Liouville Theorems for Harmonic Sections and Applications

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1 Introduction

This paper, the sequel to [9] and [10], is the last in a series of three papers announced in [5]. The purpose is to prove several generalized Liouville theorems for harmonic sections of Hermitian vector bundles over a complete metric space and to give several applications of these results. After stating the main results and discussing their proofs, we will explain how the results of this paper fit in with the other papers in the series.

We first recall the definition of polynomial-growth harmonic sections of a bundle.

DEFINITION 1.1 If M is a manifold, E is a Hermitian vector bundle over M, and $p \in M$ is fixed, we will define $\mathcal{H}_d(M, E)$ to be the *linear space of harmonic sections of E with polynomial growth of order at most d*. That is, $\eta \in \mathcal{H}_d(M, E)$ if η is harmonic, and there exists some $C < \infty$ so that $|\eta| \leq C(1 + r^d)$ where r is the distance to p.

We will also need to recall the definition of the doubling property.

DOUBLING PROPERTY We say that M^n has the *doubling property* if there exists $C_D < \infty$ such that for all $p \in M^n$ and $r > 0$

$$\operatorname{Vol}(B_{2r}(p)) \leq 2^{C_D} \operatorname{Vol}(B_r(p)) . \tag{1.1}$$

If E is a Hermitian bundle with nonnegative curvature over a manifold M, then for any harmonic section η the Bochner formula yields $\Delta|\eta|^2 \geq 2|\nabla \eta|^2$, and hence $|\eta|^2$ is subharmonic. This is the only reason that we will assume in this paper that E has nonnegative curvature (and that η is harmonic), and any other assumption guaranteeing that $|\eta|^2$ is subharmonic would suffice.

The first theorem of this paper is the following:

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THEOREM 1.2 *If M is a manifold with both the doubling property and a mean value inequality for subharmonic functions, then for any Hermitian bundle E^m of rank m with nonnegative curvature and for all $d \geq 1$,*

$$\dim \mathcal{H}_d(M, E) \leq C\, m\, d^{C_D - \varepsilon}\,, \tag{1.2}$$

where $\varepsilon = \frac{\log(1+32^{-C_D})}{\log 2}$ and $C < \infty$ depends only on C_D and the constant in the mean value inequality.

The ε in Theorem 1.2 is essentially given by the Hölder regularity of the volume measure (see the definition of ε-volume regularity in Section 3). We note that this value of ε is not optimal. However, we can get $\varepsilon = 1$ in some important cases; for instance, for manifolds with nonnegative Ricci curvature this follows from the relative volume comparison theorem.

The next result addresses, in the case of complex line bundles, a question raised by Yau [26, 27, 28].

THEOREM 1.3 *Let M^n be an open manifold with nonnegative Ricci curvature and E^m a rank m Hermitian vector bundle with nonnegative curvature. For all $d \geq 1$,*

$$\dim \mathcal{H}_d(M, E) \leq C\, m\, d^{n-1}\,, \tag{1.3}$$

where $C = C(n) < \infty$.

In Section 5, we will apply our general results to study the spaces of harmonic sections of polynomial growth over stationary n-rectifiable varifolds with Euclidean volume growth. These spaces are generalizations of classical minimal submanifolds (see, for instance, Simon [22] for the basic theory). As a particular case of our results, we recover some function-theoretic results for this class of stationary varifolds that generalize results of Cheng, Li, and Yau [3] on minimal submanifolds of spheres (equivalently, on minimal cones).

We will postpone the statement of our main theorem on stationary varifolds, Theorem 5.3, until after giving the necessary definitions in Section 5. However, we will mention the following application, which may be viewed as a weak Bernstein-type theorem (see (5.1) for the definition of the density).

COROLLARY 1.4 *If $\Sigma^n \subset \mathbb{R}^{n'}$ is an n-rectifiable stationary varifold with (volume) density at least 1 almost everywhere and bounded from above by V_Σ, then Σ must be contained in some affine subspace of dimension at most $3\, \frac{n}{n-1}\, 2^{n+3}\, e^8\, \mathrm{V}_\Sigma$.*

We note that Corollary 1.4 was proven for minimal cones by Cheng, Li, and Yau in [3].

The proofs of the finite dimensionality results in this paper consist of the following two independent steps. First, in Section 2 we will modify constructions of [7], [9], and [10] to reduce the problem to bounding the number of $L^2(B_r(p))$ orthonormal harmonic sections of E over M with a uniform bound on the $L^2(B_{\Omega\, r}(p))$ norm for $r > 0$ and fixed $p \in M$, $\Omega > 1$. Next, we will use the linearity of the space of harmonic sections to form a "Bergman kernel"; cf. Bergman [2]. That is, if $\eta_1, \dots, \eta_{\mathcal{N}}$ are the harmonic sections, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |\eta_i(x)|^2 \,. \tag{1.4}$$

Combining some standard linear algebra with the mean value inequality, we obtain pointwise bounds for $K(x)$ depending on Ω and the $L^2(B_{\Omega r}(p))$ norm but independent of $\mathcal{N}$. Integrating this over $B_r(p)$ gives a bound on $\mathcal{N}$ that depends on Ω.

As in [10], to bound the dimension of $\mathcal{H}_d(M, E)$ polynomially in d, we choose Ω to be approximately $1 + \frac{1}{d}$. The resulting estimates are then polynomial in d where the degree depends on both the rate of blowup at the boundary of the mean value inequality and the regularity of the volume measure (see the ε-volume regularity in Section 3). Similarly to corollary 0.10 of [10], when E has nonnegative curvature and M has nonnegative Ricci curvature, Theorem 1.3 gives the optimal dependence in d.

A case of Theorem 1.2 of particular interest is where M^n is an open manifold with the doubling property and a scale invariant 1-Neumann-Poincaré inequality. This was proven by a different method in [10]; in fact, we proved there much more general results. For manifolds with the doubling property and a 1-Neumann-Poincaré inequality, the mean value inequality for subharmonic functions follows from work of Saloff-Coste [21] and Grigor'yan [11]. Note, by the way, that Saloff-Coste and Grigor'yan showed that the doubling property together with the scale-invariant 1-Neumann-Poincaré inequality imply the classical Liouville property; that is, under these assumptions there are no nonconstant, positive harmonic functions.

Recall that a manifold is said to satisfy a scale-invariant k-Neumann-Poincaré inequality if the following holds:

k-NEUMANN-POINCARÉ INEQUALITY *Given a positive integer k, we say that M^n satisfies a scale-invariant k-Neumann-Poincaré inequality if there* *exists $C_N = C_N(k) < \infty$ such that for all $p \in M^n$ and $r > 0$,*

$$\frac{1}{C_N\, r^2} \leq \mu_k(B_r(p))\,. \tag{1.5}$$

Here $\mu_k(B_r(p))$ is the k^{th} eigenvalue for the Laplacian on $B_r(p)$ with Neumann boundary conditions (i.e., with vanishing normal derivative on $\partial B_r(p)$).

This being the last paper in the series of three papers announced in [5], we will explain how the results of this paper fit together with the two other papers in the series. In [9], we showed that the space of polynomial-growth harmonic functions was finite dimensional on a manifold with nonnegative Ricci curvature, thus settling a conjecture of Yau. In fact, this was shown to hold in a significantly more general setting (e.g., for manifolds with the doubling property and a Neumann-Poincaré inequality). The other three main results in this series are:

1. polynomial bounds for manifolds with the doubling property and a k-Neumann-Poincaré inequality (see the introduction of [10] for $k = 1$ and the section "Further Applications" in [10] for $k > 1$),

2. sharp "Weyl-type bounds" for manifolds with nonnegative Ricci curvature (see the introduction of [10]), and

3. the polynomial bounds of this paper for spaces of harmonic sections of polynomial growth and its applications to stationary varifolds with Euclidean volume growth (including our general results using the doubling property and the mean value inequality).

