15.094J: Robust Modeling, Optimization, Computation

Lecture 9: Adaptive Optimization

Outline

Central Problems of OR

George Dantzig: “Planning under uncertainty. This, I feel, is the real field that we should be all working in.”

Problem	**Current Theory**	**Proposal**
Modelling under Uncertainty	Probability Theory	RO
Optimization under Uncertainty	DP	RO
Optimization over Time	DP	AO

DP: Dynamic Programming
RO: Robust Optimization
AO: Adaptive Optimization

Optimization over time and under uncertainty

- The current method proposed by Richard Bellman in 1953, and taught in first year courses around the world, consists of two ideas:
- Describe uncertainty using probability distributions.
- To decide what to do today, have a plan for every eventuality in the future.

Criticisms of current approach

- Probability distributions do not exist in practice, and stochastic models are by and large computationally intractable.
- How do humans take decisions?
- For example: In some of the most important decisions in life (to whom to marry, which career to follow, etc.) do you enumerate every eventuality?
- Moreover, DP in most cases is computationally intractable in dimensions 3 or higher.

An example of DP

- An Apple store needs to decide the ordering mechanism for iPhones 5.
- You need to decide **today** how many iPhones 5 u_1 to order.
- You also need to decide how many iPhones 5 u_t to order **at time** t.
- There is demand uncertainty d_t, we assume that *the probability distribution of d_t is known*.
- There are cost of ordering c_t, and costs $f(x_t)$ for keeping inventory x_t at time t. For example, $f(x_t) = h \max(x_t, 0) + p \max(-x_t, 0)$.
- TIme horizon is T, and salvage value at time T is s.

Solution method

- State: Inventory x_t, $t = 1, \ldots, T$.
- Decision: Order u_t, $t = 1, \ldots, T$.
- Uncertainty: Demand d_t, $t = 1, \ldots, T$.
- Dynamics: $x_{t+1} = x_t + u_t - d_t$, x_1 known.
- Objective: $\min \sum_{t=1}^{T} (f(x_t) + c_t u_t)$.
- Bellman recursion:

$$J_T(x_T) = s \cdot x_T.$$

$$J_t(x_t) = \min_{u_t} E_{d_t}[c_t u_t + f(x_t) + J_{t+1}(x_t + u_t - d_t)], \quad t = T-1, \ldots, 1.$$

- Key observation: In deciding what to do now u_1, we need to decide $u_t(x_t)$ for every inventory level in the future.
- For the Galleria store, $x_t \in \{0, \ldots, 10,000\}$ and $T = 360$ (one year). So we need to calculate 3.6 million decisions to decide how many iPhones 5 to order today, $u_1(x_1)$.

Reflections

- Imagine also certain known unknowns: a) Samsung wins the appeal for patent infringement, b) the world enters a deeper recession.
- Imagine also certain unknown unknowns: A new Steve Jobs has been working in secrecy on a brand new technology on a voice recognition system, much superior to Siri, Google launches a brand new product that makes iPhones irrelevant, etc.
- Should we enumerate 3.6 million decisions?
- And what happens when instead of only the Galleria store we need to coordinate all Apple stores in New England that are served by the same distribution center?
- Then we need to enumerate of the order of 3.6^{10} million states in order to decide what to order from all the stores.

Adaptive Optimization

- Two time periods
- Data: $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{A}$, uncertainty set $\mathcal{U}$.
- Timing: *Here and now decisions* x
- Then uncertainty $\boldsymbol{B}$ is observed.
- Finally: *Wait and see decisions* $\boldsymbol{y}(\boldsymbol{B})$ are applied.

$$\begin{aligned} Z_{AO} = \max_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} + \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}(\boldsymbol{B})} \boldsymbol{d}'\boldsymbol{y}(\boldsymbol{B}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \leq \boldsymbol{b}, \qquad \forall \boldsymbol{B} \in \mathcal{U} \\ & \boldsymbol{x}, \boldsymbol{y}(\boldsymbol{B}) \geq \boldsymbol{0} \end{aligned}$$

- It is adaptive optimization as the $\boldsymbol{y}(\boldsymbol{B})$ adapts to the data.
- We avoid the difficulty of probability theory, but it is still computationally intractable as it still calls for a plan for all eventualities $\boldsymbol{B}$.

