University of California, Los Angeles
Department of Statistics

Statistics C183/C283

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Final exam
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Name: SOLUTIONS

Problem 1 (20 points)
Answer the following questions:

a. Suppose the variable X follows the generalized Wiener process with drift rate μ_X and variance σ_X^2, and the variable Y follows the generalized Wiener process with drift rate μ_Y and variance σ_Y^2. Initially the variable X has the value α and the variable Y the value β. What is the distribution of $X+Y$ after time Δt if:

1. The changes in X and Y in any short time interval Δt are uncorrelated?

$$X \sim N(a + \mu_X \Delta t, \sigma_X \sqrt{\Delta t}), \quad Y \sim N(b + \mu_Y \Delta t, \sigma_Y \sqrt{\Delta t})$$

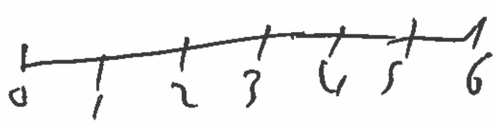
$$X+Y \sim N\left[a + b + (\mu_X + \mu_Y)\Delta t, \sqrt{(\sigma_X^2 + \sigma_Y^2)\Delta t}\right)$$

2. There is a correlation ρ between the changes in X and Y in any short time interval Δt?

$$X+Y \sim N\left(a + b + (\mu_X + \mu_Y)\Delta t, \sqrt{(\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)\Delta t}\right)$$

$dz = \varepsilon\sqrt{\Delta t} \qquad \varepsilon \sim N(0,1)$

b. Consider a variable S that follows the process $dS = \mu dt + \sigma dz$. For the first three years, $\mu = 2$ and $\sigma = 3$. For the the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable S is 5, what is the probability distribution of the variable at the end of year 6?



$$S_6 = S_0 + \Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 + \Delta S_5 + \Delta S_6$$

$E\Delta S_1 = 2 + 0 = 2$, $E\Delta S_2 = 2$, $E\Delta S_3 = 2$ || $E\Delta S_4 = 3 + 0 = 3$, $E\Delta S_5 = 3$, $E\Delta S_6 = 3$

$$E S_6 = 5 + 3(2) + 3(3) = 20$$

$$\mathrm{VAR}(S_6) = \mathrm{VAR}(\Delta S_1) + \cdots + \mathrm{VAR}(\Delta S_6) = 3 \times 3^2 + 3 \times 4^2 = 75$$

$$S_6 \sim N(20, \sqrt{75}) \text{ or } S_6 \sim N(20, 8.66)$$

Problem 2 (20 points)
Answer the following questions:

$\frac{\partial c}{\partial t} = 0$

a. Let $c = S^{-\frac{2r}{\sigma^2}}$. Does c satisfy the Black-Scholes differential equation?

$$\frac{\partial c}{\partial S} = -\frac{2r}{\sigma^2} S^{-\frac{2r}{\sigma^2}-1} \qquad \frac{\partial^2 c}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right) S^{-\frac{2r}{\sigma^2}-2}$$

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = \cdots = rc \qquad \text{YES.}$$

b. Suppose the volatility for a stock goes to zero, i.e. $\sigma \to 0$. It means the stock is riskless and must earn the risk free interest rate. Therefore, at expiration time of a call option, $S_T = S_0 e^{rt}$. What is the value of the call option at time zero (now)?

$$c = e^{-rt} * \text{MAX}\left[S_0 e^{rt} - E,\ 0\right]$$

$$\text{OR} \qquad C = \text{MAX}\left[S_0 - \frac{E}{e^{rt}},\ 0\right]$$

c. What is the result obtained by the Black-Scholes model for the situation in (b)?

