15.094J: Robust Modeling, Optimization, Computation

Lecture 4: RLO: Probabilistic Guarantees

Outline

Objectives Today

- Probabilistic Guarantees for RLO
- Insights in selecting parameters

Row-wise Ellipsoidal uncertainty

- RO:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \max_{\boldsymbol{a}_i \in U_i} \boldsymbol{a}_i'\boldsymbol{x} \leq b_i. \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

- $U_i = \{\boldsymbol{a}_i | \ \boldsymbol{a}_i = \overline{\boldsymbol{a}}_i + \boldsymbol{\Delta}_i'\boldsymbol{u}_i, \ ||\boldsymbol{u}_i||_2 \leq \rho\}, \ \boldsymbol{\Delta}_i : \ k_i \times n, \ \boldsymbol{u}_i : k_i \times 1.$
- RC:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \overline{\boldsymbol{a}}_i'\boldsymbol{x} + \rho||\boldsymbol{\Delta}_i\boldsymbol{x}||_2 \leq b_i, \qquad i = 1, \ldots, m. \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

Probabilistic Guarantee

- Suppose u_i are independent, have zero mean and have support in $[-1, 1]$.
- Suppose that $\boldsymbol{x}$ satisfies $\overline{\boldsymbol{a}}'\boldsymbol{x} + \rho||\boldsymbol{\Delta x}|| \leq b$.
- Then
$$P(\tilde{\boldsymbol{a}}'x > b) \leq e^{-\rho^2/2}.$$
- **Remarkable property**: Independent of the distributions of $\boldsymbol{u}$ (we do not even require identical distributions).
- How to select ρ: Suppose our tolerance for infeasibility is ϵ, that is $P(\tilde{\boldsymbol{a}}'\boldsymbol{x} > b) \leq \epsilon$.
- Use $\epsilon = e^{-\frac{\rho^2}{2}}$, select $\rho = \sqrt{2\log\left(\frac{1}{\epsilon}\right)}$.

ϵ	ρ
10^{-6}	5.25
10^{-5}	4.79
10^{-4}	4.29
10^{-3}	3.71
10^{-2}	3.03
10^{-1}	2.14

Proof from First Principles

- Let $X(\boldsymbol{\xi}) = w_0 + \sum_{i=1}^{k} w_i \xi_i$, where ξ_i are independent with zero mean and with support in $[-1, 1]$.
- Let $\boldsymbol{w} = (w_1, \ldots, w_k)'$. We will first show that

$$P(X(\boldsymbol{\xi}) > 0) = P\left(w_0 + \sum_{i=1}^{k} w_i \xi_i > 0\right) \leq exp\left(-\frac{w_0^2}{2||\boldsymbol{w}||^2}\right).$$

$$P(X(\boldsymbol{\xi}) > 0) = \int \chi(X(\boldsymbol{\xi}))\, dP(\boldsymbol{\xi}), \quad \chi(s) = \begin{cases} 0, & s \leq 0, \\ 1, & s > 0 \end{cases}$$

- Note that $\chi(s) \leq \gamma(s) = e^s$.
- Let $\alpha > 0$. Note also that $\chi(s) = \chi(\alpha \cdot s) \leq \gamma(\alpha \cdot s)$.
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$$P(X(\boldsymbol{\xi}) > 0) \leq E[exp(\alpha w_0 + \sum_{i=1}^{k} \alpha w_i \xi_i)] = exp(\alpha w_0) \prod_{i=1}^{k} E[exp(\alpha w_i \xi_i)].$$

Proof continued

- For every random variable ξ with zero mean and support in $[-1,1]$

$$E[e^{t\xi}] \le e^{t^2/2}.$$

- Let $f(s) = e^{ts} - \frac{e^t - e^{-t}}{2}s$.
- $f(s)$ convex in s. Maximum in $[-1,1]$ is at endpoint.
- $\max_{|s|\le 1} f(s) = f(1) = f(-1) = \frac{e^t + e^{-t}}{2}$.

$$\begin{aligned} E[e^{t\xi}] &= \int f(s)\, dP(s) && \text{[zero mean]} \\ &\le \max_{|s|\le 1} f(s) \\ &= \frac{e^t + e^{-t}}{2} \\ &\le e^{t^2/2} && \text{[Taylor series].} \end{aligned}$$

Proof continued

- For all $\alpha > 0$:

$$\begin{aligned} P(X(\boldsymbol{\xi}) > 0) &\leq exp(\alpha w_0) \prod_{i=1}^{k} E[exp(\alpha w_i \xi_i)] \\ &\leq exp\left(\alpha w_0 + \frac{\alpha^2}{2} \sum_{i=1}^{k} w_i^2\right). \end{aligned}$$

- Select α to minimize the upper bound.
- $$P(X(\boldsymbol{\xi}) > 0) \leq \min_{\alpha > 0} exp\left(\alpha w_0 + \frac{\alpha^2}{2} ||\boldsymbol{w}||^2\right).$$
- $\alpha^* = -w_0/||\boldsymbol{w}||^2$.
- $P(X(\boldsymbol{\xi}) > 0) = P\left(w_0 + \sum_{i=1}^{k} w_i \xi_i > 0\right) \leq exp\left(-\frac{w_0^2}{2||\boldsymbol{w}||^2}\right)$.

