Notes on the life and work of Alexander Grothendieck *

Piotr Pragacz**

When I was a child I loved going to school. The same instructor taught us reading, writing and arithmetic, singing (he played upon a little violin to accompany us), the archaeology of prehistoric man and the discovery of fire. I don't recall anyone ever being bored at school. There was the magic of numbers and the magic of words, signs, and sounds ... .

A. Grothendieck: *Récoltes et Semailles*

Abstract

This is a story of Alexander Grothendieck – a man who has changed the face of mathematics during some 20 years of his work on functional analysis and algebraic geometry. Last year he turned 75. This paper, written in April 2004, is based on a talk presented at the *Hommage à Grothendieck* session of Impanga[1], held at the Banach Centre in Warsaw (January 2004).

Alexander Grothendieck was born in Berlin in 1928. His father, Alexander Shapiro (1890–1942) was a Russian Jew from a Hassidic town on a now Russian-Ukrainian-Belorussian border. He was a political activist – an anarchist involved in all the major European revolutions during 1905–1939. In the 20's and 30's he lived mostly in Germany, operating in the left-wing movements against more and more powerful Nazis, and working as a street photographer. In Germany, he met Hamburg-born Hanka Grothendieck (1900–1957). (The name *Grothendieck* comes from *Plattdeutsch*, a Northern German dialect.) Hanka Grothendieck worked on and off as a journalist, but her true passion was writing. On March 28, 1928 she gave birth to their son Alexander.

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[1] *Impanga* is an algebraic geometry group, operating since 2000 at the Institute of Mathematics of the Polish Academy of Sciences. This session hosted the talks of: M. Chałupnik, *Grothendieck topologies and étale cohomology*, T. Maszczyk, *Toposes and the unity of mathematics*, J. Gorski, *Grothendieck stacks on Mazovia plains*, O. Kędzierski, *Why the derived categories?*, A. Weber, *The Weil conjectures*, G. Banaszak, *l-adic representations*, P. Krasoń, *Mordell-Weil groups of Abelian varieties.*

During 1928–1933 Alexander lived with his parents in Berlin. After Hitler's rise to power, Alexander's parents immigrated to France, leaving their son (for about 5 years) with the Heydorns, a surrogate family in Hamburg, where he went to a primary and secondary school. In 1939 Alexander joined his parents in France. His father was soon interned by the French Vichy police in the Vernet camp in the Pyrenees, and then handed out to the Nazis occupiers. He was murdered in the German concentration camp Auschwitz-Birkenau in 1942.

Very young Alexander Grothendieck

Hanka and Alexander Grothendieck did not survive the occupation without problems. In the years 1940–1942 they were interned – as "undesirable dangerous foreigners" – in the Rieucros camp near Mende in southern France. Hanka was later transferred to the Gurs camp in the Pyrenees, whilst Alexander was allowed to continue his education in Collège Cévenol in a Cévennes Mountains town of Chambon-sur-Lignon in the southern Massif Central. The college, run by local Protestants under the leadership of Pastor Trocmé, was a sanctuary to many children (mainly Jews) whose lives were endangered during the war.

Already then, Alexander asked *himself* a question that showed the uniqueness of his mind: How to accurately measure the length of a curve, area of a surface, or volume of a solid? Continuing the reflection on these problems during his university studies in Montpellier (1945–1948), he independently obtained results equivalent to Lebesgue's measure and integration theory. As expressed by J. Dieudonné in [D], the university in Montpellier – in Grothendieck's days – wasn't a "proper place" for studying great mathematical problems ... . In the fall of 1948 Grothendieck arrived in Paris, where he spent a year attending courses in the famous École Normale Supérieure (ENS), the "birthplace" of most of the French mathematical elite. In particular, he took part in Cartan's legendary seminar, that year devoted to algebraic topology. (More information about this period of Grothendieck's life, his parents, and France of those days, can be found in [C2].)

Grothendieck's interests, however, began focusing on functional analysis. Following Cartan's advice, in October 1949 he comes to Nancy, a centre of functional analysis studies, where J. Dieudonné, L. Schwartz, and others run a seminar on Fréchet spaces and their direct limits. They encounter a number of problems which they are unable to solve, and suggest Grothendieck to try and attack them. The result surpasses all expectations. In less than a year, Grothendieck manages to solve all the problems by means of some very ingenious constructions. By the time of his doctorate, Grothendieck holds 6 papers, each of which could make a very good doctoral thesis. The thesis, dedicated to his mother[2]:

