The Watrous Post-Quantum Zero-Knowledge Proof

A Crypto Reading Group Talk

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Post-Quantum ZK for NP

The model:

- ▶ Classical P and V
- ▶ ZK system for NP languages
- ▶ V^* can be quantum.
  - ▶ Modeled as a quantum polynomial-time (QPT) Turing machine.
  - ▶ equivalently (and more preferred in quantum-computing literature), poly-size quantum circuits.
  - ▶ Non-uniformity: V^* has an auxiliary quantum state that depends only on the security para. n. More accurately,

$$V^* = \{\mathsf{QC}_n, |\psi_n\rangle\}_{n\in\mathbb{N}}$$

Post-Quantum (Black-Box) ZK Is Hard

Why's **rewinding** hard?

- ▶ information gain VS state disturbance
- ▶ the no-cloning theorem

The major result in [Wat06]: a quantum rewinding lemma

Some Historical Notes

Techniques inspired by Marriot-Watrous [MW04]

- ▶ error-gap amplification for QMA using only 1 witness state

First published at STOC'06 [Wat06]

- ▶ Explicit connection to [MW04]
- ▶ Simple, ad hoc proof
- ▶ This talk mainly focuses on this version
- ▶ The notation herein is consistent with this version

Then, on SIAM Journal of Computing in 2009 [Wat09]

- ▶ Abstracts out a general quantum rewinding lemma
- ▶ Hides the connection with Marriot-Watrous
- ▶ We'll also see the high-level idea of this version

Agenda for Today

► Prove quantum ZK for the Graph Isomorphism protocol [GMW86] (in detail)
  ► Originally ad hoc [Wat06]
  ► We'll take a general perspective
► Extends to the Graph-3-coloring Protocol [GMW86] in the ideal Com model (simple)
  ► General quantum rewinding lemma
► G3C ZK with computationally-secure Com (simple-yet-tedious)
  ► Rewinding lemma in its most general form — allowing small perturbations
  ► the widely-used version in crypto literature

GMW ZK for Graph Isomorphism (GI)

Some Remarks:
- ▶ GI is not known to be NP-complete.
- ▶ the 1st message of the GMW GI protocol is perfectly uniform.

Input for P**:** statement $(G_0, G_1) \in \mathcal{G}_n \times \mathcal{G}_n$, witness $w = \sigma$ s.t. $\sigma(G_1) = G_0$
Input for V**:** (G_0, G_1)

1. P samples $\pi \leftarrow S_n$, sends $H = \pi(G_0)$
2. V sends $a \leftarrow \{0, 1\}$
3. P sends $\tau = \pi \circ \sigma^a$

V**'s decision:** accept iff $\tau(G_a) = H$

Classical Sim: guess the bit b. Set $H = \pi(G_b)$. Win if $b == a$.

Modeling in Quantum Way

Model a Quantum V^***:** circuit family $\{\mathbf{V}_H\}_{H \in \mathcal{G}_n}$, auxiliary input $|\psi\rangle$

- Receives H from P
- Perform $\mathbf{V}_H |\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{V}} |0\rangle_{\mathsf{A}} = \alpha_0 |\psi_0\rangle_{\mathsf{WV}} |0\rangle_{\mathsf{A}} + \alpha_1 |\psi_1\rangle_{\mathsf{WV}} |1\rangle_{\mathsf{A}}$
    - V: work space
    - A: single-qubit register to store V^*'s challenge.
    - Note that $\mathbf{V}_H$ operates on space $\mathsf{W} \otimes \mathsf{V} \otimes \mathsf{A}$

Modeling in Quantum Way

View the protocol through a quantum lens:

- The full space $\mathsf{W} \otimes \mathsf{X}$, where $\mathsf{X} = \mathsf{V} \otimes \mathsf{A} \otimes \mathsf{Y} \otimes \mathsf{B} \otimes \mathsf{Z}$
- Sim performs (classical Sim in superposition)

$$\mathbf{T} |0\rangle_{\mathsf{YBZ}} = \frac{1}{\sqrt{2n!}} \sum_{b \in \{0,1\}} \sum_{\pi \in S_n} |\pi(G_b)\rangle_{\mathsf{Y}} |b\rangle_{\mathsf{B}} |\pi\rangle_{\mathsf{Z}}$$

