X} & \left(\boldsymbol{c} + \dfrac{\Omega\sigma}{2\sqrt{w}}\boldsymbol{e}\right)'\boldsymbol{x} + \dfrac{\Omega\sigma\sqrt{w}}{2} & w = 1, \ldots, n \\ \min_{\boldsymbol{x} \in X} & (\boldsymbol{c} + \Omega\sigma\boldsymbol{e})'\boldsymbol{x} & w = 0. \end{cases}$$

Practical algorithm

- Until $\|x^{k+1} - \boldsymbol{x}^k\| \leq \epsilon$, set $\boldsymbol{x}^{k+1} := \arg\min_{\boldsymbol{y} \in X} (\boldsymbol{c} + \frac{\Omega}{2\sqrt{\boldsymbol{d}'\boldsymbol{x}^k}} \boldsymbol{d})' \boldsymbol{y}$
- Output $\boldsymbol{x}^{k+1}$
- Experimented on Shortest Path Problems, Uniform Matroid and Knapsack Problems, under randomly generated cost vectors in dimensions from 200 to 20,000.
- In 998 out of 1000 instances, optimal solution is found in solving less than 6 nominal problems!

Robust Network Flows

- Nominal

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{\{j:(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j:(j,i)\in\mathcal{A}\}} x_{ji} = b_i \quad \forall i \in \mathcal{N} \\
& 0 \leq x_{ij} \leq u_{ij} \qquad\qquad\qquad\qquad \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

- X set of feasible solutions flows.
- Robust

$$
\begin{aligned}
Z^* = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \max_{\{S|\ S\subseteq\mathcal{A},|S|\leq\Gamma\}} \sum_{(i,j)\in S} d_{ij}x_{ij} \\
\text{subject to} \quad & \boldsymbol{x} \in X.
\end{aligned}
$$

Theorem

For any fixed $\Gamma \leq |\mathcal{A}|$ and every $\epsilon > 0$, we can find a solution $\hat{\boldsymbol{x}} \in X$:

$$\hat{Z} = \boldsymbol{c}'\hat{\boldsymbol{x}} + \max_{\{S| \ S \subseteq \mathcal{A}, |S| \leq \Gamma\}} \sum_{(i,j) \in S} d_{ij}\hat{x}_{ij}$$

such that

$$Z^* \leq \hat{Z} \leq (1+\epsilon)Z^*$$

by solving $2\lceil \log_2(|\mathcal{A}|\overline{\theta}/\epsilon)\rceil + 3$ network flow problems, where $\overline{\theta} = \max\{u_{ij}d_{ij} \ : (i,j) \in \mathcal{A}\}$.