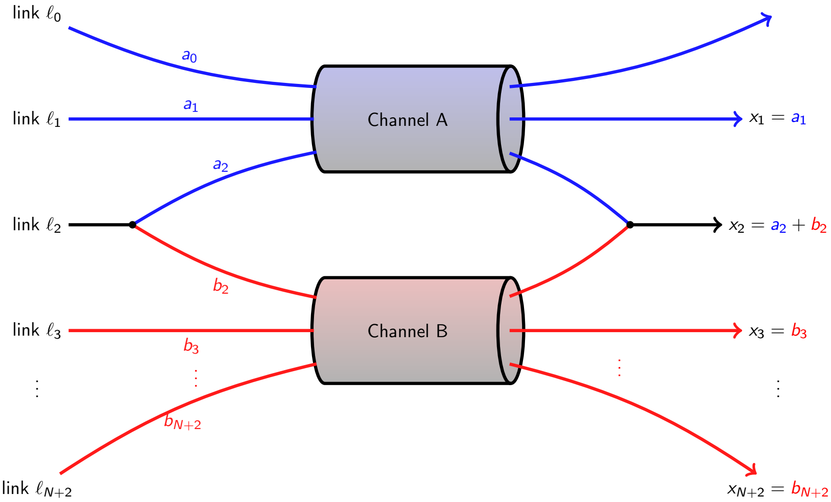
15.094J: Robust Modeling, Optimization, Computation

Lecture 14; Pareto Efficiency in Robust Optimization

Based on Iancu and Trichakis, Pareto Efficiency in Robust Optimization, 2012

Outline

A Communication Example



A Communication Example, continued

- Links $\ell_1, \ldots, \ell_{N+2}$ are for emergency purposes, whereas link ℓ_0 is used for general purposes.
- x_i transmission rate of the emergency link ℓ_i, $i = 1, 2, \ldots, N+2$. We have

$$x_1 = a_1, \quad x_2 = a_2 + b_2, \quad \text{and} \quad x_i = b_i, \quad i = 3, 4, \ldots, N+2.$$

- f_i fraction of emergency transmission routed via link ℓ_i, $i = 1, 2, \ldots, N+2$. Net emergency transmission rate

$$f'x = \sum_{i=1}^{N+2} f_i x_i.$$

- Uncertainty set

$$\mathcal{U} = \left\{ f \in \mathbb{R}_+^{N+2} : e'f = 1 \right\}.$$

- Select rates x, a and b (in case of an emergency) so as to maximize the net emergency transmission rate.

A Communication Example, continued

- RO:

$$\begin{aligned} \text{maximize} \quad & \min_{f \in U} f'x \\ \text{subject to} \quad & x_1 = a_1 \\ & x_2 = a_2 + b_2 \\ & x_i = b_i, \quad i = 3, \ldots, N+2 \\ & a_0 + a_1 + a_2 = 1 \\ & b_2 + b_3 + \ldots + b_{N+2} = 1 \\ & a,\ b \geq 0, \end{aligned}$$

- X feasible set. $Z_{Rob} = 1/N$ and the optimal set is

$$X^{RO} = \left\{ (x, a, b) \in X :\ x \geq \frac{1}{N} e \right\}.$$

- $a^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)', \quad b^* = \left(0, \frac{1}{10}, \ldots, \frac{1}{10}\right)', \quad x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{10}, \ldots, \frac{1}{10}\right)'$
- For all $f \in \mathcal{U}$, we have $f'x^* \geq 1/N = 1/10$.

A Communication Example, continued

- An associated worst-case realization of f for which the obtained performance is equal to $1/10$ is, for instance, e_3.
- Consider now a non-worst-case scenario for x^*, e.g., $f = e_1$. Then, the obtained objective value is $e_1' x^* = 1/3$.
- Contrast with

$$\overline{a} = \left(0, \frac{1}{2}, \frac{1}{2}\right)', \quad \overline{b} = \left(0, \frac{1}{10}, \ldots, \frac{1}{10}\right)', \quad \overline{x} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{10}, \ldots, \frac{1}{10}\right)'$$

- Solution is also robustly optimal, i.e., $(\overline{x}, \overline{a}, \overline{b}) \in X^{RO}$.

