15.094J: Robust Modeling, Optimization and Computation

Lecture 13: Power of Robust Policies in Adaptive Optimization

Motivation

- RO is tractable
- But how much do we lose in performance?
- Is it worse for multistage optimization?

Stochastic Model

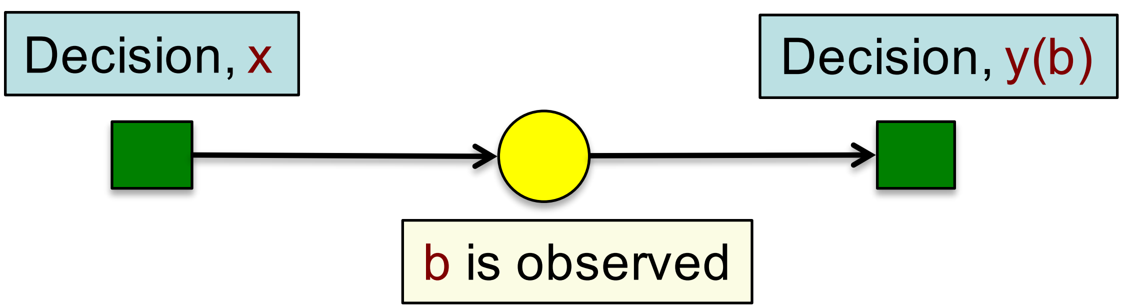
- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\textbf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] & \\
Ax + By(b) & \geq b, \forall b \in \mathcal{U} \\
x & \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) & \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

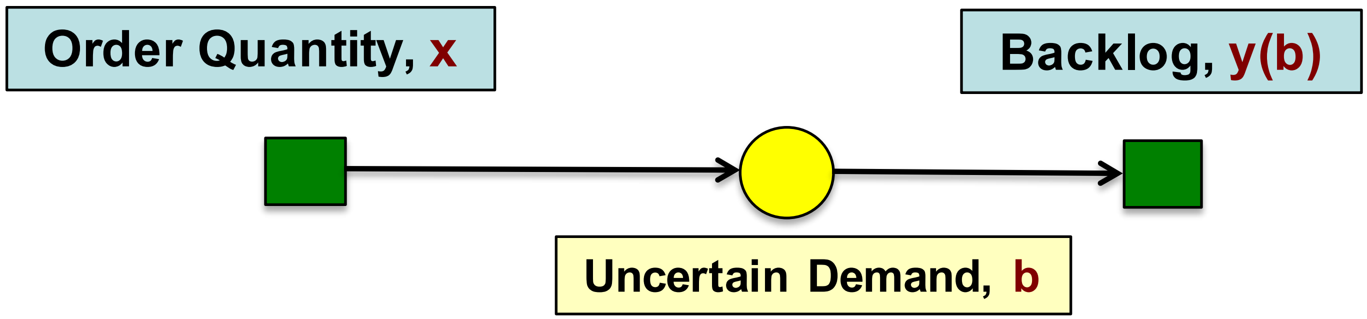
(Minimize **expected cost**)

(Uncertainty Set)

(Uncertain Right Hand Side)



Inventory Management



Minimize $\mathbb{E}_b\left[h(x-b)^+ + p(b-x)^+\right]$

$$
\begin{aligned}
\min\ \mathbb{E}_b\Big[\overbrace{h\big(x+y(b)-b\big)}^{\text{Holding Cost}} \;+\; \overbrace{p\cdot y(b)}^{\text{Backorder Penalty}}\Big] & \\
x+y(b) &\geq b,\ \forall\, b \\
x, y(b) &\geq 0
\end{aligned}
$$

Stochastic Model

- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\mathbf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] & & \\
Ax + By(b) &\geq& b,\ \forall b \in \mathcal{U} \\
x &\in& \mathbb{R}_+^n \times \mathbb{Z}_+^p \\
y(b) &\in& \mathbb{R}_+^n \times \mathbb{Z}_+^p
\end{aligned}
$$

- **Computationally intractable in general**
- **Two-stage problem is #P-hard [Dyer and Stougie (2001)]**
- **Multi-stage problem is PSPACE-hard [Dyer and Stougie (2001)]**

