16.5 重积分的应用

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课本例题

例 1 求圆锥 $z=2\sqrt{x^2+y^2}$ 在圆柱体 $x^2+y^2\le y$ 内那部分曲面的面积.

解: 根据上面的面积公式, 有

$$\Delta S=\iint\limits_{D}\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y,\qquad D=\{(x,y)\mid x^2+y^2\le y\}.$$

由于 $z=2\sqrt{x^2+y^2}$, 所以

$$z_x=\frac{2x}{\sqrt{x^2+y^2}},z_y=\frac{2y}{\sqrt{x^2+y^2}}.$$

于是

$$\Delta S=\iint\limits_{D}\sqrt{5}\mathrm{d}x\mathrm{d}y=\frac{\sqrt{5}}{4}\pi.$$

□

例 2 求球面上两条纬线 $\varphi=\varphi_1,\varphi=\varphi_2$ 和两条经线 $\psi=\psi_1,\psi=\psi_2$ 之间的曲面的面积 $(\varphi_1<\varphi_2,\psi_1<\psi_2)$.

解: 设球面方程为

$$\begin{aligned}x&=R\cos\psi\cos\varphi,\\ y&=R\cos\psi\sin\varphi,\\ z&=R\sin\psi,\end{aligned}$$

由于 $E=x_\psi^2+y_\psi^2+z_\psi^2=R^2$ 在 $f=0$ 在 $G=R^2\cos^2\psi$, 所以

$$\Delta S=\int_{\varphi_1}^{\varphi_2}d\varphi\int_{\psi_1}^{\psi_2}R^2\cos\psi d\psi=R^2(\varphi_2-\varphi_1)(\sin\psi_2-\sin\psi_1).$$

□

例 3 求密度均匀的右半椭球体的重心.

解: 设右半椭球体为

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1,\qquad y\ge 0$$

表示. 由对称性知 $\bar{x}=0$ 在 $\bar{z}=0$. 又因为 ρ 为常数, 所以

$$\bar{y}=\frac{\iiint\limits_{V}\rho y dV}{\iiint\limits_{V}\rho dV}=\frac{\iiint\limits_{V}y\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\frac{2}{3}\pi abc}=\frac{3b}{8}.$$

□

例 4 *设球体 V 具有均匀的密度 $\rho \equiv 1$, 求 V 对球外一点 A (质量为 1) 的引力 (引力系数为 k).*

解: 设球体为 $x^2+y^2+z^2 \le R^2$, 球外一点 A 的坐标为 $(0,0,a)(R<a)$. 显然, 在这种坐标设置下, $f_x = F_y = 0$, 由公式,

$$\begin{aligned} F_z &= k\int\int\int V\frac{(z-a)}{[x^2+y^2+(z-a)^2]^{3/2}}\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= k\rho\int_{-R}^{R}(z-a)\mathrm{d}z\iint\limits_{D}\frac{\mathrm{d}x\mathrm{d}y}{[x^2+y^2+(z-a)^2]^{3/2}}, \end{aligned}$$

其中 $D=\{(x,y)\mid x^2+y^2\le R^2-z^2\}$. 用柱坐标计算得

$$\begin{aligned} F_z &= k\rho\int_{-R}^{R}(z-a)\mathrm{d}z\int_0^{2\pi}d\theta\int_0^{\sqrt{R^2-z^2}}\frac{r}{[r^2+(z-a)^2]^{3/2}}dr \\ &= 2\pi k\rho\int_{-R}^{R}\left(-1-\frac{z-a}{\sqrt{R^2-2az+a^2}}\right)\mathrm{d}z \\ &= -\frac{4}{3a^2}\pi R^3\rho k. \end{aligned}$$

□

思考题

1. 如何计算曲线对其外一质点的引力.

习题

1. 求下列图形的面积:

(1) 曲面 $z=\sqrt{2xy}$ 被平面 $x+y=1, x=1$ 及 $y=1$ 所截部分.

(2) 锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $z^2=2x$ 所截部分.

