Applied Combinatorics

by Fred S. Roberts and Barry Tesman

Answers to Selected Exercises[1]

Chapter 1

1. $\begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ \hline \end{array}$;

2. $\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ \hline \end{array}$;
$\begin{array}{|ccc|} \hline 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ \hline \end{array}$;

3. Let row 1 be: $1\ 2\ \cdots\ n$. Row 2 is gotten by taking the first element (1) of row 1 and moving it to the end of the row. Row 3 is gotten from row 2 by taking the first element (2) of row 2 and moving it to the end of the row. Continue until you have n rows.

4(a). $\begin{array}{|ccc|} \hline 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ \hline \end{array}$; $\begin{array}{|ccc|} \hline a & b & c \\ b & c & a \\ c & a & b \\ \hline \end{array}$;

4(b). $\begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ \hline \end{array}$; $\begin{array}{|cccc|} \hline a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \\ \hline \end{array}$;

5. 0, 1, 00, 01, 10, 11, 000, 001, 010, 100, 011, 101, 110, 111;

[1]More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

6. See problem 5 and 0000, 0001, 0010, 0100, 1000, 0011, 0101, 0110, 1001, 1010, 1100, 0111, 1011, 1101, 1110, 1111;

7(a). no - there are only 12 such strings;

7(b). yes - there are 27 such strings;

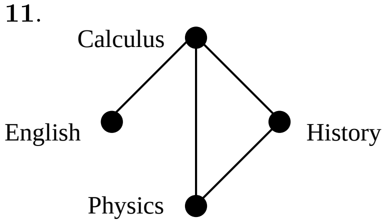
8. There are 84 such strings: there are 4 of length one; there are 16 of length two; there are 64 of length three;

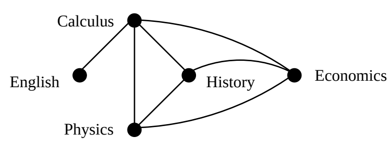
9. LLLL, LLLS, LLSL, LSLL, SLLL, LLSS, LSLS, SLLS, LSSL, SLSL, SSLL, LSSS, SLSS, SSLS, SSSL, SSSS;

10. Our conclusion would not change significantly. There are roughly 3.15×10^7 seconds per year and 100 billion equals 10^{11}. So, $3.15 \times 10^7 \times 10^{11}$ or 3.15×10^{18} networks could be analyzed in a year. Then the number of years it would take to check 6×10^{33} networks is

$$\frac{6 \times 10^{33}}{3.15 \times 10^{18}} \approx 1.9 \times 10^{15};$$

11.





;

12(a). No assignment exists. Each of Calculus, History, and Physics must get a different exam time since they overlap with one another;

12(b). time 1: English and Physics; time 2: Calculus; time 3: History;

12(c). No assignment exists. Each of Calculus, History, Physics, and Economics must get a different exam time since they overlap with one another;

12(d). time 1: English and Physics; time 2: Calculus; time 3: History; time 4: Economics;

13. If Economics is Wednesday and Transportation is Tuesday then both Housing and Health must be Thursday - this is not possible. Or, if Economics is Wednesday and Transportation is Thursday then both Housing and Health must be Tuesday - again this is not possible.

14(a). If English must be Thur. AM, then Calculus must be Wed. AM. But then History and Physics must be Tues. AM - this is not possible.

14(b). Wed. AM: English; Tues. AM: Calculus; Wed. AM: History; Thur. AM: Physics; Mon. AM: Economics.

15. Two of the four instructors can get their first choice. Assign a different morning to each of the Calculus, History, and Physics exam times. Then assign Tuesday morning to English.

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Chapter 2

***Section 2.1*.**

1. yes: $26^3 < 20,000$;

2. yes: $26^3 \cdot 10^3 > 5,000$;

3(a). $3^1 + 3^2 + 3^3$;

3(b). $3^1 + 3^2 + 3^3 + 3^4$;

3(c). $2 \cdot 3^3$;

4. $8^3 \times 10^5; 8 \times 2 \times 10 \times 8^3 \times 10^5$;

5. $n = 5$ since $2^1 + 2^2 + 2^3 + 2^4 = 30$ and $2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 62$.

6. 2^{mn};

7. $(2 \cdot 2 \cdot 2 \cdot 2 \cdot 3) - 1$;

8. $10^6 - 9^6$;

9.

Bit string x	$S_1(x)$	$S_2(x)$	$S_3(x)$	$S_4(x)$	$S_5(x)$	$S_6(x)$	$S_7(x)$	$S_8(x)$
00	0	0	0	0	0	0	0	0
10	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1
Bit string x	$S_9(x)$	$S_{10}(x)$	$S_{11}(x)$	$S_{12}(x)$	$S_{13}(x)$	$S_{14}(x)$	$S_{15}(x)$	$S_{16}(x)$
00	1	1	1	1	1	1	1	1
10	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1

10. $2^{2^{2^n}}$

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11. $2^{2^{n-1}}$;

12. 2^p; $2^p - 1$;

13(a). $2^m 2^{p-m}$;

13(b). $2^{m-1} 2^{p-m}$;

13(c). $(2^m - 1) \cdot 2^{p-m}$;

14(a). $3 \cdot 2 \cdot 3$;

14(b). $2^{3 \cdot 2 \cdot 3}$.

Section 2.2.

1. $2^3 + 2^4 + 2^5$;

2. $(8 \cdot 7) + (8 \cdot 12) + (7 \cdot 12)$;

3. $5^5 + 4^5$;

4. Yes: $10^{15} > 10,000,000$; No: $2^{15} < 10,000,000$;

5. $26^4 + 25^5$;

6. $2 + 3$;

7. $(7 \cdot 3)^{30}$;

8. $3^3 + 3 \cdot 4^3$.

Section 2.3.

1(a). 123, 132, 213, 231, 312, 321;

1(b). 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321;

2. $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$;

3. $1 \cdot n - 2 \cdot n - 3 \cdot \cdots \cdot 2 \cdot 1 \cdot 1$;

4(b). $s_6 \approx 710$ vs. $6! = 720$;

5. $2 \cdot 3 \cdot 2 \cdot 1 = 12$;

6(a). $4 \cdot 1 \cdot 3 \cdot 2 \cdot 1$;

6(b). $n - 2 \cdot 1 \cdot 1 \cdot n - 3 \cdot n - 4 \cdot \cdots \cdot 2 \cdot 1$;

7. $1 \cdot 3 \cdot 2 \cdot 1 \cdot 1$;

8(a). $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$;

8(a). $n \cdot n \cdot n - 1 \cdot n - 1 \cdot \cdots \cdot 2 \cdot 2 \cdot 1 \cdot 1$;

9. $5! \cdot 5!$ vs. $10!$.

Section 2.4.

1. $25! \cdot \frac{1}{10^9} \cdot \frac{1}{3.15 \times 10^7} \approx 4.9 \times 10^8$ years.

2. $25! \cdot \frac{1}{10^{11}} \cdot \frac{1}{3.15 \times 10^7} \approx 4.9 \times 10^6$ years.

3. There are $n!$ schedules. Each committee in each schedule must be checked to see if it received its first choice - this takes n steps per schedule. The computational complexity is $f(n) = n \cdot n!$;

5. We will start (and end) at 1.

Order	Total cost
1 2 3 4 1	1+ 3+ 11+8 = 23
1 2 4 3 1	1+ 6+ 2+ 4 = 13
1 3 2 4 1	8+ 9+ 6+ 8 = 31
1 3 4 2 1	8+ 11+ 3+ 16= 38
1 4 2 3 1	11+ 3+ 3+ 4 = 21
1 4 3 2 1	11+ 2+ 9+ 16= 38

6. best order is 2, 3, 1;

7(a). $n \times 3 \times 10^{-9}$.

7(b). $\frac{n+1}{2} \times 3 \times 10^{-9}$.

8(a). $n \times 3 \times 10^{-11}$.

8(b). $\frac{n+1}{2} \times 3 \times 10^{-11}$.

Section 2.5.

1(a). $3 \cdot 2$;

1(b). $5 \cdot 4 \cdot 3$;

1(c). $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$;

1(d). 0;

2(a). 6^3;

2(b). $6 \cdot 5 \cdot 4$;

2(c). $1 \cdot 6 \cdot 6$;

2(d). $1 \cdot 5 \cdot 4$;

3(a). 8^4;

3(b). $8 \cdot 7 \cdot 6 \cdot 5$;

3(c). $1 \cdot 8 \cdot 8 \cdot 8$;

3(d). $1 \cdot 6 \cdot 5 \cdot 1$;

4. $P(20, 5)$;

5(a). $9 \times 9 \times 8 \times 7$;

5(b). 4088. There are $1 \times 9 \times 8 \times 7$ extensions if the first digit is 1 and there are $8 \times 8 \times 8 \times 7$ extensions if the first digit is 2 - 9;

6. $40^3 = 64,000$.

Section 2.6.

1. $\{\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$;

2. 2^{35};

3. 2^7;

5. $2^{10} - 1$;

6. $2^8 - 9$;

7(a). 2^8. There are $2^3 = 8$ subsets of A and each subset can be assigned a 0 or 1;

7(b). 2^{2^n};

8(a). 2^{2^3};

8(b). 2^{2^n}.

Section 2.7.