Adaptive Optimization Model

- **Two-stage Adaptive Optimization Model**

$$\begin{aligned} \textbf{zAdapt} = \min c^T x + \boxed{\max_{b \in \mathcal{U}} \; d^T y(b)} \\ Ax + By(b) &\geq b, \; \forall b \in \mathcal{U} \\ x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\ y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \end{aligned}$$

(Minimize worst-case cost)

(Uncertainty Set)

(Uncertain Right Hand Side)

- **Still computationally intractable in general**
- **Even approximating LO within an factor of O(log m) is NP-hard [Feige et al.'07]**

Robust Optimization Model

$$
\begin{aligned}
\textbf{zRob} = \min c^T x + d^T y & \\
Ax + By &\geq b, \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
\boxed{y} &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize Cost of a **static solution**)

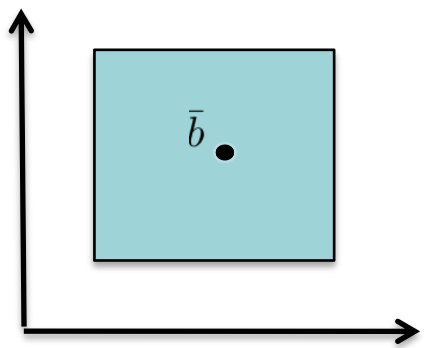
(Uncertainty Set)

(Uncertain Right Hand Side)

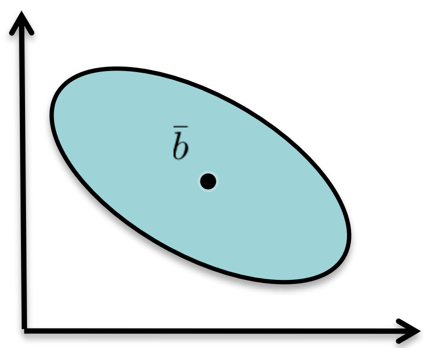
Solution **y** does **not depend on b**

- **Computationally tractable**
- **But does it give a highly conservative solution?**

Uncertainty Sets



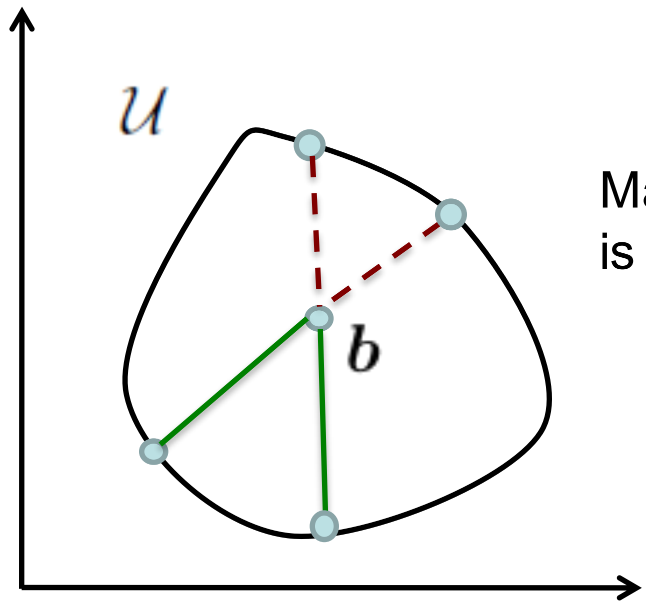
Hypercube : $||b - \bar{b}||_\infty \leq \beta$