Produits tensoriels topologiques et espaces nucléaires

HANKA GROTHENDIECK in Verehrung und Dankbarkeit gewidmet

is ready in 1953. This dissertation, published in 1955 in the *Memoirs of the Amer. Math. Soc.* [18] [3], is generally considered one of the most important events in the post-war functional analysis[4]. The years 1950–1955 mark the period of Grothendieck's most intensive work on functional analysis. In his early papers (written at the age of about 22) Grothendieck poses many questions concerning the structure of locally convex linear topological spaces, particularly the complete linear metric spaces. Some of them are related to the theory of linear partial differential equations and analytic function spaces. The Schwartz kernel theorem leads Grothendieck to distinguishing the class of *nuclear spaces*[5]. Roughly speaking, the kernel theorem asserts that "decent" operators on distributions are distributions themselves, which Grothendieck expressed abstractly as an isomorphism of certain injective and projective tensor products. The main difficulty in introducing the theory of nuclear spaces is the problem of equivalence of two interpretations of kernels: as elements of tensor products, and as linear operators (in the case of finite dimensional spaces, matrices are in one-to-one correspondence with linear transformations). This leads to the so-called *approximation problem* (a version of which was first posed in S. Banach's famous monograph [B]), whose deep study takes a considerable part of the Red Book. Grothendieck discovers many beautiful equivalences (some of the implications were earlier known to S. Banach and S. Mazur); in particular, he shows that the approximation problem is equivalent to problem 153 from the *Scottish Book* [Ma] posed by Mazur, and that, for reflexive spaces, the approximation property

[2] Grothendieck was exceptionally attached to his mother, with whom he spoke in German. She wrote poems and novels (presumably her best known work is an autobiographical novel *Eine Frau*).

[3] A complete list of Grothendieck's mathematical publications is contained in [C-R], vol. 1, pp. xiii–xx. When citing a Grothendieck's publication here, we refer to an item on that list.

[4] And called *Grothendieck's (little) Red Book*.

[5] All his life Grothendieck has been a fervent pacifist. He believed that the term "nuclear" should only be used to describe abstract mathematical objects. During the Vietnam war he taught a course on the theory of categories in a forest near Hanoi the same time that Americans were bombarding the city.

is equivalent to the so-called metric approximation property. Nuclear spaces are also related to the following 1950 Dvoretzky–Rogers theorem (solving problem 122 from [Ma]): In every infinite dimensional Banach space, there exists an unconditionally convergent series that is not absolutely convergent. Grothendieck showed that the nuclear spaces are precisely those for which unconditional convergence is equivalent to the absolute convergence of a series (see [Ma, problem 122 and remarks]). The fundamental importance of nuclear spaces comes from the fact that almost all non-Banach locally convex spaces naturally occurring in analysis are nuclear. We mean here various spaces of smooth functions, distributions, or holomorphic functions with their natural topologies – in many cases their nuclearity was shown by Grothendieck himself.

Another important result of the Red Book is the equivalence of the product definition of nuclear spaces with their realization as inverse limits of Banach spaces with morphisms being nuclear or absolutely summable operators (which Grothendieck calls *left semi-integral* operators). His study of various classes of operators (Grothendieck has been the first to define them in a functorial way, in the spirit of the theory of categories) yields deep results that gave rise to the modern, so-called *local theory* of Banach spaces. The results are published in two important papers [22, 26] in *Bol. Soc. Mat. São Paulo*, during his stay in that city (1953–1955). He shows there, in particular, that operators from a measure space into a Hilbert space are absolutely summable (a fact analytically equivalent to the so-called *Grothendieck inequality*), and makes a conjecture concerning a central problem in the theory of convex bodies, solved by A. Dvoretzky in 1959. Many very difficult questions posed in those papers were later solved by: P. Enflo (negative resolution of the approximation problem, in 1972), B. Maurey, G. Pisier, J. Taskinen («problème des topologies» on bounded sets in tensor products), U. Haagerup (non-commutative analogue of Grothendieck's inequality for C^*-algebras), J. Bourgain – a Fields medalist, and indirectly influenced the results of another "Banach" Fields medalist, T. Gowers. Supposedly, of all the problems posed by Grothendieck in functional analysis, there is only one left open to these days, see [PB, 8.5.19].

To sum up, Grothendieck's contributions to functional analysis include: nuclear spaces, topological tensor products, Grothendieck inequality, relations with absolutely summable operators, and ... many other dispersed results.[6]

In 1955 Grothendieck's mathematical interests shift to homological algebra. This is a time of the triumph of homological algebra as a powerful tool in algebraic topology, due to the work of H. Cartan and S. Eilenberg. During his stay at the University of Kansas in 1955, Grothendieck constructs his axiomatic theory of *Abelian categories*. His main result asserts that the sheaves of modules form an Abelian category with sufficiently many injective objects, which allows one to define cohomology with values in such a sheaf without any constraints on the sheaf or the base space (the theory appears in [28]).

After homological algebra, Grothendieck's curiosity directs towards alge-

[6]The above information about Grothendieck's contribution to functional analysis comes mostly from [P].

braic geometry – to a large extent due to the influence of C. Chevalley and J-P. Serre. Grothendieck considers the former a great friend of his, and in later years participates in his famous seminar in the ENS, giving a number of talks on algebraic groups and intersection theory [81–86]. He also exploits J-P. Serre's extensive knowledge of algebraic geometry, asking him numerous questions (recently, the French Mathematical Society published an extensive selection of their correspondence [CS]; this book can teach more algebraic geometry than many monographs). Serre's paper [S1], building the foundations of the theory of sheaves and their cohomology in algebraic geometry, is of key importance to Grothendieck.

