Minimum Spanning Trees

Problem

- Given a connected undirected weighted graph (G, w) with $G = (V, E)$, the goal of the minimum spanning tree (MST) problem is to **find a spanning tree of the smallest cost**.

- How to implement Prim's algorithm in $\mathbf{O((|V| + |E|) \cdot \log|V|)}$ **time**?

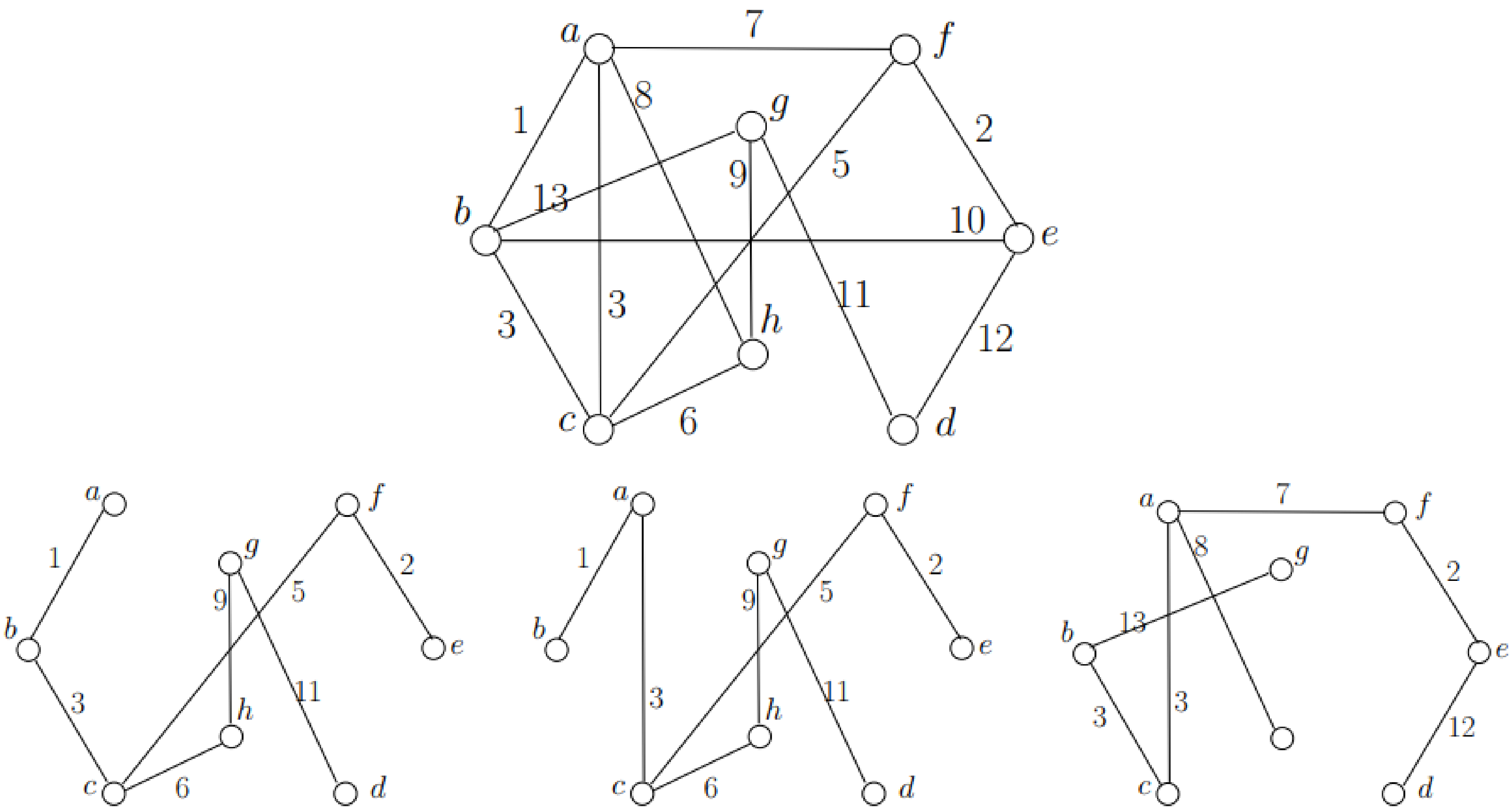
Let $G = (V, E)$ be a connected undirected graph. Let w be a function that maps each edge e of G to a positive integer $w(e)$ called the weight of e.

A spanning tree T is a **tree** satisfying the following conditions:

- The vertex set of T is V.
- Every edge of T is an edge in G.

The **cost** of T is the sum of the weights of all the edges in T.

Example



The second row shows three spanning trees. The cost of the first two trees is 37, and that of the right tree is 48.

Prim's algorithm

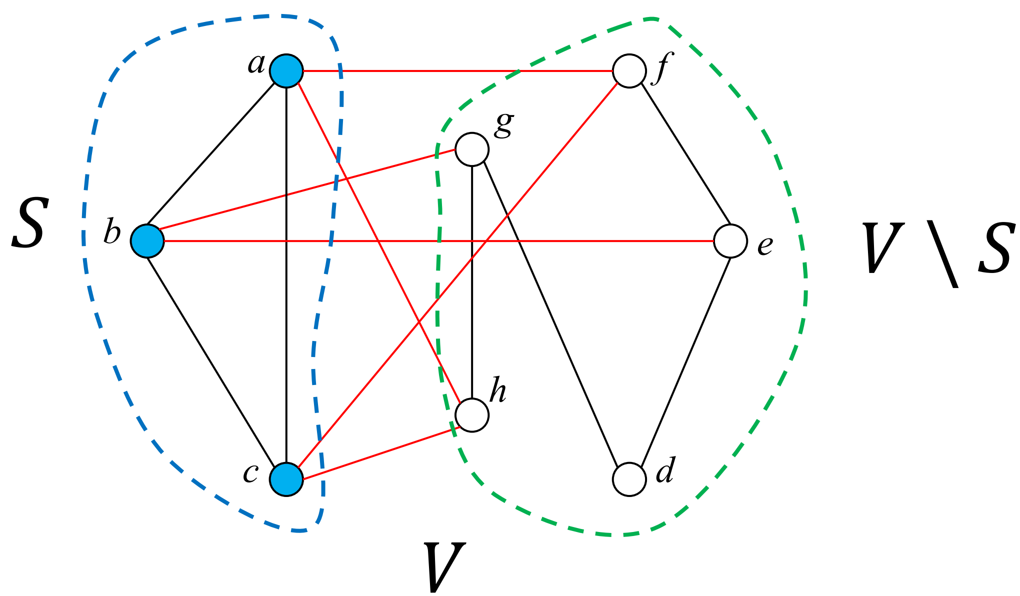
The algorithm grows a tree T_{mst} by **including one vertex at a time**.

