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Statistics C183/C283 **Instructor: Nicolas Christou**

Multigroup model
Short sales allowed with riskless lending and borrowing- Theory
From: "Simple Rules for Optimal Portfolio Selection: The Multi Group Case"
Elton, E., Gruber, M., Padberg, M. (1977), Journal of Financial and Quantitative Analysis

The stocks are grouped by industry. Let's examine the simple case of two industries. The assumption here is that the correlations within the first group are the same for all the pairs in group 1 (called it ρ_{11}), similarly within the second group correlations are the same for all pairs (called it ρ_{22}), and the correlations for all pairs of stocks between the first group and second group is the same (called it ρ_{12}). For example, suppose stocks 1, 2, 3 belong to industry 1 (automobile stocks), and stocks 4, 5, 6 belong to industry 2 (chemical stocks). Then the correlations are as follows:
Group 1: $\rho_{12} = \rho_{13} = \rho_{23} = \rho_{11}$,
Group 2: $\rho_{45} = \rho_{46} = \rho_{56} = \rho_{22}$,
Between stocks of group 1 and group 2: $\rho_{14} = \rho_{15} = \rho_{16} = \rho_{24} = \rho_{25} = \rho_{26} = \rho_{34} = \rho_{35} = \rho_{36} = \rho_{12}$.

$$\boldsymbol{\rho} = \left(\begin{array}{ccc|ccc} 1 & \rho_{11} & \rho_{11} & \rho_{12} & \rho_{12} & \rho_{12} \\ \rho_{11} & 1 & \rho_{11} & \rho_{12} & \rho_{12} & \rho_{12} \\ \rho_{11} & \rho_{11} & 1 & \rho_{12} & \rho_{12} & \rho_{12} \\ \hline \rho_{21} & \rho_{21} & \rho_{21} & 1 & \rho_{22} & \rho_{22} \\ \rho_{21} & \rho_{21} & \rho_{21} & \rho_{22} & 1 & \rho_{22} \\ \rho_{21} & \rho_{21} & \rho_{21} & \rho_{22} & \rho_{22} & 1 \end{array}\right)$$

The solution was found and it is the point of tangency to the efficient frontier. But now the stocks are grouped into industries. Assume two stocks and two industries (two per industry). The solution as always is given by the following system of equations:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14} \quad (1)$$
$$\bar{R}_2 - R_f = z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + z_4\sigma_{24} \quad (2)$$
$$\bar{R}_3 - R_f = z_1\sigma_{31} + z_2\sigma_{32} + z_3\sigma_3^2 + z_4\sigma_{34} \quad (3)$$
$$\bar{R}_4 - R_f = z_1\sigma_{41} + z_2\sigma_{42} + z_3\sigma_{43} + z_4\sigma_4^2 \quad (4)$$

The equations (1), (2), (3), and (4) using $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, and the argument at the beginning of the page can be written as follows:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\rho_{11}\sigma_1\sigma_2 + z_3\rho_{12}\sigma_1\sigma_3 + z_4\rho_{12}\sigma_1\sigma_4$$
$$\bar{R}_2 - R_f = z_1\rho_{11}\sigma_1\sigma_2 + z_2\sigma_2^2 + z_3\rho_{12}\sigma_2\sigma_3 + z_4\rho_{12}\sigma_2\sigma_4$$
$$\bar{R}_3 - R_f = z_1\rho_{12}\sigma_1\sigma_3 + z_2\rho_{12}\sigma_2\sigma_3 + z_3\sigma_3^2 + z_4\rho_{22}\sigma_3\sigma_4$$
$$\bar{R}_4 - R_f = z_1\rho_{12}\sigma_1\sigma_4 + z_2\rho_{12}\sigma_2\sigma_4 + z_3\rho_{22}\sigma_3\sigma_4 + z_4\sigma_4^2$$

To solve for $\mathbf{Z}$:

$$\mathbf{Z} = \boldsymbol{\Sigma}^{-1}\mathbf{R}$$

The elements of $\boldsymbol{\Sigma}$ are computed based on the assumption that stocks in the same industry have equal correlations, etc.

