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ON GENERALIZED RIEMANN MATRICES

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In the following I intend to give a simpler and generalized formulation of the problem of complex multiplication of Riemann matrices, recently treated with such conclusive success by A. A. Albert.[1] All known propositions remain untouched by this generalization, which in my opinion is required by the nature of the subject.

§1. Foundations: Transfer of the function theoretical problem into the algebraic one

On a Riemann surface of connectivity $n = 2p$ (genus p), we take a basis α of the closed curves. Every closed curve ξ is homologous to a linear combination of the n basic curves, $\xi \sim \sum_\alpha x_\alpha \cdot \alpha$, by means of integers x_α. The "characteristic" $[\xi\eta]$ of any two closed curves ξ and $\eta \sim \sum_\alpha y_\alpha \cdot \alpha$ which gives the number of times ξ crosses η in summa in the positive sense is a non-degenerate skew-symmetric bilinear form

$$[\xi\eta] = \sum_{\alpha,\beta} c_{\alpha\beta}\, x_\alpha\, y_\beta\, ; \qquad c_{\alpha\beta} = [\alpha\beta]$$

with integer coefficients. Transition to a new basis α of curves is performed by a unimodular integral transformation U. The construction of the integrals of the first kind through Dirichlet's principle naturally leads to associating a differential of the first kind dw_α with every closed curve α such that, for every closed curve β the real part $\Re \int_\beta dw_\alpha = c_{\alpha\beta} = [\alpha\beta]$ equals the characteristic of the two curves α and β.[2] Homologous curves α are associated with the same dw_α, addition of curves α leads to addition of the corresponding differentials dw_α. In this manner the basis of curves α gives rise to a "real basis" dw_α for the differentials of the first kind, consisting of n terms; every differential of the first kind dw can be uniquely expressed as a linear combination

$$dw = \sum_\alpha x_\alpha \cdot dw_\alpha \tag{1}$$

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[1] Rend. Circ. Mat. di Palermo **55** (1931), p. 57; Trans. Amer. Math. Soc. **33** (1931), p. 219; Annals of Math. **35** (1934), p. 1, to be continued. All work prior to 1928, in particular by Scorza, Rosati and Lefschetz, is reported in the latter's Report of the Committee on Rational Transformations, Bulletin of the National Research Council, **63** (1928), pp. 310–392.

[2] Weyl, Idee der Riemannschen Fläche, 2[te] Auflage, Leipzig, 1923, p. 98 and pp. 172–174. A general topological proof of the fact that the characteristic form is non-degenerate and even unimodular, which remains valid for higher dimensions and does not pass through the explicit construction of a canonical basis: Weyl, Revista Mat. Hispano-Americana, 1923, Theorem 10.

with constant real coefficients x_α. The basis dw_α transforms cogrediently with the basis α of the curves themselves. Whereas the real parts of the periods

$$\Re \int_\beta dw_\alpha = c_{\alpha\beta}$$

form an integral, non-singular, skew-symmetric matrix C, the imaginary parts

$$\Im \int_\beta dw_\alpha = s_{\alpha\beta}$$

are symmetric and the coefficients of a positive definite quadratic form. The definite character of the quadratic form $S = \Sigma\, s_{\alpha\beta}\, x_\alpha\, x_\beta$ may be described as the property of being non-singular in every real partial space of the total n-dimensional vector space. This is meant if we say that S is *totally regular in the field of real numbers.*[3]

When one operates with a real basis it is natural to ask how the differentials multiplied by i are expressed by the differentials themselves in a real manner:

(2) $$i dw_\alpha = \sum_\gamma r_{\gamma\alpha}\, dw_\gamma \qquad (r_{\gamma\alpha} \text{ real constants}).$$

By integrating over β and taking the real part, one obtains the equations

$$-s_{\alpha\beta} = \sum_\gamma r_{\gamma\alpha}\, c_{\gamma\beta} \qquad \text{or} \qquad s_{\beta\alpha} = \sum_\gamma c_{\beta\gamma}\, r_{\gamma\alpha}.$$

Consequently the relation

(3) $$S = C \cdot R, \qquad R = C^{-1}S$$

holds for the matrices

$$C = \| c_{\alpha\beta} \|, \qquad S = \| s_{\alpha\beta} \| \qquad \text{and} \qquad R = \| r_{\alpha\beta} \|.$$

The transformation R has, according to its significance (2), the property

(4) $$R^2 = -1.$$

C and S occur in the problem of complex multiplication only in this combination R; and the only assumption concerning R that really matters is that R arises according to (3) from an arbitrary rational non-singular skew-symmetric matrix C and an arbitrary real symmetric totally-regular matrix S. The equation (4) does not play any part and will be discarded. Our generalization in comparison with the formulation used before consists exactly in wiping out this restriction $R^2 = -1$ (compare Appendix, §6).

The question of complex multiplication arises, for instance, when we consider an arbitrary (μ, ν)-valued correspondence $P \rightarrow Q$ on our Riemann surface. P determines the point Q ν-valued: $Q_1, \cdots, Q_\nu$.

$$dw_\alpha(Q_1) + \cdots + dw_\alpha(Q_\nu)$$

is a differential of the first kind with respect to P, let us say

$$= \sum_\gamma h_{\alpha\gamma} \cdot dw_\gamma(P).$$

[3] See Weyl loc. cit., p. 116. The form S is the Dirichlet integral of the general differential of the first kind (1) and hence positive.