One of Grothendieck's first results in algebraic geometry is a classification of holomorphic bundles over the Riemann sphere [25]. It says that every such bundle is the direct sum of a certain number of tensor powers of the tautological line bundle. Some time after this publication it turned out that other "incarnations" of this result were much earlier known to mathematicians such as G. Birkhoff, D. Hilbert, as well as R. Dedekind and H. Weber (1892). This story shows, on the one hand, Grothendieck's enormous intuition for important problems in mathematics, but on the other hand, also his lack of knowledge of the classical literature. Indeed, Grothendieck wasn't a bookworm; he preferred to learn mathematics through discussions with other mathematicians. Nonetheless, this work of Grothendieck initiated systematic studies on the classification of bundles over projective spaces and other varieties.

Algebraic geometry absorbs Grothendieck throughout the years 1956–1970. His main motive at the beginning of this period is transformation of "absolute" theorems (about varieties) into "relative" results (about morphisms). Here is an example of an absolute theorem[7]:

If X is a complete variety and $\mathcal{F}$ is a coherent sheaf on X, then $\dim H^j(X,\mathcal{F}) < \infty$.

And this is its relative version:

If $f\colon X \to Y$ is a proper morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{R}^j f_* \mathcal{F}$ is coherent on Y.

Grothendieck's main accomplishment of that period is concerned with the *relative Hirzebruch-Riemann-Roch* theorem. The original problem motivating the work on this topic can be formulated as follows: given a connected smooth projective variety X and a vector bundle E over X, calculate the dimension $\dim H^0(X,E)$ of the space of global sections of E. The great intuition of Serre told him that the problem should be reformulated using higher cohomology groups as well. Namely, Serre conjectured that the number

$$\sum (-1)^i \dim H^i(X,E)$$

[7] In the rest of this paper we will use some standard algebraic geometry notions and notation (see [H]). Unless otherwise implied, by a *variety* we will mean a complex algebraic variety. Cohomology groups of such a variety – unless otherwise specified – will have coefficients in the field of rational numbers.

could be expressed in terms of topological invariants related to X and E. Naturally, Serre's point of departure was a reformulation of the classical Riemann-Roch theorem for a curve X: given a divisor D and its associated line bundle $\mathcal{L}(D)$,

$$\dim H^0(X, \mathcal{L}(D)) - \dim H^1(X, \mathcal{L}(D)) = \deg D + \frac{1}{2}\chi(X)\,.$$

(An analogous formula for surfaces was also known.)

The conjecture was proved in 1953 by F. Hirzebruch, inspired by earlier ingenious calculations of J.A. Todd. Here is the formula discovered by Hirzebruch for an n-dimensional variety X:

$$\sum (-1)^i \dim H^i(X, E) = \deg(\mathrm{ch}(E)\mathrm{td}\, X)_{2n}\,, \tag{*}$$

where $(-)_{2n}$ denotes the degree $2n$ component of an element of the cohomology ring $H^*(X)$, and

$$\mathrm{ch}(E) = \sum e^{a_i} \quad , \quad \mathrm{td}\, X = \prod \frac{x_j}{1 - e^{-x_j}}$$

(where the a_i are the Chern roots of E [8], and the x_j are the Chern roots of the tangent bundle TX).

To formulate a relative version of this result, let a proper morphism $f\colon X \to Y$ between smooth varieties be given. We want to understand the relationship between

$$\mathrm{ch}_X(-)\mathrm{td}X \quad \text{and} \quad \mathrm{ch}_Y(-)\mathrm{td}Y,$$

"induced" by f. In the case of $f\colon X \to Y = \mathit{point}$, we should obtain the Hirzebruch-Riemann-Roch theorem. The relativization of the right-hand side of (*) is easy: there exists a well defined additive mapping of cohomology groups $f_*\colon H(X) \to H(Y)$, and $\deg(z)_{2n}$ corresponds to $f_*(z)$ for $z \in H(X)$. What about the left-hand side of (*)? The relative version of the $H^j(X, \mathcal{F})$ are the coherent modules $\mathcal{R}^j f_* \mathcal{F}$, vanishing for $j \gg 0$. In order to construct a relative version of the alternating sum, Grothendieck defines the following group $K(Y)$ (now called the *Grothendieck group*): It is the quotient group of a "very large" free Abelian group generated by the isomorphism classes $[\mathcal{F}]$ of coherent sheaves on Y, modulo the relation

$$[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$$

for each exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \tag{**}$$

The group $K(Y)$ has the following universal property: every mapping φ from $\bigoplus \mathbb{Z}[\mathcal{F}]$ to an Abelian group, satisfying

$$\varphi([\mathcal{F}]) = \varphi([\mathcal{F}']) + \varphi([\mathcal{F}'']), \tag{***}$$

[8] These are the classes of divisors associated with line bundles, splitting E (see [H, p. 430]).

factors through $K(Y)$. In our situation, we define

$$\varphi([\mathcal{F}]) := \sum(-1)^j[\mathcal{R}^j f_* \mathcal{F}] \in K(Y)\,.$$

