The Watrous Post-Quantum Zero-Knowledge Proof

A Crypto Reading Group Talk

by

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Git repo for these slides and LaTeX source code: https://github.com/xiao-liang/Talk-for-Watrous

Post-Quantum ZK for NP

The model:

- Classical P and V
- ZK system for NP languages
- V^* can be quantum.
  - Modeled as a quantum polynomial-time (QPT) Turing machine.
  - equivalently (and more preferred in quantum-computing literature), poly-size quantum circuits.
  - Non-uniformity: V^* has an auxiliary quantum state that depends only on the security para. n. More accurately,

$$V^* = \{\mathsf{QC}_n, |\psi_n\rangle\}_{n\in\mathbb{N}}$$

Post-Quantum (Black-Box) ZK Is Hard

Why's **rewinding** hard?

- information gain VS state disturbance
- the no-cloning theorem

The major result in [Wat06]: a quantum rewinding lemma

Some Historical Notes

Techniques inspired by Marriot-Watrous [MW04]

- error-gap amplification for QMA using only 1 witness state

First published at STOC'06 [Wat06]

- Explicit connection to [MW04]
- Simple, ad hoc proof
- This talk mainly focuses on this version
- The notation herein is consistent with this version

Then, on SIAM Journal of Computing in 2009 [Wat09]

- Abstracts out a general quantum rewinding lemma
- Hides the connection with Marriot-Watrous
- We'll also see the high-level idea of this version

Agenda for Today

- Prove quantum ZK for the Graph Isomorphism protocol [GMW86] (in detail)
  - Originally ad hoc [Wat06]
  - We'll take a general perspective
- Extends to the Graph-3-coloring Protocol [GMW86] in the ideal Com model (simple)
  - General quantum rewinding lemma
- G3C ZK with computationally-secure Com (simple-yet-tedious)
  - Rewinding lemma in its most general form — allowing small perturbations
  - the widely-used version in crypto literature

GMW ZK for Graph Isomorphism (GI)

Some Remarks:

- GI is not known to be NP-complete.
- the 1st message of the GMW GI protocol is perfectly uniform.

Input for P: statement $(G_0, G_1) \in \mathcal{G}_n \times \mathcal{G}_n$, witness $w = \sigma$ s.t. $\sigma(G_1) = G_0$
Input for V: (G_0, G_1)

1. P samples $\pi \leftarrow S_n$, sends $H = \pi(G_0)$
2. V sends $a \leftarrow \{0, 1\}$
3. P sends $\tau = \pi \circ \sigma^a$

V's decision: accept iff $\tau(G_a) = H$

Classical Sim: guess the bit b. Set $H = \pi(G_b)$. Win if $b == a$.

Modeling in Quantum Way

Model a Quantum V^*: circuit family $\{\mathbf{V}_H\}_{H\in\mathcal{G}_n}$, auxiliary input $|\psi\rangle$

- Receives H from P
- Perform $\mathbf{V}_H |\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{V}} |0\rangle_{\mathsf{A}} = \alpha_0 |\psi_0\rangle_{\mathsf{WV}} |0\rangle_{\mathsf{A}} + \alpha_1 |\psi_1\rangle_{\mathsf{WV}} |1\rangle_{\mathsf{A}}$
  - V: work space
  - A: single-qubit register to store V^*'s challenge.
  - Note that $\mathbf{V}_H$ operates on space $\mathsf{W} \otimes \mathsf{V} \otimes \mathsf{A}$

Modeling in Quantum Way

View the protocol through a quantum lens:

- The full space $\mathsf{W} \otimes \mathsf{X}$, where $\mathsf{X} = \mathsf{V} \otimes \mathsf{A} \otimes \mathsf{Y} \otimes \mathsf{B} \otimes \mathsf{Z}$
- Sim performs (classical Sim in superposition)

$$\mathbf{T}\,|0\rangle_{\mathsf{YBZ}} = \frac{1}{\sqrt{2n!}} \sum_{b\in\{0,1\}} \sum_{\pi\in S_n} |\pi(G_b)\rangle_{\mathsf{Y}}\,|b\rangle_{\mathsf{B}}\,|\pi\rangle_{\mathsf{Z}}$$

