University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Homework 2 - Solutions

Exercise 1

a. Read the data:

```
a<- read.table("http://www.stat.ucla.edu/~nchristo/datac183c283/
    statc183c283_10stocks.txt", header=T)
```

Convert the prices into returns:

```
r1 <- (a$P1[-length(a$P1)]-a$P1[-1])/a$P1[-1]
r2 <- (a$P2[-length(a$P2)]-a$P2[-1])/a$P2[-1]
r3 <- (a$P3[-length(a$P3)]-a$P3[-1])/a$P3[-1]
r4 <- (a$P4[-length(a$P4)]-a$P4[-1])/a$P4[-1]
r5 <- (a$P5[-length(a$P5)]-a$P5[-1])/a$P5[-1]
```

b. Compute the mean and variance covariance matrix:

```
returns <- as.data.frame(cbind(r1,r2,r3,r4,r5))
mean(returns)
          r1           r2           r3           r4           r5
0.0027625075 0.0035831363 0.0066229478 0.0004543727 0.0045679106

cov(returns)
            r1          r2          r3          r4          r5
r1 0.005803160 0.001389264 0.001666854 0.000789581 0.001351044
r2 0.001389264 0.009458804 0.003944643 0.002281200 0.002578939
r3 0.001666854 0.003944643 0.016293581 0.002863584 0.001469964
r4 0.000789581 0.002281200 0.002863584 0.009595202 0.003210827
r5 0.001351044 0.002578939 0.001469964 0.003210827 0.009242440
```

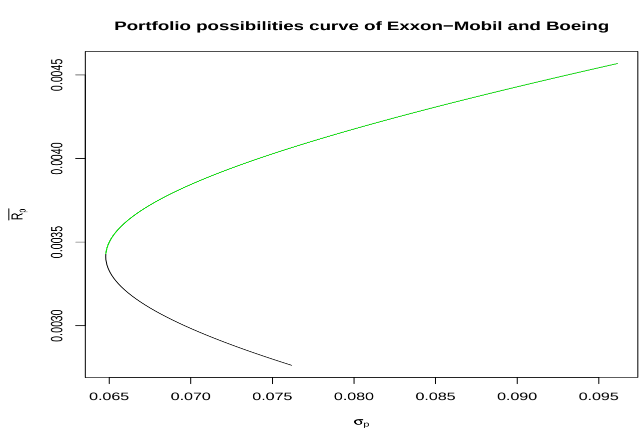
c. Use only Exxon-Mobil and Boeing stocks: For these 2 stocks find the composition, expected return, and standard deviation of the minimum risk portfolio.

```
x1_min <- (var(r5)-cov(r1,r5))/(var(r1)+var(r5)-2*cov(r1,r5))
x2_min <- 1-x1_min

> x1_min
[1] 0.6393153
> x2_min
```

```
[1] 0.3606847

rp_bar_min <- x1_min*mean(r1)+x2_min*mean(r5)
var_p_min <- x1_min^2*var(r1)+x2_min^2*var(r5)+2*x1_min*x2_min*cov(r1,r5)
sd_min <- var_p_min^0.5

> rp_bar_min
[1] 0.003413689
> sd_min
[1] 0.06478695
```