Robust Optimization

- Consider $\boldsymbol{y}(\boldsymbol{B}) = \boldsymbol{y}$ for all $\boldsymbol{B}$.
- Then we obtain RO

$$\begin{aligned} Z_{RO} = \max_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} + \min_{\boldsymbol{B} \in \mathcal{U}} \max_{\boldsymbol{y}} \boldsymbol{d}'\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}, \qquad \forall \boldsymbol{B} \in \mathcal{U} \\ & \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

- In deciding $\boldsymbol{x}$, we create a plan for the future $\boldsymbol{y}$ for all uncertainties $\boldsymbol{B}$.
- RO computationally tractable if $\mathcal{U}$ is computationally tractable.

Finitely Adaptive Optimization

- Partition the uncertainty set in k convex subsets (cutting by hyperplanes for example) $\mathcal{U}_1, \ldots, \mathcal{U}_k$.
- Set

$$\boldsymbol{y}(\boldsymbol{B}) = \begin{cases} \boldsymbol{y}_1, & \text{if } \boldsymbol{B} \in \mathcal{U}_1 \\ \boldsymbol{y}_2, & \text{if } \boldsymbol{B} \in \mathcal{U}_2 \\ \vdots & \vdots \\ \boldsymbol{y}_k, & \text{if } \boldsymbol{B} \in \mathcal{U}_k \end{cases}$$

- Intuition: Aggregate uncertainty and find an aggregate adaptive plan imitating human thinking and planning.
- FAO (FAO=RO, if $k = 1$):

$$\begin{aligned} Z_{FAO} = \max_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} + \min_{i=1,\ldots,k} \max_{\boldsymbol{y}_i \in \mathcal{U}_i} \boldsymbol{d}'\boldsymbol{y}_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_i\boldsymbol{y}_i \le \boldsymbol{b}, \qquad \forall \boldsymbol{B}_i \in \mathcal{U}_i \\ & \boldsymbol{x}, \boldsymbol{y}_i \ge \boldsymbol{0},\ i = 1, \ldots, k. \end{aligned}$$

- In deciding $\boldsymbol{x}$, we create a few plans $\boldsymbol{y}_i$ if the uncertainty is in set $\mathcal{U}_i$.
- FAO computationally tractable if $\mathcal{U}_i$ are computationally tractable, like RO.
- Readily extends under MIO conditions.

Affinely Adaptive Optimization

- Set $\boldsymbol{y}(\boldsymbol{B}) = \boldsymbol{q} + \boldsymbol{P}\zeta$ where $\zeta = \mathrm{vec}(\boldsymbol{B})$.
- Let

$$
\begin{aligned}
Z_{AAO} = \max_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} + \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{P},\boldsymbol{q}} \boldsymbol{d}'(\boldsymbol{q} + \boldsymbol{P}\zeta) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{q} + \boldsymbol{P}\zeta) \leq \boldsymbol{b}, \qquad \forall \boldsymbol{B} \in \mathcal{U} \\
& \boldsymbol{q} + \boldsymbol{P}\zeta \geq \boldsymbol{0}, \qquad \forall \boldsymbol{B} \in \mathcal{U} \\
& \boldsymbol{x} \geq \boldsymbol{0},
\end{aligned}
$$

- AAO computationally tractable if $\mathcal{U}$ is computationally tractable and we restrict $\boldsymbol{P}$ to semidefinite matrices, so that the constraint is convex.

