$\mathcal{Q}$-Set is not generally a topos

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This talk is based on a joint work with Lili Shen (申力立):

- X. Hu and L. Shen. $\mathcal{Q}$-**Set** is not generally a topos. *Fuzzy Sets and Systems, 517:109484*, 2025.

Main result

The category of Q-sets is a topos if and only if Q is a frame.

Plan

1. What is a quantale set? What is the category of quantale sets?
2. What is a topos?
3. Why being topos forces a quantale to be a frame.

What is a quantale set

In a nutshell, a quantale set is a set X equipped with a quantale-valued relation. We always consider a divisible quantale, i.e.,

Definition : divisible quantale

A *commutative quantale* Q is a commutative monoid on a complete lattice such that multiplication $\&$ has a right adjoint $\rightarrow$. A quantale Q is said to be *divisible* if for all $p, q \in \mathsf{Q}$ we have

$$p \,\&(p \rightarrow q) = p \wedge q$$

We assume the quantale Q is always divisible.

Two examples of divisible quantales

Example : frames

Any frame Ω with binary meet $\wedge$ is a commutative divisible idempotent quantale. By an idempotent quantale, we mean

$$q \mathbin{\&} q = q$$

for all $q \in \mathsf{Q}$. And any commutative divisible idempotent quantale is a frame in the sense that $\& = \wedge$.

Example : extended reals

The extended real line $[0, \infty]^{\mathsf{op}}$ with *addition* $+$ is a divisible quantale, which is which is clearly not idempotent since $x + x \neq x$ for all $x \in (0, \infty)$.

What is a quantale set

Definition of Q-set

A quantale set is a set X equipped with a function

$$X \times X \to \mathsf{Q}$$

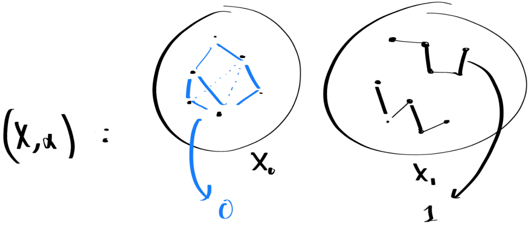
satisfying the following for $x, y, z \in X$:

1. $\alpha(x, y) \leqslant \alpha(x, x) \wedge \alpha(y, y)$
2. $\alpha(x, y) = \alpha(y, x)$
3. $\alpha(x, y) \& (\alpha(y, y) \to \alpha(y, z)) \leqslant \alpha(x, z)$

Two examples of quantale sets

Example: $\{0,1\}$-sets

Suppose (X, α) is a $\{0,1\}$-set then X is a disjoint union of X_0 and X_1, where $X_i = \{\, x \in X \,|\, \alpha(x,x) = i \,\}$. And $\alpha|_{X_1}$ is a partition.



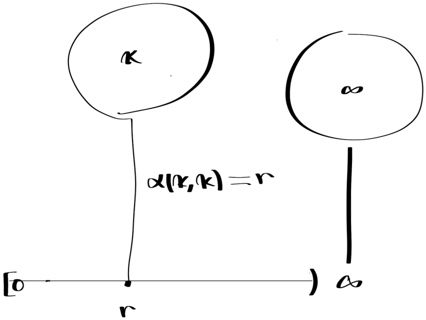
$\alpha|_{X_0} \equiv 0$ $\qquad$ $\alpha|_{X_1}$: partition

Two examples of quantale sets

Example: $[0,\infty]^{\mathrm{op}}$-sets

Boiling down the definition, a $[0,\infty]$-set is a set X together with a function $p\colon X \times X \to [0,\infty]$ such that

1. $p(x,x), p(y,y) \leqslant p(x,y)$, $p(x,y) = p(y,x)$, and
2. $p(x,z) + p(y,y) \leqslant p(x,y) + p(y,z)$.



What is the category of quantale sets

To define a category, we need to know those morphisms between quantale sets.

(A complicated) Definition : left adjoint distributor

A morphism between quantale sets (X, α) and (Y, β) is a *left adjoint distributor*. That means we have a function $\varphi\colon X \times Y \to \mathsf{Q}$ satisfies the following

1. there exists a function $\psi\colon Y \times X \to \mathsf{Q}$ called the right adjoint of φ such that $\varphi(x, y), \psi(y, x) \leqslant \alpha(x, x) \wedge \beta(y, y)$
2. $\alpha(x', x)\&(\alpha(x, x) \to \varphi(x, y)) \leqslant \varphi(x', y)$, $\varphi(x, y)\&(\beta(y, y) \to \beta(y, y')) \leqslant \varphi(x, y')$
3. $\beta(y', y)\&(\beta(y, y) \to \psi(y, x)) \leqslant \psi(y', x)$, $\psi(y, x)\&(\alpha(x, x) \to \alpha(x, x')) \leqslant \psi(y, x')$
4. $\alpha(x, x') \leqslant \bigvee_{y \in Y} \varphi(x, y)\&(\beta(y, y) \to \psi(y, x'))$ for all $x, x' \in X$
5. $\bigvee_{x \in X} \psi(y, x)\&(\alpha(x, x) \to \varphi(x, y') \leqslant \beta(y, y')$ for all $y, y' \in Y$.