Ellipsoid : $||\Sigma(b - \bar{b})||_2 \leq \beta$

Norm-Ball : $||\Sigma(b - \bar{b})||_p \leq \beta$

Symmetry of $\mathcal{U}$



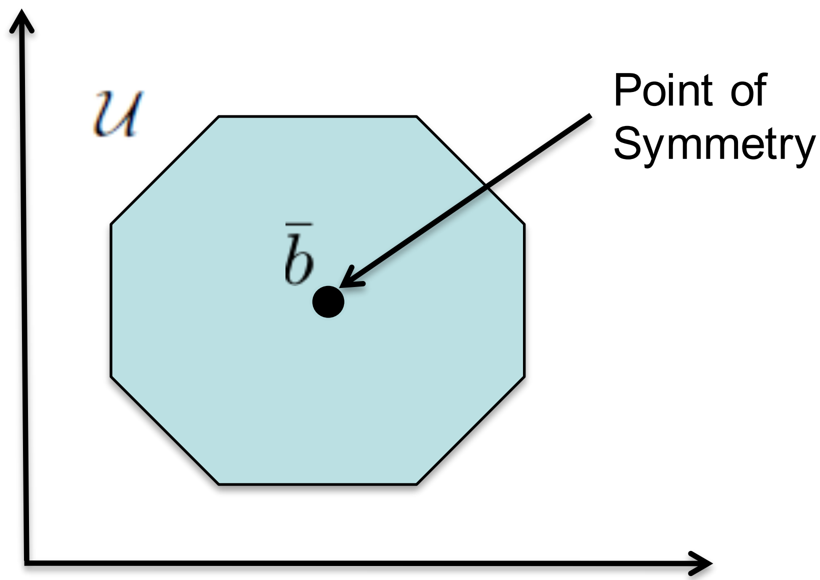
Maximum symmetry point
is **the point of symmetry of** $\mathcal{U}$

$\mathsf{sym}(b, \mathcal{U})$: minimum ratio of red and green segments

$$\mathsf{sym}(b, \mathcal{U}) = \max\{\alpha \mid b + \alpha \cdot (b - b') \in \mathcal{U}, \ \forall b' \in \mathcal{U}\}$$

$$\mathsf{sym}(\mathcal{U}) = \max_{b \in \mathcal{U}} \ \mathsf{sym}(b, \mathcal{U})$$

Example (s=1)

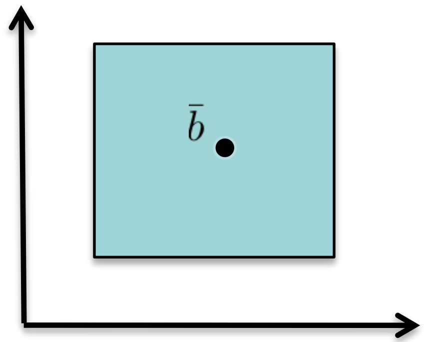


$$(\bar{b} - \delta) \in \mathcal{U} \Leftrightarrow (\bar{b} + \delta) \in \mathcal{U}, \ \forall \delta$$

$$\textbf{sym}(\mathcal{U}) = 1$$

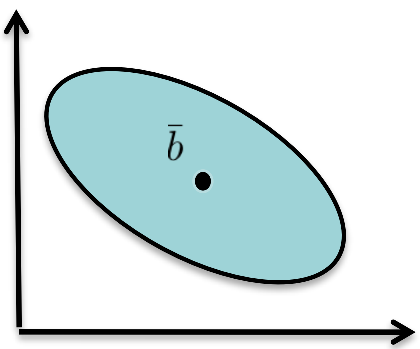
More Examples (s=1)

$\mathsf{sym}(\mathcal{U}) = 1$



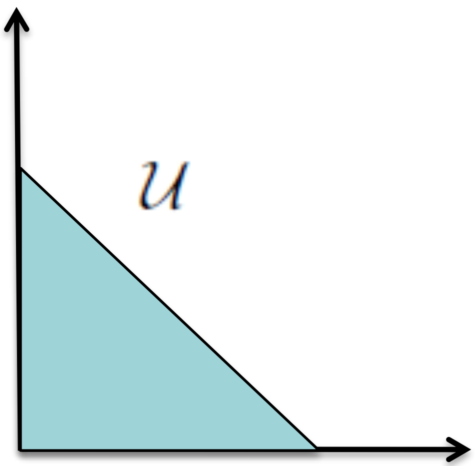
$\boxed{\textbf{Hypercube} : ||b - \bar{b}||_\infty \leq \beta}$