LO Formulation of AAO

- AAO under right hand side uncertainty.
- Consider the two stage AO problem

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}'\mathbf{x} + \min\limits_{\mathbf{b}\in\mathcal{U}} \max\limits_{\mathbf{y}(\mathbf{b})} \mathbf{d}'\mathbf{y}(\mathbf{b}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{b}) \le \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\ & \mathbf{y}(\mathbf{b}) \ge \mathbf{0}, \quad \forall \mathbf{b} \in \mathcal{U}, \\ & \mathbf{x} \ge \mathbf{0}, \end{array}$$

where $\mathcal{U} = \{\mathbf{b} \,|\, \mathbf{G}\mathbf{b} \le \mathbf{f}\}$.

- Suppose we restrict ourselves to recourse functions that are affine, that is,

$$\mathbf{y}(\mathbf{b}) = \mathbf{P}\mathbf{b} + \mathbf{q}.$$

LO Formulation of AAO, continued

- By substituting $\mathbf{y}(\mathbf{b}) = \mathbf{Pb} + \mathbf{q}$, we obtain the following static RO

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} + \min_{\mathbf{b}\in\mathcal{U}} \max_{\mathbf{P},\mathbf{q}} \mathbf{d}'(\mathbf{Pb}+\mathbf{q}) \\ \text{s.t.} \quad & \mathbf{Ax} + \mathbf{B}(\mathbf{Pb}+\mathbf{q}) \le \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\ & \mathbf{Pb}+\mathbf{q} \ge 0, \quad \forall \mathbf{b} \in \mathcal{U}, \\ & \mathbf{x} \ge \mathbf{0}, \end{aligned}$$

- By using the techniques introduced in previous lectures, this problem can be reformulated into a single linear optimization problem.
- In particular, let the matrices $\boldsymbol{W}, \boldsymbol{V}$ be the dual variables introduced to model the constraints $\mathbf{Ay} + \mathbf{B}(\mathbf{Pb}+\mathbf{q}) \le \mathbf{b}$, $\mathbf{Pb}+\mathbf{q} \ge 0$, $\forall \boldsymbol{b} \in \mathcal{U}$ respectively.

LO Formulation of AAO, continued

- The single linear optimization formulation is given by

$$
\begin{aligned}
\min_{\mathbf{y},\mathbf{F},\mathbf{q},\eta,\mathbf{W},\mathbf{V},\mathbf{w}} \quad & \mathbf{c}'\mathbf{y} + \eta \\
\text{s.t.} \quad & \mathbf{f}'\mathbf{w} \le \eta - \mathbf{d}'\mathbf{q}, \\
& \mathbf{G}'\mathbf{w} = \mathbf{P}'\mathbf{d}, \\
& \mathbf{W}'\mathbf{f} \le \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{q}, \\
& \mathbf{G}'\mathbf{W} = \mathbf{I} - \mathbf{P}'\mathbf{B}', \\
& \mathbf{V}'\mathbf{f} \le \mathbf{q}, \\
& \mathbf{G}'\mathbf{V} = -\mathbf{P}', \\
& \mathbf{y}, \mathbf{w}, \mathbf{V}, \mathbf{W} \ge \mathbf{0}.
\end{aligned}
$$

Supply Chain Management

- Single product, two echelon, multi-period supply chain.
- Inventories managed periodically over T time periods.
- Retailer : chooses commitments $\boldsymbol{w} = (w_1, \ldots, w_T)$
  - These serve as forecasts for the supplier.
  - Helps the supplier determine its production capacity.
- At the beginning of period t
  - retailer has inventory x_t
  - orders a quantity q_t at a unit cost of c_t
  - Demand d_t is realized

Costs in the Model

- Therefore, the following direct costs are incurred
  - holding cost of $h_t \max[x_t + q_t - d_t, 0]$, h_t : unit holding cost
  - shortage cost of $p_t \max[d_t - x_t - q_t, 0]$, p_t : unit shortage cost
- Contractual costs incurred
  - penalty due to deviations between committed and actual orders:

    $$\alpha_t^+ \max[q_t - w_t, 0] + \alpha_t^- \max[w_t - q_t, 0]$$

    where α_t^+, α_t^- are unit penalties for positive and negative deviations.
  - penalties on deviations between successive commitments:

    $$\beta_t^+ \max[w_t - w_{t-1}, 0] + \beta_t^- \max[w_{t-1} - w_t, 0],$$

    where β_t^+, β_t^- are associated unit penalties.
- Inventory x_{T+1} left at the end has a unit salvage value of s.