- V apply $\mathbf{V} = \sum_{H\in\mathcal{G}} \mathbf{V}_H \otimes |H\rangle\langle H|_{\mathsf{Y}} \otimes \mathbb{1}_{\mathsf{BZ}}$ on the full space $\mathsf{W} \otimes \mathsf{X}$
  - recall that $\mathbf{V}_H$ operates on $|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{V}}\,|0\rangle_{\mathsf{A}}$
  - corresponding to the exec. in super-position
  - Output format:

$$\alpha_{00}\,|\psi_{00}\rangle\,|00\rangle_{\mathsf{AB}} + \alpha_{01}\,|\psi_{01}\rangle\,|01\rangle_{\mathsf{AB}} + \alpha_{10}\,|\psi_{10}\rangle\,|10\rangle_{\mathsf{AB}} + \alpha_{11}\,|\psi_{11}\rangle\,|11\rangle_{\mathsf{AB}}$$

In summary, the protocol up to step 2 is:

$$\underbrace{\mathbf{VT}}_{\text{on } \mathsf{W}\otimes\mathsf{X}}\,\left(|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}=\mathsf{VAYBZ}}\right) \Leftrightarrow \underbrace{\mathbf{VT}(\mathbb{1}_{\mathsf{W}} \otimes |0\rangle_{\mathsf{X}})}_{\text{only on } \mathsf{W}}\,|\psi\rangle \qquad (1)$$

Measuring the Guess

Define a binary-outcome measurement on the full space $\mathsf{W} \otimes \mathsf{X}$:

- $\mathbf{\Pi}_0 = |00\rangle\langle 00|_{\mathsf{AB}} + |11\rangle\langle 11|_{\mathsf{AB}}\,, \quad \mathbf{\Pi}_1 := \mathbb{1}_{\mathsf{AB}} - \mathbf{\Pi}_0$
- work on the full space $\mathsf{W} \otimes \mathsf{X}$. Just tensor identities on registers other than AB

Performing $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ on $\mathbf{V}\mathbf{T}\,|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}}$:

- w.p. $\mathrm{Tr}\left(\langle\psi|\,\mathbf{Q}\,|\psi\rangle\right)$, the outcome is 0.
- w.p. $\mathrm{Tr}\left(\langle\psi|\,(\mathbb{1}_{\mathsf{W}} - \mathbf{Q})\,|\psi\rangle\right)$, the outcome is 1.

where $\mathbf{Q} = (\mathbb{1}_{\mathsf{W}} \otimes \langle 0|_{\mathsf{X}})\mathbf{T}^\dagger\mathbf{V}^\dagger\mathbf{\Pi}_0\mathbf{T}\mathbf{V}(\mathbb{1}_{\mathsf{W}} \otimes |0\rangle_{\mathsf{X}})$. (See Expression (1).)

Two important facts:

- $\{\mathbf{Q}, \mathbb{1}_{\mathsf{W}} - \mathbf{Q}\}$ form a POVM
- $\mathrm{Tr}\left(\langle\psi|\,\mathbf{Q}\,|\psi\rangle\right) = \mathrm{Tr}\left(\langle\psi|\,(\mathbb{1}_{\mathsf{W}} - \mathbf{Q})\,|\psi\rangle\right) = \frac{1}{2}$, independent of $|\psi\rangle$. (Cuz 1st msg. of GI prot. is perfectly uniform.)

$\Rightarrow \quad \mathbf{Q} = \mathbb{1}_{\mathsf{W}} - \mathbf{Q} = \frac{1}{2}\mathbb{1}_{\mathsf{W}}$

An Important Lemma

Let $\mathbf{\Delta}_0 := \mathbb{1}_{\mathsf{W}} \otimes |0\rangle\langle 0|_{\mathsf{X}}$.

- $\mathbf{\Delta}_0$ projects register X to all-0 qubits.
- $\mathbf{\Delta}_0 = \mathbf{\Delta}_0^\dagger$
- $\mathbf{\Delta}_1 := \mathbb{1}_{\mathsf{WX}} - \mathbf{\Delta}_0$. The $\{\mathbf{\Delta}_0, \mathbf{\Delta}_1\}$ form a POVM.

Lemma 1:

For all $|\psi\rangle \in \mathcal{H}(\mathsf{W})$, $|\gamma_0\rangle = |\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}}$ is an eigenvector of $\underbrace{\mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0}_{:=\mathbf{M}}$ with corresponding eigenvalue $\lambda = 1/2$.