At any moment, it divides the vertex set V into two parts:

- The set S of vertices that are already in T_{mst}.
- The set of other vertices: $V \setminus S$.

At the end of the algorithm, $\boldsymbol{S = V}$.

If an edge connects a vertex in V and a vertex in $V \setminus S$, we call it an **cross edge**.



Implementing Prim's algorithm

To implement the algorithm efficiently, we will **enforce the following invariant**:

- For every vertex $v \in V \setminus S$, refer to the cross edge of v with the smallest weight as the **lightest cross edge** of v and denote it as ***best-cross*(*v*)**.

Implementing Prim's algorithm

1. $\{u, v\} \leftarrow$ the edge with the smallest weight among all edges.

2. $S \leftarrow \{u, v\}$. Initialize a tree T_{mst} with only one edge $\{u, v\}$.

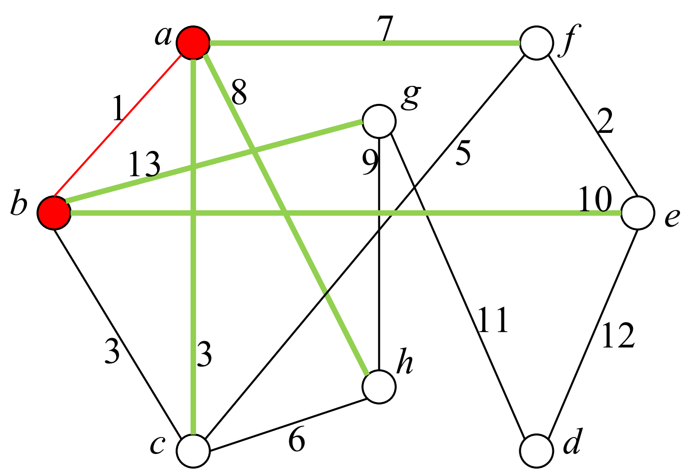
3. Enforce our invariant:

   For every vertex z of $V \setminus S$

   - $\mathit{best\text{-}cross}(z) \leftarrow$ the lighter edge between $\{z, u\}$ and $\{z, v\}$.
   - If an edge does not exist, treat its weight as infinity.

Example

Edge $\{a, b\}$ is the lightest of all. So, in the beginning $S = \{a, b\}$. The MST now has one edge $\{a, b\}$.



vertex v	best-cross and weight
a	n / a
b	n / a
c	{c, a}, 3
d	nil, ∞
e	{e, b}, 10
f	{a, f}, 7
g	{g, b}, 13
h	{a, h}, 8

Implementing Prim's algorithm

4. Repeat the following until $S = V$:

    5. Find a cross edge $\{u, v\}$ with the smallest weight.

        /* Without loss of generality, suppose $u \in S$ and $v \notin S$ */

    6. Add v into S, and add edge $\{u, v\}$ into T_{mst}.

        /* Next, restore the invariant. */

    7. Enforce the invariant again:

        For every edge $\{v, z\}$ of v

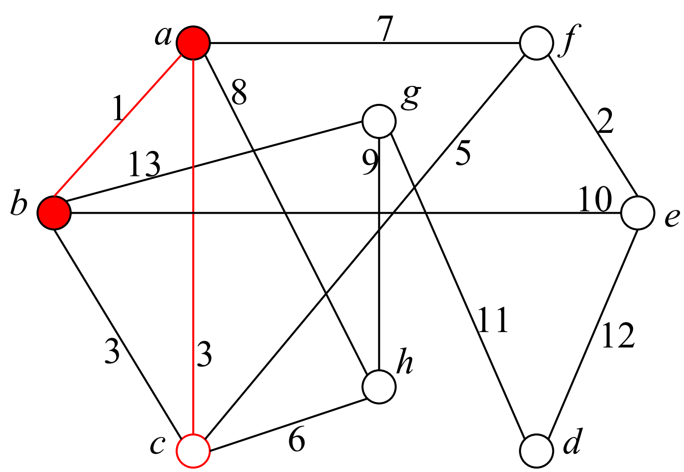
            If $z \notin S$ then

                If *best-cross*(z) is heavier than edge $\{v, z\}$ then

                    Set *best-cross*(z) as edge $\{v, z\}$.

Example

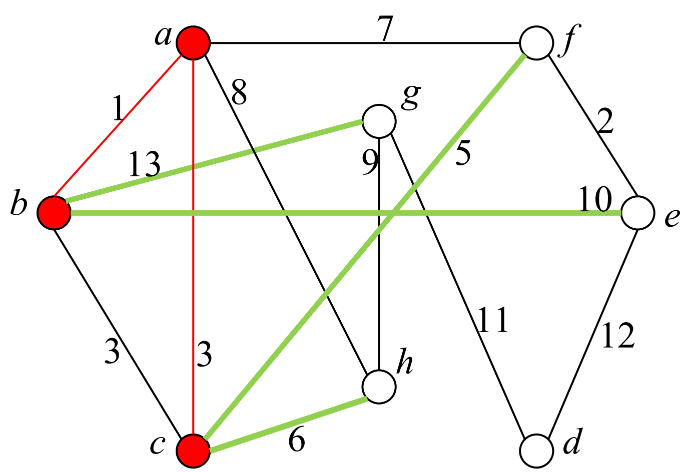
Edge $\{c, a\}$ is the lightest cross edge. So, we add **c** to S, which is now $S = \{a, b, c\}$. Add edge $\{c, a\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	{c, a}, 3
d	nil, ∞
e	{e, b}, 10
f	{a, f}, 7
g	{g, b}, 13
h	{a, h}, 8