As discussed above, the proofs of the finite dimensionality results in this paper, as in [9] and [10], consist of two independent steps. We have two similar, and complementary, approaches to conclude the second step, which apply in different settings; the first approach is explored in [10] and the second approach in this paper. The situation is analogous to the two possible approaches to proving Weyl's asymptotic formula for eigenvalues: covering arguments and Poincaré inequalities on the one hand and asymptotic estimates for the heat kernel on the other. Just as is the case for Weyl's asymptotic formula, both approaches have their own advantages and apply to different situations. This will be discussed below. In this paper the "asymptotics" approach seems more appropriate, whereas in other situations the covering argument and Poincaré inequality approach seems to be better suited (cf. [9] and see, in particular, section 6 of [10]).

We want to reemphasize that in [9] and [10] we showed that the doubling property together with a scale-invariant k-Neumann-Poincaré inequality for some k implied finite dimensionality of the spaces of harmonic functions of polynomial growth; in this case the approach of this paper does not apply. The bound on the dimension given there depends linearly on k. In fact, we gave polynomial bounds for L-harmonic functions of polynomial growth on these manifolds, where the second-order elliptic operator L is assumed to be in in divergence form and quasi-uniformly elliptic. We refer to [9] or [10] for the definitions, but we note here that this more general class of operators is not covered by the approach of this paper. Moreover, the approach of [10] can yield stronger results. An example of this was given in section 6 of [10], where we showed that for M^n, an open manifold with nonnegative Ricci curvature, the dimension of $\mathcal{H}_d(M)$ is bounded by $C_1 \, \mathrm{V}_M \, d^{n-1} + C_2 o(d^{n-1})$, where C_1, C_2, and $o(t)$ depend only on n. Here $\mathrm{V}_M = \lim_{r\to\infty} r^{-n} \operatorname{Vol}(B_r(p))$. In particular, this gives a partial converse to Yau's conjecture; namely, it shows that if for $\varepsilon > 0$, $\dim \mathcal{H}_d(M^n) \geq \varepsilon \, \dim \mathcal{H}_d(\mathbb{R}^n)$ for some sufficiently large d, then M has Euclidean volume growth (see [10] for the exact statements).

The approach of this paper, which requires a mean value inequality together with the doubling property, has its own advantages. For instance, it applies in some cases where no k-Neumann-Poincaré inequality is known. Recall that stationary varifolds with Euclidean volume growth, which is the subject of Corollary 1.4, need *not* satisfy a scale-invariant Neumann-Poincaré inequality. It also applies to give finite dimensionality for spaces of harmonic sections of bundles with nonnegative curvature, Theorem 1.2 and Theorem 1.3. On the other hand, there are manifolds with the doubling property that do not satisfy a mean value inequality but do satisfy a k-Neumann-Poincaré inequality (see examples below).

Let us close this discussion by giving a few more examples to illustrate the relationship between our two approaches. It is easy to construct manifolds with the doubling property and some k-Neumann-Poincaré inequality on large balls, and hence for which the generalized Liouville theorems of [9] and [10] hold, but which admit nonconstant bounded harmonic functions. For instance, one can connect k copies of $\mathbb{R}^n$, $n \geq 3$, by $k-1$ necks and then construct bounded harmonic functions (see Tam [23] and cf. Li and Tam [15, 17]). Furthermore, it is easy to see that such a manifold has the doubling property and a scale-invariant k-Neumann-Poincaré inequality. On the other hand, if one connects $\mathbb{R}^4$ and $\mathbb{R}^3 \times \mathbb{S}^1$ by a neck, then the resulting manifold does not have the doubling property, nor does it satisfy the mean value inequality for subharmonic functions (in fact, one can construct a harmonic function that goes to 0 on the end corresponding to $\mathbb{R}^4$ and 1 on $\mathbb{R}^3 \times \mathbb{S}^1$; see [15]). However, the 2-Neumann-Poincaré inequality is still satisfied, and it is easy to see that the methods of [10] apply. We plan to discuss this further elsewhere.

The organization of this paper is as follows:

Given $\Omega > 1$ and a linearly independent set of harmonic sections whose L^2 norms on balls centered at some fixed $p \in M$ grow polynomially in the distance from p, in Section 2 we show how to produce infinitely balls $B_r(p)$ and a set of sections contained in the span of the given sections that are $L^2(B_{\Omega r}(p))$-orthonormal and have an uniform positive lower bound on their $L^2(B_r(p))$ norms. This construction is a very slight modification of a construction from [9, 10]. In the end of Section 2, we discuss the rate of blowup in certain mean value inequalities as one approaches the boundary.

In Section 3, we use mean value inequalities together with some standard linear algebra to bound certain "Bergman kernels" associated to linear spaces of harmonic sections on a ball. Combined with the results of Section 2, we get some generalized Liouville theorems that will be applied in Sections 4 and 5.

In Section 4, we give applications of the previous section to get generalized Liouville theorems for harmonic sections on various classes of manifolds. In particular, we prove Theorems 1.2 and 1.3. We also apply our results to Cartan-Hadamard manifolds with Euclidean volume growth and manifolds that have the doubling property and have Ricci curvature bounded from below by $-(n-1)\,\Lambda\,(1+r)^{-2}$.

The general results of Section 3 are applied to generalized minimal submanifolds in Section 5, where we also mention some geometric applications.

We refer to the announcement [6] for further references.

Most of the results of this paper were announced in [5]. See also [9, 10] for related results. Using similar ideas, P. Li obtained, in the paper "Harmonic Sections of Polynomial Growth," results similar to those in this, the last paper announced in [5].

2 General Constructions

Let M be a complete noncompact metric space equipped with a Radon measure, and let E^m be a Hermitian vector bundle over M. We are particularly interested in the cases where M is either a Riemannian manifold or a (generalized) minimal submanifold of $\mathbb{R}^n$.

For $p \in M$ fixed, $r > 0$, $B_r = B_r(p)$, and η_1 and η_2 sections of E, let

$$I_{\eta_1}(r) = \int_{B_r} |\eta_1|^2 \tag{2.1}$$

and

$$J_r(\eta_1, \eta_2) = \int_{B_r} \langle \eta_1 \, , \eta_2 \rangle \, . \tag{2.2}$$

Note that J_r is a semidefinite inner product and $I(r)$ is the corresponding quadratic form.

DEFINITION 2.1 We let $\mathcal{P}_d(M, E)$ be the *linear space of sections* η *such that* $I_\eta(r)$ *has polynomial growth of order at most* $2d$*.*

Note that for a manifold M with polynomial volume growth of degree n_0, we have $\mathcal{H}_d(M, E) \subset \mathcal{P}_{d+n_0}(M, E)$.

Given a linearly independent set of sections in $\mathcal{P}_d(M, E)$, we will construct functions of one variable that reflect the growth and independence properties of this set. These results are slight generalizations of results in [9]; we will omit the proofs, which are obvious modifications of those in [9] (compare also [7]).

We begin with two definitions. The first constructs the functions whose growth properties will be studied.

