13.5 复合函数的微分法

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课本例题

例 1 设函数 $z=(\sin t)^{\ln t}$. 求 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解: 引入两个中间变量: $x=\sin t$ 和 $y=\ln t$, 则原来的函数可以写成

$$z=x^y,\qquad 其中\quad x=\sin t,\quad y=\ln t.$$

则由上面的定理可以求得:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= yx^{y-1}\cdot\cos t+x^y\ln x\cdot\frac{1}{t}\\ &= (\ln t)(\sin t)^{\ln t-1}\cdot\cos t+(\sin t)^{\ln t}\frac{\ln\sin t}{t}.\end{aligned}$$

□

例 2 设 f 是可微函数且 $F(x,y,z)=f(x^2+y^2+z^2,xyz)$, 求 $\dfrac{\partial F}{\partial x}$ 在 $\dfrac{\partial F}{\partial y}$ 和 $\dfrac{\partial F}{\partial z}$.

解: 这是三个自变量, 两个中间变量的情形. 很容易将上面的定理推广到包含本例的情形. 记 $u=x^2+y^2+z^2, v=xyz$. 为方便起见, 将 $\dfrac{\partial f}{\partial u}$ 在 $\dfrac{\partial f}{\partial v}$ 分别简记为 f_1' 和 f_2'. 则

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=2xf_1'+yzf_2',\\ \frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=2yf_1'+xzf_2',\\ \frac{\partial F}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}=2zf_1'+xyf_2'.\end{aligned}$$

□

例 3 设 f 是可微函数且 $F=f(y,u,v)$, 其中

$$u=u(x,y),\quad v=v(x,w),\qquad w=w(x,y)$$

均可微, 求 $\dfrac{\partial F}{\partial x}$ 在 $\dfrac{\partial F}{\partial y}$.

解: 这里所遇到的是更复杂的两层复合关系, 有的中间变量本身也是自变量.

注意到 x 和 y 是相互独立的自变量, 故 $\dfrac{\partial y}{\partial x}=0$, $\dfrac{\partial x}{\partial y}=0$.

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial v}{\partial w}\frac{\partial w}{\partial x}\right)\\ &= f_2'\frac{\partial u}{\partial x}+f_3'\left(v_1'+v_2'\frac{\partial w}{\partial x}\right);\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\\ &= \frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial v}{\partial w}\frac{\partial w}{\partial y}\right)\\ &= f_1'+f_2'\frac{\partial u}{\partial y}+f_3'v_2'\frac{\partial w}{\partial y}.\end{aligned}$$

□

例 4 讨论 $z=f(x,y)$ 在点 $(0,0)$ 的二阶混合偏导数:

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

解: 为用定义计算两个混合偏导, 需先求出 $f_x(0,y)$ 和 $f_y(x,0)$. 注意 $f(x,y)$ 是分片定义的函数, 故在 $(0,0)$ 的偏导数需要用定义来求, 而对于原点以外的点, 直接套用求导的公式即可.

$$f_x(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2}-\dfrac{2x^4y}{(x^2+y^2)^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0);\end{cases}$$

$$f_y(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}-\dfrac{2x^3y^2}{(x^2+y^2)^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

由此得到

$$f_x(0,y)=0,\qquad f_y(x,0)=x.$$

继而再直接分别求导:

$$f_{xy}(0,y)=0,\qquad f_{yx}(x,0)=1,$$

即

$$f_{xy}(0,0)=0,\qquad f_{yx}(0,0)=1.$$

□

例 5 设 f 是二阶连续可微函数, 且 $f(x,y,z)=f(x^2+y^2+z^2,xyz)$, 求 f 的二阶偏导数 f_{xx} 在 f_{xz} 和 f_{zx}.

解: 记两个中间变量分别为 $u = x^2 + y^2 + z^2$ 和 $v = xyz$, 又为方便起见, 记

$$\frac{\partial f}{\partial u} = f_1', \quad \frac{\partial f}{\partial v} = f_2', \quad \frac{\partial^2 f}{\partial u^2} = f_{11}'', \quad \frac{\partial^2 f}{\partial v^2} = f_{22}'', \quad \frac{\partial^2 f}{\partial u \partial v} = f_{12}''.$$

由所给的二阶连续可微条件知 f_{12}''=f_{21}''. 在例 **??** 中已经求出了 $f_x = 2xf_1' + f_2'yz$, 在此基础上接着求二阶偏导数:

$$\begin{aligned}
F_{xx} &= (2xf_1' + f_2'yz)_x \\
&= 2f_1' + 2x(f_{11}''2x + f_{12}''yz) + yz(f_{21}''2x + f_{22}''yz) \\
&= 2f_1' + 4x^2 f_{11}'' + 4xyz f_{12}'' + (yz)^2 f_{22}''; \\
F_{xz} &= (2xf_1' + f_2'yz)_z \\
&= 2x(f_{11}''2z + f_{12}''xy) + f_2'y + yz(f_{21}''2z + f_{22}''xy) \\
&= yf_2' + 4xzf_{11}'' + 2y(x^2 + z^2)f_{12}'' + xy^2 z f_{22}'',
\end{aligned}$$

这里已利用了 $f_{12}'' = f_{21}''$. 又 f 显然也是二阶连续可微的, 因此

$$F_{zx} = F_{xz} = yf_2' + 4xzf_{11}'' + 2y(x^2 + z^2)f_{12}'' + xy^2 z f_{22}''.$$

□

思考题

1. 将链式法则用向量的形式写出.