Example

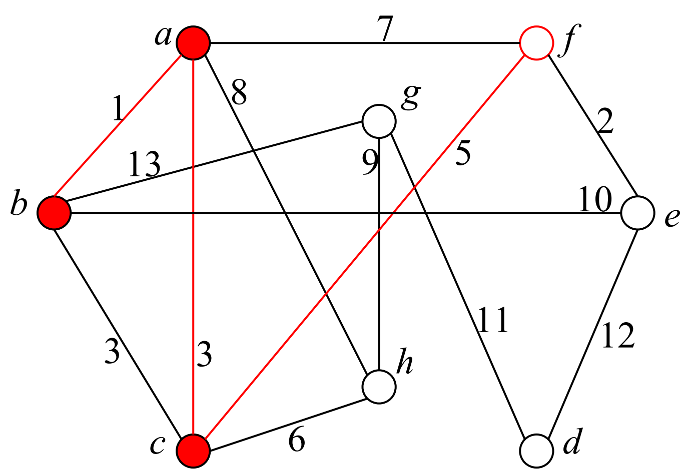
Restore the invariant.



vertex v	best-cross and weight
a	n / a
b	n / a
c	**{c, a}, 3 => n / a**
d	nil, ∞
e	{e, b}, 10
f	**{a, f}, 7 => {c, f}, 5**
g	{g, b}, 13
h	**{a, h}, 8 => {c, h}, 6**

Example

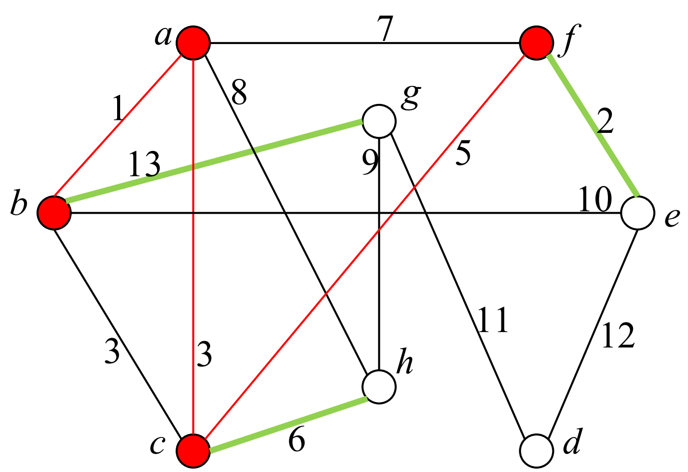
Edge $\{c, f\}$ is the lightest cross edge. So, we add f to S, which is now $S = \{a, b, c, f\}$. Add edge $\{c, f\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	nil, ∞
e	{e, b}, 10
f	{c, f}, 5
g	{g, b}, 13
h	{c, h}, 6

Example

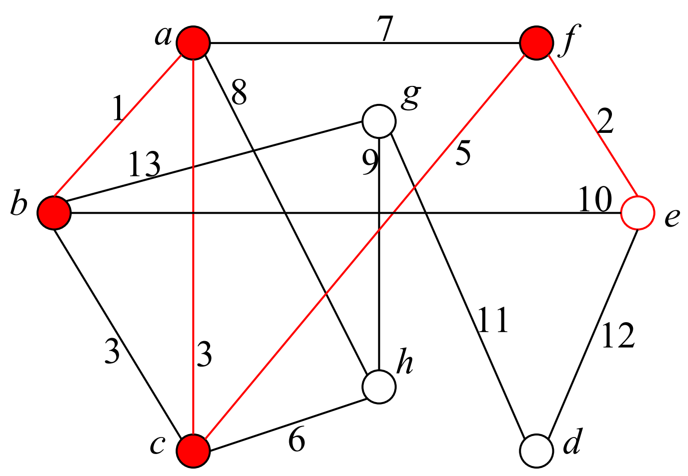
Restore the invariant.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	nil, ∞
e	**{e, f}, 2**
f	**n / a**
g	{g, b}, 13
h	{c, h}, 6

Example

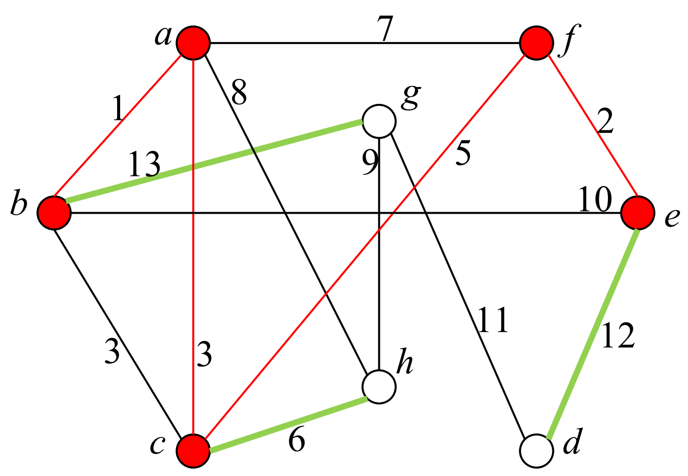
Edge $\{e, f\}$ is the lightest cross edge. So, we add ***e*** to S, which is now $S = \{a, b, c, f, e\}$. Add edge $\{e, f\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	nil, ∞
e	{e, f}, 2
f	n / a
g	{g, b}, 13
h	{c, h}, 6

