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Statistics C183/C283 | **Instructor: Nicolas Christou**

Ito's lemma, lognormal property of stock prices
Black Scholes Model

From *Options Futures and Other Derivatives* by John Hull, Prentice Hall 6th Edition, 2006.

A. Ito's lemma:

Ito's lemma gives a derivative chain rule of random variables. Let G be a function of (S, t). Ito's lemma states that G follows the generalized Wiener process as follows:

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2}\right) dt + \frac{\partial G}{\partial S}\sigma S \epsilon \sqrt{dt} \tag{1}$$

B. The lognormal property of stock prices:

Let $G = lnS$, and let us apply Ito's lemma to this function (here G is only function of S). Therefore

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

And (1) will be:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \epsilon \sqrt{dt}$$

Therefore, G follows the generalized Wiener process with drift rate $\mu - \frac{\sigma^2}{2}$ and variance σ^2. As always, $\epsilon \sim N(0, 1)$. Examine the change in G from time t (now) to time T: This can be expressed as: $\Delta G = ln(S_T) - ln(S_0)$. What is the distribution of this change? Reminder: ΔG follows the generalized Wiener process.

Example:
Let $S = \$40$, and $\mu = 0.16, \sigma = 0.20$ per year.

a. Find the distribution of lnS_T in 6 months.
b. Find $P(a < S_T < b) = 0.95$.

Mean and variance of the stock price at time T:
Use moment generating functions to find $E(S_T)$ and $var(S_T)$. Reminder: The mgf of a normal random variable X with mean μ and standard deviation σ is

$$M_X(t) = E(e^{t^*X}) = e^{\mu t^* + \frac{1}{2}\sigma^2 t^{*2}}$$

Here we use t^* so that we don't confuse it with t which stands for time. In our case the random variable is $Y = lnS_T$. Therefore the mgf of Y is:

$$M_Y(t^*) = E(e^{t^*Y}) = E(e^{t^* lnS_T}) = E(e^{lnS_T{}^{t^*}}) = E(S_T)^{t^*} \tag{2}$$

But earlier we found that:

$$Y = lnS_T \sim N\left(lnS + (\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right)$$

Therefore using the mgf of the normal random variable (2) will be:

$$M_Y(t^*) = E(S_T)^{t^*} = e^{\left(lnS + (\mu - \frac{\sigma^2}{2})(T-t)\right)t^* + \frac{1}{2}\sigma^2(T-t)t^{*2}}$$

Therefore when $t^* = 1$ we will get $E(S_T)$ and when $t^* = 2$ we will get $E(S_T^2)$.

$$E(S_T) = e^{lnS}e^{\mu(T-t)} = Se^{\mu(T-t)}$$

and

$$E(S_T^2) = S^2 e^{2\mu(T-t)+\sigma^2(T-t)}$$

Now combining $E(S_T)$ and $E(S_T^2)$ we can find the variance of S_T:

$$var(S_T) = E(S_T^2) - (E(S_T))^2 = s^2 e^{2\mu(T-t)}\left[e^{\sigma^2(T-t)} - 1\right]$$

Example:
A stock has a current price \$20, and $\mu = 0.20, \sigma = 0.40$ per year. Find its expected price and variance in 1 year from now.

On the next page we see the calculations for estimatimng volatility σ.

