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NOTE ON MATRIC ALGEBRAS

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1. **Preliminaries.** In §1 and Appendix 7 of my paper, *Generalized Riemann Matrices and Factor Sets*,[1] I established the theory of matric algebras by operating with the matrices themselves and their vector space rather than with abstract elements, and getting along without the discussion of the radical and its influence upon the structure of the algebra. One can treat the $\times$-multiplication in the same style and thus complete the theory, as I propose to show briefly in this note, in which I make use of the same notations and nomenclature as in the cited chapter. At the bottom we have a (commutative) field k; the words "in k" should tacitly be supplied to all terms like "matrix," "algebra," "irreducible."

The set of all d-rowed matrices (transformations in a d-dimensional vector space) is denoted by $\mathfrak{M}_d$ (complete matric algebra). $E = E_d$ is the unit matrix. The linear closure of a given matric set $\mathfrak{A} = \{A\}$ consisting of all possible linear combinations of the elements A of $\mathfrak{A}$ with coefficients in k will be indicated by

$$[\mathfrak{A}] \qquad \text{or} \qquad [A]_{A \text{ in } \mathfrak{A}}.$$

The set of matrices

$$\begin{Vmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A \end{Vmatrix} \quad (u \text{ rows}) \text{ and } \quad \begin{Vmatrix} A_{11} & \cdots & A_{1u} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A_{u1} & \cdots & A_{uu} \end{Vmatrix}$$

where A or the A_{ik} vary independently in $\mathfrak{A}$ is called $u \cdot \mathfrak{A}$ and $\mathfrak{A}_u$ respectively; they are algebras if $\mathfrak{A}$ is such. By transition to a new coördinate system a set $\mathfrak{A}$ of linear mappings A passes into what is called an equivalent set ($\sim \mathfrak{A}$). When δ variables x_ι undergo a linear substitution A and d variables y_k undergo a linear substitution B, then the δd products $x_\iota y_k$ undergo the substitution $A \times B$. If the x_ι and y_k are looked upon as components of vectors x, y in a δ- and d-dimensional vector space $\mathfrak{x}$ and $\mathfrak{y}$ respectively, then $z_{\iota k} = x_\iota y_k$ are the components of a vector $z = xy$ in the δd-dimensional product space $\mathfrak{x}\mathfrak{y}$. If A ranges over a set $\mathfrak{A}$ and B over $\mathfrak{B}$, we mean by $\mathfrak{A} \times \mathfrak{B}$ the set of all matrices

$$A \times B, \qquad A \text{ in } \mathfrak{A}, \qquad B \text{ in } \mathfrak{B}.$$

[1] Annals of Math. 37 (1936), pp. 709–745.

This is not an algebra even if $\mathfrak{A}$ and $\mathfrak{B}$ are such. The linear closure

$$[A \times B]_{A \text{ in } \mathfrak{A}, B \text{ in } \mathfrak{B}}$$

is to be denoted by $[\mathfrak{A} \times \mathfrak{B}]$ and will be called the *algebra product* of the algebras $\mathfrak{A}$ and $\mathfrak{B}$. We have

$$\mathfrak{A}_v = [\mathfrak{M}_v \times \mathfrak{A}],$$

hence

$$[\mathfrak{A}_v \times \mathfrak{B}] = [\mathfrak{A} \times \mathfrak{B}]_v .$$

LEMMA (1-A). *Let $R = \{C\}$ be an algebra of transformations in a d-dimensional vector space $\mathfrak{r}$, containing E_d. The product $\rho\mathfrak{r}$ of $\mathfrak{r}$ with a vector space ρ of dimensionality δ may be considered as the substratum of the transformations of $\mathfrak{M}_\delta \times R$. Then each of its invariant subspaces is of form $\rho\mathfrak{r}'$ where $\mathfrak{r}'$ is an invariant subspace of $\mathfrak{r}$. In particular, irreducibility of R entails the same for R_δ.*

Of this lemma I made use in the proof of the criterion, Theorem (1.3-D), l.c. Its demonstration is fairly obvious. Let $\epsilon_1, \cdots, \epsilon_\delta$ be a basis of ρ. Each vector z in $\rho\mathfrak{r}$ may be decomposed according to

$$z = \epsilon_1 z_1 + \cdots + \epsilon_\delta z_\delta \qquad (z_\iota \text{ vector in } \mathfrak{r}).$$

$E_{\iota\kappa}$ being the δ-rowed matrix which has a 1 at the crossing point of the ι^{th} row and the κ^{th} column and 0 elsewhere, application of the operation $E_{\iota\iota} \times E$ shows that our invariant subspace $\bar{\mathfrak{r}}$ of $\rho\mathfrak{r}$ contains the parts $\epsilon_\iota z_\iota$ of each z in $\bar{\mathfrak{r}}$. The operations like $E_{12} \times E$ prove that with $\epsilon_1 z_1$ (z_1 in $\mathfrak{r}$) also $\epsilon_2 z_1$ lies in $\bar{\mathfrak{r}}$. Consequently $\bar{\mathfrak{r}}$ is of the form $\rho\mathfrak{r}'$, and the operators $E \times C$ force the subspace $\mathfrak{r}'$ of $\mathfrak{r}$ to be invariant.

