ON ALGEBRAIC GROUPS OF TRANSFORMATIONS.*

By André Weil.

In my *Variétés abéliennes* (Hermann, Paris, 1948; quoted hereafter as VA), I gave the rudiments of a theory of algebraic group-varieties. As these have become wholly inadequate to the present state of growth of algebraic geometry, a fuller treatment of this topic will be given here.

To define a group in algebraic geometry, one simply takes over the usual definition and adds the condition that all the objects entering into it must have a meaning from the point of view of the algebraic geometer. This means that the elements of the group must be the points of algebraic varieties in finite number, that the mappings $(x, y) \rightarrow xy$ and $x \rightarrow x^{-1}$ which define the group-structure are mappings in the sense of algebraic geometry, i.e. that their graphs consist of algebraic varieties, and finally that these mappings are everywhere defined in the sense of algebraic geometry. The same can be done in an obvious manner for groups of transformations and for homogeneous spaces.

For simplicity, we consider only groups and spaces consisting of a single variety; this corresponds to the assumption of connectedness in the theory of topological groups. In § I, we shall deal with those properties of groups and transformation-spaces which are birationally invariant, giving what will eventually prove to be a birationally invariant characterization of such spaces; this is obtained by writing down the basic axioms for groups and transformation-spaces at generic points only. Our main purpose is, starting from such objects, to derive from them birationally equivalent objects which are groups and transformation-spaces in the full sense described above. The method which will be followed is very simple, and, I believe, the most natural one which could be imagined; it derives from the observation that the varieties from which one starts, even though they may not be true groups or transformation-spaces, nevertheless contain large pieces or "chunks" (more precisely, open subsets) of such spaces; to isolate these is the purpose of § II. In § III, they are pieced together, by the technique of "abstract varieties," so as to achieve the desired result; the way in which this is done at first

* Received December 27, 1954.

requires an enlargement of the groundfield. Then a modified procedure is given whereby our spaces can be exhibited as varieties over the original groundfield; according to an idea which was first applied to similar problems by Matsusaka, and again quite recently by Chow, this is done by building up suitable symmetric products by means of the Chow points of 0-cycles. Some auxiliary results, belonging to the foundations of algebraic geometry, which would have interrupted the treatment of the main topic, are dealt with in an Appendix. A further paper, to appear in this same volume, will contain various applications of the general theory.

§ I. Some birationally invariant results.

The very convenient language of the Zariski topology will be used freely in the following manner. By a *closed subset of a variety* V, or by a *closed set on* V, we shall understand any union of subvarieties of V, *other than* V; thus V itself is not a closed set on V; there is some impropriety in this (the proper words for our concept being "non-dense closed set"), but this seemed preferable to endless repetitions. An *open set on* V is defined as the complement of a closed set on V (this should properly be called a "non-empty open set"). If k is a field of definition for V, we say that a subset of V is *k-closed* if it is closed and its components are algebraic over k, and if moreover it is invariant by all automorphisms over k of the algebraic closure $\bar{k}$ of k. A *k-open set on* V is the complement of a k-closed set.

By a variety, we mean an abstract variety unless the contrary is stipulated. The final models for our groups and transformation-spaces will be constructed in §§ III-IV as abstract varieties; until then, little or nothing would be lost (and nothing would be gained) if we confined our attention to projective or to affine varieties.

1. Let V be a variety, defined over a field k. Let f be a mapping of $V \times V$ into V, defined over k. Consider the following condition on f:

(G1) *If x, y are independent generic points of V over k, and $z = f(x, y)$, then $k(x,y) = k(x,z) = k(y,z)$.*

This is equivalent to saying that $k(x) \subset k(z,y)$ and $k(y) \subset k(x,z)$. It implies that any two of the points x, y, z are independent generic points of V over k and determine the third one uniquely.

Let x, y, t be independent generic points of V over k; (G1) implies that

$(f(x,y),t)$ and $(x,f(y,t))$ are two pairs of independent generic points of V over k. Thus, if (G1) is assumed, the following condition is meaningful:

(G2) *If x, y, t are independent generic points of V over k, then:*

$$f(f(x,y),t)=f(x,f(y,t)).$$

This is of course the associativity condition (but postulated only at independent generic points) for f.

If (G1), (G2) are satisfied, we say that f is a *normal* (internal) *law of composition* on V, and that V, with this law, is a *pre-group*; f will then mostly be written as a multiplication, i. e. as xy instead of $f(x,y)$. If V, with the law f, is a pre-group over k, it is so, a fortiori, over every field K containing k. Let a variety V' be birationally equivalent to V over such a field K; let x,y be independent generic points of V over K, put $z=xy$, and call x',y',z' the generic points of V' over K which correspond to x, y, z respectively. Then $K(x')$, $K(y')$, $K(z')$ are respectively the same as $K(x)$, $K(y)$, $K(z)$, and thus, since $K(z')\subset K(x',y')$, we may write $z'=f'(x',y')$, where f' is a mapping of $V'\times V'$ into V', defined over K. One sees at once that f' satisfies (G1) and (G2); we say that it is the law of composition on V', derived from f by transfer; and we say that V', with the law f', is a pre-group *birationally equivalent* to the pre-group V with the law f. This shows that the concept of a pre-group is invariant under arbitrary birational correspondences; a pre-group can thus be studied on any model, e. g. on an affine or a projective model.

PROPOSITION 1. *Let V be a pre-group, defined over k. There is a uniquely determined mapping ϕ of V into V, which is defined over k and is such that, if we put $s^{-1}=\phi(s)$ for every s on V at which ϕ is defined, the following conditions are fulfilled whenever x, y are independent generic points on V over k:*

(i) $k(x^{-1})=k(x)$; (ii) $(x^{-1})^{-1}=x$; (iii) $y=(x^{-1})(xy)$;

(iv) $x=(xy)(y^{-1})$; (v) $(xy)^{-1}=(y^{-1})(x^{-1})$.

If we put $z=xy$, we have $k(y)\subset k(x,z)$, and therefore there is a mapping λ of $V\times V$ into V, defined over k, such that $y=\lambda(x,z)$; similarly there is a mapping μ of $V\times V$ into V, defined over k, such that $x=\mu(z,y)$. Take t generic on V over $k(x,y,z)$; put $y'=yt$, $z'=zt$; by associativity, the latter relation gives $z'=xy'$. Put $u=\mu(y,z)$; by the definition of μ, this is equivalent to $y=uz$. By (G1), this implies that $k(u,z)=k(y,z)$, and so u, z, t are independent generic points on V over k; therefore, by (G2), we have $(uz)t = u(zt)$, which can be written as $y' = uz'$; as this also shows that u, z' are generic and independent over k, the latter relation implies, by (G1), that $k(u) \subset k(y', z')$. Therefore $k(u)$ is contained in $k(y, z)$, i.e. in $k(x, y)$, and also in $k(y', z')$, i.e. in $k(x, y')$. But, by (G1), y and $y' = yt$ are generic and independent over $k(x)$, and so $k(x, y)$ and $k(x, y')$ are independent regular extensions of $k(x)$; their intersection is therefore $k(x)$, and so we have $k(u) \subset k(x)$, so that we may write $u = \phi(x)$, where ϕ is a mapping of V into V, defined over k. As x, u are no other than $\mu(z, y)$ and $\mu(y, z)$, the relation between them is symmetrical, and we have $x = \phi(u)$ and $k(x) \subset k(u)$, and so $k(x) = k(u)$. We have thus verified (i), (ii), (iii). Also, by (G1), any two of the points x, y, z in $z = xy$ determine the third one uniquely provided they are generic and independent over k; from this it follows that $u = \phi(x)$ is uniquely determined by the relation $y = uz$, and so the function ϕ is uniquely determined by (iii). From now on, write x^{-1} instead of $\phi(x)$.

Let now v be generic on V over $k(x, y)$; put $s = (xy)v = x(yv)$. As x, yv are generic and independent over k, $s = x(yv)$ is equivalent to $yv = x^{-1}s$, by (iii); and again by (iii), this is equivalent to $v = y^{-1}(x^{-1}s)$ since y and yv are generic and independent over k by (G1). By (i), the points x^{-1}, y^{-1} and v are generic and independent over k, and therefore the last relation, by (G2) can be written as $v = (y^{-1}x^{-1})s$. As $s = (xy)v$, and xy, v are generic and independent over k, this shows that $y^{-1}x^{-1} = (xy)^{-1}$, which is (v). This, with $z = xy$, can be written as $z^{-1} = y^{-1}x^{-1}$, which, by (iii), is equivalent to $x^{-1} = y(z^{-1})$; applying (v) to the latter relation, we get $x = (z^{-1})^{-1}(y^{-1})$, which, in view of (ii), gives (iv). This completes the proof.

With the same notations as above, we have $\lambda(x, z) = x^{-1}z$ and $\mu(z, y) = zy^{-1}$. One should observe, however, that the function λ may be defined at a point (s, t) of $V \times V$ without the expression $s^{-1}t$ being defined; in fact, λ may be defined at (s, t) without ϕ being defined at s. A similar remark applies to μ.

COROLLARY. *With the notations of* Prop. 1, *assume that the function* $f(x, y) = xy$ *is defined at* (x^{-1}, x). *Then the function* $x \rightarrow (x^{-1})x$ *is a constant* e, *rational over* k. *If, moreover,* f *is defined at* (e, x), *then* $ex = x$; *if it is defined at* (x, e), *then* $xe = x$.

With x, y and $z = xy$ as before, the assumption implies that $z^{-1}z$, i.e. (by Prop. 1(v)) $(y^{-1}x^{-1})z$ is defined. In the relation $(uv)z = u(vz)$, with u, v generic and independent over $k(z)$, specialize (u, v) to (y^{-1}, x^{-1}) over $k(z)$; by our assumption, the left-hand side is defined; also, u and vz get specialized to y^{-1} and $x^{-1}z$, the latter being defined and equal to y by Prop. 1 (iii). Since, by our assumption, f is defined at (y^{-1}, y), this gives $z^{-1}z = y^{-1}y$; in other words, the function $x^{-1}x$ has the same value at z and at y. As y, z are independent generic points of V over k, this implies that the function is a constant; as it is defined over k, its constant value must then be rational over k. Putting $e = x^{-1}x$ and replacing x by x^{-1}, we get, in view of Prop. 1 (i)-(ii), $e = x(x^{-1})$. Taking t generic over $k(x, y)$, specialize t to x^{-1} in the relation $t(xy) = (tx)y$; the left-hand side becomes y by Prop. 1 (iii), and the right-hand side becomes ey provided this is defined. Specializing y to x^{-1} in the same relation, we get $te = t$ provided te is defined.

2. Let V and W be two varieties, defined over a field k. Let f, g be two mappings, both defined over k, of $V \times V$ into V and of $V \times W$ into W, respectively. Consider the following conditions:

(TG1) *For a generic x over k on V, the mapping $u \rightarrow g(x, u)$ of W into W is a birational correspondence between W and W.*

This is equivalent to saying that, if x, u are independent generic points of V and of W, respectively, over k, then $k(x, g(x, u)) = k(x, u)$.

(TG2) *If x, y, u are independent generic points of V, V and W, respectively, over k, then $g(f(x, y), u) = g(x, g(y, u))$.*

If (TG1) is fulfilled, (TG2) is meaningful, since in that case $g(y, u)$ is generic on W over $k(x)$, while $f(x, y)$ is generic on V over $k(u)$ by (G1).

When (G1, 2) and (TG1, 2) are satisfied, we shall say that g is a *normal* (external) *law of composition* on W with respect to the pre-group V, and that W, with this law, is a *pre-transformation space* with respect to V; g will then mostly be written as a multiplication, i. e. as $g(x, u) = xu$; then (TG1), (TG2) appear as $k(x, xu) = k(x, u)$ and $(xy)u = x(yu)$. Just as before, we note that the concept of a normal law is independent of the field of definition, which may be enlarged at will, and that it is birationally invariant; if V' is birationally equivalent to V, and W' to W, the laws f, g can be transferred in an obvious manner to V', W'; the pair V', W', with the laws f', g' obtained from f, g by transfer, is said to be *birationally equivalent* to the pair V, W with the laws f, g. In particular, W, just as V, may be replaced by an affine model.

Take x, y, u as in (TG2); put $z = xy$, $v = yu$. By (TG1), v is generic on W over $k(x)$; so is xv, again by (TG1); therefore $x^{-1}(xv)$ is defined. But (TG2) can be written as $zu = xv$, and so we have $x^{-1}(zu) = x^{-1}(xv)$.

As x^{-1}, z and u are independent generic points of V, V and W over k, we can apply (TG2) to the left-hand side, which is therefore equal to $(x^{-1}z)u$, i. e. to yu by Prop. 1 (iii), i. e. to v. This proves $x^{-1}(xv) = v$; as x, v are independent generic points of V, W over k, this must therefore remain true for any pair of such points.

The conditions stated above may be strengthened by assuming "generic transitivity," which means the following condition:

(H) *If x, u are independent generic points of V and of W, respectively, over k, then $g(x,u)$ is a generic point of W over $k(u)$.*

In that case, we say that W is a *pre-homogeneous space* with respect to the pre-group V. This condition is equivalent to saying that the graph of g on $V \times W \times W$ has the projection $W \times W$ on $W \times W$ (in the sense of my *Foundations*; the set-theoretic projection then contains a open subset of $W \times W$, by Prop. 10 of the Appendix).

3. The following result shows that a normal law of composition may be obtained from a mapping satisfying much weaker conditions than those stated above.