解: 设向量函数 $\boldsymbol{r} = f(u, v)$ 可微, 且 $u = \phi(x, y)$ 和 $v = \psi(x, y)$ 都存在偏导数, 则复合向量函数 $\boldsymbol{r} = f(\phi(x, y), \psi(x, y))$ 也存在偏导数, 并且

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial x} &= \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial v}{\partial x} \\
&= \boldsymbol{r}_u(\phi(x,y), \psi(x,y))\phi_x(x,y) + \boldsymbol{r}_v(\phi(x,y), \psi(x,y))\psi_x(x,y),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial y} &= \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial v}{\partial y} \\
&= \boldsymbol{r}_u(\phi(x,y), \psi(x,y))\phi_y(x,y) + \boldsymbol{r}_v(\phi(x,y), \psi(x,y))\psi_y(x,y),
\end{aligned}$$

□

2. 当 x, y 是自变量时, 有 $\mathrm{d}x = \Delta x$, $\mathrm{d}y = \Delta y$, 当 x, y 是其他变量的函数时, 这些关系是否仍成立?

解: 当 x, y 是其他变量的函数时, $\mathrm{d}x = \Delta x$, $\mathrm{d}y = \Delta y$ 不成立. 例如, 令

$$x = u + v, \quad y = uv,$$

则

$$\mathrm{d}x = \mathrm{d}u + \mathrm{d}v, \quad \mathrm{d}y = v\mathrm{d}u + u\mathrm{d}v.$$

□

3. 设 $f(x_1, x_2, \cdots, x_n)$ 二阶连续可导在 f 最多有多少个不同的二阶混合偏导数?

解: f 最多有 $n+C_n^2$ 个不同的二阶混合偏导数. □

习题

1. 求以下复合函数所指定的 (偏) 导数:

(1) $z=\mathrm{e}^{xy}$, $y=\arctan x$, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$;

(2) $z=\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\sin\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$, 求 $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$;

(3) $z=\mathrm{e}^{x}+\sin x\cos y+\mathrm{e}^{y}$, $x=t^2$, $y=1+t$, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$;

(4) $z=\ln x\ln y$, $x=u+v$, $y=u-v$, 求 $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$.

解: **(1)** 由定理 13.5.1 可得

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= y\mathrm{e}^{xy}+x\mathrm{e}^{xy}\cdot\frac{1}{1+x^2}\\ &= \mathrm{e}^{xy}\left(y+\frac{x}{1+x^2}\right)\\ &= \mathrm{e}^{x\arctan x}\left(\arctan x+\frac{x}{1+x^2}\right).\end{aligned}$$

(2) 引入两个中间变量: $u=\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}$ 和 $v=\sin\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$, 则原来的函数可以写成

$$z=uv,\qquad 其中\quad u=\mathrm{e}^{\frac{1}{x}+\frac{1}{y}},\quad v=\sin\left(\frac{1}{x}+\frac{1}{y}\right).$$

则由定理 13.5.2 可得:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ &= v\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\left(-\frac{1}{x^2}\right)+u\cdot\cos\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(-\frac{1}{x^2}\right)\\ &= \sin\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\left(-\frac{1}{x^2}\right)+\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\cos\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(-\frac{1}{x^2}\right)\\ &= \left(-\frac{1}{x^2}\right)\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\left[\sin\left(\frac{1}{x}+\frac{1}{y}\right)v+\cos\left(\frac{1}{x}+\frac{1}{y}\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\\ &= v\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\left(-\frac{1}{y^2}\right)+u\cdot\cos\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(-\frac{1}{y^2}\right)\\ &= \sin\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\left(-\frac{1}{y^2}\right)+\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\cdot\cos\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(-\frac{1}{y^2}\right)\\ &= \left(-\frac{1}{y^2}\right)\cdot\mathrm{e}^{\frac{1}{x}+\frac{1}{y}}\left[\sin\left(\frac{1}{x}+\frac{1}{y}\right)v+\cos\left(\frac{1}{x}+\frac{1}{y}\right)\right]\end{aligned}$$

(3) 由定理 13.5.1 可得

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= (\mathrm{e}^x+\cos x\cos y)\cdot 2t+(\mathrm{e}^y+\sin x\sin y)\cdot 1\\ &= 2t\mathrm{e}^x+2t\cos x\cos y+\mathrm{e}^y-\sin x\sin y\\ &= 2t\mathrm{e}^{t^2}+2t\cos t^2\cos(1+t)+\mathrm{e}^{1+t}-\sin t^2\sin(1+t).\end{aligned}$$

(4) 由定理 13.5.2 可得

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= \ln y\cdot\left(\frac{1}{x}\right)\cdot 1+\ln x\cdot\left(\frac{1}{y}\right)\cdot 1\\ &= \frac{\ln y}{x}+\frac{\ln x}{y}\\ &= \frac{\ln(u-v)}{u+v}+\frac{\ln(u+v)}{u-v}.\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= \ln y\cdot\left(\frac{1}{x}\right)\cdot 1+\ln x\cdot\left(\frac{1}{y}\right)\cdot(-1)\\ &= \frac{\ln y}{x}-\frac{\ln x}{y}\\ &= \frac{\ln(u-v)}{u+v}-\frac{\ln(u+v)}{u-v}.\end{aligned}$$

□

2. 设 $z=\arctan\dfrac{y}{x}$, 证明: z 满足方程 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$.