DEFINITION 2.2 ($\eta_{i,r}$ and f_i) Suppose that $\theta_1, \ldots, \theta_s$ are linearly independent sections of E. For each $r > 0$ we will now define an J_r-orthogonal spanning set $\eta_{i,r}$ and functions f_i. Set $\eta_{1,r} = \eta_1 = \theta_1$ and $f_1(r) = I_{\eta_1}(r)$. Define $\eta_{i,r}$ by requiring it to be orthogonal to $\theta_j | B_r$ for $j < i$ with respect to the inner product J_r and so that

$$\theta_i = \sum_{j=1}^{i-1} \lambda_{ji}(r) \, \theta_j + \eta_{i,r} \, . \tag{2.3}$$

Note that $\lambda_{i,j}(r)$ is not uniquely defined if the $\theta_i | B_r$ are linearly dependent. However, since the θ_i are linearly independent on M, then for r sufficiently large we see that $\lambda_{i,j}(r)$ will be uniquely defined. In any case, for all $r > 0$ the following quantity is well-defined (and, in fact, positive for r sufficiently large):

$$f_i(r) = \int_{B_r} |\eta_{i,r}|^2 \, . \tag{2.4}$$

In the next proposition we will record some key properties of the functions f_i from Definition 2.2.

PROPOSITION 2.3 (Properties of f_i) *If $\theta_1, \dots, \theta_s \in \mathcal{P}_d(M, E)$ are linearly independent, then the f_i from Definition 2.2 have the following four properties: There exists a constant $K > 0$ (depending on the set $\{\theta_i\}$) such that for $i = 1, \dots, s$*

$$f_i(r) \leq K(r^{2d} + 1)\,, \tag{2.5}$$

$$f_i \textit{ is a nondecreasing function}, \tag{2.6}$$

$$f_i \textit{ is nonnegative and positive for } r \textit{ sufficiently large}, \tag{2.7}$$

and

$$f_i(r) = I_{\eta_{i,r}}(r) \quad \textit{and} \quad f_i(t) \leq I_{\eta_{i,r}}(t) \quad \textit{for } t < r. \tag{2.8}$$

The next result is a consequence of corollary 3.9 of [9] and Proposition 2.3. It will be used to get control of the growth in the proof of Theorem 1.3.

COROLLARY 2.4 *Suppose that $\theta_1, \dots, \theta_{2s} \in \mathcal{P}_d(M, E)$ are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exists $m \geq m_0$ and a subset $f_{\alpha_1}, \dots, f_{\alpha_s}$ such that for $i = 1, \dots, s$*

$$0 < f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{4d} f_{\alpha_i}(\Omega^m)\,. \tag{2.9}$$

Note in particular that the sections $\eta_{\alpha_i, \Omega^{m+1}} | B_{\Omega^{m+1}}$ given by Corollary 2.4 are linearly independent. Using the definitions and (2.8), we get the following proposition:

PROPOSITION 2.5 (Cf. proposition 4.16 of [9]) *Suppose that*

$$\theta_1, \dots, \theta_s \in \mathcal{P}_d(M, E)$$

are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exist $m \geq m_0$, an integer $\ell \geq \frac{1}{4}\Omega^{-4d}\, s$, and sections $\eta_1, \dots, \eta_\ell$ in the linear span of the θ_i such that for $i, j = 1, \dots, \ell$,

$$J_{\Omega^{m+1}}(\eta_i, \eta_j) = \delta_{i,j} \tag{2.10}$$

and

$$\frac{1}{2}\Omega^{-4d} \leq I_{\eta_i}(\Omega^m)\,. \tag{2.11}$$

In proving our finite dimensionality results in this paper, it will be important to estimate the rate of blowup in certain mean value inequalities as one approaches the boundary. In the remainder of this section we will show how to do this on spaces that satisfy the ordinary mean value inequality and the doubling property.

Let M be an open manifold and $\varepsilon \geq 0$.

ε–MEAN VALUE INEQUALITY *We say that M has an ε–mean value inequality if for all $r > 0$, $p \in M$, and functions u with*

$$-\varepsilon r^{-2}\, u \leq \Delta u \ \text{ on } B_r(p)\,, \tag{2.12}$$

we have $C_0 = C_0(\varepsilon, M)$ such that

$$u^2(p) \leq \frac{C_0}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} u^2\,. \tag{2.13}$$

We will now state the strong ε–mean value inequality and give its elementary proof. This lemma gives a bound on the (polynomial) rate of blowup in the mean value inequality near the boundary.

LEMMA 2.6 (Strong ε–Mean Value Inequality) *If M has an ε–mean value inequality and the doubling property, then there exists $C_1 = C_1(C_0, C_D) < \infty$ such that for $p \in M$, $r > R_0$, $0 < s \leq \frac{1}{2}$, and any function u with*

$$-\varepsilon r^{-2}\, u \leq \Delta u \quad \text{on } B_r(p)\,, \tag{2.14}$$

we have

$$\sup_{B_{(1-s)r}(p)} u^2 \leq \frac{C_1\, s^{-C_D}}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} u^2\,. \tag{2.15}$$

PROOF: Let $x \in \overline{B_{(1-s)r}(p)}$ be any point where u achieves its maximum. By the triangle inequality, $B_{sr}(x) \subset B_r(p)$; hence the ε–mean value inequality yields

$$u^2(x) \leq \frac{C_0}{\mathrm{Vol}(B_{sr}(x))} \int_{B_{sr}(x)} u^2 \leq \frac{C_0}{\mathrm{Vol}(B_{sr}(x))} \int_{B_r(p)} u^2\,. \tag{2.16}$$

Now combine the triangle inequality and the doubling property to get

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_{2r}(x)) \leq 2^{2C_D} s^{-C_D} \mathrm{Vol}(B_{sr}(x))\,. \tag{2.17}$$

The inequalities (2.16) and (2.17) yield the result with $C_1 = 2^{2\,C_D}\, C_0$. ∎

3 Finite Dimensionality

In this section, we will show how to bound the dimension of the space of polynomial growth sections of E that satisfy a strong mean value inequality. The bound on the dimension will be polynomial in the rate of growth with the exponent determined by the boundary blowup of the strong mean value inequality. Next, we will give a sharpening of this estimate when the measure has a regularity property (see the ε-volume regularity defined below). At the end of this section, we will state a corollary that will be used in the applications described in the next two sections.

Let (M, dist) be a complete, noncompact metric space equipped with a Radon measure, and let E^m be a rank m Hermitian vector bundle over M. We suppose that Γ is a linear subspace of the space of sections of E that are locally in L^2 and satisfy the following strong mean value inequality:

STRONG MEAN VALUE INEQUALITY *There exist positive constants C_1, C_Γ, and R_0 such that for any $r > R_0$, $0 < s \leq \frac{1}{2}$, and $\eta \in \Gamma$,*

$$\sup_{B_{(1-s)r}(p)} |\eta|^2 \leq \frac{C_1\, s^{-C_\Gamma}}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |\eta|^2 \,. \tag{3.1}$$

We will use the following proposition in combination with the results of Section 2:

PROPOSITION 3.1 *Let $(M, \mathrm{dist}), E$, and Γ be as above. Suppose that $0 < a < 1$ is fixed, $r > R_0$, and $v_1, \dots, v_{\mathcal{N}} \in \Gamma$ are J_r-orthonormal. Given any $d \geq 1$ such that for all i,*

$$a \leq I_{v_i}((1 - (2d)^{-1})r) \,, \tag{3.2}$$

then

$$\mathcal{N} \leq C\, m\, d^{C_\Gamma} \,, \tag{3.3}$$

where $C = C_1\, a^{-1}\, 2^{C_\Gamma}$.