Proof. Recall $\mathbf{Q} = (\mathbb{1}_{\mathsf{W}} \otimes \langle 0|_{\mathsf{X}})\mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T}(\mathbb{1}_{\mathsf{W}} \otimes |0\rangle_{\mathsf{X}}) = \frac{1}{2}\mathbb{1}_{\mathsf{W}}$.

$$\Rightarrow \quad \mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0 = (\mathbb{1}_{\mathsf{W}} \otimes |0\rangle_{\mathsf{X}})\mathbf{Q}(\mathbb{1}_{\mathsf{W}} \otimes \langle 0|_{\mathsf{X}}) = \frac{1}{2}\mathbb{1}_{\mathsf{W}} \otimes |0\rangle\langle 0|_{\mathsf{X}}$$

$$\Rightarrow \quad \forall\,|\psi\rangle\,, \mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0 \underbrace{|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}}}_{|\gamma_0\rangle} = \left(\frac{1}{2}\mathbb{1}_{\mathsf{W}} \otimes |0\rangle\langle 0|_{\mathsf{X}}\right) \underbrace{|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}}}_{|\gamma_0\rangle} = \frac{1}{2}\underbrace{|\psi\rangle_{\mathsf{W}}\,|0\rangle_{\mathsf{X}}}_{|\gamma_0\rangle}$$

Marriot-Watrous Lemma

Lemma 2: Marriot-Watrous [MW04]

Given unitary $\mathbf{U}$, proj. mnt. $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ and $\{\mathbf{\Delta}_0, \mathbf{\Delta}_1\}$. Assume $|\gamma_0\rangle$ is an evec. of $\mathbf{\Delta}_0 \mathbf{U}^\dagger \mathbf{\Pi}_0 \mathbf{U} \mathbf{\Delta}_0$ with eval. λ. Define

$$|\delta_0\rangle := \frac{\mathbf{\Pi}_0 \mathbf{U} |\gamma_0\rangle}{\sqrt{\lambda}}, \quad |\delta_1\rangle := \frac{\mathbf{\Pi}_0 \mathbf{U} |\gamma_0\rangle}{\sqrt{1-\lambda}}, \quad |\gamma_1\rangle := \frac{\mathbf{\Delta}_1 \mathbf{U}^\dagger |\delta_0\rangle}{\sqrt{1-\lambda}}.$$

Then, $\langle \gamma_0 | \gamma_1 \rangle = \langle \delta_0 | \delta_1 \rangle = 0$ and

$$\begin{aligned} \mathbf{U}|\gamma_0\rangle &= \sqrt{\lambda}|\delta_0\rangle + \sqrt{1-\lambda}|\delta_1\rangle & \qquad \mathbf{U}^\dagger|\delta_0\rangle &= \sqrt{\lambda}|\gamma_0\rangle + \sqrt{1-\lambda}|\gamma_1\rangle \\ \mathbf{U}|\gamma_1\rangle &= \sqrt{1-\lambda}|\delta_0\rangle - \sqrt{\lambda}|\delta_1\rangle & \qquad \mathbf{U}^\dagger|\delta_1\rangle &= \sqrt{1-\lambda}|\gamma_0\rangle - \sqrt{\lambda}|\gamma_1\rangle \end{aligned}$$

(draw the evolution diagram)

$$\begin{array}{cccccc} |\gamma_0\rangle & |\delta_0\rangle & |\gamma_0\rangle & |\delta_0\rangle & |\gamma_0\rangle & \cdots \\ & |\delta_1\rangle & |\gamma_1\rangle & |\delta_1\rangle & |\gamma_0\rangle & \cdots \end{array}$$

In Our Setting: Marriot-Watrous + Post-Mnt. Selection

In our setting, we have $\mathbf{U} = \mathbf{V}\mathbf{T}$, $\lambda = 1/2$, and $|\gamma_0\rangle = |\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{X}}$

Lemma 2 $\Rightarrow$ $|\gamma_0\rangle = \frac{1}{\sqrt{2}} |\delta_0\rangle + \frac{1}{\sqrt{2}} |\delta_1\rangle$, and the following:

$$|\delta_0\rangle = \sqrt{2}\mathbf{\Pi}_0\mathbf{V}\mathbf{T}\,|\gamma_0\rangle\,, \quad \mathbf{T}^\dagger\mathbf{V}^\dagger\,|\delta_1\rangle = \frac{1}{\sqrt{2}}|\gamma_0\rangle - \frac{1}{\sqrt{2}}|\gamma_1\rangle\,, \quad \mathbf{V}\mathbf{T}(\frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle) = |\delta_0\rangle$$