证明. 引入中间变量: $t=\dfrac{y}{x}$, 则原来的函数可以写成

$$z=\arctan t,\qquad 其中t=\frac{y}{x},$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\\ &= \frac{1}{1+t^2}\cdot\left(-\frac{y}{x^2}\right)\\ &= -\frac{y}{x^2+y^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\\ &= \frac{1}{1+t^2}\cdot\frac{1}{x}\\ &= \frac{x}{x^2+y^2}.\end{aligned}$$

由此,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{xy}{x^2+y^2}+-\frac{xy}{x^2+y^2}=0.$$

∎

3. 设 $z=(x^2+y^2)^n$, 证明: z 满足方程

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4n^2z^{\frac{2n-1}{n}}.$$

证明. 由 $z=(x^2+y^2)^n$ 可得

$$\frac{\partial z}{\partial x}=n(x^2+y^2)^{n-1}\cdot 2x=2nx(x^2+y^2)^{n-1},$$
$$\frac{\partial z}{\partial y}=n(x^2+y^2)^{n-1}\cdot 2y=2ny(x^2+y^2)^{n-1},$$

由此,

$$\begin{aligned}(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2 &= 4n^2x^2(x^2+y^2)^{2n-2}+4n^2y^2(x^2+y^2)^{2n-2}\\ &= 4n^2(x^2+y^2)^{2n-2}(x^2+y^2)\\ &= 4n^2(x^2+y^2)^{2n-1}\\ &= 4n^2z^{\frac{2n-1}{n}}.\end{aligned}$$

∎

4. 设 $z=xy+x\mathrm{e}^{\frac{y}{x}}$, 证明: z 满足方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z.$$

证明. 由 $z=xy+x\mathrm{e}^{\frac{y}{x}}$ 可得

$$\frac{\partial z}{\partial x}=y+e^{\frac{y}{x}}+xe^{\frac{y}{x}}\cdot\left(-\frac{y}{x^2}\right)=y+e^{\frac{y}{x}}\left(1-\frac{y}{x}\right),$$
$$\frac{\partial z}{\partial y}=x+xe^{\frac{y}{x}}\cdot\left(\frac{1}{x}\right)=x+e^{\frac{y}{x}},$$

由此,

$$\begin{aligned}x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} &= xy+e^{\frac{y}{x}}(x-y)+xy+ye^{\frac{y}{x}}\\ &= xy+xy+xe^{\frac{y}{x}}\\ &= xy+z.\end{aligned}$$

∎

5. 求下列复合函数的偏导数 (设所涉及的函数都具有连续的偏导数):

(1) $z=f(x,x+y,xy)$, 求 z_x, z_y.

(2) $z=f(r\cos\theta,r\sin\theta)$, 求 z_r, z_θ.

(3) $u=f\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x},xyz\right)$, 求 u_x, u_y 和 u_z.

(4) $u=f(\sqrt[3]{x^2+y^2+z^2})$, 求 u_x, u_y 和 u_z.

解: **(1)** 这是两个自变量, 三个中间变量的情形, 记

$$u = x, \quad v = x + y, \quad w = xy,$$

为方便起见, 将 $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ 分别简记为 f_u, f_v 和 f_w, 则

$$\begin{aligned} z_x = \frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ &= f_u \cdot 1 + f_v \cdot 1 + f_w \cdot y \\ &= f_u + f_v + y f_w, \end{aligned}$$

$$\begin{aligned} z_y = \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \\ &= f_u \cdot 0 + f_v \cdot 1 + f_w \cdot x \\ &= f_v + x f_w, \end{aligned}$$

(2)] 这是两个自变量, 两个中间变量的情形, 记

$$u = r\cos\theta, \quad v = r\sin\theta,$$

为方便起见, 将 $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ 分别简记为 f_u 和 f_v, 则

$$\begin{aligned} z_r = \frac{\partial z}{\partial r} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial r} \\ &= f_u \cdot \cos\theta + f_v \cdot \sin\theta, \end{aligned}$$

$$\begin{aligned} z_\theta = \frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial \theta} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \theta} \\ &= f_u \cdot (-r\sin\theta) + f_v \cdot (r\cos\theta) \\ &= -r f_u \sin\theta + r f_v \cos\theta. \end{aligned}$$

(3) 这是三个自变量, 两个中间变量的情形, 记

$$u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \quad v = xyz,$$

为方便起见, 将 $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ 分别简记为 f_u 和 f_v, 则

$$\begin{aligned} u_x = \frac{\partial u}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ &= f_u \cdot \left(\frac{1}{y} - \frac{z}{x^2}\right) + f_v \cdot yz \\ &= \left(\frac{1}{y} - \frac{z}{x^2}\right) f_u + yz f_v, \end{aligned}$$

$$\begin{aligned} u_y = \frac{\partial u}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= f_u \cdot \left(-\frac{x}{y^2} + \frac{1}{z}\right) + f_v \cdot xz \\ &= \left(\frac{1}{z} - \frac{x}{y^2}\right) f_u + xz f_v, \end{aligned}$$

$$\begin{aligned} u_z = \frac{\partial u}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} \\ &= f_u \cdot \left(-\frac{y}{z^2} + \frac{1}{x}\right) + f_v \cdot xy \\ &= \left(\frac{1}{x} - \frac{y}{z^2}\right) f_u + xyf_v. \end{aligned}$$