Proof of the key guarantee

- Suppose that $\boldsymbol{x}$ satisfies $\overline{\boldsymbol{a}}'\boldsymbol{x} + \rho||\boldsymbol{\Delta x}|| \leq b$.
- Then

$$P(\tilde{\boldsymbol{a}}'\boldsymbol{x} > b) = P(\overline{\boldsymbol{a}}'\boldsymbol{x} + \boldsymbol{u}'\boldsymbol{\Delta x} > b) \leq P(\boldsymbol{u}'\boldsymbol{\Delta x} > -\rho||\boldsymbol{\Delta x}||).$$

- Select $w_0 = -\rho||\boldsymbol{\Delta x}||$ and $\boldsymbol{w} = \boldsymbol{\Delta x}$, we obtain

$$P(\tilde{\boldsymbol{a}}'x > b) \leq e^{-\frac{\rho^2}{2}}.$$

Guarantees for non-independent distributions

- RO:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \tilde{\boldsymbol{A}}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in P \\ & \forall \tilde{\boldsymbol{A}} \in \mathcal{U} = \left\{ \tilde{\boldsymbol{A}} \mid ||\boldsymbol{M}(\text{vec}(\tilde{\boldsymbol{A}}) - \text{vec}(\overline{\boldsymbol{A}}))|| \leq \Delta \right\}. \end{aligned}$$

- RC:

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \overline{\boldsymbol{a}}_i\boldsymbol{x} + \Delta||\boldsymbol{M}^{-1}\boldsymbol{x}_i||^* \leq \boldsymbol{b}_i, \quad i = 1, \ldots, m \\ & \boldsymbol{x} \in P, \end{aligned}$$

- $\tilde{\boldsymbol{A}} \sim (\overline{\boldsymbol{A}}, \Sigma)$.
- Let $\boldsymbol{M} = \Sigma^{-\frac{1}{2}}$.

Probabilistic Guarantees

- $$P\left(\tilde{\boldsymbol{a}}_i'\boldsymbol{x}^* > b_i\right) \leq \frac{1}{1+\Delta^2\left(\frac{\|\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i^*\|^*}{\|\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i^*\|_2}\right)^2}.$$
- If L_p norm used in $\mathcal{U}$, then
$$P\left(\tilde{\boldsymbol{a}}_i'\boldsymbol{x}^* > b_i\right) \leq \frac{1}{1+\Delta^2\min\left\{\frac{1}{p^2},\frac{1}{n}\right\}}.$$
- If L_2 used in $\mathcal{U}$, then
$$P\left(\tilde{\boldsymbol{a}}_i'\boldsymbol{x}^* > b_i\right) \leq \frac{1}{1+\Delta^2}.$$
- Remark: Arbitrary Dependence structure.
- How to select Δ?

Proof

- Optimal robust solution satisfies

$$(\mathrm{vec}(\overline{\boldsymbol{A}}))'\boldsymbol{x}_i + \Delta||\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i||^* \leq b_i,$$

- Thus

$$P\left((\mathrm{vec}(\tilde{\boldsymbol{A}}))'\boldsymbol{x}_i > b_i\right) \leq P\left((\mathrm{vec}(\tilde{\boldsymbol{A}}))'\boldsymbol{x}_i^* \geq (\mathrm{vec}(\overline{\boldsymbol{A}}))'\boldsymbol{x}_i^* + ||\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i^*||^*\right).$$

- Bertsimas and Popescu: if S is a convex set, and $\tilde{\boldsymbol{X}} \sim (\overline{\boldsymbol{X}}, \Sigma)$, then

$$P\left(\tilde{\boldsymbol{X}} \in S\right) \leq \frac{1}{1+d^2},$$

  where

$$d^2 = \inf_{\tilde{\boldsymbol{X}} \in S} \left(\tilde{\boldsymbol{X}} - \overline{\boldsymbol{X}}\right)' \Sigma^{-1} \left(\tilde{\boldsymbol{X}} - \overline{\boldsymbol{X}}\right).$$

Proof continued

- $S_i = \left\{ \text{vec}(\tilde{\boldsymbol{A}}) \mid (\text{vec}(\tilde{\boldsymbol{A}}))'\boldsymbol{x}_i \geq (\text{vec}(\overline{\boldsymbol{A}}))'\boldsymbol{x}_i + \Delta ||\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i||^* \right\}.$
- $d_i^2 = \inf_{\text{vec}(\tilde{\boldsymbol{A}}) \in S_i} \left(\text{vec}(\tilde{\boldsymbol{A}}) - \text{vec}(\overline{\boldsymbol{A}}) \right)' \boldsymbol{\Sigma}^{-1} \left(\text{vec}(\tilde{\boldsymbol{A}}) - \text{vec}(\overline{\boldsymbol{A}}) \right).$
- Optimal solution (KKT):

$$\text{vec}(\overline{\boldsymbol{A}}) + \Delta \left(\frac{||\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i||^*}{\|\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i\|_2} \right)^2 \boldsymbol{\Sigma}\boldsymbol{x}_i,$$

- $$d^2 = \Delta^2 \left(\frac{||\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i||^*}{\|\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{x}_i\|_2} \right)^2.$$

On the interplay of probability and optimization

- Use Probability theorems to select parameters.
- Use optimization ideas to find best possible results in probability.
- In exercise we will explore other bounds.
- Use RO to solve problems under uncertainty computationally.