14.2 多元函数 Taylor 公式

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课本例题

例 1 *在原点 $(0,0)$ 的邻域中将函数 $f(x,y)=\mathrm{e}^x\ln(1+y)$ 展开为一阶带 Lagrange 型余项的 Taylor 公式.*

解: 首先算出 $f(x,y)$ 的各阶偏导数及在 $(0,0)$ 点的值: 显然 $f(0,0)=0$,

$$\begin{aligned}
f_x(x,y)&=\mathrm{e}^x\ln(1+y), & f_x(0,0)&=0;\\
f_y(x,y)&=\mathrm{e}^x\frac{1}{1+y}, & f_y(0,0)&=1;\\
f_{xx}(x,y)&=\mathrm{e}^x\ln(1+y), & f_{xx}(0,0)&=0;\\
f_{xy}(x,y)&=\mathrm{e}^x\frac{1}{1+y}, & f_{xy}(0,0)&=1;\\
f_{yy}(x,y)&=-\mathrm{e}^x\frac{1}{(1+y)^2}, & f_{yy}(0,0)&=-1.
\end{aligned}$$

根据公式 (**??**), 可以得到

$$\mathrm{e}^x\ln(1+y)=y+\frac{1}{2!}\left[x^2\ln(1+\theta y)+2\frac{xy}{1+\theta y}-\frac{y^2}{(1+\theta y)^2}\right]\mathrm{e}^{\theta x}.$$

□

思考题

1. 如果在一个区域 D 中恒有 $\mathrm{d}f\equiv 0$, 问对 f 能得出什么结论?

解: 如果在一个区域 D 中恒有 $\mathrm{d}f\equiv 0$, 则 f 在区域 D 恒为常数. □

2. 中值定理为什么要求区域是凸的?

解: 中值定理之所以要求区域是凸的, 是为了保证点 $(a+\theta h, b+\theta k)$ 依然是这个区域内的点. □

3. 三元或多元函数的 Taylor 公式应该具有什么形式?

解: 设 $m(m\geq 3)$ 元函数 $f(x_1,x_2,\cdots,x_m)$ 在点 $P_0(x_1^0,x_2^0,\cdots,x_m^0)$ 的某个邻域 $U(P_0)$ 内有直到 $n+1$ 阶的连续偏导数, 则对于任意一点

$$P(x_1^0+k_1,x_2^0+k_2,\cdots,x_m^0+k_m)\in U(P_0),$$

存在相应的 $\theta\in(0,1)$, 使得

$$\begin{aligned}
& f(x_1^0+k_1, x_2^0+k_2, \cdots, x_m^0+k_m) \\
= & f(x_1^0, x_2^0, \cdots, x_m^0) + \left(k_1\frac{\partial}{\partial x_1}+k_2\frac{\partial}{\partial x_2}+\cdots+k_m\frac{\partial}{\partial x_m}\right) f(x_1^0, x_2^0, \cdots, x_m^0) \\
& +\frac{1}{2!}\left(k_1\frac{\partial}{\partial x_1}+k_2\frac{\partial}{\partial x_2}+\cdots+k_m\frac{\partial}{\partial x_m}\right)^2 f(x_1^0, x_2^0, \cdots, x_m^0) \\
& +\cdots \\
& +\frac{1}{n!}\left(k_1\frac{\partial}{\partial x_1}+k_2\frac{\partial}{\partial x_2}+\cdots+k_m\frac{\partial}{\partial x_m}\right)^n f(x_1^0, x_2^0, \cdots, x_m^0) \\
& +\frac{1}{(n+1)!}\left(k_1\frac{\partial}{\partial x_1}+k_2\frac{\partial}{\partial x_2}+\cdots+k_m\frac{\partial}{\partial x_m}\right)^{n+1} f(x_1^0+\theta k_1, x_2^0+\theta k_2, \cdots, x_m^0++\theta k_m).
\end{aligned}$$

其中每一项都含有形式上的符号运算, 即

$$\begin{aligned}
& \left(k_1\frac{\partial}{\partial x_1}+k_2\frac{\partial}{\partial x_2}+\cdots+k_m\frac{\partial}{\partial x_m}\right)^m f(x_1^0, x_2^0, \cdots, x_m^0) \\
= & \sum_{\alpha_1+\alpha_2+\cdots+\alpha_m=m} \frac{m!}{\alpha_1!\alpha_2!\cdots\alpha_m!}\cdot\frac{\partial^m f(x_1, x_2, \cdots, x_m)}{\partial {x_1}^{\alpha_1}\partial {x_2}^{\alpha_2}\cdots\partial {x_m}^{\alpha_m}}\Big|_{(x_1^0, x_2^0, \cdots, x_m^0)} {k_1}^{\alpha_1}{k_2}^{\alpha_2}\cdots{k_m}^{\alpha_m},
\end{aligned}$$

□

习题

1. 在点 $(1,-2)$ 的邻域中用 Taylor 公式展开函数 $f(x,y)=2x^2-xy-3y^2-7x+y+1$.

