Mathematics: Queen and Servant of the Sciences*

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DICHOTOMIES IN MATHEMATICS

A 250th Anniversary, especially for a Society such as this, is an appropriate occasion for a philosophical talk. It provides an opportunity for perennial questions to be reexamined in the light of modern developments. Moreover, a talk about mathematics is much more easily conveyed to a general audience than a mathematical talk per se. For all these reasons I propose to discuss the nature of mathematics. The difficulties and ambivalences in this task are clear when we consider the dichotomies that are present. Is Mathematics an Art or a Science? Universities are uncertain about the answer since some award mathematicians a BA degree while others insist on a BSc. Then there is the traditional divide between Pure and Applied Mathematics, but this is complicated by the ever widening scope of the applications of mathematics so that even the purest parts are finding unexpected applications. For example, prime numbers are usually regarded as the purest and most useless form of mathematics but recently they have found application in the construction of security codes for banks and other organizations. Algebraic geometry has recently established links with high energy physics and mathematical logic is of increasing importance in computer science.

Lying even deeper is the traditional question: are mathematical theorems inventions or discoveries? Is mathematics a creation of the human mind or a reflection of physical reality?

The title of my lecture: "Queen and Servant of the Sciences" is another variant, presented in more poetic form and playing on our prejudices. Let me begin, in the best traditions of analytical philosophy, by subjecting this title to some textual analysis.

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* Read 29 April 1993.

It is perhaps useful to remember that all royal families have (ultimately) humble origins: the queen has evolved from the servant. Moreover their roles are sometimes confused, when the monarch is viewed as a servant of his people. This aspect is amusingly captured by the Gilbert and Sullivan song which concludes with the Gondolier princes singing:

> But the privilege and pleasure
> That we treasure beyond measure
> Is to run on little errands
> For the Ministers of State.

In our own time we are left only with constitutional monarchs having ceremonial duties but little authority. So, if we refer to mathematics as "queen of the sciences," do we have in mind a decorative symbol or a source of power?

Dichotomies are useful devices to provoke thought. Like paradoxes they highlight the difficulties but they do not provide solutions. A more constructive metaphor is to view mathematics as the language of science, and it is this idea that I would like to develop, beginning with a brief look at natural language.

Natural Language

How did natural language evolve and what is its function? Human beings have perceptions of the "real world" (which includes themselves), they reflect on this, and produce descriptions and explanations which they then communicate (to others) via language. Concepts have to be developed, names given and manipulated logically (grammatically) to produce sentences. We might say language is the "externalization of thought." Note that primitive thinking is not verbal but visual; this becomes clear if we consider animals or young children.

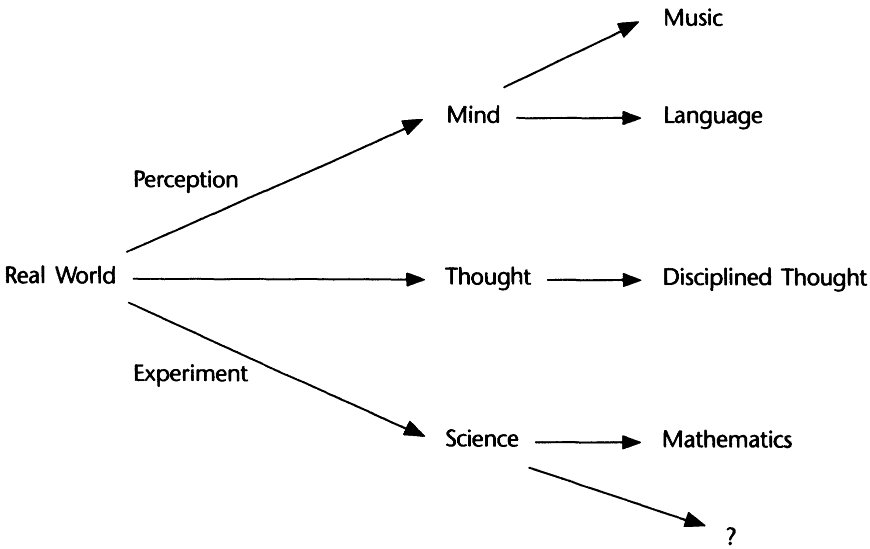
But language as it has developed has many different layers and serves many different purposes. There is a spectrum, from poetry (and other forms of literature) to the lower forms of information communication as in newspapers. There are also specialized forms of language dealing with fields such as Law. Beyond this there is the study of language itself as in linguistics or philology. We recognize that poetry has an aesthetic and creative component, which frequently transcends the strict rules of grammar, and that it is constantly searching to extend the scope and meaning of language. Moreover, in time, this creative aspect influences the more utilitarian forms of language, so that Shakespearean quotations now abound, even in newspapers.