- V apply $\mathbf{V} = \sum_{H \in \mathcal{G}} \mathbf{V}_H \otimes |H\rangle\langle H|_{\mathsf{Y}} \otimes \mathbb{1}_{\mathsf{BZ}}$ on the full space $\mathsf{W} \otimes \mathsf{X}$
  - recall that $\mathbf{V}_H$ operates on $|\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{V}} |0\rangle_{\mathsf{A}}$
  - corresponding to the exec. in super-position
  - Output format:

$$\alpha_{00} |\psi_{00}\rangle |00\rangle_{\mathsf{AB}} + \alpha_{01} |\psi_{01}\rangle |01\rangle_{\mathsf{AB}} + \alpha_{10} |\psi_{10}\rangle |10\rangle_{\mathsf{AB}} + \alpha_{11} |\psi_{11}\rangle |11\rangle_{\mathsf{AB}}$$

In summary, the protocol up to step 2 is:

$$\underbrace{\mathbf{VT}}_{\text{on } \mathsf{W} \otimes \mathsf{X}} (|\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{X=VAYBZ}}) \;\; \Leftrightarrow \;\; \underbrace{\mathbf{VT}(\mathbb{1}_{\mathsf{W}} \otimes |0\rangle_{\mathsf{X}})}_{\text{only on } \mathsf{W}} |\psi\rangle \tag{1}$$

Measuring the Guess

Define a binary-outcome measurement on the full space $W \otimes X$:

- $\mathbf{\Pi}_0 = |0\rangle\langle 0|_{AB}$, $\quad \mathbf{\Pi}_1 \coloneqq \mathbb{1}_{AB} - |0\rangle\langle 0|_{AB} = |1\rangle\langle 1|_{AB}$
- work on the full space $W \otimes X$. Just tensor identities on registers other than AB

Performing $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ on $\mathbf{VT}\,|\psi\rangle_W\,|0\rangle_X$:

- w.p. $\mathrm{Tr}\left(\langle\psi|\,\mathbf{Q}\,|\psi\rangle\right)$, the outcome is 0.
- w.p. $\mathrm{Tr}\left(\langle\psi|\,(\mathbb{1}_W - \mathbf{Q})\,|\psi\rangle\right)$, the outcome is 1.

where $\mathbf{Q} = (\mathbb{1}_W \otimes \langle 0|_X)\mathbf{T}^\dagger\mathbf{V}^\dagger\mathbf{\Pi}_0\mathbf{TV}(\mathbb{1}_W \otimes |0\rangle_X)$. (See Expression (1).)

Two important facts:

- $\{\mathbf{Q}, \mathbb{1}_W - \mathbf{Q}\}$ form a POVM
- $\mathrm{Tr}\left(\langle\psi|\,\mathbf{Q}\,|\psi\rangle\right) = \mathrm{Tr}\left(\langle\psi|\,(\mathbb{1}_W - \mathbf{Q})\,|\psi\rangle\right) = \frac{1}{2}$, independent of $|\psi\rangle$. (Cuz 1st msg. of GI prot. is perfectly uniform.)

$\Rightarrow \mathbf{Q} = \mathbb{1}_W - \mathbf{Q} = \frac{1}{2}\mathbb{1}_W$

An Important Lemma

Let $\mathbf{\Delta}_0 := \mathbb{1}_W \otimes |0\rangle\langle0|_X$.

▶ $\mathbf{\Delta}_0$ projects register X to all-0 qubits.

▶ $\mathbf{\Delta}_0 = \mathbf{\Delta}_0^\dagger$

▶ $\mathbf{\Delta}_1 := \mathbb{1}_{WX} - \mathbf{\Delta}_0$. The $\{\mathbf{\Delta}_0, \mathbf{\Delta}_1\}$ form a POVM.

LEMMA 1:

For all $|\psi\rangle \in \mathcal{H}(W)$, $|\gamma_0\rangle = |\psi\rangle_W |0\rangle_X$ is an eigenvector of $\underbrace{\mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0}_{:=\mathbf{M}}$ with

corresponding eigenvalue $\lambda = 1/2$.