An abstract *division algebra* $\rho = \{\gamma\}$ of order δ is irreducibly represented by associating with γ the substitution

$$\gamma^*: \qquad \xi' = \gamma\xi \qquad (\xi \text{ varying in } \rho)$$

(regular representation). $\rho^* = \{\gamma^*\}$. The substitution

$$\xi' = \xi\gamma$$

may be denoted by γ_* and the set of all γ_* by ρ_*. A division algebra is *normal* provided the centrum consists of the multiples of the unit only. In the Appendix I proved the following statement concerning normal division algebras by way of a simple application of Burnside's criterion:

LEMMA (1-B). *Let ρ be a normal division algebra. The δ^2 substitutions*

$$\xi' = \alpha\xi\beta \cdot$$

one obtains by letting α and β run independently over a basis $\epsilon_1, \cdots, \epsilon_\delta$ of ρ yield a basis for the complete matric algebra $\mathfrak{M}_\delta$.

One might put this down in the following formula

$$[\alpha^* \beta_*]_{\alpha, \beta \text{ in } \rho} = \mathfrak{M}_\delta .$$

2. **The basic argument.** We are now going to consider the product $\rho\mathfrak{r}$ of a normal division algebra ρ of order δ and a d-dimensional vector space $\mathfrak{r}$ subject to the transformations C of a given irreducible matric algebra $R = \{C\}$. The space $\rho\mathfrak{r}$ is considered the substratum of the transformation set

$$\rho^* \times R = \{\gamma^* \times C\}_{\gamma \text{ in } \rho,\, C \text{ in } R}.$$

We then maintain that $\rho\mathfrak{r}$ splits into a number u of *irreducible* invariant subspaces in each of which $\gamma^* \times C$ induces the *same* transformation; or that we have an equivalence

$$\rho^* \times R \sim u \cdot \mathfrak{H}, \qquad \mathfrak{H} \text{ irreducible.} \tag{2.1}$$

This statement and its proof are the backbone of our whole discussion; the rest is mere juggling around and interpretation of the result. We proceed as follows.

The vectors x of $\rho\mathfrak{r}$ are expressed in terms of a basis $e_1, \cdots, e_d$ of $\mathfrak{r}$ as:

$$x = \xi_1 e_1 + \cdots + \xi_d e_d \qquad (\xi_i \text{ in } \rho). \tag{2.2}$$

Considering ρ as a "quasi-field" in which the coefficients ξ_i vary freely, we define

$$\gamma x = (\gamma\xi_1)e_1 + \cdots + (\gamma\xi_d)e_d, \qquad x\gamma = (\xi_1\gamma)e_1 + \cdots + (\xi_d\gamma)e_d.$$

An invariant subspace $\mathfrak{l}$ of $\rho\mathfrak{r}$ is certainly a subset of vectors x of form (2.2), closed with respect to addition and front multiplication ($x \to \gamma x$); for the latter operation is what we formerly denoted by $\gamma^* \times E$. Hence $\mathfrak{l}$ has a ρ-basis $l_1, \cdots, l_n$ in terms of which every x in $\mathfrak{l}$ is uniquely expressible as

$$x = \eta_1 l_1 + \cdots + \eta_n l_n \qquad (\eta_i \text{ in } \rho),$$

and the dimensionality δn of $\mathfrak{l}$ is a multiple of δ.