PROPOSITION 2. *Let V, W be two varieties, defined over a field k. Let g be a mapping of $V \times W$ into W, defined over k, satisfying* (TG1) *and the following condition:*

(TG2′) *There are two independent generic points x, y of V over k and a generic point z of V over k such that $g(z,u) = g(x, g(y,u))$ for u generic on W over $k(x,y,z)$.*

Let $k(\bar{x})$ be the smallest field of definition containing k for the mapping $u \rightarrow g(x,u)$ of W into W, $\bar{x}$ being a generic point over k of a variety $\bar{V}$. Then one can write $\bar{x} = \phi(x)$ and $g(x,u) = \bar{g}(\bar{x},u)$, where ϕ is a mapping from V to $\bar{V}$ and $\bar{g}$ a mapping from $\bar{V} \times W$ to W, both defined over k; putting $\bar{y} = \phi(y)$, $\bar{z} = \phi(z)$, we have $k(\bar{z}) \subset k(\bar{x},\bar{y})$ and may write $\bar{z} = \bar{f}(\bar{x},\bar{y})$, where $\bar{f}$ is a mapping from $\bar{V} \times \bar{V}$ to $\bar{V}$, defined over k. Finally, $\bar{f}$ and $\bar{g}$ satisfy the conditions (G1, 2), (TG1, 2) *and define $\bar{V}$ as a pre-group and W as a pre-transformation space with respect to $\bar{V}$.*

In the first place, the smallest field of definition containing k for $u \rightarrow g(x,u)$ is contained in $k(x)$ and is therefore, by Prop. 3 of the Appendix, a finitely generated regular extension of k; this may always be written as $k(\bar{x})$, e. g. by taking $\bar{x}$ as a suitable point in an affine space, which has then a locus $\bar{V}$ over k and may be written as $\phi(x)$. As $g(x,u)$ is then rational over $k(\bar{x},u)$, it may be written as $\bar{g}(\bar{x},u)$, or more briefly as $\bar{x}u$; (TG2′) can then be written as $g(z,u)=\bar{x}(\bar{y}u)$, which shows that the function $u\rightarrow g(z,u)$ is defined over $k(\bar{x},\bar{y})$, so that $k(\bar{z})\subset k(\bar{x},\bar{y})$; we may then write $\bar{z}=\bar{f}(\bar{x},\bar{y})$, or more briefly $\bar{z}=\bar{x}\bar{y}$. It is clear that $\bar{g}$, $\bar{f}$ satisfy (TG1, 2); we have to show that $\bar{f}$ satisfies (G1, 2). By (TG1), if $v=g(x,u)$, the mapping $u\rightarrow v$ is a birational correspondence between W and W, defined over $k(\bar{x})$; its inverse must then be defined over the same field, so that we have $k(u)\subset k(\bar{x},v)$ and may write $u=h(\bar{x},v)$. Notations being as before, put $w=g(y,u)$; as this, by (TG1), is generic over $k(x)$ on W, the relation in (TG2′), which can be written as $\bar{z}u=\bar{x}w$, is equivalent to $w=h(\bar{x},\bar{z}u)$. This shows that the mapping $u\rightarrow w$ is defined over $k(\bar{x},\bar{z})$; since its smallest field of definition is $k(\bar{y})$, we get $k(\bar{y})\subset k(\bar{x},\bar{z})$. Similarly, we have $w=\bar{y}u$ and therefore $u=h(\bar{y},w)$; then the relation in (TG2′) can be written as $\bar{x}w=\bar{z}h(\bar{y},w)$, from which we conclude in the same manner that $k(\bar{x})\subset k(\bar{z},\bar{y})$. This shows that f satisfies (G1).

Now, if $\bar{x}_1$, $\bar{x}_2$ are any two generic points of $\bar{V}$ over k, there is an isomorphism σ of $k(\bar{x}_1)$ onto $k(\bar{x}_2)$ over k which maps $\bar{x}_1$ onto $\bar{x}_2$. Take u generic on W over $k(\bar{x}_1,\bar{x}_2)$, and put $u_1=\bar{x}_1u$, $u_2=\bar{x}_2u$. Then σ maps the graph of $u\rightarrow u_1$ onto the graph of $u\rightarrow u_2$. If $u_1=u_2$, these two functions coincide, and therefore, by Prop. 4 of F-IV$_2$, σ must induce the identity on the smallest field of definition of the first function. As this field is $k(\bar{x}_1)$, we have thus shown that $u_1=u_2$ implies $\bar{x}_1=\bar{x}_2$. Now let $\bar{x}$, $\bar{y}$, $\bar{t}$, u be independent generic points over k on $\bar{V}$, $\bar{V}$, $\bar{V}$, W; put $\bar{x}_1=(\bar{x}\bar{y})\bar{t}$ and $\bar{x}_2=\bar{x}(\bar{y}\bar{t})$, these being defined because $\bar{f}$ satisfies (G1). We have to show that $\bar{x}_1=\bar{x}_2$; by (G1), they are both generic over k on $\bar{V}$, and u is generic over $k(\bar{x}_1,\bar{x}_2)$ on W, so that we need only show that $\bar{x}_1u=\bar{x}_2u$. By (TG1), $\bar{x}(\bar{y}(\bar{t}u))$ is defined; by (TG2), this is the same as $(\bar{x}\bar{y})(\bar{t}u)$, which, again by (TG2), is the same as $\bar{x}_1u$ since $\bar{x}\bar{y}$, $\bar{t}$, u are independent generic points of $\bar{V}$, $\bar{V}$, W over k by (G1). Similarly $\bar{x}(\bar{y}(\bar{t}u))$ is the same as $\bar{x}((\bar{y}\bar{t})u)$ by (TG2), and this is the same as $\bar{x}_2u$ by (TG2) and (G1). This concludes the proof.

The external normal law of composition $\bar{g}$ constructed in Prop. 2 satisfies, in addition to (TG1, 2), the following condition:

(TG3) *If x is generic over k on V, $k(x)$ is the smallest field of definition containing k for the mapping $u\rightarrow g(x,u)$ of W into W.*

Whenever (TG3) is satisfied in addition to (G1, 2) (TG1, 2), we will say that V operates *faithfully* on W by g.

If V, W and a mapping g of $V \times W$ into W are given, and g satisfies (TG1, 2′, 3), then Prop. 2 shows that $(x, y) \rightarrow z$ (where x, y, z are the points of V which appear in (TG2′)) is a normal internal law of composition on V and that g is a normal external law with respect to the pre-group defined by f on V. In particular, if V, W, f and g are given and f, g satisfy (TG1, 2, 3), then f satisfies (G1, 2).

§ II. Construction of chunks.

4. Let V, W be a pre-group and a pre-transformation space and f, g the internal and external normal laws belonging to them, these being all defined over a field k; we will mostly write f, g multiplicatively, as has already been done in § I. Instead of saying that f is defined at a point (s, t) of $V \times V$, we shall frequently say that st is defined; similarly, when we say for instance that $s^{-1}((st)a)$ is defined, for s, t on V and a on W, this will mean the following: (i) f is defined at (s, t), with the value st; (ii) g is defined at (st, a), with the value $(st)a$; (iii) $x \rightarrow x^{-1}$ is defined at s, with the value s^{-1}; (iv) g is defined at $(s^{-1}, (st)a)$, with the value $s^{-1}((st)a)$. We recall that two expressions, built up from functions which are defined over k, coincide for all values of the variables for which they are both defined provided they are defined and coincide when the variables are given independent generic values over k. This applies for instance to the formulas in (G2) and (TG2) which express the associativity of f and g.

We say that V is a *group-variety* or a *group* if f is everywhere defined on $V \times V$ and $x \rightarrow x^{-1}$ is everywhere defined on V; then the corollary of Prop. 1 shows that there is a neutral element e on V with the usual properties. If V is a group, W will be called a *transformation-space* with respect to V if g is everywhere defined on $V \times W$; if, moreover, V operates transitively on W in the usual sense, i. e. if to every pair a, b on W there is an $s \in V$ such that $b = sa$, then W is called a *homogeneous space* with respect to V.

In § III, it will be shown that, to every pre-group V, there is a birationally equivalent group V', and that, to every pre-transformation space W with respect to V, there is a birationally equivalent transformation-space W' with respect to V'. The proof of this will include a proof of the fact that W is biregularly equivalent to an open subset of W' if and only if it fulfills the following condition:

(C) *If a is any point of W, and x a generic point of V over $k(a)$, then xa and $x^{-1}(xa)$ are defined.*

A pre-transformation space W with respect to V which fulfills this condition will be called a *chunk of transformation-space,* or more briefly a *chunk.*

For similar reasons, if W is a pre-homogeneous space, i. e. if g satisfies (H), we say that W is a *homogeneous chunk* if it satisfies (C) and the following:

(HC) *If a and x are as in* (C), *xa is generic over $k(a)$ on W.*

Finally, V itself will be called a *group-chunk* if it is a homogeneous chunk with respect to left-translations and if x^{-1} is everywhere defined on it, or in other words if it satisfies the following:

(GC1) *If s is any point on V, and x a generic point of V over $k(s)$, then xs and $x^{-1}(xs)$ are defined and xs is generic over $k(s)$ on V.*

(GC2) *For every s on V, s^{-1} is defined.*

PROPOSITION 3. *Call Ω the set of those points a on W such that xa and $x^{-1}(xa)$ are defined for x generic over $k(a)$ on V. Then Ω is a k-open subset of W; Ω and all k-open subsets of Ω are chunks; if $a \in \Omega$, we have $x^{-1}(xa) = a$, $k(x,a) = k(x,xa)$, and a is a point of the locus of xa over $k(a)$ on W.*

Call F the set of points on $V \times W$ where g is not defined; by Prop. 8 of the Appendix, this is a k-closed subset of $V \times W$. Let Γ be the graph of the mapping $(x,u) \rightarrow x^{-1}u$ of $V \times W$ into W, i. e. the locus of $(x,u,x^{-1}u)$ over k for x, u generic and independent over k on V, W. Call F' the k-closed subset of $V \times W \times W$ consisting of all points (x,u,v) with $(x,v) \in F$; let F'' be the union of the projections of the components of $\Gamma \cap F'$ on the product of the first two factors of $V \times W \times W$ (this being understood as in F-IV$_3$ and F-VII$_3$; F'' is the closure, in the Zariski topology, of the set-theoretic projection of $\Gamma \cap F'$ on $V \times W$; cf. Appendix, Prop. 10). It will now be shown that Ω is the same as the set Ω_1 of the points a on W such that $V \times a$ is not contained in $F \cup F''$. In fact, for xa to be defined, it is necessary and sufficient that $V \times a$ should not be contained in F; let Ω_0 be the set of points a with this property; it contains both Ω and Ω_1. If $a \in \Omega_0 - \Omega$, $x^{-1}(xa)$ is not defined; as x^{-1} is generic over $k(a)$ at the same time as x, this is equivalent to saying that $x(x^{-1}a)$ is not defined; as $x^{-1}a$ is defined, the point $(x,a,x^{-1}a)$ is then in $\Gamma \cap F'$, and therefore (x,a) is in F'', so that $V \times a \subset F''$ and $a \notin \Omega_1$. Conversely, if $a \in \Omega_0 - \Omega_1$, then (x,a) is in the projection of one of the components of $\Gamma \cap F'$, and so, if (y,u,v) is a generic point of that component over $\bar{k}$, (x,a) is a specialization of (y,u) over $\bar{k}$. As (y,u,v) is on Γ and $x^{-1}a$ is defined, v has then the unique specialization $x^{-1}a$ over $(y,u)\rightarrow(x,a)$ with respect to $\bar{k}$; therefore $(x,a,x^{-1}a)$ is in F', and so $x(x^{-1}a)$ is not defined and a is not in Ω. This proves that $\Omega=\Omega_1$; the latter set being k-open by Prop. 7 of the Appendix, Ω is k-open.

As $x^{-1}(xu)=u$ for x, u generic and independent over k on V, W, we must have $x^{-1}(xa)=a$ whenever the left-hand side is defined, and so for $a\in\Omega$ and x generic over $k(a)$; this implies that $k(a)\subset k(x,xa)$, so that $k(x,a)=k(x,xa)$. Let y be generic over $k(x,a)$ on V; then (x^{-1},xa) is a specialization of (y,xa) over $k(x,a)$; as the former point is not in F, and F is k-closed, (y,xa) is not in F, and so $y(xa)$ is defined. As yx is generic over $k(a)$ by (G1) and is therefore a generic specialization of x over $k(a)$, $(yx)a$ is defined and is a generic specialization of xa over $k(a)$. By associativity, we have $y(xa)=(yx)a$ since both sides are defined; as $x^{-1}(xa)$ is defined, it is a specialization of $y(xa)$, and therefore also of $(yx)a$ and of xa, over $k(a)$. This shows that a is a specialization of xa over $k(a)$, i.e. that it is a point of the locus of xa over $k(a)$ on W. If now Ω' is any k-open subset of Ω, then the set $C=W-\Omega'$ is k-closed, and so, if xa is in C, a must be in C; in other words, if a is in Ω', so is xa; it is then clear that Ω' is a chunk.

The locus of xa over $k(a)$ could be described as the closure of the orbit of a under V on W.

COROLLARY. *Notations being as in Prop. 3, call Ω_h the set of the points a of Ω such that W is the locus of xa over $k(a)$. Then Ω_h is k-open or empty according as W is pre-homogeneous or not. In the former case, Ω_h and all k-open subsets of Ω_h are homogeneous chunks; and, if a, b are any two points of Ω_h, there are two generic points x, y of V over $k(a,b)$ such that $xa=yb$.*

Except for the last assertion, this is an immediate consequence of Prop. 11 of the Appendix, applied to the k-open set Ω of Prop. 3. Let now a, b be in Ω_h; take x, y generic on V over $k(a,b)$, and put $u=xa$, $v=yb$. Then the loci of u and of v over $k(a,b)$ are W, and so there is an isomorphism of $k(a,b,u)$ onto $k(a,b,v)$ over $k(a,b)$ which maps u onto v; this can be extended to an isomorphism σ of $k(a,b,x)$ onto some extension of $k(a,b,v)$; then x^σ is generic on V over $k(a,b)$ and we have $u^\sigma=x^\sigma a$, i.e. $x^\sigma a=yb$, so that x^σ and y satisfy the conditions stated in the corollary.

