CSCI3160 Design and Analysis of Algorithms (2025 Fall)

Approximation Algorithms 1: Vertex Cover and MAX-3SAT

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[1]These slides are primarily based on materials prepared by Prof. Yufei Tao (please refer to Prof. Tao's version from 2024 Fall for the original content). Some modifications have been made to better align with this year's teaching progress, incorporating student feedback, in-class interactions, and my own teaching style and research perspective.

Motivation

We have learned several algorithms that help us solve problems efficiently:

- Sorting in time $O(n \log n)$
- Matrix multiplication in time $O(n^{2.81})$
- FFT-based polynomial multiplication in time $O(n \log n)$
- Activity selection in time $O(n \log n)$
- ...
- All-Pairs Shortest Paths in time $O\big(|V|(|V| + |E|) \log(|V|)\big)$

However: there are still many problems **of practical significance** for which no efficient (i.e., polynomial-time) algorithms are currently known to us humans.

Example 1: Graph-3-Coloring

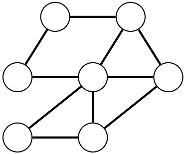
A **graph coloring** assigns colors to the vertices of a graph so that no two adjacent vertices share the same color.

In the **Graph-3-Coloring** problem, we ask:

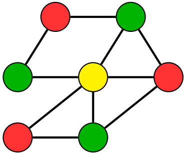
Can the vertices of a given graph be colored with at most 3 colors such that adjacent vertices have different colors?

Extension: 3-coloring is a special case of the general k-coloring problem.

Exemplary Graphs



(a) A 3-colorable graph



(b) A 3-coloring scheme

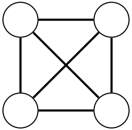


Figure: A non-3-colorable graph

Application 1: Map Coloring

- Each region on a map can be represented as a vertex in a graph.
- An edge connects two vertices if the corresponding regions share a common border.
- The goal is to color each region so that no two adjacent regions have the same color.
- If the map can be colored with 3 colors, the corresponding graph is 3-colorable.
- This has applications in geography, political boundary planning, and resource distribution.

Application 2: Scheduling Exams

- Vertices represent courses.
- An edge connects two courses if they have students in common.
- Goal: assign each course to one of 3 time slots, avoiding conflicts.
- 3-coloring determines if this is possible.
- If there are k courses under consideration, then it corresponds to the k-coloring problem.

Application 3: Frequency Assignment

- Nodes represent transmitters in a communication network.
- Edges represent interference (proximity).
- Assign 3 frequencies to avoid interference between neighbors.
- 3-coloring tells us if this can be done using only 3 frequencies.
- In general k-coloring tells us if this can be done using only k frequencies.

Subset Sum Problem and Applications

Problem Statement:

- Given a set of integers $S = \{x_1, x_2, \ldots, x_n\}$ and a target integer T,
- Does there exist a subset $S' \subseteq S$ such that the sum of elements in S' is exactly T?

Example 1:

- $S = \{3, 34, 4, 12, 5, 2\}, \quad T = 9$
- Yes: $\{4, 5\}$ or $\{3, 4, 2\}$ sum to 9.

Example 2:

- $S = \{3, 5, 9, 13\}, \quad T = 7$
- No!

Application 1: Budget Allocation

Scenario:

- A company has a list of proposed projects, each with a known cost.
- The total available budget is a fixed amount T.
- The goal is to determine if there is a combination of projects whose total cost exactly matches the budget.

Example:

- Projects: $\{P_1 : \$30k,\ P_2 : \$50k,\ P_3 : \$20k,\ P_4 : \$40k\}$
- Budget: $\$90k$
- Is there a subset of projects that costs exactly \$90k?
- Yes: $\{P_2, P_3, P_4\} \rightarrow 50k + 20k + 20k = 90k$

Relevance:

- Subset sum helps in decision support for finance and planning.
- Used in automated budget optimization tools and resource allocation systems.

Application 2: Packing and Logistics

Scenario:

- A shipping company needs to fill containers with items of different weights or volumes.
- Goal: Select a subset of items that perfectly fills a container of limited capacity.