β being a given quantity in ρ, the space $\mathfrak{l}\beta$ containing all vectors $x\beta$ (x in $\mathfrak{l}$) is invariant with respect to $\gamma^* \times C$ as well as $\mathfrak{l}$, and $\gamma^* \times C$ induces therein the same transformation as in $\mathfrak{l}$. By making use of a basis $\epsilon_1 = 1, \cdots, \epsilon_\delta$ of ρ we apply the "typical argument" to the row of irreducible invariant subspaces

$$\mathfrak{l}_1 = \mathfrak{l}\epsilon_1 = \mathfrak{l}, \mathfrak{l}_2 = \mathfrak{l}\epsilon_2, \cdots, \mathfrak{l}_\delta = \mathfrak{l}\epsilon_\delta$$

and thus succeed in picking out a number among them which by a proper arrangement may be denoted by $\mathfrak{l}_1, \cdots, \mathfrak{l}_u$ such that 1) $\mathfrak{l}_1, \cdots, \mathfrak{l}_u$ are linearly independent, and 2) each $\mathfrak{l}_\iota$ ($\iota = 1, \cdots, \delta$) is contained in the sum

$$\mathfrak{l}_1 + \cdots + \mathfrak{l}_u = (\mathfrak{l}).$$

The latter fact shows that $(\mathfrak{l})$ is also invariant with respect to back multiplications: $(\mathfrak{l})\epsilon_\iota$ is contained in $(\mathfrak{l})$ for $\iota = 1, \cdots, \delta$. Consequently $(\mathfrak{l})$ is invariant with respect to all transformations of the type

$$\alpha^*\beta_* \times C, \qquad \alpha \text{ and } \beta \text{ varying over a basis of } \rho,\ C \text{ in } R,$$

and thus, according to Lemma (1-B), with respect to $\mathfrak{M}_s \times R$. Lemma (1-A) then proves ($\mathfrak{l}$) to be the total space $\rho\mathfrak{r}$, and this remark finishes our demonstration, at the same time yielding the equation

$$d = nu:$$

u is a divisor of d.

3. **Exploitation.** Here we restate Theorem (1.3-B) of my former paper as

LEMMA (3-A). *An irreducible matric algebra R is $\sim r_v^*$ where r is a division algebra (and v a natural number).*

From (2.1) there follow the equations

$$[\rho^* \times R] \sim u[\mathfrak{H}], \qquad [\rho_v^* \times R] \sim u[\mathfrak{H}]_v .$$

According to Lemma (1-A), the algebra $[\mathfrak{H}]_v$ is irreducible as well as $[\mathfrak{H}]$; and in view of Lemma (3-A) we may put our result into the equivalence

$$(3.1) \qquad [\mathrm{P} \times R] \sim u \cdot \mathfrak{P}, \qquad \mathfrak{P} \text{ irreducible,}$$

holding for any two irreducible matric algebras P and R the first of which is normal. Our result implies the abstract statement:

THEOREM (3-B). *The algebra product of two simple algebras one of which is normal, is a simple algebra again.*

From this we could infer our concrete proposition that $[\mathrm{P} \times R]$ is a certain multiple u of an irreducible $\mathfrak{P}$ by means of the general fact mentioned on page 714, l.c., that every representation of a simple algebra is a multiple of its irreducible representation. Our proof here, however, aimed directly at this concrete statement and yielded the further result that u is a divisor of the degree d of R.

Lemma (3-A) makes transition from division algebras r to simple algebras R so easy that it is perhaps convenient to specialize our result (2.1) to the case $R = r^*$ rather than to generalize it to (3.1). Hence let us write down the (special $\times$ special)-equation

$$(3.2) \qquad \rho^* \times r^* \sim u \cdot \mathfrak{H}$$

This leads back to the (general $\times$ general)-result (3.1) in the form

$$(3.3) \qquad [\rho_v^* \times r_w^*] \sim u \cdot [\mathfrak{H}]_{vw} .$$

Transition from ρ^* and r^* to $\mathrm{P} = \rho_v^*$ and $R = r_w^*$ leaves the multiplicity u unchanged while replacing $[\mathfrak{H}]$ by the likewise irreducible $[\mathfrak{H}]_{vw}$.

Concerning the (special $\times$ special)-case (3.2), I feel bound to make two additional remarks.

First remark. An invariant subspace of $\rho\mathfrak{r}$ has a basis $l_1, \cdots, l_n$ relative to the quasi-field of coefficients in ρ. However, we may exchange the rôles of ρ and r and look upon ρ as a vector space and on r as a quasi-field of multiplicators or coefficients. $\mathfrak{l}$ will then have an r-basis $\lambda_1, \cdots, \lambda_\nu$ in terms of which

$$y_1\lambda_1 + \cdots + y_\nu\lambda_\nu$$

describes $\mathfrak{l}$ with the coefficients y_ι ranging over r. The dimensionality of $\mathfrak{l}$ is

$$n\delta = \nu d, \qquad \text{therefore} \qquad d:\delta = n:\nu.$$

As $d = nu$, we obtain the further relation $\delta = \nu u$ and thus realize that *u is a common divisor of d and δ*. The same relationship prevails in the (general $\times$ general)-case (3.1). For in passing from ρ^* to $\mathrm{P} = \rho^*_v$ and from r^* to $R = r^*_w$, the degrees δ and d change into δv and dw respectively, while u stays put, eq. (3.3). With this additional information on hand we give the concrete counterpart of Theorem (3-B) as follows:

THEOREM (3-C). *The algebra product of two irreducible matric algebras* P *and* R *one of which is normal, decomposes into a number u of equal irreducible components* $\mathfrak{P}$ *according to the equivalence*

$$[\mathrm{P} \times R] \sim u \cdot \mathfrak{P}.$$

The multiplicity u is a common divisor of the degrees of both factors.

