CSCI3160: Tutorial 3

- Problem 1
  - $O(n\log n)$-time algorithm for finding the number of inversions.
- Problem 2
  - $O(n\log n)$-time algorithm to solve the dominance counting problem.

Review: Counting inversions

- Problem: Given an array A of n distinct integers, count the number of inversions.
- An inversion is a pair of (i, j) such that
  - $1 \leq i < j \leq n$.
  - $A[i] > A[j]$.

Example: Consider $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$.
Then $(1, 2)$ is an inversion because $A[1] = 10 > A[2] = 3$. So are $(1, 3), (3, 4), (4, 5)$, and so on.
There are in total 31 inversions.

Review: Counting inversions

- Let: $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$
  - $A_1 = (10, 3, 9, 8, 2)$, $A_2 = (5, 4, 1, 7, 6)$.
  - The counts of inversions in A_1 and A_2 are known by solving the "counting inversion" problem recursively on A_1 and A_2.

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- We need to count the number of crossing inversion (i, j) where i is in A_1 and j in A_2.

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- We need to count the number of crossing inversion (i, j) where i is in A_1 and j in A_2.
- Binary search
  - Sort A_1 and A_2, and conduct $n/2$ binary searches ($O(n\log n)$).

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- We need to count the number of crossing inversion (i, j) where i is in A_1 and j in A_2.
- Binary search
  - Sort A_1 and A_2, and conduct $n/2$ binary searches ($O(n\log n)$).
  - Let $f(n)$ be the worst-case running time of the algorithm on n numbers.
    - ✓ $f(n) \leq 2f(\lceil n/2 \rceil) + O(n\log n)$
    - ✓ which solves to $f(n) = O(n\log^2 n)$.

Counting inversions: a faster algorithm

- Strategy: ask a harder question, and exploit it in the conquer phase.

Counting inversions and sorting

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- Given an array A of n distinct integers, output the number of inversions and produce an array to store the integers of A in ascending order.

Counting inversions and sorting

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- $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$
  - $A_1 = (2,3,8,9,10)$, 8 invs; $A_2 = (1,4,5,6,7)$, 4 invs.

Counting inversions and sorting

- Strategy: ask a harder question, and exploit it in the conquer phase.
- Given an array A of n distinct integers, output the number of inversions and produce an array to store the integers of A in ascending order.
- $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$
  - $A_1 = (2,3,8,9,10)$, 8 invs; $A_2 = (1,4,5,6,7)$, 4 invs.
- Exploit subproblem property
  - Subarrays A_1, A_2 are sorted
    - Count crossing inversions in O(n) time.
    - Merge 2 sorted arrays in O(n) time.

Counting crossing inversions

- Let S_1 and S_2 be two disjoint sets of n integers. Assume that S_1 is stored in an array A_1, and S_2 in an array A_2. Both A_1 and A_2 are sorted in ascending order. Design an algorithm to find the number of such pairs (a, b) satisfying the following conditions:
  - ✓ $a \in S_1$,
  - ✓ $b \in S_2$,
  - ✓ $a > b$.
  - ✓ Your algorithm must finish in O(n) time.

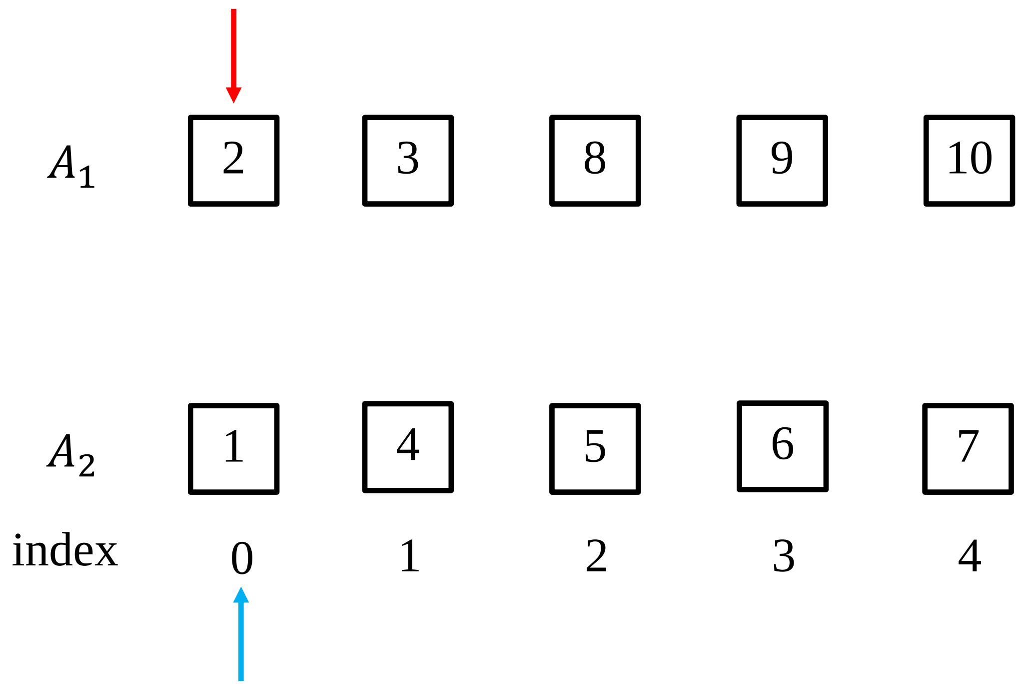
Counting crossing inversions

- Method
  - Merge A_1 and A_2 into one sorted list A.
- Let: $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$
  - $A_1 = (2,3,8,9,10)$, $A_2 = (1,4,5,6,7)$

A_1	2	3	8	9	10
A_2	1	4	5	6	7

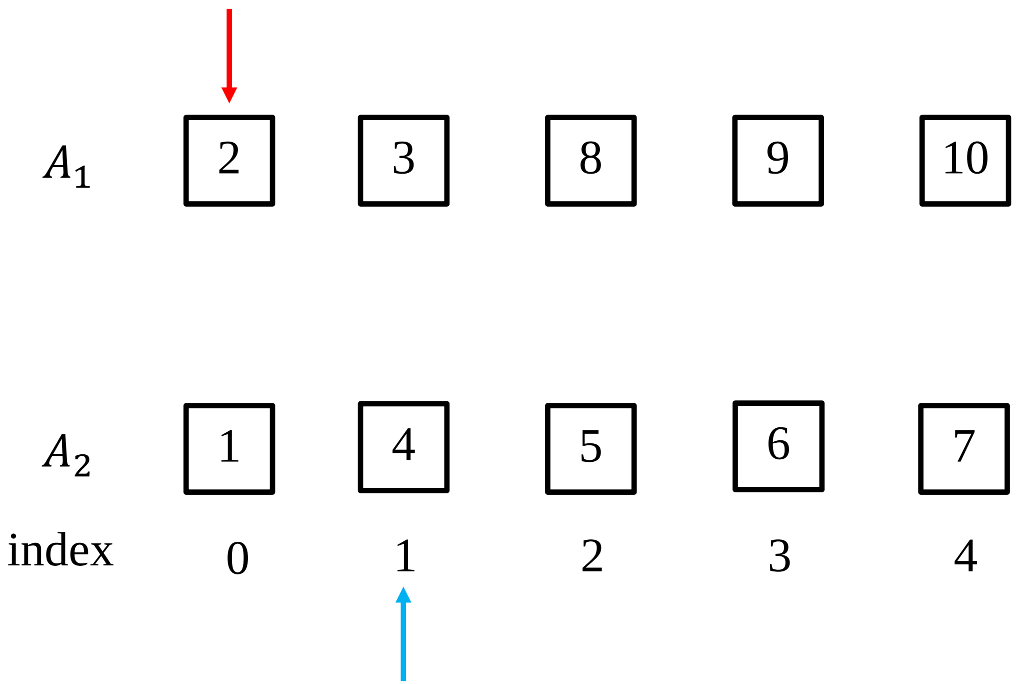
- We will merge them together and in the meantime maintain the count of crossing inversions.