Proof. Recall $\mathbf{Q} = (\mathbb{1}_W \otimes \langle0|_X)\mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V}\mathbf{T}(\mathbb{1}_W \otimes |0\rangle_X) = \frac{1}{2}\mathbb{1}_W$.

$$\Rightarrow \quad \mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V}\mathbf{T}\mathbf{\Delta}_0 = (\mathbb{1}_W \otimes |0\rangle_X)\mathbf{Q}(\mathbb{1}_W \otimes \langle0|_X) = \frac{1}{2}\mathbb{1}_W \otimes |0\rangle\langle0|_X$$

$$\Rightarrow \quad \forall |\psi\rangle, \mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V}\mathbf{T}\mathbf{\Delta}_0 \underbrace{|\psi\rangle_W |0\rangle_X}_{|\gamma_0\rangle} = \left(\frac{1}{2}\mathbb{1}_W \otimes |0\rangle\langle0|_X\right) \underbrace{|\psi\rangle_W |0\rangle_X}_{|\gamma_0\rangle} = \frac{1}{2}\underbrace{|\psi\rangle_W |0\rangle_X}_{|\gamma_0\rangle}$$
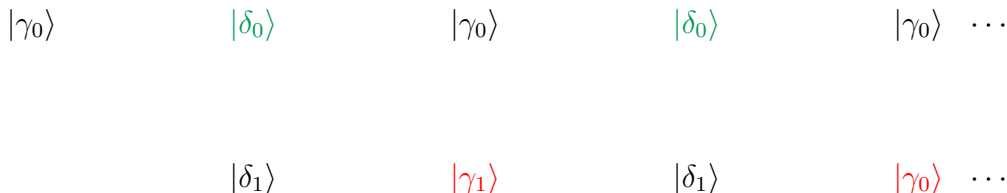
Marriot-Watrous Lemma

Given unitary $\mathbf{U}$, proj. mnt. $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ and $\{\mathbf{\Delta}_0, \mathbf{\Delta}_1\}$. Assume $|\gamma_0\rangle$ is an evec. of $\mathbf{\Delta}_0\mathbf{U}^\dagger\mathbf{\Pi}_0\mathbf{U}\mathbf{\Delta}_0$ with eval. λ. Define

$$|\delta_0\rangle := \frac{\mathbf{\Pi}_0\mathbf{U}\,|\gamma_0\rangle}{\sqrt{\lambda}}, \quad |\delta_1\rangle := \frac{\mathbf{\Pi}_0\mathbf{U}\,|\gamma_0\rangle}{\sqrt{1-\lambda}}, \quad |\gamma_1\rangle := \frac{\mathbf{\Delta}_1\mathbf{U}^\dagger\,|\delta_0\rangle}{\sqrt{1-\lambda}}.$$

Then, $\langle\gamma_0|\gamma_1\rangle = \langle\delta_0|\delta_1\rangle = 0$ and

$$\mathbf{U}\,|\gamma_0\rangle = \sqrt{\lambda}\,|\delta_0\rangle + \sqrt{1-\lambda}\,|\delta_1\rangle \qquad \mathbf{U}^\dagger\,|\delta_0\rangle = \sqrt{\lambda}\,|\gamma_0\rangle + \sqrt{1-\lambda}\,|\gamma_1\rangle$$

$$\mathbf{U}\,|\gamma_1\rangle = \sqrt{1-\lambda}\,|\delta_0\rangle - \sqrt{\lambda}\,|\delta_1\rangle \qquad \mathbf{U}^\dagger\,|\delta_1\rangle = \sqrt{1-\lambda}\,|\gamma_0\rangle - \sqrt{\lambda}\,|\gamma_1\rangle$$

(draw the evolution diagram)

$|\gamma_0\rangle$ $\qquad\qquad$ $|\delta_0\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\qquad\qquad$ $|\delta_0\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\cdots$

$\qquad\qquad\quad$ $|\delta_1\rangle$ $\qquad\qquad$ $|\gamma_1\rangle$ $\qquad\qquad$ $|\delta_1\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\cdots$

In Our Setting: Marriot-Watrous + Post-Mnt. Selection

In our setting, we have $\mathbf{U} = \mathbf{VT}$, $\lambda = 1/2$, and $|\gamma_0\rangle = |\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{X}}$

