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Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

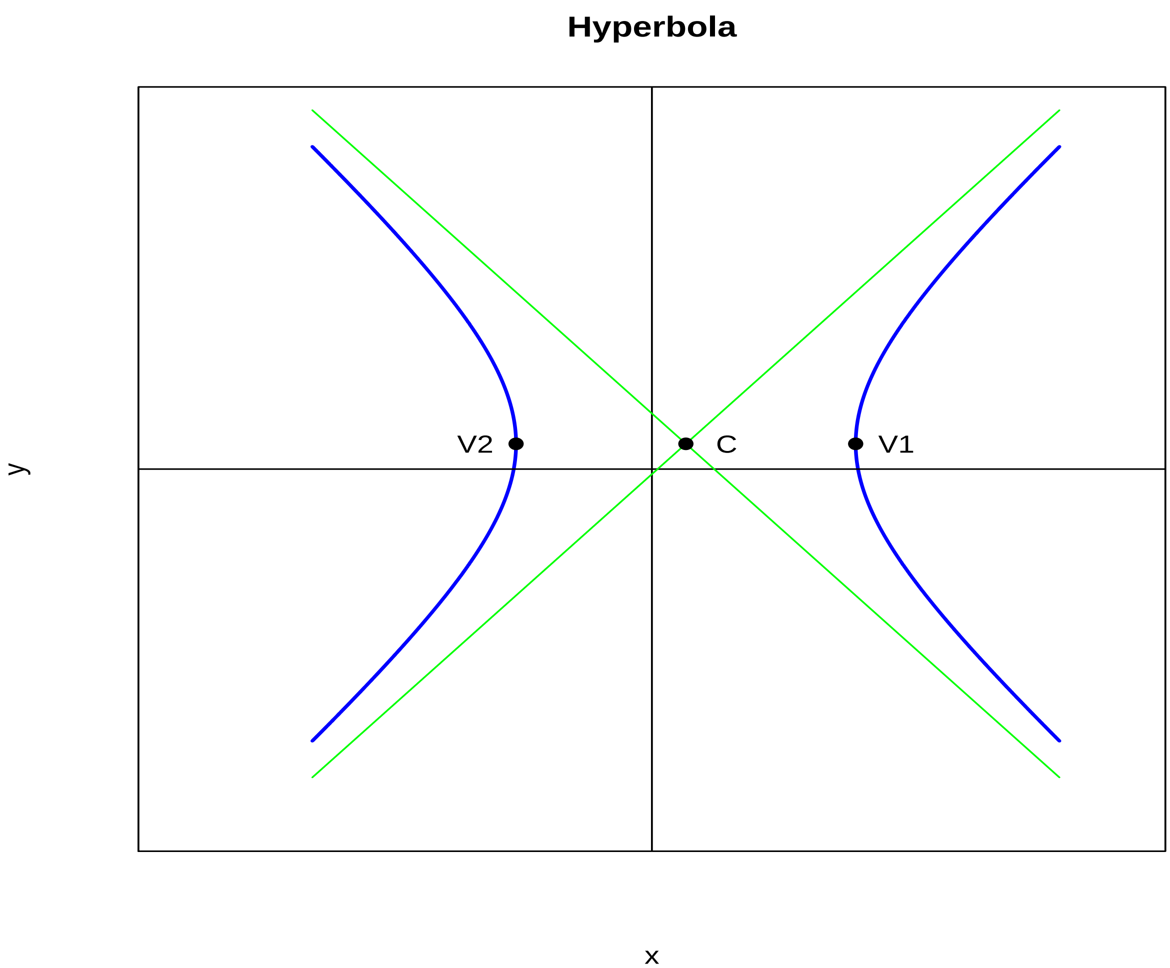
Hyperbola

Equation of a hyperbola:

$$\begin{aligned}
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1, \quad \text{opens right and left, or east-west.} \\
\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} &= 1, \quad \text{opens up and down, or north-sout.}
\end{aligned}$$

Let's examine the east-west hyperbola:

$$\begin{aligned}
\text{Center} &= (h, k) \\
\text{Vertices} &= (h+a, k) \text{ and } (h-a, k) \\
\text{Slopes of asymptotes} &= \pm\frac{b}{a} \\
\text{Equations of asymptotes } y &= k \pm \frac{b}{a}(x-h).
\end{aligned}$$



Refer to equation (12) page 1854 of the paper "An Analytic Derivation of the Efficient Frontier", by Robert C. Merton, *The Journal of Financial and Quantitative Analysis,* Vol. 7, No. 4:

$$
\begin{aligned}
\sigma^2 &= \frac{CE^2 - 2AE + B}{D} \\
\sigma^2 - \frac{C}{D}\left(E^2 - 2\frac{A}{C}E\right) &= \frac{B}{D} && \text{Note: Add on both sides: } \frac{C}{D}\frac{A^2}{C^2} \text{ to get} \\
\sigma^2 - \frac{C}{D}\left(E - \frac{A}{C}\right)^2 &= \frac{B}{D} - \frac{C}{D}\frac{A^2}{C^2} \\
\sigma^2 - \frac{C}{D}\left(E - \frac{A}{C}\right)^2 &= \frac{BC - A^2}{DC} && \text{From page 1853 : } \quad D = BC - A^2 \\
\sigma^2 - \frac{C}{D}\left(E - \frac{A}{C}\right)^2 &= \frac{1}{C} && \text{Divide both sides by } \frac{1}{C} \text{ to get} \\
\frac{\sigma^2}{1/C} - \frac{(E - A/C)^2}{D/C^2} &= 1 && \text{Finally} \\
\frac{(\sigma - 0)^2}{1/C} - \frac{(E - A/C)^2}{D/C^2} &= 1 && ***
\end{aligned}
$$

This is a hyperbola with:

$$
\begin{aligned}
\text{Center} &= \left(0, \frac{A}{C}\right) \\
\text{Vertices} &= \left(\frac{1}{C}, \frac{A}{C}\right) \text{ and } \left(-\frac{1}{C}, \frac{A}{C}\right) \\
\text{Slopes of asymptotes} &= \pm\sqrt{\frac{D}{C}} \\
\text{Equations of asymptotes } E &= \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma
\end{aligned}
$$

From (***) above we get the equation for E as a function of σ:

$$E = \frac{A}{C} \pm \frac{1}{C}\sqrt{D(C\sigma^2 - 1)}$$

The equation of the *efficient* frontier is

$$E = \frac{A}{C} \pm \frac{1}{C}\sqrt{D(C\sigma^2 - 1)}$$

or

$$E = E_{min} + \frac{1}{C}\sqrt{DC(\sigma^2 - \sigma^2_{min})}$$

Note: $E_{min} = \frac{A}{C}$ and $\sigma^2_{min} = \frac{1}{C}$.