(4) 这是三个自变量, 一个中间变量的情形, 记

$$v = \sqrt[3]{x^2+y^2+z^2},$$

为方便起见, 将 $\dfrac{\partial f}{\partial v}$ 分别简记为 f_v, 则

$$\begin{aligned} u_x = \frac{\partial u}{\partial x} &= \frac{\partial f}{\partial v}\frac{\partial u}{\partial x} \\ &= f_v \cdot \frac{1}{3}(x^2+y^2+z^2)^{-\frac{2}{3}} \cdot 2x \\ &= \frac{2}{3}x(x^2+y^2+z^2)^{-\frac{2}{3}} f_v, \end{aligned}$$

$$\begin{aligned} u_y = \frac{\partial u}{\partial y} &= \frac{\partial f}{\partial v}\frac{\partial u}{\partial y} \\ &= f_v \cdot \frac{1}{3}(x^2+y^2+z^2)^{-\frac{2}{3}} \cdot 2y \\ &= \frac{2}{3}y(x^2+y^2+z^2)^{-\frac{2}{3}} f_v, \end{aligned}$$

$$\begin{aligned} u_z = \frac{\partial u}{\partial z} &= \frac{\partial f}{\partial v}\frac{\partial u}{\partial x} \\ &= f_v \cdot \frac{1}{3}(x^2+y^2+z^2)^{-\frac{2}{3}} \cdot 2z \\ &= \frac{2}{3}z(x^2+y^2+z^2)^{-\frac{2}{3}} f_v. \end{aligned}$$

□

6. 求下列复合函数的全微分 (设所涉及的函数都具有连续的偏导数):

(1) $z = f(x+y, xy, x/y)$, 求 $\mathrm{d}z$; (2) $u = f\left(\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}\right)$, 求 $\mathrm{d}u$.

解: (1) 先求出 $\dfrac{\partial z}{\partial x}$ 与 $\dfrac{\partial z}{\partial y}$, 这是两个自变量, 三个中间变量的情形, 记

$$u = x+y, \quad v = xy, \quad w = \frac{x}{y},$$

为方便起见, 将 $\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}$ 与 $\dfrac{\partial f}{\partial w}$ 分别简记为 f_u, f_v 和 f_w, 则

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ &= f_u \cdot 1 + f_v \cdot y + f_w \cdot \frac{1}{y} \\ &= f_u + yf_v + \frac{1}{y}f_w, \end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}\\ &= f_u\cdot 1+f_v\cdot x+f_w\cdot(-\frac{x}{y^2})\\ &= f_u+xf_v-\frac{x}{y^2}f_w.\end{aligned}$$

由函数 z 的两个偏导在定义域内存在且连续, 根据定理 13.4.4 可知函数 z 可微, 且有

$$\begin{aligned}\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y\\ &= \left(f_u+yf_v+\frac{1}{y}f_w\right)\mathrm{d}x+\left(f_u+xf_v-\frac{x}{y^2}f_w\right)\mathrm{d}y\end{aligned}$$

(2) 先求出 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ 与 $\frac{\partial u}{\partial z}$, 这是三个自变量, 一个中间变量的情形, 记

$$v=\frac{x}{y}+\frac{y}{z}+\frac{z}{x},$$

为方便起见, 将 $\frac{\partial f}{\partial v}$ 简记为 f_v , 则

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &= f_v\cdot\left(\frac{1}{y}-\frac{z}{x^2}\right)\\ &= \left(\frac{1}{y}-\frac{z}{x^2}\right)f_v,\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &= f_v\cdot\left(-\frac{x}{y^2}+\frac{1}{z}\right)\\ &= \left(\frac{1}{z}-\frac{x}{y^2}\right)f_v,\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\\ &= f_v\cdot\left(-\frac{y}{z^2}+\frac{1}{x}\right)\\ &= \left(\frac{1}{x}-\frac{y}{z^2}\right)f_v,\end{aligned}$$

由函数 u 的两个偏导在定义域内存在且连续, 根据定理 13.4.4 可知函数 u 可微, 且有

$$\begin{aligned}\mathrm{d}u &= \frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z\\ &= \left(\frac{1}{y}-\frac{z}{x^2}\right)f_v\mathrm{d}x+\left(\frac{1}{z}-\frac{x}{y^2}\right)f_v\mathrm{d}y+\left(\frac{1}{x}-\frac{y}{z^2}\right)f_v\mathrm{d}z.\end{aligned}$$

□

7. 设 $f(\tau)$ 是可微函数.

(1) 证明: $z=F(x^2+y^2)$ 满足方程:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

(2) 证明: $u=F(x^2+y^2+z^2)$ 满足方程:

$$\left(1-\frac{y}{x}\right)\frac{\partial u}{\partial x}+\left(1-\frac{z}{y}\right)\frac{\partial u}{\partial y}+\left(1-\frac{x}{z}\right)\frac{\partial u}{\partial y}=0.$$

证明. (1) 由 $z=F(x^2+y^2)$ 可知, 这是两个自变量, 一个中间变量的情形, 记

$$u=x^2+y^2,$$

为方便起见, 将 $\dfrac{\partial F}{\partial u}$ 简记为 F_u , 则

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\\ &= F_u\cdot 2x\\ &= 2xF_u,\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\\ &= F_u\cdot 2y\\ &= 2yF_u,\end{aligned}$$

由此,

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=2xyF_u-2xyF_u=0.$$

(2) 由 $u=F(x^2+y^2+z^2)$ 可知这是三个自变量, 一个中间变量的情形, 记

$$v=x^2+y^2+z^2,$$

为方便起见, 将 $\dfrac{\partial F}{\partial v}$ 简记为 F_v , 则

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\\ &= F_v\cdot 2x\\ &= 2xF_v,\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}\\ &= F_v\cdot 2y\\ &= 2yF_v,\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\\ &= F_v\cdot 2z\\ &= 2z.F_v,\end{aligned}$$

由此,

$$
\begin{aligned}
& \left(1-\frac{y}{x}\right)\frac{\partial u}{\partial x}+\left(1-\frac{z}{y}\right)\frac{\partial u}{\partial y}+\left(1-\frac{x}{z}\right)\frac{\partial u}{\partial y} \\
= & \left(1-\frac{y}{x}\right)\cdot 2xF_v+\left(1-\frac{z}{y}\right)\cdot 2yF_v+\left(1-\frac{x}{z}\right)\cdot 2zF_v \\
= & 2F_v(x-y+y-z+z-x) \\
= & 0.
\end{aligned}
$$