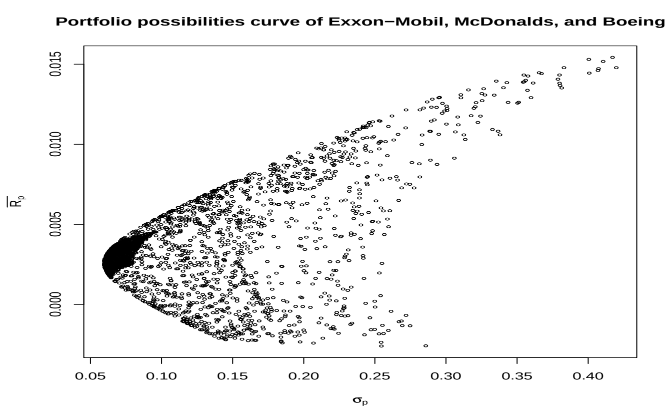
d. Plot and print the portfolio possibilities curve and identify the efficient frontier on it (no short sales).

```
xa <- seq(0,1,0.01)
xb <- 1-xa

r1_bar <- mean(r1)
r5_bar <- mean(r5)
var1 <- var(r1)
var5 <- var(r5)
cov_15 <- cov(r1,r5)

rp_bar <- xa*mean(r1)+xb*mean(r5)
var_p <- xa^2*var(r1)+xb^2*var(r5)+2*xa*xb*cov(r1,r5)
sd_p <- var_p^.5

plot(sd_p,rp_bar, type="l", xlab=expression(sigma[p]),
ylab=expression(bar(R[p])), main="Portfolio possibilities curve of
Exxon-Mobil and Boeing")

#Identify the efficient frontier:
aa <- as.data.frame(cbind(sd_p,rp_bar))
points(aa[aa$rp_bar>rp_bar_min,], col="green", type="l")
```



e. Use only Exxon-Mobil, McDonalds and Boeing stocks and assume short sales are allowed.

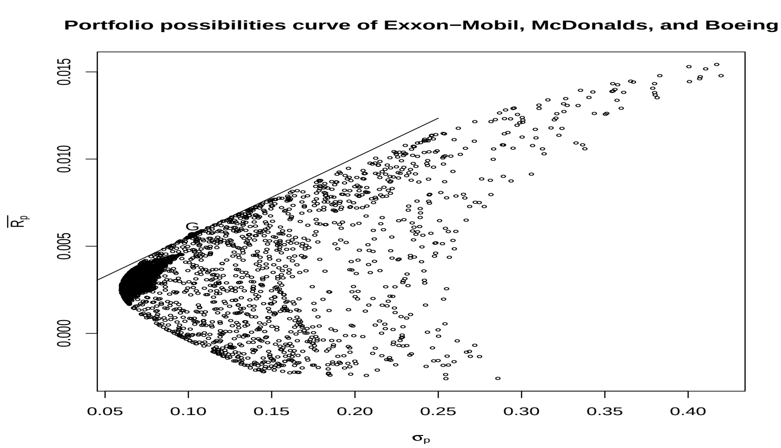
```
#Read the a, b, c combinations of the three stocks:
data <- read.table("http://www.stat.ucla.edu/~nchristo/datac183c283/
   statc183c283_abc.txt", header=T)

#Compute the standard deviation of each portfolio:
sigma_p <- (data$a^2*var(r1)+data$b^2*var(r4)+data$c^2*var(r5)+
2*data$a*data$b*cov(r1,r4)+2*data$a*data$c*cov(r1,r5)+
2*data$b*data$c*cov(r4,r5))^.5

#Compute the expected return of each portfolio:
rp_bar <- data$a*mean(r1)+data$b*mean(r4)+data$c*mean(r5)

plot(sigma_p, rp_bar, xlab=expression(sigma[p]),
ylab=expression(bar(R[p])),
main="Portfolio possibilities curve of Exxon-Mobil,
McDonalds, and Boeing", cex=0.5)
```



f. Assume $R_f = 0.001$ and that short sales are allowed. Find the composition, expected return and standard deviation of the portfolio of the point of tangency G and draw the tangent to the efficient frontier of question (e).