Finally, in order to construct a group-chunk from a given pre-group V, one need only observe that the graph V_1 of the function $x\rightarrow x^{-1}$ is a subvariety of $V\times V$, birationally equivalent to V, and that, if we transfer that function to V_1, we get an everywhere biregular birational correspondence between V_1 and itself since it is the same as the function induced on V_1 by the mapping $(x, y) \rightarrow (y, x)$ of $V \times V$ onto itself. Therefore we may assume that we have started from a pre-group V on which x^{-1} was everywhere defined; had that not been the case, one would merely have had to replace V by V_1 to make it so. Call now Ω_h the set of the points $s \in V$ with the property stated in (GC1); by the corollary of Prop. 3, this is a k-open subset of V; as $x \rightarrow x^{-1}$ is an everywhere biregular mapping of V onto itself, it transforms Ω_h into a k-open set Ω_h^{-1}; then $\Omega_h \cap \Omega_h^{-1}$ is a group-chunk.

Thus we have constructed chunks for the three kinds of objects under consideration, viz., transformation-spaces, homogeneous spaces and groups. If W is a pre-transformation space, defined over k, the set W' of simple points on W is a k-open set on W; by applying Prop. 3 to W', we obtain a non-singular chunk. Similarly, one would get an everywhere normal chunk by taking for W' the set of points where W is normal, this being k-open by Corollary 3 of Prop. 8 of the Appendix. It will presently be seen that homogeneous chunks and in particular group-chunks are always non-singular, so that no special procedure is required to make them such.

By Corollary 2 of Prop. 8 of the Appendix, if one has constructed a chunk, one can at once derive from it a birationally equivalent chunk which is an affine variety; this also applies to homogeneous chunks. As to group-chunks, starting from a pre-group which we take to be an affine variety, and replacing it by the graph of the function x^{-1} on it, we get for our pre-group an affine model V on which x^{-1} is everywhere defined. Let V' be a k-open set on V which is a homogeneous chunk; let $x = (x_1, \cdots, x_m)$ be a generic point of V over k; take a polynomial P with coefficients in k which is 0 on $V - V'$ but not on V; as x^{-1} and $P(x)$ are everywhere defined functions on V, so is $P(x^{-1})$. Call V'' the locus of

$$(x_1, \cdots, x_m, 1/P(x), 1/P(x^{-1}))$$

over k in affine space; this is biregularly equivalent to the k-open subset determined on V by the inequalities $P(x) \neq 0$, $P(x^{-1}) \neq 0$. This is a group-chunk. We have thus proved the following:

PROPOSITION 4. *To every pre-homogeneous space* (resp. *pre-group*) *defined over k, there is a birationally equivalent homogeneous chunk* (resp. *group-chunk*) *which is an affine variety, defined over k. To every pre-transformation space W and every point a on W with the property stated in* (C), *there is a birationally equivalent chunk W' which is an affine variety and is such that the birational correspondence between W and W' is biregular at a; if a is simple on W, W' may be taken non-singular; if W is normal at a, W' may be taken everywhere normal.*

5. PROPOSITION 5. *Let V be a group-chunk and W a pre-transformation space with respect to V; let k be a field of definition for V, W. Then, if s is any point of V, and u a generic point of W over $k(s)$, su and $s^{-1}(su)$ are defined; the mapping $u \rightarrow su$ is a birational correspondence between W and itself; and $k(s,u) = k(s,su)$.*

Take x, y generic and independent on V over $k(s,u)$; put $y' = yx^{-1}$ and $u' = xu$; by (G1) and (TG1), y' and u' are generic and independent over k on V, W, so that $y'u'$ is defined; as we have shown $x^{-1}u'$ to be defined and equal to u, we get, by associativity, $y'u' = yu$. We now show that the expression obtained by substituting s for y in $y'u'$, i.e. in $(yx^{-1})(xu)$, is defined. In fact, since V is a group-chunk, the mapping $(z,t) \rightarrow t^{-1}z^{-1}$, where z, t are generic and independent over $k(s)$ on V, is defined at (s,t), and its value $t^{-1}s^{-1}$ at that point is generic over $k(s)$ on V; this implies that the mapping $(z,t) \rightarrow (t^{-1}z^{-1})^{-1}$ is also defined there; as this is only another expression for the mapping $(z,t) \rightarrow zt$, we conclude that the latter is defined at (s,t), i.e. that st is defined, and that st is generic over $k(s)$; substituting x^{-1} for t, this shows that $x' = sx^{-1}$ is defined and generic over $k(s)$ on V, and a fortiori that the mapping $y \rightarrow yx^{-1}$ of V into V is defined at s, with the value x'. The mapping $u \rightarrow u'$ is defined at u, with the value u' which is generic over $k(x,s)$ on W by (TG1). So x' and u' are generic and independent over k on V, W, and $x'u'$ is defined; more precisely, we have shown that the mapping $(y,u) \rightarrow y'u' = (yx^{-1})(xu)$ of $V \times W$ into W, which is defined over $k(x)$, is defined at (s,u). As this is only another expression for the mapping $(y,u) \rightarrow yu$, this implies that the latter is defined at (s,u), i.e. that su is defined, and that these mappings have the same value there, i.e. that $x'u' = su$. By (TG1), $x'u'$ is generic on W over $k(x,s)$; therefore su is generic on W over $k(s)$. But then our assumptions on s, u are also satisfied by s^{-1}, su, so that it follows from what we have already proved that $s^{-1}(su)$ is defined; its value must then be u, since $x^{-1}(xu) = u$, and so we have $k(u) \subset k(s,su)$, and therefore $k(s,u) = k(s,su)$; this means that $u \rightarrow su$ is a birational correspondence between W and W.

COROLLARY. *Assumptions being as in Prop. 5, assume also that V operates faithfully on W; let s, s' be any two points of V, and let u be generic on W over $k(s,s')$. Then $su = s'u$ implies $s = s'$.*

Take x generic over $k(s,s',u)$ on V. Since V is a group-chunk, xs is defined and generic over $k(u)$ on V; and su is defined and generic over $k(x)$ on W by Prop. 5; by associativity, this gives $(xs)u = x(su)$. Similarly we have $(xs')u = x(s'u)$. Therefore $su = s'u$ implies $(xs)u = (xs')u$. But then we can repeat the argument used in the proof of Prop. 2; there is an isomorphism σ of $k(xs)$ onto $k(xs')$, mapping xs onto xs'; as this transforms the graph of the function $u \to (xs)u$ into itself, and as $k(xs)$ is the smallest field of definition for this graph because of the assumption of faithfulness, σ must be the identity, and $xs = xs'$. As $s = x^{-1}(xs)$ and $s' = x^{-1}(xs')$, this gives $s = s'$.

PROPOSITION 6. *Let V be a group-chunk and W a chunk of transformation-space with respect to V. Let s be any point on V and (a, b) any point on the graph of the birational correspondence $u \to su$ between W and itself; then the latter is biregular at (a, b), sa and $s^{-1}b$ are defined, and we have $sa = b$, $s^{-1}b = a$.*

We first show that sa is defined. Take x, y, u generic and independent on V, V, W over $k(a, b, s)$; and consider the mapping $(x, u) \to y^{-1}((yx)u)$, defined over $k(y)$, of $V \times W$ into W. By (GC1), $x \to yx$ is defined at s, with a value ys which is generic on V over $k(s, a)$. By (C), the mapping $(x, u) \to xu$ is defined at (ys, a), and so the mapping $(x, u) \to (yx)u$ is defined at (s, a), with the value $(ys)a$. At the same time, $(ys)u$ is defined since u is generic on W over $k(y, s)$, and $y(su)$ is defined because su is defined and generic over $k(y)$ by Prop. 5; by associativity, this gives $(ys)u = y(su)$. As (a, b) is on the graph of $u \to su$, and (u, su) is a generic point of that graph over $k(y, s)$, (a, b) is a specialization of (u, su) over $k(y, s)$; but then the relation $(ys)u = y(su)$ implies $(ys)a = yb$. By Prop. 3, the mapping $v \to y^{-1}v$ is defined at yb, i.e. at $(ys)a$, with the value b. We have thus proved that $(x, u) \to y^{-1}((yx)u)$ is defined at (s, a), with the value b; as this is but an expression for $(x, u) \to xu$, this shows that sa is defined and equal to b. Interchanging a, s with b, s^{-1}, and making use of Prop. 5, we see from this that $s^{-1}b$ is defined and equal to a. This implies a fortiori that the mappings $u \to su$, $u \to s^{-1}u$ are respectively defined at a, b, with values b, a; this means that the birational correspondence $u \to su$ is biregular at (a, b).

COROLLARY. *Every homogeneous chunk is non-singular.*

Let W be such a chunk with respect to a pre-group V; replace V by a birationally equivalent group-chunk. For any a on W, take x generic on V over $k(a)$; then $u \to xu$ is a birational correspondence between W and itself, transforming a into the generic point xa of W over $k(a)$, and biregular at a. As xa is simple on W, a must therefore be simple on W.

§ III. Construction of spaces.

From now on, until the end of § III, *V and W will denote respectively a group-chunk and a chunk of transformation-space with respect to V, both being at the same time assumed to be affine varieties; k will denote a common field of definition for V and W and for the normal laws given on them.*

6. Let n, n' be the dimensions of V, W, and take $N > 4n$ and also $> 3n + n'$; take N independent generic points $t_1, \cdots, t_N$ over k on V; put $K = k(t_1, \cdots, t_N)$. Let u be a generic point of W over K; put $S_\alpha = W$ and $u_\alpha = t_\alpha u$ for $1 \leqq \alpha \leqq N$. Take the u_α as the corresponding generic points of the varieties S_α over K; this defines birational correspondences $T_{\beta\alpha}$ between any two of the S_α; as we may write, by associativity, $u_\beta = (t_\beta t_\alpha^{-1}) u_\alpha$, $T_{\beta\alpha}$ is the birational correspondence $u \rightarrow (t_\beta t_\alpha^{-1}) u$ between W and itself. Proposition 6 of § II shows that $T_{\beta\alpha}$ is biregular at any pair of points on its graph. Therefore the varieties S_α (with empty "frontiers") and the $T_{\beta\alpha}$ may be used to define an abstract variety $\boldsymbol{S}$. Call $\bar{u}$ the generic point of $\boldsymbol{S}$ over K with the representatives u_α and write $\bar{u} = \Phi(u)$, $u = \Psi(\bar{u})$; Φ is a birational correspondence between W and $\boldsymbol{S}$, and Ψ is its inverse; both are defined over K. Let a be any point of W; by Prop. 4 of the Appendix, there is an α such that t_α is generic over $k(a)$; as W is a chunk, $t_\alpha a$ is then defined; this means that Φ is defined at a, $\Phi(a)$ being the point of $\boldsymbol{S}$ with the representative $t_\alpha a$ on S_α. As $t_\alpha^{-1}(t_\alpha a)$ is then defined and has the value a, Ψ is defined at the point $\Phi(a)$ with the value a. This shows that Φ is a biregular mapping of W onto its set-theoretic image $\Phi(W)$ on $\boldsymbol{S}$; as the latter is the set of points of $\boldsymbol{S}$ where Ψ is defined, it is K-open on $\boldsymbol{S}$ by Prop. 8 of the Appendix. Once and for all, we will agree to denote by $\bar{a}$ the image $\Phi(a)$ of $a \in W$ by Φ in $\Phi(W)$.

All this can be applied to the case when W is taken to be the same as V, V acting upon itself by left-translations. Let $\boldsymbol{G}$ be the abstract variety thus obtained from V; call Φ_0 the birational correspondence between V and $\boldsymbol{G}$ which takes the place of the mapping Φ defined above; and call Ψ_0 its inverse. We transfer to $\boldsymbol{G}$ the normal law on V by means of Φ_0; in other words, for x, y generic and independent on V over K and $\bar{x} = \Phi_0(x)$, $\bar{y} = \Phi_0(y)$, we define $\bar{x}\bar{y} = \Phi_0(xy)$, which implies $\bar{x}^{-1} = \Phi_0(x^{-1})$, and prove that this makes $\boldsymbol{G}$ into a group. In fact, the representative of $\bar{x}^{-1}$ on G_β is $t_\beta x^{-1} = (t_\beta x_\alpha^{-1}) t_\alpha$; if $\bar{s}$ is a point of $\boldsymbol{G}$ with a representative s_α on G_α, we can choose β such that t_β is generic on V over $k(s_\alpha, t_\alpha)$. Then, since V is a group-chunk, $t_\beta s_\alpha^{-1}$ is defined and generic on V over $k(s_\alpha, t_\alpha)$, and so $x \rightarrow (t_\beta x^{-1}) t_\alpha$ is defined at s_α; this means that $\bar{s}^{-1}$ is defined and has a representative on G_β. Similarly, if we write $t = t_\gamma t_\alpha^{-1}$, the representative of $\bar{x}\bar{y}$ on G_γ is $t_\gamma xy = ((tx_\alpha)t_\beta^{-1})y_\beta$; let $\bar{r}$, $\bar{s}$ be two points of $\boldsymbol{G}$ with representatives r_α, s_β on G_α, G_β respectively; by Prop. 4 of the Appendix, we can choose γ so that t_γ is generic on V over $k(r_\alpha, s_\beta, t_\alpha, t_\beta)$; the same will then be true of t, and also of tr_α and of $(tr_\alpha)t_\beta^{-1}$ since V is a group-chunk; for a similar reason, this implies that $(x, y) \rightarrow ((tx)t_\beta^{-1})y$ is defined at (r_α, s_β), and this completes the proof that $\boldsymbol{G}$ is a group.