1. $C(10, 5)$;

2. $C(50, 7)$;

3(a). $\frac{6!}{3!(6-3)!} = 20$;

3(b). $\frac{7!}{4!3!} = 35$;

3(c). $\frac{5!}{1!4!} = 5$;

3(d). 0;

4. $\frac{n!}{1!(n-1)!} = n$;

5. $\frac{5!}{2!3!} = 10$; $\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}$;

6. $\frac{6!}{2!4!} = 15$; $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$, $\{a, f\}$, $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{b, f\}$, $\{c, d\}$, $\{c, e\}$, $\{c, f\}$, $\{d, e\}$, $\{d, f\}$, $\{e, f\}$;

7(a). $C(7, 2) = \frac{7!}{2!(7-2)!} = \frac{7!}{2!5!}$ and $C(7, 5) = \frac{7!}{5!(7-5)!} = \frac{7!}{5!2!}$;

7(b). $C(6,4) = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!}$ and $C(6,2) = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!}$;

8. 1 7 21 35 35 21 7 1;

9. $C(5,3) = \frac{5!}{3!2!} = 10$, $C(4,2) = \frac{4!}{2!2!} = 6$, $C(4,3) = \frac{4!}{3!1!} = 4$, and $10 = 6 + 4$;

10. $C(7,5) = \frac{7!}{5!2!} = 21$, $C(6,4) = \frac{6!}{4!2!} = 15$, $C(6,5) = \frac{6!}{5!1!} = 6$, and $21 = 15 + 6$;

11(a). $C(8,4)$;

11(b). $C(8,4)/2$;

11(c). $2^8 - 2$;

12. $C(6,3) + C(6,2) + C(6,1) + C(6,0)$;

13(a). $C(10,5)$;

13(b). $2^{10} - 2$;

14. $C(21,5)C(5,3)8! + C(21,4)C(5,4)8! + C(21,3)C(5,5)8!$;

15. Calculate the sum of those odd numbers with distinct digits with no 0's, a 0 in the tens place, or a 0 in the hundreds place. No 0's: 5 choices for the ones place, then $8 \cdot 7 \cdot 6$ choices for the other three places; 0 in the tens place: 5 choices for the ones place and 1 choice for the tens place, then $8 \cdot 7$ choices for the other two places; 0 in the hundreds place: 5 choices for the ones place and 1 choice for the hundreds place, then $8 \cdot 7$ choices for the other two places;
$(5 \cdot 8 \cdot 7 \cdot 6) + (5 \cdot 1 \cdot 8 \cdot 7) + (5 \cdot 1 \cdot 8 \cdot 7) = 2240$;

16(a). $C(7,3) \cdot C(4,2)$;

16(b). $C(7,1) \cdot C(4,1) + C(7,2) \cdot C(4,2) + C(7,3) \cdot C(4,3) + C(7,4) \cdot C(4,4)$;

16(c). $C(10,3)$;

16(d). $C(7,2) \cdot C(4,2) - C(6,1) \cdot C(3,1)$;

17(a)(i). $C(9,4)$;

17(a)(ii). 6;

17(a)(iii). 2;

17(a)(iv). 1;

17(b). JAVA, JAVA, JAVA, JAVA, JAVA, C++, C++, C++, C++;

17(c)(i). $9!$;

17(c)(ii). $6 \cdot 4! \cdot 5!$;

17(c)(iii). $2 \cdot 4! \cdot 5!$;

17(c)(iv). $5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$;

18(a). $[C(3,1)C(27,2) + C(3,2)C(27,1) + C(3,3)C(27,0)] \times [C(12,1)C(138,2) + C(12,2)C(138,1) + C(12,3)C(138,0)]$;

18(b). $C(30,3) \cdot C(150,3) - C(27,3) \cdot C(138,3)$;

19(a). $\binom{n}{m}\binom{m}{k} = \frac{n!}{m!(n-m)!}\frac{m!}{k!(m-k)!} = \frac{n!}{(n-m)!k!(m-k)!}$;
$\binom{n}{k}\binom{n-k}{m-k} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(m-k)!((n-k)-(m-k))!} = \frac{n!}{k!(m-k)!(n-m)!}$;

20. Picking r items from n is the same as not picking $n-r$ items from n.

21. Sum the entries in the row labeled n, i.e., the $(n+1)^{st}$ row since the labels start with 0. For $n=2$: $1+2+1=4$; For $n=3$: $1+3+3+1=8$; For $n=4$: $1+4+6+4+1=16$; in general, 2^n.

22. use repeated applications of Theorem 2.2 and the fact that $\binom{n}{0} = \binom{n+1}{0}$;

23. A set consists of n male items and m female items for a total of $n+m$ items. Left side: number of ways to pick a subset of size r from this set of $n+m$ items. Right side: Picking a subset of size r can be done by picking 0 males & r females, or 1 male and $r-1$ females, or $\ldots$, or r males & 0 females.

24(a). $\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r}$ and $\left\langle {n \atop r-1} \right\rangle + \left\langle {n-1 \atop r} \right\rangle = \binom{n+r-2}{r-1} + \binom{n+r-2}{r}$ are equal using equation (2.3);

25.
$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$
and
$$\frac{n}{r}\left\langle {n+1 \atop r-1} \right\rangle = \frac{n}{r}\binom{n+r-1}{r-1} = \frac{n}{r}\frac{(n+r-1)!}{(r-1)!n!} = \frac{(n+r-1)!}{r!(n-1)!}$$
and
$$\frac{n+r-1}{r}\left\langle {n \atop r-1} \right\rangle = \frac{n+r-1}{r}\binom{n+r-2}{r-1} = \frac{n+r-1}{r}\frac{(n+r-2)!}{(r-1)!(n-1)!} = \frac{(n+r-1)!}{r!(n-1)!};$$

26(a). In the sequence of n 1's, t could be any integer from 0 to n, inclusive;

26(b). The sequence is strictly increasing when $\binom{n}{i} < \binom{n}{i+1}$ or $\frac{n!}{i!(n-i)!} < \frac{n!}{(i+1)!(n-i-1)!}$ or $i!(n-i)! > (i+1)!(n-i-1)!$ or $n-i > i+1$ or $i < \frac{n-1}{2}$. Similarly, it is strictly decreasing when $i > \frac{n-1}{2}$.

26(c). By 26(b), the largest entry in the sequence occurs when $i = \lfloor \frac{n}{2} \rfloor$. That is, at $\binom{n}{\lfloor n/2 \rfloor}$;

Section 2.8.

1(a). No;

1(b). No;

1(c). Yes;

1(d). No;

1(e). No;

2(a). 1/2;

2(b). 1/36;

2(c). 1/3;

3(a). $\frac{3}{8}$;

3(b). $\frac{1}{2}$;

3(c). $\frac{3}{4}$;

4. $\dfrac{C(3,2)\cdot 2 + C(3,3)}{3^3} = \dfrac{7}{27}$;

5. $\frac{15}{16}$;

6. $\frac{16}{4^5}$;

7. $\frac{2}{13}$;

8. $\frac{4}{52}\cdot\frac{4}{52}$;

9. $\frac{5}{16}$;

10. $\frac{5}{8}$;

11. $\dfrac{C(4,3)+C(4,4)}{4^2} = \dfrac{5}{16}$;

12. $\frac{1}{4}$;

13. $\frac{21}{32}$;

14. $\frac{\frac{20\cdot 18\cdot 16\cdot 14}{4!}}{C(20,4)}$;

15(a). $C(6,2)/2^6 + C(6,3)/2^6$.

15(b). $C(6,2)/2^6 + C(6,4)/2^6$.

15(c). $C(6,2)/2^6 + 2^5/2^6 - C(5,1)/2^6$.

15(d). $C(6,0)/2^6 + C(6,2)/2^6 + C(6,4)/2^6 + C(6,6)/2^6$.

15(e). $C(5,1)/2^6 + C(5,3)/2^6 + C(5,5)/2^6$.

16(a). probability of $E = \frac{n(E)}{n(S)}$ and probability of $E^c = \frac{n(S)-n(E)}{n(S)} = 1 - \frac{n(E)}{n(S)}$.

16(b). E and F disjoint implies $n(E\cup F) = n(E) + n(F)$.
So, probability of $E\cup F = \frac{n(E\cup F)}{n(S)} =$
$\frac{n(E)+n(F)}{n(S)}$ and probability of $E +$ probability of $F = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)}$.

16(c). E and F not disjoint implies $n(E \cup F) = n(E) + n(F) - n(E \cap F)$.
So, probability of $E \cup F = \frac{n(E \cup F)}{n(S)} = \frac{n(E)+n(F)-n(E \cap F)}{n(S)}$ and probability of $E +$ probability of $F -$ probability of $E \cap F = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$.

17(a). $2^2/2^4 + 2^2/2^4 - 1/2^4$.

17(b). $2^3/2^4 + 2/2^4 - 1/2^4$.

Section 2.9.

1(a). *aaaa*, *aaab*, *aaba*, *abaa*, *baaa*, *aabb*, *abab*, *abba*, *baab*, *baba*, *bbaa*, *abbb*, *babb*, *bbab*, *bbba*, *bbbb*;

1(b). $aa, ab, ac, ba, bb, bc, ca, cb, cc$;

1(c). $\{a, a, a, a\}, \{a, a, a, b\}, \{a, a, b, b\}, \{a, b, b, b\}, \{b, b, b, b\}$;

1(d). $\{a, a\}, \{a, b\}, \{a, c\}, \{b, b\}, \{b, c\}, \{c, c\}$;

2(a). $P^R(2, 4) = 2^4$;

2(b). $P^R(3, 2) = 3^2$;

2(c). $C^R(2, 4) = C(2 + 4 - 1, 4) = 5$;

2(d). $C^R(3, 2) = C(3 + 2 - 1, 2) = 6$;

3(a). $P^R(3, 7) = 3^7$;

3(b). $C^R(4, 7) = C(4 + 7 - 1, 7)$;

4. $C^R(4, 8) = C(4 + 8 - 1, 8)$;

5. $C^R(5, 12) = C(5 + 12 - 1, 12) = 1820$;

6. $P^R(5, 8) - P^R(5, 7) = 5^8 - 5^7$;

7(a). $C^R(4, 2) = C(4 + 2 - 1, 2) = 10$;

7(b). 1560;

8(a). $C^R(3, 82) = C(3 + 82 - 1, 82) = 3486$;

8(b). $C^R(2, 82) = C(2 + 82 - 1, 82) = 83$;

9. $P^R(4, 12) = 4^{12}$;

10. $\displaystyle\sum_{j=0}^{400} C^R(3, j)$;

11. $\displaystyle\sum_{j=0}^{nl} C^R(m, j)$.