Example

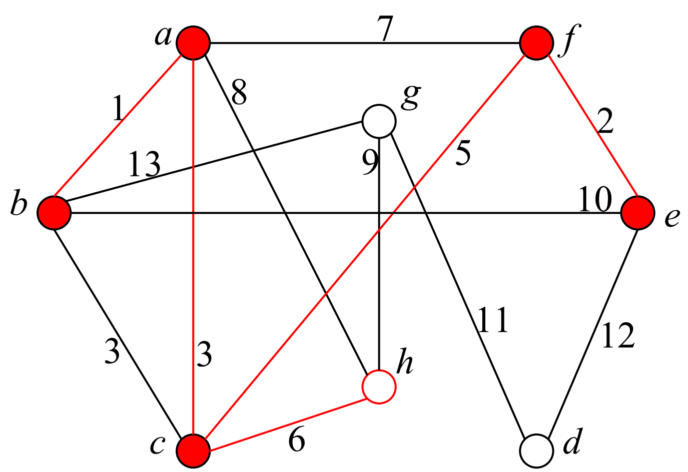
Restore the invariant.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	**{e, d}, 12**
e	**n / a**
f	n / a
g	{g, b}, 13
h	{c, h}, 6

Example

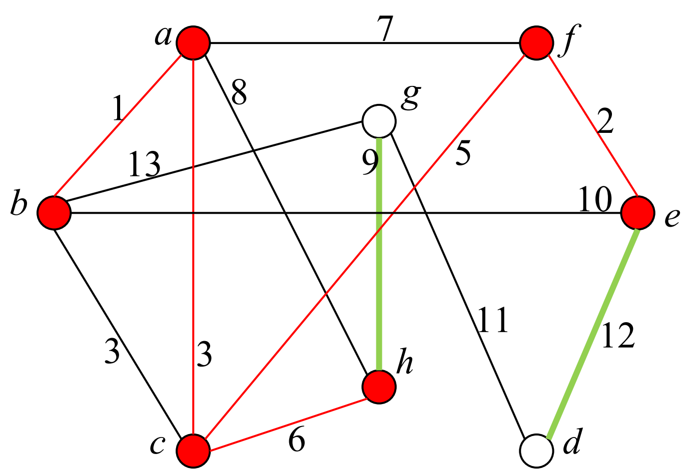
Edge $\{c, h\}$ is the lightest cross edge. So, we add $\boldsymbol{h}$ to S, which is now $S = \{a, b, c, f, e, h\}$. Add edge $\{c, h\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	{e, d}, 12
e	n / a
f	n / a
g	{g, b}, 13
h	{c, h}, 6

Example

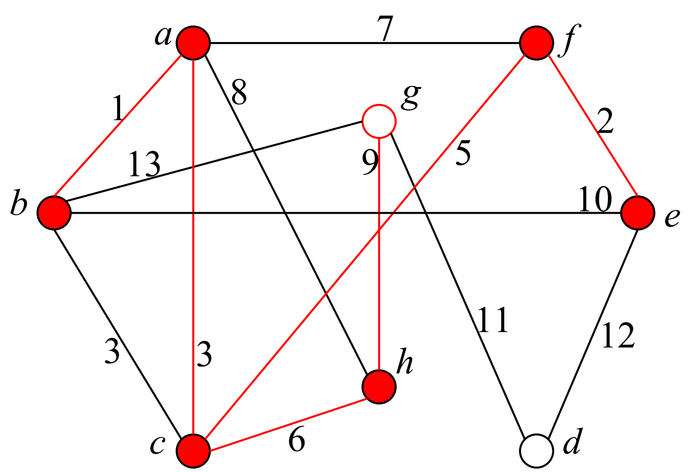
Restore the invariant.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	{e, d}, 12
e	n / a
f	n / a
g	**{g, h}, 9**
h	**n / a**

Example

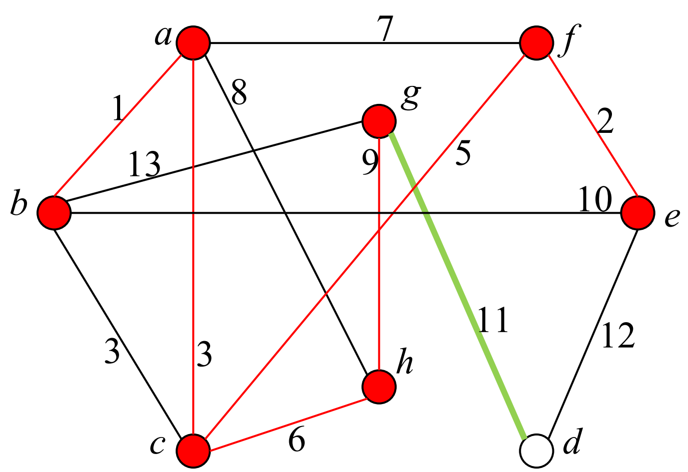
Edge $\{g, h\}$ is the lightest cross edge. So, we add g to S, which is now $S = \{a, b, c, f, e, h, g\}$. Add edge $\{g, h\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	{e, d}, 12
e	n / a
f	n / a
g	{g, h}, 9
h	n / a

Example

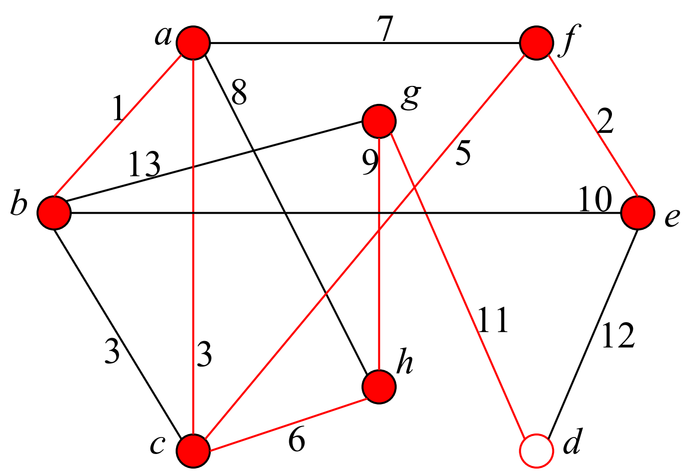
Restore the invariant.