Now, going back to the space $\boldsymbol{S}$ constructed before, we transfer to $\boldsymbol{G}$, $\boldsymbol{S}$, by means of the birational correspondences Φ_0, Φ, the normal law given for V, W; in other words, for x, u generic and independent over K on V, W, and for $\bar{x} = \Phi_0(x)$, $\bar{u} = \Phi(u)$, we define $\bar{x}\bar{u} = \Phi(xu)$, and prove that this makes $\boldsymbol{S}$ into a transformation-space with respect to $\boldsymbol{G}$. In fact, the representative of $\bar{x}\bar{u}$ on S_γ is $((tx_\alpha)t_\beta^{-1})u_\beta$, where $t = t_\gamma t_\alpha^{-1}$ as before; the rest of the proof is then quite similar to the proof given above.

Naturally, if W is non-singular, $\boldsymbol{S}$ is non-singular; if W is everywhere normal, $\boldsymbol{S}$ is everywhere normal. Finally, if W is a homogeneous chunk, $\boldsymbol{S}$ is a homogeneous space. In fact, in that case, let $\bar{a}$, $\bar{b}$ be any two points of $\boldsymbol{S}$, with representatives a_α, b_β in S_α, S_β respectively. Take x generic over $K(\bar{a}, \bar{b})$ on V; put $\bar{x}' = \bar{x}\bar{a}$, $\bar{x}'' = \bar{x}\bar{b}$. For u generic over $K(x)$ on W, we have $\Psi(\bar{x}\bar{u}) = (xt_\alpha^{-1})u_\alpha$; as W is a homogeneous chunk, $x' = (xt_\alpha^{-1})a_\alpha$ is defined and generic over $K(\bar{a}, \bar{b})$ on W, and therefore we have $x' = \Psi(\bar{x}')$; similarly we have $x'' = \Psi(\bar{x}'')$ with $x'' = (xt_\beta^{-1})b_\beta$ generic over $K(\bar{a}, \bar{b})$ on W. That being so, there is an isomorphism of $K(\bar{a}, \bar{b}, x')$ onto $K(\bar{a}, \bar{b}, x'')$ over $K(\bar{a}, \bar{b})$ which maps x' onto x''; this can be extended to an isomorphism σ of $K(\bar{a}, \bar{b}, \bar{x})$ onto some extension of $K(\bar{a}, \bar{b}, x'')$. Then we have $\bar{x}^\sigma\bar{a} = \bar{x}'' = \bar{x}\bar{b}$, and so $\bar{b} = \bar{x}^{-1}\bar{x}^\sigma\bar{a}$.

7. From now on, it will be assumed that W and consequently $\boldsymbol{S}$ are everywhere normal. With this assumption, we shall construct an abstract variety $\boldsymbol{S}'$, defined over k, and a birational correspondence F between $\boldsymbol{S}'$ and W, also defined over k, so that the birational correspondence $\Phi \circ F$, defined over K, between $\boldsymbol{S}'$ and $\boldsymbol{S}$ is an everywhere biregular mapping of $\boldsymbol{S}'$ onto $\boldsymbol{S}$. This construction can then be applied to V itself, giving a variety $\boldsymbol{G}'$ and a birational correspondence F_0 between $\boldsymbol{G}'$ and V, both defined over k, such that $\Phi_0 \circ F_0$ is biregular between $\boldsymbol{G}'$ and $\boldsymbol{G}$. Transferring the normal laws for V, W to $\boldsymbol{G}'$, $\boldsymbol{S}'$ by means of F, F_0, we see that we have thus constructed a group $\boldsymbol{G}'$ and a transformation-space $\boldsymbol{S}'$, birationally equivalent to V, W over k; if W is pre-homogeneous and we have constructed $\boldsymbol{S}$ as a homogeneous space, $\boldsymbol{S}'$ will be a homogeneous space.

In constructing $\boldsymbol{S}'$, we may assume that V operates faithfully on W; in fact, if this were not so, one could replace V by another pre-group $\bar{V}$ satisfying this condition, according to Prop. 2 of § I, no. 3.

Notations will now be the same as in no. 6, with the additional assumptions that W and consequently $\boldsymbol{S}$ are everywhere normal, and that V acts faithfully on W, so that $\boldsymbol{G}$ acts faithfully on $\boldsymbol{S}$.

Let k' be any field containing k. Let $\sum_{i=1}^{r}(s_i)$ be a cycle of dimension 0 on V, rational over k', and assume that $s_i \neq s_j$ whenever $i \neq j$. Then, if we put $k'' = k'(s_1, \cdots, s_r)$, k'' is a Galois extension of k', i. e. separably algebraic and normal over k'. Call K'' the compositum of K and k''; let u be a generic point of W over K'', and put $w_i = s_i u$. If m is the dimension of the ambient affine space to W, we write $w_i = (w_{i1}, \cdots, w_{im})$. Put now

$$y(T, U) = \prod_{i=1}^{r} \left(T - \sum_{\mu=1}^{m} w_{i\mu} U_\mu\right) \tag{1}$$

where $T, U_1, \cdots, U_m$ are indeterminates; let y be the point, in an affine space of suitable dimension, whose coordinates are all the coefficients of the homogeneous polynomial $y(T, U)$ except that of T^r; this is the so-called "Chow point" of the cycle $\sum_i (w_i)$, and $y(T, U)$ is its "Chow form."

As V acts faithfully on W, and the s_i have been assumed to be distinct, the corollary of Prop. 5, § II, no. 5, shows that the w_i are all distinct. We can therefore apply to them the following general result:

Lemma. *If in* (1) *we take the w_i to be any set of distinct points, and k_0 is the prime field, then the w_i are separably algebraic over $k_0(y)$.*

By F-I$_5$, Th. 1, we need only show that a derivation D of the field $k_0(w_1, \cdots, w_r)$ over $k_0(y)$ must be trivial. In fact, applying D to (1), we get:

$$0 = \sum_{i=1}^{r} \left(T - \sum_{\mu} w_{i\mu} U_\mu\right)^{-1} \sum_{\mu} D w_{i\mu} U_\mu;$$

as the w_i are all distinct, this cannot be an identity in $T, U_1, \cdots, U_m$ unless all the $Dw_{i\mu}$ are 0.

Proposition 7. *Notations being as defined above, we have $k'(y) = k'(u)$ provided the s_i are all distinct and satisfy the following condition:*

(S) *The set of points $\bar{s}_i = \Phi_0(s_i)$ on $\boldsymbol{G}$ is not mapped onto itself by any right-translation.*

The cycle $\sum (w_i)$ is the image of the cycle $\sum (s_i)$ by the mapping

$x \to xu$ of V into W; it is therefore rational over $k'(u)$. By the main theorem on symmetric functions (VA, no. 7, Th. 1), this implies that y is rational over $k'(u)$, i. e. that $k'(y) \subset k'(u)$. On the other hand, the lemma shows that the w_i are separably algebraic over $k''(y)$; as we have $u = s_i^{-1}w_i$ by Prop. 5 of § II, no. 5, u is therefore separably algebraic over $k''(y)$, hence also over $k'(y)$. Let σ be any automorphism over $k'(y)$ of the algebraic closure of $k'(y)$; as it induces an isomorphism of $k'(u)$ onto $k'(u^\sigma)$ over k', u^σ is generic on W over k', so that s_iu^σ is defined by Prop. 5. This gives $(s_iu)^\sigma = s_i^\sigma u^\sigma$, i. e. $w_i^\sigma = s_i^\sigma u^\sigma$. But the decomposition of the homogeneous polynomial $y(T, U)$ into linear factors is uniquely determined; applying σ to (1), we see thus that the w_i^σ must be the same as the w_i except for a permutation, i. e. that there is a permutation $i \to \sigma(i)$ such that $w_i^\sigma = w_{\sigma(i)}$. This can be written as $s_i^\sigma u^\sigma = s_{\sigma(i)}u$; as the s_i^σ are the same as the s_i except for a permutation, we can write them as $s_i^\sigma = s_{\tau(i)}$, where $i \to \tau(i)$ is a permutation. Then we have $\Phi(s_{\tau(i)}u^\sigma) = \Phi(s_{\sigma(i)}u)$, which can be written as $\bar{s}_{\tau(i)}\Phi(u^\sigma) = \bar{s}_{\sigma(i)}\bar{u}$, i. e. $\Phi(u^\sigma) = \bar{s}_{\tau(i)}^{-1}\bar{s}_{\sigma(i)}\bar{u}$. As $\boldsymbol{G}$ acts faithfully on $\boldsymbol{S}$, the corollary of Prop. 5 shows that all the elements $\bar{s}_{\tau(i)}^{-1}\bar{s}_{\sigma(i)}$ of $\boldsymbol{G}$, for $1 \leqq i \leqq r$, must coincide; if $\bar{t}$ is their common value, we have $\bar{s}_{\sigma(i)} = \bar{s}_{\tau(i)}\bar{t}$, which shows that the right-translation $\bar{t}$ maps the set $\bar{s}_i$ onto itself. By (S), this implies that $\bar{t}$ is the neutral element of $\boldsymbol{G}$, so that $\Phi(u^\sigma) = \bar{u}$, and therefore $u^\sigma = u$. As u is separably algebraic over $k'(y)$, this shows that $k'(u) \subset k'(y)$.

8. Proposition 7 shows that we may write $y = f(u)$, where f is a birational correspondence, defined over k', between W and the locus Y of y over k' in affine space. If $k'[y]$ is the ring generated over k' by the coordinates of y, it is well-known that the integral closure of $k'[y]$ in $k'(y)$ is a finitely generated ring over k', i. e. that it can be written as $k'[y^*]$, where y^* is a point in a suitable affine space; call Y^* the locus of y^* over k' in that affine space. As we have $k'(y^*) = k'(y) = k'(u)$, we may write $y^* = f^*(u)$, f^* being a birational correspondence between W and Y^*, defined over k'. It is usual to say that Y^* is derived from Y by "normalization" over k'. By Prop. 14 of the Appendix, since k'' is separably algebraic over k', $k''[y^*]$ is integrally closed in $k''(y^*)$.

PROPOSITION 8. *With the notations explained above, y^* and $\bar{u}$ are corresponding generic points over K'' on Y^* and $\boldsymbol{S}$ in a birational correspondence between Y^* and $\boldsymbol{S}$ which maps Y^* biregularly onto the K''-open set $\Omega = \bigcap_i \bar{s}_i^{-1}\Phi(W)$ on $\boldsymbol{S}$.*

In the first place, we prove that the coordinates $w_{i\mu}$ of the w_i are all in $k''[y^*]$; as they are in $k''(y)$ because of the relations $w_i = s_i u$ and $k'(u) = k'(y)$, it will be enough to show that they are integral over the ring $k''[y]$, or in other words (e.g. by F-App. II, Prop. 6) that they are everywhere finite on W. In fact, let π be any place of $k''(y)$ such that $y(\pi)$ is finite. Take r independent variables $\lambda_1, \cdots, \lambda_r$ over $k''(y)$, and extend π to a place π' of $k''(y, \lambda_1, \cdots, \lambda_r)$ at which every one of the r points $(\lambda_i, \lambda_i w_{i1}, \cdots, \lambda_i w_{im})$ is finite and $\neq (0, \cdots, 0)$. The relation (1), by which y was defined, can be written

$$\lambda_1 \cdots \lambda_r y(T, U) = \prod_{i=1}^{r} (\lambda_i T - \sum_{\mu} (\lambda_i w_{i\mu}) U_\mu).$$

Taking the values of both sides at π', we see that the right-hand side does not become identically 0 at that place; as $y(\pi)$ is finite, this implies that no λ_i can become 0 at π'; but then $w_{i\mu}(\pi)$ can be written as $(\lambda_i w_{i\mu})(\pi')/\lambda_i(\pi')$ and is finite. This proves the assertion about the $w_{i\mu}$.

We have thus shown that the mappings $y^* \rightarrow w_i$ of Y^* into W are everywhere defined on Y^*; as we have $\bar{u} = \bar{s}_i^{-1}\Phi(w_i)$, this implies that $y^* \rightarrow \bar{u}$ is everywhere defined and maps Y^* into the set Ω defined in Prop. 8. Conversely, the definition of y can be written

$$y(T, U) = \prod_{i=1}^{r} (T - \sum_{\mu} \Psi_\mu(\bar{s}_i \bar{u}) U_\mu)$$

if we call $\Psi_\mu(\bar{u})$ the coordinates of $\Psi(\bar{u})$. As Ψ is everywhere defined on $\Phi(W)$, this shows that the mapping $\bar{u} \rightarrow y$ is defined at every point of the set Ω. As $k'[y^*]$ is the integral closure of $k'[y]$ in $k'(y)$, it is therefore contained in the integral closure of the specialization-ring of every point of Ω on $\boldsymbol{S}$. But we have assumed that W and consequently $\boldsymbol{S}$ are normal, i.e. that the specialization-ring of every point of $\boldsymbol{S}$ (over any field of definition for $\boldsymbol{S}$) is integrally closed. This proves that $\bar{u} \rightarrow y^*$ is everywhere defined on the set Ω. In view of what we have proved above, Ω is therefore the set of points of $\boldsymbol{S}$ where this mapping is defined, and is K''-open by Prop. 8 of the Appendix; more precisely, it is K'-open if K' is the compositum of K and k'. This completes the proof.