∎

8. 称 $u=f(x,y,z)$ 是 n 次齐次函数, 如果对任何的 $t\in\mathbb{R}$, 成立

$$f(tx,ty,tz)=t^n f(x,y,z).$$

证明: 因为 $u=f(x,y,z)$ 是可微的 n 次齐次函数, 则 u 满足:

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

证明. 如果 $u=f(x,y,z)$ 是 n 次齐次函数, 故对任意的 $t\in\mathbb{R}$, 成立

$$f(tx,ty,tz)=t^n f(x,y,z), \tag{1}$$

对上面的等式两边关于 t 求偏导,

$$
\begin{aligned}
\frac{\partial f(tx,ty,tz)}{\partial t} &= x{f_1}'(tx,ty,tz)+y{f_2}'(tx,ty,tz)+z{f_3}'(tx,ty,tz) \qquad (2)\\
&= nt^{n-1}f(x,y,z)
\end{aligned}
$$

其中, ${f_1}'(tx,ty,tz), {f_2}'(tx,ty,tz), {f_3}'(tx,ty,tz)$ 分别表示左端 $f(tx,ty,tz)$ 对第一个中间变量 tx , 第二个中间变量 ty 与第三个中间变量 tz 的偏导数.

在 (**??**)令 $t=1$, 得

$$x{f_1}'(x,y,z)+y{f_2}'(x,y,z)+z{f_3}'(x,y,z)=nf(x,y,z),$$

又 $u=f(x,y,z)$, 故有

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

∎

9. 对所指定的变量和阶数求偏导数和微分:

$$(1)\quad z=\mathrm{e}^x\cos y, 求\ \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\partial y};\quad (2)\quad z=x^3+y^3-3xy, 求\ \frac{\partial^2 z}{\partial x^2}, \frac{\partial^4 z}{\partial x^2\partial y^4};$$

解: (1) 由

$$\frac{\partial z}{\partial x}=\mathrm{e}^x\cos y,$$

$$\frac{\partial z}{\partial y}=-\mathrm{e}^x\sin y,$$

得

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(\mathrm{e}^x\cos y)=\mathrm{e}^x\cos y,$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}(\mathrm{e}^x\cos y)=-\mathrm{e}^x\sin y.$$

(2) 由

$$\frac{\partial z}{\partial x}=3x^2-3y,$$
$$\frac{\partial z}{\partial y}=3y^2-3x,$$

得

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(3x^2-3y)=6x,$$

$$\frac{\partial^4 z}{\partial x^2\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial^3 z}{\partial x^2\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(6x)\right)=0$$

□

10. 求所指定的偏导数 (以下均假定所涉及的函数具有所需要阶数的连续偏导数).

(1) $z=f(x+y,xy)$, 求 $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x\partial y}$, $\dfrac{\partial^2 z}{\partial y^2}$;

(2) $u=f\left(\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}\right)$, 求 $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial x\partial y}$, $\dfrac{\partial^2 u}{\partial y^2}$;

(3) $z=f(x,y)$, $y=\varphi(x)$, 求 $\dfrac{d^2 z}{\mathrm{d}x^2}$;

(4) $z=f(x,x^2,x^3)$, 求 $\dfrac{d^2 z}{\mathrm{d}x^2}$.

解: (1) 记两个中间变量为

$$u=x+y,\quad v=xy,$$

为方便起见, 记

$$\frac{\partial f}{\partial u}=f_u,\quad \frac{\partial f}{\partial v}=f_v,\quad \frac{\partial^2 f}{\partial u^2}=f_{uu},\quad \frac{\partial^2 f}{\partial v^2}=f_{vv},\quad \frac{\partial^2 f}{\partial u\partial v}=f_{uv},\quad \frac{\partial^2 f}{\partial v\partial u}=f_{vu}$$

根据所给的二阶连续可微条件知 $f_{uv}=f_{vu}$, 则由

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=f_u\cdot 1+f_v\cdot y=f_u+yf_v,$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=f_u\cdot 1+f_v\cdot x=f_u+xf_v,$$

得

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \\
&= \frac{\partial}{\partial x}(f_u + y f_v) \\
&= f_{uu}\frac{\partial u}{\partial x} + f_{uv}\frac{\partial v}{\partial x} + y\left(f_{vu}\frac{\partial u}{\partial x} + f_{vv}\frac{\partial v}{\partial x}\right) \\
&= f_{uu}\cdot 1 + f_{uv}\cdot y + y(f_{vu}\cdot 1 + f_{vv}\cdot y) \\
&= f_{uu} + 2y f_{uv} + y^2 f_{vv},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \\
&= \frac{\partial}{\partial y}(f_u + y f_v) \\
&= f_{uu}\frac{\partial u}{\partial y} + f_{uv}\frac{\partial v}{\partial y} + y\left(f_{vu}\frac{\partial u}{\partial y} + f_{vv}\frac{\partial v}{\partial y}\right) \\
&= f_{uu}\cdot 1 + f_{uv}\cdot x + y(f_{vu}\cdot 1 + f_{vv}\cdot x) \\
&= f_{uu} + (x+y) f_{uv} + x y^2 f_{vv},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial y}(f_u + x f_v) \\
&= f_{uu}\frac{\partial u}{\partial y} + f_{uv}\frac{\partial v}{\partial y} + x\left(f_{vu}\frac{\partial u}{\partial y} + f_{vv}\frac{\partial v}{\partial y}\right) \\
&= f_{uu}\cdot 1 + f_{uv}\cdot x + x(f_{vu}\cdot 1 + f_{vv}\cdot x) \\
&= f_{uu} + 2x f_{uv} + x^2 f_{vv}.
\end{aligned}$$