$\mathsf{sym}(\mathcal{U}) = 1$

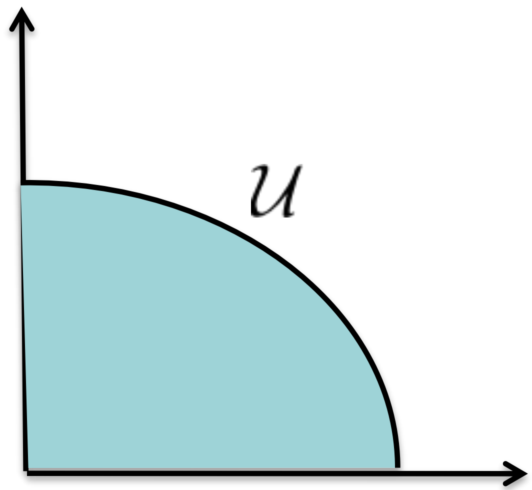


$\boxed{\textbf{Ellipsoid} : ||\Sigma(b - \bar{b})||_2 \leq \beta}$

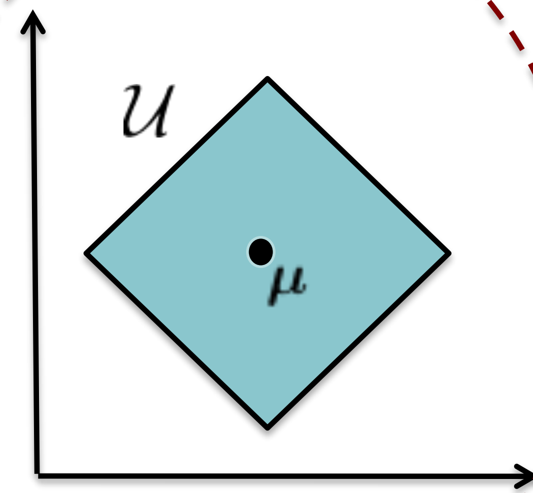
Examples (s ≤ 1)



$$\mathbf{sym}(\mathcal{U}) = \frac{1}{n}$$



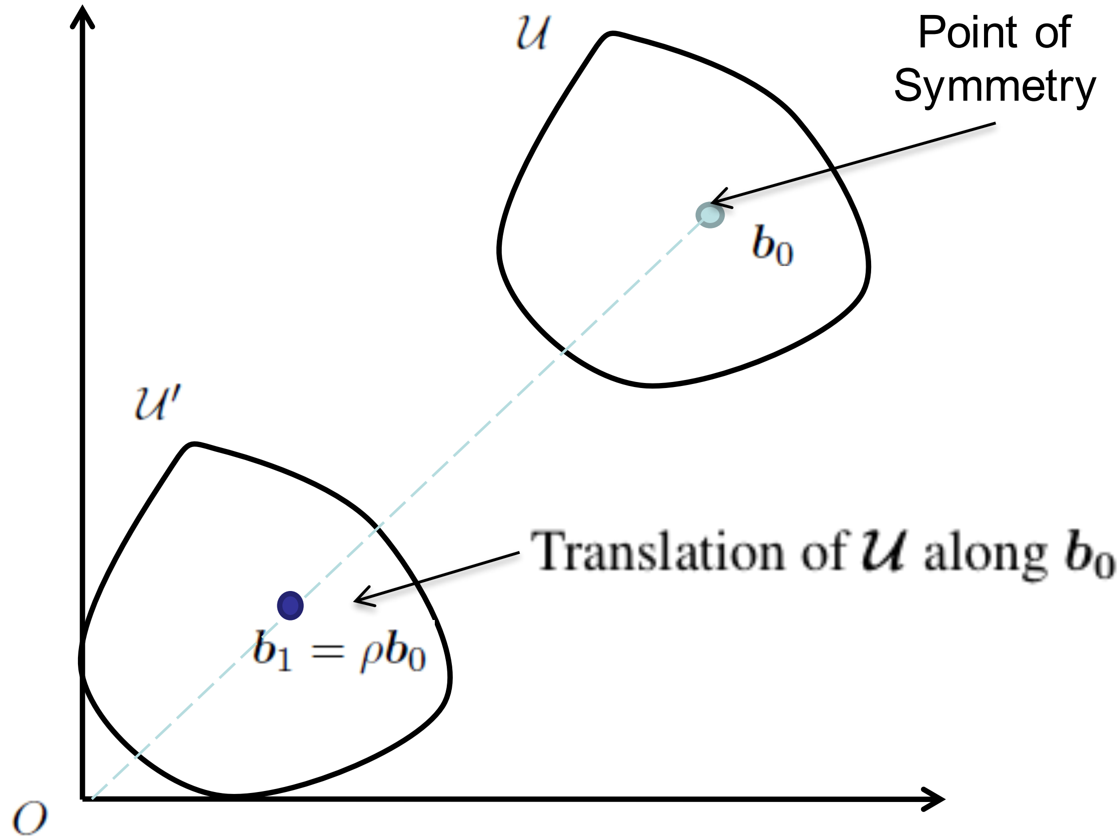
$$\mathbf{sym}(\mathcal{U}) = \frac{1}{\sqrt{n}}$$