Lemma 2 $\Rightarrow$ $|\gamma_0\rangle = \frac{1}{\sqrt{2}} |\delta_0\rangle + \frac{1}{\sqrt{2}} |\delta_1\rangle$, and the following:

$$|\delta_0\rangle = \sqrt{2}\mathbf{\Pi}_0\mathbf{VT} |\gamma_0\rangle, \quad \mathbf{T}^{\dagger}\mathbf{V}^{\dagger} |\delta_1\rangle = \frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle, \quad \mathbf{VT}(\frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle) = |\delta_0\rangle$$

Starting with $|\gamma_0\rangle \rightarrow \mathbf{VT} |\gamma_0\rangle \rightarrow$ measurement $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$:

- ▶ w.p. $1/2$, it is $|\delta_0\rangle$ — we are done!
- ▶ w.p. $1/2$, it is $|\delta_1\rangle$
    - ▶ Key observation: $\mathbf{T}^{\dagger}\mathbf{V}^{\dagger} |\delta_1\rangle = \frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle$
    - ▶ If we can flip the phase of the 2nd term $\Rightarrow$ $\frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$.
    - ▶ Then, simply do $\mathbf{VT}(\frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle) = |\delta_0\rangle$

Yes, we can! (next slide)

Phase Flip for the 2nd Term

We want: $\frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle \quad \rightarrow \quad \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$

Recall the following

- $|\gamma_0\rangle = |\psi\rangle_\mathsf{W} |0\rangle_\mathsf{X}$ and $\boldsymbol{\Delta}_0 = \mathbb{1}_\mathsf{W} \otimes |0\rangle\langle 0|_\mathsf{X}$
- $\Rightarrow \quad \boldsymbol{\Delta}_0 |\gamma_0\rangle = |\gamma_0\rangle$
- Lemma 2 says $|\gamma_1\rangle = \sqrt{2}\boldsymbol{\Delta}_1 \mathbf{T}^\dagger \mathbf{V}^\dagger |\delta_0\rangle \quad \Rightarrow \quad \boldsymbol{\Delta}_0 |\gamma_1\rangle = 0$

Therefore, it is not hard to come up with the following idea:

$$\underbrace{(2\boldsymbol{\Delta}_0 - \mathbb{1}_\mathsf{WX})}_{=\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_1}(\frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle) = \frac{2}{\sqrt{2}}\boldsymbol{\Delta}_0 |\gamma_0\rangle - \frac{2}{\sqrt{2}}\boldsymbol{\Delta}_0 |\gamma_1\rangle - \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$$

$$= \frac{2}{\sqrt{2}} |\gamma_0\rangle - 0 - \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$$

$$= \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$$

Summarizing the Watrous Simulator

- ▶ Start with $|\gamma_0\rangle_{\mathsf{XW}} = |\psi\rangle_{\mathsf{X}} |0\rangle_{\mathsf{W}}$
- ▶ Perform $\mathbf{VT} |\gamma_0\rangle_{\mathsf{XW}}$
- ▶ Perform measurement $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$
  - ▶ If outcome is 0 — guessed correctly (in $|\delta_0\rangle$). Go next step.
  - ▶ Otherwise, we are in $|\delta_1\rangle = \sqrt{2}\mathbf{\Pi}_1\mathbf{VT} |\gamma_0\rangle$.
    - ▶ Perform $\mathbf{T}^\dagger\mathbf{V}^\dagger |\delta_1\rangle = \frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle$
    - ▶ Perform $(2\mathbf{\Delta}_0 - \mathbb{1}_{\mathsf{WX}})(\frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle) = \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$
    - ▶ Perform $\mathbf{VT}(\frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle) = |\delta_0\rangle$. Go next step.
- ▶ Sim can finish the last round as the honest prover.

Extending to G3C—Idealized Com Model (1/3)

- ▶ The graph-3-coloring (G3C) problem is NP-complete
- ▶ Start point: the G3C classical ZK proof from [GMW86]

Caveats:

- ▶ $\Pr[\text{Guess correctly}] = \frac{1}{m}$, where $m = \#$ edges.
- ▶ $\Pr[\text{Guess correctly}] \quad \perp \quad |\psi\rangle$?
    - ▶ 1st msg using perfect-binding (PB) Com
    - ▶ What about binding? — Collapse-binding suffices [Unr16]
    - ▶ We assume an ideal Com for simplicity: perfect-hiding and perfectly-binding
    - ▶ Extends to comp.-hiding Com later