Observe that (***) follows from the long exact sequence of derived functors

$$\cdots \longrightarrow \mathcal{R}^j f_* \mathcal{F}' \longrightarrow \mathcal{R}^j f_* \mathcal{F} \longrightarrow \mathcal{R}^j f_* \mathcal{F}'' \longrightarrow \mathcal{R}^{j+1} f_* \mathcal{F}' \longrightarrow \cdots\,,$$

associated with the short exact sequence (**) (see [H, Chap. III]). Thus, we obtain an additive mapping

$$f_! : K(X) \to K(Y).$$

Now the *relative* Hirzebruch-Riemann-Roch theorem, discovered by Grothendieck ([102], [BS]) and being a sign of his genius, asserts the commutativity of the diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{f_!} & K(Y) \\ {\scriptstyle \mathrm{ch}_X(-)\mathrm{td}\,X}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_Y(-)\mathrm{td}\,Y} \\ H(X) & \xrightarrow{f_*} & H(Y)\,. \end{array}$$

(Note that due to its additivity, the Chern character $\mathrm{ch}(-)$ is well defined in K-theory.) More information about various aspects of the intersection theory, of which the ultimate result is the above Grothendieck-Riemann-Roch theorem, can be found in [H, Appendix A] [9]. The theorem has been applied in many specific calculations of characteristic classes.

Grothendieck's group K spurred the development of K-theory, marked with the works of D. Quillen and many others. Note that K-theory plays an important role in many areas of mathematics, from the theory of differential operators (the Atiyah-Singer theorem) to the modular representation theory of finite groups (the Brauer theorem).

Following this spectacular result, Grothendieck is proclaimed a "superstar" of algebraic geometry, and invited to the International Congress of Mathematicians in Edinburgh in 1958, where he sketches a program to define a cohomology theory for positive characteristics that should lead to a proof of the Weil conjectures, see [32]. The Weil conjectures [W] suggested deep relations between the arithmetic of algebraic varieties over finite fields, and the topology of complex algebraic varieties. Let $k = \mathbb{F}_q$ be a finite field with q elements, and let $\bar{k}$ be its algebraic closure. Fix a finite collection of homogeneous polynomials in $n+1$ variables with coefficients in k. Let X (resp. $\bar{X}$) be the zero-set of this collection in the n-dimensional projective space over k (resp. $\bar{k}$). Denote by N_r the number of points in $\bar{X}$ whose coordinates lie in the field $\mathbb{F}_{q^r}$ with q^r elements,

[9] In fact, the Grothendieck-Riemann-Roch theorem was proved for varieties over any algebraically closed field (of arbitrary characteristic) by taking the values of the Chern character in the Chow rings (cf. [102], [BS]).

$r = 1, 2, \ldots$. "Organize" the N_r into a generating function, called the *zeta function of* X:

$$Z(t) := \exp\Big(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\Big).$$

The Weil conjectures, for a smooth variety X, concern the properties of $Z(t)$, as well as the relations with the classical Betti numbers of the complex variety "associated" with X. The formulation of the Weil conjectures can be found in 1.1–1.4 of [H, Appendix C], or W1–W5 of [M, Chap. VI, § 12] (both lists begin with the conjecture on *rationality* of the zeta function $Z(t)$). The above sources also contain some introductory information about the Weil conjectures, as well as an account of the struggle for their proof, which (besides Weil and the Grothendieck school) involved mathematicians such as B. Dwork, J-P. Serre, S. Lubkin, S. Lang, Yu. Manin, and many others.

The Weil conjectures become the main motivation for Grothendieck's work in algebraic geometry during his stay at the IHES[10]. He begins working at the IHES in 1959, and soon under his charismatic leadership, emerges the *Séminaire de Géométrie Algébrique du Bois-Marie* (after the wood surrounding the IHES). For the next decade, the seminar will become the world's "capital" of algebraic geometry. Working on mathematics 12 hours a day, Grothendieck generously shares his ideas with his co-workers. The atmosphere of this exceptional seminar has been captured in an interview [Du] with one of Grothendieck's students, J. Giraud. Let us concentrate now on the main ideas explored by Grothendieck at the IHES[11].

Schemes are objects that allow for unification of geometry, commutative algebra, and number theory. Let X be a set, and let F be a field. Consider the ring

$$F^X = \{\text{functions } f : X \to F\}$$

with multiplication defined pointwise. For $x \in X$, define $\alpha_x \colon F^X \to F$ by $f \mapsto f(x)$. The kernel of α_x being a maximal ideal, we can identify X with the set of all maximal ideals in F^X. Thus, we replace a simpler object, X by a more complicated one, which is the set of all maximal ideals in F^X. Variants of this idea appeared in the work of M. Stone on the theory of Boolean lattices, as well as in papers of I.M. Gelfand on commutative Banach algebras. In commutative algebra, similar ideas were first exploited by M. Nagata and E. Kähler. In the late 50's, many mathematicians in Paris (Cartan, Chevalley, Weil, ...) intensively searched for a generalization of the concept of variety over an algebraically closed field.