But there is another approach, similar to the C^* cut-off point...

Multigroup model
Short sales allowed
Example

The stocks are grouped into industries. Here we have nine stocks as follows:

Auto manufacturers	Drug	Beverage
Ford	Johnston&Johnston	Coca Cola
GM	Merck	Hansen
Toyota	Pfizer	Pepsico

The correlation matrix is given below.

$$\boldsymbol{\rho}=\begin{pmatrix} 1.0000 & 0.4980 & 0.3029 & -0.0283 & 0.1803 & 0.1915 & 0.2285 & 0.0612 & 0.1147 \\ 0.4980 & 1.0000 & 0.4212 & -0.1537 & -0.0083 & 0.0906 & 0.1779 & 0.0643 & -0.1178 \\ 0.3029 & 0.4212 & 1.0000 & -0.1157 & 0.1016 & 0.1045 & 0.1231 & 0.0035 & -0.0059 \\ -0.0283 & -0.1537 & -0.1157 & 1.0000 & 0.4387 & 0.4164 & -0.0211 & -0.0403 & 0.3426 \\ 0.1803 & -0.0083 & 0.1016 & 0.4387 & 1.0000 & 0.5349 & 0.0517 & -0.0382 & 0.4162 \\ 0.1915 & 0.0906 & 0.1045 & 0.4164 & 0.5349 & 1.0000 & 0.0526 & -0.0035 & 0.3876 \\ 0.2285 & 0.1779 & 0.1231 & -0.0211 & 0.0517 & 0.0526 & 1.0000 & 0.1524 & 0.0757 \\ 0.0612 & 0.0643 & 0.0035 & -0.0403 & -0.0382 & -0.0035 & 0.1524 & 1.0000 & 0.0597 \\ 0.1147 & -0.1178 & -0.0059 & 0.3426 & 0.4162 & 0.3876 & 0.0757 & 0.0597 & 1.0000 \end{pmatrix}$$

The multigroup model requires the following inputs: the average return and standard deviation of the return for each stock, the average correlation within each industry and between industries. These are summarized below:

	ford	gm	toyota	jnj	merck	pfizer	coke	hansen	pepsico
mean	0.0169	0.0142	0.0051	0.0009	0.0089	0.0043	0.0002	0.0545	0.0091
sd	0.1314	0.1065	0.0726	0.0823	0.0857	0.0607	0.0649	0.1745	0.0525

Within and between group correlations:

$$\bar{\boldsymbol{\rho}}=\begin{pmatrix} 0.4073 & 0.0403 & 0.0722 \\ 0.0403 & 0.4633 & 0.1275 \\ 0.0722 & 0.1275 & 0.0960 \end{pmatrix}$$

For example, $\rho_{11}=0.4073, \rho_{12}=0.0403$, etc.

To find the optimum portfolio (point of tangency) we will use the following equations for the z_i's:

$$z_i=\frac{1}{\sigma_i(1-\rho_{kk})}\left[\frac{\bar{R}_i-R_f}{\sigma_i}-\sum_{g=1}^{p}\rho_{kg}\Phi_g\right], \text{ for } k=1,\cdots,p \tag{5}$$

where, p is the number of industries (in our example 3). The unknowns are the Φ's. These can be found from

$$\boldsymbol{\Phi}=\boldsymbol{A}^{-1}\boldsymbol{C}$$

where,

$$\boldsymbol{A}=\begin{pmatrix} 1+\frac{N_1\rho_{11}}{1-\rho_{11}} & \frac{N_1\rho_{12}}{1-\rho_{11}} & \frac{N_1\rho_{13}}{1-\rho_{11}} \\ \frac{N_2\rho_{21}}{1-\rho_{22}} & 1+\frac{N_2\rho_{22}}{1-\rho_{22}} & \frac{N_2\rho_{23}}{1-\rho_{22}} \\ \frac{N_3\rho_{31}}{1-\rho_{33}} & \frac{N_3\rho_{32}}{1-\rho_{33}} & 1+\frac{N_3\rho_{33}}{1-\rho_{33}} \end{pmatrix}$$