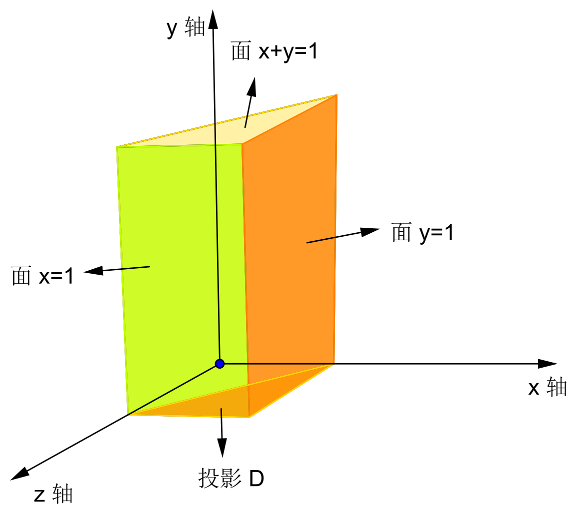
解: **(1)** 区域 D 的图像如下图所示

根据上面的面积公式, 有

$$\Delta S=\iint\limits_{D}\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y, \qquad D=\{(x,y)\mid 1-x\leqslant y\leqslant 1, 0\leqslant x\leqslant 1\}.$$

由于 $z=\sqrt{2xy}$, 所以

$$z_x=\frac{\sqrt{2}}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}, z_y=\frac{\sqrt{2}}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}.$$



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于是

$$
\begin{aligned}
\Delta S &= \iint\limits_D \sqrt{1+\frac{1}{2}x^{-1}y+\frac{1}{2}xy^{-1}}\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_D \frac{x+y}{\sqrt{2xy}}\mathrm{d}x\mathrm{d}y \\
&= \frac{\sqrt{2}}{2}\int_0^1 \mathrm{d}x \int_{1-x}^1 \left(x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2}\int_0^1 (2x^{\frac{1}{2}}y^{\frac{1}{2}}+\frac{2}{3}x^{-\frac{1}{2}}y^{\frac{3}{2}}\Big|_{1-x}^{1})\mathrm{d}x \\
&= \frac{\sqrt{2}}{2}\int_0^1 (2x^{\frac{1}{2}}+\frac{2}{3}x^{-\frac{1}{2}}-2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}-\frac{2}{3}x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}})\mathrm{d}x \\
&= \sqrt{2}\int_0^1 x^{\frac{1}{2}}\mathrm{d}x+\frac{\sqrt{2}}{3}\int_0^1 x^{-\frac{1}{2}}\mathrm{d}x-\sqrt{2}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\mathrm{d}x+\frac{\sqrt{2}}{3}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}\mathrm{d}x \qquad (1)
\end{aligned}
$$

因为

$$\begin{aligned}\int_0^1 x^{\frac{1}{2}}\mathrm{d}x &= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{2}{3}; \end{aligned} \tag{2}$$

$$\begin{aligned}\int_0^1 x^{-\frac{1}{2}}\mathrm{d}x &= 2x^{\frac{1}{2}}|_0^1 \\ &= 2; \end{aligned} \tag{3}$$

记 $I_1=\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\mathrm{d}x=\int_0^1\sqrt{1-(2x-1)^2}\mathrm{d}x$, 于是令 $2x-1=\cos t$, 则 $x=\frac{1}{2}(\cos t+1)$, 且 $\mathrm{d}x=-\frac{1}{2}\sin t\mathrm{d}t$, 则

$$\begin{aligned}I_1 &= \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\mathrm{d}x=\int_0^1\sqrt{1-(2x-1)^2}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\pi}^0\sqrt{1-\cos^2 t}(-\frac{1}{2}\sin t)\mathrm{d}t \\ &= \frac{1}{4}\int_{-\pi}^0|\sin t|(-\sin t)\mathrm{d}t \\ &= \frac{1}{4}\int_{-\pi}^0\sin^2 t\mathrm{d}t \\ &= \frac{1}{4}\int_0^{\pi}\sin^2 t\mathrm{d}t \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 t\mathrm{d}t \\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{8};\end{aligned}$$

记 $I_4=\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}\mathrm{d}x$, 则由 $I_3=\frac{\pi}{8}$, 可得

$$I_4=\int_0^1\sqrt{\frac{1-x}{x}}\mathrm{d}x-\int_0^1\sqrt{x-x^2}\mathrm{d}x=\int_0^1\sqrt{\frac{1-x}{x}}\mathrm{d}x-\frac{\pi}{8}$$