Since working with distributors directly is difficult, we look for a category equivalent to the category of distributors, which leads us to *Cauchy completion*.

Making distributor to (non)-expanding map

(Non)-expanding maps

By an *expanding map*, we mean a function $f\colon X \to Y$ such that for all $x, x' \in X$ we have

$$\alpha(x,x) = \beta(f(x), f(x)) \quad \alpha(x,x') \leqslant \beta(f(x), f(x'))$$

An *embedding* is a injective expanding map $f\colon X \to Y$ such that

$$\alpha(x,x') = \beta(f(x), f(x')).$$

Definition: Cauchy complete

A Cauchy completion of a Q-set (Y, β) is a quantale set $(\widehat{Y}, \widehat{\beta})$ together with a expanding map

$$\iota\colon (Y,\beta) \to (\widehat{Y}, \widehat{\beta})$$

such that for every distributor $\varphi\colon (X,\alpha) \to (Y,\beta)$, there exists an expanding map $(X,\alpha) \to (\widehat{Y}, \widehat{\beta})$. (Y,β) is said to be Cauchy complete, if we can take $\iota = \mathrm{id}_Y$.

Examples of Cauchy complete quantale sets

In the category of Cauchy complete quantale sets, morphisms are easier to describe.

Definition : the category of Cauchy complete quantale sets

A morphism between two Cauchy complete quantale sets is an expanding map, which is categorical equivalent to the category of quantale sets.

This category is called DQ-$\mathbf{CcSymCat}$ in a fancy way.

Cauchy complete $\{0,1\}$-set

In fact, every $\{0,1\}$-set is Cauchy complete, and is isomorphic to a pointed set X_*. And since our functor is type-preserving,i.e.,the only element maps to the point $*$ is $*$ is itself. Hence the category of Cauchy complete $\{0,1\}$ set is equivalent to $\mathbf{Set}$ the category of ordinary sets.

Examples of Cauchy complete quantale sets

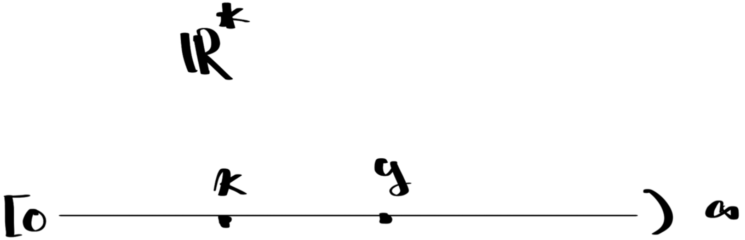
Cauchy complete $[0,\infty]^{\mathsf{op}}$-set

A $[0,\infty]^{\mathsf{op}}$-set X is Cauchy complete, if and only if X has ∞ and every Cauchy sequence $\{x_n\}$ converges.

very special case : if $\alpha(x,x) = 0$ for $x \neq \infty$. Then $X_0 = \{x \,|\, \alpha(x,x) = 0\}$ is a metric space with metric $d = \alpha|_{X_0}$, and X is Cauchy complete iff X_0 is Cauchy complete in the usual sense. Here is a simple example:

Examples of Cauchy complete quantale sets

Consider the set $\mathbb{R}^* := [0, \infty]$ with α defining by



$$\alpha(x, y) = |x - y|, \quad \alpha(x, \infty) = \infty$$

$\mathbb{R}^*$ is Cauchy complete.

Examples of Cauchy complete quantale sets

There are some special Cauchy complete Q-set that need to be mentioned:

Special Cauchy complete Q-set

1. $C_q = \{p \leqslant q \mid p \,\&\, (q \to p) = p\}$
2. $(\mathrm{Q}, \wedge)$

C_p is the Cauchy completion of a point $\{p\}$, and $(\mathrm{Q}, \wedge)$ is the terminal object in the category of Cauchy complete quantale sets.
The inclusion $C_p \subseteq C_q$ induces a new partial order

$$p \sqsubseteq q \iff p \leqslant q \text{ and } p \,\&\, (q \to p) = p$$

which has a binary meet $p \sqcap q$ in Q.