The u equal parts into which the generic matrix $\Gamma \times C$ of $\mathrm{P} \times R$ decomposes will occasionally be denoted by $\Pi(\Gamma, C)$. In the special case $\mathrm{P} = \rho^*$ we simply write $\Pi(\gamma, C)$ instead of $\Pi(\gamma^*, C)$, and similarly when R is specialized into r^*.

Second remark. In the (special $\times$ special)-relation (3.2) or in

$$[\rho^* \times r^*] \sim u[\mathfrak{H}]$$

we apply Lemma (3-A) to the irreducible $[\mathfrak{H}]$ and infer from it that

$$[\mathfrak{H}] = \mathfrak{p}^*_v$$

where the abstract division algebra $\mathfrak{p}$, called the R. Brauer product, is uniquely determined by the factors ρ and r. Comparison of degrees and orders in the ensuing equivalence

$$[\rho^* \times r^*] \sim u \cdot \mathfrak{p}^*_v$$

leads to the relations

$$\delta d = uv\mathfrak{d}, \qquad \delta d = v^2\mathfrak{d},$$

$\mathfrak{d}$ being the degree of $\mathfrak{p}^*$ = order of $\mathfrak{p}$. Hence $v = u$ and

$$d = nu, \qquad \delta = \nu u, \qquad \mathfrak{d} = n\nu:$$

THEOREM (3-D). *The algebra product of the regular representations ρ^* and r^* of two division algebras ρ and r of orders δ and d decomposes according to*

$$[\rho^* \times r^*] \sim u \cdot \mathfrak{p}^*_u$$

provided ρ is normal. Putting

$$d = nu, \qquad \delta = \nu u, \qquad (n \text{ and } \nu \text{ integers})$$

the order of the division algebra $\mathfrak{p}$ equals $n\nu$.

4. **Adjunction.** We have not as yet evaluated to the full the idea involved in our backbone proof that an invariant subspace $\mathfrak{l}$ of $\rho\mathfrak{r}$ can be referred to a ρ-basis $l_1, \cdots, l_n$. Let us now consider its implications for the case which is the other way around: $\mathrm{P} \times r^*$, P being a normal irreducible matric algebra, r an arbitrary division algebra. Let $\mathfrak{l}$ be an invariant subspace of the vector space $\mathfrak{r}r$ upon which the operators $\Gamma \times c^*$ of $\mathrm{P} \times r^*$ work (Γ are operators in the δ-dimensional space $\mathfrak{r}$, r is of order d). $\mathfrak{l}$ has an r-basis $\mathfrak{l}_1, \cdots, \mathfrak{l}_\nu$ such that each $\mathfrak{x}$ of $\mathfrak{l}$ is uniquely expressible as

$$\mathfrak{x} = y_1\mathfrak{l}_1 + \cdots + y_\nu\mathfrak{l}_\nu \qquad (y_\iota \text{ in } r). \tag{4.1}$$

We now look upon $\mathfrak{r}r = \mathfrak{r}_r$ as the vector space $\mathfrak{r}$ under extension of its field of multiplicators k into the quasi-field r. The elements of $\mathfrak{r}_r$ are rows $\mathfrak{x}$ of δ quantities $x_1, \cdots, x_\delta$ in r. Addition is defined in the obvious manner, multiplication by a quantity c of r as:

$$c(x_1, \cdots, x_\delta) = (cx_1, \cdots, cx_\delta).$$

A linear subspace $\mathfrak{l}_r$ of $\mathfrak{r}_r$ is a subset closed with respect to addition and multiplication by any c in r. The subspace $\mathfrak{l}_r$ has a basis $\mathfrak{l}_1, \cdots, \mathfrak{l}_\nu$ as indicated by eq. (4.1).