在上式中记 $\sqrt{\frac{1-x}{x}}=t$, 则 $x=\frac{1}{t^2+1}$, 于是有

$$\begin{aligned}I_4 &= \int_0^1\sqrt{\frac{1-x}{x}}\mathrm{d}x-\frac{\pi}{8} \\ &= \int_{+\infty}^0 t\mathrm{d}\frac{1}{t^2+1}-\frac{\pi}{8} \\ &= t\cdot\frac{1}{t^2+1}\Big|_{+\infty}^0+\int_0^{+\infty}\frac{1}{t^2+1}\mathrm{d}t-\frac{\pi}{8} \\ &= \frac{\pi}{2}-\frac{\pi}{8} \\ &= -\frac{3\pi}{8};\end{aligned}$$

把 (2)—— (4) 代入 (1)式可得

$$\Delta S=\sqrt{2}\cdot\frac{2}{3}+\frac{\sqrt{2}}{3}\cdot\sqrt{2}-\sqrt{2}\cdot\frac{\pi}{8}-\frac{\sqrt{2}}{3}\cdot\frac{3\pi}{8}=\frac{\sqrt{2}}{4}\pi.$$

(2) 联立方程组 $\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases}$ 消去 z, 可得

$$x^2+y^2=4x,$$

即

$$(x-2)^2+y^2=4,$$

从而得到曲面在 xy 平面上的投影为 $D=\{(x,y)|\ (x-2)^2+y^2\leqslant 4\}$, 由面积公式可得

$$\Delta S=\iint\limits_D\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y,$$

由于 $z=\sqrt{x^2+y^2}$, 可得

$$z_x=\frac{x}{\sqrt{x^2+y^2}},\quad z_y=\frac{y}{\sqrt{x^2+y^2}},$$

于是有

$$\begin{aligned}\Delta S &= \iint\limits_D\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\mathrm{d}x\mathrm{d}y \\ &= \sqrt{2}\iint\limits_D\mathrm{d}x\mathrm{d}y \\ &= \sqrt{2}\cdot 4\pi \\ &= 4\sqrt{2}.\end{aligned}$$

□

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2. 求下列均匀平面薄板的重心:
   (1) 半椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1, y\geq 0$;
   (2) 高为 h, 底分别为 A 和 b 的等腰梯形.

解: (1) 由题可设 $\rho=c$, 其中 c 为常数。设其重心坐标为 $(\bar{x},\bar{y})$, 由对称性可知 $\bar{y}=0$, 记 $D=\{(x,y)|\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\}$, 由公式可得

$$\begin{aligned}\bar{x} &= \frac{\iint\limits_D\rho x\mathrm{d}x\mathrm{d}y}{\iint\limits_D\rho\mathrm{d}x\mathrm{d}y} \\ &= \frac{\iint\limits_D x\mathrm{d}x\mathrm{d}y}{\frac{ab\pi}{2}} \\ &= \frac{2}{ab\pi}\iint\limits_D x\mathrm{d}x\mathrm{d}y,\end{aligned}$$

做极坐标变换 $\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$ 则 xy 的平面区域 D 与 $\triangle = \{(r,\theta)|\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, 0 \leqslant r \leqslant 1\}$ 一一对应，于是有

$$\begin{aligned} barx &= \frac{2}{ab\pi}\iint\limits_D x\mathrm{d}x\mathrm{d}y \\ &= \frac{2}{ab\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 ar\cos\theta \cdot abr\mathrm{d}r \\ &= \frac{2}{ab\pi}\cdot 1 \cdot \frac{a^2br^3}{3}\Big|_0^1 \\ &= \frac{4a}{3\pi}. \end{aligned}$$

所以重心坐标为 $(\frac{4a}{3\pi}, 0)$.

