15.094J/1.142J Robust Modeling, Optimization and Computation	**Project Proposal** **By: Hanzhang Qin**

Robust Univariate Optimization: stronger results in 1D?

My interest is intrigued by a recent result on the famous S-Lemma. It is shown in [1] that in one dimension the S-Lemma can be strengthened to have a stronger form. Namely, the lemma is stated as follows, which says under some regularity condition, the S-Lemma holds for any univariate polynomials (with arbitrary degrees). This is in contradiction to multivariate polynomials where we know a similar result only holds for quadratic functions.

Lemma 1. *Assume $gcd(g, g') = 1$ (regularity condition). Then the following two statements are equivalent:*

(i) $\forall\ x \in \mathbb{R},\ f(x) \geq 0$ *implies* $\forall\ x \in \mathbb{R},\ g(x) \geq 0$*;*

(ii) There exists nonnegative polynomials $h_1(x) \geq 0, h_2(x) \geq 0$ *and* $h_3(x) \geq 0$ *such that* $h_1(x)f(x) = h_2(x)g(x) + h_3(x)$.

Moreover, if we have the decomposition $f(x) = f_0(x)\Pi_{i=1}^{I} f_i(x)$ *and* $g(x) = g_0(x)\Pi_{j=1}^{J} g_j(x)$*, where* $f_i(x)$ *and* $g_j(x)$ *have degree* ≤ 2 $(\forall\ i, j)$ *and denote* $degf_0(x) = 2s$ *and* $deg\ g_0(x) = 2r$ *then* $degh_1(x) = 2s + 2\lceil \frac{degf(x)-2r}{2} \rceil \left(\frac{degg(x)-2s}{2} - 1 \right)$, $degh_2(x) = 2r + 2\lceil \frac{degg(x)-2s}{2} \rceil \left(\frac{degf(x)-2r}{2} - 1 \right)$.

It is well known S-lemma is useful in deriving robust/adaptive counterparts for mathematical programs, so it is then natural to think we may have stronger results in the univariate case than the high-dimensional case. And this is often true. For example, in [2], the authors are able to characterize exactly the optimal probability bounds in the univariate case (can be solved via semidefinite programming) while they show in high dimension such a characterization is NP-hard. The key in their univariate construction is the usage of the linear matrix inequalities (LMI) for nonnegative univariate polynomials.

I believe combining LMI and the univariate S-Lemma might bring about new insights into the robust univariate optimization problems, where similar ideas have been adopted by [3] to derive new counterparts for data-driven chance constrained problems. During the project, I would attempt to explore in depth on this issue. Not sure if it would be a serious research attempt in the future, but at least I think it will be much fun to put my efforts into it.

References

[1] Simai He, Man Hong Wong, and Shuzhong Zhang. "On the S-Lemma for Univariate Polynomials."

[2] Dimitris Bertsimas, and Ioana Popescu. "Optimal inequalities in probability theory: A convex optimization approach." SIAM Journal on Optimization 15.3 (2005): 780-804.

[3] Ruiwei Jiang, and Yongpei Guan. "Data-driven chance constrained stochastic program." Mathematical Programming 158.1-2 (2016): 291-327.