Counting crossing inversions



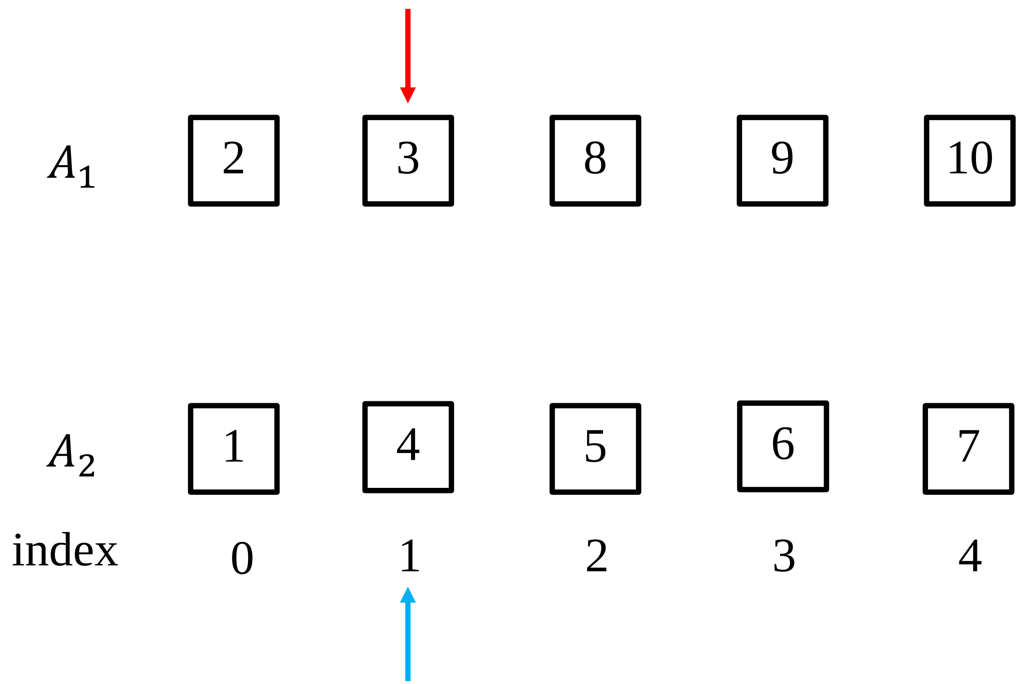
- Ordered list produced: Nothing yet
- The count of crossing inversions : 0

Counting crossing inversions



- Ordered list produced: 1
- The count of crossing inversions : 0

Counting crossing inversions

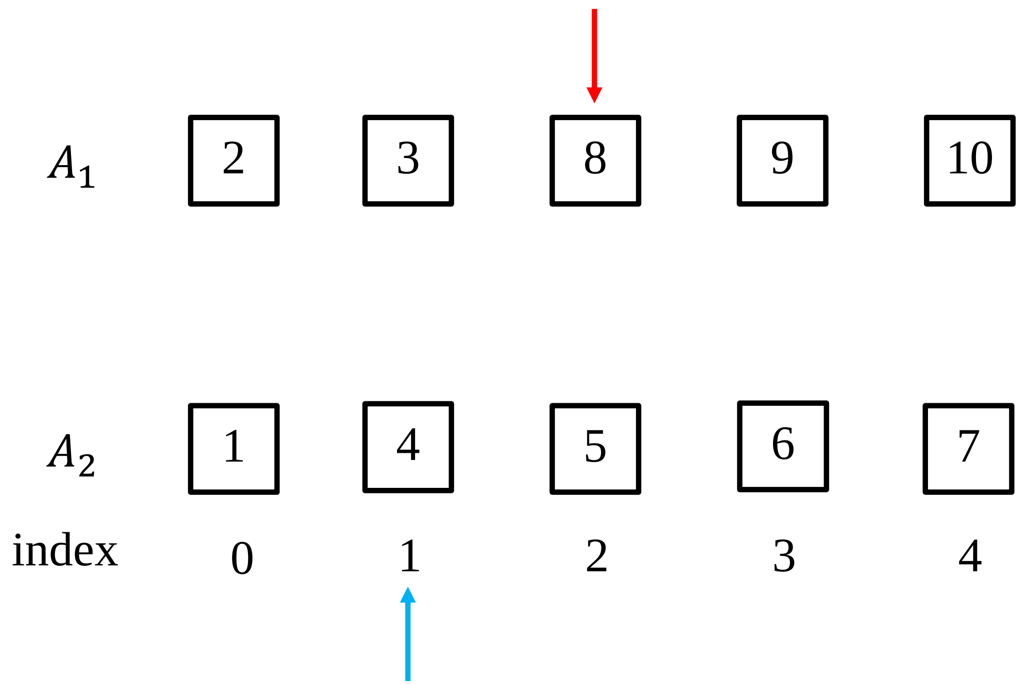


- Ordering produced: 1, 2
- The count of crossing inversions : $0 + 1 = 1.$

Last count

Newly added: (2,1) is a crossing inversion

Counting crossing inversions

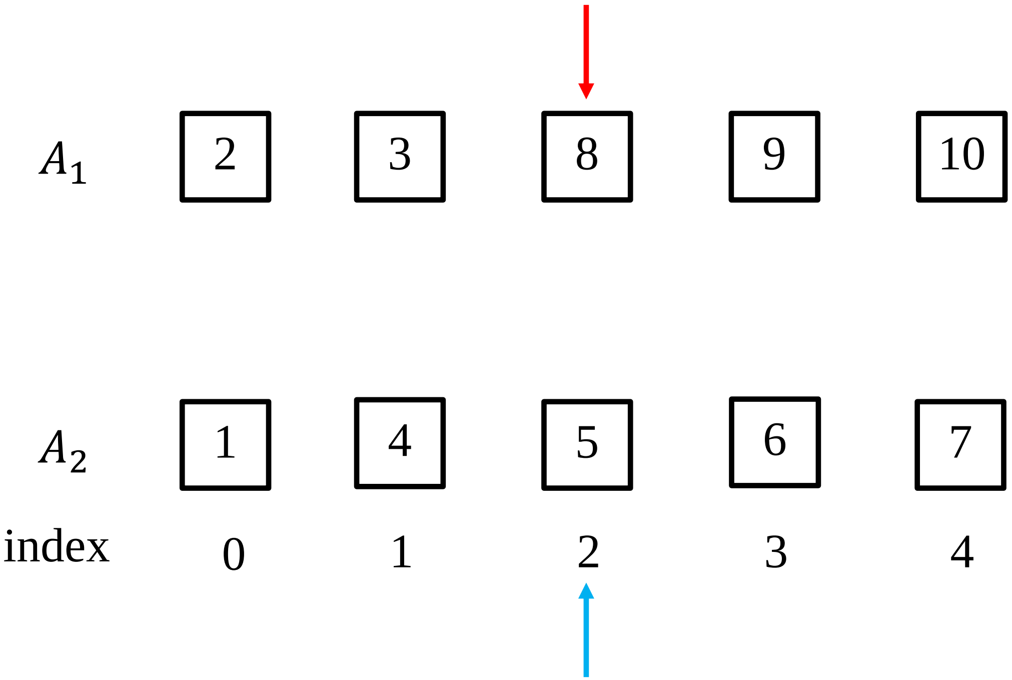


- Ordering produced: 1, 2, 3
- The count of crossing inversions : $1 + 1 = 2$.