vertex v	best-cro(v) and weight
a	n / a
b	n / a
c	n / a
d	**{d, g}, 11**
e	n / a
f	n / a
g	**n / a**
h	n / a

Example

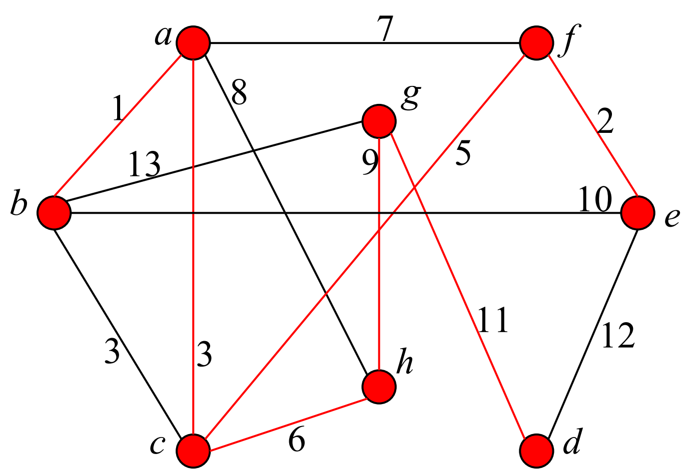
Finally, **edge $\{d, g\}$** is the lightest cross edge. So, we add **d** to S, which is now $S = \{a, b, c, f, e, h, g, d\}$. Add edge $\{d, g\}$ into the MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	{d, g}, 11
e	n / a
f	n / a
g	n / a
h	n / a

Example

We have obtained our final MST.



vertex v	best-cross and weight
a	n / a
b	n / a
c	n / a
d	n / a
e	n / a
f	n / a
g	n / a
h	n / a

Data structure

For a fast implementation, we need a good data structure.

Let P be a set of n tuples of the form $(id, weight, data)$. Design a data structure to support the following operations:

- ✓ **Find**: given an integer t, find the tuple $(id, weight, data)$ from P where $t = id$; return nothing if the tuple does not exist.
- ✓ **Insert**: add a new tuple $(id, weight, data)$ to P.
- ✓ **Delete**: given an integer t, delete the tuple $(id, weight, data)$ from P where $t = id$.
- ✓ **DeleteMin**: remove from P the tuple with the smallest weight.

We can build this structure so it requires $O(n)$ space and supports all these essential operations in **$\mathbf{O(\log \boldsymbol{n})}$ time**.

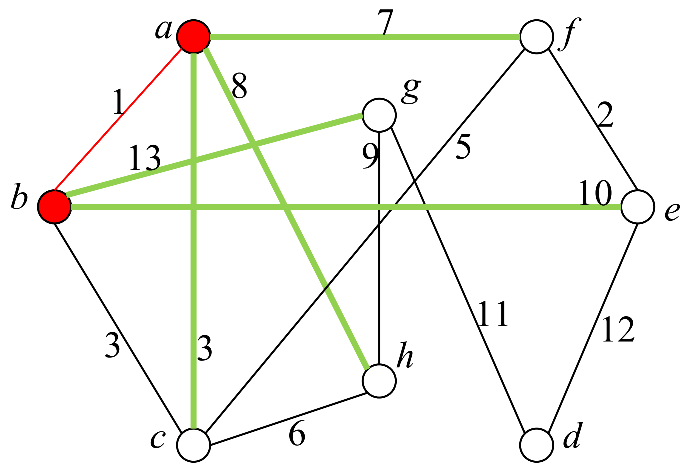
Data structure

Maintain P in two binary search trees T_1 and T_2, where the tuples are **indexed on ids** in T_1, and **on weights** in T_2. It supports the following operations:

- ✓ **Find**: search the tuple in T_1.
- ✓ **Insert**: insert the new tuple into both T_1 and T_2.
- ✓ **Delete**: first find the tuple with id t in T_1, from which we know the weight. Now, delete the tuple from both T_1 and T_2.
- ✓ **DeleteMin**: find the tuple with the smallest weight from T_2(which can be found by continuously descending into left child nodes). Now we have its id t as well. Remove the tuple from both T_1 and T_2.

Example

Edge $\{a, b\}$ is the lightest of all. $S = \{a, b\}$.



P

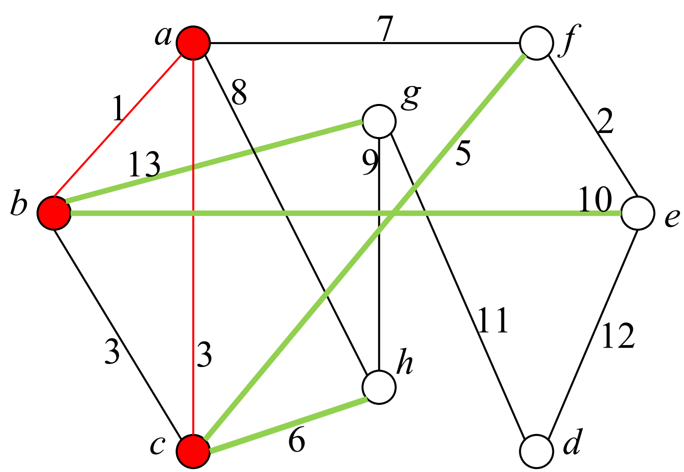
vertex	weight	best-cross
c	3	{c, a}
d	∞	nil
e	10	{e, b}
f	7	{a, f}
g	13	{g, b}
h	8	{a, h}

6 (*id*, *weight, data*) insertions into P.

In general, $|V| - 2$ insertions in $\boldsymbol{O(|V| \cdot \log|V|)}$ time.

Example

Edge $\{c, a\}$ is the lightest cross edge. So, we add c to S, which is now $S = \{a, b, c\}$. Add edge $\{c, a\}$ into the MST.