Starting with $|\gamma_0\rangle \rightarrow \mathbf{V}\mathbf{T}\,|\gamma_0\rangle \rightarrow$ measurement $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$:

- w.p. $1/2$, it is $|\delta_0\rangle$ — we are done!
- w.p. $1/2$, it is $|\delta_1\rangle$
  - Key observation: $\mathbf{T}^\dagger\mathbf{V}^\dagger\,|\delta_1\rangle = \frac{1}{\sqrt{2}}|\gamma_0\rangle - \frac{1}{\sqrt{2}}|\gamma_1\rangle$
  - If we can flip the phase of the 2nd term $\Rightarrow$ $\frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle$.
  - Then, simply do $\mathbf{V}\mathbf{T}(\frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle) = |\delta_0\rangle$

Yes, we can! (next slide)

Phase Flip for the 2nd Term

We want: $\frac{1}{\sqrt{2}}|\gamma_0\rangle - \frac{1}{\sqrt{2}}|\gamma_1\rangle \quad \rightarrow \quad \frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle$

Recall the following

- $|\gamma_0\rangle = |\psi\rangle_{\mathsf{W}}|0\rangle_{\mathsf{X}}$ and $\mathbf{\Delta}_0 = \mathbb{1}_{\mathsf{W}} \otimes |0\rangle\langle 0|_{\mathsf{X}}$
- $\Rightarrow \quad \mathbf{\Delta}_0|\gamma_0\rangle = |\gamma_0\rangle$
- Lemma 2 says $|\gamma_1\rangle = \sqrt{2}\mathbf{\Delta}_1\mathbf{T}^\dagger\mathbf{V}^\dagger|\delta_0\rangle \quad \Rightarrow \quad \mathbf{\Delta}_0|\gamma_1\rangle = 0$

Therefore, it is not hard to come up with the following idea:

$$
\begin{aligned}
\underbrace{(2\mathbf{\Delta}_0 - \mathbb{1}_{\mathsf{WX}})}_{=\mathbf{\Delta}_0 - \mathbf{\Delta}_1}\left(\frac{1}{\sqrt{2}}|\gamma_0\rangle - \frac{1}{\sqrt{2}}|\gamma_1\rangle\right) &= \frac{2}{\sqrt{2}}\mathbf{\Delta}_0|\gamma_0\rangle - \frac{2}{\sqrt{2}}\mathbf{\Delta}_0|\gamma_1\rangle - \frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle \\
&= \frac{2}{\sqrt{2}}|\gamma_0\rangle - 0 - \frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle \\
&= \frac{1}{\sqrt{2}}|\gamma_0\rangle + \frac{1}{\sqrt{2}}|\gamma_1\rangle
\end{aligned}
$$

Summarizing the Watrous Simulator

- Start with $|\gamma_0\rangle_{\mathsf{XW}} = |\psi\rangle_{\mathsf{X}} |0\rangle_{\mathsf{W}}$
- Perform $\mathbf{VT} |\gamma_0\rangle_{\mathsf{XW}}$
- Perform measurement $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$
  - If outcome is 0 — guessed correctly (in $|\delta_0\rangle$). Go next step.
  - Otherwise, we are in $|\delta_1\rangle = \sqrt{2}\mathbf{\Pi}_1\mathbf{VT} |\gamma_0\rangle$.
    - Perform $\mathbf{T}^\dagger\mathbf{V}^\dagger |\delta_1\rangle = \frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle$
    - Perform $(2\mathbf{\Delta}_0 - \mathbb{1}_{\mathsf{WX}})(\frac{1}{\sqrt{2}} |\gamma_0\rangle - \frac{1}{\sqrt{2}} |\gamma_1\rangle) = \frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle$
    - Perform $\mathbf{VT}(\frac{1}{\sqrt{2}} |\gamma_0\rangle + \frac{1}{\sqrt{2}} |\gamma_1\rangle) = |\delta_0\rangle$. Go next step.
- Sim can finish the last round as the honest prover.