PROOF: For $t > 0$ set $B_t = B_t(p)$. For each $x \in B_r$, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |v_i|^2(x) \,. \tag{3.4}$$

Note that since each $v_i \in L^2_{\mathrm{loc}}(M, E)$, $K(x)$ must be finite by the mean value inequality. By construction, $K(x)$ is the trace of the (Hermitian) bilinear form

$$(v, w) \to \langle v, w \rangle(x) \tag{3.5}$$

for any v and w in the span of the v_i.

Recall that a Hermitian matrix can always be diagonalized by a unitary change of basis. Therefore, given $x \in B_r$, we can choose a new orthonormal basis $\{w_i\}$ for the inner product space $(\mathrm{span}\{v_i\}, J_r)$ such that at most m (the rank of the bundle) of the w_i, say $w_1, \dots, w_m$, are nonvanishing at x. Using the invariance of the trace under an orthogonal change of basis, we have that

$$K(x) = \sum_{i=1}^{\mathcal{N}} |w_i|^2(x) = \sum_{i=1}^{m} |w_i|^2(x) \,. \tag{3.6}$$

Now, since each w_i has an L^2 norm 1 on B_r, the strong mean value inequality gives for $0 < s \leq \frac{1}{2}$ and any $x \in B_{(1-s)r}$

$$|w_i|^2(x) \leq \frac{C_1\, s^{-C_\Gamma}}{\mathrm{Vol}(B_r)} \,. \tag{3.7}$$

Combining (3.6) and (3.7),

$$\mathrm{Vol}(B_r)\, K(x) = \mathrm{Vol}(B_r) \left(\sum_{i=1}^{m} |w_i|^2(x) \right) \leq C_1\, m\, s^{-C_\Gamma} \tag{3.8}$$

for any $x \in B_{(1-s)r}$. Setting $s = \frac{1}{2d}$ and integrating (3.8) yields

$$\int_{B_{(1-\frac{1}{2d})r}} K \leq C_1\, m\, 2^{C_\Gamma}\, d^{C_\Gamma} \,. \tag{3.9}$$

On the other hand, by assumption (3.2) and the definition of K,

$$\mathcal{N} a \leq \sum_{i=1}^{\mathcal{N}} I_{v_i}((1-(2d)^{-1})r) = \int_{B_{(1-\frac{1}{2d})r}} K \,. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\mathcal{N} \leq C_1\, a^{-1}\, 2^{C_\Gamma}\, m\, d^{C_\Gamma} \,. \tag{3.11}$$

∎

We are now prepared to state and prove our first general result.

PROPOSITION 3.2 *Let (M, dist), E, and Γ be as above. For any $d \geq 1$,*

$$\dim\left(\mathcal{P}_d(M,E) \cap \Gamma\right) \leq C_2\, m\, d^{C_\Gamma} \,, \tag{3.12}$$

where $C_2 = C_1\, 2^{C_\Gamma + 3}\, e^8$.

PROOF: Set $\Omega = (1 - \frac{1}{2d})^{-1}$ and choose a positive integer m_0 such that $\Omega^{m_0} \geq R_0$. Suppose that $\theta_1, \dots, \theta_s \in (\mathcal{P}_d(M,E) \cap \Gamma)$ are linearly independent.

Applying Proposition 2.5, there exist $m \geq m_0$, an integer $\ell \geq \frac{1}{4}\Omega^{-4d}\, s$, and sections $\eta_1, \dots, \eta_\ell$ in the linear span of the θ_i such that

$$J_r(\eta_i, \eta_j) = \delta_{i,j} \tag{3.13}$$

and

$$\frac{1}{2}\Omega^{-4d} \leq I_{\eta_i}(\Omega^{-1}\, r)\,, \tag{3.14}$$

where we have set $r = \Omega^{m+1}$. For any $b \geq 2$,

$$\left(1 - \frac{1}{b}\right)^{-b} = \left(1 + \frac{1}{b-1}\right)^{b} \leq e^{\frac{b}{b-1}} \leq e^2\,. \tag{3.15}$$

Therefore, $e^{-4} \leq \Omega^{-4d}$; hence

$$\frac{e^{-4}}{4}\, s \leq \ell\,, \tag{3.16}$$

and (3.14) yields for $i = 1, \dots, \ell$

$$\frac{e^{-4}}{2} \leq I_{\eta_i}((1 - (2d)^{-1})r)\,. \tag{3.17}$$

Since the η_i are J_r-orthonormal and satisfy (3.17), Proposition 3.1 (with $a = \frac{e^{-4}}{2}$) implies that

$$\ell \leq C_1\, 2^{C_\Gamma + 1}\, e^4\, m\, d^{C_\Gamma}\,. \tag{3.18}$$

Combining (3.16) and (3.18),

$$s \leq C_1\, 2^{C_\Gamma + 3}\, e^8\, m\, d^{C_\Gamma}\,, \tag{3.19}$$

and since s was arbitrary

$$\dim(\mathcal{P}_d(M,E) \cap \Gamma) \leq C_1\, 2^{C_\Gamma + 3}\, e^8\, m\, d^{C_\Gamma}\,. \tag{3.20}$$

∎

We will next give a sharpening of the above estimates when the measure has an additional regularity property.

We saw above that harmonic functions with a high rate of growth on a ball concentrate near the boundary of the ball (this can be made precise; see section 4 of [10]). It is therefore useful to get sharper bounds for the volumes of thin annuli. This amounts to showing some regularity properties for the volume measure. The relevant property is analogous to Hölder regularity, and we shall call it the ε-volume regularity property.

ε-VOLUME REGULARITY *We will say M has the ε-volume regularity property if for $0 < \varepsilon \leq 1$ and $1 \leq C_W < \infty$, given any $0 < \delta \leq \frac{1}{2}$ we get an $R_0 > 0$ such that for all $r \geq R_0$ and $p \in M$,*

$$\operatorname{Vol}(B_r(p) \setminus B_{(1-\delta)r}(p)) \leq C_W\, \delta^{\varepsilon} \operatorname{Vol}(B_r(p))\,. \tag{3.21}$$

The classical relative volume comparison theorem implies immediately that manifolds with nonnegative Ricci curvature satisfy (3.21) for $\varepsilon = 1$. Later we will see that this property is also satisfied for $\varepsilon = 1$ by n-rectifiable stationary varifolds with Euclidean volume growth and by Cartan-Hadamard manifolds with Euclidean volume growth. In general, for a manifold M that has the doubling property, we have the following weaker statement:

LEMMA 3.3 *If M is a complete manifold with the doubling property with doubling constant C_D, then M has the ε-volume regularity property where*

$$\varepsilon = \frac{\log(1 + 32^{-C_D})}{\log 2} \quad \textit{and} \quad C_W = (1 + 32^{-C_D})^2\,.$$

PROOF: Fix $p \in M$. First, for any $r > 0$ and $0 < s \leq \frac{1}{2}$, we let $\{B_{\frac{3s}{2}r}(x_i)\}_{i\in I}$ be a maximal disjoint set of balls with radius $\frac{3s}{2}r$ and centers contained in $\partial B_{(1-\frac{s}{2})r}(p)$. Observe that by maximality

$$\partial B_{(1-\frac{s}{2})r}(p) \subset \bigcup_i B_{3sr}(x_i)\,,$$

and hence by the triangle inequality

$$B_r(p) \setminus B_{(1-s)r}(p) \subset \bigcup_i B_{4sr}(x_i)\,.$$

By the doubling property

$$\begin{aligned} \operatorname{Vol}(B_r(p) \setminus B_{(1-s)r}(p)) &\leq \sum_{i\in I} \operatorname{Vol}(B_{4sr}(x_i)) \\ &\leq 8^{C_D} \sum_{i\in I} \operatorname{Vol}(B_{\frac{s}{2}r}(x_i))\,. \end{aligned} \tag{3.22}$$