解: 由已知, 得

$$f(1,-2)=2\times 1+2-3\times 4-7-2+1=-16;$$

$$\begin{aligned}
f_x(x,y)=4x-y-7, &\qquad f_x(1,-2)=-1; \\
f_y(x,y)=-x-6y+1, &\qquad f_y(1,-2)=12; \\
f_{xx}(x,y)=4, &\qquad f_{xx}(1,-2)=0; \\
f_{xy}(x,y)=-1, &\qquad f_{xy}(1,-2)=-1; \\
f_{yy}(x,y)=-6, &\qquad f_{yy}(1,-2)=-6.
\end{aligned}$$

且

$$\frac{\partial^n}{\partial^i\partial y^{n-i}}f(x,y)\equiv 0, \quad n\geq 3,$$

所以

$$\begin{aligned}
f(x,y) &= 2x^2-xy-3y^2-7x+y+1 \\
&= -16+[-(x-1)+12(y+2)] \\
&\quad +\frac{1}{2!}[4(x-1)^2-2(x-1)(y+2)-6(y+2)^2] \\
&= 2(x-1)^2-(x-1)(y+2)-3(y+2)^2-(x-1)+12(y+2)-16.
\end{aligned}$$

□

2. 在原点的邻域中将下列函数展开至二阶 Taylor 公式.

(1) $f(x,y)=\mathrm{e}^{xy}$;

(2) $f(x,y)=\sin(x+y)$;

(3) $f(x,y)=\mathrm{e}^{x^2}\ln(1+y^2)$.

解: (1) 首先算出 $f(x,y)$ 的各阶偏导数及在 $(0,0)$ 点的值: 显然 $f(0,0)=1$,

$$\begin{aligned}
f_x(x,y)&=y\mathrm{e}^{xy}, & f_x(0,0)&=0;\\
f_y(x,y)&=x\mathrm{e}^{xy}, & f_y(0,0)&=0;\\
f_{xx}(x,y)&=y^2\mathrm{e}^{xy}, & f_{xx}(0,0)&=0;\\
f_{xy}(x,y)&=\mathrm{e}^{xy}(1+y^2), & f_{xy}(0,0)&=1;\\
f_{yy}(x,y)&=x^2\mathrm{e}^{xy}, & f_{yy}(0,0)&=0.
\end{aligned}$$

根据公式 14.2.4, 可以得到

$$\begin{aligned}
f(x,y)=e^{xy} &= f(0,0)+\left(x\frac{\partial}{\partial x}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial}{\partial x}\right)^2 f(0,0)+o(\rho^2)\\
&= f(0,0)+(x\cdot 0+y\cdot 0)+\frac{1}{2!}\left(x^2\cdot 0+2xy\cdot 1+y^2\cdot 0\right)+o(\rho^2)\\
&= 1+xy+o(\rho^2),\quad \rho\to 0.
\end{aligned}$$

(2) 首先算出 $f(x,y)$ 的各阶偏导数及在 $(0,0)$ 点的值: 显然 $f(0,0)=0$,

$$\begin{aligned}
f_x(x,y)&=\cos(x+y), & f_x(0,0)&=1;\\
f_y(x,y)&=\cos(x+y), & f_y(0,0)&=1;\\
f_{xx}(x,y)&=-\sin(x+y), & f_{xx}(0,0)&=0;\\
f_{xy}(x,y)&=-\sin(x+y), & f_{xy}(0,0)&=0;\\
f_{yy}(x,y)&=-\sin(x+y), & f_{yy}(0,0)&=0.
\end{aligned}$$

根据公式 14.2.4, 可以得到

$$\begin{aligned}
f(x,y)=\sin(x+y) &= f(0,0)+\left(x\frac{\partial}{\partial x}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial}{\partial x}\right)^2 f(0,0)+o(\rho^2)\\
&= f(0,0)+(x\cdot 1+y\cdot 1)+\frac{1}{2!}\left(x^2\cdot 0+2xy\cdot 0+y^2\cdot 0\right)+o(\rho^2)\\
&= x+y+o(\rho^2),\quad \rho\to 0.
\end{aligned}$$