Serre showed that the notion of localization of a commutative ring leads to a sheaf over the maximal spectrum Specm of an (arbitrary) commutative ring. Note that the mapping $A \to \mathrm{Specm}(A)$ is not a functor (the inverse image of a

[10] IHES = Institut des Hautes Études Scientifiques: mathematical research institute in Bures-sur-Yvette near Paris – a fantastic location for doing mathematics, also thanks to its lovely canteen that will probably never run out of bread and wine.

[11] See also [D] for a more detailed account of the theory of schemes.

maximal ideal need not be maximal). On the other hand,

$$A \to \mathrm{Spec}(A) := \{\text{prime ideals in } A\}$$

is a functor. It seems that it was P. Cartier who in 1957 first proposed the following: *a proper generalization of the classical algebraic variety is a ringed space* $(X, \mathcal{O}_X)$ *locally isomorphic to* $\mathrm{Spec}(A)$ (although it was a result of speculations of many algebraic geometers). Such an object was called a *scheme*.

The music pavilion of the IHES, Bures-sur-Yvette;
venue of the first algebraic geometry seminars.

Grothendieck was planning to write a 13-volume course in algebraic geometry EGA[12] based on the concept of schemes and culminating in the proof of the Weil conjectures. He managed to publish 4 volumes, written together with Dieudonné. But in fact, most of the material to appear in the later volumes was covered by SGA[13] – publications of the algebraic geometry seminar at the IHES. (The text [H], to which we often refer here, is a didactic recapitulation of the most useful parts of EGA concerning schemes and cohomology.)

Let us now turn to constructions in algebraic geometry that make use of representable functors. Fix an object X in the category $\mathcal{C}$. We associate with it a contravariant functor from $\mathcal{C}$ to the category of sets,

$$h_X(Y) := \mathrm{Mor}_{\mathcal{C}}(Y, X).$$

At first sight, it is hard to see any use of such a simple assignment. However, the knowledge of this functor gives us a unique (up to isomorphism) object X that "represents" it (a fact known as the *Yoneda Lemma*). It is thus natural to

[12] EGA –*Éléments de Géométrie Algébrique*, published by the Publ. IHES and Springer Verlag [57–64].

[13] SGA – *Séminaire de Géometrie Algébrique*, published by the Springer Lecture Notes in Mathematics and (SGA 2) by North-Holland [97–103].

make the following definition: A contravariant functor from $\mathcal{C}$ to the category of sets is called *representable* (by X) if it is of the form h_X for some object X in $\mathcal{C}$. Grothendieck masterfully exploits the properties of representable functors to construct various *parameter spaces.* Such spaces are often encountered in algebraic geometry, a key example being the *Grassmannian* parametrizing linear subspaces of a given dimension in a given projective space. A natural question is whether there exist more general schemes parametrizing subvarieties of a given projective space, and having certain fixed numerical invariants.

Let S be a scheme over a field k. A *family of closed subschemes of* $\mathbb{P}^n$ *with the base* S is a closed subscheme $X \subset \mathbb{P}^n \times_k S$ together with the natural morphism $X \to S$. Fix a numerical polynomial P. Grothendieck considers the functor Ψ^P from the category of schemes to the category of sets, that assigns to S the set $\Psi^P(S)$ of flat families of closed subschemes of $\mathbb{P}^n$ with base S and Hilbert polynomial P. If $f\colon S' \to S$ is a morphism, then

$$\Psi^P(f)\colon \Psi^P(S) \to \Psi^P(S')$$

assigns to a family $X \to S$ the family $X' = X \times_S S' \to S'$. Grothendieck proves that the functor Ψ^P is representable by a scheme (called a *Hilbert scheme*) that is projective [74][14]. This is a (very) ineffective result – for example, estimating the number of irreducible components of the Hilbert scheme of curves in 3-dimensional projective space, with a given genus and degree, is still an open problem. Nonetheless, in numerous geometric considerations it suffices to know that such an object exists, which makes this theorem of Grothendieck useful in many applications. More generally, Grothendieck constructs a so-called *Quot-scheme* parametrizing (flat) quotient sheaves of a given coherent sheaf, with a fixed Hilbert polynomial [73]. Quot-schemes enjoy many applications in constructions of moduli spaces of vector bundles. Yet another scheme, constructed by Grothendieck in the same spirit, is the *Picard scheme* [75, 76].

In 1966 Grothendieck receives the Fields Medal for his contributions to functional analysis, the Grothendieck-Riemann-Roch theorem, and the work on the theory of schemes (see [S2]).

The most important subject of Grothendieck's research at the IHES is, however, the theory of *étale cohomology.* Recall that, for the purpose of the Weil conjectures, the issue is to construct an analogue of the cohomology theory of complex varieties for algebraic varieties over a field of positive characteristic (but with coefficients in a field of characteristic zero, so that one could count the fixed points of a morphism as a sum of traces in cohomology groups, à la Lefschetz). Earlier efforts to exploit the classical topology used in algebraic geometry – the Zariski topology (closed subsets = algebraic subvarieties), turned out unsuccessful, the topology being "too poor" for homological needs. Grothendieck observes that a "good" cohomology theory can be built by considering a *variety together with all its unramified coverings* (see [32] for details on the context of

[14] In fact, Grothendieck proves a much more general result for projective schemes over a base Noetherian scheme.

this discovery). This is the beginning of the theory of étale topology, developed together with M. Artin and J-L. Verdier. Grothendieck's brilliant idea was the revolutionary generalization of the notion of topology, differing from the classical topological space in that the "open sets" need not be all contained in the same set, but do have some basic properties that allow one to build a "satisfactory" cohomology theory of sheaves.