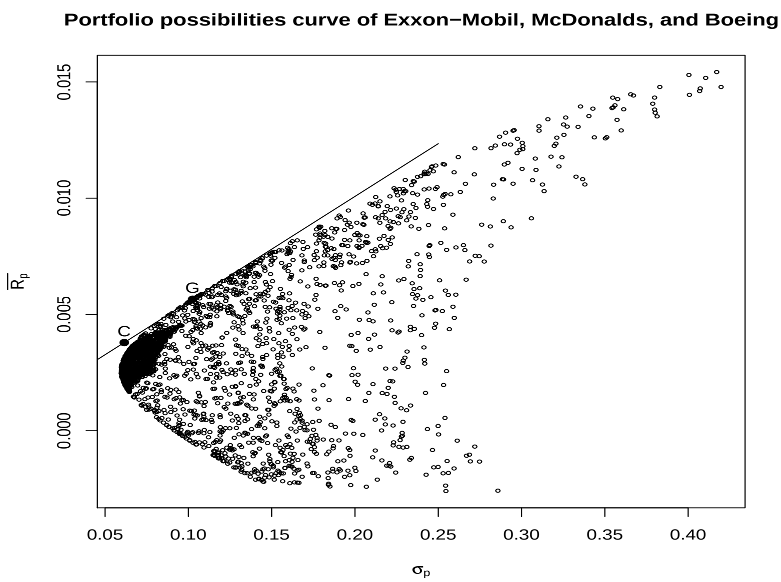
```
a1 <- as.data.frame(cbind(r1,r4,r5))
R_ibar <- as.matrix(mean(a1))

R <- R_ibar-0.001
var_covar <- cov(a1)

var_covar_inv <- solve(var_covar)

z <- var_covar_inv %*% R
```

```
x <- z/sum(z)

> x
          [,1]
r1  0.5284782
r4 -0.4955882
r5  0.9671100

R_Gbar <- t(x) %*% R_ibar

var_G <- t(x) %*% var_covar %*% x
sd_G <- (t(x) %*% var_covar %*% x)^.5

> R_Gbar
            [,1]
[1,] 0.005652415
> sd_G
          [,1]
[1,] 0.1025256

slope <- (R_Gbar-0.001)/(sd_G)

#Find a third point on CAL:
r11 <- .001+slope*.25

segments(0,.001, sd_G, R_Gbar)
segments(sd_G, R_Gbar, .25, r11)

#Identify point G:
points(sd_G, R_Gbar, cex=1, pch=19)
text(sd_G, R_Gbar+0.0005, "G")
```



g. Find the expected return and standard deviation of the portfolio that consists of 60% G 40% risk free asset. Show this position on the capital allocation line (CAL).

```
Rc_bar <- 0.60*R_Gbar + 0.40*0.001
sd_c <- 0.60*sd_G

> Rc_bar
            [,1]
[1,] 0.003791449
> sd_c
           [,1]
[1,] 0.06151535


points(sd_c, Rc_bar, cex=1, pch=19)
text(sd_c, Rc_bar+0.0005, "C")
```



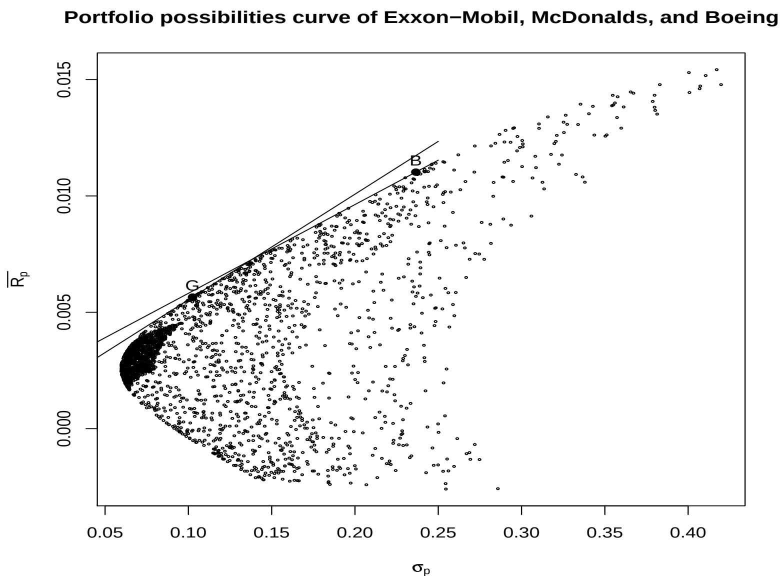
h. Using $R_{f1} = 0.001$ and $R_{f2} = 0.002$ find the composition of two portfolios A and B (tangent to the efficient frontier).