$$\mathbf{sym}(\mathcal{U}) = 1$$

$$\boxed{\mathcal{U} = \left\{ b \in \mathbb{R}^n_+ : \left| \frac{\sum_{i \in S} b_i - |S|\mu}{\sqrt{|S|}} \right| \leq 2, \ \forall S \subseteq N := \{1, \ldots, n\} \right\}}$$

Translation Factor of $\mathcal{U}$



Translation factor of $\mathcal{U}$, $\rho(\mathcal{U}) = \frac{||\boldsymbol{b_1}||}{||\boldsymbol{b_0}||}$

Results: Robust Solutions

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_b d^T y(b) & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Theorem 1 *Let* $\rho = \rho(\mathcal{U})$ *and* $s = \mathsf{sym}(\mathcal{U})$. *Then,*

$$\frac{z_{\mathsf{Rob}}}{z_{\mathsf{Stoch}}} \leq \left(1 + \frac{\rho}{s}\right)$$

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Our Results: Implications

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) & \geq b, \ \forall b \in \mathcal{U} \\ x, y(b) & \geq 0 \end{aligned}$$

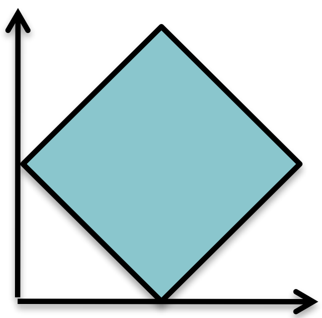
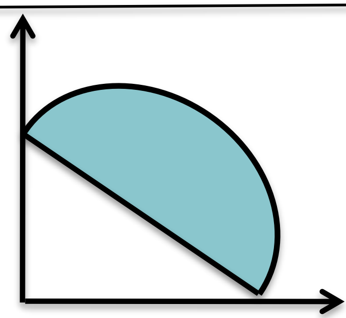
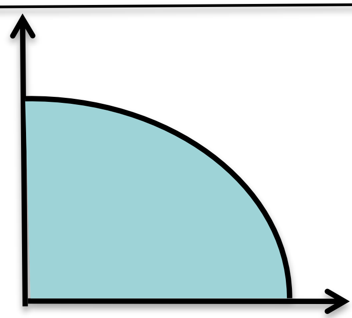
Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_b d^T y(b) & \\ Ax + By(b) & \geq b, \ \forall b \in \mathcal{U} \\ x, y(b) & \geq 0 \end{aligned}$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$(1+\rho) \leq 2$	$(1+\rho) \leq 2$
General ($1/n < s \leq 1$)	$(1+\rho/s)$	$(1+\rho/s)$

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Bounds for different Sets

$\mathcal{U}(\rho = 1)$	**sym($\mathcal{U}$)**	**Stochasticity Gap**
	$\mathbf{1}$	$\mathbf{2}$
	$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$	$(\mathbf{1} + \sqrt{\mathbf{2}})$
	$\frac{\mathbf{1}}{\sqrt{n}}$	$(\mathbf{1} + \sqrt{n})$

Integer Variables

Stochastic (zStoch)

$$
\begin{aligned}
\min\ & c^T x + \mathbb{E}[d^T y(b)] \\
Ax + By(b) &\geq b,\ \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+
\end{aligned}
$$