Extending to G3C—Idealized Com Model (2/3)

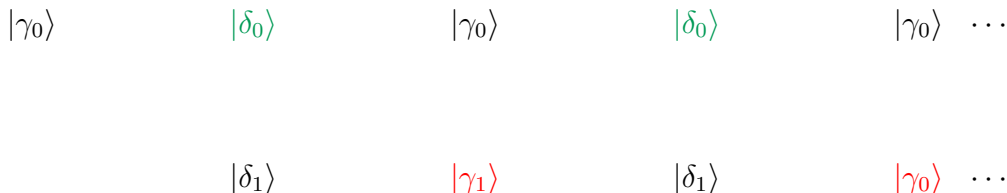Key ingredients for the GI simulator:

- Define an operator: $\mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0 \ (=: \mathbf{M})$
- An technical Lemma 1: $\lambda = \frac{1}{2} \quad (\perp \quad |\psi\rangle)$
- Invoke Marriot-Watrous Lemma 2 with $\lambda = \frac{1}{2}$:
    - Voilà 😊! We can get $|\delta_0\rangle$ within ≤ 2 steps

What will change for the G3C protocol?

- $\mathbf{M}$ defined as before ($\mathbf{T}$ modified in the natural way)
- Lemma 1: $\lambda = \frac{1}{m} \quad (\perp \quad |\psi\rangle)$
- Invoke Marriot-Watrous Lemma 2 with $\lambda = \frac{1}{m}$:
    - 🙁! no guarantee for $|\delta_0\rangle$ within ≤ 2 steps
- Solution: use the full power of Matrriot-Watrous analysis (next slide).

Extending to G3C—Idealized Com Model (3/3)

(draw the evolution diagram in the current setting)

$|\gamma_0\rangle$ $\qquad\qquad$ $|\delta_0\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\qquad\qquad$ $|\delta_0\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\;\cdots$

$\qquad\qquad\quad$ $|\delta_1\rangle$ $\qquad\qquad$ $|\gamma_1\rangle$ $\qquad\qquad$ $|\delta_1\rangle$ $\qquad\qquad$ $|\gamma_0\rangle$ $\;\cdots$

The main take-away:

▶ $\mathbf{U}|\gamma_0\rangle = \sqrt{\lambda}|\delta_0\rangle + \sqrt{1-\lambda}|\delta_1\rangle$ $\quad$ $\mathbf{U}^\dagger|\delta_0\rangle = \sqrt{\lambda}|\gamma_0\rangle + \sqrt{1-\lambda}|\gamma_1\rangle$, where $\lambda = 1/m$.
  $\mathbf{U}|\gamma_1\rangle = \sqrt{1-\lambda}|\delta_0\rangle - \sqrt{\lambda}|\delta_1\rangle$ $\quad$ $\mathbf{U}^\dagger|\delta_1\rangle = \sqrt{1-\lambda}|\gamma_0\rangle - \sqrt{\lambda}|\gamma_1\rangle$

▶ Measure $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ at each $|\delta\rangle$, if results in $|\delta_1\rangle$:
  ▶ $\mathbf{U}(2\mathbf{\Delta}_0 - \mathbb{1})\mathbf{U}^\dagger|\delta_1\rangle = 2\sqrt{p(1-p)}|\delta_0\rangle + (1-2p)|\delta_1\rangle$

▶ Measure $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$. Go to $|\delta_1\rangle$ w.p. $(1-2p)$.

▶ Keep failing after t iteration: $(1-p)(1-2p)^t$. Can be negligible by setting t properly.

The General Quantum Rewinding Lemma (Exact)

LEMMA 3: EXACT QUANTUM REWINDING [WAT09]

$\mathbf{Q}$ is a QC works on $|\psi\rangle$ and with $\Pr[\text{success}] = p \ (\perp \ |\psi\rangle)$ outputs $|\delta_0\rangle$. Then, for any $\varepsilon > 0$, there exists another QC $\mathbf{R}$ of size

$$O\left(\frac{\log(1/\varepsilon)}{p(1-p)} \cdot \text{size}(\mathbf{Q})\right)$$

such that for every input $|\psi\rangle$, the output ρ of $\mathbf{R}$ satisfies $\text{Tr}(\rho|\delta_0\rangle\langle\delta_0|) \geq 1 - \varepsilon$.