(2) 记中间变量为

$$v = \frac{x}{y} + \frac{y}{z} + \frac{z}{x},$$

为方便起见, 记

$$\frac{\partial f}{\partial v} = f_v, \quad \frac{\partial^2 f}{\partial v^2} = f_{vv},$$

由

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_v\left(\frac{1}{y} - \frac{z}{x^2}\right),$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_v\left(\frac{1}{z} - \frac{x}{y^2}\right),$$

得

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(f_v\left(\frac{1}{y}-\frac{z}{x^2}\right)\right) \\
&= f_{vv}\frac{\partial v}{\partial x}\cdot\left(\frac{1}{y}-\frac{z}{x^2}\right)+f_v\cdot\frac{\partial}{\partial x}\left(\frac{1}{y}-\frac{z}{x^2}\right) \\
&= f_{vv}\left(\frac{1}{y}-\frac{z}{x^2}\right)\cdot\left(\frac{1}{y}-\frac{z}{x^2}\right)+f_v\cdot\frac{2z}{x^3} \\
&= f_{vv}\left(\frac{1}{y}-\frac{z}{x^2}\right)^2+\frac{2z}{x^3}f_v,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(f_v\left(\frac{1}{y}-\frac{z}{x^2}\right)\right) \\
&= f_{vv}\frac{\partial v}{\partial y}\cdot\left(\frac{1}{y}-\frac{z}{x^2}\right)+f_v\cdot\frac{\partial}{\partial y}\left(\frac{1}{y}-\frac{z}{x^2}\right) \\
&= f_{vv}\left(\frac{1}{z}-\frac{x}{y^2}\right)\cdot\left(\frac{1}{y}-\frac{z}{x^2}\right)+f_v\cdot\left(-\frac{1}{y^2}\right) \\
&= f_{vv}\cdot\left(\frac{1}{z}-\frac{x}{y^2}\right)\left(\frac{1}{y}-\frac{z}{x^2}\right)-\frac{1}{y^2}f_v,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(f_v\left(\frac{1}{z}-\frac{x}{y^2}\right)\right) \\
&= f_{vv}\frac{\partial v}{\partial y}\cdot\left(\frac{1}{z}-\frac{x}{y^2}\right)+f_v\cdot\frac{\partial}{\partial y}\left(\frac{1}{z}-\frac{x}{y^2}\right) \\
&= f_{vv}\left(\frac{1}{z}-\frac{x}{y^2}\right)\cdot\left(\frac{1}{z}-\frac{x}{y^2}\right)+f_v\cdot\frac{2x}{y^3} \\
&= f_{vv}\left(\frac{1}{z}-\frac{x}{y^2}\right)^2+\frac{2x}{y^3}f_v.
\end{aligned}
$$

(3) 为方便起见, 记

$$
\frac{\partial f}{\partial x}=f_x,\quad \frac{\partial f}{\partial y}=f_y,\quad \frac{\partial^2 f}{\partial x^2}=f_{xx},\quad \frac{\partial^2 f}{\partial y^2}=f_{yy},\quad \frac{\partial^2 f}{\partial x\partial y}=f_{xy},\quad \frac{\partial^2 f}{\partial y\partial x}=f_{yx}
$$

根据所给的二阶连续可微条件知 $f_{xy}=f_{yx}$, 则由

$$
\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=f_x\cdot 1+f_y\cdot\phi'(x)=f_x+\phi'(x)f_y,
$$

得

$$\begin{aligned}\frac{\mathrm{d}^2z}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}(f_x+\phi'(x)f_y)\\ &= f_{xx}\frac{\mathrm{d}x}{\mathrm{d}x}+f_{xy}\frac{\mathrm{d}y}{\mathrm{d}x}+\phi'(x)\left(f_{yx}\frac{\mathrm{d}x}{\mathrm{d}x}+f_{yy}\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)\cdot f_y\\ &= f_{xx}\cdot 1+f_{xy}\cdot\phi'(x)+\phi'(x)(f_{yx}\cdot 1+f_{yy}\cdot\phi'(x))+\phi''(x)f_y\\ &= f_{xx}+2\phi'(x)f_{xy}+(\phi'(x))^2f_{yy}+\phi''(x)f_y.\end{aligned}$$

(4) 记三个中间变量为

$$u=x,\quad v=x^2,\quad w=x^3$$

为方便起见, 记

$$\begin{aligned}&\frac{\partial f}{\partial u}=f_u,\quad \frac{\partial f}{\partial v}=f_v,\quad \frac{\partial f}{\partial w}=f_w,\\ &\frac{\partial^2 f}{\partial u^2}=f_{uu},\quad \frac{\partial^2 f}{\partial v^2}=f_{vv},\quad \frac{\partial^2 f}{\partial w^2}=f_{ww},\\ &\frac{\partial^2 f}{\partial u\partial v}=f_{uv},\quad \frac{\partial^2 f}{\partial v\partial u}=f_{vu},\quad \frac{\partial^2 f}{\partial u\partial w}=f_{uw},\\ &\frac{\partial^2 f}{\partial w\partial u}=f_{wu},\quad \frac{\partial^2 f}{\partial v\partial w}=f_{vw},\quad \frac{\partial^2 f}{\partial w\partial v}=f_{wv},\end{aligned}$$

根据所给的二阶连续可微条件知 $f_{uv}=f_{vu},\quad f_{uw}=f_{wu},\quad f_{vw}=f_{wv}$, 则由

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}=f_u\cdot 1+f_v\cdot 2x+f_w\cdot 3x^2=f_u+2xf_v+3x^2f_w,$$

