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Department of Statistics

Statistics C183/C283 Instructor: Nicolas Christou

Exam 1
22 April 2016

Name: SOLUTION UCLA ID: ____________

Answer the following questions:

a. The betas of 30 stocks were obtained using simple regression in two successive periods: 2007-12-31 to 2011-12-31 (period 1) and 2011-12-31 to 2016-01-31 (period 2). There are 48 months in each period. Suppose we use the unadjusted betas in the first period as predictions of the betas in the second period. We can then compute the PRESS to evaluate the performance of these unadjusted betas. The following information is obtained from these data:

$\sum_{i=1}^{30} \beta_{i1} = 31.31761$	Sum of the betas in period 1.	$\bar{P} = 1.04392$
$\sum_{i=1}^{30} \beta_{i2} = 29.69676$	Sum of the betas in period 2.	$\bar{A} = 0.989892$
$var(\beta_1) = 0.4558062$	Variance of the betas in period 1.	$S_P^2 = 0.4558062$
$var(\beta_2) = 0.3235921$	Variance of the betas in period 2.	$S_A^2 = 0.3235921$
$cov(\beta_1, \beta_2) = 0.1700069$	Covariance between the betas in the two periods.	

Find the value of the prediction sum of squares (PRESS) using its decomposition presented in the paper "The Adjustment of Beta Forecasts", by Robert C. Klemkosky and John D. Martin, *The Journal of Finance*, Vol. 30, No. 4 (Sep., 1975).

$$PRESS = (\bar{A}-\bar{P})^2 + (1-\beta_1)^2 S_P^2 + (1-r_{AP}^2) S_A^2$$

$$r = \frac{cov(\beta_1,\beta_2)}{(S_P^2 \cdot S_A^2)^{1/2}} = \frac{0.1700069}{(0.455 \times 0.323)^{1/2}} = 0.4426673 \; , \quad \beta = \frac{cov(\beta_1,\beta_2)}{var(\beta_1)} = 0.3729807$$

$$PRESS = 0.4423036$$

b. Refer to question (a). Suppose now we use the Vasicek adjustment procedure to adjust the betas in period 1 in order to be better predictions for betas in period 2. Which one of the three components of PRESS do you expect to decrease using the adjustment betas? Explain.

inefficiency part
due to adjustment of betas

c. Consider the single index model. Show that the covariance between the returns of two stocks as computed by the classical Markowitz method (i.e. $cov(R_1, R_2) = \frac{1}{n-1}\sum_{t=1}^{n}(R_{t1} - \bar{R}_1)(R_{t2} - \bar{R}_2))$ is equal to $\hat{\beta}_1\hat{\beta}_2 var(Rm) + cov(e_1, e_2)$, where n is the number of months, $\hat{\beta}_1$ is the beta coefficient of the regression of R_1 on R_m, $\hat{\beta}_2$ is the beta coefficient of the regression of R_2 on R_m, $var(R_m) = \frac{1}{n-1}\sum_{t=1}^{n}(R_{mt} - \bar{R}_m)^2$, and e_1 and e_2 are the residuals of the two regressions. Please comment on this result from the portfolio point of view.

$$cov(R_1,R_2) = \frac{1}{n-1}\sum\left[R_{t1}-\bar{R}_1\right]\left[R_{t2}-\bar{R}_2\right] = \frac{1}{n-1}\sum\left[R_{t1}-\hat{R}_{t1}+\hat{R}_{t1}-\bar{R}_1\right]\left[R_{t2}-\hat{R}_{t2}+\hat{R}_{t2}-\bar{R}_2\right]$$

$$= \frac{1}{n-1}\sum(R_{t1}-\hat{R}_{t1})(R_{t2}-\hat{R}_{t2}) + \frac{1}{n-1}\sum(R_{t1}-\hat{R}_{t1})(\hat{R}_{t2}-\bar{R}_2) + \frac{1}{n-1}\sum(\hat{R}_{t1}-\bar{R}_1)(R_{t2}-\hat{R}_{t2})$$

$$+ \frac{1}{n-1}\sum(\hat{R}_{t1}-\bar{R}_1)(\hat{R}_{t2}-\bar{R}_2)$$

USE $\left.\begin{aligned}\hat{R}_{t1} &= \bar{R}_1 + \hat{\beta}_1(R_{mt}-\bar{R}_m)\\ \hat{R}_{t2} &= \bar{R}_2 + \hat{\beta}_2(R_{mt}-\bar{R}_m)\end{aligned}\right\}$

. . . . .

$$= cov(e_1, e_2) + \hat{\beta}_1\hat{\beta}_2 var(R_m)$$

ASIDE: $\bar{e}_1 = 0$

$\bar{e}_2 = 0$

d. In the paper "An Analytic Derivation of the Efficient Frontier," *The Journal of Financial and Quantitative Analysis*, Vol. 7, No. 4, Robert Merton gives on page 1854 the proportion of the k_{th} risky asset held in the frontier portfolio with expected return E by

$$x_k = \frac{E\sum_{j=1}^{m} v_{kj}(CE_j - A) + \sum_{j=1}^{m} v_{kj}(B - AE_j)}{D}, \quad k = 1, \ldots, m. \tag{1}$$

On the same page it is shown that the expected return of the minimum risk portfolio is $\bar{E} = \frac{A}{C}$. Using equation (1) above show that the proportion of the k_{th} risky asset of the minimum risk portfolio is $x_k = \frac{\sum_{j=1}^{m} v_{kj}}{C}$, $k = 1, \ldots, m$.

$$X_K = \frac{\frac{A}{C}C\sum V_{Kj}E_j - A\frac{A}{C}\sum V_{Kj} + B\sum V_{Kj} - A\sum V_{Kj}E_j}{D}$$

$$= \frac{B - \frac{A^2}{C}}{D}\sum V_{Kj} = \frac{\frac{BC-A^2}{C}}{BC-A^2}\sum V_{Kj}$$

$D = BC - A^2 \rightarrow \boxed{X_K = \frac{\sum V_{Kj}}{C}}$

e. Consider a portfolio consisting of n risky assets. Describe three methods to trace out the efficient frontier when short sales allowed.