Last count

Newly added: (3,1) is a crossing inversion.

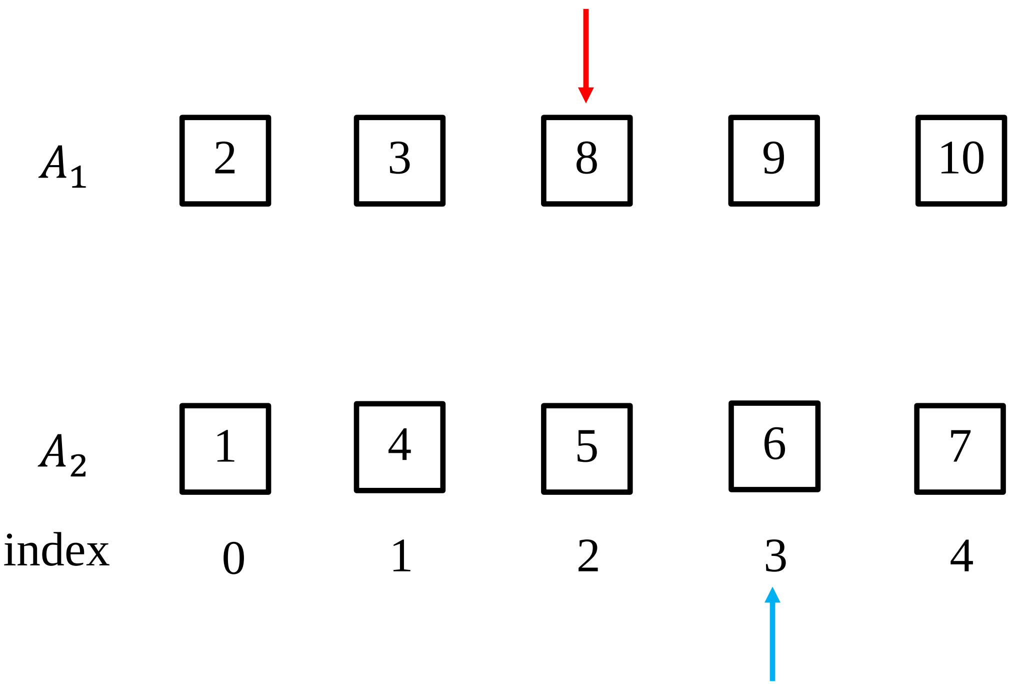
Counting crossing inversions



- Ordering produced: 1, 2, 3, 4
- The count of crossing inversions : 2

Last count

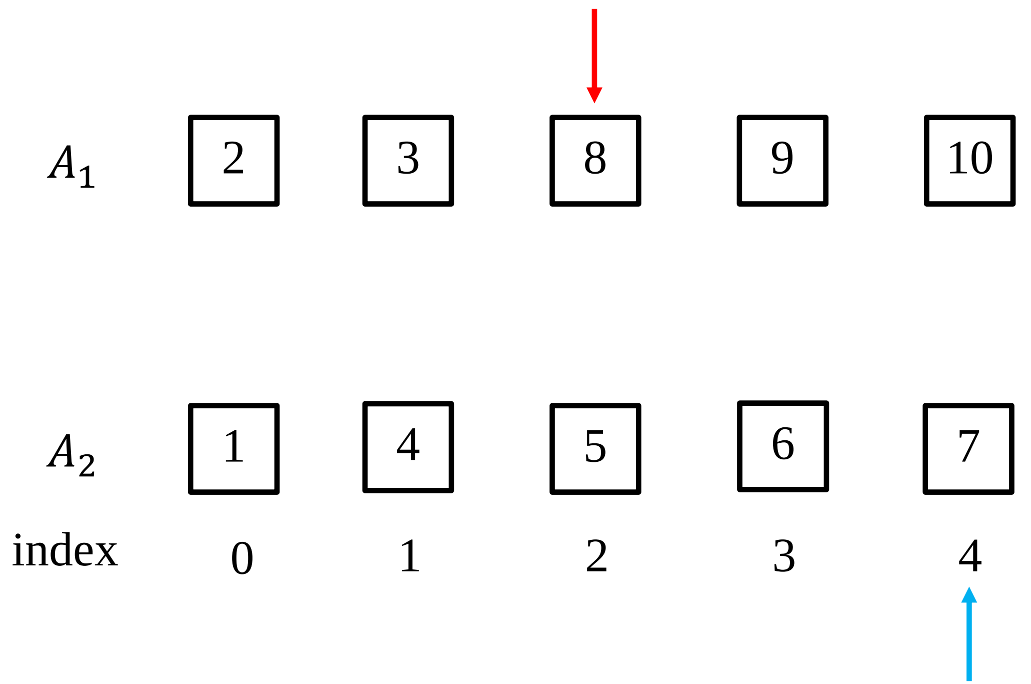
Counting crossing inversions



- Ordering produced: 1, 2, 3, 4, 5
- The count of crossing inversions : 2

Last count

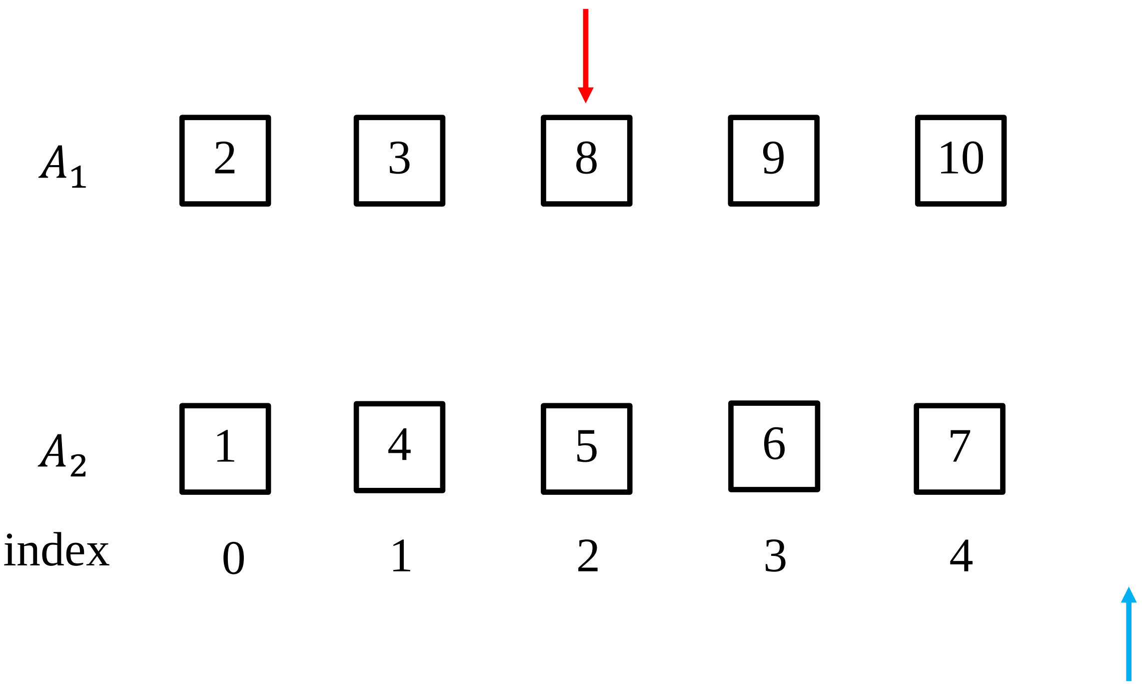
Counting crossing inversions



- Ordering produced: 1, 2, 3, 4, 5, 6
- The count of crossing inversions : 2.