1. Portfolio A is exactly as portfolio G above. For portfolio B we follow the same procedure.

```
a1 <- as.data.frame(cbind(r1,r4,r5))
R_ibar <- as.matrix(mean(a1))

R2 <- R_ibar-0.002
var_covar <- cov(a1)
```

```
var_covar_inv <- solve(var_covar)

z2 <- var_covar_inv %*% R2

x2 <- z2/sum(z2)

> x2
         [,1]
r1  0.5312205
r4 -1.8026632
r5  2.2714427

R_Bbar <- t(x2) %*% R_ibar

var_B <- t(x2) %*% var_covar %*% x2
sd_B <- (t(x2) %*% var_covar %*% x2)^.5

> R_Bbar
           [,1]
[1,] 0.01102417
> sd_B
          [,1]
[1,] 0.2365542

slope2 <- (R_Bbar-0.002)/(sd_B)

#Find a third point on CAL:
r22 <- .002+slope2*.30

segments(0,.002, sd_B, R_Bbar)
segments(sd_B, R_Bbar, .25, r22)

#Identify point B:
points(sd_B, R_Bbar, cex=1, pch=19)
text(sd_B, R_Bbar+0.0005, "B")

#Plot the cloud of points:
plot(sigma_p, rp_bar, xlab=expression(sigma[p]),
ylab=expression(bar(R[p])),
main="Portfolio possibilities curve of Exxon-Mobil,
McDonalds, and Boeing", cex=0.1)

#Draw the tangent when Rf=0.001:
segments(0,.001, sd_G, R_Gbar)
segments(sd_G, R_Gbar, .25, r11)
```

```
#Identify point G:
points(sd_G, R_Gbar, cex=1, pch=19)
text(sd_G, R_Gbar+0.0005, "G")

#Draw the tangent when Rf=0.002:
segments(0,.002, sd_B, R_Bbar)
segments(sd_B, R_Bbar, .25, r22)

#Identify point B:
points(sd_B, R_Bbar, cex=1, pch=19)
text(sd_B, R_Bbar+0.0005, "B")
```



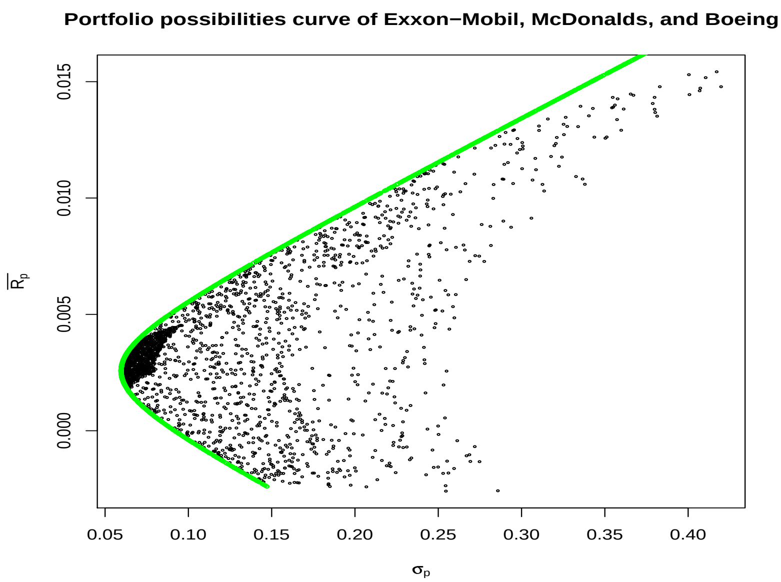
2. Covariance between portfolios A and B:

```
cov_AB <- t(x) %*% var_covar %*% x2

> cov_AB
           [,1]
[1,] 0.02264823
```