A Communication Example, continued

- It has the same qualities as (x^*, a^*, b^*) in protecting us from worst-case realizations of f.
- However, if $f = e_1$ realizes, this solution yields strictly better performance, $e_1'\overline{x} = 1/2 > e_1'x^* = 1/3$.
- $\overline{x}$ performs at least equally well, compared to x^* for *any* realization of f drawn from U!
- We have

$$\begin{array}{rcll} f'\overline{x} & \geq & f'x^*, & \forall f \in \mathcal{U}, \quad \text{and} \\ f'\overline{x} & > & f'x^*, & \forall f \in \mathcal{U} \cap \{f : f_1 + f_2 > 0\}. \end{array}$$

Pareto Robustly Optimal (PRO) solutions

- A solution x is called a Pareto Robustly Optimal (PRO) solution for the RO:

$$\text{maximize}_{x \in X} \quad \min_{p \in \mathcal{U}} p'x,$$

if it is robustly optimal, i.e., $x \in X^{RO}$, and $\nexists\, \bar{x} \in X$ such that

$$\begin{aligned} p'\bar{x} &\geq p'x, \quad \forall p \in \mathcal{U}, \quad \text{and} \\ \bar{p}'\bar{x} &> \bar{p}'x, \ \text{for some } \bar{p} \in \mathcal{U}. \end{aligned}$$

- $\bar{x}$ *Pareto dominates* x.
- RO can be viewed as a multiobjective optimization problem with an infinite number of objectives (corresponding to uncertainty scenarios).
- The solution $(\overline{x}, \overline{a}, \overline{b})$ Pareto dominates the given RO solution.
- $(\overline{x}, \overline{a}, \overline{b})$ is a PRO solution. In fact, if we denote the set of all PRO solutions with X^{PRO}, then

$$X^{PRO} = \left\{ (x, a, b) \in X : x \geq \frac{1}{N} e, \quad x_1 + x_2 = 1 \right\}.$$

Key Questions

- Given a robustly optimal solution $x \in X^{RO}$, how do we check if x is also PRO?
- If it is not, how do we find a PRO $\bar{x} \in X^{RO}$ that Pareto dominates x?
- How do we optimize over the set of PRO solutions X^{RO}?

Finding PRO Solutions

- Given a robustly optimal solution $x \in X^{RO}$, how do we check if x is also PRO?
- Given $x \in X^{RO}$, consider an arbitrary point $\bar{p} \in ri(\mathcal{U})$, and the problem:

$$\begin{array}{ll} \max\limits_{y} & \bar{p}' y \\ \text{subject to} & y \in \mathcal{U}^* \\ & x + y \in X. \end{array}$$

- Then, either the optimal value in the problem is zero, in which case $x \in X^{PRO}$,
- or the optimal value is strictly positive, $\bar{x} = x + y^\star$ Pareto dominates x and $\bar{x} \in X^{PRO}$, for any optimal $y^\star$.
- $\mathcal{U}^* = \{y \in \mathbb{R}^n$ such that $y' p \geq 0, \forall\, p \in \mathcal{U}\}$ is the dual cone of $\mathcal{U}$.

Proof

- $y = 0$ is always feasible. Hence, the optimal value is nonnegative. The discussion separates in two disjoint cases.
- *The optimal value is zero.* Assume that $x \notin X^{PRO}$, i.e., there exists $\tilde{x} \in X$ that Pareto dominates x:
  - $p'\tilde{x} \geq p'x, \forall\, p \in \mathcal{U}$,
  - $\exists\, \hat{p} \in \mathcal{U}$ such that $\hat{p}'\tilde{x} > \hat{p}'x$.
- We can take $\hat{p} \in ext(\mathcal{U})$, e.g., as a vertex solution to $\max_{p \in \mathcal{U}} (\tilde{x} - x)'p$.
- Note that $y = \tilde{x} - x$ is readily feasible. We claim that $\bar{p}'(\tilde{x} - x) > 0$.
- To prove this, recall that any $\bar{p} \in ri(\mathcal{U})\ \exists\, \lambda \in \mathbb{R}^{|ext\mathcal{U}|}$ such that $\lambda > 0$, $e'\lambda = 1$ and $\bar{p} = \sum_{i \in \mathcal{I}} \lambda_i p_i$, where $ext(\mathcal{U}) = \{p_i\ :\ i \in \mathcal{I}\}$. ,

$$\bar{p}'(\tilde{x} - x) = \underbrace{\lambda_{\hat{p}}\hat{p}'(\tilde{x} - x)}_{>0} + \sum_{i \in \mathcal{I}\,:\,p_i \neq \hat{p}} \underbrace{\lambda_i p_i'(\tilde{x} - x)}_{\geq 0} > 0.$$

- Contradicting the fact that optimal solution is zero.