Constraints

- Balance equations, that link the inventories, order quantities and realized demand.
- Upper and lower bounds.
- Nominal Optimization Problem

$$
\begin{aligned}
\min \quad & \Big\{ -s\,[x_{T+1}]^{+} + \sum_{t=1}^{T} \big[c_t q_t + h_t\,[x_{t+1}]^{+} + p_t\,[-x_{t+1}]^{+} + \alpha_t^{+}\,[q_t - w_t]^{+} \\
& \quad + \alpha_t^{-}\,[w_t - q_t]^{+} + \beta_t^{+}\,[w_t - w_{t-1}]^{+} + \beta_t^{-}\,[w_{t-1} - w_t]^{+} \Big\} \\
\text{s.t.} \quad & x_{t+1} = x_t + q_t - d_t, \quad \forall t, \\
& L_t \le q_t \le U_t, \quad \forall t, \\
& \hat{L}_t \le \sum_{\tau=1}^{t} q_\tau \le \hat{U}_t, \quad \forall t.
\end{aligned}
$$

Optimization under Uncertain Demand

- For a given ordering policy, events preceding time t are determined by past demands.
- That is,

$$q_t = q_t\left(\boldsymbol{d}^{t-1}\right),$$

  where

$$\boldsymbol{d}^{t-1} = (d_1, \ldots, d_{t-1}).$$

- On the other hand, $\boldsymbol{w} = (w_1, \ldots, w_T)$ must be determined before any realization of demand data.
  - These are the "here and now" decisions.
- Let the demand vector $\boldsymbol{d}^T = (d_1, \ldots, d_T)$ come from the uncertainty set

$$\mathcal{U}^T = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_T,$$

  where $\mathcal{U}_t$ is the uncertainty of demand d_t at period t.

Adaptive Robust Optimization Problem

- The Min-Max problem is given by

$$\min_{x_t(),q_t(),w_t} \quad \Bigg\{-s\left[x_{T+1}\left(\boldsymbol{d}^T\right)\right]^+ + \sum_{t=1}^{T}\Bigg[c_t q_t\left(\boldsymbol{d}^{t-1}\right) + h_t\left[x_{t+1}\left(\boldsymbol{d}^t\right)\right]^+ + p_t\left[-x_{t+1}\left(\boldsymbol{d}^t\right)\right]^+$$

$$+\alpha_t^+\left[q_t\left(\boldsymbol{d}^{t-1}\right) - w_t\right]^+ + \alpha_t^-\left[w_t - q_t\left(\boldsymbol{d}^{t-1}\right)\right]^+$$

$$+\beta_t^+\left[w_t - w_{t-1}\right]^+ + \beta_t^-\left[w_{t-1} - w_t\right]^+\Bigg\}$$

$$\text{s.t.} \qquad \forall \boldsymbol{d}^t \in \mathcal{U}^t = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_t,\ \ t = 1, \ldots, T:$$

$$x_{t+1}\left(\boldsymbol{d}^t\right) = x_t\left(\boldsymbol{d}^{t-1}\right) + q_t\left(\boldsymbol{d}^{t-1}\right) - d_t, \quad \forall t,$$

$$L_t \le q_t\left(\boldsymbol{d}^{t-1}\right) \le U_t, \quad \forall t,$$

$$\hat{L}_t \le \sum_{\tau=1}^{t} q_\tau\left(\boldsymbol{d}^{\tau-1}\right) \le \hat{U}_t, \quad \forall t.$$

Difficult to Solve

- Solution using Dynamic Programming.
- Difficulties
  - the objective function is not smooth
- Even for simple polyhedral uncertainty sets, the problem is NP-hard in general.
- **Core difficulty**: The functional dependence of $q_t\left(\boldsymbol{d}^{t-1}\right)$ is not known.
- How about affine functions?