Language is both grammar and literature. It embodies concepts and meaning and it is hard to separate words from their meaning. Ideas create words and words enable us to formulate new and more complex ideas.

Mathematics

Mathematics starts with ordinary language but digs deeper, with greater precision (starting from Numbers) and develops further concepts and rules. It can be viewed as a specialized superlanguage. George Boole explained the relation between grammar and the algebra of symbolic logic (now the foundation of computer science) in which equations play the role of sentences.

We can illustrate the analogy between mathematics and language by the following diagram:



Just as primitive thought (e.g., in animals) is non-verbal, so primitive science (e.g., associating heat and light) is non-mathematical. Moreover, the mind has produced several types of language in which to express itself, music being one example. At present mathematics is by its depth and scope the pre-eminent language of science, but it remains to be seen whether other types of language (indicated by ?) will be needed.

If we accept this analogy then we can begin to understand how mathematics has a creative/aesthetic side, like poetry, where the imagination is being stretched, and a more utilitarian side, as used by engineers in routine calculations. Also mathematics has its own internal analysis, like linguistics and philology. Finally, mathematics is constantly being enlarged by the addition of new concepts in response to the advancing needs of science. This may be compared with the growth and development of language, having to deal with the needs of sophisticated modern society.

As with language, where thought and word interact with one another, so science and mathematics interact with each other. It is difficult to separate contents and framework: each influences the other in a complex symbiosis. It is for this reason that I have no difficulty in describing mathematics as the language of science. Some of my colleagues might feel that this gives mathematics too humble a status, that of the "servant," and they would prefer the loftier position of the "queen" from whom all authority and beauty emanates. But if we reflect on the power of words, and the role they play in organizing, refining, and transmitting ideas, then we see that the role is an honourable one. Ideas without words remain vague and ineffective, and science without mathematics remains similarly handicapped.

Mathematics and Physics

Let us consider as a key example the particular relation between Mathematics and Physics. This is the oldest and most intimate. Its most famous embodiment is in Newton's Theory of Gravity. The inverse square law, that all material particles attract each other with a force inversely proportional to the square of their distance apart, proved a spectacular triumph. It explained the planetary system and much else, but the notion of "action at a distance" was philosophically controversial. It was not physically detectable (on small scales) and it was essentially a "mathematical fiction." In due course however it became accepted, attitudes changed, and gravitational force is now accepted as a physical phenomenon, with mathematics only acknowledged as a tool when one works out the consequences of the basic laws. In fact historically, philosophically, and logically mathematics is there at the beginning in providing the formulation of the basic laws.

In the first half of the twentieth century two major new physical theories emerged: Einstein's Theory of General Relativity and Quantum Mechanics. In each case we have very sophisticated mathematical theories which are philosophically difficult to grasp. The only way that they can be understood is in mathematical terms: ordinary language is totally inadequate.

As the twentieth century has progressed this process has moved on inexorably. The search for the ultimate building blocks of matter, and the ultimate forces that bind them, delves deeper and deeper into ever more esoteric mathematics. As we get further away from common experience and as experiments become more and more costly, mathematical coherence becomes the predominant criterion. Producing a mathematical theory which embraces all known experimental facts and is internally consistent becomes the real driving force in theoretical physics.

Current ideas involve radical notions which question the fundamental position of space-time, postulate quantum "super-strings" instead of particles, and conceive of abstract higher-dimensional spaces. "Reality" becomes a purely mathematical structure: the only debate concerns rival mathematical theories. As telescopes probe to outer space and microscopes to minute sizes, so mathematics is our intellectual probe to physical reality.

But the precise type of mathematics needed is itself developed in response to the requirements of physics. At present an interesting mixture is being explored, partly rigorous mathematics, partly infused with physical intuition. Like mystical poetry, some grammar ignored, part intuition, part language, searching for the deepest truths, a new language is being developed. The aim remains eventually to build a logical structure consistent with experiment. This would represent the final takeover of the physical world by mathematics.