The use of complex integers in the continued fraction (3) offers no difficulty. We are led to the geometric picture of the continued fraction as a chain of spheres, beginning with the plane $z=1$, and proceeding thence from sphere to tangent sphere. We have here the same possibilities of convergence or divergence as before. At the worst we can set up continued fractions whose convergents are present in every small region of the complex plane. There is no immediate generalization of the simple continued fraction to the complex case. There are, however, schemes for developing a complex number in a continued fraction, the best processes being due to Hurwitz.*

The analogue of Hurwitz's theorem on rational approximations to an irrational number was first discovered by the present author.† The inequality (1) is replaced by

$$\left| \frac{p}{q} - \omega \right| < \frac{k}{q\bar{q}},$$

and the minimum k for which infinitely many fractions always satisfy the inequality is $k=1/\sqrt{3}$. The proof of this, however, is not elementary.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

MATHEMATICAL EDUCATION IN GERMANY BEFORE 1933

RICHARD COURANT, New York University

At the suggestion of the editors I shall try to give a brief account of trends in the teaching of mathematics in Germany, particularly at German universities, in the period from the World War until 1933.

The situation of mathematics in Germany, since the early part of the 19th century, has been pivoted around a definite connection between university and high school. In a sweeping reform following the French Revolution, the institution of the German "humanistisches Gymnasium" was established. Teachers at these institutions were required to undergo a very thorough academic preparation, and the task of preparing them was entrusted to the philosophical faculties of the universities. The teachers' training was in no way of an elementary char-

* Acta Mathematica, vol. 11, 1887, pp. 187–200.

† L. R. Ford, Transactions of the American Mathematical Society, vol. 27, 1925, pp. 146–154.

acter. First rate mathematical scholars—following the example of classical philologists—gathered a crowd of enthusiastic students and lectured on the most advanced subjects on the boundary line of research. The pace was set by Jacobi, who discussed in his class, ten hours a week, his newest discoveries on elliptic functions.

The requirements of the state board examination for high school teachers were excessively high; they called for full-fledged scholars and probably could never be enforced even approximately. Still, published yearly reports of the better high schools sometimes contained important mathematical contributions from the staff, as for instance the famous paper on Abelian functions by Weierstrass. A considerable number of the famous German mathematicians of the 19th century started their careers as more or less successful teachers in high schools.

For the development of mathematics at universities it certainly was fortunate that a rather stationary and responsive audience was assured. But the young teacher, coming from the university to the school, which he had left five or six years ago, found little use for his specialized knowledge of Abelian functions or other sophisticated subjects, while he had more or less forgotten the material to be covered in the high school curriculum.

However, on the whole, the system worked quite satisfactorily as long as the number of teachers and of high school students was comparatively small, and the latter somewhat restricted to a group seriously striving for a higher scholastic education. But gradually this situation changed with the progress of democratization in education. New types of high schools sprang up, drawing away from the rigorous humanistic ideal and tending more to useful lines. After the war of 1870 this process led to an enormous increase in the demand for teachers, not all of whom were able to attain high scholastic standards. But nothing was changed in the fundamental system of four to six years of graduate study as preparation for school teachers. The number of mathematics chairs at universities increased considerably; even more the number of students in the courses. The contradictory character of the relationship between university and high school became a burning question. What could be done to preserve the university standards and still meet the real needs of a broadening school system?

Under Felix Klein's inspiring leadership progress was made in bridging the gap from both sides. The curriculum of the high schools was modernized. The concepts of function and of limit, the elements of calculus, analytic geometry, projective geometry, *etc.*, conquered much space in the high school curriculum, superseding antiquated routine material. On the other hand, in the university, survey courses were introduced, Felix Klein's lectures on elementary mathematics from a higher viewpoint being the outstanding example. It was very fortunate that also other academic teachers of highest rank, *e.g.*, Hilbert, followed this example enthusiastically. Hilbert's courses in foundations of geometry and in intuitive geometry, and his lectures on the problem of squaring the circle are famous instances.

The process of mutual adaptation was interrupted by the World War, the aftermath of which created intensified problems. The number of high school students leaped to unknown heights. A desperate demand for large numbers of teachers was voiced officially. Thus the classrooms of the universities became overcrowded with students who were not driven by thirst for higher learning but attracted by the prospect of a secure job. Soon the teaching jobs were filled, but the inert stream of students in the universities preparing for the teaching profession continued.

At the University of Berlin in some years more than 600 students registered for a single course in number theory, and about as many in calculus and analytic geometry. At Göttingen courses on the theory of functions of a complex variable had an attendance of up to 330 students. In highly specialized and advanced courses 50–100 students could often be found, many of whom were also preparing for a teaching job. While at the high schools the reform work proceeded organically, the universities were facing a more complex problem. How could they give the masses a really worth-while instruction without hopelessly lowering the general level and without giving up the ideals of academic freedom?

