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HODGE'S GENERAL CONJECTURE IS FALSE FOR TRIVIAL REASONS

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§1

The startling title is somewhat misleading, as everybody will think about the part of the Hodge conjecture which is most generally remembered, namely the part concerned with a criterion for a cohomology class (on a projective smooth connected scheme X over **C**) to be "algebraic", i.e. to come from an algebraic cycle with rational† coefficients. This conjecture is plausible enough, and (as long as it is not disproved!) should certainly be regarded as the deepest conjecture in the "analytic" theory of algebraic varieties. However, in [6, p. 184], Hodge gave a more general formulation of his conjecture in terms of filtrations of cohomology spaces, and the main aim of my note is to show that for a rather trivial reason, this formulation has to be slightly corrected.

Consider on the complex cohomology

$$H^i(X^{an}, \mathbf{C}) = H^i(X^{an}\ \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$

(X^{an} denotes the analytic space associated to the scheme X) the "Hodge filtration" Filt^p, which can be defined in terms of the Hodge bigraduation as the sum of all $H^{p',q}$ with $p' + q = i, p' \geqslant p$. This filtration of course is not "rational over **Q**" i.e. does not come from a filtration on $H^i(X^{an}, \mathbf{Q})$, except in trivial cases. However, there is on the rational cohomology a very significant filtration, which might be called the "arithmetic" filtration, as it embodies deep arithmetic properties of the scheme X, which we will denote by Filt'^p, where Filt'^p is the space of cohomology classes for which there exists a Zariski closed subset T of X, of codimension $\geqslant p$, such that the given class vanishes on $X - T$. We denote by the same notation Filt'^p the corresponding filtration of the complex cohomology. Both Filt^p and Filt'^p are decreasing filtrations on $H^i(X^{an}, \mathbf{C})$, and it is well-known that the second is finer than the first, which means

$$(*) \qquad \mathrm{Filt}'^p\, H^i(X^{an}, \mathbf{Q}) \subset \mathrm{Filt}^p\, H^i(X^{an}, \mathbf{C}) \cap H^i(X^{an}, \mathbf{Q}).$$

† In fact, Hodge states his conjecture for integral cohomology. That this is too optimistic was proved in [1].

If we assume for simplicity $i \leqslant \dim X$, then *Hodge's conjecture* (translated from homology into cohomology) *states that the inclusion* $(*)$ *is an equality.*

Let us remark that the complex space $\mathrm{Filt}'^p\, H^i(X^{\mathrm{an}}, \mathbf{C})$ generated by the left hand side of $(*)$ is a sub-Hodge structure of $H^i(X^{\mathrm{an}}, \mathbf{C})$, i.e. is stable under the decomposition into types p, q. This fact, which is probably "well-known", follows from the fact that Filt'^p can be also described as the space generated by the images of the Gysin homomorphisms

$$H^{i-2q}(Y^{\mathrm{an}}, \mathbf{Q}) \to H^i(X^{\mathrm{an}}, \mathbf{Q})$$

for desingularizations Y of closed subschemes T of X which are of pure codimension $q \geqslant p$. I will skip the proof of this fact, already stated and used in [5, 10.1]†. As the previous homomorphisms are compatible with the Hodge structures, the assertion follows. Now equality in $(*)$ would imply a highly non trivial intrinsic condition‡ on the Hodge structure $H^i(X^{\mathrm{an}}, \mathbf{C})$, namely that the $\mathbf{C}$-vector-subspace generated by the right hand side of $(*)$ is a sub-Hodge structure; If i is odd, this would imply for instance that the dimension over $\mathbf{Q}$ of that space is even. It is evident that if $i \leqslant 2$, or if $i = 2p$, the condition thus obtained is trivially satisfied. However, already for $i = 3$, $p = 1$, it becomes non empty, and may in fact not be satisfied for the threefold product of an elliptic curve with itself.

To see this, let us take more generally τ elliptic curves over $\mathbf{C}$, with lattice periods generated by $1, \tau_\alpha (1 \leqslant \alpha \leqslant i)$. The rank of $\mathrm{Filt}'^1\, H^i(X, \mathbf{Q})$, where X is the product of the elliptic curves, is immediately computed, it is equal to $2^i - N$, where N is the rank of the vector space over $\mathbf{Q}$ generated by all j-fold products, $0 \leqslant j \leqslant i$, of τ_α's with distinct indices, i.e. by the coefficients of the polynomial $\prod_\alpha (1 + \tau_\alpha T)$. If i is odd, this rank may well be odd; for instance if $i = 3$, and all τ_i equal to the same τ, this will happen exactly when τ is cubic over $\mathbf{Q}$.