Last count

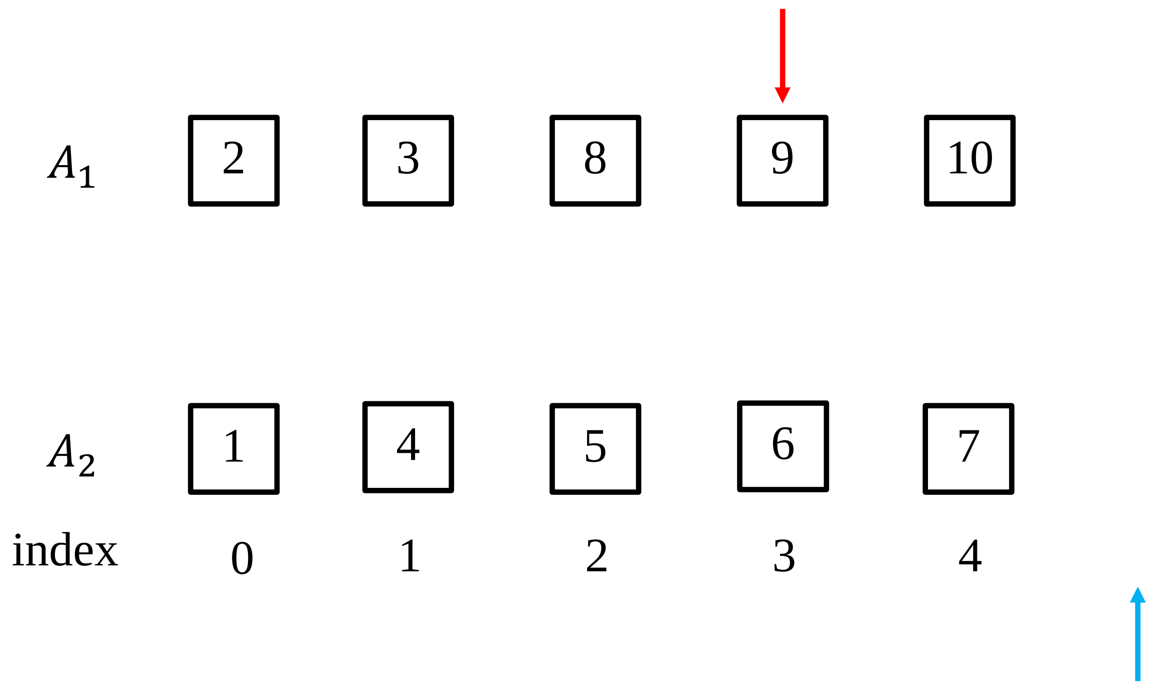
Counting crossing inversions



- Ordering produced: 1, 2, 3, 4, 5, 6, 7
- The count of crossing inversions : 2

Last count

Counting crossing inversions

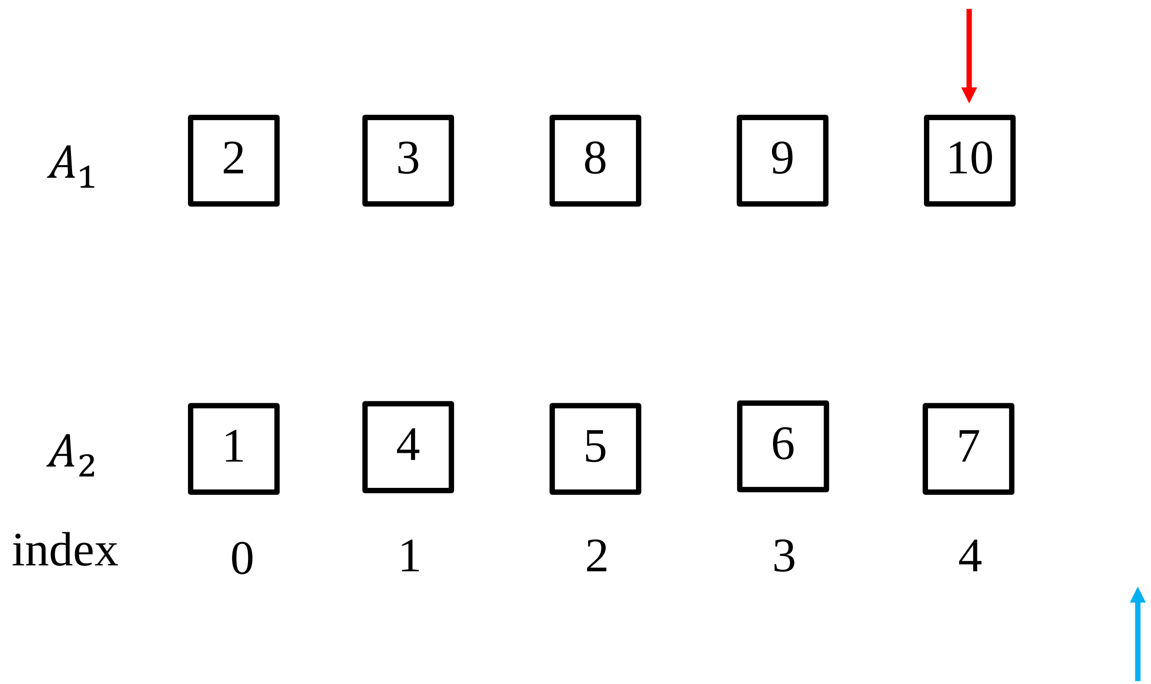


- Ordering produced: 1, 2, 3, 4, 5, 6, 7, 8
- The count of crossing inversions : $2 + 5 = 7$.

Last count

Newly added count:
(8,1), (8,4), (8,5), (8,6), (8,7)

Counting crossing inversions



- Ordering produced: 1, 2, 3, 4, 5, 6, 7, 8, 9
- The count of crossing inversions : $7 + 5 = 12.$

Last count

Newly added count:
(9,1), (9,4), (9,5), (9,6), (9,7)

Counting crossing inversions

A_1	2	3	8	9	10
A_2	1	4	5	6	7
index	0	1	2	3	4

- Ordering produced: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
- The count of crossing inversions : $12 + 5 = 17$.

Last count

Newly added count: #integers from A_2 already in the ordered list produced

Counting inversions

- Analysis
  - Let $f(n)$ be the worst-case running time of the algorithm on n numbers.

Then

- $f(n) \leq 2f(\lceil n/2 \rceil) + O(n)$,
- which solves to $f(n) = O(n\log n)$.