Extending to G3C—Idealized Com Model (1/3)

- The graph-3-coloring (G3C) problem is NP-complete
- Start point: the G3C classical ZK proof from [GMW86]

Caveats:

- $\Pr[\text{Guess correctly}] = \frac{1}{m}$, where m = # edges.
- $\Pr[\text{Guess correctly}] \quad \perp \quad |\psi\rangle$?
  - 1st msg using perfect-binding (PB) Com
  - What about binding? — Collapse-binding suffices [Unr16]
  - We assume an ideal Com for simplicity: perfect-hiding and perfectly-binding
  - Extends to comp.-hiding Com later

Extending to G3C—Idealized Com Model (2/3)

Key ingredients for the GI simulator:

- Define an operator: $\mathbf{\Delta}_0^\dagger \mathbf{T}^\dagger \mathbf{V}^\dagger \mathbf{\Pi}_0 \mathbf{V} \mathbf{T} \mathbf{\Delta}_0$ (=: $\mathbf{M}$)
- An technical Lemma 1: $\lambda = \frac{1}{2} \quad (\perp \quad |\psi\rangle)$
- Invoke Marriot-Watrous Lemma 2 with $\lambda = \frac{1}{2}$:
  - Voilà 🙂! We can get $|\delta_0\rangle$ within ≤ 2 steps

What will change for the G3C protocol?

- $\mathbf{M}$ defined as before ($\mathbf{T}$ modified in the natural way)
- Lemma 1: $\lambda = \frac{1}{m} \quad (\perp \quad |\psi\rangle)$
- Invoke Marriot-Watrous Lemma 2 with $\lambda = \frac{1}{m}$:
  - 🙁! no guarantee for $|\delta_0\rangle$ within ≤ 2 steps
- Solution: use the full power of Matrriot-Watrous analysis (next slide).

Extending to G3C—Idealized Com Model (3/3)

(draw the evolution diagram in the current setting)

$$\begin{array}{cccccc} |\gamma_0\rangle & |\delta_0\rangle & |\gamma_0\rangle & |\delta_0\rangle & |\gamma_0\rangle & \cdots \\ & & & & & \\ & |\delta_1\rangle & |\gamma_1\rangle & |\delta_1\rangle & |\gamma_0\rangle & \cdots \end{array}$$

The main take-away:

- $\begin{array}{ll} \mathbf{U}\,|\gamma_0\rangle = \sqrt{\lambda}\,|\delta_0\rangle + \sqrt{1-\lambda}\,|\delta_1\rangle & \mathbf{U}^\dagger\,|\delta_0\rangle = \sqrt{\lambda}\,|\gamma_0\rangle + \sqrt{1-\lambda}\,|\gamma_1\rangle \\ \mathbf{U}\,|\gamma_1\rangle = \sqrt{1-\lambda}\,|\delta_0\rangle - \sqrt{\lambda}\,|\delta_1\rangle & \mathbf{U}^\dagger\,|\delta_1\rangle = \sqrt{1-\lambda}\,|\gamma_0\rangle - \sqrt{\lambda}\,|\gamma_1\rangle \end{array}$, where $\lambda = 1/m$.
- Measure $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$ at each $|\delta\rangle$, if results in $|\delta_1\rangle$:
  - $\mathbf{U}(2\mathbf{\Delta}_0 - \mathbb{1})\mathbf{U}^\dagger\,|\delta_1\rangle = 2\sqrt{p(1-p)}\,|\delta_0\rangle + (1-2p)\,|\delta_1\rangle$
- Measure $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1\}$. Go to $|\delta_1\rangle$ w.p. $(1-2p)$.
- Keep failing after t iteration: $(1-p)(1-2p)^t$. Can be negligible by setting t properly.

The General Quantum Rewinding Lemma (Exact)

Lemma 3: Exact Quantum Rewinding [Wat09]

$\mathbf{Q}$ is a QC works on $|\psi\rangle$ and with $\Pr[\mathsf{success}] = p \ \ (\perp \ \ |\psi\rangle)$ outputs $|\delta_0\rangle$. Then, for any $\varepsilon > 0$, there exists another QC $\mathbf{R}$ of size

$$O\left(\frac{\log(1/\varepsilon)}{p(1-p)} \cdot \mathsf{size}(\mathbf{Q})\right)$$

such that for every input $|\psi\rangle$, the output ρ of $\mathbf{R}$ satisfies $\mathrm{Tr}(\rho|\delta_0\rangle\langle\delta_0|) \geq 1 - \varepsilon$.