Alexander Grothendieck

The origins of these ideas are sketched in the following discussion of Cartier [C1]. When using sheaves on a variety X or studying cohomology of X with coefficients in sheaves, the key role is played by the lattice of open subsets of X (the points of X being of secondary importance). In our considerations, we can thus, without any harm, "replace" the variety by the lattice of its open subsets. Grothendieck's idea was to adapt B. Riemann's concept of multivalued holomorphic functions that actually "live" not on open subsets of the complex plane, but rather on suitable Riemann surfaces that cover it (Cartier uses a suggestive term «les surfaces de Riemann étalées»). Between these Riemann surfaces there are projections, and hence they form objects of a certain category. A lattice is an example of a category in which between any two objects there is at most one morphism. Grothendieck suggests then to replace the lattice of open sets with the category of open étale sets. Adapted to algebraic geometry, this concept allows one to resolve the fundamental difficulty of the lack of an implicit function theorem for algebraic functions. Also, it allows us to view the étale sheaves in a functorial way.

To continue our discussion in a more formal way, suppose that a category $\mathcal{C}$ is given, which admits fibre products. A *Grothendieck topology* on $\mathcal{C}$ is an assignment to every object $X \in C$ of a set $\mathrm{Cov}(X)$ of a families of morphisms $\{f_i\colon X_i \to X\}_{i\in I}$, called the *coverings* of X, satisfying the following conditions:

1) $\{id\colon X \to X\} \in \mathrm{Cov}(X)$;

2) if $\{f_i\colon X_i \to X\} \in \mathrm{Cov}(X)$, then, induced by a base change $Y \to X$, the family $\{X_i \times_X Y \to Y\}$ belongs to $\mathrm{Cov}(Y)$;

3) if $\{X_i \to X\} \in \mathrm{Cov}(X)$ and, for all i, $\{X_{ij} \to X_i\} \in \mathrm{Cov}(X_i)$, then the bi-indexed family $\{X_{ij} \to X\}$ belongs to $\mathrm{Cov}(X)$.

If $\mathcal{C}$ admits direct sums – and let us suppose so – then a family $\{X_i \to X\}$ can be replaced with a single morphism

$$X' = \coprod_i X_i \to X\,.$$

Having coverings, one can consider sheaves and their cohomology. A contravariant functor F from $\mathcal{C}$ to the category of sets is called a *sheaf of sets* if, for every covering $X' \to X$, have

$$F(X) = \{s' \in F(X') : p_1^*(s') = p_2^*(s')\}\,,$$

where p_1, p_2 are the two projections from $X' \times_X X'$ onto X'. A *canonical topology* in the category $\mathcal{C}$ is the topology "richest in coverings" in which all the representable functors are sheaves. If in turn, every sheaf in a canonical topology is a representable functor, then the category $\mathcal{C}$ is called a *topos*. More information about the Grothendieck topologies can be found for instance in [BD].

Let us return to geometry. Very importantly: the above f_i need not be embeddings! The most significant example of a Grothendieck topology is the *étale topology*, where the $f_i\colon X_i \to X$ are étale morphisms[15] that induce a surjection $\coprod_i X_i \to X$. Grothendieck's cohomological machinery applied to this topology yields the construction of the étale cohomology $H^i_{\text{ét}}(X, -)$. Although the basic ideas are relatively simple, the verification of many technical details regarding the properties of étale cohomology required many years of hard work, which involved the "cohomological" students of Grothendieck: P. Berthelot, P. Deligne, L. Illusie, J-P. Jouanolou, J-L. Verdier, and others, successively filling up the details of new results sketched by Grothendieck. The results of the Grothendieck school's work on étale cohomology are published in [100][16].

The proof of the Weil conjectures required a certain variant of étale cohomology – the *l-adic cohomology*. Its basic properties, particularly a *Lefschetz-type formula*, allowed Grothendieck to prove some of the Weil conjectures, but the most difficult one – the analogue of the Riemann Hypothesis – remained unsolved. In the process of proving the conjecture, Grothendieck has played a role similar to that of the biblical Moses, who led the Israelis off Egypt and towards the Promised Land, was their guide for the most part of the trip, but was not supposed to reach the goal himself. In the case of the Weil-Riemann conjecture, the goal was reached by Grothendieck's most brilliant student – Deligne [De]. (Grothendieck's plan to prove the Weil-Riemann conjecture by first proving the so-called *standard conjectures* has not been realized to these days – the conjectures are discussed in [44].)

[15] These are smooth morphisms of relative dimension zero. For smooth varieties, étale morphisms are precisely those that induce isomorphisms of the tangent spaces at all points – naturally, such morphisms need not be injective. A general discussion of étale morphisms can be found in [M].

[16] A didactic exposition of étale cohomology can be found in [M].