9. Denote now by S any cycle $\sum_i (s_i)$ on V, *rational over the ground-field* k, consisting of distinct points s_i and satisfying condition (S). From such a set S, and taking $k' = k$, we can derive as above a point y, which we now write as y_S, and furthermore a point y_S^* such that $k[y_S^*]$ is the integral closure of $k[y_S]$ in $k(y_S)$; as above, we call $Y_S{}^*$ the locus of $y_S{}^*$ over k; we write Ω_S for the open subset of $\mathbf{S}$ denoted by Ω in Prop. 8. If we allow S to run through any finite set of cycles with the properties stated above, then all the varieties $Y_S{}^*$ will be birationally equivalent to W and to each other, and we can take the points $y_S{}^*$ to be corresponding generic points of these varieties over k. It is then an immediate consequence of Prop. 8 that the affine varieties $Y_S{}^*$ (with empty "frontiers"), and the birational correspondences between them for which the $y_S{}^*$ are corresponding generic points of the $Y_S{}^*$ over k, determine an abstract variety $\mathbf{S}'$, and that this is biregularly equivalent over a suitable field (as a matter of fact, over K itself) with the union of the open sets Ω_S on $\mathbf{S}$. In order to prove that $\mathbf{S}'$ will be biregularly equivalent to $\mathbf{S}$ itself for a suitable choice of the cycles S, it is therefore enough, in view of the well-known "compactoid" property of open sets in the Zariski topology, to show that the family of all open sets Ω_S is a covering of $\mathbf{S}$. In other words, we have to prove the following:

PROPOSITION 9. *Given any point $\bar{a}$ on $\mathbf{S}$, there is a cycle $S = \sum_i (s_i)$ on V, rational over k, consisting of distinct points s_i and satisfying condition* (S), *and such that $\bar{s}_i\bar{a} \in \Phi(W)$ for all i.*

Assume that $\bar{a}$ has a representative a_α on S_α; take x generic over $K(\bar{a}) = K(a_\alpha)$ on V, and put $u = (xt_\alpha{}^{-1})a_\alpha$, this being defined because W is a chunk. If we put, as usual, $\bar{x} = \Phi_0(x)$ and $\bar{u} = \Phi(u)$, we have then $\bar{u} = \bar{x}\bar{a}$, so that $u = \Psi(\bar{x}\bar{a})$. As the mapping $x \rightarrow \bar{x}\bar{a}$ is everywhere defined on V, this shows that the mapping $x \rightarrow u$ of V into W is defined at the points s of V such that $\bar{s}\bar{a} \in \Phi(W)$, and at those points only. Let F be the closed subset of V where the mapping $x \rightarrow u$ is not defined; by Prop. 12 of the Appendix, there is a maximal $\bar{k}$-closed subset F_0 of V contained in F; then an algebraic point of V over k is in F if and only if it is in F_0. Call F_1 the union of the conjugates over k of all the components of F_0; this is a k-closed set on V, and its definition shows that the cycle S on V will satisfy the last one of the conditions stated in Prop. 9 if and only if it lies in $V - F_1$.

Now assume first that the field k is infinite. Applying Prop. 13 of the Appendix to the variety $V' = V - F_1$, and to the empty subset of $V' \times V'$, we obtain a separably algebraic point s_1 over k on V'; call $s_1, \cdots, s_d$ all the distinct conjugates of s_1 over k; if this set satisfies condition (S), which will be the case in particular if $d = 1$, then it solves our problem. Suppose that this is not so, and therefore that $d > 1$. For any $r > d$, let $s_{d+1}, \cdots, s_r$ be any set of $r - d$ points on $V - F_1$, distinct from one another and from $s_1, \cdots, s_d$; put $S' = \{\bar{s}_1, \cdots, \bar{s}_d\}$ and $S'' = \{\bar{s}_{d+1}, \cdots, \bar{s}_r\}$. If the set

$S' \cup S''$ is mapped into itself by a right-translation τ other than the identity, one of the following circumstances must occur: (i) τ maps each one of the sets S', S'' onto itself; then τ is of the form $s'^{-1}t'$, with s', t' in S', and there must be two elements s'', t'' of S'' such that $t'' = s''\tau$; (ii) τ maps S' into S''; as $d > 1$, we can choose two distinct elements s', t' in S', and then $s'' = s'\tau$, $t'' = t'\tau$ are in S'', so that we have $t'' = (t's'^{-1})s''$; (iii) τ maps some $s' \epsilon S'$ onto some $s'' \epsilon S''$ and some $t' \epsilon S'$ onto some $t_1' \epsilon S'$; then $s'' = s't'^{-1}t_1'$. Thus, in order to satisfy the requirements of Prop. 9, it is enough to take as $s_{d+1}, \cdots, s_r$ the conjugates over k of a point $s = s_{d+1}$ of $V - F_1$, separably algebraic over k, satisfying the following conditions: (a) no $\bar{s}_l$, for $d + 1 \leqq l \leqq r$, coincides with any of the points $\bar{s}_i$ or $\bar{s}_i\bar{s}_j^{-1}\bar{s}_h$ for $1 \leqq i, j, h \leqq d$; (b) no pair of distinct conjugates of s over k lies on the graph of any of the birational correspondences $x \rightarrow \Psi(\bar{x}\bar{s}_i^{-1}\bar{s}_j)$, $x \rightarrow \Psi(\bar{s}_i\bar{s}_j^{-1}\bar{x})$ for $1 \leqq i, j \leqq d$. As to (a), it will be satisfied provided we take s on $V - F_2$, where F_2 is the union of F_1, of the set $s_1, \cdots, s_d$, and of the set of all conjugates over k of those algebraic points on V whose image on $\boldsymbol{G}$ coincides with one of the points $\bar{s}_i\bar{s}_j^{-1}\bar{s}_h$. Then our result follows at once by applying Prop. 13 of the Appendix to the variety $V - F_2$ and to the union of the graphs of the birational correspondences in (b).

If k is finite, we have to proceed differently. Take any algebraic point s_1 over k on $V - F_1$; call $s_1, \cdots, s_d$ its distinct conjugates over k; if this set satisfies condition (S), it solves our problem. If not, we use a result of Lang-Weil (this Journal, vol. 76 (1954), p. 819) which says that, if l is sufficiently large, there must be a point s on $V - F_1$ which is rational over the (unique) extension of k of degree l. We take l prime and $> d$. If s is rational over k, the cycle (s) solves our problem; if not, it is of degree l over k; call $s_{d+1}, \cdots, s_{d+l}$ its distinct conjugates over k; they are distinct from $s_1, \cdots, s_d$, since the latter are of degree d over k. The set $s_{d+1}, \cdots, s_{d+l}$ may solve our problem. If it does not, the group $\mathfrak{g}$ of right-translations mapping the set $\{\bar{s}_{d+1}, \cdots, \bar{s}_{d+l}\}$ onto itself is of order $\nu > 1$; as that set must be the union of cosets with respect to $\mathfrak{g}$, ν must divide l, and so $\mathfrak{g}$ is cyclic of order l; call τ a generator of $\mathfrak{g}$. Let τ' be a right-translation mapping onto itself the set $\{\bar{s}_1, \cdots, \bar{s}_{d+l}\}$. If τ' is not the identity and maps some element of the set $\{\bar{s}_{d+1}, \cdots, \bar{s}_{d+l}\}$ into an element of the same set, it must be of the form τ^i, and therefore of order l; but this cannot be, since $d + l$ is not a multiple of l. Therefore τ' must map the set $\{\bar{s}_{d+1}, \cdots, \bar{s}_{d+l}\}$ into the set $\{\bar{s}_1, \cdots, \bar{s}_d\}$. As $d < l$, this is also impossible. Therefore the set $s_1, \cdots, s_{d+l}$ solves our problem.

This completes the proof of the results announced at the beginning of no. 7. Writing now G, S instead of $\boldsymbol{G'}$, $\boldsymbol{S'}$, we may restate them in a somewhat more complete form as follows:

THEOREM. (i) *To every pre-group V, defined over a field k, there is a birationally equivalent group G, also defined over k; this is uniquely determined up to an isomorphism.*

(ii) *To every pre-homogeneous space W with respect to V, defined over k, there is a birationally equivalent homogeneous space with respect to G, also defined over k; this is uniquely determined up to an isomorphism.*

(iii) *Let W be a pre-transformation space with respect to V, defined over k; let a be a point of W such that W is normal at a and that, if x is generic over $k(a)$ on V, xa and $x^{-1}(xa)$ are defined. Then there is a transformation-space S with respect to G, birationally equivalent to W over k in such a way that the birational correspondence between them is biregular at a; S may be taken everywhere normal, and it may be taken to be non-singular if a is simple on W. Moreover, S is uniquely determined up to a birational correspondence which is biregular at every point of the form $\bar{s}\bar{a}$, where $\bar{a}$ is the point corresponding to a on S and $\bar{s}$ is any point of G.*

Except for the statements about unicity, all this has been proved above. As to unicity, the statements in (i) and (ii) are special cases of the statement in (iii); and the latter is an immediate consequence of the fact that the operations of G are everywhere biregular mappings of S onto S.

Appendix.

If X is any cycle, we denote by $|X|$ the *support* of X, i. e. the closed set which is the set-theoretic union of the components of X.

PROPOSITION 1. *Let $k(x)$ be a regular extension of a field k, and $k(x, y)$ a regular extension of $k(x)$. Then $k(x, y)$ is a regular extension of k.*

This is an immediate consequence of F-I_7, Th. 5.

PROPOSITION 2. *Let $k(x)$ be a regular extension of a field k; let K be an overfield of k, linearly disjoint from $k(x)$ over k; let k' be the algebraic closure of k in K. Then $k'(x)$ is the algebraic closure of $k(x)$ in $K(x)$.*

Let y be an element of $K(x)$, algebraic over $k(x)$; we may take x to be a generic point over K of a variety V, defined over k, in an affine space; and then we may write $y = F(x)$, where F is a function on V, defined over K; call Γ the graph of F. As y is algebraic over $k(x)$, there is a polynomial $P \in k[X, Y]$ such that $P(x, Y) \neq 0$ and $P(x, y) = 0$; then P induces on the product $V \times D$ of V and of the affine space D of dimension 1 a function which is not 0 on $V \times D$ and is 0 on Γ. As Γ has the same dimension as V, it must be a component of the divisor (P) of P, and is therefore algebraic over k. The smallest field of definition of Γ containing k must then be contained in k', so that F is defined over k'; this implies that y is in $k'(x)$.

COROLLARY. *If K is primary over k, $K(x)$ is primary over $k(x)$.*

In fact, the assumption means that k' is purely inseparable over k; this implies that $k'(x)$ is purely inseparable over $k(x)$.

PROPOSITION 3. *Let $k(x)$ be a finitely generated extension of a field k; then every field K such that $k \subset K \subset k(x)$ is finitely generated over k.*

Let $t = (t_1, \cdots, t_n)$ be a maximal set of algebraically independent elements of K over k; then K is algebraic over $k(t)$. Replacing k by $k(t)$, we see that it is enough to prove our proposition in the case when K is algebraic over k. This being assumed, call k' the smallest field of definition containing k for the locus of x over the algebraic closure $\bar{k}$ of k; then k' is a finite algebraic extension of k and is algebraically closed in $k'(x)$ since $k'(x)$ is regular over k'. But then k' is the algebraic closure of k in $k'(x)$ and therefore contains the algebraic closure of k in $k(x)$, so that K is contained in k'.

COROLLARY. *If $k(x)$ is regular over k, so is K.*

PROPOSITION 4. *Let t be a point, k a field, and let $t_1, \cdots, t_N$ be N independent generic specializations of t over k. Let x be a point of dimension $d < N$ over k and such that $k(x)$, $k(t)$ are linearly disjoint over k. Then there is an α such that t_α is a generic specialization of t over $k(x)$.*

Call n the dimension of $k(t)$ over k. By F-I_6, Th. 3, every t_α is a specialization of t over $k(x)$; if none is generic, every t_α must have over $k(x)$ a dimension $\leqq n - 1$; but then $(x, t_1, \cdots, t_N)$ has over k a dimension $\leqq d + N(n-1) < Nn$, which is impossible, since $(t_1, \cdots, t_N)$ has the dimension Nn over k.

PROPOSITION 5. *Let V be a variety, defined over a field k; let K be an overfield of k and x a point of V. Let A and A' be the prime rational cycles,* *over k and over K respectively, with the generic point x. Then A is the same as A' if and only if K and $k(x)$ are linearly disjoint over k.*

We may replace V by any representative of V on which x has a representative, so that it is enough to prove our result for cycles in the affine n-space. For A to be the same as A', it is at any rate necessary that they should have the same dimension, so that K and $k(x)$ must be independent over k; assume from now on that this is so. Among the coordinates of x, let $(x_1, \cdots, x_r)$ be a maximal set of independent variables over k and therefore also over K; write y for the point $(x_1, \cdots, x_r)$ and z for $(x_{r+1}, \cdots, x_n)$. By F-VII$_6$, Th. 12, $A = A'$ if and only if $A \cdot (y \times S^{n-r})$ is the same as $A' \cdot (y \times S^{n-r})$; by F-VI$_3$, Th. 12, this is so if and only if z has the same complete set of conjugates over $K(y)$ as over $k(y)$, and therefore, by F-I$_4$, Prop. 12 and F-I$_2$, Prop. 6, if and only if $K(y)$ and $k(y, z) = k(x)$ are linearly disjoint over $k(y)$. The latter condition means that there is no relation $\sum_i u_i \Phi_i(y) = 0$ in which the u_i are linearly independent elements of $k(x)$ over $k(y)$ and the $\Phi_i(y)$ are in $K[y]$ and not all 0. Assume that there is such a relation; we may write $\Phi_i(y) = \sum_j P_{ij}(y) \xi_j$, where the ξ_j are linearly independent elements of K over k and the $P_{ij}(y)$ are in $k[y]$ and not all 0. Then we have $\sum_j v_j \xi_j = 0$ with $v_j = \sum_i u_i P_{ij}(y)$; as the v_j are in $k(x)$ and not all 0 because of the assumptions on the u_i and $P_{ij}(y)$, this shows that, when that is so, K and $k(x)$ are not linearly disjoint over k. Conversely, assume that there is a relation $\sum_j v_j \xi_j = 0$ in which the ξ_j are linearly independent elements of K over k and the v_j are in $k(x)$ and not all 0; as the ξ_j are then also linearly independent elements of $K(y)$ over $k(y)$, this implies that $K(y)$ and $k(x)$ are not linearly disjoint over $k(y)$.