If P runs over the cycle β, then $Q_1, \cdots, Q_\nu$ together travel over a certain cycle $\sum_\gamma a_{\gamma\beta}\cdot\beta$ (a integers). Hence as $\int_\beta dw_\alpha = c_{\alpha\beta} + is_{\alpha\beta}$, we have in an obvious notation:

$$(C + iS)A = H(C + iS).$$

This splits into the two real equations

$$CA = HC \qquad \text{or} \qquad H = CAC^{-1} \text{ and}$$

$$SA = HS.$$

By substituting H from the first equation, the latter furnishes

$$C^{-1}SA = AC^{-1}S \qquad \text{or} \qquad RA = AR.$$

Let us replace the field of real numbers by any field P and the field of rational numbers by a subfield ρ of P; ρ is considered as the basic domain of rationality. Then we are concerned with the following problem:

Given a matrix R in P arising by equation (3) from a symmetric totally-regular matrix S in P and a skew-symmetric, non-singular matrix C in ρ, the algebra $\mathfrak{A}$ of all matrices A in ρ commuting with R is to be investigated. (In particular we should like to know how a "*Riemann matrix*" R looks, whose "*commutator algebra*" $\mathfrak{A}$ does not consist merely of the multiples of the unit matrix. Hence the problem is to investigate the structure of $\mathfrak{A}$ independently of R and then to find Riemann matrices R corresponding to a given $\mathfrak{A}$ of the ascertained structure.) One may now forget all except this problem.

Only transformations U whose coefficients lie in ρ ("rational transformations") of the coordinate systems are admissible. U carries C, S, R over into

$$U'CU, \qquad U'SU, \qquad U^{-1}RU.$$

§2. Poincaré's and Schur's theorems

Poincaré's theorem: *A Riemann matrix R of the reduced form* (5) *can be completely decomposed into its parts R_1 and R_2 by means of an appropriate rational transformation U*:

$$R = \left|\begin{array}{c|c} R_1 & 0 \\ \hline Q & R_2 \end{array}\right| \rightarrow \left|\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array}\right| \text{ by } U = \left|\begin{array}{c|c} 1 & 0 \\ \hline B & 1 \end{array}\right|. \tag{5}$$

("Rational" always means: lying in ρ.)

One has to prove that Q can be brought into the form

$$Q = BR_1 - R_2B \tag{6}$$

with a rational B. As a hypothesis to start with, we have the equation (3) or

$$\left|\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right| \cdot \left|\begin{array}{c|c} R_1 & 0 \\ \hline Q & R_2 \end{array}\right| = \left|\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right|$$

at our disposal. This contains in particular:

$$C_{22} R_2 = S_{22}. \tag{7}$$

S_{22} is non-singular as S is totally regular:

$$\det (S_{22}) \neq 0, \qquad \det (C_{22}) \neq 0.$$

Since C_{22} is skew-symmetric and S_{22} is symmetric, (7) yields by going over to the transposed matrices: $-R_2'C_{22} = S_{22}$,

$$R_2' = -S_{22} C_{22}^{-1} = -C_{22} R_2 C_{22}^{-1}. \tag{8}$$

Furthermore, we have

$$S_{12} = C_{12} R_2, \qquad S_{21} = C_{21} R_1 + C_{22} Q. \tag{9}$$

The symmetry of S and the skew symmetry of C imply

$$S_{12}' = S_{21} \qquad \text{and} \qquad C_{12}' = -C_{21}.$$

Hence, according to (9):

$$-R_2' C_{21} = C_{21} R_1 + C_{22} Q.$$

In replacing R_2' here by the expression (8) we get, after cancellation of the factor C_{22} in front:

$$R_2 C_{22}^{-1} C_{21} = C_{22}^{-1} C_{21} R_1 + Q.$$

Hence $B = -C_{22}^{-1} C_{21}$ satisfies the desired equation (6).

The proof has not even made use of the fact that S is totally regular in P but only in ρ; that is to say, the quadratic form S is supposed to be non-singular in any partial space spanned by vectors the components of which are numbers in ρ.

R is reducible if it can be rationally transformed into a reduced matrix like (5). $R \sim$ (equivalent) R' means that the matrix R' can be brought into coincidence with R through rational transformation. Following Poincaré's theorem, R may be decomposed into irreducible constituents $R_1, R_2, \cdots$, and it is allowed to assume that the equivalent ones among them are equal; in this manner R breaks up into "blocks" of equal irreducible parts. Some information about the rational commutators A of the splitting Riemann matrix R is provided by

SCHUR'S LEMMA: 1) *R_1 and R_2 being irreducible and inequivalent, zero is the only matrix A in ρ (of the right number of rows and columns) which satisfies the equation*

$$R_1 A = A R_2. \tag{10}$$

2) *Matrices A in ρ which commute with an irreducible R, are either zero or non-singular.*

While A. A. Albert has to use an analogue of Schur's lemma in his treatment, which he proves in a manner similar to Schur's, we are led directly to a particular case of Schur's lemma as it is known in the theory of representations where the single matrix R is replaced by a whole system, in particular a group of matrices.