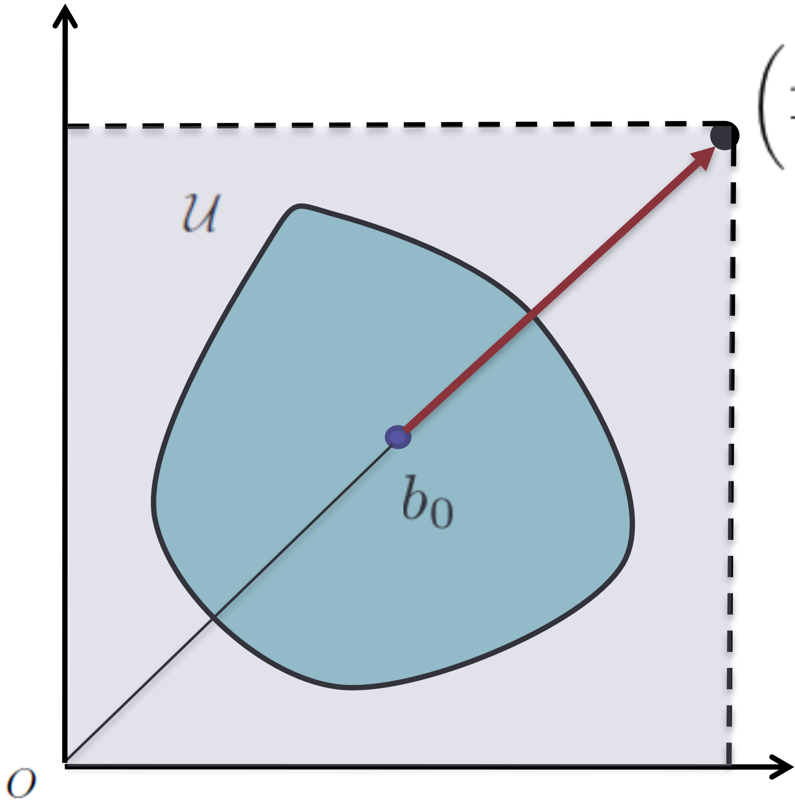
Adaptive (zAdapt)

$$
\begin{aligned}
\min\ & c^T x + \max_b d^T y(b) \\
Ax + By(b) &\geq b,\ \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$\lceil(1+\rho)\rceil = 2$	$\lceil(1+\rho)\rceil = 2$
General ($1/n < s \leq 1$)	$\lceil(1+\rho/s)\rceil$	$\lceil(1+\rho/s)\rceil$

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Proof



$$\left(1+\frac{\rho}{s}\right) b_0 = \tau \cdot b_0$$

$$\boxed{b \le \tau \cdot b_0, \ \forall b \in \mathcal{U}}$$

Optimal Stochastic Solution: $x^*, y^*(b), \ \forall b \in \mathcal{U}$

Feasible Static Solution: $(\tau x^*, \tau y^*(b_0))$

$$A(\tau x^*) + B(\tau y^*(b_0)) \ge \tau b_0 \ge b, \forall b \in \mathcal{U}$$

Cost Analysis

$$z_{\text{Rob}} \leq \tau(c^T x^* + d^T y^*(b_0))$$

$$z_{\text{Stoch}} = c^T x^* + \mathbb{E}_b[d^T y^*(b)]$$

$$\begin{aligned} Ax^* + By^*(b) &\geq b \\ \mathbb{E}_b\big[Ax^* + By^*(b)\big] &\geq \mathbb{E}_b[b] \\ Ax^* + B\mathbb{E}_b[y^*(b)] &\geq b_0 \end{aligned}$$

$\mathbb{E}_b[y^*(b)]$ is a feasible solution for $b_0 \Rightarrow d^T y^*(b_0) \leq d^T \mathbb{E}_b[y^*(b)]$

$$z_{\text{Rob}} \leq \tau \cdot z_{\text{Stoch}} = \left(1 + \frac{\rho}{s}\right) z_{\text{Stoch}}$$

Our Results: Cost, RHS uncertainty

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}_{(b,d)}[d^T y(b,d)] \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_{(b,d)} \ d^T y(b,d) \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)		
General (1/n ≤ s ≤ 1)		

Assume: $E_{b,d}[(b,d)] = (\bar{b},\bar{d})$ where $(\bar{b},\bar{d})$ is the **point of symmetry**

Our Results: Cost, RHS uncertainty

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}_{(b,d)}[d^T y(b,d)] & \\ Ax + By(b,d) & \geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) & \in \mathbb{R}^n_+ \end{aligned}$$

Adaptive (zAdapt)