Monomorphisms

Fact

A morphism in the category of Cauchy complete quantale sets is monic iff it is injective.

$$\begin{tikzcd} C_p \arrow[r, shift left, "e_k"] \arrow[r, shift right, "e_{k'}"'] & X \arrow[r, tail, "f"] & Y \end{tikzcd}$$

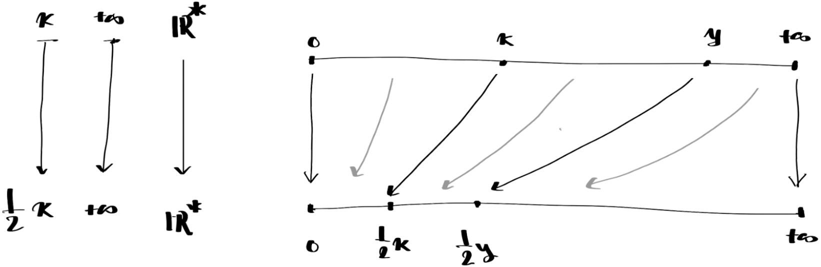
$$p \longmapsto k, k' \longmapsto f(k) = f(k')$$

or in a fancy way: since C_p are generator.

A monic morphisms may not be an embedding, here is an example:

There are monomorphism which is not embedding in some quantale!

Consider $\mathbb{R}^* = [0, \infty]$ The map $x \mapsto \frac{1}{2}x$ is monic (since its injective), but not an embedding.



What is a topos

We only consider those topos are neither too small nor too large.

(Informal) Definition of (Grothendieck) topos

A Grothendieck topos is a category of sheaves.

A sheaf can be regarded as a class of *formal functions* on a space satisfying the gluing lemma.

Sheaves on topological space

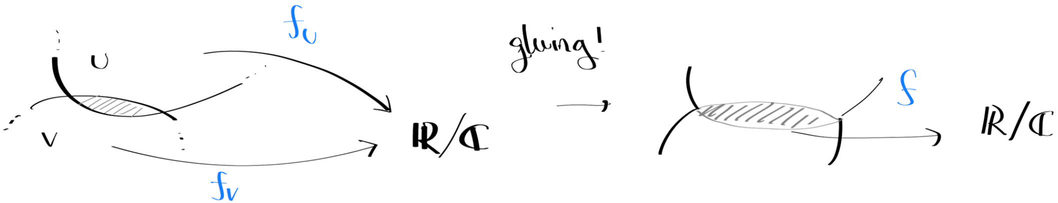
Consider a topological space X, let $\Gamma(U)$ be the set of functions,(or continuous functions) define on the open set U. Then the restriction $\Gamma(U) \to \Gamma(V), V \subseteq U$ makes Γ a contravariant functor from the lattice $\mathcal{O}(X)$ of open subsets to the category of sets. Such Γ is a sheaf.

Gluing functions

Suppose we have two functions f_U, f_V are defined on two open subsets U, V respectively, and they coincide in the intersection of their domain, i.e.,

$$f_U|_{U \cap V} = f_V|_{U \cap B}$$

we can glue them to get a function defined on a larger open subset $U \cup V$



by defining $f(x) = f_U(x)$ if $x \in U$ and $f(x) = f_V(x)$ if $x \in V$.

Basic properties about topos

Property I

Every monomorphism in a topos is an embedding.

Property II

A *subobject* of X is an isomorphic class of monomorphism with codomain X. In a topos subobjects of an object always form a lattice.
The intersection $U \cap V$ is pullback of $U \to X$ along $V \to X$.
The union $U \cup V$ of two subobjects is given by the pullback of pushout.

So there are some divisible quantale (e.g. $[0, \infty]^{\mathsf{op}}$) that the category of quantale sets are not topos. If we already know the category of quantale sets is a topos, what do we know about the quantale?

Being topos forces quantale to be a frame

We are trying to use those basic properties to show that our quantale is a frame, i.e., a quantale that every elements are idempotent.

Recipe

Step 1: Every monomorphisms is embedding: the intersection of C_p and C_q is of the form $C_{p \sqcap q}$.
Step 2: The union of C_p and C_q is the Cauchy completion of (Z, γ) such that $Z(p, q) = p \sqcap q$
Step 3: Since the Cauchy completion is a embedding we have that $p \sqcap q = p \wedge q$.

That means

$$q = q \wedge \top = q \sqcap \top \sqsubseteq \top$$

$$q \sqcap \top \iff q \& (\top \to q) = q$$

$$q \,\&\, q = q \,\&\, (\top \to q) = q$$

Thank you!