Section 2.10.

1(a).

Distribution

Cell	1	2	3	4	5	6	7	8
1	abc		ab	c	ac	b	bc	a
2		abc	c	ab	b	ac	a	bc

1(d).

Distribution

Cell	1	2	3	4
1	aaa		aa	a
2		aaa	a	aa

3(b). $2^4 = 16$;

3(c). $4^2 = 16$;

3(e). $C(2+4-1,4) = 5$;

3(f). $C(4+2-1,2) = 10$;

4(a). $S(3,1) + S(3,2) = 4$;

4(d). Number of partitions of 3 into two or fewer parts = 2;

5(a). $2!S(3,2) = 6$;

5(d). $C(2,1) = 2$;

6(a). $S(3,2) = 3$;

6(d). Number of partitions of 3 into exactly two parts = 1;

7(a). $\{1,1,1,1\}, \{1,1,2\}, \{2,2\}, \{4\}, \{1,3\}$;

9(a). 1;

9(b). 1;

9(c). 1;

9(d). $\binom{n}{2}$;

9(e). 1;

10. 12 indistinguishable balls (misprints), 5 distinguishable cells (kinds of misprints), cells can be empty: $C(5+12-1,12)$;

11. 25 indistinguishable balls (misprints), 75 distinguishable cells (pages), cells can be empty: $C(75+25-1,25)$;

12. 30 distinguishable balls (errors), 100 indistinguishable cells (codewords), cells can be empty: $S(30,1)+S(30,2)+S(30,3)+\cdots+S(30,100)$;

13. 9 distinguishable balls (passengers), 5 indistinguishable cells (floors), cells can be empty: $S(9,1)+S(9,2)+S(9,3)+S(9,4)+S(9,5)$;

14. 10 indistinguishable balls (lasers), 5 indistinguishable cells (tumors), cells can be empty: Number of partitions of 10 into 5 or fewer pieces;

15. $C(6+30-1,30)$;

16. 10 balls (customers), 7 cells (salesmen), cells can't be empty: answer depends on the choice (interpretation) of distiniguishable balls and/or cells;

17. $\dfrac{10!}{2^5 5!} = 945$.

18. $4!S(6,4)$ assuming both jobs and workers are distinguishable;

20. $1/C(11,8)$;

22(a). $S(n,k)$ counts the number of ways to place n distinguishable balls into k indistinguishable cells with no cell empty. Consider the n^{th} ball. Either it is by itself in a cell OR it is with other balls. $S(n-1,k-1)$ counts the number of ways where the n^{th} ball is by itself: To assure it is by itself, place the remaining $n-1$ balls in $k-1$ cells with no cell empty and then put the n^{th} ball in its own k^{th} cell. $kS(n-1,k)$ counts the number of ways where the n^{th} ball is with other balls in some cell: To assure that it is with other balls, place the remaining $n-1$ balls in k cells *with no cell empty* and then put the n^{th} ball in one of the k (nonempty) cells (i.e., there are k choices for the n^{th} ball);

22(c). 90;

24(a). 16. One ordering of $\{1,1,1,1,1\}$, 4 orderings of $\{1,1,1,2\}$, 3 orderings of $\{1,2,2\}$, 3 orderings of $\{1,1,3\}$, 2 orderings of $\{2,3\}$, 2 orderings of $\{1,4\}$, and 1 ordering of $\{5\}$;

24(b). 4. $\{2,3\}$, $\{3,2\}$, $\{1,4\}$, $\{4,1\}$;

24(c). The number of partitions of n into exactly k parts is Case 4b in Table 2.9, page 54. Saying that "order matters" is the same as saying that the cells are now distinguished. Thus, we are now in Case 2b in Table 2.9, page 54 which is counted by $C(n-1,k-1)$.

Section 2.11.

1(a). 630;

1(d). 90/729;

2. $C(n;1,1,1,\ldots,1) = \binom{n}{1,1,1,\ldots,1} = \dfrac{n!}{1!1!1!\cdots 1!} = n!$;

6(a). $C(7;4,1,2) = 105$;

9(a). $C(9;3,2,1,1,1,1) = 30,240$;

9(b). $\dfrac{C(9;3,2,1,1,1,1)}{26^9} = \dfrac{30,240}{26^9} \approx 5.56957 \times 10^{-9}$;

10. $4C(10;4,2,2,2) + C(4,2)C(10;3,3,2,2)$;

11. 4!. Since there are 5 a's and 4 non-a's, any such permutation must start with an a and alternate a's and non-a's. Thus, we need only count how to order the four non-a's;

13(a). $C(5;3,1,1)$;

16(b). $P(4;3,1)$;

16(d). $P(4;2,1,1) = 12$.

***Section 2.12*.**

1(a). $3! = 6$;

1(b). $\dfrac{5!}{2} = 60$;

1(c). CAAGCUGGUC;

2. $C(10;2,3,3,2)$;

3(a). $\dfrac{4!}{3!} = 4$;

3(b). $4!$;

3(c). GUCGGGUU and GGGUCGUU;

7(a). $\dfrac{3!}{2!} = 3$;

7(b). $\dfrac{5!}{3!2!} = 10$;

7(c). 00101010, 01001010 and 01010010;

8. GCGUGU and GUGCGU;

10. yes: 00101010 and 01001010 have the same breakup.

***Section 2.13*.**

1. $9/C(12,4)$;

3(a). The given *order* has a probability of $\frac{1}{C(13;8,5)}$ of being observed;

3(b). $\frac{C(9;8,1)}{C(13;8,5)}$. Group the 5 sick trees together as one unit S^*. Then the number of orders of 1 S^* and 8 W's is $C(9;8,1)$;

5(a). Group the 6 infested houses together as one unit I^*. Then the number of orders of 1 I^* and 5 noninfested houses is $C(6;5,1)$;

5(b). Only 1: start with an infested and alternate;

8(a). $7/4^4$;

9. $C(17;13,4)$;

10. $\binom{15}{5,4,5,1}$. This is equivalent to RNA chains of length 15 having five A's, four U's, five H's and one G, where H = "CG."

Section 2.14.

1(a). $\binom{5}{0}x^5 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}y^5$;

1(b). $\binom{3}{0}a^3 + \binom{3}{1}a^2 2b + \binom{3}{2}a(2b)^2 + \binom{3}{3}(2b)^3 = a^3 + 6a^2b + 12ab^2 + 8b^3$;

1(c). $\binom{4}{0}(2u)^4 3v^0 + \binom{4}{1}(2u)^3 3v^1 + \binom{4}{2}(2u)^2 3v^2 + \binom{4}{3}(2u)^1 3v^3 + \binom{4}{4}(2u)^0 3v^4 = 16u^4 + 96u^3v + 216u^2v^2 + 216u^1v^3 + 81v^4$;

2(a). $4368 = \binom{16}{11}1^5$;

2(b). $2912 = \binom{14}{11}2^3$;

2(c). $2^{11} = \binom{11}{0}2^{11}$;

3. $C(12,9)C(4,0) + C(12,8)C(4,1) + \cdots + C(12,5)C(4,4)$;

4. $C(10,8)C(6,0) + C(10,7)C(6,1) + \cdots + C(10,2)C(6,6)$;

5. $\displaystyle\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k} a^i b^j c^k$;

6. $\binom{6}{2,2,2} = 90$;

7. $\binom{5}{1,1,3} = 20$;

8. $\displaystyle\sum_{\substack{n_i\geq 0,\forall i\\ n_1+n_2+\cdots+n_k=n}} \binom{n}{n_1,n_2,\ldots,n_k} a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$;

9. $\binom{6}{3,1,2} = 60$;

10. $\binom{6}{1,2,2,1} \cdot 2 = 360$;

11. $\binom{12}{3,2,1,6} \cdot 5^2 \cdot 2 \cdot 2^6 = 177,408,000$;

13. $\frac{1}{2} \cdot 2^{12} = 2048$;

15. $\frac{1}{2} \cdot 2^{n} = 2^{n-1}$;

16(a). 3^n;

16(b). 5^n;

16(c). $(1+x)^n$;

16(d). $n(n-1)2^{n-2}$;

16(e). $n2^{n-1}$;

17(a). Start with $(x+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} x^k$, differentiate. and then let $x = 2$;

17(b). Start with $(x+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} x^k$, differentiate. and then let $x = 1$.

Section 2.15.

1(a). $\{1,2\}, \{1,3\}, \{1,2,3\}$;

2(a). $\{1,2,5\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{4,5\}$;

3(b). $\frac{1}{5}$ for each;

6(a). yes - $[5; 5, 2, 1]$;

11(b). nonpermanent: $\frac{9!9!}{3!16!}$.

Section 2.16.

1(a). 2143 precedes 3412;

2(b). 0101;

3(c). $\{1,3,5,6\}$;

4(c). 152634;

11. $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$;

12. $0111 \cdots 1$;

21(b). 4.

Section 2.17.

3. 456123;

5(c). $n12\cdots(n-1)$;

5(d). 2;

7(b). 5: $15423 \to 14523 \to 14253 \to 14235 \to 12435 \to 12345$;

9(a). 2: first, transpose 7 and 46 to get 5123467; then transpose 5 and 1234;

10(a). 2.

Section 2.18.

1(b). yes;

1(d). yes;

4. $f(n) \leq k_1 g(n)$ for some positive constant k_1 and $n \geq r_1$; $g(n) \leq k_2 h(n)$ for some positive constant k_2 and $n \geq r_2$; then $f(n) \leq k_1 k_2 h(n)$ for positive constant $k_1 k_2$ and $n \geq \max\{r_1, r_2\}$;

8(a). yes;

9(a). yes;

9(f). yes;

9(i). yes;

12. The third is "big oh" of the other two; the first is "big oh" of the second.

Section 2.19.