The complexity of the situation was enhanced by different factors: While the great majority of mathematics students still was preparing for the state board examination for teachers (incidentally at the same time studying two minor subjects, usually physics and another science), an increasing number of students was considering other professions, *e.g.*, the actuarial career, and at the principal mathematical centers there was always a group of deeply interested young scholars, partly from abroad, with outspoken academic ambitions. This smaller elite group was apt to form a separate stratum in closer contact with the staff and rather separated from the bulk of the student body. Naturally the professors were more interested in this top group than in the crowd.

Another difficulty arose from the traditional academic freedom in learning. There were no official examinations and no marks all through the years between high school and the state board examination (except examinations for tuition scholarships). It happened that candidates, after five years of attending advanced classes and having had little contact with instructors and fellow students, came up for examination, and to the dismay of both sides it turned out that they knew literally nothing about mathematics and had stupidly wasted their years at the university. The absolute freedom for the student to select his courses and to conduct his studies to his own taste was sometimes a handicap for the majority, although of the highest benefit to the top group. A similar effect came from the academic freedom in teaching. The professor was absolutely free in the selection and presentation of his material, not bound to any curriculum. Thus it actually happened that elementary calculus was taught in a class of several hundred students on the basis of Brouwer's intuitionism, a very interesting experiment for the lecturer and three or four students but less useful for the others.

During the period after 1919 the technological institutes ("Technische Hochschulen") began to compete with the universities in the task of training high school teachers in mathematics and physics. The mathematics classes at these institutions were terribly overcrowded by the enormous number of prospective engineers. But the professors of mathematics were anxious to have also some students interested in mathematics as such, and the technological institutes succeeded in being opened also for teacher training. However, this competition of the technological institutes did not amount to much.

Since there was little uniformity in the different universities, I shall, in some detail, describe only the situation which had developed at Göttingen.

The beginner, entering the university and facing a confusing variety of possibilities, found some advice in a printed guide and, if he desired, could consult with assistants assigned to advisory work. He was not bound to heed suggestions, but usually in the first year he took differential and integral calculus, 4 hours a week, analytic geometry, also 4 hours, and the *Anfängerpraktikum*, practical exercises for beginners, 3–5 consecutive hours a week. (In addition a 5 hour lecture course on experimental physics, and other courses.) The courses never followed closely a definite textbook. The students were supposed to take and work out notes or to study the official record of the lectures in the reading room. The level of the calculus courses corresponded more or less to an advanced calculus course at American universities.

It was a strict rule in the science faculty that these beginners' courses be given by experienced, responsible professors and not by young instructors. But a course was hardly ever repeated by the same professor in consecutive years. Thus entirely different, even antagonistic tendencies in presentation were offered to the student, who was urged to supplement his information by independent study of the literature easily accessible in the large mathematical reference library. While the lecture courses followed more or less the old tradition of separation between the talking professor and the passive audience, the five hour period of practical exercises was an important innovation for coping with the problem of mass instruction. The students, more than 200, were given in advance a mimeographed sheet with problems, some mere exercises, some requiring a certain amount of inventive thinking. They were supposed to come to class after having already given some thought to the problems, while the professor in charge, prior to the class meeting, held a conference with a group of assistants (about one for every 15–20 students) concerning the problems, the different methods and aspects of the solutions, and the probable difficulties of the students. The class room consisted of two large halls with big tables and blackboards, and the 5 hours (or less if the student preferred) were spent not only in finding the solutions, but mainly in personal discussion between students and assistant instructors. There was ample opportunity for anyone who cared to clarify any subject connected with the beginners' courses in discussion with competent people. The solutions were later written up by the students, handed in, criticized in writing by a conference of the assistants, and returned and individually discussed at the next meeting. Records with marks were taken. This institution proved astonishingly efficient. A considerable percentage of the students really learned something; others realized that it was best for them to give up mathematics, and soon the few with talents were discovered and given particular consideration. Forming of study groups among the students was encouraged, and collusion in solving the problems was not discouraged.

After the beginners' courses, which were sometimes supplemented by a course on descriptive geometry, the period of the normal full courses, "Kursusvorlesungen," started. A great variety was offered here: Theory of functions of a complex variable, differential equations, functions of real variables, potential theory, partial differential equations of physics, analytic dynamics, algebra, elementary number theory, differential geometry, projective and non-euclidean geometry, calculus of variations, group theory, mechanics of continuous bodies, and many other subjects were covered. The average student took two or three courses a term, altogether often more than 12 of these before his examination. Theory of functions of a complex variable was always included, but generally there was much leeway left in the selection.

In the same category belonged the full courses on elementary mathematics from a higher viewpoint, and other survey courses. Many of these "Kursusvorlesungen" again were attended by hundreds of students, and again an attempt was made to reach at least a part of them personally in additional two hour periods of exercises ("Übungen"), often held by one or more assistants. Here problems of a more advanced type were discussed, sometimes in writing, but not on such a systematic basis as in the beginners' "Praktikum."