(2) 由题可设 $\rho = c$, 其中 c 为常数。联立方程组 $\begin{cases} y = 2x^2 \\ x + y = 1 \end{cases}$ 可解得

$$x = \frac{1}{2}, \text{或} x = -1,$$

从而得到曲面在 xy 平面上的区域为 $D = \{(x,y)|\ 1 - x \leqslant y \leqslant 2x^2, -1 \leqslant x\frac{1}{2}\}$, 区域 D 的如下图所示

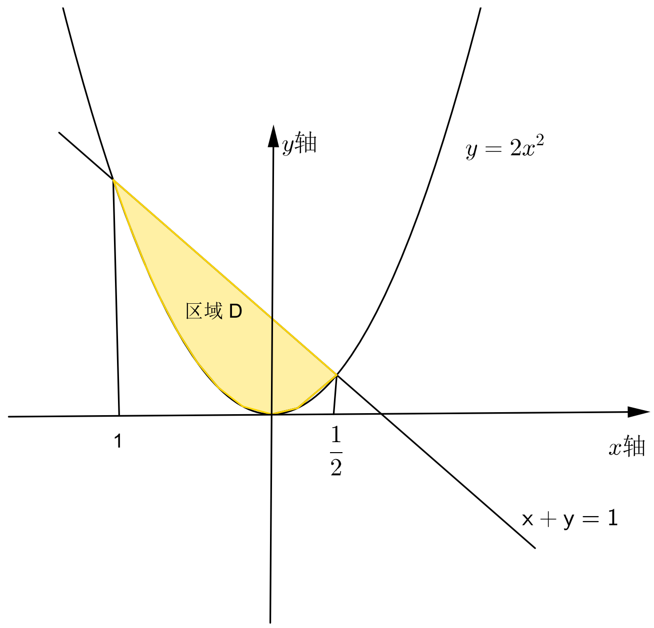
由面积公式可得 D 的面积为

$$\begin{aligned} \Delta S &= \iint\limits_D \mathrm{d}x\mathrm{d}y \\ &= \int_{-1}^{\frac{1}{2}}(1 - x - 2x^2)\mathrm{d}x \\ &= (x - \frac{x^2}{2} - \frac{2}{3}x^3)_{-1}^{\frac{1}{2}} \\ &= \frac{9}{8} \end{aligned}$$

设其重心坐标为 $(\bar{x}, \bar{y})$, 则

$$\begin{aligned} \bar{x} &= \frac{\iint\limits_D \rho x\mathrm{d}x\mathrm{d}y}{\iint\limits_D \rho\mathrm{d}x\mathrm{d}y} \\ &= \frac{\iint\limits_D x\mathrm{d}x\mathrm{d}y}{\frac{9}{8}} \\ &= \frac{8}{9}\int_{-1}^{\frac{1}{2}}\mathrm{d}x\int_{1-x}^{2x^2} x\mathrm{d}y \\ &= \frac{8}{9}\int_{-1}^{\frac{1}{2}}(x - x^2 - 2x^3)\mathrm{d}x \\ &= \frac{8}{9}(\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2})\Big|_{-1}^{\frac{1}{2}} \\ &= -\frac{1}{4}, \end{aligned}$$

所以重心坐标为 $(-\frac{1}{4}, \frac{4}{5})$.



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$$
\begin{aligned}
\bar{y} &= \frac{\iint\limits_{D} \rho y \mathrm{d}x\mathrm{d}y}{\iint\limits_{D} \rho \mathrm{d}x\mathrm{d}y} \\
&= \frac{\iint\limits_{D} y \mathrm{d}x\mathrm{d}y}{\frac{9}{8}} \\
&= \frac{8}{9}\int_{-1}^{\frac{1}{2}} \mathrm{d}x \int_{1-x}^{2x^2} x\mathrm{d}y \\
&= \frac{8}{9}\int_{-1}^{\frac{1}{2}} (\frac{(1-x)^2}{2} - 2x^4)\mathrm{d}x \\
&= \frac{8}{9}(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{6} - \frac{2x^5}{5})\Big|_{-1}^{\frac{1}{2}} \\
&= \frac{4}{5},
\end{aligned}
$$

所以重心坐标为 $(-\frac{1}{4}, \frac{4}{5})$.

□

3. 求下列均匀立体的重心:
(1) 由 $z = x^2 + y^2$, 平面 $x + y = 1$ 及三个坐标面围成;
(2) 由坐标面及平面 $x + 2y - z = 1$ 所围的四面体.