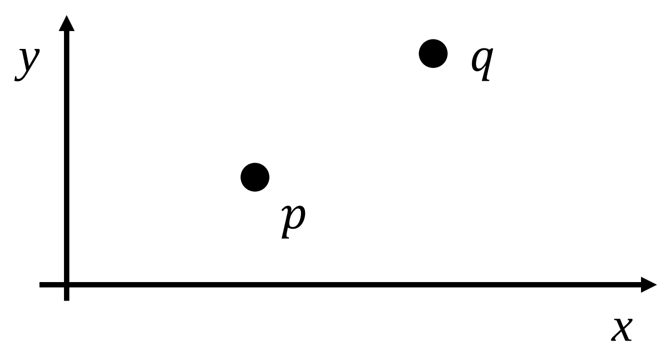
Dominance counting

Problem

- Give an $O(n\log n)$-time algorithm to solve the dominance counting problem discussed in the class.

Point dominance definition

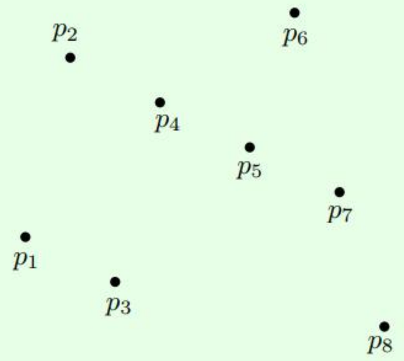
- Denote by $\mathbb{N}$ the set of integers. Given a point p in two-dimensional space $\mathbb{N}^2$, denote by $p[1]$ and $p[2]$ its x- and y-coordinates, respectively.
- Given two distinct points p and q, we say that q dominates p if $p[1] \leq q[1]$ and $p[2] \leq q[2]$.



Dominance counting

- Let P be a set of n points in $\mathbb{N}^2$. Find, for each point $p \in P$, the number of points in P that are dominated by p.

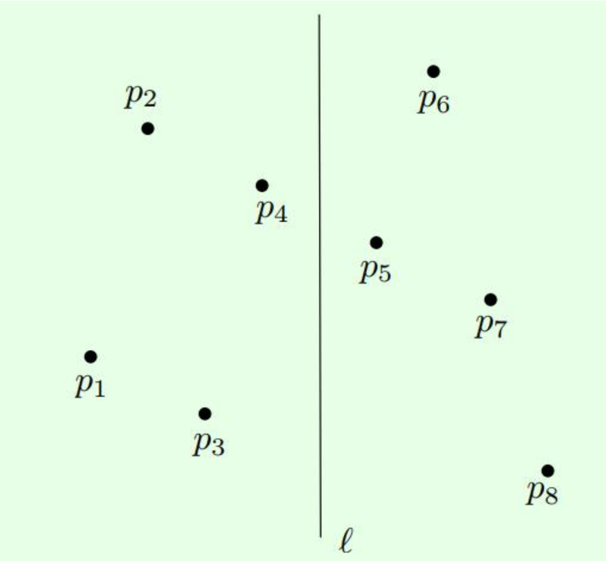
Example:



We should output: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2), (p_5, 2), (p_6, 5)$, $(p_7, 2), (p_8, 0)$.

Dominance counting

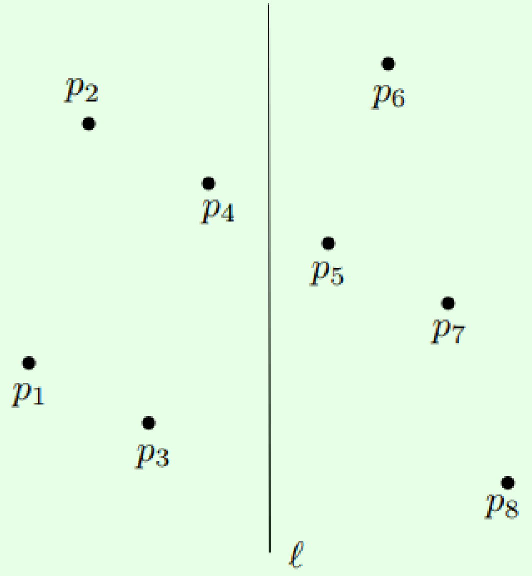
- Divide: Find a vertical line l such that P has $\lceil n/2 \rceil$ points on each side of the line. (k-selection, $O(n)$ time).



Dominance counting

- Divide:
  - P_1 = the set of points of P on the left of l.
  - P_2 = the set of points of P on the right of l.

Example:



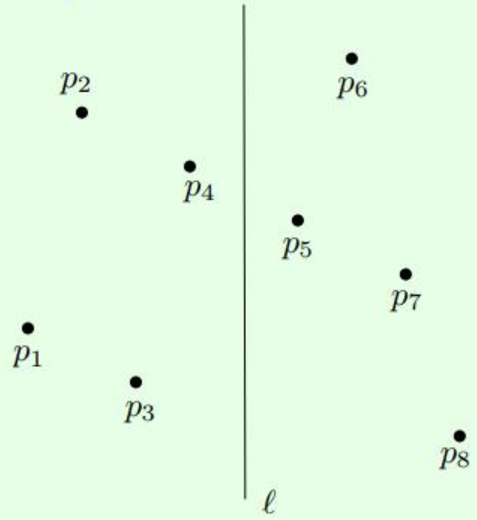
$$P_1 = \{p_1, p_2, p_3, p_4\}$$

$$P_2 = \{p_5, p_6, p_7, p_8\}.$$

Dominance counting

- Divide:
  - Solve the dominance counting problem on P_1 and P_2 separately.

Example:



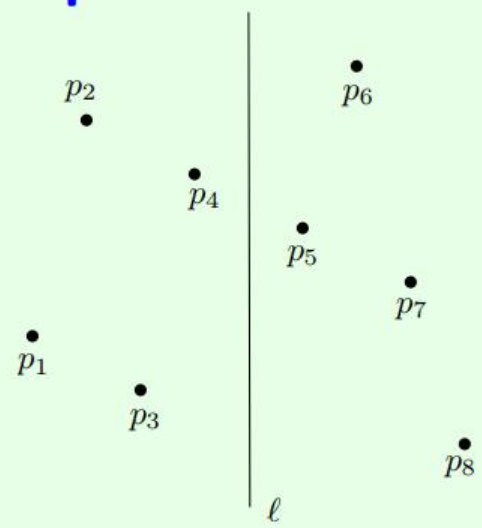
On P_1, we have obtained: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2)$.

On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

Dominance counting

- Divide:
  - Solve the dominance counting problem on P_1 and P_2 separately.
  - It remains to obtain, for each point $p \in P_2$, how many points in P_1 it dominates.

Example:



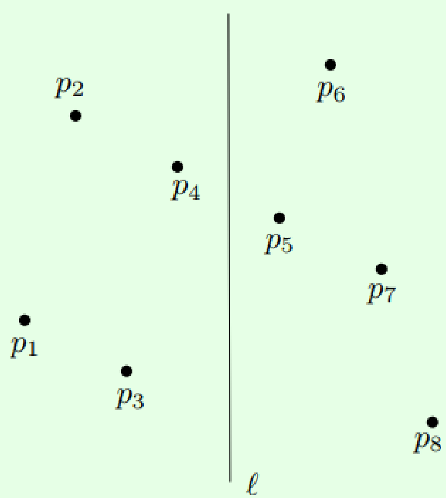
On P_1, we have obtained: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2)$.