3. Trace out the efficient frontier: We have so far the mean and variances of portfolios A, B and their covariance. By allowing short sales and using many combinations of xa, xb we will be able to trace the efficient frontier.

```
xa <- runif(2000, -1.5,2.5)
xb <- 1-xa

sd_t <- (xa^2*var_G + xb^2*var_B + 2*xa*xb*cov_AB)^0.5
R_tbar <- xa*R_Gbar + xb*R_Bbar
```

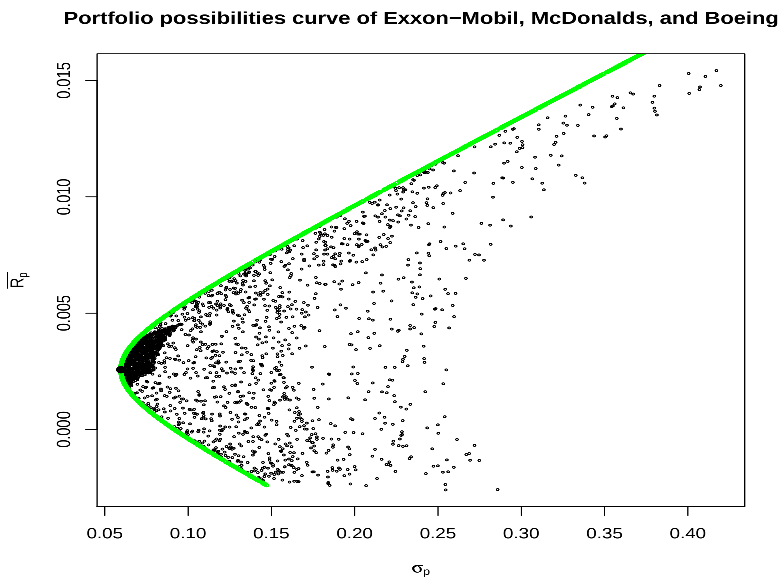
```
#Plot again the cloud of points:
plot(sigma_p, rp_bar, xlab=expression(sigma[p]),
ylab=expression(bar(R[p])),
main="Portfolio possibilities curve of Exxon-Mobil, McDonalds,
and Boeing", cex=0.1)

#Trace out the efficient frontier:
points(sd_t, R_tbar, col="green", cex=0.5)
```



4. Composition of the minimum risk portfolio in terms of Exxon-Mobil, McDonalds, Boeing. We find first the minimum risk portfolio in terms of A, B:

```
xA_min <- (var_B - cov_AB)/(var_B + var_G - 2*cov_AB)
xB_min <- 1-xA_min
#Composition in terms of Exxon-Mobil, McDonalds, and Boeing:
x_xom <- xA_min *  x[1] + xB_min * x2[1]
x_mcd <- xA_min *  x[2] + xB_min * x2[2]
x_boe <- xA_min *  x[3] + xB_min * x2[3]

> x_xom
          [,1]
[1,] 0.5269063
> x_mcd
          [,1]
[1,] 0.2536533
> x_boe
          [,1]
[1,] 0.2194404
```

```
#Verify:
rbar_3 <- x_xom*mean(r1) + x_mcd*mean(r4) + x_boe*mean(r5)

sd_3 <- (x_xom^2*var(r1) + x_mcd^2*var(r4) +
x_boe^2*var(r5)+2*x_xom*x_mcd*cov(r1,r4)+2*x_xom*x_boe*cov(r1,r5) +
2*x_mcd*x_boe*cov(r4,r5))^0.5

> rbar_3
           [,1]
[1,] 0.00257322
> sd_3
           [,1]
[1,] 0.05961942
```



Exercise 2

The solution is based on finding the Z values. Note: You can also solve this problem using the results of exercise 2.

a. Since $X_A = X_B = \frac{1}{2}$ it follows that $Z_A = Z_B$. We get the following system:

$$\begin{aligned} 0.12 - 0.04 &= 0.04Z_A + 0.0016Z_B \\ \bar{R}_B - 0.04 &= 0.0016Z_A + 0.0064Z_B \end{aligned}$$

Solve for Z_A to get $Z_A = 1.9231$. Therefore, $\bar{R}_B = 0.04 + (0.0016 + 0.0064)1.9231 \Rightarrow \bar{R}_B = 0.055385$.

b. Stock B will not be held implies that $X_B = 0$ therefore, $Z_B = 0$.

$$\begin{aligned} 0.12 - 0.04 &= 0.04Z_A + 0.0016Z_B \\ \bar{R}_B - 0.04 &= 0.0016Z_A + 0.0064Z_B \end{aligned}$$

But, $Z_B = 0$, therefore, $Z_A = 2$. It follows that $\bar{R}_B = 0.0432$.