得

$$\begin{aligned}\frac{\mathrm{d}^2z}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}(f_u+2xf_v+3x^2f_w)\\ &= f_{uu}\frac{\mathrm{d}u}{\mathrm{d}x}+f_{uv}\frac{\mathrm{d}v}{\mathrm{d}x}+f_{uw}\frac{\mathrm{d}w}{\mathrm{d}x}\\ &\quad +2f_v+2x\left(f_{vu}\frac{\mathrm{d}u}{\mathrm{d}x}+f_{vv}\frac{\mathrm{d}v}{\mathrm{d}x}+f_{vw}\frac{\mathrm{d}w}{\mathrm{d}x}\right)\\ &\quad +6xf_w+3x^2\left(f_{wu}\frac{\mathrm{d}u}{\mathrm{d}x}+f_{wv}\frac{\mathrm{d}v}{\mathrm{d}x}+f_{vw}\frac{\mathrm{d}w}{\mathrm{d}x}\right)\\ &= f_{uu}\cdot 1+f_{uv}\cdot 2x+f_{uw}\cdot 3x^2\\ &\quad +2f_v+2x(f_{vu}\cdot 1+f_{vv}\cdot 2x+f_{vw}\cdot 3x^2)\\ &\quad +6xf_w+3x^2(f_{wu}\cdot 1+f_{wv}\cdot 2x+f_{vw}\cdot 3x^2)\\ &= f_{uu}+4xf_{uv}+6x^2f_{uw}+4x^2f_{vv}+12x^3f_{vw}+9x^4_{ww}+2f_v+6xf_w\end{aligned}$$

□

11. 证明: 如果 $z=x^\alpha y^\beta$ ($\alpha+\beta=1$ 在 $x>0,y>0$), 则 z 满足

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2.$$

证明. 由

$$\frac{\partial z}{\partial x} = \alpha x^{\alpha-1}y^{\beta},$$

$$\frac{\partial z}{\partial y} = \beta x^{\alpha}y^{\beta-1},$$

得

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\\ &= \frac{\partial}{\partial y}(\alpha x^{\alpha-1}y^{\beta})\\ &= \alpha(\alpha-1)x^{\alpha-2}y^{\beta},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\\ &= \frac{\partial}{\partial y}(\alpha x^{\alpha-1}y^{\beta})\\ &= \alpha\beta x^{\alpha-1}y^{\beta-1},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial}{\partial y}(\beta x^{\alpha}y^{\beta-1})\\ &= \beta(\beta-1)x^{\alpha}y^{\beta-2}.\end{aligned}$$

由此,

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} &= \alpha(\alpha-1)x^{\alpha-2}y^{\cdot}\beta(\beta-1)x^{\alpha}y^{\beta-2}\\ &= \alpha\beta(\alpha-1)(\beta-1)x^{2\alpha-2}y^{2\beta-2}\\ &= \alpha\beta[\alpha\beta-(\alpha+\beta)+1]x^{2\alpha-2}y^{2\beta-2},\end{aligned}$$

又 $\alpha+\beta=1$, 故

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} &= \alpha\beta\alpha\beta x^{2\alpha-2}y^{2\beta-2}\\ &= \alpha^2\beta^2x^{2\alpha-2}y^{2\beta-2}\\ &= \left(\frac{\partial^2 z}{\partial x\partial y}\right)^2.\end{aligned}$$

$\blacksquare$

12. 证明: 函数 $z=\ln\sqrt{(x-a)^2+(y-b)^2}$ 满足 Laplace 方程:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0.$$

证明. 由

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\cdot\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}}=\frac{x-a}{(x-a)^2+(y-b)^2},$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}\cdot\frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}}=\frac{y-b}{(x-a)^2+(y-b)^2},$$

得

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\\ &= \frac{\partial}{\partial y}\left(\frac{x-a}{(x-a)^2+(y-b)^2}\right)\\ &= \frac{(x-a)^2+(y-b)^2-2(x-a)^2}{[(x-a)^2+(y-b)^2]^2}\\ &= \frac{(y-b)^2-(x-a)^2}{[(x-a)^2+(y-b)^2]^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial}{\partial y}\left(\frac{x-a}{(y-b)^2+(y-b)^2}\right)\\ &= \frac{(x-a)^2+(y-b)^2-2(y-b)^2}{[(x-a)^2+(y-b)^2]^2}\\ &= \frac{(x-a)^2-(y-b)^2}{[(x-a)^2+(y-b)^2]^2},\end{aligned}$$

由此,

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{(y-b)^2-(x-a)^2}{[(x-a)^2+(y-b)^2]^2}+\frac{(x-a)^2-(y-b)^2}{[(x-a)^2+(y-b)^2]^2}=0.$$

∎

---

13. 证明复合函数的可微性定理 (即定理 **??**).