For each $i \in I$ let γ_i be a minimal geodesic with unit speed connecting p and x_i. Set $y_i = \gamma_i((1 - \frac{3}{2}s)r)$. Observe that again by the triangle inequality the balls $B_{\frac{s}{2}r}(y_i)$ are pairwise disjoint, $B_{\frac{s}{2}r}(x_i) \subset B_{\frac{3s}{2}r}(y_i) \subset B_{2sr}(y_i)$, and $B_{\frac{s}{2}r}(y_i) \subset B_{(1-s)r}(p) \setminus B_{(1-2s)r}(p)$. Therefore by (3.22) and the doubling property

$$\begin{aligned} \text{Vol}(B_r(p) \setminus B_{(1-s)r}(p)) &\leq 8^{C_D} \sum_{i \in I} \text{Vol}(B_{2sr}(y_i)) \qquad (3.23) \\ &\leq 32^{C_D} \sum_{i \in I} \text{Vol}(B_{\frac{s}{2}r}(y_i)) \\ &\leq 32^{C_D} \text{Vol}(B_{(1-s)r}(p) \setminus B_{(1-2s)r}(p)) \, . \qquad (3.24) \end{aligned}$$

For each positive integer j, set

$$v_j = \text{Vol}(B_r(p) \setminus B_{(1-2^{-j})r}(p)) \, . \qquad (3.25)$$

From (3.23) with $s = 2^{-j}$,

$$(1 + \mu) \, v_j \leq v_{j-1} \, , \qquad (3.26)$$

where $\mu = 32^{-C_D}$. By iterating (3.26), we get

$$(1 + \mu)^{j-1} \, v_j \leq v_1 \leq \text{Vol}(B_r(p)) \, . \qquad (3.27)$$

Finally, given any $0 < \delta \leq \frac{1}{2}$, let j be the positive integer with $2^{-j-1} < \delta \leq 2^{-j}$. Applying (3.27) yields

$$\begin{aligned} \text{Vol}(B_r(p) \setminus B_{(1-\delta)r}(p)) \leq v_j &\leq (1 + \mu)^{2 + \frac{\log \delta}{\log 2}} \, \text{Vol}(B_r(p)) \\ &= C_W \, \delta^{\varepsilon} \, \text{Vol}(B_r(p)) \, , \end{aligned} \qquad (3.28)$$

where $\varepsilon = \frac{\log(1+32^{-C_D})}{\log 2}$ and $C_W = (1 + 32^{-C_D})^2$. ∎

Note that the ε above is not quite sharp. However, it does not seem likely that the doubling property implies 1-volume regularity.

The following proposition sharpens Proposition 3.1 when M also satisfies the ε-volume regularity property.

PROPOSITION 3.4 *Let (M, dist), E, and Γ be as above with $C_\Gamma > 1$, and suppose that M satisfies the ε-volume regularity property. Suppose that $0 < a < 1$ is fixed, $r > R_0$, and $v_1, \dots, v_{\mathcal{N}} \in \Gamma$ are J_r-orthonormal. Given any $d \geq 1$ such that for all i*

$$a \leq I_{v_i}((1 - (2d)^{-1})r) \, , \qquad (3.29)$$

then

$$\mathcal{N} \le C\, m\, d^{C_\Gamma - \varepsilon}\,, \tag{3.30}$$

where $C = C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, a^{-1}\, 2^{C_\Gamma + 2}$.

PROOF: For $s > 0$, set $B_s = B_s(p)$. Since $d \ge 1$, we can choose a positive integer N with $d \le N \le 2d$. Let K be given by (3.4). Following the proof of Proposition 3.1 through equation (3.8), we get

$$\mathrm{Vol}(B_r)\, K(x) = \mathrm{Vol}(B_r) \left(\sum_{i=1}^{m} |w_i|^2(x) \right) \le C_1\, m \left(\frac{j}{2d} \right)^{-C_\Gamma} \tag{3.31}$$

for each $j = 1, \dots, N$ and any $x \in B_{(1-\frac{j}{2d})r}$.

We break down the integral of K to get

$$\int_{B_{(1-\frac{1}{2d})r}} K = \int_{B_{(1-\frac{N}{2d})r}} K + \sum_{j=1}^{N-1} \int_{B_{(1-\frac{j}{2d})r} \setminus B_{(1-\frac{j+1}{2d})r}} K\,. \tag{3.32}$$

We will now bound each of the two terms on the right-hand side of (3.32). Since $d \le N \le 2d$, $B_{(1-\frac{N}{2d})r} \subset B_{\frac{r}{2}}$, and therefore by (3.31)

$$\int_{B_{(1-\frac{N}{2d})r}} K \le C_1\, m \left(\frac{1}{2} \right)^{-C_\Gamma} \frac{\mathrm{Vol}(B_{\frac{r}{2}})}{\mathrm{Vol}(B_r)} \le C_1\, m\, 2^{C_\Gamma}\,. \tag{3.33}$$

Bounding the integral of K on each annulus above in terms of its maximum, (3.31) yields

$$\sum_{j=1}^{N-1} \int_{B_{(1-\frac{j}{2d})r} \setminus B_{(1-\frac{j+1}{2d})r}} K \tag{3.34}$$

$$\le C_1\, m \sum_{j=1}^{N-1} \frac{\mathrm{Vol}(B_{(1-\frac{j}{2d})r} \setminus B_{(1-\frac{j+1}{2d})r})}{\mathrm{Vol}(B_r)} \left(\frac{j}{2d} \right)^{-C_\Gamma}$$

$$\le C_1\, m \sum_{j=1}^{N-1} C_W \left(\frac{1}{2d} \right)^{\varepsilon} \left(\frac{j}{2d} \right)^{-C_\Gamma}\,, \tag{3.35}$$

where the second inequality follows from the ε-volume regularity property. Using the elementary inequality

$$\sum_{j=1}^{N-1} j^{-C_\Gamma} = 1 + \sum_{j=2}^{N-1} j^{-C_\Gamma} \le 1 + \int_1^\infty s^{-C_\Gamma}\, ds = \frac{C_\Gamma}{C_\Gamma - 1}\,, \tag{3.36}$$

(3.35) implies that

$$\sum_{j=1}^{N-1} \int_{B_{(1-\frac{j}{2d})r} \setminus B_{(1-\frac{j+1}{2d})r}} K \leq C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, 2^{C_\Gamma - \varepsilon}\, m\, d^{C_\Gamma - \varepsilon}\,. \tag{3.37}$$

Substituting (3.33) and (3.37) into (3.32) (since $\varepsilon \leq 1$)

$$\int_{B_{(1-\frac{1}{2d})r}} K \leq C_1\, m\, 2^{C_\Gamma} \left(2 + C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, d^{C_\Gamma - \varepsilon} \right) \tag{3.38}$$

$$\leq C_1\, 2^{C_\Gamma + 2}\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, m\, d^{C_\Gamma - \varepsilon}\,, \tag{3.39}$$

since $d, C_W \geq 1$ and $C_\Gamma > 1$.

Combining (3.10) and (3.38), we get

$$\mathcal{N} \leq C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, a^{-1}\, 2^{C_\Gamma + 2}\, m\, d^{C_\Gamma - \varepsilon}\,. \tag{3.40}$$

∎

Arguing as in Proposition 3.2, we get the following:

PROPOSITION 3.5 *Let* (M, dist), E, *and* Γ *be as in Proposition* 3.4. *For any* $d \geq 1$,

$$\dim\left(\mathcal{P}_d(M, E) \cap \Gamma\right) \leq C_2\, m\, d^{C_\Gamma - \varepsilon}\,, \tag{3.41}$$

where $C_2 = C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, 2^{C_\Gamma + 5}\, e^8$.