The commutators A of R, lying in ρ, split up, in consequence of part 1) of Schur's lemma, into *blocks* corresponding to the blocks of equal irreducible parts of R. For an individual block, however, like

$$\left\| \begin{matrix} R_1 & & \\ & R_2 & \\ & & R_3 \end{matrix} \right\| \qquad (R_1 = R_2 = R_3 = \text{an irreducible } R^{\cdot}) \tag{11}$$

the commutators are of the shape

$$A = \left\| \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix} \right\| \tag{12}$$

where the A_{ik} are arbitrary commutators of the irreducible $R^{\cdot}$ (Scorza's theorem).

The fact that $S = CR$ is symmetric finds its expression in the equation

$$CR = R'C' = -R'C. \tag{13}$$

Corresponding to the decomposition of R into irreducible parts like (11), we write every matrix A in the form (12). Since S is totally regular, the parts S_{11}, S_{22}, S_{33} cannot be singular. According to $S = CR$ or $S_{11} = C_{11} R_1, \cdots$, the same holds for C_{11}, C_{22}, C_{33}. The equations

$$C_{11} R_1 = - R_1' C_{11}, \cdots$$

which are contained in (13) then show that $-R_i'$ is rationally equivalent to R_i. Hence, according to part 1) of Schur's lemma and to equation (13), the matrix C splits up into blocks in the manner described before for the commutators A. For an individual block C has the form

$$\left\| \begin{matrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{matrix} \right\| .$$

C stays skew-symmetric and non-singular, and the corresponding $S = CR$ stays symmetric and totally regular if one cancels the lateral terms C_{ik} $(i \neq k)$, i.e. if one now chooses C equal to

$$\left\| \begin{matrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{33} \end{matrix} \right\| .$$

Following these results we may restrict ourselves for further investigation to an *irreducible* R. The rational commutators A of an irreducible R form according to part 2) of Schur's lemma, a *division algebra* $\mathfrak{A}$.

§3. The involution of the commutator algebra $\mathfrak{A}$.

Any rational commutator A of the irreducible R gives rise by dint of the equation $CA = B$, to a matrix B satisfying the equation

$$BR = -R'B. \tag{14}$$

Conversely, any rational B satisfying this equation is of the form CA, A being a commutator. Transposition of (14) yields this same relation for B'. Hence B' must equal CA^* where the second factor A^* again is a commutator:

$$CA^* = (CA)' = A'C' = -A'C.$$

Thus, after canceling the minus sign, we find that every commutator A is associated with a "dual" one

$$A^* = C^{-1}A'C. \tag{15}$$

The transition $A \to A^*$ is first an *anti-automorphism*:

$$(A_1 A_2)^* = A_2^* A_1^*,$$

second, an *involution*. For the equation $CA^* = A'C$ changes by transposition into

$$-A^{*\prime}C = -CA \qquad \text{or} \qquad A^{**} = A.$$

We have thus proved the theorem of *Rosati*:[4]

The division algebra $\mathfrak{A}$ permits an anti-automorphic involution $A \to A^*$.

It may be observed that all previous results remain valid if C is a *symmetric* rather than an anti-symmetric, non-singular matrix in ρ; in this case the number of dimensions n need not be even.

The matrix algebra $\mathfrak{A}$ can be considered as the representation of an abstract division algebra $\mathfrak{a}$ in ρ the fundamental operations of which are: addition and multiplication of quantities of $\mathfrak{a}$, multiplication of a quantity a of $\mathfrak{a}$ by a number of ρ, and involution $a \to a^*$. The latter has the properties

$$(a + b)^* = a^* + b^*, \qquad (\lambda a)^* = \lambda a^*, \qquad (ab)^* = b^*a^* \quad (\lambda = \text{number in } \rho).$$

We call this an *involutorial division algebra.* $\mathfrak{a}$ has only one irreducible representation $\mathfrak{A}$: $a \to A = A(a)$ in ρ. The most general representation of $\mathfrak{a}$, and hence in particular the matrix algebra with which we dealt and which may be denoted by $\underline{\mathfrak{A}}$ for the rest of this section, is a multiple of $\mathfrak{A}$; the matrix $\underline{\mathfrak{A}}$ associated with the quantity a in $\underline{\mathfrak{A}}$ splits into t times the matrix A.

[4] Rend. Circ. Mat. di Palermo **53** (1929), p. 79–134.

Hence we proceed as follows: we start with an involutorial division algebra $\mathfrak{a}$ in ρ and its irreducible representation $\mathfrak{A}$. In $\mathfrak{A}$ the dual element a^* may be associated with the matrix A^*. $a^* \to A'$ as well as $\mathfrak{A}$: $a^* \to A^*$ defines a representation of $\mathfrak{a}$; so the former must be equivalent to $\mathfrak{A}$, i.e. there exists a non-singular matrix C_0 in ρ satisfying the equation

$$A(a^*) = C_0^{-1} A'(a)C_0$$

identically with respect to a. The most general matrix C which fulfills the same equation

$$CA^* = A'C \tag{16}$$

identically in a is of the form $C = C_0L$ where L commutes with all matrices A of $\mathfrak{A}$. The algebra of these L may be denoted by $\mathfrak{L}$, more exactly by $\mathfrak{L}_\rho$ or $\mathfrak{L}_{\mathrm{P}}$ according to whether we suppose that L lies in ρ or P. $\mathfrak{L}_\rho$ is a division algebra on account of the irreducibility of $\mathfrak{A}$, and consequently any C satisfying the equation (16) is either 0 or non-singular.