On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

Dominance counting

- Review: Binary search
  - Sort P_1 by y-coordinate. ($O(n\log n)$)
  - Then, for each point $p \in P_2$, we can obtain the number of points in P_1 dominated by p using binary search. ($O(n\log n)$)

Example:



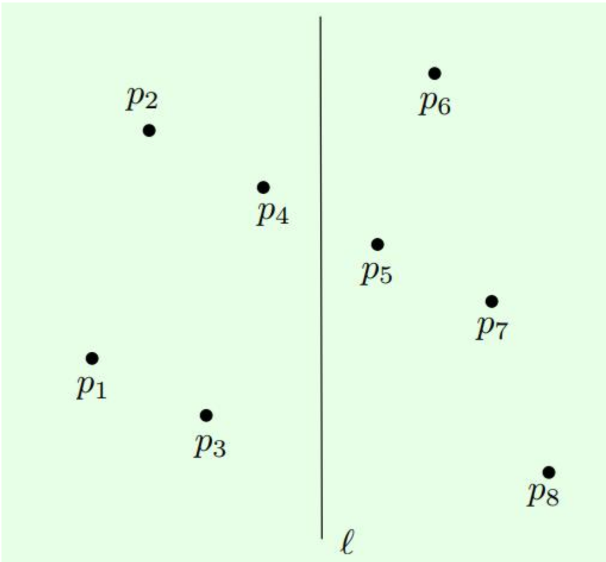
P_1 in ascending of y-coordinate: p_3, p_1, p_4, p_2.

How to perform binary search to obtain the fact that p_5 dominates 2 points in P_1?

- Search using the y-coordinate of p_5.

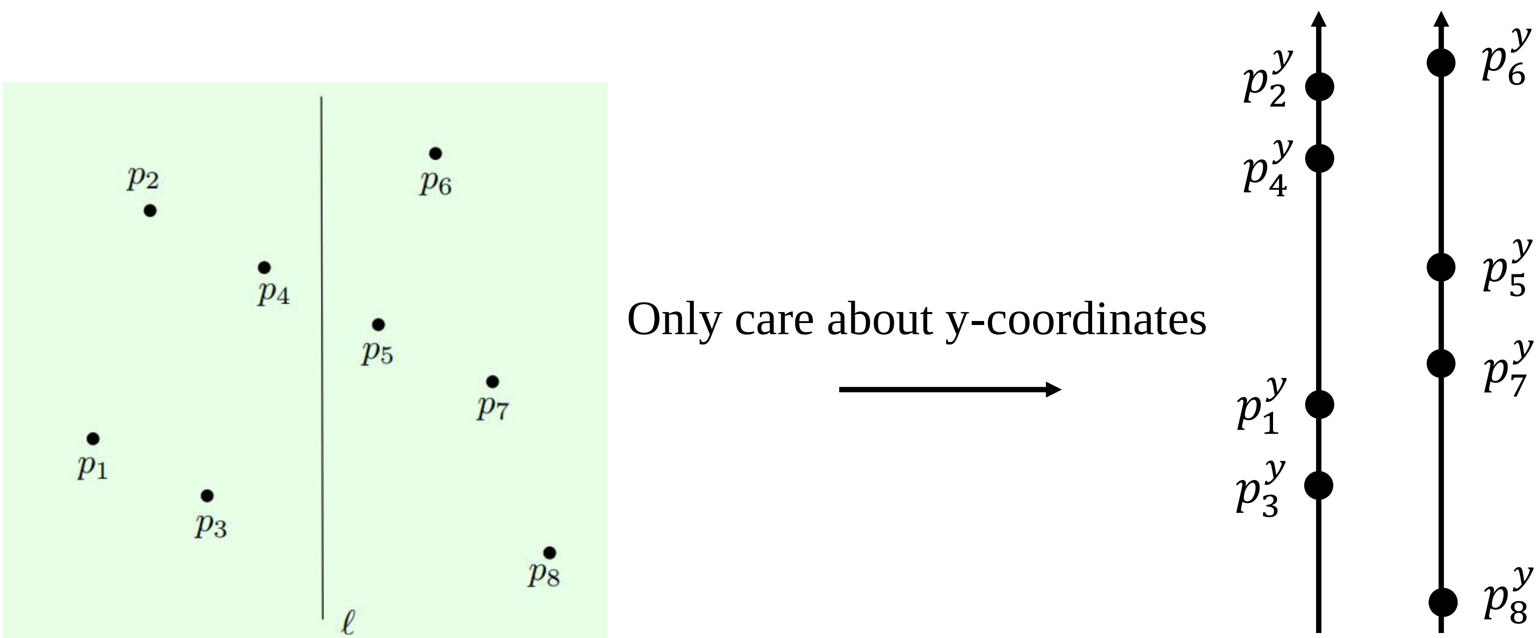
Dominance counting: a faster algorithm

- Ask a harder question:
  - Output the dominance counts and sort P by y-coordinate.
- Scan the point from P_1 by y-coordinate in ascending order, and scan P_2 in the same way synchronously.
  - Merge the following two sorted arrays, based on y-coordinates and obtain the number of points in P_1 dominated by p.
  - $P_1 \;= (p_3 \,, p_1 \,, p_4 \,, p_2 \,)$
  - $P_2 \;= (p_8 \,, p_7 \,, p_5 \,, p_6 \,)$



Dominance counting

- Scan the points from P_1 by y-coordinate in ascending order. Do the same on P_2.
  - $P_1 \; = (p_3 \,, p_1 \,, p_4 \,, p_2 \,)$
  - $P_2 \; = (p_8 \,, p_7 \,, p_5 \,, p_6 \,)$

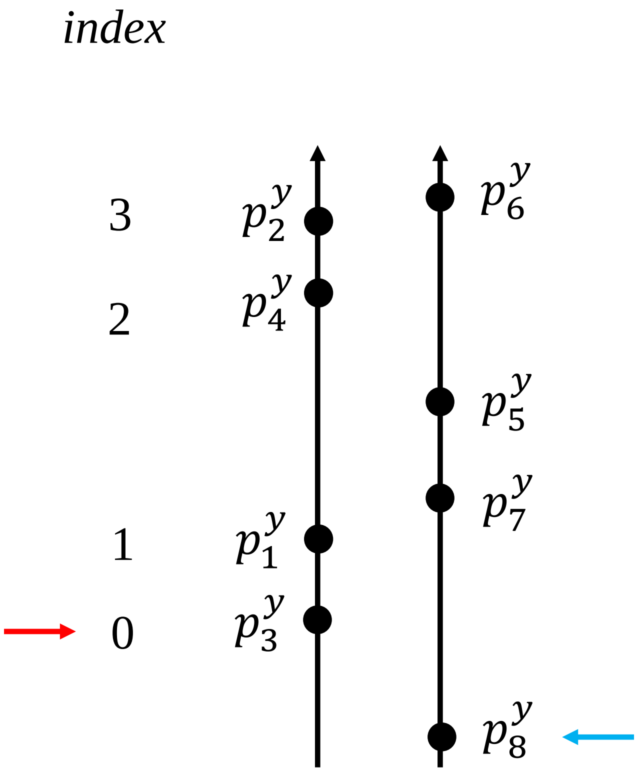


Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- $\bar{P} = ()$
  - All the points will be stored in this array in ascending order of y-coordinate.
  - To be produced by merging P_1 and P_2.