PROOF: Set $\Omega = (1 - \frac{1}{2d})^{-1}$ and choose a positive integer m_0 such that $\Omega^{m_0} \geq R_0$. Suppose that $\theta_1, \dots, \theta_s \in (\mathcal{P}_d(M, E) \cap \Gamma)$ are linearly independent. Applying Proposition 2.5, there exist $N \geq m_0$, an integer ℓ with

$$\frac{e^{-4}}{4}\, s \leq \ell\,, \tag{3.42}$$

and J_r-orthonormal sections $\eta_1, \dots, \eta_\ell$ in the linear span of the θ_i for $i = 1, \dots, \ell$

$$\frac{e^{-4}}{2} \leq I_{\eta_i}((1 - (2d)^{-1})r)\,, \tag{3.43}$$

where we have set $r = \Omega^{N+1}$. Proposition 3.4 (with $a = \frac{e^{-4}}{2}$) implies that

$$\ell \leq C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, 2^{C_\Gamma + 3}\, e^4\, m\, d^{C_\Gamma - \varepsilon}\,. \tag{3.44}$$

Combining (3.42) and (3.44),

$$\dim\left(\mathcal{P}_d(M,E)\cap\Gamma\right) \leq C_1\, C_W\, \frac{C_\Gamma}{C_\Gamma - 1}\, 2^{C_\Gamma+5}\, e^8\, m\, d^{C_\Gamma-\varepsilon}\,. \tag{3.45}$$

∎

In the applications, we will be primarily interested in the case where Γ is the linear space of harmonic sections of E, where E^m is a Hermitian vector bundle with nonnegative curvature over M. For future reference, we will now apply the results of this section to this case and record this in Corollary 3.6 below.

Since E has nonnegative curvature, the Bochner formula yields, for any $\eta \in \Gamma$,

$$2\,|\nabla\eta|^2 \leq \Delta|\eta|^2\,. \tag{3.46}$$

We conclude that $|\eta|^2$ is subharmonic. Hence if M satisfies the strong mean value inequality for nonnegative subharmonic functions, then Γ satisfies the strong mean value inequality with the same exponent. Therefore, Proposition 3.5 yields the following corollary:

COROLLARY 3.6 *Let M be an open (generalized) manifold that satisfies the strong mean value inequality for nonnegative subharmonic functions and has the ε-volume regularity property. If E^m is a Hermitian vector bundle with nonnegative curvature over M, then for all $d \geq 1$,*

$$\dim\mathcal{H}_d(M,E) \leq C\, m\, d^{C_\Gamma-\varepsilon} \tag{3.47}$$

where $C = C(C_\Gamma, C_1, C_W) < \infty$.

In order to apply the above corollary we need M to have a notion of subharmonicity. Usually we will take M to be a manifold, but in Section 5 we will allow M to be a rectifiable varifold (see [22]).

4 Applications to Function Theory

In this section, we will give some applications of Corollary 3.6. First, we will prove Theorem 1.2 for open manifolds M^n that have the doubling property and satisfy a mean value inequality for subharmonic functions. We will then consider two cases where we can get sharper bounds on the dimension. Namely, we will prove Theorem 1.3 for manifolds with nonnegative Ricci curvature and Corollary 4.1 for Cartan-Hadamard manifolds with Euclidean volume growth.

As before, E^m will be a Hermitian vector bundle over M, and $p \in M$ will be fixed.

When M is in one of the above classes of manifolds and the curvature of the bundle E is nonnegative, we will observe that the harmonic sections satisfy a strong mean value inequality. It follows from the results of the prior sections that the space $\mathcal{H}_d(M, E)$ is finite-dimensional.

First, note that it follows from Saloff-Coste [21] and Grigor'yan [11] that the scale-invariant 1-Neumann-Poincaré inequality together with the doubling property imply the mean value inequality for nonnegative subharmonic functions. Hence these properties together imply the strong mean value inequality by Lemma 2.6.

Recall that Li and Schoen [14] had previously proven the mean value inequality (with $\varepsilon = 0$) for manifolds with Ricci curvature bounded from below (see [16] for the case $\varepsilon > 0$).

Fixing a Hermitian vector bundle with nonnegative curvature, E^m, over M, let Γ denote the linear space of harmonic sections of E. Since E has nonnegative curvature, the Bochner formula (see (3.46)) implies that $|\eta|^2$ is subharmonic and hence that Γ satisfies the strong mean value inequality with $C_\Gamma = C_D$. Therefore, combining Lemma 3.3 and Corollary 3.6 yields Theorem 1.2.

If M^n actually has nonnegative Ricci curvature, then $C_D = n$. Furthermore, for any $0 < \delta \leq \frac{1}{2}$, the relative volume comparison theorem implies that

$$\mathrm{Vol}(B_s) \leq (1-\delta)^{-n} \mathrm{Vol}(B_{(1-\delta)s}) \leq \left(1 + n\, 2^{n+1}\, \delta\right) \mathrm{Vol}(B_{(1-\delta)s}) \,, \tag{4.1}$$

where the second inequality follows from the elementary inequality

$$\left| \frac{d}{dt} (1-t)^{-n} \right| \leq n\, 2^{n+1} \tag{4.2}$$

for $0 < t \leq \frac{1}{2}$. From this we see for any $0 < \delta \leq \frac{1}{2}$,

$$\mathrm{Vol}(B_s \setminus B_{(1-\delta)s}) \leq n\, 2^{n+1}\, \delta\, \mathrm{Vol}(B_{(1-\delta)s}) \,. \tag{4.3}$$

We conclude that any manifold with nonnegative Ricci curvature has the 1-volume regularity property (with the constant depending only on the dimension). Therefore, Corollary 3.6 applies to yield Theorem 1.3.

It is easy to see that these results continue to hold for manifolds that have Ricci curvature bounded from below by $-(n-1)\, \Lambda\, (1+r)^{-2}$ and that satisfy the doubling property.

Namely, let M^n be an open manifold with Ricci curvature bounded from below by $-(n-1)\Lambda(1+r)^{-2}$, where r is the distance to $p \in M$. Suppose also that M has the doubling property with doubling constant $C_D < \infty$. Note that the doubling property is not automatically satisfied even for a manifold with nonnegative Ricci curvature outside of a compact set. Fix $r > 1$ and $0 < s \leq \frac{1}{4}$. By the maximum principle, if u^2 is a subharmonic function, there exists $x \in \partial B_{(1-s)r}(p)$ such that

$$u^2(x) = \sup_{B_{(1-s)r}(p)} u^2 \,. \tag{4.4}$$

By the triangle inequality, $B_{s\,r}(x) \subset B_r(p) \setminus B_{\frac{r}{2}}(p)$; hence the Li-Schoen mean value inequality yields

$$u^2(x) \leq \frac{C_1}{\mathrm{Vol}(B_{s\,r}(x))} \int_{B_{s\,r}(x)} u^2 \leq \frac{C_1}{\mathrm{Vol}(B_{s\,r}(x))} \int_{B_r(p)} u^2 \tag{4.5}$$

where $C_1 = C_1(n, \Lambda)$. Combine the triangle inequality and the doubling property to get

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_{2r}(x)) \leq 4^{C_D} \mathrm{Vol}(B_{\frac{r}{2}}(x)) \,. \tag{4.6}$$

Since the Ricci curvature is bounded from below by $-4(n-1)\Lambda r^{-2}$ on $B_{\frac{r}{2}}(x)$,

$$\mathrm{Vol}(B_{\frac{r}{2}}(x)) \leq C_2\, s^{-n} \mathrm{Vol}(B_{s\,r}(x)) \tag{4.7}$$

where $C_2 = C_2(n, \Lambda)$. Combining (4.5), (4.6), and (4.7) yields the strong mean value inequality for subharmonic functions.