At Göttingen this method of supplementing lecture courses by "Übungen" for larger groups of students was made possible by the institution of the mathematical assistant. Before the war, assistantships in mathematics hardly existed except for a few such positions at Göttingen and Berlin. Future professors were recruited from the ranks of the "Privatdozenten," voluntary academic teachers admitted to teaching only after a very careful process of selection, but not paid except by student fees. They came mostly from the moderately well-to-do middle class and were more or less able to support themselves. Only at some larger institutions, like Göttingen and Berlin, did income from student fees suffice to support a few of the less wealthy "Privatdozenten," and, in certain exceptional cases, government fellowships, *etc.*, made academic careers possible for promising students without means. After the destruction of middle class wealth by war and inflation, it became imperative to put the financial care for the next academic generation on a more systematic basis. Thus most "Privatdozenten" were paid some modest salary by making them assistants or giving them some official duties in the teaching process, and it was only by this definite innovation that the problem of instruction for larger masses could be tackled. At the university of Göttingen, for example, where, before the war, only one full-time and one or two part-time assistants in mathematics existed, there were in 1929 more than 10 full-time assistants, all actual or prospective "Privatdozenten," and in addition an indefinite number of part-time assistants according to the needs of the "Praktikum" and "Übungen." Generally also these part-time assistants were fully trained young scholars with the doctor's degree and some record of independent research.

After a student had taken a few of the "Kursusvorlesungen," he usually started attending some courses of the highest category, so-called special lectures and seminars. These were mostly two hour courses, and covered an enormous variety of subjects according to the personal taste and interest of the professor or instructor. Very often these special courses concerned the particular field in which the instructor was doing research and led directly to or beyond the boundary line of general knowledge. It was mostly in connection with these special courses and seminars that actual research was done by teachers and students. Little special consideration was given to students who were merely sitting in and did not cooperate. Also in these courses and seminars, a considerable number of prospective school teachers were active. The state board examination required written theses and very often these theses originated in such advanced courses or seminars and were the nuclei of doctoral theses. Subjects and standards of these state board theses varied greatly. A few examples: Reciprocity theorems in number theory; Morse's relations for stationary points; Morse's theory of conjugate points in the calculus of variations; the role of semicontinuity in the calculus of variations; theory of characteristics for hyperbolic systems of differential equations; characteristic values of partial differential equations in their dependence on the domain; finding explicit formulas for the conformal mapping of given Riemann surfaces on plane domains; critical survey of certain current high school textbooks of mathematics from the viewpoint of mathematical correctness; and so on.

Not infrequently these theses contained some modest original contribution, and in most cases they at least proved that the student had studied some field with real understanding. Sometimes, it is true, results were very poor and unsatisfactory.

At other universities different attempts were made to cope with the mass problem; *e.g.*, at the University of Berlin there was a systematic organization of many smaller study groups of students under the guidance of advanced students and younger instructors. At smaller universities with a much smaller number of students and in the absence of a coherent group of ambitious future scholars, it was easier to maintain personal contact between responsible professors and a large portion of the student body. But almost nowhere was it possible to restore the idyllic pre-war situation, where the whole mathematics group at a university was somewhat like a family with mutual personal confidence and contacts. Where this mathematical "family life" was preserved and even developed after the war, as in Göttingen, it was really restricted to the comparatively small, though by no means exclusive, top group. This class division could not fail to create among the less fortunate majority the psychological atmosphere of an inferiority complex, which so easily becomes dangerous for scientific institutions and which was, in connection with the hopeless outlook regarding jobs, an important factor in the events at the universities in 1933.

While the period between the war and 1933 did not, in spite of honest attempts, solve burning problems of mathematical instruction at the universities, it certainly produced more mathematical progress than most previous periods of similar length.

Since 1933, the total revolution within the system of German universities has so drastically curtailed the number of students that the mass problem of the previous period simply no longer exists. It seems premature to forecast the trend of the future development of mathematics at German universities.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A LINKAGE FOR DESCRIBING CURVES PARALLEL TO THE ELLIPSE

R. C. YATES, University of Maryland

Any point P of a bar AB with its ends moving along two perpendicular lines describes an ellipse. If this bar be joined at its midpoint to the origin (the intersection of the lines) by another bar, the half of the original bar not containing P may be discarded.

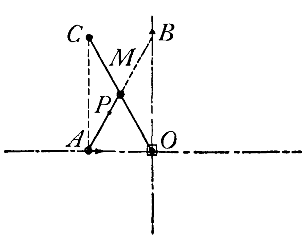

FIG. 1

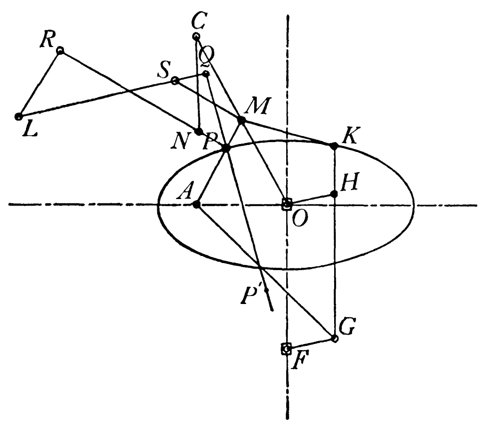

FIG. 2

In any position of the jointed bars the direction of motion of M in Fig. 1 is perpendicular to OM while A has always a horizontal direction. Thus the instantaneous center of rotation of the bar AM is the point C where the vertical line through A meets OM extended. The normal to the ellipse at P then must