If one changes the equation (16) for $C = C_0$ to the transposed one, and exchanges A with A^*—which one is justified in doing because of the involutorial character of the mapping $a \to a^*$—one sees that C_0' satisfies the same equation. For this reason one can choose C_0 *either symmetric or skew-symmetric.* For if C_0 is not symmetric, one forms $C_0 - C_0' = C^0$; this is a solution of (16), $\neq 0$ and hence, according to our above remark, non-singular, so we can use C^0 instead of C_0.

Let C_0 be symmetric or skew-symmetric from now on. The fact that the equation (16) is satisfied by C' if by C, shows that the matrix $L^* = C_0^{-1} L' C_0$ always lies in $\mathfrak{L}$ if L does. L may be called *even* or *odd* according as $L^* = \pm L$. Any L is the sum of an even and an odd L. If we start with the irreducible matrix algebra $\mathfrak{A}$, the corresponding matrices C and S have to be of the form

$$C = C_0L, \qquad S = C_0M,$$

where L and M are even or odd matrices in $\mathfrak{L}_\rho$ and $\mathfrak{L}_{\mathrm{P}}$ respectively,—even or odd according to whether or not C and S are to be of the same parity as C_0.

$$R = C^{-1} S = L^{-1} M$$

is a matrix in $\mathfrak{L}_{\mathrm{P}}$ which can be obtained in this manner from an even or odd matrix L in $\mathfrak{L}_\rho$ and an even or odd matrix M in $\mathfrak{L}_{\mathrm{P}}$.

If $\underline{\mathfrak{A}}$ equals t times the irreducible representation $\mathfrak{A}$, we choose as our $\mathbf{C}_0$ the matrix that decomposes into t times C_0. The most general matrix $\mathbf{L}$ of $\underline{\mathfrak{L}}$ is of shape

$$\left\| \begin{matrix} L_{11} & \cdots & L_{1t} \\ \cdots & \cdots & \cdots \\ L_{t1} & \cdots & L_{tt} \end{matrix} \right\|$$

where the L_{ik} are arbitrary matrices in $\mathfrak{L}$. $\mathbf{L}^* = \mathbf{C}_0^{-1}\,\mathbf{L}'\,\mathbf{C}_0$. $\mathbf{L}'$ will be even: $\mathbf{L} = \mathbf{L}^*$, if the L_{ii} along the main diagonal are even, and if equations like $L_{21} = L_{12}^*$ hold for the lateral terms. $\mathbf{R} = \mathbf{L}^{-1}\,\mathbf{M}$; $\mathbf{L}$ in $\mathfrak{L}_\rho$, $\mathbf{M}$ in $\mathfrak{L}_P$, $\mathbf{L}$ and $\mathbf{M}$ even or odd.

The commutator algebra of the $\mathbf{R}$, to which we are led in this way, comprises the given $\underline{\mathfrak{A}}$ without being necessarily identical with it. The general solution $\mathbf{R}$, however, will depend on certain parameters, and one might expect that the commutator algebra will not embrace more than $\underline{\mathfrak{A}}$ if these parameters avoid certain special conditions.

The question as to whether the rational non-singular C_0, which by means of (15) effects the transition $A \to A^*$ is to be chosen symmetrically or skew-symmetrically, can be decided by more refined tools only,—at least in the case of even dimensionality (compare section 5). For odd dimensionality, of course, only the case of a symmetric C_0 can occur. For a two-term *reducible* representation $\underline{\mathfrak{A}}$ ($t = 2$) both possibilities may be arrived at: for one may put

$$\mathbf{C}_0 = \begin{Vmatrix} 0 & C_{12} \\ C_{21} & 0 \end{Vmatrix}$$

and take as C_{12} a rational non-singular solution of (16) and $C_{21} = \pm C_{12}'$ according as one wishes a symmetric or a skew-symmetric $\mathbf{C}_0$. The same remark holds in general for even t.

The result of these considerations is: that *our problem can be reduced essentially to the construction of all involutorial division algebras in ρ.*

For more detailed analysis one will have recourse to the *splitting fields* of the division algebra $\mathfrak{a}$ and the corresponding factor systems according to I. Schur and R. Brauer.[5] This method even before it yields the algebra $\mathfrak{a}$ and its representation $\mathfrak{A}$, leads to the algebra $\mathfrak{L}$ of the matrices commutable with $\mathfrak{A}$ in which the Riemann matrix R lies.

§4. Adjunction of the centrum

From now on we discard the use of the bold-face symbols: unless otherwise stated, $\mathfrak{A}$ denotes either the irreducible representation of $\mathfrak{a}$ or a multiple of it. Following Schur (loc. cit.) one may proceed as follows. Let A be a matrix of the rational commutator algebra $\mathfrak{A}$. The characteristic polynomial $|\lambda 1 - A|$ of A (1 = unit matrix, λ the variable) shall be decomposed into its irreducible factors in ρ: $\prod \varphi(\lambda)$. For an individual factor $\varphi(\lambda)$ of degree h, one has $|\varphi(A)| = 0$ and hence $\varphi(A) = 0$. Let us start in the vector space of the transformations A with a vector $e \neq 0$ and then form the series $e, Ae, A^2e, \cdots$. They span an h-dimensional partial space invariant with respect to A, in which the transformation A has the characteristic polynomial $\varphi(\lambda)$ with its roots

[5] Schur, Trans. Amer. Math. Soc. (2) **15** (1909), p. 159. Brauer, Math. Zschr. **28** (1928), p. 67.