Furthermore, since these manifolds satisfy the 1-volume regularity property (this follows easily from lemma 1.3 of [18]), we see by Theorem 1.3 that for $d \geq 1$,

$$\dim \mathcal{H}_d(M, E) \leq C\, m\, d^{n-1} \tag{4.8}$$

where $C = C(C_D, n, \Lambda)$.

We note that for a manifold M^n with Ricci curvature bounded from below by $-(n-1)\,\Lambda\,(1+r)^{-2}$ where r is the distance to $p \in M$, the doubling property is equivalent to the following (VC) condition of Li and Tam [18].

(VC) *There exists* $c_0 > 0$ *such that if* $x \in \partial B_r(p)$*, then*

$$\mathrm{Vol}(B_r(p)) \leq c_0 \mathrm{Vol}(B_{\frac{r}{2}}(x)) \,. \tag{4.9}$$

By the standard relative volume comparison theorem for all $x \in M$ and $s \leq \frac{r(x)}{4}$

$$\mathrm{Vol}(B_{2s}(x)) \leq 2^{C_1} \mathrm{Vol}(B_s(x))\,. \tag{4.10}$$

Further, according to lemma 1.3 of [18], there exists $C_2 = C_2(\Lambda, n)$ such that for all $r > 0$

$$\mathrm{Vol}(B_{2r}(p)) \leq 2^{C_2} \mathrm{Vol}(B_r(p))\,. \tag{4.11}$$

Set $C_3 = \max\{C_1, C_2\}$.

The equivalence of the doubling property and the (VC) condition for these manifolds can be seen as follows: Suppose first that M^n as above has the doubling property with doubling constant C_D. For $r > 0$ and $x \in \partial B_r(p)$, the triangle inequality and the doubling property yield

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_{2r}(x)) \leq 4^{C_D} \mathrm{Vol}(B_{\frac{r}{2}}(x))\,. \tag{4.12}$$

Therefore, we may take $c_0 = 4^{C_D}$.

Conversely, suppose that M^n has Ricci curvature bounded from below by $-(n-1)\,\Lambda\,(1+r)^{-2}$ where r is the distance to $p \in M$ and satisfies condition (VC). We want to see that M has the doubling property. By (4.10) and (4.11), it suffices to consider $x \neq p$ and $\frac{r(x)}{4} < s$. If we first suppose that $\frac{r(x)}{4} < s \leq 4r(x)$,

$$\mathrm{Vol}(B_{2s}(x)) \leq \mathrm{Vol}(B_{9\,r(x)}(p)) \leq 2^{4\,C_3}\, \mathrm{Vol}(B_{r(x)}(p)) \tag{4.13}$$

$$\leq 2^{4C_3}\, c_0\, \mathrm{Vol}(B_{\frac{r(x)}{2}}(x)) \leq 2^{5\,C_3}\, c_0\, \mathrm{Vol}(B_{\frac{r(x)}{4}}(x))\,, \tag{4.14}$$

and the result follows since $B_{r(x)/4}(x) \subset B_s(x)$. On the other hand, if $4r(x) < s$, then $B_{3s/4}(p) \subset B_s(x)$ and $B_{2s}(x) \subset B_{9s/4}(p)$. Therefore, (4.11) implies that

$$\mathrm{Vol}(B_{2s}(x)) \leq 4^{C_3}\, \mathrm{Vol}(B_s(x))\,. \tag{4.15}$$

Function theory on this class of manifolds has been studied by Li, Tam, and Wang (see [17, 18, 24]).

Finally, we will give bounds on the dimension of the space of polynomial-growth harmonic sections of a bundle with nonnegative curvature on Cartan-Hadamard manifolds with Euclidean volume growth.

Recall that if M^n is a Cartan-Hadamard manifold (i.e., a simply connected open manifold with nonpositive curvature), then the exponential map is a global diffeomorphism. Moreover, the Laplacian comparison theorem implies that $\Delta\, r \geq \frac{n-1}{r}$. These two facts imply that if u is a nonnegative subharmonic function, then

$$r^{1-n} \int_{\partial B_r(p)} u \tag{4.16}$$

is monotonically nondecreasing. In particular, setting $u = 1$ in (4.16) shows that

$$r^{1-n}\, \mathrm{Vol}(\partial B_r(p)) \tag{4.17}$$

is monotonically nondecreasing.

Suppose, in addition, that M^n has Euclidean volume growth; that is, there exists $\mathrm{V}_M < \infty$ such that for some $x \in M$ (and hence for all $x \in M$), $r > 0$,

$$\frac{\mathrm{Vol}(B_r(x))}{\mathrm{V}_0^n(1)\, r^n} \leq \mathrm{V}_M\,. \tag{4.18}$$

Combining (4.17) and (4.18), we get that for any $x \in M$ and $r > 0$,

$$\mathrm{V}_1^{n-1}(\pi)\, r^{n-1} \leq \mathrm{Vol}(\partial B_r(x)) \leq \mathrm{V}_M\, \mathrm{V}_1^{n-1}(\pi)\, r^{n-1} \tag{4.19}$$

and

$$\mathrm{V}_0^n(1)\, r^n \leq \mathrm{Vol}(B_r(x)) \leq \mathrm{V}_M\, \mathrm{V}_0^n(1)\, r^n\,. \tag{4.20}$$

Therefore, M has the 1-volume regularity property with $C_W = n\, \mathrm{V}_M$.

As an immediate consequence of (4.16) and (4.19), we have the following well-known mean value inequality for any nonnegative subharmonic function u (cf. [12]):

$$u(x) \leq \frac{1}{\mathrm{V}_0^n(1)\, r^n} \int_{B_r(x)} u \leq \frac{\mathrm{V}_M}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} u\,. \tag{4.21}$$

Combining (4.20) and (4.21), we get for any $r > 0$, $0 < s \leq \frac{1}{2}$, $x \in B_{(1-s)r}(p)$, and u a nonnegative subharmonic function,

$$u^2(x) \leq \frac{1}{\mathrm{V}_0^n(1)\, s^n\, r^n} \int_{B_{s\,r}(x)} u^2 \leq \mathrm{V}_M\, \frac{s^{-n}}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} u^2\,. \tag{4.22}$$

We can now apply Corollary 3.6 to M with the Riemannian distance and the Riemannian measure to get the following corollary:

COROLLARY 4.1 *Let M^n be a simply connected, open manifold with non-positive sectional curvature and Euclidean volume growth, and suppose that E^m is a Hermitian vector bundle with nonnegative curvature. There exists $C = C(n, \mathrm{V}) < \infty$ such that for any $d \geq 1$, $\dim \mathcal{H}_d(M, E) \leq C\, m\, d^{n-1}$.*

We note that a version of Corollary 4.1 was proven previously by Kasue for harmonic functions under the additional assumption that the Riemann curvature tensor decays quadratically (see [13]).

5 Applications to Minimal Submanifolds

In this section, we will give some applications to the study of harmonic sections of vector bundles over stationary, n-rectifiable varifolds of arbitrary codimension in Euclidean space. As a special case, we will get function-theoretic results for these generalized minimal submanifolds; these results generalize earlier results of [9] and [3]. In addition, we will give some geometric applications of our function-theoretic results that extend results of [3].