1(a). 27;

2(a). 53;

3. 6;

4(a). 4;

6. Yes; use Corollary 2.15.1 and the fact that the average number of seats per car is $\frac{465}{95} \approx 4.89$;

9(a). 9;

9(b). 20;

13(a). longest increasing is 6, 7 or 5, 7 and longest decreasing is 6, 5, 4, 1;

14. $13, 14, 15, 16, 9, 10, 11, 12, 5, 6, 7, 8, 1, 2, 3, 4$;

15. let the pigeons be the 81 hours and the five holes be days 1 and 2, 3 and 4, 5 and 6, 7 and 8, and 9 and 10;

24. if there are n people, each has at most $n - 1$ acquaintances;

29(a). $X = \{\{a, b\}, \{b, c\}, \{c, d\}\}, Y = \{\{a, c\}, \{b, d\}, \{a, d\}\}$;

30(a). 2.

Additional Exercises for Chapter 2.

No answers provided.

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Answers to Selected Exercises[1]

Chapter 3

Section 3.1.

1(a). $V =$ {Chicago (C), Springfield (S), Albany (A), New York (N), Miami (M)};

1(b). $A =$
{(C, S), (S, C), (C, N), (N, C), (A, N), (N, A), (C, M), (M, C), (N, M), (M, N)};

4(b). In $G_3, E = \{\{u, v\}, \{v, w\}, \{u, w\}, \{x, y\}, \{x, z\}, \{y, z\}\}$;

17(a). yes;

17(b). no;

19. 15;

20. 32;

23. no;

24. yes.

Section 3.2.

1(d). yes;

5. D_4: no; D_6: yes;

6(a). no;

6(e). yes;

9(b). D_4: yes; D_8: no;

10(c). D_4: yes; D_8: yes;

13. D_8: $\{p, q, r, s, t\}, \{u\}, \{v\}, \{w\}$;

18. *Hint:* Use induction on the number of vertices;

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

25(a). yes;

30(a). 9;

30(b). $2\binom{n-1}{2} + (n-1) = (n-1)^2$.

Section 3.3.

3(a). (b): yes;

3(b). (b): 3;

5. no;

8. 4;

18. no;

21. (b):yes;

41(a). 4;

54(b). Z_n, n odd, $n \geq 3$;

55(a). $\omega(G) = \alpha(G^c)$;

54(b). Z_5 plus a vertex adjacent to two consecutive vertices of Z_5.

Section 3.4.

1(a). $x(x-1)^3$;

2(a). 24;

2(c). 48;

6. $[P(I_2, x) - P(I_1, x)][P(I_1, x) - 2]^2$;

9(a). $5 \cdot 4!$; $\binom{5}{2} \cdot 4!$;

11(a). 2;

11(c). (a):0;

13(d). $P(x) \neq x^n$ and the sum of the coefficients is not zero;

20(b). yes;

25(c). $(-1)^{n-1}(n-1)!$.

Section 3.5.

4(a). 11;

4(b). 9;

9. $n-k$;

13. 16;

16. There are too few edges to have a spanning tree; alternatively, the deleted edge was the only simple chain between its end vertices;

19. 2 if $n \geq 2$, since $2(2-1)^{n-1} > 0$ and $1(1-1)^{n-1} = 0$; 1 if $n = 1$;

26. *Hint:* The sum of the degrees is $4k + m$ and we have a tree;

29(b). yes;

31(a). 6;

32(b). $\binom{8}{2}6! = 20,160$.

Section 3.6.

2(a). a: 0; b, c: 1; d, e, f, g: 2; h, i, j, k: 3;

3(a). 3;

4(a). $\{d, e, h, i\}$;

7(a). The children of vertex 1 are 2 and 3 and the children of vertex 2 are 4 and 5;

11(a). 6;

21. $[1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + (h+1)2^h]/n$, where h is the height;

26. 3 4 1 2, 3 1 4 2, 3 1 2 4, 1 3 2 4, 1 2 3 4, 1 2 3 4;

31. 10;

43(b). 240.

Section 3.7.

1(a). For D_1 : $\begin{array}{c} \\ u \\ v \\ w \\ x \end{array}\begin{array}{c} \begin{array}{cccc} u & v & w & x \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$;

2(h). $\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} \{a,b\} & \{b,c\} & \{a,d\} & \{b,e\} & \{d,e\} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$;

7. For G_1 : $\begin{array}{c|ccc} & u & v & w \\ \hline u & 0 & 1 & 0 \\ v & 1 & 0 & 1 \\ w & 0 & 1 & 0 \end{array}$; **13**. For D_4 : $\begin{array}{c|cccccc} & u & v & w & x & y & z \\ \hline u & 1 & 1 & 1 & 1 & 1 & 1 \\ v & 1 & 1 & 1 & 1 & 1 & 1 \\ w & 1 & 1 & 1 & 1 & 1 & 1 \\ x & 0 & 0 & 0 & 1 & 1 & 1 \\ y & 0 & 0 & 0 & 1 & 1 & 1 \\ z & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$;

21(a). There is a path from i to j if and only if there is a path of length at most $n-1$;

24(a). j is in the strong component containing i if and only if $r_{ij} = 1$ and $r_{ji} = 1$;

30. it gives the number of vertices that edges i and j have in common;

33. yes: take Z_4 as in Figure 3.22 and append x adjacent to a and b and y adjacent to b and c; repeat with Z_4 and x as above, but take y adjacent to c and d; relabel edges.

Section 3.8.

4. 7;

5(a). $\{a, c, e\}$;

5(f). $\{a, b, d\}$;

8(a). Let 4 "red edges from one vertex" be $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$. If any one of the edges $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{c, d\}$, $\{c, e\}$, $\{d, e\}$ is red then there will be 3 vertices all joined by red edges. If they are all blue then vertices b, c, d, e are all joined by blue edges;

9(a). Yes;

11(c). 4;

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Answers to Selected Exercises[1]

Chapter 4

Section 4.1.

2. Consider a brother and a sister;

4. Less than;

6(a). It is a complete graph with loops at every vertex;

8(b). Suppose that R^c is not symmetric. Then, for some $a, b \in X$, aR^cb but $\sim bR^ca$. Therefore, R is not symmetric since bRa but $\sim aRb$;

10. The relation $(X, R \cap S)$ is reflexive, symmetric, asymmetric, antisymmetric, and transitive;

16. Consider the binary relation (X, R) and suppose that aRa for some $a \in X$. By asymmetry, it must be the case that $\sim aRa$, which is a contradiction;

19(a). $X = \{a, b, c\}$ and $R = \{(a, c)\}$;

20. (a), (d), (e), and (g) are equivalence relations;

25. Reflexive and Symmetric hold;

26(e)i. aSb iff $\sim aRb$ & $\sim bRa$ iff $a \not> b$ & $b \not> a$ iff $a = b$;

27. 2^{n^2}.

Section 4.2.

1(b). Yes.

2(b). Yes.

3(b). No.

4(b). No.

5(b). No.

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

6(b). No.

7(b). No.

8(b). No.

9(b). No.

10(b). No.

13(c). $K = \{(1,3),(2,3),(3,4)\}$.

19(a). $L_{S^{-1}} = [x_n, x_{n-1}, \ldots, x_1]$.

19(b). $L_S \cap L_{S^{-1}} = \emptyset$.

21. Transitive and complete, but not asymmetric: $X = \{a\}$ and $R = \{(a,a)\}$. Transitive and asymmetric, but not complete: $X = \{a,b,c\}$ and $R = \{(a,b),(a,c)\}$. Complete and asymmetric, but not transitive: $X = \{a,b,c\}$ and $R = \{(a,b),(b,c),(c,a)\}$.

25(a). Yes.

30(a). No.

33(a). w_1 and m_2 are both better off by leaving their assigned partners and marrying each other.

Section 4.3.

1. No.

4(a). 3.

4(e). 4.

5(c). 2.

9(a). $[\hat{1}, x, y, d, z, a, b, c, \hat{0}]$.

12. Strict partial order (c) of Figure 4.23 has dimension 2.

17. $[a,x]$, $[b,y]$, $[c,z]$, $[d,w]$.

18. $\{u\}$, $\{y,w\}$, $\{z,v\}$, $\{x\}$.

Section 4.4.

1(a). No for both strict partial orders

1(b). No for both strict partial orders

2(b). (a): $\hat{0}$; (b): d; (c): $\hat{0}$.

3(a). (a): Not a lattice; (b): Not a lattice.

11.

a	b	c	$a \wedge (b \vee c)$	$(a \wedge b) \vee (a \wedge c)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
1	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

12. (a): No.

16(c).

p	q	q'	$p \wedge q'$	$(p \wedge q') \to q$
F	F	T	F	T
F	T	F	F	T
T	F	T	T	F
T	T	F	F	T

17(b). $p =$ Pete loves Christine; $q =$ Christine loves Pete; $p \wedge q =$ Pete and Christine love each other.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

18(b). Equivalent.

20(a).

x_1	x_2	$x_1' \wedge x_2$	$x_1 \vee x_2$	$(x_1' \wedge x_2) \vee (x_1 \vee x_2)$
1	1	0	1	1
1	0	0	1	1
0	1	1	1	1
0	0	0	0	0

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Answers to Selected Exercises[1]

Chapter 5

Section 5.1.

1(b). $\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$;

1(e). $1 + x + \frac{3x^2}{2!} + \sum_{k=3}^{\infty} \frac{x^k}{k!}$;

2(b). $\sum_{k=0}^{\infty} x^{k+2}$;

2(c). $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{8k+4} = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \cdots$;

3(a). $1 + x + x^2$;

3(d). $-x - x^2 + \frac{1}{1-x}$;

3(j). $-1 - x + e^x$;

4(c). $(0, 0, 0, 1, 1, 1, \ldots)$;

4(h). $\left(6, \frac{2^1}{1!}, \frac{2^2}{2!}, \frac{2^3}{3!}, \ldots, \frac{2^k}{k!}, \ldots\right)$;

4(k). $(1, 0, 1, 0, 1, 0, \ldots)$;

5(a). 1;

6(a). 0;

11. 3^n;

13(b). $1 + 8x + 12x^2$;

14(b). $1 + 16x + 72x^2 + 96x^3 + 24x^4$.