解: (1) 由题可设 $\rho = c$, 其中 c 为常数。由 $z = x^2 + y^2$, 平面 $x + y = 1$ 及三个坐标面围成的立体 V 在 xy 平面上的投影为 $D = \{(x, y)|\ 0 \leqslant y \leqslant 1 - x, 0 \leqslant x \leqslant 1\}$, 于是 V 的体积为

$$\begin{aligned} V &= \iiint\limits_V \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iint\limits_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \mathrm{d}x \int_{1-x}^0 (x^2 + y^2)\mathrm{d}y \\ &= \int_0^1 (x^2 - x^3 + \frac{(1-x)^3}{3})\mathrm{d}x \\ &= (\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12})\Big|_0^1 \\ &= \frac{1}{6}, \end{aligned}$$

设其重心坐标为 $(\bar{x}, \bar{y}, \bar{z})$, 于是有

$$\begin{aligned} \bar{x} &= \frac{\iiint\limits_V cx dV}{\iiint\limits_V c dV} \\ &= \frac{\iiint\limits_V x dV}{\iiint\limits_V dV} \\ &= 6 \cdot \iiint\limits_V x dV \\ &= 6 \cdot \iint\limits_D x(x^2 + y^2)\mathrm{d}x\mathrm{d}y \\ &= 6 \int_0^1 \mathrm{d}x \int_0^{1-x} (x^3 + xy^2)\mathrm{d}y \\ &= 6 \int_0^1 x^3(1-x) + \frac{x}{3}(1-x)^3\mathrm{d}x \\ &= 6\left(\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^2}{6} - \frac{x^5}{15} + \frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{2}{5}, \end{aligned}$$

$$\begin{aligned}
\bar{y} &= \frac{\iiint\limits_V cydV}{\iiint\limits_V cdV} \\
&= \frac{\iiint\limits_V ydV}{\iiint\limits_V dV} \\
&= 6\cdot\iiint\limits_V ydV \\
&= 6\cdot\iint\limits_D y(x^2+y^2)\mathrm{d}x\mathrm{d}y \\
&= 6\int_0^1 \mathrm{d}x\int_0^{1-x}(yx^2+y^3)\mathrm{d}y \\
&= 6\int_0^1 \frac{x^2(1-x)^2}{2}+\frac{1}{4}(1-x)^4\mathrm{d}x \\
&= 6\left(\frac{x^3}{6}-\frac{x^4}{4}+\frac{x^5}{10}-\frac{(1-x)^5}{20}\right)\Big|_0^1 \\
&= \frac{2}{5},
\end{aligned}$$

$$\begin{aligned}
\bar{z} &= \frac{\iiint\limits_V czdV}{\iiint\limits_V cdV} \\
&= \frac{\iiint\limits_V zdV}{\iiint\limits_V dV} \\
&= 6\cdot\iiint\limits_V zdV \\
&= 6\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{x^2+y^2} z\mathrm{d}z \\
&= 6\int_0^1 \mathrm{d}x\int_0^{1-x}\frac{(x^2+y^2)^2}{2}\mathrm{d}y \\
&= 6\int_0^1 \mathrm{d}x\int_0^{1-x}\frac{x^4+2x^2y^2+y^4}{2}\mathrm{d}y \\
&= 6\int_0^1 \frac{x^4-x^5}{2}+\frac{x^2(1-x)^3}{3}+\frac{(1-x)^5}{10}\mathrm{d}x \\
&= 6\left(-\frac{5}{6}\frac{x^6}{6}+\frac{3}{2}\frac{x^5}{5}-\frac{x^4}{4}-\frac{1}{3}\frac{x^3}{3}-\frac{1}{10}\frac{(1-x)^5}{6}\right)\Big|_0^1 \\
&= \frac{7}{30},
\end{aligned}$$

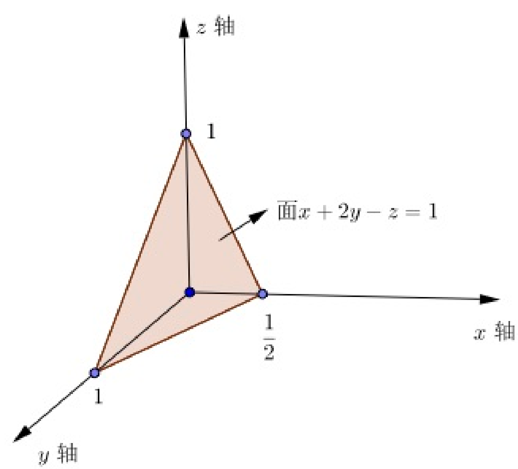
所以其重心坐标为 $(\frac{2}{5},\frac{2}{5},\frac{7}{30})$.