$\alpha_1, \cdots, \alpha_h$ all different from each other. By repeating the construction a second time for a vector e' not contained in the partial space thus found, a third time, $\cdots$, finally a ν^{th} time, one breaks up the whole vector space in a number of partial spaces of the described kind. Hence the characteristic polynomial in the total vector space equals $\{\varphi(\lambda)\}^{\nu}$, and in an appropriate co-ordinate system A becomes a diagonal matrix along the main diagonal of which appears t times the number α_1, then t times the number α_2, and so on. As one readily sees from the equation $RA = AR$, the matrix R splits up in this (non-rational!) co-ordinate system into h matrices of order ν:

$$R = \begin{Vmatrix} R_1 & & \\ & \ddots & \\ & & R_h \end{Vmatrix}. \tag{17}$$

Let P be again, and from now on, the field of all real numbers. By performing the transformation into the co-ordinate system just introduced, possibly not real, the matrices C and S shall be treated as the coefficient matrices of bilinear forms of the variables $\bar{x}_\alpha$ and y_α:

$$\sum\nolimits_{\alpha\beta} c_{\alpha\beta}\, \bar{x}_\alpha\, y_\beta, \qquad \sum\nolimits_{\alpha\beta} s_{\alpha\beta}\, \bar{x}_\alpha\, y_\beta,$$
$$C \to \bar{U}'CU, \qquad S \to \bar{U}'SU.$$

$\sum s_{\alpha\beta}\, \bar{x}_\alpha\, x_\beta$ then remains a definite Hermitian form; the conditions of symmetry read $\bar{C}' = -C$, $\bar{S}' = S$. In agreement with (17) we write, with reference to the new co-ordinate system,

$$C = \begin{Vmatrix} C_{11} & \cdots & C_{1h} \\ \cdots & \cdots & \cdots \\ C_{h1} & \cdots & C_{hh} \end{Vmatrix}, \qquad S = \begin{Vmatrix} S_{11} & \cdots & S_{1h} \\ \cdots & \cdots & \cdots \\ S_{h1} & \cdots & S_{hh} \end{Vmatrix}. \tag{18}$$

We have $C_{ii} R_i = S_{ii}$. Here S_{ii} is non-singular and hence C_{ii} has to be non-singular too. The relation (15) defining the anti-automorphic involution now reads as follows:

$$A^* = C^{-1} \bar{A}' C, \qquad CA^* = \bar{A}'C,$$

since in this shape it is invariant with respect to arbitrary, even complex co-ordinate transformations.

We want to prove (*Rosati's theorem*):

If A is even or odd, the matrix C splits up like R corresponding to the numerically different roots α_i of A: $C_{ik} = 0$ for $i \neq k$ in (18). *The roots α are real for an even A; they are pure imaginary for an odd A.*

Indeed by using the even A in its diagonal normal form, the equation

$$A = C^{-1} \bar{A}'C \qquad \text{or} \qquad CA = \bar{A}'C$$

takes on the form

$$C_{ik}(\bar{\alpha}_i - \alpha_k) = 0.$$

Putting $i = k$ yields $\bar{\alpha}_i = \alpha_i$ as C_{ii} is non-singular and hence $\neq 0$. For $i \neq k$ we then get $C_{ik} = 0$ on account of $\alpha_i = \bar{\alpha}_i \neq \alpha_k$. The proof runs along similar lines for an odd A.

The centrum $\mathfrak{z}$ of the algebra $\mathfrak{a}$ is isomorphic to a number field k over ρ of degree h. We are going to replace ρ by k and hence to consider $\mathfrak{a}$ as an algebra over k (the fundamental operations in $\mathfrak{a}$ are then: addition and multiplication of the quantities in $\mathfrak{a}$, multiplication of a quantity in $\mathfrak{a}$ by a number λ in k). Let us apply Rosati's theorem to the matrices A of the centrum only. It proves to be natural, following Lefschetz and Albert, to distinguish two cases.

1) All quantites a of $\mathfrak{z}$ are even: $a^* = a$. By using the determining quantity a_0 of the field $\mathfrak{z}$ and its corresponding matrix A_0, one realizes that k is a totally real field; that is to say, k and all its conjugate fields with respect to ρ are real. In the co-ordinate system in which A_0 is a diagonal matrix, not only R, C, and S, but all matrices A of the algebra $\mathfrak{A}$, break up into parts corresponding to the h numerically different roots α_i of A_0; for every A commutes with A_0. Incidentally, $\rho(\alpha_i) = k_i$ are the h conjugate fields of k. Our problem in ρ reduces to the analogous problem in the "central field" k, the dimensionality n being lowered to $\nu = n/h$.