and

$$\boldsymbol{C}=\begin{pmatrix} \sum_{i=1}^{N_1}\frac{\bar{R}_i-R_f}{\sigma_i(1-\rho 11)} \\ \sum_{i=1}^{N_2}\frac{\bar{R}_i-R_f}{\sigma_i(1-\rho 22)} \\ \sum_{i=1}^{N_3}\frac{\bar{R}_i-R_f}{\sigma_i(1-\rho 33)} \end{pmatrix}$$

In our example, $N_1 = N_2 = N_3 = 3$, and $R_f = 0.002$, and the solution is:

$$\mathbf{\Phi} = \begin{pmatrix} 3.0616 & 0.2040 & 0.3654 \\ 0.2253 & 3.5897 & 0.7127 \\ 0.2396 & 0.4231 & 1.3186 \end{pmatrix}^{-1} \begin{pmatrix} 0.4566 \\ 0.1957 \\ 0.4517 \end{pmatrix} = \begin{pmatrix} 0.1112 \\ -0.0176 \\ 0.3280 \end{pmatrix}.$$

Therefore, $\Phi_1 = 0.1112, \Phi_2 = -0.0176, \Phi_3 = 0.3280$. The cut-off points for the three groups (this is the summation in the square bracket of equation (1)) are: 0.0683, 0.0381, and 0.0373 respectively. For example, the cut-off point for the first group is:

$$C_1^* = \sum_{g=1}^{p} \rho_{1g}\Phi_g = \rho_{11}\Phi_1 + \rho_{12}\Phi_2 + \rho_{13}\Phi_3 = 0.4073(0.1112) + 0.0403(-0.0176) + 0.0722(0.3280) = 0.0683.$$

Using (1) we can find the z_i's:

$$\begin{aligned}
z_1 &= \frac{1}{0.1314(1-0.4073)}\left[\frac{0.0169-0.002}{0.1314} - 0.0683\right] = 0.5782 \\
z_2 &= \frac{1}{0.1065(1-0.4073)}\left[\frac{0.0142-0.002}{0.1065} - 0.0683\right] = 0.7294 \\
z_3 &= \frac{1}{0.0726(1-0.4073)}\left[\frac{0.0051-0.002}{0.0726} - 0.0683\right] = -0.6018 \\
z_4 &= \frac{1}{0.0823(1-0.4633)}\left[\frac{0.0009-0.002}{0.0823} - 0.0381\right] = -1.1757 \\
z_5 &= \frac{1}{0.0857(1-0.4633)}\left[\frac{0.0089-0.002}{0.0857} - 0.0381\right] = 0.9311 \\
z_6 &= \frac{1}{0.0607(1-0.4633)}\left[\frac{0.0043-0.002}{0.0607} - 0.0381\right] = 0.0175 \\
z_7 &= \frac{1}{0.0649(1-0.0960)}\left[\frac{0.0002-0.002}{0.0649} - 0.0373\right] = -1.182 \\
z_8 &= \frac{1}{0.1745(1-0.0960)}\left[\frac{0.0545-0.002}{0.1745} - 0.0373\right] = 1.6693 \\
z_9 &= \frac{1}{0.0525(1-0.0960)}\left[\frac{0.0091-0.002}{0.0525} - 0.0373\right] = 2.0622
\end{aligned}$$

The sum of the z_i's is: $\sum_{i=1}^{9} z_i = 3.092$, and the fractions invested in each stock will be $\frac{z_i}{\sum_{i=1}^{9} z_i}$. We have:
$x_1 = 0.187, x_2 = 0.236, x_3 = -0.195,$
$x_4 = -0.380, x_5 = 0.301, x_6 = 0.006,$
$x_7 = -0.362, x_8 = 0.540, x_9 = 0.667.$