(2) 由题可设 $\rho=c$, 其中 c 为常数。由 $z=x^2+y^2$, 平面 $x+y=1$ 及三个坐标面围成的立体 $V=\{(x,y,z)|\ x+2y-1\leqslant z\leqslant 0, 0\leqslant y\leqslant\frac{1-x}{2}, 0\leqslant x\leqslant 1\}$, 于是 V 的体积为

解法一

$$
\begin{aligned}
V &= \iiint\limits_{V} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 \mathrm{d}y \int_{x+2y-1}^0 \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 x+2y-1\mathrm{d}y \\
&= \int_0^1 (\frac{(1-x)^2}{4})\mathrm{d}x \\
&= --\frac{(1-x)^3}{12} - \Big|_0^1 \\
&= \frac{1}{12},
\end{aligned}
$$

解法二平面 $x+2y-z=1$ 的如下图所示



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因为平面 $x+2y-z=1$ 与坐标平面 xy, yz, xz 的交点分别为 $(1,1,0), (0,\frac{1}{2},0), (1,0,0)$, 所围四面体的体积为

$$
\frac{1}{3}\cdot\frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1 = \frac{1}{12}.
$$

设其重心坐标为 $(\bar{x},\bar{y},\bar{z})$, 于是有

$$
\begin{aligned}
\bar{x} &= \frac{\iiint\limits_V cxdV}{\iiint\limits_V cdV} \\
&= \frac{\iiint\limits_V xdV}{\iiint\limits_V dV} \\
&= 12 \cdot \int_0^1 x\mathrm{d}x \int_{\frac{1-x}{2}}^0 \mathrm{d}y \int_{x+2y-1}^0 \mathrm{d}z \\
&= 12 \cdot \int_0^1 (x\frac{(1-x)^2}{4})\mathrm{d}x \\
&= 3\int_0^1 (x-2x^2+x^3)\mathrm{d}x \\
&= 3\left(\frac{x^2}{2}-\frac{2x^2}{3}+\frac{x^4}{4}\right)\Big|_0^1 \\
&= \frac{1}{4},
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= \frac{\iiint\limits_V cydV}{\iiint\limits_V cdV} \\
&= \frac{\iiint\limits_V ydV}{\iiint\limits_V dV} \\
&= 12 \cdot \int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 y\mathrm{d}y \int_{x+2y-1}^0 \mathrm{d}z \\
&= 12\int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 y(1-x-2y)\mathrm{d}y \\
&= 12\int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 y(1-x)-2y^2\mathrm{d}y \\
&= 12 \cdot \int_0^1 \left(\frac{(1-x)^3}{8}-\frac{1}{3}\frac{(1-x)^3}{4}\right)\mathrm{d}x \\
&= 3\int_0^1 \frac{(1-x)^3}{24})\mathrm{d}x \\
&= 6\left(-\frac{(1-x)^4}{24\cdot 4}\right)\Big|_0^1 \\
&= \frac{1}{8},
\end{aligned}
$$

$$\begin{aligned}
\bar{z} &= \frac{\iiint\limits_V czdV}{\iiint\limits_V cdV} \\
&= \frac{\iiint\limits_V zdV}{\iiint\limits_V dV} \\
&= 12 \cdot \iiint\limits_V zdV \\
&= 12\int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 \mathrm{d}y \int_{x+2y-1}^0 z\mathrm{d}z \\
&= 12\int_0^1 \mathrm{d}x \int_{\frac{1-x}{2}}^0 \left(\frac{x^2}{2}+2y^2+\frac{1}{2}+2xy-2y-2x\right)\mathrm{d}y \\
&= 12\int_0^1 \left(\frac{x^2(1-x)}{4}+\frac{2}{3}\frac{(1-x)^3}{8}+x\frac{(1-x)^2}{4}-\frac{(1-x)^2}{4}-2x\frac{1-x}{2}\right)\mathrm{d}x \\
&= 12\int_0^1 \left(\frac{x^2}{4}-\frac{x^3}{4}+x\frac{(1-x)^2}{4}-\frac{(1-x)^3}{6}\right)\mathrm{d}x \\
&= -\frac{1}{4},
\end{aligned}$$

所以其重心坐标为 $(\frac{1}{4},-\frac{1}{8},-\frac{1}{4})$.