2) The set $\mathfrak{z}_0$ of the even quantities of $\mathfrak{z}$ does not exhaust the whole $\mathfrak{z}$. Under such circumstances $\mathfrak{z}$ arises from the field $\mathfrak{z}_0$ by adjoining an odd quantity b_0, the square of which b_0^2 lies in $\mathfrak{z}_0$. As $\mathfrak{z}_0$ is isomorphic to a totally real number-field k_0, the centrum $\mathfrak{z}$ is isomorphic to a quadratic extension k of k_0 arising from k_0 by adjoining the square root of a totally negative number γ_0 in k_0. Let us first apply Rosati's theorem to a determining quantity a_0 of $\mathfrak{z}_0$ resulting in the decomposition effected by the transition from ρ to k_0 and then apply it to b_0 for the transition from k_0 to k. The result is the same as in case 1), with the difference, however, that the involution $a \rightarrow a^*$ is not reflected as the identity $\lambda \rightarrow \lambda$ in the central field k, but as the change $\lambda \rightarrow \bar{\lambda}$ to the conjugate complex. The field K into which P extends, by adjoining the numbers of k, coincides with the field of all real numbers in case 1), and with the field of all complex numbers in case 2). The partial matrices R_i of R, (17), lying in K are irreducible in the conjugate fields k_i, for any reduction of them would result in a corresponding reduction of R in ρ. Our problem now has been reduced to the following:

1) Let k be a totally-real number-field, or a number field originating from such a field by adjoining the square root of a totally negative number γ_0 in it. Construct the most general involutorial division algebra over k, with k as its centrum; the involutorial correspondence is supposed to have the properties

$$(a + b)^* = a^* + b^*, \qquad (ab)^* = b^*a^*, \qquad (\lambda a)^* = \bar{\lambda}a^*,$$

where λ is any number of the central field k.

2) Let $\mathfrak{A}$: $a \rightarrow A$ be one of the representations of $\mathfrak{a}$ in k (the irreducible one, or one of its multiples), and let C_0 be a symmetric or skew-symmetric non-

singular matrix in k, by means of which the given involution $A \to A^*$ is expressed by the relation

$$A^* = C_0^{-1} \bar{A}' C_0 .$$

Construct first the algebra $\mathfrak{L}$ ($\mathfrak{L}_k$ or $\mathfrak{L}_K$) of all matrices in k or K commutable with $\mathfrak{A}$ and then in the most general way a Riemann matrix R (lying in $\mathfrak{L}_K$ and) possessing $\mathfrak{A}$ as its commutator algebra.

§5. Splitting field

The order of the division algebra $\mathfrak{a}$ is a square m^2. With respect to an appropriate splitting field $k(\vartheta)$ over k of degree m, and a corresponding co-ordinate system, the general matrix A of the irreducible representation $a \to A$ of $\mathfrak{a}$ splits up into conjugate m-rowed matrices A_α lying in the m conjugate fields $k(\vartheta_\alpha)$. The individual set $\mathfrak{A}_\alpha = \{A_\alpha\}$ is absolutely irreducible and not only irreducible in $k(\vartheta_\alpha)$. The different A_α are equivalent to each other, since k is the centrum, and accordingly there exist definite non-singular "conjugate" matrices $P_{\alpha\beta}$ in the fields $k(\vartheta_\alpha, \vartheta_\beta)$ satisfying the relations

$$P_{\alpha\beta} A_\beta = A_\alpha P_{\alpha\beta} \qquad (P_{\alpha\alpha} = \text{unit matrix}) \tag{19}$$

for every a. The $P_{\alpha\beta}$ in their turn satisfy equations of the form

$$P_{\alpha\beta} P_{\beta\gamma} = c_{\alpha\beta\gamma} \cdot P_{\alpha\gamma} . \tag{20}$$

The numbers $c_{\alpha\beta\gamma}$ form the *factor set*. With respect to the same co-ordinate system (which is irrational in k), the most general matrix L commutable with all A is of the form

$$\| z_{\alpha\beta} P_{\alpha\beta} \| \qquad (z_{\alpha\beta} \text{ conjugate numbers}).$$

In case 1) $a^* \to A'_\alpha(a)$ as well as $a^* \to A_\alpha(a^*) = A^*_\alpha$, is a representation of $\mathfrak{a}$ in $k(\vartheta_\alpha)$. Hence an equation like

$$A^*_\alpha = C_\alpha^{-1} A'_\alpha C_\alpha \qquad \text{or} \qquad C_\alpha A^*_\alpha = A'_\alpha C_\alpha \tag{21}$$

necessarily holds with fixed non-singular conjugate matrices C_α in $k(\vartheta_\alpha)$. C_α is uniquely determined by this equation but for a numerical factor, as the set of the A_α is absolutely irreducible. Consequently

$$C'_\alpha = \mu_\alpha C_\alpha .$$

This equation leads at once to the condition $\mu_\alpha^2 = 1$ and hence the numerical factor μ_α equals $+1$ or -1. Thus the distinction, C_α symmetric or skew-symmetric, is urged upon us. The matrix C splitting up into the C_α is rational in the original co-ordinate system, and it brings about the transition $A \to A^*$ by means of (15).

The matrices $\check{P}_{\alpha\beta} = P'^{-1}_{\alpha\beta}$, contragredient to the $P_{\alpha\beta}$, satisfy the same relations (19) for A'_α which the $P_{\alpha\beta}$ satisfy for A_α. Hence we must have $\check{P}_{\alpha\beta} = y_{\alpha\beta} P_{\alpha\beta}$

on account of (21). Since the factor set of the $\check{P}_{\alpha\beta}$ equals $1/c_{\alpha\beta\gamma}$ by (20), there follows

$$c^2_{\alpha\beta\gamma} = \frac{y_{\alpha\gamma}}{y_{\alpha\beta}\, y_{\beta\gamma}} \qquad \text{or} \qquad c^2_{\alpha\beta\gamma} \sim 1:$$

the "exponent" of the factor set, and consequently—due to a rather profound proposition concerning division algebras over an algebraic number-field,[6] — the Schur index m must equal 1 or 2.