COROLLARY. *Let V be a variety, defined over a field k. Let A be a prime rational cycle on V over an overfield K of k. Then, if K' is any field such that $k \subset K' \subset K$ over which A is rational, A is prime rational over K'; of all such fields K', there is one smallest one K_0; and an automorphism σ of K over k transforms A into itself if and only if it induces the identity on K_0.*

As in the proof of Prop. 5, it is enough to consider cycles in an affine space. Assume that A is prime rational over K and rational over $K' \subset K$, and write it as $A = \sum_i n_i A_i$, where the A_i are distinct prime rational cycles over K'. Let Z be a component of A_1; it is algebraic over K', and so every conjugate of Z over K is a fortiori such over K', so that every component of

A is a component of A_1; therefore we must have $A = n_1 A_1$. By F-I_8, Prop. 26, the coefficient of Z in A is at most equal to its coefficient in A_1; therefore we have $A = A_1$. That being so, it follows from Prop. 5 and from F-I_6, Th. 3 and F-I_7, Lemma 2, that there is a smallest field K_0 with the properties stated in our corollary; in fact, if x is a generic point of A over K, and if $\mathfrak{P}$ is the prime ideal in $K[X]$ consisting of all polynomials in $K[X]$ which are 0 at x, K_0 is the smallest subfield of K such that $\mathfrak{P}$ has a set of generators in $K_0[X]$. As $\mathfrak{P}$ is also the ideal in $K[X]$ whose set of zeros is the support $|A|$ of A, the last assertion follows from F-I_7, Lemma 2.

PROPOSITION 6. *Let V be a variety, defined over a field k, and A a cycle on V; assume either that A is a divisor on V or that the coefficients in A of all the components of A are $\not\equiv 0 \bmod p$, p being the characteristic. Then, of all the overfields of k over which A is rational, there is one smallest one k_0, k_0 is finitely generated over k; and an isomorphism σ of k_0 over k onto some extension of k leaves A invariant if and only if it leaves every element of k_0 invariant.*

Except for the last statement, this result is due to Chow. Let A be any cycle on V; for every representative V_α of V, call A_α the sum of the terms in the reduced expression for A which pertain to components with representatives in V_α; then A is rational over an overfield K of k if and only if every A_α is rational over K; and an isomorphism of K which leaves A invariant must leave all the A_α invariant. Therefore it is enough to deal with cycles on an affine variety V. For such a cycle A, put $A = \sum_n nA_n$, where A_n is the sum of the terms with the coefficient n in the reduced expression for A; then A is rational if and only if every cycle nA_n is rational; and an isomorphism which leaves A invariant must leave all the A_n invariant. Finally, if $n = p^\nu n'$ with n' prime to p, nA is rational if and only if $p^\nu A$ is rational. Therefore it will be enough to deal with the following two cases: (i) A is a cycle in affine space, consisting of a sum of distinct components; (ii) A is a divisor on an affine variety V and of the form $A = qA_0$, where q is a power of p and A_0 is a sum of distinct components.

(i) Let $\mathfrak{A}$ be the ideal of all polynomials (with coefficients in the universal domain) which are 0 on the support $|A|$ of A; this is the intersection of the prime ideals determined similarly by the components of A. The first assertion in our proposition will then be a consequence of F-I_7, Lemma 2, if we prove that A is rational over a field K if and only if $\mathfrak{A}$ has a set of generators in $K[X]$, i.e. if it is the extension to the universal domain of the ideal $\mathfrak{A} \cap K[X]$; the second assertion in our proposition also follows from the same lemma, provided one observes that, if k_0 is the smallest field such that $\mathfrak{A}$ has a set of generators in $k_0[X]$, an isomorphism which leaves A invariant must map k_0 onto k_0, i. e. it must induce an automorphism in k_0, so that the lemma in question is applicable.

If $\mathfrak{A}$ has a set of generators (P_ν) in $K[X]$, the support $|A|$ of A is the set of zeros of the P_ν and is therefore K-closed. On the other hand, $|A|$ must also be K-closed if A is rational over K. In order to prove the equivalence of those two properties, one may then begin by assuming that $|A|$ is K-closed. Consider first the case in which all the components of A are the conjugates of one of them, say Z, over K; let x be a generic point of Z over $\bar{K}$; then A is rational over K if and only if $K(x)$ is separable over K. Put $K' = K^{p^{-\infty}}$, this being the smallest "perfect" field containing K. Put:

$$\mathfrak{P} = \mathfrak{A} \cap K[X], \qquad \mathfrak{P}' = \mathfrak{A} \cap K'[X],$$

and call $\mathfrak{Q}'$ the extension of $\mathfrak{P}$ to $K'[X]$. By F-IV$_2$, Th. 4, and F-II$_1$, Prop. 3, $\mathfrak{P}$ and $\mathfrak{P}'$ consist of the polynomials, in $K[X]$ and in $K'[X]$ respectively, which are 0 at x; they are prime ideals; moreover, if $P' \in \mathfrak{P}'$, some power P'^n of P' is in $\mathfrak{P}'$ and hence in $\mathfrak{Q}'$; as $\mathfrak{Q}' \subset \mathfrak{P}'$, this implies that $\mathfrak{Q}'$ is primary and belongs to the prime ideal $\mathfrak{P}'$. By F-I$_6$, Th. 3, and F-I$_7$, Prop. 19, we see that $\mathfrak{P}' = \mathfrak{Q}'$ if and only if $K(x)$ is separable over K, and therefore, as we have shown, if and only if A is rational over K. But, if $\mathfrak{A}$ is the extension of $\mathfrak{P}$ to the universal domain, $\mathfrak{P}'$ must a fortiori be the extension of $\mathfrak{P}$ to $K'[X]$. Conversely, if $\mathfrak{Q}' = \mathfrak{P}'$, the extension of $\mathfrak{P}$ to the universal domain is the same as that of $\mathfrak{P}'$; but it is well-known and easily verified that the latter must be a "radical" ideal, i. e. one consisting of all the polynomials which are 0 on a closed set; then one sees at once that it must be the same as $\mathfrak{A}$. This completes the proof in the special case we were considering.

Now assume that $|A|$ is any K-closed set; then we can write A as the sum of cycles A_i such that the components of each A_i are mutually conjugate over K, and $\mathfrak{A}$ is the intersection of the ideals $\mathfrak{A}_i$ similarly determined by the A_i. Put:

$$\mathfrak{P}_i = \mathfrak{A}_i \cap K[X], \qquad \mathfrak{P}_i' = \mathfrak{A}_i \cap K'[X],$$

and call $\mathfrak{Q}_i'$ the extension of $\mathfrak{P}_i$ to $K'[X]$. If A is rational over K, all the A_i must be so, so that, as shown above, the $\mathfrak{A}_i$ must be the extensions of the $\mathfrak{P}_i$ to the universal domain. It is then easily seen that $\mathfrak{A}$ is the extension of the intersection of the $\mathfrak{P}_i$, i. e. of $\mathfrak{A} \cap K[X]$. Assume, on the other hand, that A is not rational over K; then we have $\mathfrak{Q}_i' \neq \mathfrak{P}_i'$ for at least one i; from the unicity of the decomposition of an ideal into an intersection of primary ideals, it follows then that the intersection of the $\mathfrak{Q}_i'$, which is the extension of $\mathfrak{A} \cap K[X]$ to $K'[X]$, cannot be the same as the intersection of the $\mathfrak{P}_i'$, which is $\mathfrak{A} \cap K'[X]$. A fortiori, $\mathfrak{A}$ cannot then be the extension of $\mathfrak{A} \cap K[X]$ to the universal domain. This completes the proof for case (i).

(ii) Let V be a variety, defined over k, in an affine space; let A_0 be a divisor on V and the sum of distinct components; let q be a power of the characteristic $p \neq 0$; put $A = qA_0$. If P is any polynomial which is not 0 on V, denote by $(P)_V$ the divisor of the function induced by P on V. Call $\mathfrak{A}$ the ideal of all the polynomials P, with coefficients in the universal domain, such that either $P = 0$ on V or $(P)_V \succ A$. If A is rational over an overfield K of k, $\mathfrak{A}$ is then the extension of $\mathfrak{A} \cap K[X]$ to the universal domain, as follows at once from F-VIII$_3$, Th. 10. Conversely, assume that $\mathfrak{A}$ is the extension of $\mathfrak{A} \cap K[X]$ to the universal domain; we will prove that A is then rational over K; our proposition will then follow from this as in case (i). As a polynomial P is 0 on $|A|$ if and only if some power P^n of P is in $\mathfrak{A}$, our assumption on A implies that A is K-closed, and therefore that A_0 is rational over $K' = K^{p^{-\infty}}$. Let Z be a component of A. As well-known, there is a polynomial P such that $(P)_V = A_0 + B$, where B has no component in common with A_0; write P as $P = \sum_i \xi_i P_i$, where the ξ_i are linearly independent over K' and the P_i are in $K'[X]$; by F-VIII$_3$, Th. 10, we have $(P_i)_V \succ A_0$ for all i; and Z must have the coefficient 1 in at least one of the P_i, since otherwise it would occur in B; if we call that polynomial P', P' is then in $K'[X]$, Z has the coefficient 1 in $(P')_V$, and we have $(P')_V \succ A_0$. But then P'^q is in $\mathfrak{A}$, and therefore, by hypothesis, may be written as $\sum_j \eta_j Q_j$, where the Q_j are in $\mathfrak{A} \cap K[X]$. The latter fact implies that Z has at least the coefficient q in all the $(Q_j)_V$; as it has the coefficient q in P'^q, it must have the coefficient q in one at least of the divisors $(Q_j)_V$; as these divisors are rational over K, this implies that, if A_1 is the sum of Z and its conjugates over K, qA_1 is rational over K. As this is so for every component Z of A, A is therefore rational over K.

PROPOSITION 7. *Let U, V be two varieties, defined over a field k; let F be a k-closed subset of $U \times V$. Then the set A of the points a on U such that $a \times V \subset F$ is k-closed.*

Let $W_1, \cdots, W_m$ be those components of F which have the "projection" V on V (in the sense of F-IV$_3$, F-VII$_3$); if v is a generic point of V over k, W_i has a generic point over $\bar{k}$ of the form (u_i, v); and $a \in A$ if and only if, for v' generic over $k(a)$ on V, (a, v') is a specialization of some (u_i, v) over $\bar{k}$. Let V_1 be any representative of the abstract variety V; let v_1 be the representative of v on V_1; the ambient affine space for V_1 being embedded in a projective space, let V_0 be the locus of v_1 over k in that projective space. Let F_0 be the union of the loci of the points (u_i, v_1) over $\bar{k}$ in $U \times V_0$; F_0 is k-closed on $U \times V_0$. Then A is the set of the points a on U such that $F_0 \cap (a \times V_0)$ has a component of dimension $\geqq \dim(V_0)$. As V_0 is complete, our conclusion is now contained in Lemma 7 of my paper in *Math. Ann.*, vol. 128 (1954), p. 104.

PROPOSITION 8. *Let ϕ be a mapping of a variety U into a variety V; let k be a field of definition for U, V and ϕ. Then the set of points of U where ϕ is defined is k-open.*

(i) Assume first that U is an affine variety and V is the affine space of dimension 1. Let x be a generic point of U over k; put $y = \phi(x)$. Let $\mathfrak{A}$ be the set of all polynomials P in $k[X]$ such that $P(x)y$ is in $k[x]$; this is an ideal in $k[X]$, containing the ideal $\mathfrak{P}$ of those polynomials which are 0 at x and therefore on V. Since y may be written as $Q(x)/P(x)$, with P, Q in $k[X]$ and $P(x) \neq 0$, we have $\mathfrak{A} \neq \mathfrak{P}$. As the points where ϕ is not defined are the zeros of $\mathfrak{A}$, the set of such points is k-closed.

(ii) Take V as in (i), and assume that U is an abstract variety, with the representatives U_α, on each of which a "frontier" F_α (i. e. a k-closed set) is given, according to the definitions in F-VII$_1$. Call F_α' the k-closed subset of U_α where ϕ is not defined; the set F of the points of U where ϕ is not defined is then the union of the images of the sets $F_\alpha' \cap (U_\alpha - F_\alpha)$ by the canonical birational mappings of the U_α into U. It is easily seen that F must be k-closed provided the following assertion is true: if x is a point of U with a representative x_α on some U_α which is a generic point over $\bar{k}$ of a component of F_α', then every specialization x' of x over k is in F. In fact, let β be such that x' has a representative x_β' on U_β; then x must also have a representative x_β on U_β, and, from the biregularity of the correspondence between U_α, U_β at (x_α, x_β), it follows that x_β must be in F_β'; as x_β' is a specialization of x_β over k, and as F_β' is k-closed, x_β' must then be in F_β', so that x' is in F. This proves our result for this case.

It follows trivially from this that our result remains true when U is an abstract variety and V is an affine space or more generally an affine variety.

(iii) Let U be an abstract variety and let V be a k-open subset of an affine variety V_1; let V_0 be the projective variety whose part "at finite distance" is V_1; then $V_0 - V$ is a k-closed subset F_0 of V_0. Call Γ the graph of ϕ on $U \times V_0$; the set F of the points of U where ϕ, considered as a mapping of U into V, is not defined, is then the union of the set F_1 of the points of U where ϕ is not defined as a mapping of U into V_1 and of the set-theoretic projection of $\Gamma \cap (U \times F_0)$ on U. As V_0 is complete, the latter set coincides with the "projection" in the sense of F-IV$_3$ and F-VII$_3$ and is k-closed (e. g. by F-VII$_4$, Prop. 10 and 11); and F_1 is k-closed, as shown in (ii). Therefore F is k-closed.