Proof, continued

- *The optimal value is positive.*
- Then, $\bar{x}$ Pareto dominates x.
- To see that $\bar{x} \in X^{PRO}$, consider testing this by solving LOP, i.e., with x replaced by $\bar{x} = x + y^\star$, and the same $\bar{p}$.
- We claim that the optimal solution to this problem must be zero (which, in turn, implies that $\bar{x} \in X^{PRO}$).
- Otherwise, if an optimal solution $\tilde{y}$ existed, with $\bar{p}'\tilde{y} > 0$,
  then $y^\star + \tilde{y}$ would be feasible
  and provide a higher objective value than $y^\star$
  when solving LOP to test whether x was PRO,
  contradicting the fact that $y^\star$ was an optimal solution.

Tractability

- $X = \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \mathcal{U} = \{p \in \mathbb{R}^n : Dp \geq d\},$
- $\mathcal{U}^* = \{y \in \mathbb{R}^n \ : \ y'p \geq 0, \forall\, p \in \mathcal{U}\}$

$$= \{y \in \mathbb{R}^n \ : \ \exists\, \lambda \in \mathbb{R}_+^{m_U} \text{ such that } D'\lambda = y,\ d'\lambda \geq 0\}.$$

- PRO solutions always exist.
- How to find them?
- $\bar{p} \in ri(\mathcal{U})$. Theorem: all optimal solutions to $\mathrm{maximize}_{x \in X^{RO}}\ \bar{p}'x$ are PRO.
- Simple way to generate solutions in X^{PRO}:
  - Sample different values $\bar{p}$ from $ri(\mathcal{U})$, and
  - Solve $\mathrm{maximize}_{x \in X^{RO}}\ \bar{p}'x$.
  - Theorem For any $x \in X^{PRO}$, there exists $\bar{p} \in ri(\mathcal{U})$ such that $x \in argmax_{y \in X^{RO}}\ \bar{p}'y$.

Are all Robust solutions PRO?

- Consider any $\bar{p} \in ri(\mathcal{U})$, and the following optimization problem:

$$
\begin{array}{ll}
\max\limits_{x,y} & \bar{p}' y \\
\text{subject to} & x \in X^{RO} \\
& y \in \mathcal{U}^* \\
& x + y \in X.
\end{array}
$$

- Then, $X^{PRO} = X^{RO}$ if and only the optimal value is zero.

Non-convexity of the set of PRO solutions

$$X = \{x \in \mathbb{R}^4 \,:\, x_1 \leq 1,\, x_2 + x_3 \leq 6,\, x_3 + x_4 \leq 5,\, x_2 + x_4 \leq 5\},$$
$$\mathcal{U} = conv\Big(\big\{e_i,\, i \in \{1, \ldots, 4\}\big\}\Big).$$

- $Z_{Rob} = 1$, and $X^{RO} = \{x \in X \,:\, x \geq e\}$.
- $x^1 = \begin{bmatrix}1 & 2 & 4 & 1\end{bmatrix}'$ and $x^2 = \begin{bmatrix}1 & 4 & 2 & 1\end{bmatrix}'$ are both PRO solutions (they are the optimal solutions to the problems of maximizing $\begin{bmatrix}\epsilon & \epsilon & 1-3\epsilon & \epsilon\end{bmatrix}'$ and $\begin{bmatrix}\epsilon & 1-3\epsilon & \epsilon & \epsilon\end{bmatrix}'$ over X^{RO}, respectively, for some small $\epsilon > 0$).
- $0.5\,x^1 + 0.5\,x^2 = \begin{bmatrix}1 & 3 & 3 & 1\end{bmatrix}' \notin X^{PRO}$, since it is Pareto dominated by $\begin{bmatrix}1 & 3 & 3 & 2\end{bmatrix}' \in X^{RO}$.