证明. 由假设 $u=\phi(x,y), y=\psi(x,y)$ 在点 (x,y) 可微, 于是

$$\triangle u=\frac{\partial u}{\partial x}\triangle x+\frac{\partial u}{\partial y}\triangle y+\alpha_1\triangle x+\beta_1\triangle y, \tag{1}$$

$$\triangle v=\frac{\partial v}{\partial x}\triangle x+\frac{\partial v}{\partial y}\triangle y+\alpha_2\triangle x+\beta_2\triangle y, \tag{2}$$

其中当 $\triangle x,\triangle y$ 趋于零时, $\alpha_1,\alpha_2,\beta_1,\beta_2$ 都趋向于零. 有由 $z=f(u,v)$ 在点 (u,v) 可微, 所以

$$\triangle z=\frac{\partial z}{\partial u}\triangle u+\frac{\partial z}{\partial v}\triangle v+\alpha\triangle u+\beta\triangle v, \tag{3}$$

其中当 $\triangle u,\triangle v,\to 0$ 时, $\alpha,\beta\to 0$ (我们补充 α,β 之定义使当 $\triangle u=0,\triangle v=0$ 时, $\alpha=\beta=0$), 将 (**??**), (**??**) 代入 (**??**), 得

$$\begin{aligned}\triangle z &= \left(\frac{\partial z}{\partial u}+\alpha\right)\left(\frac{\partial u}{\partial x}\triangle x+\frac{\partial u}{\partial y}\triangle y+\alpha_1\triangle x+\beta_1\triangle y\right)\\ &\quad+\left(\frac{\partial z}{\partial v}+\beta\right)\left(\frac{\partial v}{\partial x}\triangle x+\frac{\partial v}{\partial y}\triangle y+\alpha_2\triangle x+\beta_2\triangle y\right).\end{aligned}$$

整理后

$$\triangle z=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\triangle x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\triangle y+\bar{\alpha}\triangle x+\bar{\beta}\triangle y, \tag{4}$$

其中

$$\bar{\alpha}=\frac{\partial z}{\partial u}\alpha_1+\frac{\partial z}{\partial v}\alpha_2+\frac{\partial u}{\partial x}\alpha+\frac{\partial v}{\partial x}\alpha+\alpha\alpha_1+\beta\alpha_2, \tag{5}$$

$$\bar{\beta}=\frac{\partial z}{\partial u}\beta_1+\frac{\partial z}{\partial v}\beta_2+\frac{\partial u}{\partial y}\alpha+\frac{\partial v}{\partial y}\alpha+\alpha\beta_1+\beta\beta_2. \tag{6}$$

由于 $\phi(x,y),\psi(x,y)$ 在点 (x,y) 可微, 因此它们在点 (x,y) 都连续, 即当 $\triangle x,\triangle y\to 0$ 时, 有 $\triangle u,\triangle v\to 0$. 从而也有 $\alpha\to 0,\beta\to 0$, 以及 $\alpha_1,\alpha_2,\beta_1,\beta_2\to 0$. 于是在 (**??**), (**??**) 式中, 当 $\triangle x,\triangle y\to 0$, 有 $\bar{\alpha},\beta\to 0$. 故由 (**??**) 式推得复合函数 $z=f[\phi(x,y),\psi(x,y)]$ 在 (x,y) 也可微, 并求得 z 关于 x 和 y 的偏导数为

$$\frac{\partial z}{\partial x}|_{(x,y)}=\frac{\partial z}{\partial u}|_{(u,v)}\frac{\partial u}{\partial x}|_{(x,y)}+\frac{\partial z}{\partial v}|_{(u,v)}\frac{\partial v}{\partial x}|_{(x,y)},$$

$$\frac{\partial z}{\partial y}|_{(x,y)}=\frac{\partial z}{\partial u}|_{(u,v)}\frac{\partial u}{\partial y}|_{(x,y)}+\frac{\partial z}{\partial v}|_{(u,v)}\frac{\partial v}{\partial y}|_{(x,y)}.$$

从而其全微分为

$$\begin{aligned}\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y\\ &= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y.\end{aligned}$$

∎

14. 设 $z=f(x,y)$ 具有连续的一阶偏导数. 证明

(1) 如果满足方程 $xf_x(x,y)+yf_y(x,y)=0$, 则可以写 $f(r\cos\theta,r\sin\theta)=F(\theta)$;

(2) 如果满足方程 $yf_x(x,y)-xf_y(x,y)=0$, 则可以写 $f(r\cos\theta,r\sin\theta)=G(r)$.

证明. (1) 即证 $\frac{\partial f}{\partial r}=0$, 因为若 $\frac{\partial f}{\partial r}=0$, 则有 $f(r\cos\theta,r\sin\theta)=F(\theta)$, 事实上, 引入中间变量

$$x=r\cos\theta,\quad y=r\sin\theta,$$

记

$$\frac{\partial f}{\partial x}=f_x(x,y),\quad \frac{\partial f}{\partial y}=f_y(x,y),$$

则

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\ &= f_x(x,y)\cdot\cos\theta+f_y(x,y)\cdot\sin\theta\\ &= f_x(x,y)\cdot\frac{x}{r}+f_y(x,y)\cdot\frac{y}{r}\\ &= \frac{1}{r}(xf_x(x,y)+yf_y(x,y)),\end{aligned}$$

由已知, $f(x,y)$ 满足方程 $xf_x(x,y)+yf_y(x,y)=0$, 故

$$\frac{\partial f}{\partial r}=\frac{1}{r}\cdot 0=0,\quad \text{证得结论.}$$

(2) 即证 $\dfrac{\partial f}{\partial \theta}=0$, 因为若 $\dfrac{\partial f}{\partial \theta}=0$, 则有 $f(r\cos\theta, r\sin\theta)=G(r)$, 由 (1), 得

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= f_x(x,y)\cdot(-r\sin\theta)+f_y(x,y)\cdot(r\cos\theta) \\
&= -\frac{y}{\sin\theta}f_x(x,y)+\frac{x}{\cos\theta}f_y(x,y) \\
&= -yf_x(x,y+xf_y(x,y),
\end{aligned}$$

由已知, $f(x,y)$ 满足方程 $yf_x(x,y)-xf_y(x,y)=0$, 故

$$\frac{\partial f}{\partial \theta}=0,\quad \text{证得结论.}$$

$\blacksquare$

$$\|\cdot\|_1 \lesssim \|\cdot\|_2$$