Section 5.2.

1(a). $\left(0, 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right)$;

1(b). $\left(1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, \ldots\right)$;

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[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

1(e). $(9, 8, 5, 6, 5, 5, 5, \ldots)$;

2(a). $(0, 1, 2, 3, \ldots)$;

3(b). $a_k = \sum_{i=0}^{k} i$;

4(a). $a_k = 15(k+1)$;

5. 5;

6. 38;

7(a). $(1, 4, 9, 27, 81, \ldots)$;

8(a). $A(x) + (11 - a_3)x^3$;

10(a). $\frac{x}{(1-x)^2} + \frac{3}{1-x} = \frac{3-2x}{(1-x)^2}$;

11(b). $\frac{2x}{(1-x)^3}$;

17. $R(x, B) = 1 + 6x + 11x^2 + 8x^3 + 2x^4$.

***Section 5.3*.**

1(a). $(1 + x + x^2)^2(1 + x + x^2 + x^3)^2$, coefficient of x^5;

1(e). $(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)^2(x^4 + x^5 + x^6 + x^7)$, coefficient of x^{12};

1(i). $(1 + x + x^2 + x^3 + x^4 + \cdots + x^{12})^8$, coefficient of x^{12};

1(m). $(1 + x + x^2 + x^3 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)(1 + x^{10} + x^{20} + x^{30} + \cdots)(1 + x^{25} + x^{50} + x^{75} + \cdots)$, coefficient of x^{100};

2. $(1 + x)^3(1 + x + x^2 + \cdots)$, coefficient of x^5;

8. $(1 + x + x^2 + x^3 + x^4)^3(1 + x + x^2 + x^3)(1 + x + \cdots + x^7)(1 + x + \cdots + x^{12})$;

13(c). $(1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^k)$.

***Section 5.4*.**

1(b). -10;

2(a). 35;

3. $-\frac{1}{9}$;

6. $\binom{16}{11}$;

11. $\binom{12}{7}$;

13(b). $\binom{p+k-1}{k}$;

18(b). $\binom{n-1}{r-1}$;

21. 37.

Section 5.5.

1(b). e^{3x};

1(c). $e^x - x - \frac{x^2}{2!}$;

2(a). $a_k = 4k!$;

2(g). $a_k = 2^k + 5^k$;

6(a). $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)^2 (1+x)^2$, coefficient of $\frac{x^3}{3!}$;

6(d). $(1 + x + x^2 + \cdots + x^{2n})^3$, coefficient of x^{3n};

12. $(e^x - 1)^p$;

14. $\frac{1}{2}[5^k - 3^k]$;

19(b). $\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right] 3!$.

Section 5.6.

1(a). $G(x) = \frac{1}{3} + \frac{1}{3}x + \frac{1}{3}x^2, E = 1, V = \frac{2}{3}$;

7(a). $\frac{px}{1+px-x}$;

8(b). $E = \frac{qm}{p}, V = \frac{q^2m}{p^2} + \frac{qm}{p}$.

Section 5.7.

1(d). Coleman: $0, \frac{4}{8}, \frac{4}{8}, \frac{4}{8}$; Banzhaf: $0, \frac{4}{12}, \frac{4}{12}, \frac{4}{12}$;

2(b). [5; 4, 2, 1, 1];

4(c). $\frac{5}{12}, \frac{3}{12}, \frac{3}{12}, \frac{1}{12}$.

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Answers to Selected Exercises[1]

Chapter 6

Section 6.1.

1. $a_5 = 1535$, $a_6 = 6143$,;

2. $S_4 = 9000$;

3. $S_4 = 10,368$;

4(b). 528;

4(c). 00, 11, 12, 13, 21, 22, 23, 31, 32, 33;

9. $b_4 = F_5 = 8$: XOOX, XOXO, OXOX, XOOO, OXOO, OOXO, OOOX, OOOO;

11. 4123, 4312, 4321, 3142, 3412, 3421, 2143, 2341, 2413;

14(a). $7! - D_7$;

14(b). 1;

19(a). $(D_4)^2$;

19(b). $(4!)^2$;

24. $f(n+1) = (2n+1)f(n)$;

31. $F_{n+2} - 1$;

34. $f(n+1) = f(n) + 2n, n \geq 1, f(1) = 2$;

38(a). $n!$;

40. $C_3 = 4$.

Section 6.2.

1(a). linear;

1(e). not linear;

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

2(a). not homogeneous;

2(e). homogeneous;

3(a). has constant coefficients;

3(f). does not have constant coefficients;

5(a). $x^2 + 2x + 1 = 0$;

5(f). $x^2 - 2x - 3 = 0$;

6(a). -1, -1;

6(i). 1, 2, -2;

10(a). a solution;

11(c). not a solution;

12(a). $a_n = 2 \cdot (-1)^n - 4n \cdot (-1)^n$;

12(e). $h_n = \frac{7}{3} \cdot 3^n + \frac{5}{3} \cdot (-3)^n$;

17(c). yes;

17(d). $a_n = (-\frac{i}{2})(i)^n + (\frac{i}{2})(-i)^n$;

22. 3 is a characteristic root of multiplicity 3;

26(a). $a_n = \frac{8}{9}n(3^n) + (-\frac{5}{9})n^2(3^n)$;

27(a). $a_n = 5^n - \frac{3}{5}n5^n$.

Section 6.3.

2(a). $a_k = 3k + 1$;

3(a). $a_n = 2 \cdot (-1)^n - 4n \cdot (-1)^n$;

3(e). $h_n = \frac{7}{3} \cdot 3^n + \frac{5}{3} \cdot (-3)^n$;

4(a). $-\frac{1}{2} + \frac{3}{2} \cdot 3^k$;

4(e). $\left(\frac{1}{2}\right)^k - \left(\frac{1}{3}\right)^k$;

9. $G_n = \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$;

15. $G(x) = \frac{2(x-2)}{(x-1)(x^2-2x+2)}$;

19. $G(x) = \frac{x^3}{(1-2x)(x^2+1)}$;

27. $1 + 7x + 13x^2 + 6x^3$.

Section 6.4.

2. $u_5 = 42$;

4. $R_4 = 4$;

5. $h_4 = 36$;

8(a). $C(x) - x^2 = A(x) + B(x) - x$;

8(b). $A(x) = 4xC(x)$;

8(c). $B(x) = x\{[C(x)]^2 + 1\}$;

8(d). $C(x) - x^2 = 4xC(x) + x[C(x)]^2$;

15(a). $q_3 = 5$;

19(i). $B(x) = x[U(x)]^2$;

22(c). $A(x)B(x) = \frac{x^3}{1-2x}$.

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Answers to Selected Exercises[1]

Chapter 7

***Section 7.1*.**

1. 65;

8. 15;

11. 6233;

14. $\left(7^9 - \binom{7}{1}(7-1)^9 + \binom{7}{2}(7-2)^9 \mp \cdots + \binom{7}{6}(7-6)^9 - 0\right)/7^9$;

18(d). $P(G,x) = x^4 - 4x^3 + 5x^2 - 2x$;

21(a). 19;

21(b). 10;

21(c). 15;

26(b). 3 times;

28. 28;

29. 456;

34. $b_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! \pm \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!$.

***Section 7.2*.**

1. 22%;

4. 7;

9. $\frac{1}{2}$;

11. $\binom{n}{m}D_{n-m}$;

12(a). $\frac{6}{16}$;

14(a). 630;

[1]More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

18(a). 7/24;

18(b). 1/24;

18(c). 0/24;

22. 512;

32. 9;

36. 11.

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Answers to Selected Exercises[1]

Chapter 8

***Section 8.1*.**

1(c). reflexivity, transitivity;

1(h). symmetry;

2. Yes;

3(a). $\{a, b\}, \{c, d\}$;

7. $\{bb\}, \{rr\}, \{pp\}, \{bp, pb\}, \{br, rb\}, \{pr, rp\}$;

9. see Figure 8.10 in the text;

13(a). there are 3 others;

16. 3.

***Section 8.2*.**

1(b). $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$;

2(a). $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$;

5(a). all;

5(f). **G1, G2**;

8(a). no;

9. $C(1) = \{1, 7\}$;

10(a). $\{1, 5\}, \{2, 4\}, \{3\}$;

11(a). reflection in a diagonal from upper right to lower left;

12(a). $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$;

[1]More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

21. no, for example, associativity may not hold. Let $p = 6$, then $(2 \circ 3) \circ 5 \neq 2 \circ (3 \circ 5)$;

23(a). $\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \right\}$.

Section 8.3.

5(a). $\frac{1}{2}(5+1) = 3 : \{1,5\}, \{2,4\}, \{3\}$;

6(a). (a) $\mathrm{St}(1) = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \right\}$, **(b)** $C(1) = \{1,5\}$;

12. no, G is not a group of permutations.

13(b). 12.

Section 8.4.

2. 3^8;

4(a). black;

5(a). r;

6(a). no;

6(g). 8;

7(a). no;

7(f). 4;

10(e). 6;

21(c). if π_i is $\begin{pmatrix} \{1,2\} & \{1,3\} & \{2,3\} \\ \{1,2\} & \{2,3\} & \{1,3\} \end{pmatrix}$, then $\mathrm{Inv}(\pi_i^*)$ is 4.

Section 8.5.

1(a). (2)(5)(17)(364);

1(b). (2)(1534);

1(c). (164827)(35);

3(a). $\mathrm{cyc}(\pi)$ is 5 and 3, respectively;

4(a). x_1^5 and $x_1x_2^2$, respectively;

5(a). $\frac{1}{2}(x_1^5 + x_1x_2^2)$;

9(a). $\frac{1}{2}\left[2^4 + 2^2\right] = 10$;

12(a). $\frac{1}{2}\left[2^3 + 2^2\right] = 6$;

22(b). x_2^4;

26. (16)(15)(14)(13)(12);

31(a). $D_5 = (2-1)(3-2)(3-1)(4-3)(4-2)(4-1)(5-4)(5-3)(5-2)(5-1)$;

31(a). $\pi D_5 = (3-4)(5-3)(5-4)(2-5)(2-3)(2-4)(1-2)(1-5)(1-3)(1-4)$;

Section 8.6.