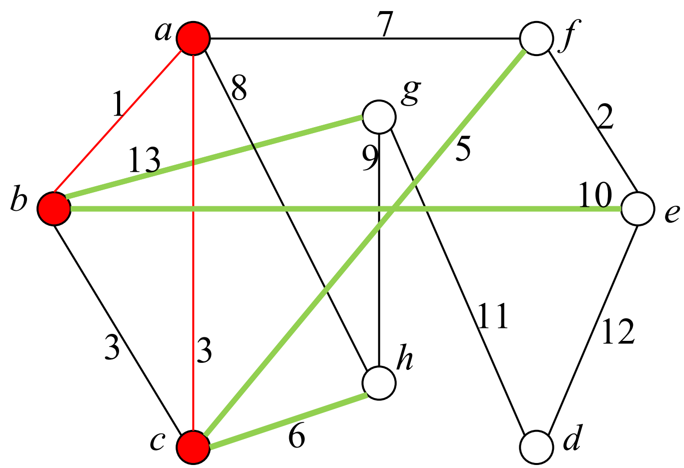
P

vertex	weight (key)	best-cross
~~c~~	~~3~~	~~{c, a}~~
d	∞	nil
e	10	{e, b}
f	7	{a, f}
g	13	{g, b}
h	8	{a, h}

Perform DeleteMin to obtain $\{c, a\}$ in $\mathbf{O(\log|V|)}$ time.

Example

Restore the invariant.



P

vertex	weight	best-cross
d	∞	nil
e	10	{e, b}
f	**7 => 5**	**{a, f} => {c, f}**
g	13	{g, b}
h	**8 => 6**	**{a, h} => {c, h}**

For edge $\{c, b\}$, perform a find op. using the id of b => b has no tuple in P.

For edge $\{c, a\}$, perform a find op. => a has no tuple in P.

For edge $\{c, f\}$, perform a find op. => f has a tuple with weight 7.

As $\{c, f\}$ is lighter, delete (f, 7, $\{a, f\}$) from P and insert (f, 5, $\{c, f\}$).

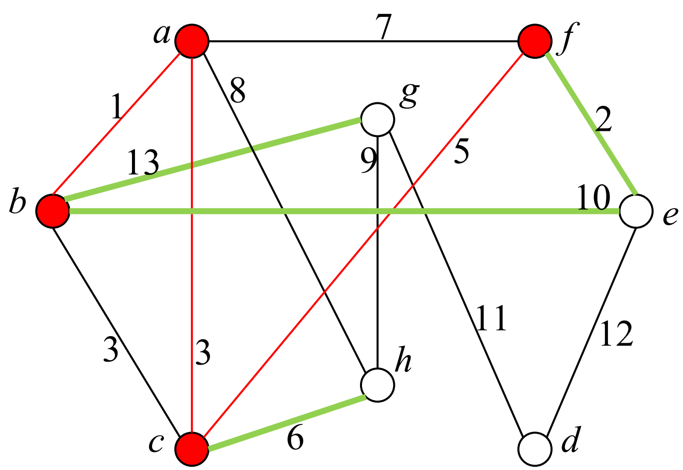
For edge $\{c, h\}$, perform a find op. => h has a tuple with weight 8.

As $\{c, h\}$ is lighter, delete (h, 8, $\{a, h\}$) from P and insert (h, 6, $\{c, h\}$).

Time: $\boldsymbol{O(d_c \log|V|)}$ time where d_c is the degree of c.

Example

Edge $\{c, f\}$ is the lightest cross edge. So, we add f to S, which is now $S = \{a, b, c, f\}$. Add edge $\{c, f\}$ into the MST.



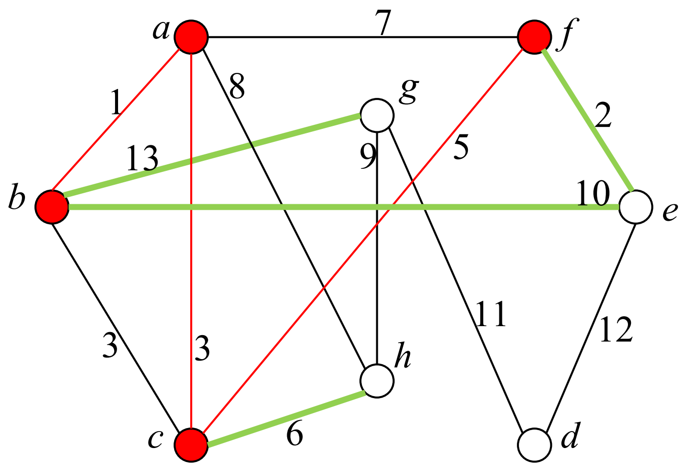
P

vertex	weight	best-cross
d	∞	Nil
e	10	{e, b}
~~f~~	~~5~~	~~{c, f}~~
g	13	{g, b}
h	6	{c, h}

Perform DeleteMin to obtain $\{f, c\}$ in $O(\log|V|)$ time.

Example

Restore the invariant.



P

vertex	weight	best-cross
d	∞	Nil
e	**10=>2**	**{e, b}=>{e, f}**
g	13	{g, b}
h	6	{c, h}

For edge $\{f, a\}$, perform a find op. using the id of a => a has no tuple in P.

For edge $\{f, c\}$, perform a find op. => c has no tuple in P.

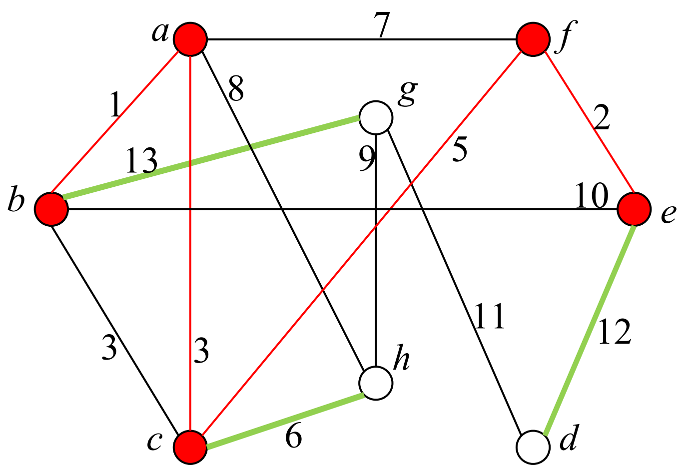
For edge $\{f, e\}$, perform a find op. => e has a tuple with weight 2.