$m = 1$ is the trivial case where the division algebra $\mathfrak{a}$ over k coincides with k, and where the t-dimensional reducible representation $\mathfrak{A}$ consists of the multiples of the unit matrix lying in k.

In the case $m = 2$, $\mathfrak{a}$ is a quaternion algebra over k; its quantities have the form

$$a = c_0 + c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3; \qquad (c_i \text{ numbers in } k)$$

$\epsilon_1^2 = e$ and $\epsilon_2^2 = b$ lie in k,[7] $\epsilon_3 = \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1$. Let us write the quantities a of $\mathfrak{a}$ as

$$a = \rho + \bar{\sigma}\epsilon_2 = (\rho, \sigma)$$

where ρ and σ are numbers in $k(\sqrt{e})$.

$$\rho = c_0 + c_1\epsilon_1 \sim \rho = c_0 + c_1\sqrt{e}, \qquad \bar{\rho} = c_0 - c_1\sqrt{e}; \qquad \sigma = c_2 + c_3\sqrt{e}.$$

The multiplication $(\xi', \eta') = (\rho, \sigma)(\xi, \eta)$ is then expressed as the linear substitution

$$\xi' = \rho\xi + b\bar{\sigma}\eta,$$
$$\eta' = \sigma\xi + \bar{\rho}\eta.$$

Thus the quantities of k can be looked upon as the matrices of the form

$$A_1 = \begin{Vmatrix} \rho, & b\bar{\sigma} \\ \sigma, & \bar{\rho} \end{Vmatrix} \qquad \text{in} \qquad k(\sqrt{e}).$$

The irreducible representation of $\mathfrak{a}$ in k if properly normalized, splits up by the substitution

$$\text{(22)} \qquad \begin{Vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{e} & 0 & -\sqrt{e} & 0 \\ 0 & \sqrt{e} & 0 & -\sqrt{e} \end{Vmatrix}$$

[6] R. Brauer, H. Hasse, E. Noether, Journ. f. reine u. angew. Mathem. **167**, 1931, p. 401. Cf. A. A. Albert, Annals of Math. **33**, 1932, pp. 311–318.

[7] e and b are such that the equation $ex^2 + by^2 = 1$ cannot be solved by numbers x, y in k.

into the two conjugate parts A_α ($\alpha = 1, 2$) in the splitting field $k(\sqrt{e})$:

$$A = \left|\begin{array}{cc|cc} \rho, & b\bar{\sigma} & & \\ \sigma, & \bar{\rho} & & \\ \hline & & \bar{\rho}, & b\sigma \\ & & \bar{\sigma}, & \rho \end{array}\right|.$$

From

$$P_{12} = \left\| \begin{matrix} 0 & b \\ 1 & 0 \end{matrix} \right\|$$

one finds as the most general matrix commuting with all A:

$$L = \left|\begin{array}{cc|cc} \alpha & 0 & 0 & b\beta \\ 0 & \alpha & \beta & 0 \\ \hline 0 & b\bar{\beta} & \bar{\alpha} & 0 \\ \bar{\beta} & 0 & 0 & \bar{\alpha} \end{array}\right|$$

α, $\bar{\alpha}$; β, $\bar{\beta}$ must be two pairs of conjugate numbers in $k(\sqrt{e})$ (or conjugate complex numbers), if this matrix is to lie in k (or the field K of real numbers respectively) after undoing the co-ordinate transformation (22).

Two typical anti-automorphic involutions $a \to a^*$ are the following ones:

$$\epsilon_1 \to -\epsilon_1, \quad \epsilon_2 \to \epsilon_2, \quad \epsilon_3 \to \epsilon_3 \quad \text{or} \quad (\rho, \sigma) \to (\bar{\rho}, \sigma);$$

$$\epsilon_1 \to -\epsilon_1, \quad \epsilon_2 \to -\epsilon_2, \quad \epsilon_3 \to -\epsilon_3 \quad \text{or} \quad (\rho, \sigma) \to (\bar{\rho}, -\sigma).$$

We are going to treat both involutions simultaneously, the upper sign always referring to the first, the lower to the second. A C_α satisfying the equation (21) is given by

$$C_1 = C_2 = \left\| \begin{matrix} 0 & 1 \\ \pm 1 & 0 \end{matrix} \right\| :$$

$$\left\| \begin{matrix} 0, & 1 \\ \pm 1, & 0 \end{matrix} \right\| \cdot \left\| \begin{matrix} \bar{\rho}, & \pm b\bar{\sigma} \\ \pm\sigma, & \rho \end{matrix} \right\| = \left\| \begin{matrix} \rho, & \sigma \\ b\bar{\sigma}, & \bar{\rho} \end{matrix} \right\| \cdot \left\| \begin{matrix} 0, & 1 \\ \pm 1, & 0 \end{matrix} \right\|.$$