1. The second has weight 192;

2(a). g of part (a) has weight x^2y^2;

4. $2a^3b + 2ab^3 + a^2b^2$;

8. 3280;

16. 11;

19(e). $1 + x$;

20(c). if the entries of (x_1, x_2, x_3) give the colors of a, b, c, respectively, then equivalence classes are $\{(0,0,0)\}$, $\{(0,0,1)\}$, $\{(0,1,0),(1,0,0)\}$, $\{(0,1,1),(1,0,1)\}$, $\{(1,1,0)\}$, $\{(1,1,1)\}$.

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Answers to Selected Exercises[1]

Chapter 9

***Section 9.1*.**

1. $\boxed{\begin{array}{cccc} 1 & 4 & 3 & 2 \\ 1 & 4 & 3 & 2 \\ 1 & 4 & 3 & 2 \\ 1 & 4 & 3 & 2 \end{array}}$;

2(b). it must be a multiple of 6;

6(a). 21.

***Section 9.2*.**

1(a). no;

3(a). no;

4(a). yes;

5(a). cannot be sure;

5(c). cannot be sure;

8. yes;

9. no;

11(a). no;

11(b). it is at most 7.

17. $A^{(1)} = \boxed{\begin{array}{ccc} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{array}}$, $A^{(2)} = \boxed{\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}}$;

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

19. $$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 4 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 2 & 4 & 3 & 4 \\ 3 & 1 & 3 & 1 \\ 3 & 2 & 4 & 4 \\ 3 & 3 & 1 & 3 \\ 3 & 4 & 2 & 2 \\ 4 & 1 & 4 & 2 \\ 4 & 2 & 3 & 3 \\ 4 & 3 & 2 & 4 \\ 4 & 4 & 1 & 1 \end{bmatrix};$$

22(a). yes;

22(b). no;

22(c). yes.

Section 9.3.

1(a). 2;

2(b). $a + b = 9, a \times b = 8$;

5(c). no;

7(b). 258;

10(a).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

11(b). 2;

13(a). 3; 7;

15(a). no.

Section 9.4.

1(a). not a BIBD;

2(a). $b = 50, r = 25$;

3(a). $r(k-1) \neq \lambda(v-1)$;

7. no: $b \geq v$ fails;

9(a). $\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} \{1,2\} & \{1,3\} & \{2,4\} & \{1,2,3\} & \{2,3,4\} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$;

10(a). $\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$;

18. 737;

21. $b = 26, v = 13, r = 20, k = 10, \lambda = 15$;

26(a). this is a $(4m-1, 2m-1, m-1)$-design, $m = 2^3$;

35(a). no: $k - \lambda$ is not a square;

41. take two copies of each block of a (31, 15, 7)-design.

Section 9.5.

1(d). (P3);

2(a). There are 9 distinct points, no 3 of which lie on the same line;

4. 21;

8(a). no;

9(a). $v = 31, k = 6, \lambda = 1$;

10(a). yes (Corollary 9.27.1);

14(a). yes (but cannot be sure);

16(a). 1;

17(a). 1;

22(b). if we take
$U_3 = \{1,3,5,7\}, V_2 = \{2,3,4,13\}, W_{11} = \{3,6,8,11\}, W_{21} = \{3,9,10,12\}$, then the point 3 is associated with (3, 2) and (3, 2, 1, 1);

22(c). $a_{32}^{(1)} = 1, a_{32}^{(2)} = 1$;

23(a). (2, 3) is associated with (2, 3, 1, 2);

23(b). $W_{12} = \{(1,2),(2,1),(3,3)\}$;

23(e). W_{12} is now $\{(1,2),(2,1),(3,3),w_1\}$, the finite points are all (i,j) with $1 \leq i,j \leq 3$, and the infinite points are u,v,w_1,w_2;

23(f). m^2+m+1 lines, including the line at infinity.

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Answers to Selected Exercises[1]

Chapter 10

Section 10.1. No exercises.

Section 10.2.

1(a). (49, 96);

1(b). (65, 88);

1(c). 44, 84, 42, 56, 42, 32, 33, 48;

1(d). 29, 20, 36, 48, 25, 48, 50, 76, 10, 4, 28, 20, 49, 60, 24, 36, 28, 52;

1(e). 37, 56, 32, 20, 59, 80, 66, 92, 41, 52, 33, 48, 30, 36, 43, 56;

1(f). 5, 8, 51, 72, 60, 80, 18, 20, 31, 36, 57, 76, 39, 60, 66, 104;

2(a). 1, 1, 2, 1;

2(b). 1, 0, 1, 1;

2(c). 0, 1, 1, 0;

2(d). 0, 0, 0, 0;

3(a). 1, 1, 1, 1, 1, 1;

3(b). 1, 0, 1, 1, 0, 1;

3(c). 0, 0, 0, 0, 0, 0;

4(a). 1, 1, 1, 1, 3, 3, 3;

4(b). 1, 0, 0, 0, 0, 1, 1;

4(c). 0, 0, 0, 1, 1, 1, 1;

5(a). (8, 14, 14);

5(b). (46, 91, 96);

[1]More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

5(c). (81, 142, 161);

5(d). (69, 117, 115, 71, 119, 157, 79, 138, 147, 36, 64, 90, 47, 89, 93);

5(e). (43, 81, 91, 52, 84, 100, 36, 59, 72, 58, 97, 134, 18, 31, 42, 39, 73, 89, 43, 78, 73);

5(f). (38, 74, 72, 63, 108, 109, 53, 86, 103, 63, 106, 113, 48, 82, 78, 34, 63, 56, 97, 168, 187);

6(a). (28, 54);

6(b). (106, 170);

6(c). 29, 54, 70, 112, 112, 172, 48, 78;

6(d). 81, 124, 60, 96, 16, 30, 67, 110, 33, 50, 77, 118, 91, 144, 33, 54, 21, 38;

6(e). 50, 82, 93, 142, 96, 154, 103, 166, 73, 116, 48, 78, 57, 90, 74, 118;

6(f). 6, 10, 78, 126, 100, 160, 37, 58, 61, 96, 95, 152, 51, 84, 82, 136;

7(a). (9, 8, 8);

7(b). (65, 46, 4);

7(c). (82, 81, 80);

7(d). (73, 69, 84, 53, 71, 92, 84, 79, 80, 29, 36, 32, 61, 47, 20);

7(e). (52, 43, 20, 40, 52, 80, 28, 36, 52, 40, 58, 76, 14, 18, 20, 43, 39, 20, 55, 43, 32);

7(f). (55, 38, 8, 67, 63, 72, 41, 53, 80, 61, 63, 80, 53, 48, 56, 47, 34, 20, 97, 97, 104);

8(a). AB;

8(b). BC;

8(c). error;

8(d). AC;

8(e). error;

8(f). error;

8(g). error;

9(a). ABC;

9(b). error;

9(c). BAT;

9(d). PIRATE;

9(e). error;

10(a). AA;

10(b). error;

10(c). error;

10(d). AC;

10(e). error;

10(f). error;

10(g). error;

11(a). error;

11(b). error;

11(c). error;

11(d). error;

11(e). error;

12(a). (i) 011;

12(a). (ii) 010;

12(a). (iIi) 110;

12(b). (i) 1001;

12(b). (ii) 0011;

12(b). (iii) 1101;

12(c). (i) 10000;

12(c). (ii) 11001;

12(c). (iii) 01110;

13. 101, 000, 011, 011;

14. a double repetition code;

15. using notation from Example 10.3, let **M**= $(I_k \quad I_k \quad \cdots \quad I_k)$ where there are p copies of I_k which is the $k \times k$ identity matrix;

16. using notation from Example 10.4, let

$$\mathbf{M}_{i,j} = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{else} \end{cases} .$$

***Section 10.3*.**

1(a). 6;

1(b). 5;

1(c). 2;

1(d). 3;

2(a). $111111\underline{0}$;

2(b). $1001011\underline{0}$;

2(c). $001001001\underline{1}$;

2(d). $01010110111\underline{1}$;

3(a). detect 2, correct 1;

3(b). detect 3, correct 1;

3(c). detect 3, correct 1;

4(a). median: 00000000, mean: 00000000;

4(c). median: 000000000, mean: 000000000;

5(a). 0;

5(b). 1;

5(c). 1;

5(d). 1;

6. For example, let $C = \{001111, 110110\}$ and x_i's: $001000, 110110$;

7(a). (i) 4, **(ii)** 4, **(iii)** 4;

7(b). $d - 1$;

7(c). $\lceil (d/2) - 1 \rceil$;

8. $\frac{2^{10}}{\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}} = 2^{10}/176$;

9. $\frac{2^{11}}{\binom{11}{0}+\binom{11}{1}+\binom{11}{2}} = 2^{11}/67$;

10. $\frac{2^{7}}{\binom{7}{0}+\binom{7}{1}+\binom{7}{2}} = 2^{7}/29$;

11(a). $\binom{10}{0}(.1)^0(.9)^{10}$;

11(b). $\binom{10}{1}(.1)^1(.9)^{9}$;

11(c). $\binom{10}{2}(.1)^2(.9)^8$;

11(d). $1 - \left(\binom{10}{0}(.1)^0(.9)^{10}\right) - \left(\binom{10}{1}(.1)^1(.9)^9\right) - \left(\binom{10}{2}(.1)^2(.9)^8\right)$;

12(a). $\binom{6}{0}(.001)^0(.999)^6$;

12(b). $\binom{6}{1}(.001)^1(.999)^5$;

12(c). $\binom{6}{2}(.001)^2(.999)^4$;

12(d). $1 - \left(\binom{6}{0}(.001)^0(.999)^6\right) - \left(\binom{6}{1}(.001)^1(.999)^5\right) - \left(\binom{6}{2}(.001)^2(.999)^4\right)$;

13(a). $\binom{n}{t}(p)^t(1-p)^{n-t}$;

13(b). $\left[\binom{5}{5}(p)^5(1-p)^0\right] \cdot \left[\frac{1}{2^5}\right]$;

13(c). $\left[\binom{4}{3}(p)^3(1-p)^1\right] \cdot \left[\frac{1}{\binom{4}{3}3^3}\right]$;

14. probability of 0 errors is .729, of 0 or 1 errors is .972, so $d = 3$;

19(b). 1 error;

22(b). 2 errors (use horizontal completeness).