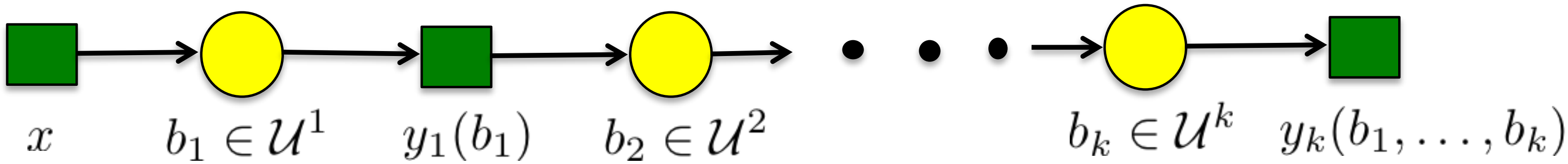
$$\begin{aligned} \min c^T x + \max_{(b,d)} \ d^T y(b,d) & \\ Ax + By(b,d) & \geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) & \in \mathbb{R}^n_+ \end{aligned}$$

Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$\Omega(m)$	$(1+\rho)^2 \leq 4$
General ($1/n \leq s \leq 1$)	$\Omega(m)$	$(1+\rho/s)^2$

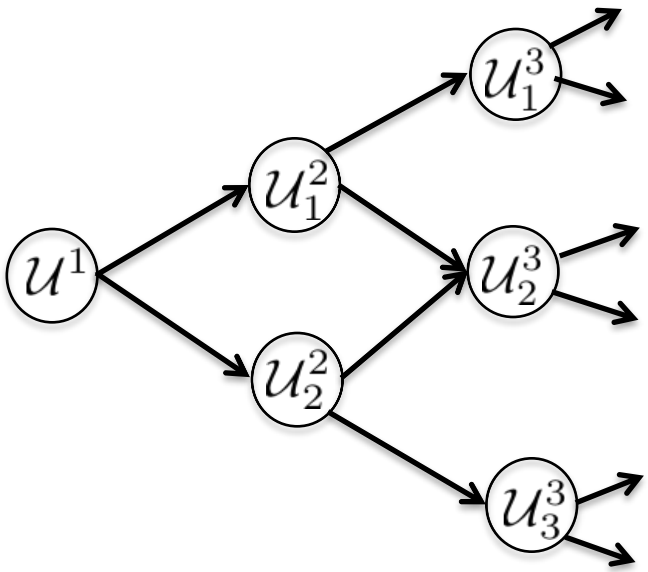
Assume: $E_{b,d}[(b,d)]$ = $(\bar{b},\bar{d})$ where $(\bar{b},\bar{d})$ is the **point of symmetry**