As $\{f, e\}$ is lighter, delete $(e, 10, \{e, b\})$ from P and insert $(e, 2, \{e, f\})$.

Time: $O(d_f \log|V|)$ time where d_f is the degree of f.

Example

Edge $\{e, f\}$ is the lightest cross edge. So, we add e to S, which is now $S = \{a, b, c, f, e\}$. Add edge $\{e, f\}$ into the MST.



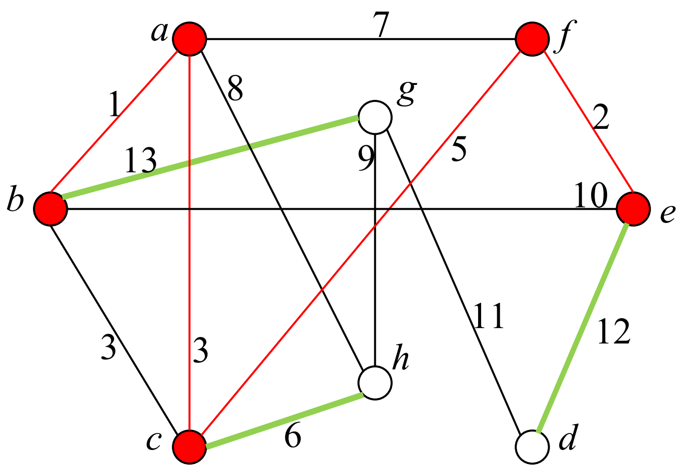
P

vertex	weight	best-cross
d	∞	Nil
~~e~~	~~2~~	~~{e, f}~~
g	13	{g, b}
h	6	{c, h}

Perform DeleteMin to obtain $\{e, f\}$ in $O(\log|V|)$ time.

Example

Restore the invariant.



P

vertex	weight	best-cross
d	**∞ => 12**	**Nil => {e,d}**
g	13	{g, b}
h	6	{c, h}

For edge $\{e, f\}$, perform a find op. using the id of f => f has no tuple in P.

For edge $\{e, b\}$, perform a find op. => b has no tuple in P.

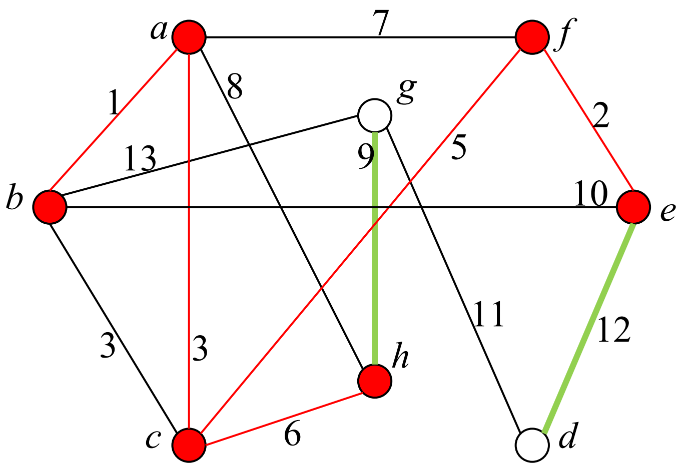
For edge $\{e, d\}$, perform a find op. => d has a tuple with weight ∞.

As $\{e, d\}$ is lighter, delete (d, ∞, Nil) from P and insert (d, 12, $\{e, d\}$).

Time: $O(d_e \log|V|)$ time where d_e is the degree of e.

Example

Edge $\{c, h\}$ is the lightest cross edge. So, we add h to S, which is now $S = \{a, b, c, f, e, h\}$. Add edge $\{c, h\}$ into the MST.



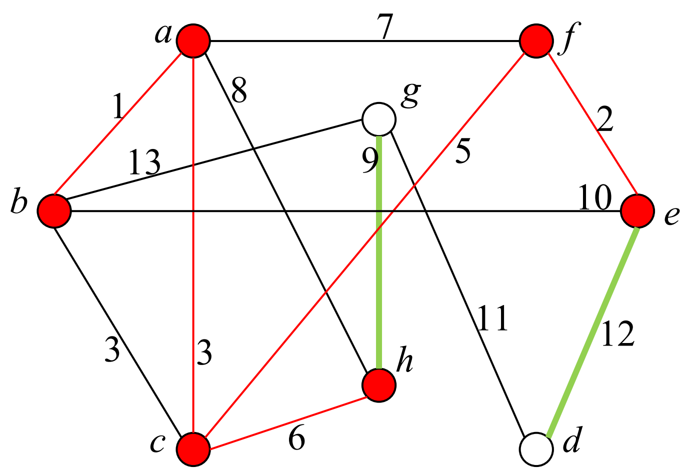
P

vertex	weight	best-cross
d	12	{e,d}
g	13	{g, b}
h	~~6~~	~~{c, h}~~

Perform DeleteMin to obtain $\{c,\ h\}$ in $O(\log|V|)$ time.

Example

Restore the invariant.