Each Γ is a linear substitution with ordinary numbers $\gamma_{\iota\kappa}$ in k:

$$x'_\iota = \sum_\kappa x_\kappa \gamma_{\iota\kappa} \qquad (\iota, \kappa = 1, \cdots, \delta)$$

and hence commutes with all the multiplications $\mathfrak{x} \to c\mathfrak{x}$. If $\mathfrak{l}_r$ is invariant with respect to the transformations Γ of P, then each Γ: $\mathfrak{x} \to \mathfrak{x}'$ carries the basic vectors $\mathfrak{l}_1, \cdots, \mathfrak{l}_\nu$ into linear combinations of themselves:

$$\mathfrak{l}'_\iota = \sum_\kappa c_{\iota\kappa}\mathfrak{l}_\kappa \qquad (\iota, \kappa = 1, \cdots, \nu).$$

Commuting as it does with the multiplications, Γ then carries (4.1) into

$$\mathfrak{x}' = y_1\mathfrak{l}'_1 + \cdots + y_\nu\mathfrak{l}'_\nu = y'_1\mathfrak{l}_1 + \cdots + y'_\nu\mathfrak{l}_\nu$$

where

$$y'_\iota = \sum_\kappa y_\kappa c_{\iota\kappa} \qquad (\iota, \kappa = 1, \cdots, \nu).$$

Here it is quite essential to write the coefficients $c_{\iota\kappa}$ *after* or to the right of the variables y_κ: we therefore speak of a *right-transformation.* Γ induces in $\mathfrak{l}_r$ the right-transformation $\| c_{\iota\kappa} \|$ and the correspondence $\Gamma \to \| c_{\iota\kappa} \|$ constitutes a *right-representation* of P in r. It seems worth while to present our chief result in this new garb:

THEOREM (4-A). *Under extension of k into a quasi-field r over k, a given normal matric algebra* P, *irreducible in k, breaks up into u equal irreducible right-representations of* P *in r.*

This mode of visualizing the situation is related to our former viewpoint in the following manner. Adopting the coördinate system here used, $\Pi(\Gamma, \mathfrak{l})$ arises from our right-representation $\Gamma \to \| c_{\iota\kappa} \|$ in replacing each $c_{\iota\kappa}$ by the matrix $(c_{\iota\kappa})_*$ (back multiplication and hence lower asterisk!) whereas $\Pi(E_\delta, c)$ is simply $E_\nu \times c^*$.

Of particular import is the case when r is a commutative field K. We then deduce from our theorem that a normal irreducible matric algebra P in k splits into u equal irreducible matric algebras in K after extending the reference field k to a finite field K over k. Let us consider again the special case $\mathrm{P} = \rho^*$. We then must have an equivalence like

$$\rho^* \text{ ext. to } K \sim u \cdot \pi_v^*$$

where π is a (normal) division algebra in K. θ being the degree = order of π^*, comparison of degrees and orders leads to the relations

$$\delta = uv\theta, \qquad \delta = v^2\theta,$$

hence

$$u = v \quad \text{and} \quad \delta = u^2\theta.$$

THEOREM (4-B). *A normal division algebra ρ in k breaks up according to the equation*

$$\rho^* \text{ ext. to } K \sim u \cdot \pi_u^*$$

under extension of the field k into a finite field K over k. Hence

$$(\text{order of } \rho) = u^2 \cdot (\text{order of } \pi).$$

An easy consequence thereof is our final

THEOREM (4-C). *The order of a normal division algebra is a square number.*

Indeed, Theorem (4-B) informs us that on successive extensions of the reference field the order of ρ shrinks by throwing off square factors. Therefore we merely have to show how to find an extension so as to effect actual reduction ($u > 1$) as long as one has not yet reached the core: $\rho = (1)$. If ρ is not of order 1 we choose an element α of ρ different from any multiple of the unit and adjoin a root z of the characteristic equation $\varphi(z) = 0$ of the substitution $\alpha_*: \xi \to \xi\alpha$. The transformation $\alpha_* - zE$ is then singular, $\neq 0$, and commutes with all γ^*; hence, according to Schur's Lemma, ρ^* must needs reduce in $K = k(z)$. What one actually does is to pick out an irreducible k-factor $\psi(z)$ of $\varphi(z)$ and then define $k(z)$ *in abstracto* as the field of all k-polymonials of the indeterminate z modulo $\psi(z)$.

Here we find ourselves at the entrance gate to the "*splitting fields*," and there we might well finish our brief journey over what seems to me a particularly smooth and open road through this well-explored territory.

It was essential to suppose one of our factors to be normal. Again, the unruly things happen in the commutative fields: the algebra product of two fields breaks up into inequivalent parts, and to secure its full reducibility at all assumptions concerning separability are needed.

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