C_1 is symmetric for the first, and skew-symmetric for the second involution! The matrix C^0 splitting up into C_1 and $C_2 = C_1$ fulfills the equation (15). The most general such matrix has the form C^0L, that is

$$C = \left\| \begin{array}{cc|cc} 0 & \alpha & \beta & 0 \\ \pm\alpha & 0 & 0 & \pm b\beta \\ \hline \bar{\beta} & 0 & 0 & \bar{\alpha} \\ 0 & \pm b\bar{\beta} & \pm\bar{\alpha} & 0 \end{array} \right\| .$$

C is symmetric for the first involution if α is arbitrary, and $\bar{\beta} = \beta$ (three free parameters in k or K respectively!); it is skew-symmetric if $\alpha = 0, \bar{\beta} = -\beta$ (one parameter). As to the second involution, C is skew-symmetric if α is arbitrary and $\bar{\beta} = -\beta$; it is symmetric if $\alpha = 0, \bar{\beta} = \beta$. This important example shows how the possibilities provided by the general theory may occur together.

The situation is much more complicated in the second case when k is obtained by quadratic imaginary extension of a totally real field. The method of factor sets, however, seems to be appropriate here also for studying the conditions prevailing in a given splitting field of minimum degree m (which need not be either cyclic or even Galois). We note only that the distinction between a symmetric or skew-symmetric C generating the involution $A \rightarrow A^* = C^{-1}\bar{A}'C$ now becomes irrelevant. For any symmetric C: $\bar{C}' = C$ gives rise to a skew-symmetric one through multiplication by the purely imaginary number $\sqrt{\gamma_0}$ which extended k_0 to yield k, and vice versa. For all further developments we refer the reader to Albert's paper in the Annals, 1931.

§6. Appendix: Relationship of the new formulation to the usual one

Again we consider the basis α of closed curves on the Riemann surface, and its characteristic form C with coefficients $c_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \cdots, n$). The matrix C is rational, non-singular and skew-symmetric. Let us choose this time the real basis $dw_\alpha = du_\alpha + idv_\alpha$ of the differentials of the first kind in an arbitrary manner independent of the basis α. The matrices $\| u_{\alpha\beta} \|$, $\| v_{\alpha\beta} \|$ of the real and imaginary parts of the periods

$$u_{\alpha\beta} = \int_\alpha du_\beta, \qquad v_{\alpha\beta} = \int_\alpha dv_\beta$$

may be designated by F and G. From the Dirichlet integral we readily obtain the fact that the bilinear form with the coefficient matrix

$$F'C^{-1}G = S \tag{23}$$

is symmetric and the corresponding quadratic form positive definite. Under the influence of an arbitrary *rational* transformation U of the basis of curves, and an arbitrary *real* transformation V of the basis dw_α of the differentials of the first kind,

$$C;\, F,\, G \quad \text{change into} \quad U'CU;\, U'FV,\, U'GV, \tag{24}$$

and thus the matrix defined by (23)

$$S \quad \text{into} \quad V'SV.$$

The real matrix $R = \| r_{\alpha\beta} \|$ effecting the transition from dw_α to $-idw_\alpha$:

$$(25) \qquad -idw_\alpha = \sum_\beta r_{\beta\alpha}\, dw_\beta$$

is obtained from the equation

$$(26) \qquad G = FR,$$

and the law by which R changes under the influence of the transformation (24) is given by

$$(27) \qquad R \to V^{-1}RV.$$

We are therefore led to consider the following general situation: C a non-singular skew-symmetric rational matrix, F and G real matrices such that the matrix S defined by (23) is symmetric and "definite." The point is to investigate relations that are invariant with respect to the transformations (24) where U is rational, V real, and both non-singular. In particular, we are concerned with the question of "complex multiplication": how do pairs of non-singular matrices A and B look, A being rational, B real, such that the two equations

$$A'F = FB, \qquad A'G = GB$$

obtain simultaneously? After introducing R by (26), the second equation yields, with respect to the first,

$$(28) \qquad BR = RB.$$

Our way of treatment amounts to the following: by an appropriate transformation V [formula (24), $U = 1$] one takes care that

$$(29) \qquad F \quad \text{becomes} \quad = C' = -C.$$

This equation is preserved under the influence of transformations (24) only if $V = U$. Now S becomes equal to G, and in the problem of complex multiplication (28), $B = C^{-1}A'C$ as well as A is a rational matrix.

The *usual* treatment introduces the assumption $R^2 = -1$. Such an R may be brought into the form

$$(30) \qquad R = \left\| \begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array} \right\|$$

by an appropriate transformation (27) (the partial squares are p-rowed, $n = 2p$). The substitution (25) is of this form, if the real basis dw_α arises from a complex basis $dw_1, \cdots, dw_p$ in the following manner:

$$dw_1, \cdots, dw_p; \qquad -idw_1, \cdots, -idw_p.$$

The normalization (30) bound to the hypothesis $R^2 = -1$ thus leads to the usual and the normalization (29) requiring no restriction leads to *our* formulation. The greater freedom afforded by the latter for the choice of R should facilitate considerably the existence proofs,—in particular the proof of the proposition that the problem does not impose any more restrictions on the structure of the commutator algebra than its involutorial character.

The only new idea in this paper is the elimination of the assumption $R^2 = -1$, but I could not avoid retelling the whole story in order to show that this hypothesis is superfluous.

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