Example:

- Items: $\{w_1 = 3kg, w_2 = 7kg, w_3 = 2kg, w_4 = 6kg\}$
- Container capacity: $9kg$
- Feasible subset: $\{w_2, w_3\} \rightarrow 7 + 2 = 9$

Relevance:

- Subset sum models the core problem in bin packing and cargo loading.
- Used in warehouse automation, shipping logistics, and supply chain optimization.

Unfortunately, we currently do not know any polynomial-time algorithms for Graph 3-Coloring, Subset Sum, or many other problems that have broad applications and significant real-world importance.

A central research theme for Theoretical Computer Science (TCS):

- **What can we do about these hard problems?**

Two branches of computer science have been developed centered around this theme:

- **Computation Complexity:** Seeks to understand the inherent difficulty of problems and classify them based on resource requirements. (Not our focus in this course.)
- **Approximation Algorithms:** Develops efficient algorithms that produce near-optimal solutions to hard problems. (This will be our focus for the remaining lectures.)

Branch 1: Computational Complexity

Goal: Understand the fundamental limits of computation.

- Classifies problems into complexity classes such as:
  - **P:** Problems solvable in polynomial time.
  - **NP:** Problems whose solutions can be verified in polynomial time.
  - **NP-complete:** The hardest problems in NP — if any one of them can be solved in polynomial time, then all of NP can.
- Addresses profound open questions like:
  - **P vs NP:** Can every efficiently verifiable problem also be efficiently solvable?
- Also studies:
  - Reductions between problems (to compare difficulty)
  - Space and time trade-offs
  - Randomized and quantum complexity classes

Takeaway: Computational complexity helps us understand *why* certain problems are hard, and how that hardness is structured.

Branch 2: Approximation Algorithms

Goal: Design efficient algorithms that find near-optimal solutions for hard optimization problems.

- When exact solutions are computationally infeasible (e.g., NP-hard problems), we aim for **good enough** solutions in **polynomial time**.
- An approximation algorithm returns a solution whose value is within a provable factor of the optimum.
- Central questions in this area:
  - How close can we get to the optimal solution in polynomial time?
  - Are there limits (hardness of approximation) beyond which we can't do better unless P = NP?
- Techniques include:
  - Greedy methods
  - LP/SDP relaxations
  - Randomized rounding

Takeaway: Approximation algorithms offer a practical path forward for solving hard problems when exact solutions are out of reach.

An introductory Journey to Approximation Algorithms

For a rigorous discussion about Approximation Algorithms, we first need to borrow some concepts from Computational Complexity.

What's the exact meaning of "no polynomial-time algorithms are known" for some problem (e.g., Graph 3-Coloring, Subset Sum)?

- We must be precise about the **computation model** we are using.

Long story short, **Turing Machine** has become the standard model of computation in complexity theory:

- We won't explain what a Turing Machine is. But in short, it is a slightly more powerful model than the RAM model we utilize in this course so far.
- It is mathematically simple yet powerful enough to simulate any "reasonable" algorithm.
- It is **robust**: polynomial-time computations on one reasonable model can be simulated in polynomial time on a Turing Machine.
- It is **closed under composition**: combining polynomial-time Turing Machines results in a polynomial-time Turing Machine.

Church-Turing Thesis

Informal Statement:

Any function that can be computed by a "reasonable" mechanical procedure (i.e., algorithm) can be computed by a Turing Machine.

Key Points:

- It is a **foundational hypothesis** in computer science — not a formal theorem.
- Supported by the equivalence of many independent computational models:
  - Turing Machines
  - Lambda calculus
  - Recursive functions
  - RAM machines
- All known models of computation that align with our intuitive notion of an algorithm are **no stronger than** in power.
- Thus, this thesis conjecture that the Turing Machine is an adequate model for defining what is computable (or an Algorithm).

A Brief History of the Church-Turing Thesis

Early 1900s: Mathematicians, including **David Hilbert**, were seeking to formalize all of mathematics.