Estimate annual volatility for `AAPL`:

```
a <- read.csv("http://real-chart.finance.yahoo.com/table.csv?s=AAPL&a=02&b=09&c=2016&
d=04&e=19&f=2016&g=d&ignore=.csv", sep=",", header=TRUE)
```

```
         Date   Open   High    Low  Close    Volume        pi    p[i-1] pi/p[i-1] u=log(pi/p[i-1])
1  2016-05-19  94.64  94.64  93.57  94.20  30342700  94.20000  94.56000 0.9961929 -3.814383e-03
2  2016-05-18  94.16  95.21  93.89  94.56  41923100  94.56000  93.49000 1.0114451  1.138008e-02
3  2016-05-17  94.55  94.70  93.01  93.49  46507400  93.49000  93.88000 0.9958458 -4.162882e-03
4  2016-05-16  92.39  94.39  91.65  93.88  61140600  93.88000  90.52000 1.0371189  3.644655e-02
5  2016-05-13  90.00  91.67  90.00  90.52  44188200  90.52000  90.34000 1.0019925  1.990502e-03
6  2016-05-12  92.72  92.78  89.47  90.34  76109800  90.34000  92.51000 0.9765430 -2.373648e-02
7  2016-05-11  93.48  93.57  92.46  92.51  28539900  92.51000  93.42000 0.9902591 -9.788665e-03
8  2016-05-10  93.33  93.57  92.11  93.42  33592500  93.42000  92.79000 1.0067895  6.766548e-03
9  2016-05-09  93.00  93.77  92.59  92.79  32855300  92.79000  92.72000 1.0007550  7.546763e-04
10 2016-05-06  93.37  93.45  91.85  92.72  43458200  92.72000  93.24000 0.9944230 -5.592583e-03
11 2016-05-05  94.00  94.07  92.68  93.24  35890500  93.24000  93.62000 0.9959410 -4.067265e-03
12 2016-05-04  95.20  95.90  93.82  94.19  41025500  93.62000  94.60401 0.9895987 -1.045580e-02
13 2016-05-03  94.20  95.74  93.68  95.18  56831300  94.60401  93.07333 1.0164460  1.631220e-02
14 2016-05-02  93.97  94.08  92.40  93.64  48160100  93.07333  93.17272 0.9989332 -1.067330e-03
15 2016-04-29  93.99  94.72  92.51  93.74  68531500  93.17272  94.25613 0.9885057 -1.156087e-02
16 2016-04-28  97.61  97.88  94.25  94.83  82242700  94.25613  97.22803 0.9694337 -3.104321e-02
17 2016-04-27  96.00  98.71  95.68  97.82 114602100  97.22803 103.71851 0.9374221 -6.462157e-02
18 2016-04-26 103.91 105.30 103.91 104.35  56016200 103.71851 104.44410 0.9930529 -6.971367e-03
19 2016-04-25 105.00 105.65 104.51 105.08  28031600 104.44410 105.04047 0.9943225 -5.693676e-03
20 2016-04-22 105.01 106.48 104.62 105.68  33683100 105.04047 105.32871 0.9972634 -2.740384e-03
21 2016-04-21 106.93 106.93 105.52 105.97  31552500 105.32871 106.48169 0.9891721 -1.088698e-02
22 2016-04-20 106.64 108.09 106.06 107.13  30611000 106.48169 106.26303 1.0020577  2.055629e-03
23 2016-04-19 107.88 108.00 106.23 106.91  32384900 106.26303 106.82958 0.9946967 -5.317419e-03
24 2016-04-18 108.89 108.95 106.94 107.48  60821500 106.82958 109.18523 0.9784252 -2.181097e-02
25 2016-04-15 112.11 112.30 109.73 109.85  46939000 109.18523 111.42161 0.9799286 -2.027553e-02
26 2016-04-14 111.62 112.39 111.33 112.10  25473900 111.42161 111.36198 1.0005355  5.353536e-04
27 2016-04-13 110.80 112.34 110.80 112.04  33257300 111.36198 109.77166 1.0144875  1.438355e-02
28 2016-04-12 109.34 110.50 108.66 110.44  27232300 109.77166 108.36025 1.0130252  1.294109e-02
29 2016-04-11 108.97 110.61 108.83 109.02  29407500 108.36025 108.00244 1.0033130  3.307542e-03
30 2016-04-08 108.91 109.77 108.17 108.66  23581700 108.00244 107.88316 1.0011056  1.105002e-03
31 2016-04-07 109.95 110.42 108.12 108.54  31801900 107.88316 110.28851 0.9781903 -2.205100e-02
32 2016-04-06 110.23 110.98 109.20 110.96  26404100 110.28851 109.14547 1.0104727  1.041820e-02
33 2016-04-05 109.51 110.73 109.42 109.81  26578700 109.14547 110.44755 0.9882109 -1.185915e-02
34 2016-04-04 110.42 112.19 110.27 111.12  37356200 110.44755 109.32438 1.0102737  1.022129e-02
35 2016-04-01 108.78 110.00 108.20 109.99  25874000 109.32438 108.33043 1.0091752  9.133314e-03
36 2016-03-31 109.72 109.90 108.88 108.99  25888400 108.33043 108.89698 0.9947974 -5.216204e-03
37 2016-03-30 108.65 110.42 108.60 109.56  45601100 108.89698 107.02836 1.0174591  1.730845e-02
38 2016-03-29 104.89 107.79 104.88 107.68  31190100 107.02836 104.55343 1.0236714  2.339560e-02
39 2016-03-28 106.00 106.19 105.06 105.19  19411400 104.55343 105.03053 0.9954576 -4.552751e-03
40 2016-03-24 105.47 106.25 104.89 105.67  26133000 105.03053 105.48774 0.9956657 -4.343715e-03
41 2016-03-23 106.48 107.07 105.90 106.13  25703500 105.48774 106.07417 0.9944715 -5.543857e-03
42 2016-03-22 105.25 107.29 105.21 106.72  32444400 106.07417 105.26908 1.0076480  7.618875e-03
43 2016-03-21 105.93 107.65 105.14 105.91  35502700 105.26908 105.27901 0.9999056 -9.436324e-05
44 2016-03-18 106.34 106.50 105.19 105.92  44205200 105.27901 105.15974 1.0011342  1.133527e-03
45 2016-03-17 105.52 106.47 104.96 105.80  34420700 105.15974 105.32871 0.9983958 -1.605495e-03
46 2016-03-16 104.61 106.31 104.59 105.97  38303500 105.32871 103.94712 1.0132913  1.320370e-02
47 2016-03-15 103.96 105.18 103.85 104.58  40067700 103.94712 101.89959 1.0200937  1.989447e-02
48 2016-03-14 101.91 102.91 101.78 102.52  25076100 101.89959 101.64117 1.0025425  2.539257e-03
49 2016-03-11 102.24 102.28 101.50 102.26  27408200 101.64117 100.55776 1.0107740  1.071636e-02
50 2016-03-10 101.41 102.24 100.15 101.17  33513600 100.55776 100.50807 1.0004944  4.942959e-04
```

```
sum(u)
[1] -0.06481782

sum(u^2)
[1] 0.01203149
```