§2

This makes clear how the Hodge conjecture should be corrected, to eliminate trivial counterexamples: namely the left hand side of $(*)$ should be the largest sub-space of the right hand side, generating a subspace of $H^i(X^{\mathrm{an}}, \mathbf{C})$ which is a sub-Hodge structure, i.e. stable under decomposition into p, q types. In other words, an element of $H^i(X^{\mathrm{an}}, \mathbf{C})$ should belong to Filt'^p if and only if all its bihomogeneous components belong to the $\mathbf{C}$-vector space generated by the right hand side of $(*)$.

This formulation may seem a little too cumbersome to inspire confidence. To make it look better, we may remark that it is equivalent to the conjunction of the usual Hodge

† (Added April 1969). It has come to the author's attention that the statement in *loc. cit.* (formula (10.7) or (9.17)) is false in the form given there. It is true however for *X proper, and constant coefficients* $\mathbf{Q}_l$ (thus neglecting torsion) *provided we admit resolution of singularities and the Weil conjectures.* Moreover in char. zero, the last statement is true, as a consequence of P. Deligue's recent extension of Hodge's theory to arbitrary complex algebraic varieties (possibly singular and non-complete).

(‡) It seems that there is no necessary intrinsic condition known for an abstract Hodge structure to be embeddable in one coming from a projective smooth scheme over $\mathbf{C}$, except the existence of a "polarization"—although (as Mumford pointed out to me) Griffiths's general transversality theorem implies (by a Baire argument) that there are many Hodge structures of given degree $\geqslant 2$ which are not "algebraical" in the previous sense. Of course, any necessary condition of algebraicity would be highly interesting!

conjecture (case $i = 2p$), and the following one: for every sub-Hodge structure M of $H^i(X^{an}, \mathbf{C})$ (namely a subspace generated by its intersection with $H^i(X^{an}, \mathbf{C})$ and stable under the decomposition into types) which is *simple* (namely does not contain another sub-Hodge structure except 0 and M), so that there exists a unique p_0 such that $\mathrm{Filt}'^{p_0}(M) = M$, $\mathrm{Filt}'^{p_0+1}(M) = 0$, the integer p_0 is the smallest integer such that $M^{p_0, i-p_0} \neq 0$.

For the reader informed about the yoga of "motives", the most striking equivalent formulation would be the following: a homogeneous motive M over the field of complex numbers is *effective*, i.e. can be imbedded in the motive-theoretic cohomology of some X as above (without twisting back à la Tate) if and only if its Hodge realization is effective, i.e. if and only if $M^{p,q} \neq 0$ implies $p, q \geqslant 0$. (The usual Hodge conjecture means that the natural functor from motives over $\mathbf{C}$ to Hodge structures is fully faithful.)

For $i = 2p$, the Hodge conjecture (which need in this case not be corrected) is just the usual Hodge conjecture, characterizing algebraic cohomology classes. The next important instance occurs for $i = 2p + 1$ (where the corrected version has to be taken). In this case, $\mathrm{Filt}'^p H^{2p+1}(X^{an}, \mathbf{Q})$ has a remarkable geometric interpretation, in terms of Weil's (or, equivalently, Griffiths's) complex torus $J^p(X)$ associated to X (whose π_1 tensored by $\mathbf{Q}$ is $H^{2p+1}(X^{an}, \mathbf{Q})$), as corresponding to the abelian subvariety of $J^{p+1}(X)$ defined by the images of the algebraic cycles of codimension $p + 1$ on X which are algebraically equivalent to zero [3], [7], [8]. Hodge had already remarked that this subspace of $H^{2p+1}(X^{an}, \mathbf{Q})$ is contained in Filt^p (i.e. is contained in $H^{p,p+1} + H^{p+1,p}$) and Hodge's conjecture provides a kind of converse to this statement, giving a characterization of the "algebraic part" of $J^{p+1}(X)$ in terms of the Hodge structure of $H^{2p+1}(X^{an}, \mathbf{C})$. It should be pointed out that in this particular case, however, and for fixed X and p, the Hodge conjecture is easily seen to be equivalent to the usual Hodge conjecture in degree $2(p + 1)$ for all products $C \times X$, where C is a proper, smooth algebraic curve over $\mathbf{C}$. This is due to the fact that an effective Hodge structure of degree 1 which admits a polarization (i.e. a "Riemann form") can be viewed as the $H^1(A^{an}, \mathbf{C})$ of an abelian variety A, which in turn can be obtained as a quotient of the jacobian of a suitable algebraic curve C.