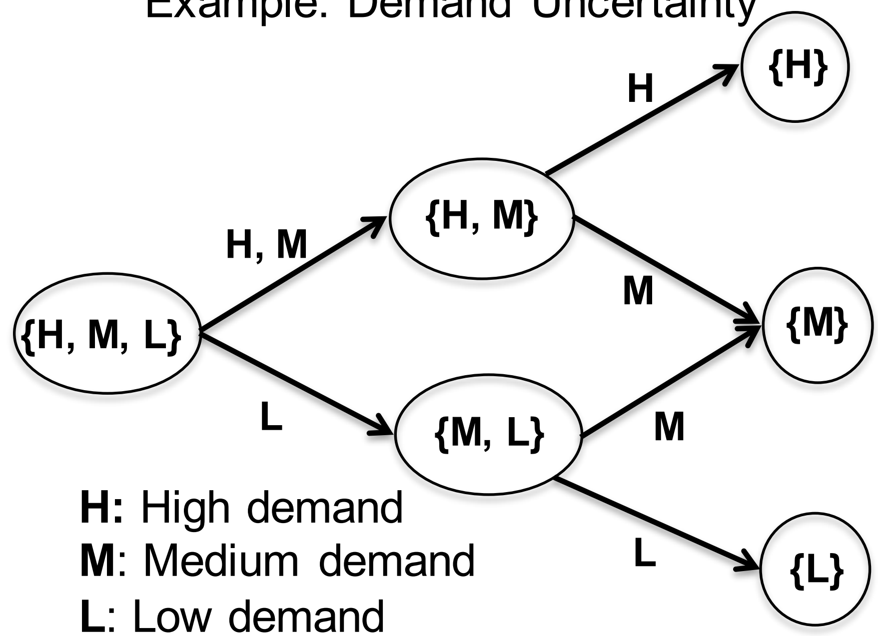
Multi-Stage Problems

Multi-Stage Stochastic Model



Example: Demand Uncertainty





Static solution is not a good approximation

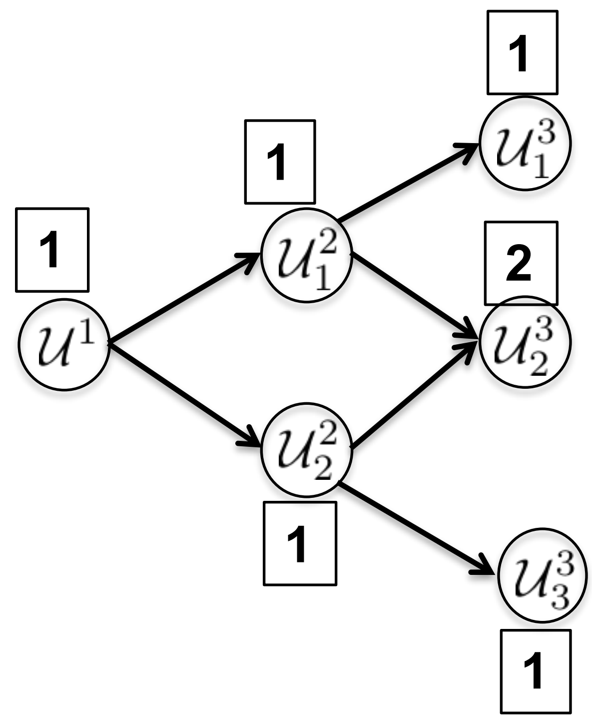
Fully-Adaptive Solution

- Requires optimal decision for each possible scenario
- Uncountable set of scenarios (typically)
- Suffers from the curse of dimensionality
- PSPACE hard to compute in general

Finitely Adaptive Solution

- Partition the scenarios into a small number of sets
- Compute a static solution for each set of scenarios in the partition
- Finite (small) number of solutions in each stage

- partition the scenarios according to the realized paths in the uncertainty network
- Number of paths is finite (small)
- In each stage k, compute a solution for each path from stage 1 to stage k
- For any path P, the solution is feasible for all possible parameter realizations on P



Performance of Finitely Adaptive Solution

Theorem 2 *Let* $\rho = \max \ \rho(\mathcal{U})$ *and* $s = \min \ \mathsf{sym}(\mathcal{U})$ *over all* $\mathcal{U}$. *Also, for all* $\mathcal{U}$, *let,*

$$\mathbb{E}[b] \geq \mathbf{b_0},$$

where $\mathbf{b_0}$ *is the point of symmetry of* $\mathcal{U}$. *Then,*

$$\textit{Cost of an optimal finitely adaptable solution} \leq \left(1 + \frac{\rho}{s}\right) z_{\mathsf{Stoch}}\ .$$

- **Finitely Adaptive solution is a good approximation of the multi-stage stochastic problem**
- **Performance bound = 2 for uncertainty sets with symmetry = 1**

Bounds for Multi-stage Problems

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$(1+\rho) \leq 2$	$(1+\rho) \leq 2$
General ($1/n < s \leq 1$)	$(1+\rho/s)$	$(1+\rho/s)$

Finitely Adaptive Solution: Multi-stage Adaptive Optimization Problem

Theorem 3 *Consider the adaptive problem with both rhs and cost uncertainty, i.e., both* $\mathbf{b}, \mathbf{d}$ *are uncertain. Let* $\rho = \max\ \rho(\mathcal{U})$ *and* $s = \min\ \mathsf{sym}(\mathcal{U})$ *over all* $\mathcal{U}$*. Then,*

$$\textit{Cost of an optimal finitely adaptable solution} \leq \left(1 + \frac{\rho}{s}\right)^2 z_{\mathsf{Adapt}}\ .$$

- **Finitely Adaptive** solution is a **good** approximation of the multi-stage adaptive problem with both rhs and cost uncertainty
- **Performance bound ≤ 4** for uncertainty sets with symmetry = 1
- Finitely adaptive solution is not a good approximation for the corresponding stochastic problem

Conclusions

- Choose uncertainty sets carefully
- Criteria: Tractability and Symmetry
- Finite Adaptability, which humans heuristically use is **near optimal** if the uncertainty set is symmetric, that is reasonable known unknowns