▶ "Exact" refers to the face that $p \ \perp \ |\psi\rangle$.
▶ The $\frac{\log(1/\varepsilon)}{p(1-p)}$: because we need to set a proper t to get negligible failing error.
▶ Only need poly-size for a negligible ε.
▶ For the trace, the closer to 1, the better.

G3C ZK with Comp.-Hiding Com

- Sim's 1st msg. $\stackrel{c}{\approx}$ Prover's 1st msg.
- V^*'s challenge $a \not\perp$ the 1st msg.
- In Lemma 3, $\Pr[\mathsf{success}] = p(|\psi\rangle)$.
  - $p(|\psi\rangle)$ jiggles within an negl. small interval.
- Need a version of Lemma 3 allowing small perturbations

The Version Allowing Small Perturbations

> **LEMMA 4: QUANTUM REWINDING WITH SMALL PERTURBATIONS [WAT09, SEC. 4.2]**
>
> Let $\mathbf{Q}$, $|\psi\rangle$, and $|\delta_0\rangle$ as before. But $\Pr[\text{success}] = p(|\psi\rangle)$ now depends on $|\psi\rangle$. Let $p_0, q \in (0,1)$ and $\varepsilon \in (0, 1/2)$ be real numbers such that
>
> $$(1). \; |p(\psi) - q| < \varepsilon \qquad (2). \; p_0 \leq p(\psi) \qquad (3). \; p_0(1 - p_0) \leq q(1 - q)$$
>
> Then, for any $\varepsilon > 0$, there exists another QC $\mathbf{R}$ of size $O\left(\frac{\log(1/\varepsilon)}{p_0(1-p_0)} \cdot \text{size}(\mathbf{Q})\right)$ such that for every input $|\psi\rangle$, the output ρ of $\mathbf{R}$ satisfies:
>
> $$\text{Tr}(\rho|\delta_0\rangle\langle\delta_0|) \geq 1 - 16\varepsilon\frac{\log^2(1/\varepsilon)}{p_0^2(1 - p_0)^2}.$$

Proof at a high-level:

▶ Consider each eigen-space separately (next slide).

▶ For detailed calculation, see [Wat09, Sec. 4.2].

Proof Sketch for Lemma 4

Proof Sketch:

- ▶ In Lemma 1, $|\gamma_0\rangle = |\psi\rangle_W |0\rangle_X$ is no longer an evec. of $\mathbf{M}$
  - ▶ The reason: $|\psi\rangle_W$ is not an evec. of $\mathbf{Q}$
- ▶ (mental exper.) Thus, decomp. $|\psi\rangle$ in the evecs $\{|\psi_i\rangle\}_{i \in [\text{dim}]}$ of $\mathbf{Q}$
- ▶ (mental exper.) For each i, we obtain Lemmas 1 and 2
- ▶ (mental exper.) In the Marriot-Watrous procedure, in each egein space:
  $$\mathbf{VT} |\psi_i\rangle_W |0\rangle_X = \sqrt{p(|\psi_i\rangle)} |\delta_0(|\psi_i\rangle)\rangle + \sqrt{1 - p(|\psi_i\rangle)} |\delta_1(|\psi_i\rangle)\rangle$$
- ▶ (mental exper.) Define a unitary $\mathbf{N}$ such that for all $i \in [\text{dim}]$:
  $$\sqrt{p(|\psi_i\rangle)} |\delta_0(|\psi_i\rangle)\rangle + \sqrt{1 - p(|\psi_i\rangle)} |\delta_1(|\psi_i\rangle)\rangle \to \sqrt{q} |\delta_0(|\psi_i\rangle)\rangle + \sqrt{1 - q} |\delta_1(|\psi_i\rangle)\rangle$$
- ▶ (mental exper.) Ready to apply the Exact Rewinding Lemma 3 (w/ p_0 as we don't know p.) (Need $p_0(1 - p_0) \le q(1 - q)$.)

In summary, this is a Sim w/ an imaginary operator $\mathbf{N}$, giving the same trace bound as in Lemma 3. But for the real Sim, there is no $\mathbf{N}$.

- ▶ Doesn't matter. $\mathbf{N}$ only affects the trace bound negligibly.
- ▶ By tedious-yet-elementary linear algebra (see [Wat09, Sec. 4.2]).

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