§3

It may be of interest to review here the few non trivial instances known to the author where the Hodge conjecture has been checked.

a) The case $p = 1$, $i = 2$, i.e. the characterization of cohomology classes coming from divisors, due to Lefschetz, which has become trivial now through sheaf cohomology and the exact sequence of the exponential.

b) The case $i = \dim X$, any p, provided we make the following two assumptions, where Y denotes a "general" hyperplane section of X: 1°) The Hodge conjecture is true for $H^{i-2}(Y^{an}, \mathbf{C})$ in filtration p^{-1} (this condition is satisfied if $i \leqslant 4$). 2°) The part of $H^{i-1}(Y^{an}, \mathbf{C})$ orthogonal to the image of $H^{i-1}(X^{an}, \mathbf{C})$ (the so called "vanishing cycles" part of $H^{i-1}(Y^{an}, \mathbf{C})$) is contained in Filt'^p (if $i = 3$ and $p = 1$, this amounts to saying that the component of type (2, 0) of the vanishing cycles subspace of $H^2(Y^{an}, \mathbf{C})$ is zero). For

$i = 3$, this case is mentioned in Hodge's exposé [6]. It is not hard to establish, using Leray's spectral sequence for the "fibering" of X by a suitable pencil of hyperplane sections, and resolution of singularities.

c) The case of a product of elliptic curves, $i = 2p$, any p. This case is due to Tate (unpublished), who proves it by observing that the "Hodge classes" in the cohomology of X are sums of products of Hodge classes of degree 2, so that a) applies.

d) The case of a general cubic threefold in P^4, $i = 3$, $p = 1$, due to Gherardelli [2]†.

e) The case of a cubic fourfold in P^5, $i = 2p, p = 2$, due to Griffiths, using e) and recent results of his [4].

§4

In most concrete examples, it seems very hard to *check* the Hodge conjecture, due to the difficulty in explicitly determining the filtration Filt′ of the cohomology, and even in determining simply the part of the cohomology coming from algebraic classes. It may be easier, for the time being, to *test* the Hodge conjectures in various non trivial cases, through various consequences of the Hodge conjectures which should be more amenable to direct verification. I would like to mention here two such consequences, which can be seen in fact to be consequences already of the *usual* Hodge conjecture.

First, if X is as before, the dimensions of the graded components of the vector space associated to the arithmetic filtration Filt′ (and indeed this very filtration itself, if we interpret complex cohomology as de Rham cohomology, which makes a purely algebraic sense) is clearly invariant if we transform X by any automorphism of the field **C**, or equivalently, if we change the topology of **C** by such an automorphism. In other words, if we have a smooth projective scheme X over a field K of char 0, then the invariants we get by different imbeddings of K into the field **C** are the same. Granting the Hodge conjecture, the same should be true if we replace the Filt′ filtration by the filtration described in §2 in terms of the Hodge structure (which is a transcendental description). What if we take for instance for X a "general" abelian variety of given dimension or powers of it, or powers of a "general" curve C of given genus? The case of genus 1 checks by Tate's result recalled in example c) above.

Secondly, and more coarsely, if we have a projective and smooth morphism $f: X \to S$ of algebraic schemes over **C**, we can for every $s \in S$ consider the complex cohomology of the fiber X_s as a Hodge structure, and look at the filtration "rational over **Q**" which it defines (and which conjecturally should be the arithmetic filtration). Hodge's conjecture would imply that the set of points $s \in S^{an}$ where the dimensions of the components of the associated graded space have fixed values has a very special structure: it should be the difference of two countable unions of Zariski-closed subsets of S, which in fact should even be definable over a fixed subfield of **C**, of finite type over the field **Q**. (A simple application of Baire's theorem, not using Hodge's conjecture, would give us only a considerably weaker

† (Added April 1969). This can be viewed also as a particular case of Hodge's result quoted in example b), and Manin has observed that this example extends to any *univational threefold X*. Cf. Manin: Correspondances, motives and monoidal transforms (in Russian), *Mat. Sbornik* 77 (1968), 475–507.

structure theorem for the set in question, where Zariski-closed subsets would be replaced by the images, under the projection of the universal covering $\tilde{S}$ of S^{an}, of analytic subsets of $\tilde{S}$†.)

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† (Added April 1969) David Lieberman has informed me that he can prove the stronger result obtained by replacing $\tilde{S}$ by S^{an} itself.