(3) 首先算出 $f(x,y)$ 的各阶偏导数及在 $(0,0)$ 点的值: 显然 $f(0,0)=0$,

$$\begin{aligned}
f_x(x,y)&=2x\mathrm{e}^{x^2}\ln(1+y^2), & f_x(0,0)&=0;\\
f_y(x,y)&=\frac{2y}{1+y^2}\mathrm{e}^{x^2}, & f_y(0,0)&=0;\\
f_{xx}(x,y)&=2\mathrm{e}^{x^2}(1+2x^2)\ln(1+y^2), & f_{xx}(0,0)&=0;\\
f_{xy}(x,y)&=\frac{2y}{1+y^2}2x\mathrm{e}^{x^2}, & f_{xy}(0,0)&=0;\\
f_{yy}(x,y)&=\frac{2-2y^2}{(1+y^2)^2}\mathrm{e}^{x^2}, & f_{yy}(0,0)&=2.
\end{aligned}$$

根据公式 14.2.4, 可以得到

$$\begin{aligned} f(x,y)=\mathrm{e}^{x^2}\ln(1+y^2) &= f(0,0)+\left(x\frac{\partial}{\partial x}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial}{\partial x}\right)^2 f(0,0)+o(\rho^2) \\ &= f(0,0)+(x\cdot 0+y\cdot 0)+\frac{1}{2!}\left(x^2\cdot 0+2xy\cdot 0+y^2\cdot 0\right)+o(\rho^2) \\ &= y^2+o(\rho^2), \quad \rho\to 0. \end{aligned}$$

□

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3. 设 $D\subset\mathbb{R}^2$ 是凸开域在 $f_x(x,y), f_y(x,y)$ 在 D 上存在且有界. 证明 $f(x,y)$ 在 D 上一致连续.

证明. 由已知, $f_x(x,y), f_y(x,y)$ 有界, 即

$\exists M>0$, 使对 $\forall(x,y)\in D$, 都有 $|f_x(x,y)|\le M, |f_y(x,y)|\le M$,

对 $\forall\epsilon>0$, $\exists\delta=\epsilon/2M$, 对 $\forall P_1(x_1,y_1), P_2(x_2,y_2)\in D$,(不妨设 $x_1<x_2, y_1<y_2$), 满足 $|x_1-x_2|<\delta, |y_1-y_2|<\delta$,

由二元函数中值公式, 得

$$\begin{aligned} |f(x_1,y_1)-f(x_2,y_2)| &= |f_x(x_1+\theta|x_1-x_2|, y_1+\theta|y_1-y_2|)\cdot|x_1-x_2| \\ &\quad +f_y(x_1+\theta|x_1-y_1|, y_1+\theta|y_1-y_2|)\cdot|y_1-y_2|| \\ &\le |f_x(x_1+\theta|x_1-y_1|, y_1+\theta|y_1-y_2|)\cdot|x_1-x_2|| \\ &\quad +|f_y(x_1+\theta|x_1-y_1|, y_1+\theta|y_1-y_2|)\cdot|y_1-y_2|| \\ &\le M(|x_1-x_2|+|y_1-y_2|)<2M\delta=\epsilon. \end{aligned}$$

故由二元函数的一致连续定义可知, $f(x,y)$ 在 D 上一致连续. ∎

4. 设 $\boldsymbol{p}$ 和 $\boldsymbol{q}$ 是 $\mathbb{R}^2$ 中线性无关的向量, $f(x,y)$ 是可微函数. 证明: 如果

$$\frac{\partial f}{\partial\boldsymbol{p}}\equiv 0, \quad \frac{\partial f}{\partial\boldsymbol{q}}\equiv,$$

则 $f(x,y)$ 是常值函数.

证明. 设 $\boldsymbol{p}=(\cos\alpha_1,\sin\alpha_1), \boldsymbol{q}=(\cos\alpha_2,\sin\alpha_2)$, 其中 $\alpha_1\ne\alpha_2$,

由于 $f(x,y)$ 在 $\mathbb{R}^2$ 上可微,

$$\frac{\partial f}{\partial\boldsymbol{p}}(x,y)=f_x(x,y)\cos\alpha_1+f_y(x,y)\sin\alpha_1\equiv 0, \tag{1}$$

$$\frac{\partial f}{\partial\boldsymbol{q}}(x,y)=f_x(x,y)\cos\alpha_2+f_y(x,y)\sin\alpha_2\equiv 0, \tag{2}$$

因为 $\boldsymbol{p}$ 和 $\boldsymbol{q}$ 是 $\mathbb{R}^2$ 中线性无关的向量, 所以

$$\begin{vmatrix} \cos\alpha_1 & \sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \end{vmatrix}\ne 0,$$

因此, 线性方程组 (**??**) 和 (**??**) 只有零解, 即

$$f_x(x,y)\equiv 0, \quad f_y(x,y)\equiv 0,$$

于是由推论 14.2.2 可知, $f(x,y)$ 是常值函数.

∎