(iv) Let U, V be arbitrary abstract varieties; let x be a generic point of U over k; let the V_α be those representatives of V on which $\phi(x)$ has a representative $\phi_\alpha(x)$, and let F_α be the "frontiers" on the V_α. Then ϕ is defined at a point of U if there is an α such that ϕ_α, considered as a mapping of U into $V_\alpha - F_\alpha$, is defined there. Therefore the set where ϕ is not defined is the intersection of the sets where the ϕ_α are not defined; as the latter sets are k-closed by (iii), this completes the proof.

Corollary 1. *Let V be an abstract variety, defined over k, with the representatives V_α. Then, for each α, the set Ω_α of the points of V which have a representative on V_α is k-open; and the canonical correspondence between V and V_α is an everywhere biregular mapping of Ω_α onto $V_\alpha - F_\alpha$ if F_α is the frontier for V_α.*

Let x be a generic point of V over k, and let x_α be its representative on V_α; if we put $x_\alpha = \phi_\alpha(x)$, ϕ_α is the "canonical correspondence" between V and V_α. Then Ω_α is the set of points where ϕ_α, considered as a mapping of V into $V_\alpha - F_\alpha$, is defined; it is k-open by Prop. 8. The rest is obvious.

Corollary 2. *Let V be an abstract variety, defined over a field k. Then there is a finite covering of V by k-open subsets of V, each of which is biregularly equivalent to an affine variety.*

Corollary 1 says that V has a covering by the k-open sets Ω_α, each of which is biregularly equivalent to the k-open subset $V_\alpha - F_\alpha$ of the affine variety V_α. It is therefore enough to prove our assertion for a k-open subset $V - F$ of an affine variety V defined over k. Let $\mathfrak{A}$ be the set of all polynomials in $k[X]$ which are 0 on F; it is an ideal in $k[X]$, and, as F is k-closed, it is the set of zeros of $\mathfrak{A}$. Let $P_1, \cdots, P_m$ be a set of generators for $\mathfrak{A}$; as $F \neq V$, they are not all 0 on V, and we may assume that $P_1, \cdots, P_r$ are not 0 on V while $P_{r+1}, \cdots, P_m$ are 0 on V, with $1 \leqq r \leqq m$. For $1 \leqq \rho \leqq r$, call Ω_ρ the k-open subset of V consisting of the points where P_ρ is not 0; the Ω_ρ are a covering of $V - F$. Let $x = (x_1, \cdots, x_n)$ be a generic point of V over k; let V_ρ be the locus of the point

$$(x_1, \cdots, x_n, 1/P_\rho(x_1, \cdots, x_n))$$

in the affine space of dimension $n+1$. Then V_ρ is biregularly equivalent to Ω_ρ.

COROLLARY 3. *Let V be a variety, defined over a field k. The set of points of V where V is normal* (resp. *relatively normal with respect to k*) *is a k-open subset of V.*

Let V^* be the variety derived from V by normalization with reference to the smallest perfect field $k' = k^{p^{-\infty}}$ containing k (resp. with reference to k); let x be a generic point of V over k; let x^* be the corresponding point of V^*, which is generic over k' (resp. over k) on V^*. We may then write $x^* = \phi(x)$, where ϕ is a mapping of V into V^*, defined over k' (resp. over k). Then the points where V is normal (resp. relatively normal) are those where ϕ is defined. As any k'-open set is also k-open, this proves the corollary.

PROPOSITION 9. *Let V be a variety, defined over a field k; let F be a closed subset of V. For F to be k-closed, it is necessary that it should contain all the specializations over k of all its points; it is sufficient that it should contain all the generic specializations over k of all its points, or also that it should be invariant under all isomorphisms over k of a common field of definition $K \supset k$ for its components.*

The necessity of the first condition follows from F-IV$_2$, Th. 4; we first prove that this condition is sufficient. In fact, it implies that, if z is a generic point over K of a component Z of F, the locus Z' of z over $\bar{k}$ is contained in F; as Z is the locus of z over $\bar{K}$, Z' contains Z; as z cannot be in any other component of F than Z, we get $Z' = Z$; thus all components of F are algebraic over k, and then F-IV$_2$, Th. 4, shows that all the conjugates of Z over k must be contained in F. Now we show that the second condition implies the first one. Let x be any point of F and let x' be a specialization of x over k. Then if V is the locus of x over $\bar{k}$, F-IV$_2$, Th. 4, shows that x' must be on a conjugate V' of V over k. Let x'' be a generic point of V' over $\bar{K}$; then x'' is a generic specialization of x over k by F-IV$_2$, Th. 4, and is therefore in F by hypothesis, and x' is a specialization of x'' over $\bar{K}$ and a fortiori over K and so is in F since F is K-closed. Finally the last condition implies the second one; for let x' be a generic specialization over k of a point x in a component Z of F; then the isomorphism of $k(x)$ onto $k(x')$ over k which maps x onto x' can be extended to an isomorphism σ of $K(x)$

onto a field $K^\sigma(x')$, and then x' is on Z^σ by F-IV$_2$, Th. 3, Coroll. 2, and is therefore in F if F is invariant under σ.

PROPOSITION 10. *Let W be a subvariety of a product $U \times V$, with the "projection" U on U; let k be a field of definition for U, V, W. Then the set-theoretic projection of W on U contains an open subset of U; and the union of all such sets is k-open.*

The assumption means that, if (u, v) is a generic point of W over k, u is generic over k on U. Let V_1 be a representative of V on which v has a representative v_1; F_1 being the corresponding frontier, put $V_1' = V_1 - F_1$, so that v_1 is in V_1'; let W_1 be the locus of (u, v_1) over k on $U \times V_1'$. Let V_0 be the projective variety whose part "at finite distance" is V_1; put $F_0 = V_0 - V_1'$; this is a k-closed set on V_0. The set $W_1 \cap (u \times V_1')$ can be written as $u \times X$, where X is either V_1' (in the trivial case $W = U \times V$) or else a $k(u)$-closed subset of V_1'; as v_1 is in X, X is not empty, so that we can choose in it a point w which is algebraic over $k(u)$. Let W' be the locus of (u, w) over $\bar{k}$ on $U \times V_0$, which has the same dimension as U; call n that dimension. Then the set $C = W' \cap (U \times F_0)$ is a $\bar{k}$-closed subset of W', so that all its components are of dimension $\leqq n - 1$. As V_0 is complete, the set-theoretic projection C' of C on U is then a $\bar{k}$-closed subset of U. Let a be any point in $U - C'$; as V_0 is complete, there is a point (a, b) on W' with the projection a on U; as a is not in C', b cannot be in F_0 and is therefore in V_1', so that (a, b) is in W_1. Therefore the $\bar{k}$-open set $U - C'$ on U is contained in the set-theoretic projection of W_1 and a fortiori in that of W. The last assertion in our proposition is then an immediate consequence of the sufficiency of the last condition in Prop. 9.

PROPOSITION 11. *Let U, V, W be three varieties and f a mapping of $U \times V$ into W, all defined over a field k. Assume that, for every $a \in U$, f is defined at (a, x) for x generic on V over $k(a)$. Let Ω be the set of those $a \in U$ such that, for x generic over $k(a)$ on V, $f(a, x)$ is generic over $k(a)$ on W. Then Ω is either empty or k-open on U.*

Let r be the dimension of W; for z generic over k on W, let $z_1, \cdots, z_r$ be r algebraically independent elements of $k(z)$ over k; put $z_i = \phi_i(z)$, where ϕ_i is a function on W, defined over k. It is clear that a point z' of W is generic over an overfield K of k if and only if the ϕ_i are all defined at z' and their values $\phi_i(z')$ are independent over K. Let u, x be independent generic points of U, V over k; we may assume that $f(u, x)$ is generic over k on W, since otherwise Ω is empty. Put $f_i = \phi_i \circ f$; Ω is then the set of those points a on U such that, for x generic over $k(a)$ on V, the $f_i(a,x)$ are all defined and are algebraically independent over $k(a)$.

Take u, x as above; assume that u is not in Ω; we prove that Ω must then be empty. In fact, the assumption on u means that there is a polynomial P with coefficients in $k(u)$ such that

$$P(f_1(u,x),\cdots,f_r(u,x))=0.$$

Write $P=\sum_\nu t_\nu M_\nu(Z)$, where the $M_\nu(Z)$ are monomials (with coefficient 1) in the indeterminates $Z_1,\cdots,Z_r$, and the t_ν are in $k(u)$ and not all 0. Let a be any point of U, and take x' generic over $K=k(a)$ on V. Take a variable quantity λ over $k(u,x)$; extend the specialization $u\rightarrow a$ over k to a $\bar{K}$-valued place π of the field $k(u,\lambda)$ such that the elements λt_ν of $k(u,\lambda)$ are all finite and not all 0 at π; call t_ν' the value of λt_ν at π. As $k(u,\lambda)$ and $k(x)$ are independent regular extensions of k, the place π of $k(u,\lambda)$ and the isomorphism of $k(x)$ onto $k(x')$ over k which maps x onto x' make up a specialization of the set of quantities $k(u,\lambda)\cup k(x)$, which can be extended to a place π' of $k(u,\lambda,x)$ at which u, x and the λt_ν have respectively the values a, x' and t_ν'. If the $f_i(a,x')$ are not all defined, a is not in Ω; if they are all defined, they are the values at π' of the elements $f_i(u,x)$ of $k(u,\lambda,x)$. In the latter case, the relation

$$\lambda P(f_1(u,x),\cdots,f_r(u,x))=0,$$

taken at π', gives an algebraic relation between the $f_i(a,x')$ whose coefficients t_ν' are in $\bar{K}$ and are not all 0; this implies that the $f_i(a,x')$ are not independent over $K=k(a)$, so that a is again not in Ω. This shows that, for $u\notin\Omega$, Ω must be empty. From now on, therefore, we may assume that u is in Ω.

We prove now that Ω must contain a k-open set. Since the assumptions and the conclusion of our proposition are not affected if V is replaced by any birationally equivalent variety to V over k (the mapping f being transferred to the latter in an obvious manner), we may take for V an affine variety; put $x=(x_1,\cdots,x_m)$. Then we can write the f_i as

$$f_i(u,x)=P_i(x)/P_0(x),$$

where $P_0,P_1,\cdots,P_r$ are polynomials in the indeterminates $X_1,\cdots,X_m$ with coefficients in $k(u)$, and $P_i(x)\neq 0$ for $0\leqq i\leqq r$. Call $M_\nu(X)$, with $0\leqq\nu\leqq N$, all the monomials in $X_1,\cdots,X_r$ which either are of degree 0 or 1 (i.e. equal to one of the monomials $1, X_1,\cdots,X_r$) or occur in one at least of the P_i; call $\bar{x}$ the point in the projective space P^N with the homogeneous coordinates $(M_\nu(x))$, and call $\bar{V}$ the locus of $\bar{x}$ over k. We may replace V by the birationally equivalent $\bar{V}$; then, writing V, x instead of $\bar{V}$, $\bar{x}$, and calling $(x_0, \cdots, x_N)$ the homogeneous coordinates for x, we see that the $f_i(u, x)$ are expressed as z_i/z_0, with

$$z_i = \sum_{\nu=0}^{N} t_{i\nu} x_\nu \qquad (0 \leqq i \leqq r),$$

where the $t_{i\nu}$ are elements of $k(u)$. If V is contained in any linear subvariety of P^N, then the smallest linear subvariety of P^N which contains V is defined over k; if this is of dimension N', we can express $N - N'$ of the coordinates x_ν linearly in terms of the others, with coefficients in k; thus we may assume that V is not contained in any linear subvariety of P^N.

We may write $t_{i\nu} = \phi_{i\nu}(u)$, where the $\phi_{i\nu}$ are functions on U, defined over k; as z_0 is not 0, we may assume that $t_{00} = 1$. By Prop. 8, the subset U' of U where all the $\phi_{i\nu}$ are defined and finite is k-open. Call n the dimension of V; as n is then the dimension of $k(u, x)$ over $k(u)$, and the $f_i(u, x)$ are independent over $k(u)$, we have $r \leqq n$. Put $z_j = \sum_\nu w_{j\nu} x_\nu$ for $r + 1 \leqq j \leqq n$, where the $w_{j\nu}$, for $r + 1 \leqq j \leqq n$, $0 \leqq \nu \leqq N$, are $(n - r)(N + 1)$ independent variables over $k(u, x)$; call S the affine space of dimension $(n - r)(N + 1)$. By F-II$_5$, Prop. 24, the $n - r$ quantities $z_{r+1}/z_0, \cdots, z_n/z_0$ are algebraically independent over the field

$$K = k(u, w, z_1/z_0, \cdots, z_r/z_0).$$

Now take any $a \in U'$; take $\bar{x}$, $\bar{w}$ generic and independent over $k(a)$ on V, S; put $\bar{t}_{i\nu} = \phi_{i\nu}(a)$, $\bar{z}_i = \sum_\nu \bar{t}_{i\nu}\bar{x}_\nu$ for $0 \leqq i \leqq r$, and $\bar{z}_j = \sum_\nu \bar{w}_{j\nu}\bar{x}_\nu$ for $r + 1 \leqq j \leqq n$. As $\bar{t}_{00} = 1$, and V is not contained in any linear variety, $\bar{z}_0$ is not 0; therefore, if we put $f_j = z_j/z_0$ for $r + 1 \leqq j \leqq n$, the functions $f_1, \cdots, f_n$ on $U' \times S$ are defined at $(a, \bar{w})$ and have the values $\bar{z}_1/\bar{z}_0, \cdots, \bar{z}_n/\bar{z}_0$ respectively. If one assumes that $\bar{z}_1/\bar{z}_0, \cdots, \bar{z}_n/\bar{z}_0$ are algebraically independent over $k(a, \bar{w})$ this implies a fortiori that $\bar{z}_1/\bar{z}_0, \cdots, \bar{z}_r/\bar{z}_0$ are so over $k(a)$, i. e. that $a \in \Omega$. Therefore, if we prove that there is a k-closed subset C of $U' \times S$ such that, with the notations just introduced, the quantities $\bar{z}_1/\bar{z}_0, \cdots, \bar{z}_n/\bar{z}_0$ are algebraically independent over $k(a, \bar{w})$ whenever $(a, \bar{w})$ is not in C, it will follow that Ω contains the set of those points a on U' such that $a \times S$ is not contained in C; and this set will be k-open by Prop. 7. In other words, as long as we merely wish to prove that Ω contains a k-open set, it is enough to prove it for $U' \times S$ and the functions $f_i = z_i/z_0$ for $1 \leqq i \leqq n$ instead of for U and for $f_1, \cdots, f_r$. This means that, writing U instead of $U' \times S$, it is enough to prove our assertion under the additional assumption $r=n$, the $\phi_{i\nu}$ being now everywhere defined on U, with $\phi_{00}=1$.