Henceforth, $\Sigma^n \subset \mathbb{R}^{n'}$ will be a stationary, rectifiable n-varifold with integer multiplicity (see [22] for the definitions). Associated to each such Σ is a Radon measure μ_Σ; with a slight abuse of notation, we will also use Σ to denote the (Hausdorff n-dimensional rectifiable) set on which the measure is supported. We shall make the standard assumption that the density is at least 1 on the support. Note that this class of generalized minimal submanifolds includes the case of embedded minimal submanifolds equipped with the intrinsic Riemannian metric. For $x, y \in \mathbb{R}^{n'}$, let $\mathrm{dist}(x, y) = |x - y|$ denote the Euclidean distance from x to y.

Let E^m be a rank m vector bundle over Σ that is the restriction to Σ of an ambient vector bundle. If Σ is a properly embedded submanifold, then every Hermitian vector bundle E arises this way. For $d > 0$, we define the spaces $\mathcal{H}_d(\Sigma, E)$ with respect to the Euclidean norm (instead of the intrinsic distance). Notice that with this definition, the coordinate functions x_i are in $\mathcal{H}_1(\Sigma, \mathbb{R})$. For example, on the catenoid Σ_c^2 in $\mathbb{R}^3$ (which is rotationally symmetric about the x_3-axis), the function x_3 grows slower than any power of the intrinsic distance; however, it is not in $\mathcal{H}_d(\Sigma_c^2, \mathbb{R})$ for any $d < 1$.

Recall the following classical facts about stationary, rectifiable varifolds in Euclidean space (see, for instance, [22]).

Given any $x \in \mathbb{R}^{n'}$ and $r > 0$, the density is defined by

$$\Theta_\Sigma(x, r) = \frac{\mathrm{Vol}(\Sigma \cap B_r(x))}{\mathrm{V}_0^n(1)\, r^n}\,. \tag{5.1}$$

LEMMA 5.1 (Monotonicity of Volume) *If Σ^n is any stationary, rectifiable n-varifold in $\mathbb{R}^{n'}$ and $x \in \mathbb{R}^{n'}$, then $\Theta_\Sigma(x,r)$ is monotonically nondecreasing in r.*

We say that a stationary, rectifiable n-varifold Σ has Euclidean volume growth if given any $x \in \mathbb{R}^{n'}$ there exists $\mathrm{V} < \infty$ such that

$$\Theta_\Sigma(x,r) \leq \mathrm{V} \tag{5.2}$$

for all $r > 0$. For any immersed minimal submanifold Σ with Euclidean volume growth, any $x \in \Sigma$, and any $r > 0$, it follows easily from Lemma 5.1 that

$$1 \leq \Theta_\Sigma(x,r) \leq \mathrm{V}\,. \tag{5.3}$$

We will now observe that these varifolds have the 1-volume regularity property. Let

$$\mathrm{V}_\Sigma = \lim_{r\to\infty} \Theta_\Sigma(x,r) < \infty\,. \tag{5.4}$$

Given $0 < \delta \leq \frac{1}{2}$, choose R_0 such that for $r \geq \frac{R_0}{2}$,

$$\mathrm{V}_\Sigma - \Theta_\Sigma(x,r) < \delta\, \mathrm{V}_\Sigma\,. \tag{5.5}$$

Therefore for $r \geq R_0$

$$\begin{aligned} \mathrm{Vol}(B_r(p) \setminus B_{(1-\delta)r}(p)) &\leq \mathrm{V}_\Sigma\, \mathrm{V}_0^n(1) \,\left(1-(1-\delta)^{n+1}\right) r^n \\ &\leq 2\,(n+1)\,\delta\, \mathrm{Vol}(B_r(p))\,. \end{aligned} \tag{5.6}$$

Michael and Simon [20] proved a mean value inequality for nonnegative subharmonic functions on a large class of generalized minimal submanifolds; we refer to 18.1 of [22] for the case of stationary, rectifiable n-varifolds. Note that these mean value inequalities, and our results as well, extend to the case where the generalized mean curvature is bounded and decays sufficiently fast. The arguments are quite similar to those used for manifolds with Ricci curvature bounded from below by $-(n-1)\,\Lambda\,(1+r)^{-2}$ that satisfy the doubling property (see Section 4). We leave the details of such a generalization to the reader.

LEMMA 5.2 (18.1 of [22])) *Let Σ^n be as above and u be a nonnegative weakly subharmonic function on Σ. Then for almost all $x \in \Sigma$ and all $r > 0$,*

$$u(x) \leq \frac{1}{\mathrm{V}_0^n(1) r^n} \int_{B_r(x)\cap\Sigma} u\,. \tag{5.7}$$

Combining Lemma 5.2 and a slight variation of the argument of Lemma 2.6, we see that Σ^n satisfies the strong mean value inequality for subharmonic functions with $C_1 = \mathrm{V}_\Sigma$ and $C_\Gamma = n$. Therefore we have the following application of Corollary 3.6:

THEOREM 5.3 *Let Σ^n be a stationary, rectifiable n-varifold with density at least one almost everywhere and bounded from above by $\mathrm{V}_\Sigma < \infty$, and let E^m be a vector bundle over Σ that has nonnegative curvature. For any $d \geq 1$, $\dim \mathcal{H}_d(\Sigma, E) \leq C\, m\, \mathrm{V}_\Sigma\, d^{n-1}$ where $C = 3\,\frac{n}{n-1}\,2^{n+3}\,e^8$.*

Note that affine algebraic varieties are examples of stationary varifolds with bounded density.

We now turn to some geometric applications of this function-theoretic result. Recall that the classical Bernstein theorem implies that, through dimension 7, graphical minimal hypersurfaces must be affine. A weaker form of this is true in all dimensions by the Allard regularity theorem [1]. Namely, there exists $\delta = \delta(n, n') > 0$ such that if $\Sigma^n \subset \mathbb{R}^{n'}$ is a stationary, rectifiable n-varifold as above with $\mathrm{V}_\Sigma \leq 1 + \delta$, then Σ is an n-plane. We note that it will follow from Corollary 1.4 that δ is independent of n'.

We will show that any upper bound on the density implies that the stationary, rectifiable n-varifold lies in some affine subspace (where the dimension of the affine space is bounded in terms n and the upper bound for the density). In fact, this subspace has dimension at most $3\,\frac{n}{n-1}\,2^{n+3}\,e^8\,\mathrm{V}_\Sigma$. When Σ is C^2, it is easy to see that the coordinate functions are harmonic; more generally, even when Σ is a stationary varifold the coordinate functions are weakly harmonic (see [22, p. 115]). Therefore, since the coordinate functions have linear growth and are thus in $\mathcal{H}_1(\Sigma, \mathbb{R})$, we get the same uniform bound on the number of independent coordinate functions and, hence, Corollary 1.4.

Recall that complex submanifolds of $\mathbb{C}^{n'} = \mathbb{R}^{2n'}$ are absolutely area minimizing (and hence minimal). By a result of Stoll, an upper bound on the density implies that the submanifold is actually affine algebraic. For projective algebraic varieties, Corollary 1.4 is a consequence of the classical Bezout theorem.

We note finally that, by considering conical minimal submanifolds, Theorem 5.3 implies lower bounds for the eigenvalues (in terms of dimension and area) of minimal submanifolds in $\mathbb{S}^n$ and also for projective algebraic varieties. These types of eigenvalue bounds were previously known in these cases (due to Cheng, Li, and Yau [3] and Li and Tian [19], respectively).

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