C. Black Scholes model:

A call option is a function of S (stock price) and t (time). Let C be the price of the call option. Then from Ito's lemma we have:

$$dC = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S}\sigma S \epsilon \sqrt{dt} \tag{3}$$

Similar to the binomial option pricing model we want to create a riskless portfolio by

- Buying the call.
- Sell n shares of the stock per call.

Then, the portfolio at time 0 is:

$$C - nS = \Pi$$

This portfolio will change from time t to time $t + dt$ as follows:

$$dC - ndS = d\Pi \tag{4}$$

But, dc is given by (3) and also S follows generalized Wiener process, that is:

$$dS = \mu S dt + \sigma S \epsilon \sqrt{dt}$$

Therefore (4) is:

$$d\Pi = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S}\sigma S \epsilon \sqrt{dt} - n\left(\mu S dt + \sigma S \epsilon \sqrt{dt}\right)$$

or

$$d\Pi = \left(\frac{\partial C}{\partial S} - n\right)\sigma S \epsilon \sqrt{dt} + \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - n\mu S\right) dt$$

This will be a riskless portfolio if we eliminate the term involving ϵ (the only random component) from the above expression. So if we choose $n = \frac{\partial C}{\partial S}$ then

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt$$

One more step: Since this is a riskless portfolio it must earn the risk free rate during time dt

$$r\Pi dt = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt$$

Also, $\Pi = C - nS$ and $n = \frac{\partial C}{\partial S}$. Putting all these together we get the Black-Scholes differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

The solution of this differential equation gives the Black-Scholes option pricing formula:

The value C of a European call option at time $t = 0$ is:

$$C = S_0\Phi(d_1) - \frac{E}{e^{rt}}\Phi(d_2)$$

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

S_0	Price of the stock at time $t = 0$
E	Exercise price at expiration
r	Continuously compounded risk-free interest
σ	Annual standard deviation of the returns of the stock
t	Time to expiration in years
$\Phi(d_i)$	Cumulative probability at d_i of the standard normal distribution $N(0, 1)$, that is, $\Phi(d_i) = P(Z \leq d_i)$

Example:
Use the Black-Scholes option pricing formula to find the price of the European call if $S_0 = \$30, E = \$29$, days to expiration $t = 40$, annual standard deviation $\sigma = 0.30$, and continuously compounded risk-free interest rate $r = 0.05$.