This being now assumed, put $z_{n+1}=\sum_\nu w_\nu x_\nu$, where the w_ν are $N+1$ independent variables over $k(u,x)$. As $k(u,w,x)$ is of dimension n over $k(u,w)$, there is a homogeneous polynomial P, with coefficients in $k(u,w)$ and not all 0, such that $P(z_0,\cdots,z_n,z_{n+1})=0$, and P is uniquely determined up to a factor in $k(u,w)$. As $z_1/z_0,\cdots,z_n/z_0$ are algebraically independent, there is at least one term in P where z_{n+1} occurs with a non-zero exponent; after multiplying P with a suitable element of $k(u,w)$, we may assume that the coefficient of this term is 1. Write all the other coefficients in P as $\psi_\rho(u,w)$, where the ψ_ρ are functions on $U\times S^{n+1}$, defined over k. We now prove our assertion about Ω by showing that Ω contains the set of all points a on U such that all the ψ_ρ are defined at $(a,\bar{w})$ for $\bar{w}$ generic on S^{n+1} over $k(a)$; this is a k-open subset of U by Prop. 8 and 7. In fact, let a be a point with that property; take $\bar{x}$ generic over $k(a,\bar{w})$ on V. Put $\bar{t}_{i\nu}=\phi_{i\nu}(a)$, these being all defined, according to our present assumptions; put $\bar{z}_i=\sum_\nu \bar{t}_{i\nu}\bar{x}_\nu$ for $0\leqq i\leqq n$, and $\bar{z}_{n+1}=\sum_\nu \bar{w}_\nu\bar{x}_\nu$. If we specialize the relation

$$P(z_0,\cdots,z_{n+1})=0$$

over the specialization $(\bar{a},\bar{w},\bar{x})$ of (u,w,x) with respect to k, we get, since the ψ_ρ are all defined at $(a,\bar{w})$, a homogeneous relation between $\bar{z}_0,\cdots,\bar{z}_{n+1}$ with coefficients in $k(a,\bar{w})$, containing $\bar{z}_{n+1}$ with a non-zero exponent in a term of coefficient 1. This shows that $\bar{z}_{n+1}/\bar{z}_0$ is then algebraic over the field $L(\bar{w})$, where we have put

$$L=k(a,\bar{z}_1/\bar{z}_0,\cdots,\bar{z}_n/\bar{z}_0).$$

Now take $n(N+1)$ independent variables $w_{i\nu}$ over $k(a)$, for $1\leqq i\leqq n$, $0\leqq\nu\leqq N$; put $y_i=\sum_\nu w_{i\nu}\bar{x}_\nu$ for $1\leqq i\leqq n$; what we have proved above shows that, for each i, $y_i/\bar{z}_0$ is algebraic over $L(w_{i0},\cdots,w_{iN})$, and therefore a fortiori over the field

$$L'=L(w_{10},\cdots,w_{nN})=k(a,w_{10},\cdots,w_{nN},\bar{z}_1/\bar{z}_0,\cdots,\bar{z}_n/\bar{z}_0).$$

On the other hand, one sees just as before, using F-II$_5$, Prop. 24, that the $y_i/\bar{z}_0$, for $1\leqq i\leqq n$, are algebraically independent over the field $k(a,w_{10},\cdots,w_{nN})$; as they are algebraic over L', this implies that L' has at least the dimension n over the latter field, so that the $\bar{z}_i/\bar{z}_0$, for $1\leqq i\leqq n$, must be algebraically independent over it. But then they must a fortiori be so over $k(a)$, which means that a is in Ω.

This completes the proof of the following statement: the assumptions being again those of Prop. 11, Ω must either be empty or contain a k-open subset of U. Now we prove Prop. 11 by induction on the dimension of U, the conclusion being trivially true if that dimension is 0. Assume that Ω is not empty; put $X = U - \Omega$; we have proved that X is contained in a k-closed subset C of U. Call U_i the components of C; they are algebraic over k, and of dimension $< \dim(U)$. By the induction assumption, $\Omega \cap U_i$ is either empty or a $\bar{k}$-open subset of U_i; in both cases its complement C_i on U_i is a $\bar{k}$-closed subset of U. As X is the union of the C_i, this shows that X is $\bar{k}$-closed. As it is obviously invariant by all automorphisms of $\bar{k}$ over k, it must then be k-closed.

PROPOSITION 12. *Let U be a variety defined over a field k; let F be a closed subset of U. Then, among all the k-closed subsets of U contained in F, there is one maximal set F_0.*

Let K be the smallest common field of definition containing k for all the components of F; let σ run through all the isomorphisms of K over k into the universal domain. As such an isomorphism σ leaves all k-closed sets invariant, every k-closed subset of U which is contained in F is contained in all the sets F^σ and therefore in their intersection F_0; F_0 is closed, since it is the intersection of closed sets; and it is k-closed, by Prop. 9. This proves the proposition.

PROPOSITION 13. *Let U be a variety defined over an infinite field k; let F be a closed subset of $U \times U$. Then there is a point a on U, separably algebraic over k and such that no pair (a', a'') of distinct conjugates of a over k is in F.*

Applying Prop. 12 to $U \times U$, F and the algebraic closure $\bar{k}$ of k, we see that there is a $\bar{k}$-closed subset F_0 of $U \times U$ such that a subvariety of $U \times U$ which is algebraic over k is contained in F if and only if it is contained in F_0; this applies in particular to algebraic points over k on $U \times U$. By replacing F by the union of all conjugates over k of all the components of F_0, we see that it is enough to prove our result in the case in which F is k-closed. We may assume that no component of F is contained in the diagonal of $U \times U$, since the omission of such components does not affect the content of our proposition. Furthermore, we may, in order to prove our proposition, replace U by any k-open subset of U; in view of this, we first replace U by the set of its simple points, and then use Corollary 2 of Prop. 8 to replace U by an affine variety. Thus we may assume that U is a non-singular affine variety, defined over k, and that F is a k-closed subset of $U \times U$, no component of which is contained in the diagonal of $U \times U$.

Let n and N be the dimensions of U and of the ambient affine space, respectively. The case $n = N$ is trivial, since in that case any rational point of U over k, e.g. 0, would solve our problem; therefore we assume $n < N$. Consider all sets of n linear equations:

$$\sum_{\nu=1}^{N} t_{i\nu} X_\nu = t_{i0} \qquad (1 \leqq i \leqq n), \tag{1}$$

and identify the set (1) with the point $t = (t_{i0}, t_{i\nu})$ in the affine space T of dimension $n(N+1)$. In the space T, we consider the following sets:

(a) Call A the set of those points t for which the left-hand sides of (1) are not linearly independent; as A can be described as the set of zeros of certain determinants, it is k_0-closed, k_0 being the prime field (one could easily see that A is in fact a variety, defined over k_0). Put $T' = T - A$; for $t \in T'$, (1) defines a linear variety $L(t)$ of dimension $N - n$.

(b) Take t generic over k on T; by F-V_1, Th. 1, $U \cap L(t)$ is not empty, and, if u is a point in it, u is algebraic over $k(t)$ and is generic on U over the field $K = k(t_{11}, \cdots, t_{nN})$. As the t_{i0} are then in $K(u)$, we have $k(u, t) = K(u)$, so that $k(u, t)$ is a regular extension of k. Let W be the locus of (u, t) over k on $U \times T$; by F-V_1, Prop. 4, if $t' \in T'$, a point u' is in $U \cap L(t')$ if and only if (u', t') is in W. By Prop. 10, there is a k-closed subset B of T such that $T - B$ is contained in the set-theoretic projection of W on T; then, if $t' \in T - (A \cup B)$, $U \cap L(t')$ is not empty.

(c) Let $P_\rho(X) = 0$, for $1 \leqq \rho \leqq r$, be a set of equations for U with coefficients in k; put $P_{\rho\nu} = \partial P_\rho / \partial X_\nu$. Let D be the subset of $U \times T$ consisting of the points where the matrix

$$\left\| \begin{matrix} t_{i\nu} \\ P_{\rho\nu}(u) \end{matrix} \right\| \qquad (1 \leqq i \leqq n, 1 \leqq \rho \leqq r; 1 \leqq \nu \leqq N)$$

is of rank $< N$; since this can be expressed by the vanishing of determinants, D is a k-closed subset of $U \times T$ (as U is non-singular, it could be shown that D is actually a variety, defined over k). As W is not contained in D, $D \cap W$ is a k-closed subset of W (also, in fact, a variety), so that its components have a dimension $< n(N+1)$. Let D' be the "projection" of $D \cap W$ on T (i.e. the closure of the set-theoretic projection); this is a k-closed subset of T. Let u' be a point in $U \cap L(t')$, for $t' \in T'$; then, if $L(t')$ is not transversal to U at u', (u', t') must be in D and therefore in

$D \cap W$, and t' must be in D'. Therefore, if $t' \in T - (A \cup D')$, $L(t')$ is transversal to U at every point of $U \cap L(t')$.

(d) Let X be any component of F; let (u, v) be a generic point of X over $\bar{k}$. As X is not in the diagonal of $U \times U$, we have $u \neq v$ and may assume for instance that $u_1 \neq v_1$. Take the $t_{i\nu}$ independent over $k(u, v)$ for $1 \leqq i \leqq n$, $2 \leqq \nu \leqq N$; as $n < N$, t will then be in T' for all choices of the t_{i1}, t_{i0}. Determine the t_{i1}, t_{i0} by the condition that $L(t)$ should contain both u and v; this determines them uniquely. As X is at most of dimension $2n - 1$, (u, v, t) is then of dimension $< n(N + 1)$ over k; therefore the locus Y of t over $\bar{k}$ is not T. For any $t' \in T'$, assume that $U \cap L(t')$ contains two distinct points u', v' such that (u', v') is in X; then there is ν such that $u_\nu' \neq v_\nu'$, which implies that $u_\nu \neq v_\nu$. It is easily seen that the $t_{i\mu}$, for $1 \leqq i \leqq n$ and all $\mu \neq \nu$ and $\neq 0$, must then be independent over $k(u, v)$, and furthermore that (u', v', t') must be a specialization of (u, v, t) over $\bar{k}$, so that t' is in Y. Therefore, if t' is in T' and is not in the union C of all the varieties Y corresponding in this manner to the components X of F, there cannot be a pair of distinct points (u', v') in $U \cap L(t')$ such that (u', v') is in F. To conjugate components X, X' of F over k, there correspond conjugate varieties Y, Y' over k in T; therefore C is a k-closed subset of T.

Now let $P(t)$ be any polynomial other than 0 in the coordinates of t, with coefficients in k, which is 0 on the union of the k-closed subsets A, B, D' and C of T. As k is infinite, there is on T a rational point t over k such that $P(t) \neq 0$. As t is not in A, it determines a linear variety $L(t)$; as t is not in B, $U \cap L(t)$ is not empty. Take for a any point in $U \cap L(t)$; as t is not in D', $L(t)$ is transversal to U at a, so that a is separably algebraic over k. As all the conjugates of a over k are in $U \cap L(t)$, and as t is not in C, no pair of distinct conjugates a', a'' of a over k can be such that (a', a'') is in F. This completes the proof.

If k is a finite field, the conclusion of Prop. 13 need not be true. In fact, take for U a variety without rational points over k (e. g. the plane non-singular curve $x^4 + y^4 + z^4 = 0$ over the field with 5 elements); q being the number of elements of k, and $(x_1, \cdots, x_n)$ being a representative of a generic point of U over k, call x' the point whose corresponding representative is $(x_1{}^q, \cdots, x_n{}^q)$. Then the conclusion of Prop. 13 is false if we take for F the locus of (x, x') over k on $U \times U$.

PROPOSITION 14. *Let V be an affine variety, defined over a field k; let x be a generic point of V over k; assume that the ring $k[x]$ is integrally closed in $k(x)$. Then, if k' is any separably algebraic extension of k, $k'[x]$ is integrally closed in $k'(x)$.*

Put $n = [k' : k]$; as k' is separably algebraic over k, there are n distinct isomorphisms σ of k' into the algebraic closure of k. Each σ can then be extended uniquely to an isomorphism, which we also denote by σ, of $k'(x)$ onto $k'^{\sigma}(x)$ over $k(x)$. Let z be an element of $k'(x)$, integral over $k'[x]$ and therefore also over $k[x]$; then all the z^{σ} are also integral over $k[x]$. If $\xi_1, \cdot \cdot \cdot, \xi_n$ are n linearly independent elements of k' over k, it is well-known that $\det(\xi_i^{\sigma})$ is not 0; therefore all the z^{σ}, and z among them, can be expressed as linear combinations of the n elements $w_i = \sum_{\sigma} \xi_i^{\sigma} z^{\sigma}$; but these are integral over $k[x]$ and are traces over $k(x)$ of elements of $k'(x)$, so that they are in $k(x)$; as $k[x]$ is integrally closed, the w_i are therefore in $k[x]$, so that z is in $k'[x]$.

THE UNIVERSITY OF CHICAGO.