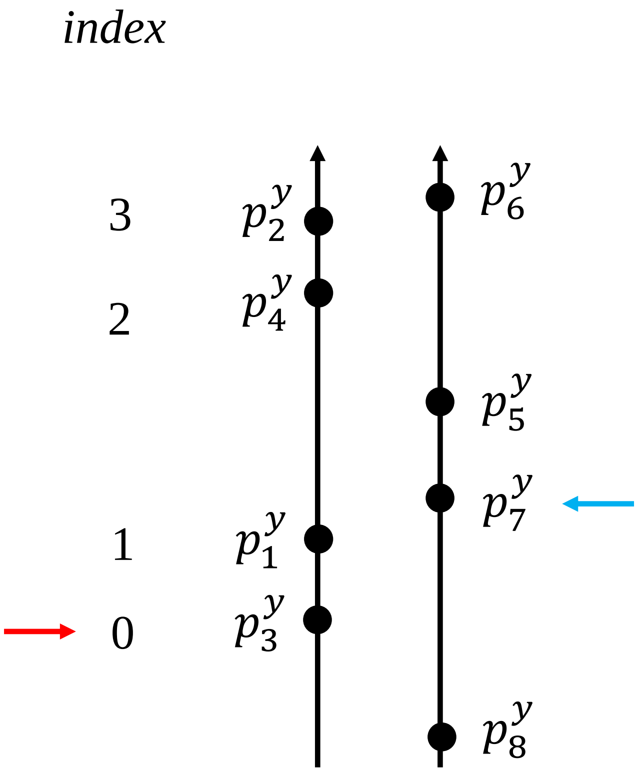
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 0$
- $\bar{P} = ()$



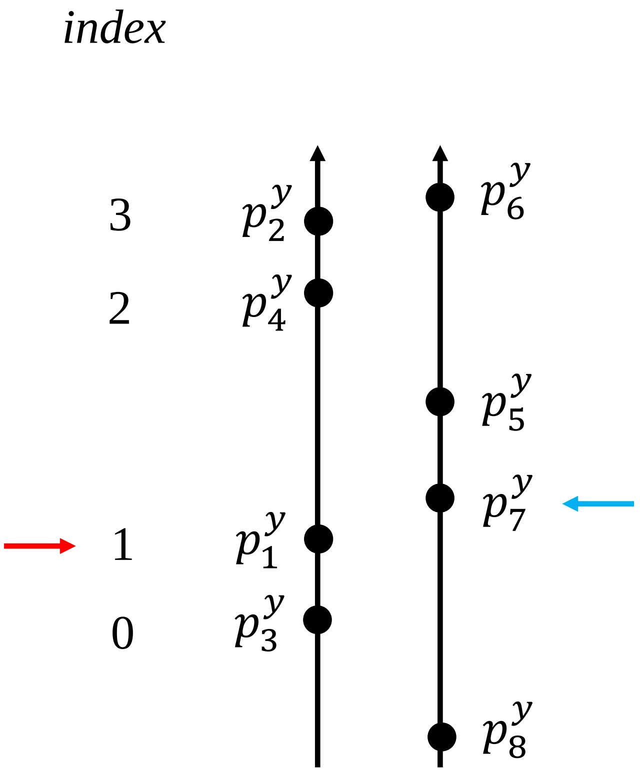
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 0$
- $\bar{P} = (p_8)$
  - p_8 dominates 0 point in P_1.



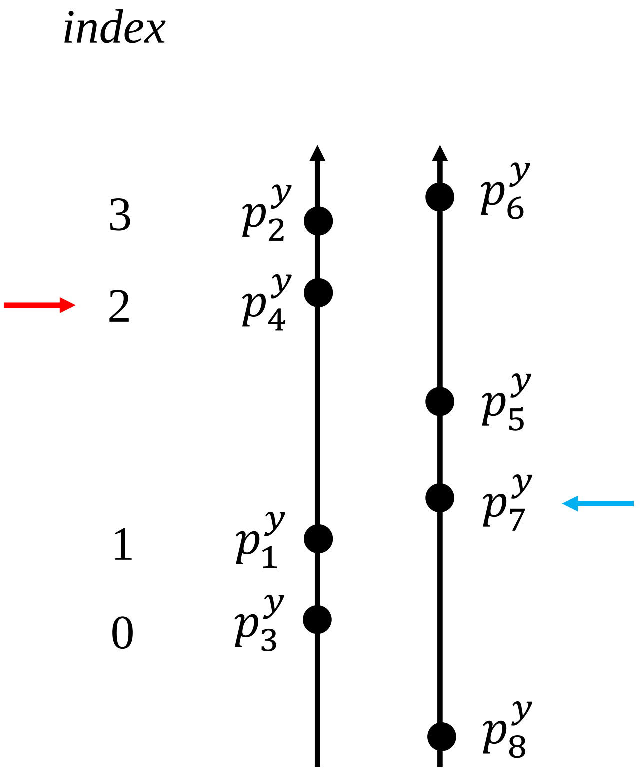
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 0$
- $\bar{P} = (p_8, p_3)$



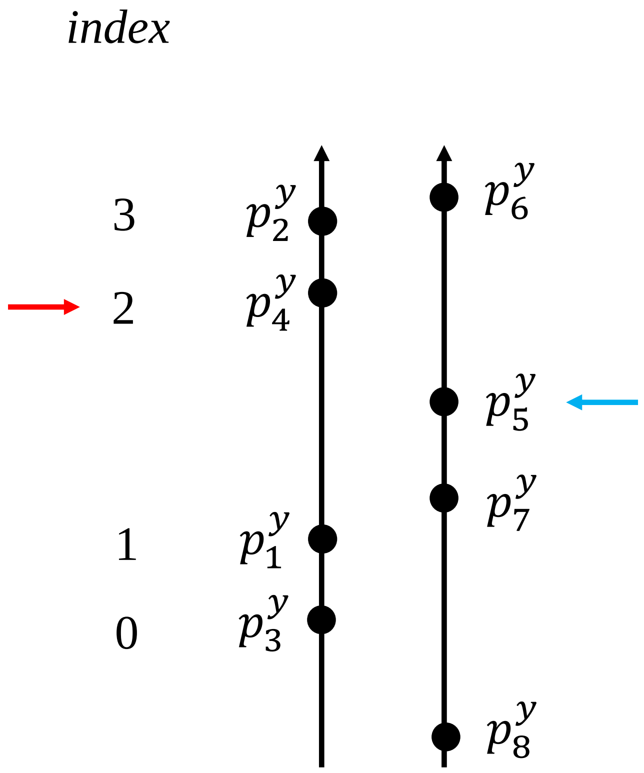
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 0$
- $\bar{P} = (p_8, p_3, p_1)$