Exercise 3

a. It is given that $var(\frac{1}{2}A + \frac{1}{2}B) = 00525$.
Therefore, $\frac{1}{4}(0.16) + \frac{1}{4}(0.25) + 2\frac{1}{2}\frac{1}{2}\sigma_{AB} = 0.0525$. It follows that $\sigma_{AB} = -0.10$.

b. $\bar{R}_C = (1-x)R_f + x\bar{R}_G = 0.11$. Therefore, $(1-x)(0.05) + x(0.60 \times 0.14 + 0.40 \times 0.10) = 0.11 \Rightarrow x = 0.81$. To obtain combination C we need to invest 81% in portfolio G and 19% in the risk free asses.

c. This is similar to part (b). We want $\bar{R}_C = 0.10$.
$(1-x)(0.05) + x(0.60 \times 0.14 + 0.40 \times 0.10) = 0.10 \Rightarrow x = 0.676$.
Invest $0.676 \times 0.60 = 41\%$ in stock A.
Invest $0.676 \times 0.40 = 27\%$ in stock B.
And $1 - 0.41 - 0.27 = 32\%$ in R_f.

Exercise 4

a. $cov(\sum_{i=1}^n x_i R_i, R_m) = cov(\frac{1}{n}\sum_{i=1}^n R_i, R_m) = cov(\frac{1}{n}\sum_{i=1}^n(\alpha_i + \beta_i R_m + \epsilon_i), R_m) = cov(\bar{\alpha} + \bar{\beta}R_m + \frac{1}{n}\sum_{i=1}^n \epsilon_i, R_m) = cov(\bar{\beta}R_m, R_m) = \bar{\beta}\sigma_m^2$.

b. $var(\sum_{i=1}^n x_i R_i) = var(\frac{1}{n}\sum_{i=1}^n R_i) = \frac{1}{n^2}\sum_{i=1}^n(\delta_0 + \delta_i R_m + \epsilon_i) = \frac{1}{n^2}var(\sum_{i=1}^n \epsilon_i) = \frac{1}{n^2}(\sum_{i=1}^n var(\epsilon_i) + \sum_{i=1}^n \sum_{j \neq i}^n cov(\epsilon_i, \epsilon_j)) = \frac{1}{n^2}(n\sigma^2 + n(n-1)k\sigma^2) = \frac{\sigma^2}{n} + \frac{n-1}{n}k\sigma^2$. As n gets larger, $\sigma_p^2 \approx k\sigma^2$.

c. Assume short sales. In general, the composition of the minimum risk portfolio is given by $\mathbf{x} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$.

d. $cov(R_p, R_i) = cov(\alpha_p + \beta_p R_m + \sum_{i=1}^n x_i \epsilon_i, \alpha_i + \beta_i R_m + \epsilon_i) = \beta_i \beta_p \sigma_m^2 + x_i \sigma_{\epsilon_i}^2$.

Exercise 5

a. The answer is given by $\mathbf{x} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$. (sum of each row of the inverse of the variance covariance matrix divided by the sum of all the elements of the inverse of the variance covariance matrix). Since we have the inverse variance covariance matrix the composition of the minimum risk portfolio is:
$x_1 = \frac{166.21139-22.40241}{166.21139-22.40241-22.40241+220.41076} = 0.4207$, and $x_2 = 1 - x_1 = 0.5793$.

b. We want a combination of A and R_f that has expected return 0.01219724. This is equal to

$$\begin{aligned} 0.01219724 &= (1-x)R_f + x\bar{R}_A \\ 0.01219724 &= (1-x)0.011 + x(0.01315856) \end{aligned}$$

We find $x = 0.5546475$. Therefore the composition of portfolio B will be 55.5% in portfolio A and 44.5% in R_f. Or 44.5% in R_f, $0.555 \times 0.4207 = 0.233$ in stock 1 and $0.555 \times 0.5793 = 0.322$ in stock 2.

c. A better strategy is to find the point of tangency and move up from point B until we reach the tangent. This point will be a combination of the point of tangency G and the risk free asset R_f.