- HYPERBOLA
- Find Two PORTFOLIO
- USE MANY R_f VALUES

f. Suppose the single index model holds and short sales are allowed. Two portfolios consisting of the same n stocks are located on the efficient frontier. Let $x_1, \ldots, x_n$ be the weights of portfolio A and $y_1, \ldots, y_n$ be the weights of portfolio B. Find the covariance between these two portfolios in terms of the betas and variances of the error terms.

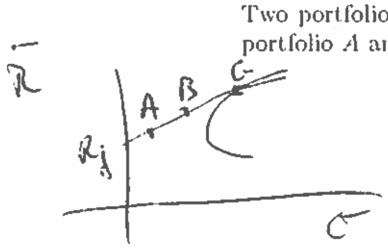
$$Cov(R_A, R_B) = Cov\left[\sum x_i R_i, \sum y_i R_i\right] = Cov\left(\sum x_i(\alpha_i + \beta_i R_m + \varepsilon_i), \sum y_i(\alpha_i + \beta_i R_m + \varepsilon_i)\right]$$

$$= Cov\left(\sum x_i \alpha_i + R_m \sum x_i \beta_i + \sum x_i \varepsilon_i, \sum y_i \alpha_i + R_m \sum y_i \beta_i + \sum y_i \varepsilon_i\right]$$

$$= Cov\left(\alpha_{PA} + \beta_{PA} R_m + \sum x_i \varepsilon_i, \alpha_{PB} + \beta_{PB} R_m + \sum y_i \varepsilon_i\right)$$

$$= \beta_{PA} \cdot \beta_{PB} \sigma_M^2 + \sum x_i y_i \sigma_{\varepsilon_i}^2$$

g. Refer to question (f). Suppose the single index model holds, short sales are allowed, and risk free asset exists. Two portfolios A and B are located on the capital asset allocation line (CAL). Find the correlation between portfolio A and B.



$$Cov(R_A, B_B) = Cov\left[x_1 R_G + (1-x_1)R_f, x_2 R_G + (1-x_2) R_f\right)$$

$$= x_1 x_2 \sigma_G^2$$

$$VAR(R_A) = VAR\left(x_1 R_G + (1-x_1) R_f\right) = x_1^2 \sigma_G^2$$

$$VAR(R_B) = VAR\left(x_2 R_G + (1-x_2) R_f\right) = x_2^2 \sigma_G^2$$

$$COR(R_A, R_B) = \frac{Cov(R_A, R_B)}{\sigma_A \sigma_B} = \frac{x_1 x_2 \sigma_G^2}{\sqrt{x_1^2 \sigma_G^2}\sqrt{x_2^2 \sigma_G^2}} = 1.$$

$$\underset{\sim}{Z} = \Sigma^{-1}\bar{R} = \Sigma^{-1}R - R_f\,\Sigma^{-1}\underset{\sim}{1}$$

h. Consider a portfolio of n risky assets, short sales are allowed, and risk free asset exists. For what value of R_f the k_{th} risky asset (a) will not be held, (b) will be held long, and (c) will be held short?

$$Z_k = \sum V_{kj}\bar{R}_j - R_f \sum V_{kj}$$

(a). $Z_k = 0 \Rightarrow R_f = \dfrac{\sum V_{kj}\bar{R}_j}{\sum V_{kj}}$

(b) $Z_k > 0 \Rightarrow R_f < \dfrac{\sum V_{kj}\bar{R}_j}{\sum V_{kj}}$

(c). $Z_k < 0 \Rightarrow R_f > \dfrac{\sum V_{kj}\bar{R}_j}{\sum V_{kj}}$

i. Consider a portfolio of $n = 3$ risky assets, short sales are allowed, and risk free asset exists, with $R_f = 0.001$. The inverse of the variance covariance matrix and the mean returns of the three stocks are given below:

`Inverse of the variance covariance matrix:`

	A	B	C
A	325.66	-96.31	-348.91
B	-96.31	889.84	-882.07
C	-348.91	-882.07	2213.58

$$Z = \Sigma^{-1}\begin{pmatrix} 0.010-0.001 \\ 0.011-0.001 \\ 0.014-0.001 \end{pmatrix} = \begin{pmatrix} -2.56499 \\ -3.43530 \\ 16.31565 \end{pmatrix}$$

`Mean returns:`

A	B	C
0.010	0.011	0.014

Find the composition of the optimal portfolio.

$$\underset{\sim}{X} = \frac{Z}{1'Z} = \begin{pmatrix} -0.2375 \\ -0.3177 \\ +1.5552 \end{pmatrix}$$

j. Refer to question (i). Find the mean return of a portfolio that consists of 50% in the risk free asset and 50% in the optimal portfolio found in question (j) and show it on the capital allocation line. Express the standard deviation of this portfolio in terms of σ_G (the standard deviation of the optimal portfolio in question (i)).

$$\bar{R}_A = \tfrac{1}{2}R_f + \tfrac{1}{2}\bar{R}_G \qquad \bar{R}_G = X'R = \underset{\sim}{X}'\underset{\sim}{R} = 0.0159$$

$$\bar{R}_A = \tfrac{1}{2}(0.001) + \tfrac{1}{2}(0.0159) = 0.00845$$

$$\sigma_A = \tfrac{1}{2}\sigma_G$$