- "Exact" refers to the face that $p \ \ \perp \ \ |\psi\rangle$.
- The $\frac{\log(1/\varepsilon)}{p(1-p)}$: because we need to set a proper t to get negligible failing error.
- Only need poly-size for a negligible ε.
- For the trace, the closer to 1, the better.

G3C ZK with Comp.-Hiding Com

- Sim's 1st msg. $\overset{c}{\approx}$ Prover's 1st msg.
- V^*'s challenge $a \not\angle$ the 1st msg.
- In Lemma 3, $\Pr[\mathsf{success}] = p(|\psi\rangle)$.
  - $p(|\psi\rangle)$ jiggles within an negl. small interval.
- Need a version of Lemma 3 allowing small perturbations

The Version Allowing Small Perturbations

Lemma 4: Quantum Rewinding with Small Perturbations [Wat09, Sec. 4.2]

Let $\mathbf{Q}$, $|\psi\rangle$, and $|\delta_0\rangle$ as before. But $\Pr[\mathsf{success}] = p(|\psi\rangle)$ now depends on $|\psi\rangle$. Let $p_0, q \in (0,1)$ and $\varepsilon \in (0, 1/2)$ be real numbers such that

$$(1).\ |p(\psi) - q| < \varepsilon \qquad (2).\ p_0 \leq p(\psi) \qquad (3).\ p_0(1-p_0) \leq q(1-q)$$

Then, for any $\varepsilon > 0$, there exists another QC $\mathbf{R}$ of size $O\left(\frac{\log(1/\varepsilon)}{p_0(1-p_0)} \cdot \mathsf{size}(\mathbf{Q})\right)$ such that for every input $|\psi\rangle$, the output ρ of $\mathbf{R}$ satisfies:

$$\mathrm{Tr}(\rho|\delta_0\rangle\langle\delta_0|) \geq 1 - 16\varepsilon\frac{\log^2(1/\varepsilon)}{p_0^2(1-p_0)^2}.$$

Proof at a high-level:

- Consider each eigen-space separately (next slide).
- For detailed calculation, see [Wat09, Sec. 4.2].

Proof Sketch for Lemma 4

Proof Sketch:

- In Lemma 1, $|\gamma_0\rangle = |\psi\rangle_{\mathsf{W}} |0\rangle_{\mathsf{X}}$ is no longer an evec. of $\mathbf{M}$
  - The reason: $|\psi\rangle_{\mathsf{W}}$ is not an evec. of $\mathbf{Q}$
- (mental exper.) Thus, decomp. $|\psi\rangle$ in the evecs $\{|\psi_i\rangle\}_{i\in[\mathsf{dim}]}$ of $\mathbf{Q}$
- (mental exper.) For each i, we obtain Lemmas 1 and 2
- (mental exper.) In the Marriot-Watrous procedure, in each egein space:
$$\mathbf{VT}|\psi_i\rangle_{\mathsf{W}}|0\rangle_{\mathsf{X}} = \sqrt{p(|\psi_i\rangle)}\,|\delta_0(|\psi_i\rangle)\rangle + \sqrt{1-p(|\psi_i\rangle)}\,|\delta_1(|\psi_i\rangle)\rangle$$
- (mental exper.) Define a unitary $\mathbf{N}$ such that for all $i \in [\mathsf{dim}]$:
$$\sqrt{p(|\psi_i\rangle)}\,|\delta_0(|\psi_i\rangle)\rangle + \sqrt{1-p(|\psi_i\rangle)}\,|\delta_1(|\psi_i\rangle)\rangle \to \sqrt{q}\,|\delta_0(|\psi_i\rangle)\rangle + \sqrt{1-q}\,|\delta_1(|\psi_i\rangle)\rangle$$
- (mental exper.) Ready to apply the Exact Rewinding Lemma 3 (w/ p_0 as we don't know p.) (Need $p_0(1-p_0) \leq q(1-q)$.)

In summary, this is a Sim w/ an imaginary operator $\mathbf{N}$, giving the same trace bound as in Lemma 3. But for the real Sim, there is no $\mathbf{N}$.

- Doesn't matter. $\mathbf{N}$ only affects the trace bound negligibly.
- By tedious-yet-elementary linear algebra (see [Wat09, Sec. 4.2]).

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