- Hilbert posed the famous **Entscheidungsproblem** (decision problem):
  - *Is there a general algorithm to determine whether a given mathematical statement is provable?*
- This led to the deeper question: **What exactly is an algorithm or effective procedure?**

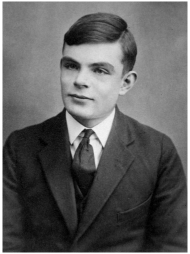
Alonzo Church (1936):

- Proposed the **lambda calculus** as a formal model of computation.
- Introduced the notion of **computable functions**.
- Argued that this model captured all effectively calculable functions.

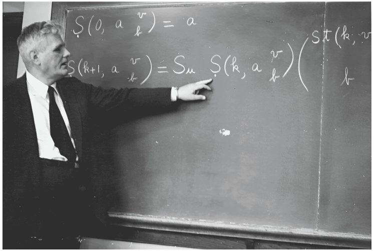
Alan Turing (1936):

- Independently tackled the same question.
- Introduced the **Turing Machine** in his groundbreaking paper:
  - "On Computable Numbers, with an Application to the Entscheidungsproblem"
- Showed that Turing Machines could simulate any mechanical computation.

Fun fact: Church was Turing's academic advisor at Princeton after this work!



(a) Alan Turing



(b) Alonzo Church

In computer science, there is a set of **NP-hard** problems such that

- nobody has found a polynomial-time algorithm for **any** of those problems;
- no polynomial-time algorithms can exist for **any** of those problems **unless** $\mathcal{P} = \mathcal{NP}$.

> - $\mathcal{P}$ = the set of problems that can be solved in polynomial time on a Turing machine
> - $\mathcal{NP}$ = the set of problems whose solution can be efficiently verified by a Turing machine
>
> Turing machines are formalized in CSCI3130 (Formal Languages and Automata Theory), and so is the notion of NP-hard.

> Whether $\mathcal{P} = \mathcal{NP}$ is still unsolved to this day.

What can we do if a problem is NP-hard?

The rest of the course will focus on a principled approach for tackling NP-hard problems: **approximation**.

In many problems, even though an optimal solution may be expensive to find, we can find **near-optimal** solutions efficiently.

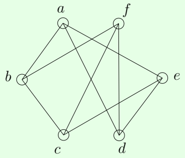
Next, we will see two examples: **vertex cover** and **MAX-3SAT**.

The Vertex Cover Problem

$G = (V, E)$ is a simple undirected graph.
A subset $S \subseteq V$ is a **vertex cover** of G if every edge $\{u, v\} \in E$ is incident to at least one vertex in S.

The V.C. Problem: Find a vertex cover of the smallest size.

Example:



An optimal solution is $\{a, f, c, e\}$.

The vertex cover problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in $|V|$.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

Approximation Algorithms

$\mathcal{A}$ = an algorithm that, given any legal input $G = (V, E)$, returns a vertex cover of G.

OPT_G = the smallest size of all the vertex covers of G.

> $\mathcal{A}$ is a ρ-**approximate algorithm** for the vertex cover problem if, for any legal input $G = (V, E)$, $\mathcal{A}$ can return a vertex cover with size at most $\rho \cdot OPT_G$.

The value ρ is the **approximation ratio**.
We say that $\mathcal{A}$ achieves an approximation ratio of ρ.

Consider the following algorithm.

Input: $G = (V, E)$

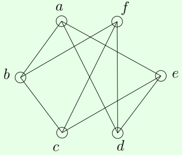
$S = \emptyset$
while E is not empty **do**
  pick an arbitrary edge $\{u, v\}$ in E
  add u, v to S
  remove from E all the edges of u and all the edges of v
return S

It is easy to show:

- S is a vertex cover of G;
- The algorithm runs in time polynomial to $|V|$ and $|E|$.

We will prove later that the algorithm is 2-approximate.

Example:



Suppose we start by picking edge $\{b, c\}$.
Then, $S = \{b, c\}$ and $E = \{\{a, e\}, \{a, d\}, \{d, e\}, \{d, f\}\}$.

Any edge in E can then be chosen. Suppose we pick $\{a, e\}$.
Then, $S = \{a, b, c, e\}$ and $E = \{\{d, f\}\}$.