P

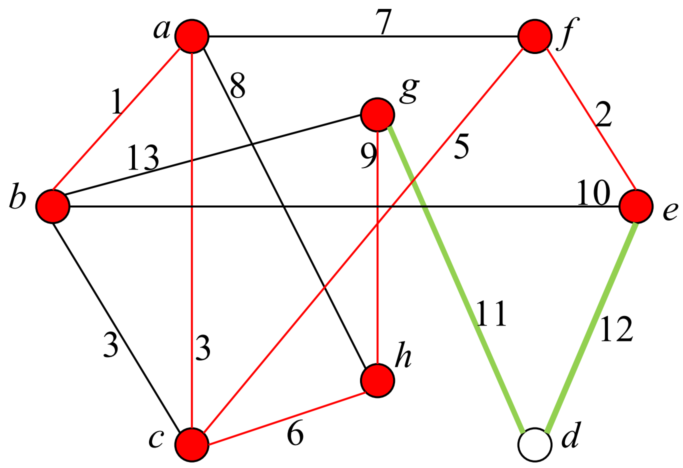
vertex	weight	best-cross
d	12	{e,d}
g	13 => 9	{g, b} => {g,h}

For edge $\{h, a\}$, perform a find op. using the id of a => a has no tuple in P.
For edge $\{h, c\}$, perform a find op. => c has no tuple in P.
For edge $\{h, g\}$, perform a find op. => g has a tuple with weight 13.
As $\{h, g\}$ is lighter, delete $(g, 13, \{g, b\})$ from P and insert $(g, 9, \{g, h\})$.

Time: $O(d_h \log|V|)$ time where d_h is the degree of h.

Example

Edge $\{g, h\}$ is the lightest cross edge. So, we add g to S, which is now $S = \{a, b, c, f, e, h, g\}$. Add edge $\{g, h\}$ into the MST.



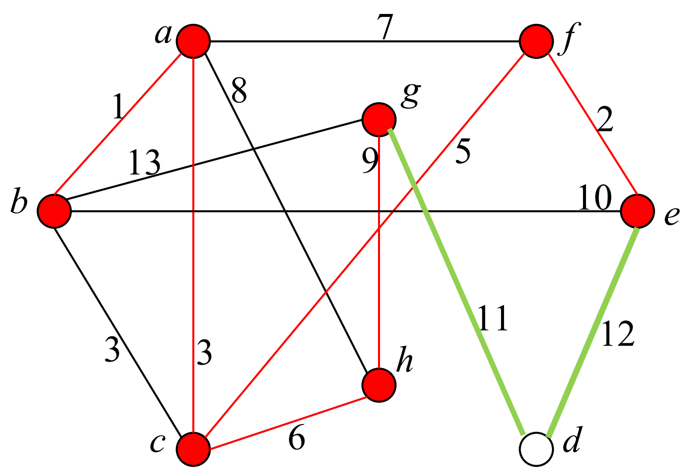
P

vertex	weight	best-cross
d	12	{e,d}
g	~~9~~	~~{g,h}~~

Perform DeleteMin to obtain $\{g, h\}$ in $O(\log|V|)$ time.

Example

Restore the invariant.



P

vertex	weight	best-cross
d	12=>11	{e,d}=>{g,d}

For edge $\{g, b\}$, perform a find op. using the id of b => b has no tuple in P.

For edge $\{g, h\}$, perform a find op. => h has no tuple in P.

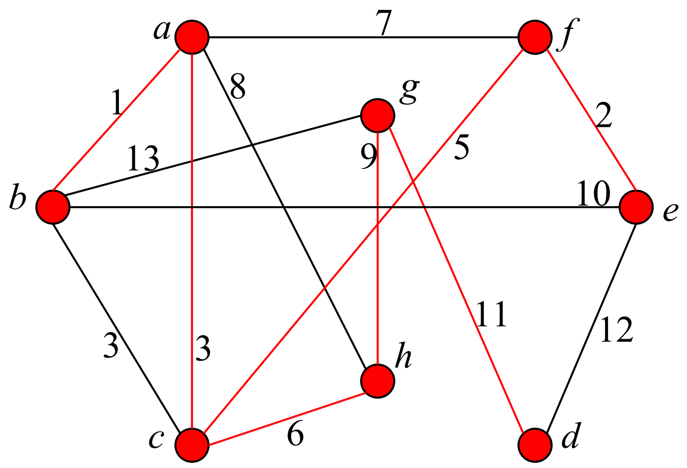
For edge $\{g, d\}$, perform a find op. => d has a tuple with weight 12.

As $\{g, d\}$ is lighter, delete $(d, 12, \{e, d\})$ from P and insert $(g, 11, \{g, d\})$.

Time: $O(d_g \log|V|)$ time where d_g is the degree of g.

Example

Finally, edge $\{g, d\}$ is the lightest cross edge. So, we add d to S, which is now $S = \{a, b, c, f, e, h, g, d\}$. Add edge $\{g, d\}$ into the MST.



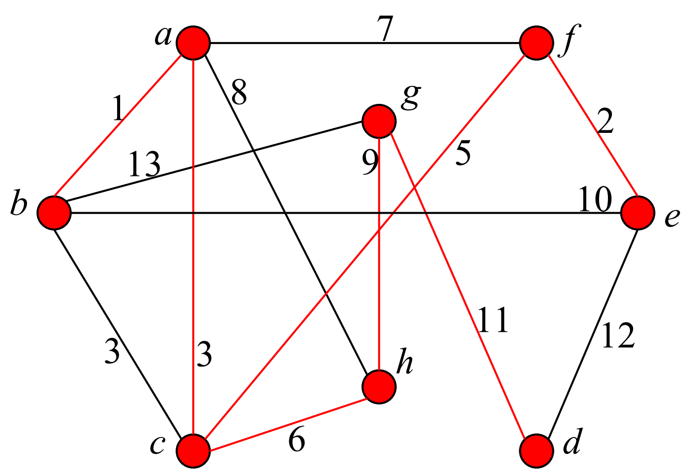
P

vertex	weight	best-cross
d	~~11~~	~~{g,d}~~

Perform DeleteMin to obtain $\{g, d\}$ in $O(\log|V|)$ time.

Example

We have obtained our final MST.



Total time:

$O(|V| \cdot \log|V| + \sum_{v\in V} \log|V| + \sum_{v\in V} d_v \log|V|)$
$= O((2|V| + 2|E|) \cdot \log|V|)$
$= O((|V| + |E|) \cdot \log|V|)$