In 1970 Grothendieck accidentally discovers that the IHES finances are partially supported by military sources, and leaves the IHES instantly. He receives a prestigious position at the Collège de France, however by that time (Grothendieck is about 42) there are things that interest him more than mathematics: one has to save the endangered world! Grothendieck cofounds an ecological group called *Survivre et Vivre (Survive and Live)*. In this group he is accompanied by two outstanding mathematicians and friends: C. Chevalley and P. Samuel. The group publishes in 1970–1975 a magazine under the same name. Typically for his temperament, Grothendieck engages wholly in this activity, and soon his lectures at the Collège de France have little to do with mathematics, concerning instead the issues like ... how to avoid the world war and live ecologically. Consequently, Grothendieck needs to find himself a new job. He receives an offer from his "home" university in Montpellier, and soon settles down on a farm near the city and works as an "ordinary" professor (with teaching duties) at the university. Working in Montpellier, Grothendieck writes a number of (long) sketches of new mathematical theories in an effort to obtain a position in the CNRS[17] and talented students from the ENS to work with. He "receives" no students, but for the last four years before retirement (at the age of 60) is employed by the CNRS. The sketches are currently being developed by several groups of mathematicians; it is a good material for a separate article.

In Montpellier Grothendieck writes also his mathematical memoirs *Récoltes et Semailles (Harvests and Sowings)* [G1], containing marvellous pieces about his perspectives on mathematics, about "male" and "female" roots in mathematics, and hundreds of other fascinating things. The memoirs contain also a detailed account of Grothendieck's relationship with the mathematical community, as well as a very critical judgement of his former students ... . But let us talk about more pleasant things. Speaking of a model mathematician, Grothendieck without hesitation names E. Galois. Of the more contemporary scientists, Grothendieck very warmly recalls J. Leray, A. Andreotti, and C. Chevalley. It is symptomatic how greatly important to Grothendieck is the human aspect of his contacts with other mathematicians. He writes in [G1]:

If, in «Récoltes et Semailles» I'm addressing anyone besides myself, it isn't what's called a "public". Rather I'm addressing that someone who is prepared to read me as a person, and as a solitary person.

Maybe it was the loneliness experienced in all his life that made him so sensitive about it?

In 1988 Grothendieck refuses to accept a prestigious Crafoord Prize, awarded to him, jointly with Deligne, by the Royal Swedish Academy of Sciences (huge money!). Here is a quote of the most important, in my opinion, part of Grothendieck's letter to the Swedish Academy in regard to the prize (see [G2]):

I am convinced that time is the only decisive test for the fertility of new ideas or views. Fertility is measured by offspring, not by honors.

[17] CNRS – Centre National de la Recherche Scientifique, French institution employing scientists without formal didactic duties.

Let us add that the letter contains also his extremely critical opinion on the professional ethic of the mathematical community of the 70's and 80's ... .

It is time for some summary. Here are the 12 most important topics of Grothendieck's work in mathematics, reproduced after [G1]:

1. Topological tensor products and nuclear spaces;

2. "Continuous" and "discrete" dualities (derived categories, the "six operations");

3. The Riemann-Roch-Grothendieck yoga (K-theory and its relationship to intersection theory);

4. Schemes;

5. Topos theory;

(Toposes, as pointed out before, realize (as opposed to schemes) a "geometry without points" – see also [C1] and [C2]. Grothendieck "admired" toposes more than schemes. He valued most the topological aspects of geometry that led to the right cohomology theories.)

6. Étale cohomology and l-adic cohomology;

7. Motives, motivic Galois groups ($\otimes$-Grothendieck categories);

8. Crystals, crystalline cohomology, yoga of the De Rham coefficients, the Hodge coefficients;

9. "Topological algebra": ∞-stacks; derivations; cohomological formalism of toposes, inspiring a new conception of homotopy;

10. Mediated topology;

11. The yoga of Anabelian algebraic geometry. Galois-Teichmüller theory;

(This point Grothendieck considered the hardest and "the deepest". Recently, important results on this subject were obtained by F. Pop.)

12. Schematic or arithmetic viewpoints on regular polyhedra and in general all regular configurations.

(This subject was developed by Grothendieck after moving from Paris to Montpellier, in his spare time at a family vineyard.)

The work of numerous mathematicians who carried on 1.–12. has made up a significant chunk of the late XX century mathematics. Many of the Grothendieck's ideas are being actively developed nowadays and will certainly have a significant impact on the mathematics of the XXI century.

Let us name the most important continuators of Grothendieck's work (among them, a few Fields medalists):

1. P. Deligne: complete proof of the Weil conjectures in 1973 (to a large extent based on techniques of SGA);

2. G. Faltings: proof of the Mordell conjecture in 1983;

3. A. Wiles: proof of Fermat's Last Theorem in 1994;

(it is hard to imagine 2. and 3. without EGA)

4. V. Drinfeld, L. Lafforgue: proof of the Langlands reciprocity for general linear groups over function fields;

5. V. Voevodsky: theory of motives and proof of Milnor's conjecture.

The last point is related to the following Grothendieck's "dream": there should exist an "Abelianization" of the category of algebraic varieties – a category of *motives* together with the *motivic cohomology*, from which one could read the Picard variety, the Chow groups, etc. A. Suslin and V. Voevodsky constructed motivic cohomology satisfying the postulates of Grothendieck.