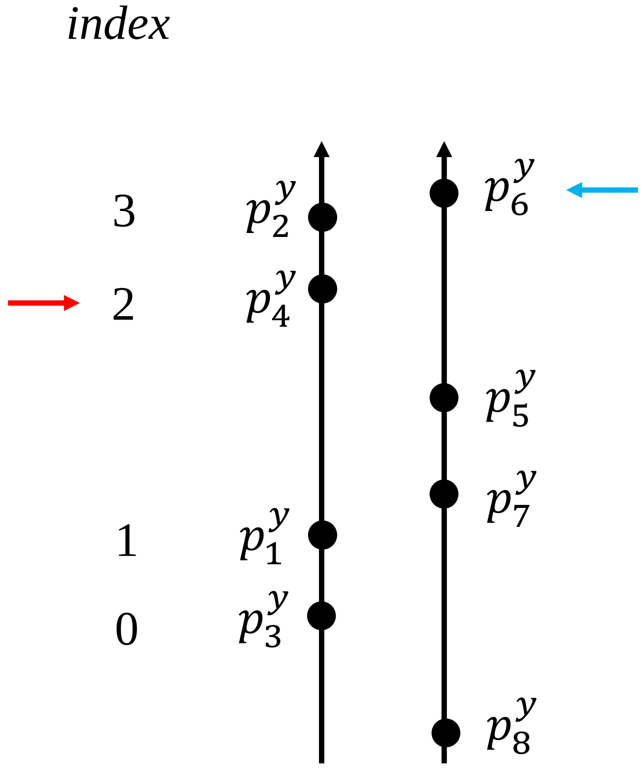
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 2$
- $\bar{P} = (p_8, p_3, p_1, p_7)$
  - p_7 dominates 2 point in P_2



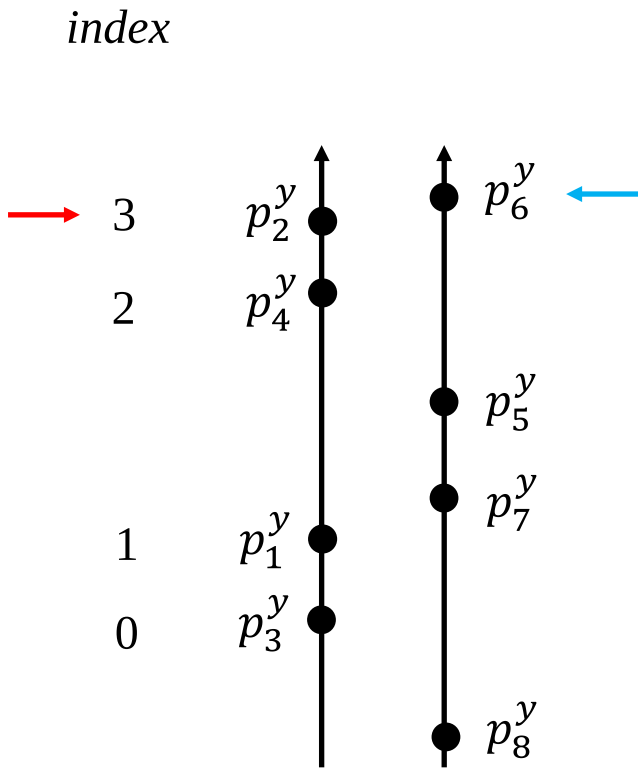
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 4$
- $\bar{P} = (p_8, p_3, p_1, p_7, p_5)$
  - p_5 dominates 2 point in P_1



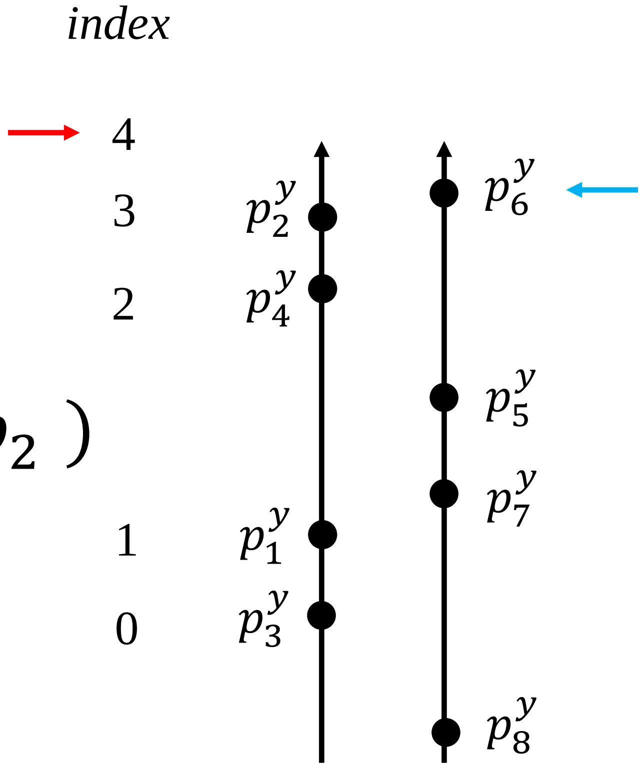
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = \left(p_8, p_7, p_5, p_6\right)$
- count $= 4$
- $\bar{P} = (p_8, p_3, p_1, p_7, p_5, p_4)$



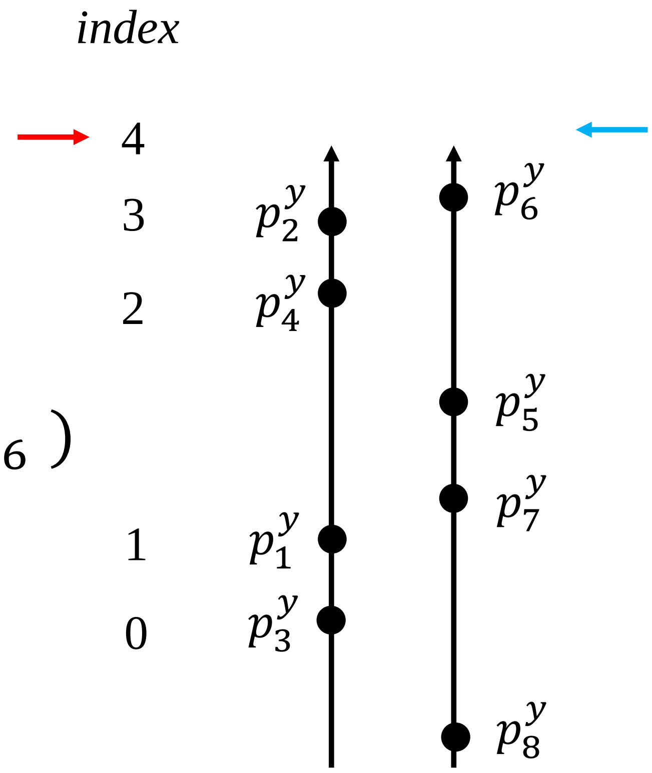
Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 4$
- $\bar{P} = (p_8, p_3, p_1, p_7, p_5, p_4, p_2)$



Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$
- $P_2 = (p_8, p_7, p_5, p_6)$
- count $= 8$
- $\bar{P} = (p_8, p_3, p_1, p_7, p_5, p_4, p_2, p_6)$
  - p_6 dominates 4 points in P_1.



Dominance counting

- $P_1 = (p_3, p_1, p_4, p_2)$.
- $P_2 = \left(p_8, p_7, p_5, p_6\right)$.
- count $= 8$
- $\bar{P} = (p_8, p_3, p_1, p_7, p_5, p_4, p_2, p_6)$.
- Current time complexity: $O(n)$.

Dominance counting

Analysis

- Let $f(n)$ be the worst-case running time of the algorithm on n points.
- $f(n) \leq 2f(\lceil n/2 \rceil) + O(n)$,
- which solves to $f(n) = O(n\log n)$.