Section 10.4.

1(a). 1101;

1(b). 1011;

1(c). 0110;

1(d). 0000;

2(a). 111111;

2(b). 101101;

2(c). 000000;

3(a). 1111111;

3(b). 1000011;

3(c). 0001111;

4(b). the $3 \to 6$ double repetition code;

5. $\mathrm{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$;

6(a). 2;

7(a). $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$;

8(a). $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$;

9(a). $x_3 = x_1 + x_2, x_4 = x_1$;

11(a). no error;

11(b). yes – an error;

12(c). yes – an error;

14(b). 111000;

15(c). not possible;

17(a). 1001100;

21. 111 if $p = 2$;

27. use $|C| = 2^{2^p-1-p}$, $t = 1$;

28(a). (i) 1401;

28(b). (ii) 102102;

29(b). $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} - \mathbf{y})$.

Section 10.5.

2(e). $2(r - \lambda) - 1$;

4(a). no;

5. yes;

9(b). 7;

16(a). $2m$;

16(b). $4m - 1$ if $i = j, 2m - 1$ otherwise;

22(b). no;

26. $\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$;

29(a). the distance between codewords of equal weight is even.

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Answers to Selected Exercises[1]

Chapter 11

***Section 11.1*.**

1(a). One example: assign labels $a = 1, e = 2, j = 3, k = 4, f = 5, g = 6, h = 7, l = 8, m = 9, i = 10, d = 11, c = 12, b = 13$ and mark the edges $\{a, e\}, \{e, j\}, \{j, k\}, \{k, f\}, \{f, g\}, \{g, h\}, \{h, l\}, \{l, m\}, \{m, i\}, \{i, d\}, \{d, c\}, \{c, b\}$;

2. not connected;

3(a). connected;

4(a). one example: use the marked edges in answer to 1(a);

6. it is a spanning forest;

8(a). no;

10. yes.

***Section 11.2*.**

1(a). none;

1(f). $\{c, e\}$;

2(a). orientation based on answer to Exercise 1(a), Section 11.1 orients 1 to 2 to 3 to ... to 13 and all other edges from higher number to lower number;

7. $V = \{x, a, b, c\}, E = \{\{x, a\}, \{x.b\}, \{x, c\}\}$;

8(a). none;

8(f). c, e;

16. for digraph (a) and measure (1): essentially the only orientations are: (i) which uses arcs $(a, b), (b, c), (c, f), (f, e), (e, d), (d, a)$, and (b, e), or (ii) which uses arcs $(a, b), (b, e), (e, d), (d, a), (e, f), (f, c)$, and (c, b); both are equally efficient; for digraph (a) and measure (3): orientation (ii) above is best; for digraph (a) and

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

measure (4): orientation (ii) above is best; for digraph (a) and measure (6): orientation (ii) above is best;

25. D_1: category 3, D_4: category 2;

26. if $V = \{a, b, c, d\}$ and $A = \{(a, b), (b, c), (c, d), (d, a)\}$, any arc is (3, 2);

27. in previous example, any vertex is (3, 2);

28(a). 1.

Section 11.3.

2. G_1 : none; $G_2 : a, c, e, b, f, e, i, j, k, l, i, k, h, g, f, d, a$; G_3 : none; $G_4 : f, e, c, d, e, b, a, b, a, d, g, f$;

3. G_1;

4. D_1: $a, b, d, c, d, f, e, c, a$; D_2: none; D_3: none; D_4: none; D_5: none;

5. $D_2 : c, a, b, d, f, e, c, d$; $D_5 : b, a, e, f, g, b, c, d, e$;

10(a). yes;

10(b). yes;

10(c). no;

10(d). yes;

12(a). $D_1 : 2$;

12(b). $D_2 : 2$; $D_5 : 2$;

13. 12;

16. no: consider D_2 of Figure 11.25.

Section 11.4.

3(a). add edge $\{c, f\}$;

3(b). add edges $\{e, g\}$ and $\{g, h\}$;

3(c). add edges $\{a, d\}$ and $\{b, c\}$;

3(d). add edges $\{a, d\}$ and $\{b, c\}$;

6(a). CAAGCUGGUC;

9(a). yes: $A_1A_1A_2A_2A_2A_3A_3A_3A_3A_3A_2A_1$;

17(a). say B is an interior extended base of a U, C (G) fragment; then both B and the preceeding extended base end in G (U, C), so B is on the second list;

18. ends in A;

20(b). if there is a second abnormal fragment, it is B alone.

Section 11.5.

1(a). a, b, d, f, e, c, a;

1(b). $i, a, b, c, d, e, f, g, h, i$;

1(c). $a, b, c, d, h, g, f, j, k, l, p, o, n, m, i, e, a$;

2(a). e, c, d, a, b;

2(b). a, b, c, d, f, e;

2(c). $a, b, c, d, e, j, g, i, f, h$;

3(a). a, b, c, d, e, f, a;

3(b). a, b, c, e, d, a;

3(c). c, a, d, b, e, f, c;

4(a). a, d, b, c;

4(b). a, b, d, c, e, f;

4(c). a, c, d, b, e;

6. no;

7(a). Z_4;

7(b). K_4;

7(c). the graph in Figure 11.50;

7(d). $K_{1,4}$;

8(a). $K_{1,3}$;

8(b). K_4;

9(a). yes;

10(a). for (a) of Figure 11.45: complete graph;

10(b). for (a) of Figure 11.45: yes;

11. $K_{1,3}$;

12. Z_5;

13. for (a) of Figure 11.4: no;

14. for (a) of Figure 11.4: no;

15(a). use Theorem 11.8.

Section 11.6.

1(b). yes;

2. for (b) of Exercise 1, Section 4.1: no;

3. for (a), the labeling is a topological order;

8. SF, B, H, LA, NY or SF, B, LA, NY H, or SF, B, NY H, LA;

10. i beats j iff $i < j$;

11(a). $a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$;

12. 3, 1, 2;

15. if C is a, b, f, d, c, e, a, then H has edges $\{b, d\}$ to $\{e, f\}$ and $\{a, c\}$ to $\{e, f\}$ and is 2-colorable;

20(b). (0, 1, 2, 3, 4);

22(a). no;

22(b). no;

23. consider the tournament on $V(D) = \{1, 2, 3, 4\}$ and $A(D) = \{(1, 2), (2, 4), (3, 1), (3, 2), (4, 1), (4, 3)\}$;

28(a). $(0, 1, 2, \ldots, n - 1)$;

31. start with any vertex x and find the longest simple path heading into x; this must start at a vertex with no incoming arcs;

32(a). $\binom{s(u)}{2}$;

32(c). use part (b) and the fact that $s(u) \geq 2$ for some vertex u in every tournament of four or more vertices.

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Answers to Selected Exercises[1]

Chapter 12

***Section 12.1*.**

1(a-c). $\{a, \alpha\}, \{b, \beta\}, \{c, \gamma\}, \{e, \delta\}$;
3(a). $\{a, b\}, \{c, d\}, \{e, f\}$;
5. find a minimum weight matching;
7(b). put a very small weight on all edges not in the graph.

***Section 12.2*.**

1(a). {Cutting, Shaping, Polishing, Packaging};
1(b). $\{1, 2, 3\}$;
1(c). $\{a, b, c, d\}$;
2(a). {Smith, Jones, Black};
2(b). {White, Cutting, Shaping, Gluing, Packaging};
2(c). $\{a, b, c, d, f, i\}$;
3. (a): no; (b): yes; (c): no; (d): no;
5(a). (a, a, a, a, b, d, a);
6(a). no SDR;
6(b). no SDR;
6(c). (a_1, a_3, a_2, a_4);
6(d). (b_5, b_1, b_3, b_4);
6(e). no SDR;
6(f). no SDR;
8(a). two;
8(b). two;
8(c). 2^5;
8(d). 2^n;
9. no;
14(a). yes;
14(b). three: (c, d, a, b, e), (d, c, a, b, e), (d, e, a, b, c);
16(a). yes: $\begin{pmatrix} 6 \\ 7 \\ 4 \\ 2 \\ 1 \end{pmatrix}$;

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

16(b). no;
18(a). n even;
18(b). n even;
18(c). n odd;
20. yes: show each likes exactly p;
22(a). since an SDR exists, either x is used or it is not. If not and S_i contains x, replace S_i's representative in the SDR with x;
22(b). no, e.g., let $S_1 = \{x\}$, $S_2 = \{a, x\}$, and $S_3 = \{b, x\}$;
23(a). for example, $(2, 3, 4, \ldots, n-1, n, 1)$;
24. no, consider the vertex/vertices of degree one.

Section 12.3.
1. yes;
2(a). yes;
2(b). yes;
2(c). no;
2(d). no;
2(e). yes;
2(f). no;
2(g). yes;
2(h). yes.