In August 1991 Grothendieck suddenly abandons his house and, without a word, leaves to an unknown location somewhere in the Pyrenees. He devotes himself to philosophical meditations (free choice, determinism, and the existence of ... the devil in the world); earlier, he wrote an interesting text *La clef des songes* describing his argument for the existence of God based on a dream analysis, and writes texts on physics. He wishes *no* contacts with the outside world.

We come to the end. Here is a handful of reflections.

The following words of Grothendieck, from [G1], describe what interested him most in mathematics:

That is to say that, if there is one thing in mathematics which (no doubt this has always been so) fascinates me more than anything else, it is neither "number", nor "magnitude" but above all "form". And, among the thousand and one faces that form chooses in presenting itself to our attention, the one that has fascinated me more than any other, and continues to fascinate me, is the structure buried within mathematical objects.

It is truly amazing that resulting from this reflection of Grothendieck on the "form" and "structure" are theories that provide tools (of unparalleled precision) for calculating *specific* numerical quantities and finding *explicit* algebraic relations. An example of such a tool in algebraic geometry is the Grothendieck-Riemann-Roch theorem. Another, less known example, is the language of *Grothendieck's* λ*-rings* [102], that allows one to treat *symmetric functions* as operators on polynomials. This in turn provides a uniform approach to numerous classical polynomials (e.g. symmetric, orthogonal) and formulas (e.g. interpolation formulas or those of the representation theory of general linear groups and symmetric groups). The polynomials and formulas are often related to the famous names such as: E. Bézout, A. Cauchy, A. Cayley, P. Chebyshev, L. Euler, C.F. Gauss, C.G. Jacobi, J. Lagrange, E. Laguerre, A-M. Legendre, I. Newton, I. Schur, T.J. Stieltjes, J. Stirling, J.J. Sylvester, J.M. Hoene-Wroński, and others. What's more, the language of λ-rings allows one to establish useful algebro-combinatorial generalizations of the results of these classics, see [L]. The work of Grothendieck shows that there is no essential dichotomy between the quantitative and qualitative aspects of mathematics.

Undoubtedly, Grothendieck's point of view explained above helped him to accomplish the enormous work towards the *unification* of important subjects in geometry, topology, arithmetic, and complex analysis. It also relates to Grothendieck's fondness for studying mathematical problems in their full generality.

Grothendieck's work style is well described in the following tale of his, from [G1]. Suppose one wants to prove a conjecture. There are two extreme methods to do this. First: by force. As with opening a nut: one cracks the shell with a nutcracker and gets to the fruit inside. But there is also another way. One can put a nut into a softening liquid and wait patiently until it suffices to gently press the shell and it opens all by itself. Anyone who read Grothendieck's works would have no doubt that it was the latter approach he used when working on mathematics. Cartier [C1] gives a yet more suggestive characterization of this method: it is the Joshua way of conquering Jericho. One wants to get to Jericho guarded by tall walls. If one compasses the city sufficiently many times, thus weakening their construction (by resonance), then eventually it will suffice to blow with the trumpets and shout with a great shout and ... the walls of Jericho shall fall down flat!

Let us share the following piece of advice, especially with young mathematicians. Grothendieck highly valued writing down his mathematical considerations. He regarded the process of writing and editing of mathematical papers itself an *integral* part of the research work, see [He].

Finally, let us listen to Dieudonné, a faithful witness of Grothendieck's work, and a mathematician of an immense encyclopedic knowledge. He wrote (see [D]) on the occasion of Grothendieck's 60'th birthday (that is, some 15 years ago):

There are few examples in mathematics of a theory that monumental and fruitful, done by a single man in such a short time.

He is accompanied by the editors of *The Grothendieck Festschrift* [C-R] (where [D] was published), who say in the introduction:

It is difficult to grasp fully the magnitude of Alexander Grothendieck's contribution to and influence on twentieth century mathematics. He has changed the very way we think about many branches in mathematics. Many of his ideas, revolutionary when introduced, now seem so natural as to have been inevitable. Indeed, there is a whole new generation of mathematicians for whom these ideas are part of the mathematical landscape, a generation who cannot imagine that Grothendieck's ideas were ever absent.

During the preparation of this article I asked a couple of my French friends whether Grothendieck was still alive. Their answers could be summarized as follows: "Unfortunately, the only news we will have about Grothendieck will be the notice of his death. Since we still haven't got any, he must be alive." On March 28, 2004 Grothendieck turned 76.

The bibliography of Grothendieck's work is huge and obviously stretches beyond the scope of this modest exposition. We cite only those bibliographical items to which we refer directly in the text. One can find there more detailed references to papers of Grothendieck and other authors writing about him and his work. We heartily recommend visiting the website of the Grothendieck Circle:

`http://www.grothendieck-circle.org/`

containing much interesting mathematical and biographical material about Grothendieck and his parents.

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Piotr Pragacz
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
e-mail: P.Pragacz@impan.gov.pl