Finally, pick $\{d, f\}$.
$S = \{a, b, c, d, e, f\}$ and $E = \emptyset$.

Theorem 1: The algorithm returns a set of at most $2 \cdot OPT_G$ vertices.

Let M be the set of edges picked.

Example: In the previous example, $M = \{\{b, c\}, \{a, e\}, \{d, f\}\}$.

Lemma 1: The edges in M do not share any vertices.

Proof: Suppose that M has edges e_1 and e_2 both incident to a vertex v. W.l.o.g., assume that e_1 was picked before e_2. After picking e_1, the algorithm deleted all the edges of v, because of which e_2 could not have been picked, giving a contradiction. □

Lemma 2: $|M| \leq OPT_G$.

Proof: Any vertex cover must include at least one vertex of each edge in M. $|M| \leq OPT_G$ follows from Lemma 1. □

Theorem 1 holds because the algorithm returns exactly $2|M|$ vertices.

The MAX-3SAT Problem

A **variable**: a boolean unknown x whose value is 0 or 1.
A **literal**: a variable x or its negation $\bar{x}$.
A **clause**: the OR of 3 literals with different variables.

S = a set of clauses
$\mathcal{X}$ = the set of variables appearing in at least one clause of S
A **truth assignment** of S: a function from $\mathcal{X}$ to $\{0, 1\}$.

A truth assignment f **satisfies** a clause in S if the clause evaluates to 1 under f.

> **The MAX-3SAT Problem:** Let S be a set of n clauses. Find a truth assignment of S to maximize the number of clauses satisfied.

Example:

$$\begin{aligned} S = \{ & x_1 \vee x_2 \vee x_3, \\ & x_1 \vee x_2 \vee \bar{x_3}, \\ & x_1 \vee \bar{x_2} \vee x_3, \\ & x_1 \vee \bar{x_2} \vee \bar{x_3}, \\ & \bar{x_1} \vee x_3 \vee x_4, \\ & \bar{x_1} \vee x_3 \vee \bar{x_4}, \\ & \bar{x_1} \vee \bar{x_3} \vee x_4. \\ & \bar{x_1} \vee \bar{x_3} \vee \bar{x_4} \}. \end{aligned}$$

$n = 8$ and $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$.

The truth assignment $x_1 = x_2 = x_3 = x_4 = 1$ satisfies 7 clauses. It is impossible to satisfy 8.

The MAX-3SAT problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in n.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

Approximation Algorithms

$\mathcal{A}$ = an algorithm that, given any legal input S, returns a truth assignment of S.

OPT_S = the largest number of clauses that a truth assignment of S can satisfy.

Z_S = the number of clauses satisfied by the truth assignment $\mathcal{A}$ returns.

- Z_S is a random variable if $\mathcal{A}$ is randomized.

> $\mathcal{A}$ is a **randomized ρ-approximate algorithm** for MAX-3SAT if $\boldsymbol{E}[Z_S] \geq \rho \cdot OPT_S$ holds for any legal input S.

The value ρ is the **approximation ratio**.
We also say that $\mathcal{A}$ achieves an approximation ratio of ρ in expectation.

Consider the following algorithm.

Input: a set S of clauses with variable set $\mathcal{X}$

for each variable $x \in \mathcal{X}$ **do**
    toss a fair coin
    if the coin comes up heads **then** $x \leftarrow 1$
    else $x \leftarrow 0$

It is clear that the algorithm runs in $O(n)$ time.

Next, we show that the algorithm achieves an approximation ratio $7/8$ in expectation.

Theorem 2: The algorithm produces a truth assignment that satisfies $\frac{7}{8}n$ clauses in expectation.

Proof: It suffices to show that each clause is satisfied with probability $7/8$. W.l.o.g., suppose that the clause is $x_1 \vee x_2 \vee x_3$. The clause is 0 if and only if x_1, x_2, and x_3 are all 0. The probability for $x_1 = x_2 = x_3 = 0$ is $1/8$. $\square$

Think: What about a clause like $x_1 \vee x_2 \vee \bar{x_3}$?