Section 12.4.
1(a). $\{1, 2, 3, 4, 5\}$;
1(b). $\{1, 2, 4, 5\}$;
2(b). {Smith, Jones, Brown, Black, White};
6(a). minimum $\{2, 6\}, \{3, 5\}, \{1, 4\}$;
6(b). minimum $\{1, 2\}, \{3, 4\}, \{5, 6\}$;
6(c). minimum $\{1, 3\}, \{2, 4\}, \{5, 8\}, \{6, 7\}$;
6(c). minimum $\{a, \alpha\}, \{b, \beta\}, \{b, \delta\}, \{c, \gamma\}$;
7(a). no;
7(b). $\left\lceil \frac{n}{2} \right\rceil$;
9. minimum edge covering;
13. If I is independent, $V - I$ is a vertex cover and if K is a vertex cover, $V - K$ is independent;
15. $|M^*| \le |K^*|$ implies $|I| \le |I^*| \le |F^*| \le |F|$.

Section 12.5.
1(a). $8, \{8, 9\}, 9, \{9, 6\}, 6$;
1(b). $8, \{8, 9\}, 9, \{9, 6\}, 6, \{6, 3\}, 3$;
1(c). add edges $\{8, 9\}$ and $\{6, 3\}$, delete edge $\{9, 6\}$;
6(a). pick vertex 8, place edge $\{5, 8\}$ in T, note 8 is unsaturated, and note $5, \{5, 8\}, 8$ is an M-augmenting chain;
9(a). use edges $\{6, 7\}, \{7, 11\}, \{11, 9\}$.

Section 12.6.

1. (a): $\delta(G) = 1, m(G) = 3$; (b): $\delta(G) = 0, m(G) = 3$; (c): $\delta(G) = 1, m(G) = 4$; (d): $\delta(G) = 1, m(G) = 4$;
2(b). $2p \leq 3|N(S)|$;
2(c). $|S| - |N(S)| \leq p - \frac{2}{3}p$;
2(d). $m(G) = |X| - \delta(G) \geq 9 - \frac{1}{3}(9) = 6$.

Section 12.7.

2(a). worker 1 to job 2, 2 to 3, 3 to 1, and 4 to 4;
2(b). worker 1 to job 2, 2 to 3, 3 to 1, and 4 to 4;
2(c). worker 1 to job 2, 2 to 5, 3 to 4, 4 to 1, and 5 to 3;
3. worker 1 to job 4, 2 to 1, 3 to 5, 4 to 2, and 5 to 3;
4. machine 1 to location 2, 2 to 1, 3 to 5, 4 to 4, and 5 to 3;
5(a). speaker 1 with speaker 3, 2 with 5, 3 with 1, 4 with 6, 5 with 2, and 6 with 4;
6. no, there are three solutions: $B_1 - H_1$, $B_4 - H_2$, $B_3 - H_3$ or $B_1 - H_2$, $B_2 - H_1$, $B_3 - H_3$ or $B_2 - H_1$, $B_4 - H_2$, $B_3 - H_3$.

Section 12.8.

1. $(n!)^{2n}$ since there are $n!$ choices for the preference list for each of the n men and n women;
2. $n!$: there are n choices for the first man, then $n-1$ choices for the second man, $\ldots$, and finally, 1 choice for the n^{th} man;
4. m_3 has w_4 higher on his preference list and w_4 has m_3 higher on her preference list;
5(a). $m_1 - w_1$, $m_2 - w_2$, $m_3 - w_3$, $m_4 - w_4$;
5(b). $m_1 - w_4$, $m_2 - w_3$, $m_3 - w_2$, $m_4 - w_1$;
11(a). there are 3 possible matchings each with a blocking pair: $p_1 - p_2, p_3 - p_4$ with blocking pair p_2, p_3; and $p_1 - p_3, p_2 - p_4$ with blocking pair p_1, p_2; and $p_1 - p_4, p_2 - p_3$ with blocking pair p_1, p_3;
11(b). no.

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Answers to Selected Exercises[1]

Chapter 13

Section 13.1.

1(a). Add edges in the order $\{b, e\}, \{d, e\}, \{a, e\}, \{c, d\}$;
1(b). Add edges in the order $\{a, b\}, \{b, c\}, \{d, e\}, \{e, f\}, \{g, h\}, \{h, i\}, \{a, d\}, \{d, g\}$;
1(c). Add edges in the order $\{c, g\}, \{a, b\}, \{a, d\}, \{f, g\}, \{d, f\}, \{e, f\}$ where ties were broken arbitrarily;
2(a). add edges $\{a, e\}, \{b, e\}, \{d, e\}, \{c, d\}$;
2(b). Add edges in the order $\{a, b\}, \{b, c\}, \{a, d\}, \{d, e\}, \{e, f\}, \{d, g\}, \{g, h\}, \{h, i\}$;
2(c). Add edges in the order $\{a, b\}, \{a, d\}, \{d, f\}, \{f, g\}, \{c, g\}, \{e, f\}$;
3(a). terminate with message disconnected; T ends up with 5 edges;
3(b). terminate with message disconnected; T ends up with 7 edges;
3(c). terminate with message disconnected; T ends up with 6 edges;
5. vat pairs: $\{7, 8\}, \{6, 7\}, \{1, 5\}, \{4, 5\}, \{5, 8\}, \{2, 3\}, \{3, 8\}$;
6. component pairs: $\{1, 4\}, \{2, 6\}, \{3, 4\}, \{1, 6\}, \{5, 6\}$;
9. (a): edges $\{b, c\}, \{c, e\}, \{c, d\}, \{a, b\}$; (b): edges $\{f, i\}, \{e, h\}, \{d, g\}, \{c, f\}, \{b, e\}, \{a, d\}, \{g, h\}, \{h, i\}$; (c): edges $\{d, e\}, \{c, d\}, \{a, e\}, \{b, c\}, \{e, f\}, \{d, g\}$;
11. yes;
12(a). edges $\{a, e\}, \{a, b\}, \{c, d\}, \{d, e\}$;
12(b). edges $\{c, e\}, \{a, b\}, \{c, d\}, \{b, d\}$;
13. in Step 1 of Kruskal's Algorithm, set $T = \{$ the set of edges specified as having to belong to the spanning tree $\}$;
16(a). for network (c): G' has edges $\{a, b\}, \{c, g\}, \{a, d\}, \{f, g\}, \{d, f\}$; in the next iteration we add $\{e, f\}$ and obtain a minimum spanning tree.

Section 13.2.

3(a). We successively add to $W : a, b, d, c, e, z$, obtaining the path a, d, e, z;
3(b). We successively add to $W : a, c, b, f, d(\text{or } e), e(\text{or } d), z$, obtaining the path a, b, e, z;
3(c). We successively add to $W : a, b, c, e, d, z$, obtaining the path a, c, e, z;
3(d). We successively add to $W : a, b, e, h, c, f, i, j, g, d, z$, obtaining the path a, h, i, j, z;
4. the path a, b, d, g, i;

[1] More solutions to come. Comments/Corrections would be appreciated and should be sent to: Barry Tesman (tesman@dickinson.edu) or Fred Roberts (froberts@dimacs.rutgers.edu).

5. grind, weigh, polish, inspect;
6. buy in year 1, sell in year 3, buy in year 3 and then sell in year 6;
10. $(n-2)!$;
12(b). $1, 4, 6, 7$;
12(c). first line: words $1, 2, 3$; second line: words $4, 5$; third (last) line: word 6;
14. it will take the boatman seven trips across the river. First the boatman takes the goat across the river. He then goes back and takes the wolf across. He drops the wolf off, but at the same time puts the goat back into the boat. He takes the goat back across the river. The boatman drops the goat off, but at the same time puts the cabbage in the boat. He takes the cabbage across and then goes back and gets the goat;
15. 7: draw a digraph D where
$V(D) = \{x = (x_1, x_2, x_3) : x_i =$ number of gallons in jug $i\}$, and
$A(D) = \{(x, y) : y$ is attainable from x by pouring one jug into another$\}$ and find a shortest path from (8, 0, 0) to (4, 4, 0);
19. no;
21. let $\bar{d}(i, j)$ be the distance from i to j and let $\mathbf{A}$ = adjacency matrix; then $\bar{d}(i, j)$ is the smallest k such that the i, j entry of $\mathbf{A}^k$ is nonzero.

Section 13.3.
1. for network (c): **(a)** feasible, **(b)** value 4;
2. (a): 8;
3. (a): $s_{sa} = 3, s_{sb} = 0, s_{ba} = 1, s_{at} = 0, s_{bt} = 2$;
4(a). yes;
5(b). no;
6. (a): augmenting chain s, a, b, t;
17(b). for graph (a), let the flow be 1 on arcs $(s, a), (s, b), (s, d), (a\alpha), (b, \beta), (d, \gamma), (\alpha, t), (\beta, t), (\gamma, t)$, and 0 otherwise; the matching is $\{a, \alpha\}, \{b, \beta\}, \{d, \gamma\}$;
17(c). for graph (a), the (s, t) cut $S = \{s, c, d, \alpha, \beta, \gamma\}, T = \{t, a, b\}$ has corresponding covering $\{a, b, \alpha, \beta, \gamma\}$;
20. (a): $x_{sa} = 3, x_{sb} = 2, x_{at} = 1, x_{ab} = 2, x_{bt} = 4, x_{sc} = 3, x_{ct} = 3, \text{rest} = 0$;
22. (a): $F(0011) = 1, F(0010) = 0$, and so on;
23. (a): $x_{ad} = 11, x_{bd} = 1, x_{df} = 12, x_{ce} = 5, x_{eh} = 5, \text{rest} = 0$;
36. argue by induction on the length of C that if C is flow-augmenting, it contains a simple flow-augmenting chain.

Section 13.4.
1. (a):
$x_{sa} = x_{at} = 2, x_{sb} = x_{bt} = 1, x_{ba} = 0, \text{or } x_{sa} = x_{at} = 1, x_{sb} = x_{bt} = 2, x_{ba} = 0$;
10(a). (13.13) and (13.14) give us
$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \leq \sum_{i=1}^{n} a_i < \sum_{j=1}^{m} b_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij}$;
11(a). In (a), such a flow has $x_{sa} = 1, x_{at} = 2, x_{sb} = 2, x_{ba} = 1, x_{bt} = 1$ and negative cost augmenting circuit is t, a, b, t.