Affine Adaptability leads to Tractability

- q_t is an affine function of realized demands, that is,

$$q_t = q_t^0 + \sum_{\tau=0}^{t-1} q_t^\tau d_\tau .$$

- With q_t being affine functions, this enforces the variables x_t to be affine too!

$$x_{t+1}\left(\boldsymbol{d}^t\right) = x_{t+1}^0 + \sum_{\tau=1}^{t} x_{t+1}^\tau d_\tau .$$

- The problem now reduces to finding the parameters

$$\{q_t^\tau, x_t^\tau\} \ \forall \tau < t, \forall t = 1, \ldots, T$$

.

- Leads to a linear optimization formulation.

Performance of Fully Adaptive and Affine-adaptive solutions

- The fully adaptive problem is intractable (solved using Dynamic Programming), whereas the affinely adaptive problem can be reformulated as a linear optimization problem.
- In Ben-Tal et.al. [2005], experiments were performed on three kinds of datasets to compare the performance of Fully adaptive, affinely adaptaiveand static robust solutions.
- In almost all the cases, the affinely adaptive solution achieved the same objective as a fully adaptive solution.

Performance

Table 2 Opt(Min-Max), AARC, and RC Solutions for Data Sets *A*12, *D*2, and *W*12 (in Parentheses: Excess Over the Opt(Min-Max) Solution)

Data	Uncertainty (in %)	Opt(min-max)	AARC	RC
*D*2	10	40,750.0	40,750.0 (+0.0%)	40,750.0 (+0.0%)
	20	44,150.0	44,150.0 (+0.0%)	44,150.0 (+0.0%)
	30	47,550.0	47,550.0 (+0.0%)	47,550.0 (+0.0%)
	40	50,950.0	50,950.0 (+0.0%)	50,950.0 (+0.0%)
	50	54,350.0	54,350.0 (+0.0%)	54,350.0 (+0.0%)
	60	57,760.0	57,760.0 (+0.0%)	57,760.0 (+0.0%)
	70	61,170.0	61,170.0 (+0.0%)	61,170.0 (+0.0%)
*A*12	10	913.128	913.128 (+0.0%)	1,002.941 (+9.8%)
	20	1,397.440	1,397.440 (+0.0%)	1,397.440 (+0.0%)
	30	2,190.620	2,190.620 (+0.0%)	2,190.620 (+0.0%)
	40	3,087.540	3,087.540 (+0.0%)	3,087.540 (+0.0%)
	50	4,006.040	4,006.040 (+0.0%)	4,006.040 (+0.0%)
	60	4,934.680	4,934.680 (+0.0%)	4,934.680 (+0.0%)
	70	5,863.320	5,863.320 (+0.0%)	5,863.320 (+0.0%)
*W*12	10	13,531.8	13,531.8 (+0.0%)	15,033.4 (+11.1%)
	20	15,063.5	15,063.5 (+0.0%)	18,066.7 (+19.9%)
	30	16,595.3	16,595.3 (+0.0%)	21,100.0 (+27.1%)
	40	18,127.0	18,127.0 (+0.0%)	24,300.0 (+34.1%)
	50	19,658.7	19,658.7 (+0.0%)	27,500.0 (+39.9%)
	60	21,190.5	21,190.5 (+0.0%)	30,700.0 (+44.9%)
	70	22,722.2	22,722.2 (+0.0%)	33,960.0 (+49.5%)

Conclusions

- Philosophically replacing DP with FAO or AAO is closer to human decision making.
- FAO or AAO are tractable.
- High quality performance in an interesting application.