□

4. 计算密度为 ρ 的均匀柱体 $x^2+y^2 \le a^2, 0 \le z \le h$ 对于点 $P(0,0,b)(b>h)$ 处的单位质量的引力.

解: 由对称性可知, $F_x = F_y = 0$, 由公式,

$$\begin{aligned}
F_z &= k\int\int\int V \frac{(z-b)}{[x^2+y^2+(z-a)^2]^{3/2}}\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= k\rho\int_0^h \mathrm{d}z \iint\limits_D \frac{(z-b)}{[x^2+y^2+(z-a)^2]^{3/2}}\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

其中 $D=\{(x,y) \mid x^2+y^2 \le a^2\}$. 做极坐标变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ 则 xy 的平面区域 D 与 $\triangle = \{(r,\theta)| \, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant a\}$ 一一对应，所以

$$\begin{aligned}
F_z &= k\rho\int_0^h (z-b)\mathrm{d}z \iint\limits_{\triangle} \frac{1}{[r^2+(z-b)^2]^{3/2}}\mathrm{d}\theta\mathrm{d}r \\
&= k\rho\int_0^{2\pi} \mathrm{d}\theta \int_0^a r\mathrm{d}r \int_0^h \frac{(z-b)}{[r^2+(z-b)^2]^{3/2}}\mathrm{d}z \\
&= k\rho\int_0^{2\pi} \mathrm{d}\theta \int_0^a r\mathrm{d}r \int_0^h \frac{1}{2}\frac{1}{[r^2+(z-b)^2]^{3/2}}\mathrm{d}(z-b)^2 \\
&= k\rho\int_0^{2\pi} \mathrm{d}\theta \int_0^a r\left((r^2+b^2)^{-\frac{1}{2}}-(r^2+(h-b)^2)^{-\frac{1}{2}}\right)\mathrm{d}r \\
&= k\rho\int_0^{2\pi} \mathrm{d}\theta \int_0^a \left((r^2+b^2)^{-\frac{1}{2}}\mathrm{d}r^2-\frac{1}{2}(r^2+(h-b)^2)^{-\frac{1}{2}}\mathrm{d}(r^2+(h-b)^2)\right) \\
&= 2\pi k\rho\left[h-\sqrt{a^2+(h-b)^2}+\sqrt{a^2+b^2}\right].
\end{aligned}$$

所以，密度为 ρ 的均匀柱体 $x^2+y^2\le a^2, 0\le z\le h$ 对于点 $P(0,0,b)(b>h)$ 处的单位质量的引力为 $F_x=F_y=0, F_z 2\pi k\rho\left[h-\sqrt{a^2+(h-b)^2}+\sqrt{a^2+b^2}\right]$. □

5. 求螺旋面

$$x=r\cos\varphi, y=r\sin\varphi, z=b\varphi, \quad 0\le r\le a, 0\le\varphi\le 2\pi$$

的面积.

解: 由于 $E=x_r^2+y_r^2+z_r^2=1, F=x_r x_\phi+y_r y_\phi+z_r z_\phi=0, G=x_\phi^2+y_\phi^2+z_\phi^2=r^2+h^2$ 所以曲面的面积为

$$\begin{aligned} S &= \int_0^a \mathrm{d}r\int_0^{2\pi}\sqrt{EG-F^2}\mathrm{d}\phi \\ &= \int_0^a \mathrm{d}r\int_0^{2\pi}\sqrt{r^2+h^2}\mathrm{d}\phi \\ &= \int_0^a 2\pi\sqrt{r^2+h^2}\mathrm{d}r \\ &= 2\pi\int_0^a\sqrt{r^2+h^2}\mathrm{d}r. \end{aligned}$$

由公式

$$\int\sqrt{x^2+a^2}\mathrm{d}x=\frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln\frac{|x+\sqrt{x^2+a^2}|}{h}\right)+C,$$

可得

$$S=2\pi\cdot\frac{1}{2}\left(a\sqrt{a^2+h^2}+h^2\ln\frac{a+\sqrt{a^2+h^2}}{h}\right)=\pi\left(a\sqrt{a^2+h^2}+h^2\ln\frac{a+\sqrt{a^2+h^2}}{h}\right).$$

□