Optimizing over PRO solutions

- For any $r \in \mathbb{R}^n$ and any $\bar{p} \in ri(\mathcal{U})$, let $(x^\star, \mu^\star, \eta^\star, z^\star) \in \mathbb{R}^n \times \mathbb{R}^{m_x} \times \mathbb{R} \times \{0,1\}^{m_x}$ be an optimal solution of the following MIO

$$
\begin{array}{rl}
\max\limits_{x,\mu,\eta,z} & r'x \\
\text{subject to} & x \in X^{RO} \\
& \mu \leq M(1-z) \\
& b - Ax \leq Mz \\
& DA'\mu - d\,\eta \geq D\bar{p} \\
& \mu \geq 0,\ \eta \geq 0,\ z \in \{0,1\}^{m_x}\ ,
\end{array}
$$

- M is a sufficiently large value. Then, $x^\star$ is an optimal solution of the problem $\max_{x \in X^{PRO}}\ r'x$.

Proof

- Need to show that $x \in X^{RO}$ is PRO if and only if there exist μ, η, z that satisfy MIO.
- $X = \{x \in \mathbb{R}^n : Ax \leq b\}$, $\mathcal{U} = \{p \in \mathbb{R}^n : Dp \geq d\}$, $\mathcal{U}^* = \{y \in \mathbb{R}^n \ : \ \exists \, \lambda \geq 0, \text{ such that } D'\lambda = y, \ d'\lambda \geq 0\}$..
- Let $x \in X^{RO}$. Then, x is PRO if and only if, for $\bar{p} \in ri(U)$, the optimal value in the problem $\max_y \{\bar{p}'y \ : \ y \in \mathcal{U}^*, \ x + y \in X\}$ is zero. Equivalently, x is PRO if and only if the optimal value in the following primal-dual pair is zero:

$$
\begin{array}{llcll}
\max\limits_{\lambda} & \bar{p}'D'\lambda & = & \min\limits_{\mu,\eta} & \mu'(b - Ax) \\
\text{subject to} & d'\lambda \geq 0 & & \text{subject to} & DA'\mu - d\,\eta \geq D\bar{p} \\
 & A\,D'\lambda \leq b - Ax & & & \mu \geq 0 \\
 & \lambda \geq 0 & & & \eta \geq 0.
\end{array}
$$

- Optimal value is zero if and only if there exist μ, η satisfying the constraints of the dual, and such that $\mu'(b - Ax) = 0$ (complementary slackness). The latter constraint can be modeled using binary variables z.

Sampling Procedure

1. Given an Inequality description for X and $\mathcal{U}$ (i.e., A, b, D and d). Number of steps N.
2. $\hat{X}^{PRO} := \emptyset$
3. For $i := 1, \ldots, N$
4. Sample a point $\bar{p} \in ri(\mathcal{U})$.
5. Set $\hat{X}^{PRO} := \hat{X}^{PRO} \cup argmax_{x \in X^{RO}} \ \bar{p}'x$.
6. Solve $\max_{x \in \hat{X}^{PRO}} r'x$.

Portfolio Optimization Example

$$
\begin{array}{ll}
\text{maximize} & \displaystyle\min_{\zeta \in U} \sum_{i=1}^{n+1} r_i \, x_i \\
\text{subject to} & r_i = \mu_i + \sigma_i \, \zeta_i, \quad i = 1, \ldots, n \\
& r_{n+1} = \mu_{n+1} \\
& \displaystyle\sum_{i=1+\frac{k\,N}{4}}^{\frac{(k+1)\,N}{4}} x_i \leq 0.25, \quad k = 0, 1, 2, 3 \\
& e' x = 1 \\
& x \geq 0,
\end{array}
$$

with variables $r \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$ and $\mathcal{U} = \{\zeta \in \mathbb{R}^n : -e \leq \zeta \leq e, \; e'\zeta = 0\}$.

Portfolio Optimization Example, continued

- 10,000 instances of problems of size $n = 8$.
- $\{\mu_i\}_{i=1}^n$ are set to 3%,
- $\{\sigma_i\}_{i=1}^n \sim$ U(1% ,10%).
- The risk-free return $\mu_{n+1} \sim$ U(1% ,10%).
- In 31% of the 10,000 instances, solution was not PRO.
- For these solutions contrast contrast PRO and RO solutions.
- Median performance gap recorded was 12%
- Maximum performance gap was as high as 74%.