Exercise 6

a. $var(X) = 0.1 - 0.3^2 = 0.01 \Rightarrow sd(X) = 0.1$.
$var(Y) = 0.29 - 0.2^2 = 0.25 \Rightarrow sd(Y) = 0.5$.
$\sigma_{XY} = E(XY) - (EX)(EY) = 0 - (0.3)(0.2) \Rightarrow \sigma_{XY} = -0.06$. Finally, $\rho = \frac{\sigma_{XY}}{sd(X)sd(Y)} = -\frac{0.06}{0.1\times 0.5} = -1.2$, but always $-1 \leq \rho \leq 1$, therefore X and Y cannot possibly have these properties.

b. $\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \times \sigma_B} = \frac{0.79\times 1.120.0022}{\sqrt{0.79^2\times 0.0022+0.027}\times\sqrt{1.12^2 0.0022+0.006}} = 0.12$.

c. $\hat{\beta}_B \pm t_{\frac{\alpha}{2};n-2}\frac{\hat{\sigma}_{\epsilon B}}{\sqrt{\sum_{t=1}^{m}(R_{mt}-\bar{R}_m)^2}}$,

$1.12 \pm 2.000\frac{\sqrt{0.006}}{\sqrt{0.13}}$,

1.12 ± 0.43 or $0.69 \leq \beta_B \leq 1.55$.

d. $R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$, where, $SST = SSR + SSE$. We can find SST using, $SST = \hat{\beta}_A^2 \sum_{t=1}^{m}(R_{Bt} - \bar{R}_B)^2 + (m-2)\hat{\sigma}_{\epsilon A}^2 = 0.79^2(0.13) + 58(0.027) = 1.647$. Therefore, $R^2 = \frac{0.79^2(0.13)}{1.647} = 0.049$.

Exercise 7

a. $\beta_p = \sum_{i=1}^{3} x_i \beta_i = 0.30(1.08) + 0.50(0.80) + 0.20(1.22) = 0.968.$
Also, $\alpha_p = \sum_{i=1}^{3} x_i \alpha_i = 0.30(0.01) + 0.50(0.04) + 0.20(0.08) = 0.039.$

b. The excepted value and variance of the point of tangency G are:
$\bar{R}_p = \alpha_p + \beta_p \bar{R}_m = 0.1358$, where $\alpha_p = \sum_{i=1}^{3} x_i \alpha_i$ and $\beta_p = \sum_{i=1}^{3} x_i \beta_i$ and
$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{3} x_i^2 \sigma_{\epsilon_i}^2 = 0.003684048$ and therefore $\sigma_p = 0.06069636$.
$\bar{R}_C = (1-x)R_f + x\bar{R}_p = -0.6(0.002) + 1.6(0.1358) = 0.21608.$
$\sigma_C = x\sigma_p = 1.6(0.06069636) = 0.097.$

c. $cov(R_p, R_m) = \beta_p \sigma_m^2 = 0.968(0.002) = 0.001936.$

d. Here, $x = 0.60, 1 - x = 0.40$.
$\bar{R}_C = 0.40(0.002) + 0.60(0.1358) = 0.08228.$
$\sigma_C = 0.60\sigma_p = 0.60(0.060696) = 0.03642.$

e. $cov(R_1, R_m) = cov(\alpha_1 + \beta_1 R_m + \epsilon_1, R_m) = \beta_1 \sigma_m^2 = 1.08(0.002) = 0.00216.$