SUPPOSE $S_0 > E/e^{rt} \rightarrow \ln S_0/E > -rt \Rightarrow \ln\frac{S_0}{E} + rt > 0$

SO $d_1 = \frac{\ln S_0/E + (r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{>0}{0} = \infty \rightarrow d_2 = \infty$

$\therefore \Phi(d_1) = \Phi(d_2) = 1 \Rightarrow c = S_0 - E/e^{rt}$ BLACK-SCHOLES

SUPPOSE, $S_0 < E/e^{rt} \Rightarrow \ln S_0/E + rt < 0$

SO $d_1 = \frac{<0}{0} = -\infty \rightarrow d_2 = -\infty$

$\Phi(d_1) = \Phi(d_2) = 0 \rightarrow C = 0$ BLACK-SCHOLES

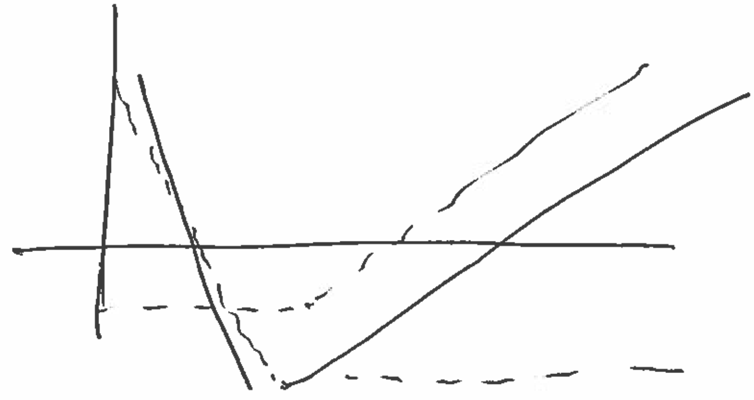
d. A *straddle* is an option trading strategy where the investor buys a put and a call with the same expiration date and exercise price.

1. Construct a table that shows the *payoff* of the put, the call, and the total. Please do not use numbers. Use E, S_T, etc.

S_T	LONG CALL	LONG PUT	TOTAL
$S < E$	0	$E-S$	$E-S$
$S > E$	$S-E$	0	$S-E$

2. Draw the diagram that shows the *profit* of the put, the call, and the total. Again, no numbers!

PROFIT



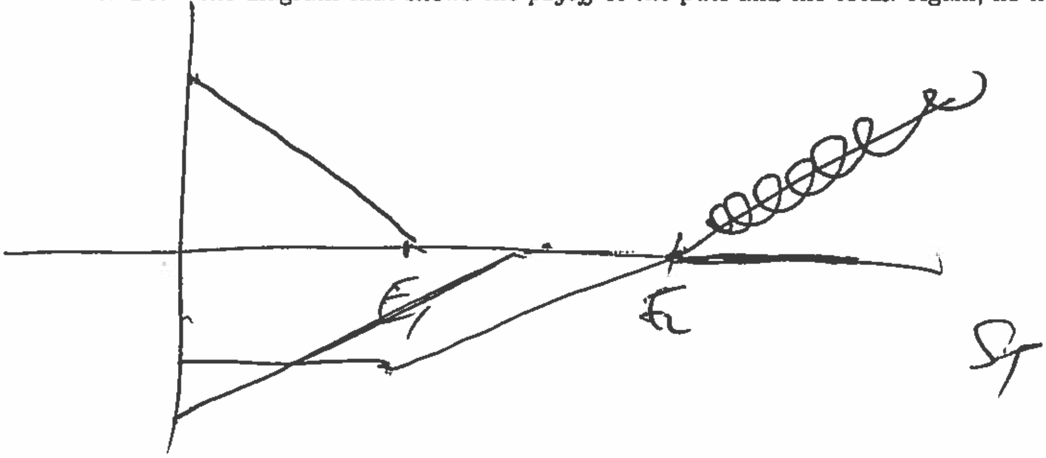
S_T

Problem 2 (20 points) BUY PUT WITH E_1 || SELL PUT WITH E_2
Part A:
Consider a bull spread when puts with exercise prices E_1 and E_2, with $E_2 > E_1$, are used.

a. Construct a table that shows the *payoff* of the puts and the total. Please do not use numbers. Use E, S_T, etc.

S_T	LONG PUT	SHORT PUT	TOTAL
$S_T > E_2$	0	0	0
$E_1 < S_T < E_2$	0	$S_T - E_2$	$S_T - E_2$
$S_T < E_1$	$E_1 - S_T$	$S_T - E_2$	$E_1 - E_2$

b. Draw the diagram that shows the *payoff* of the puts and the total. Again, no numbers!

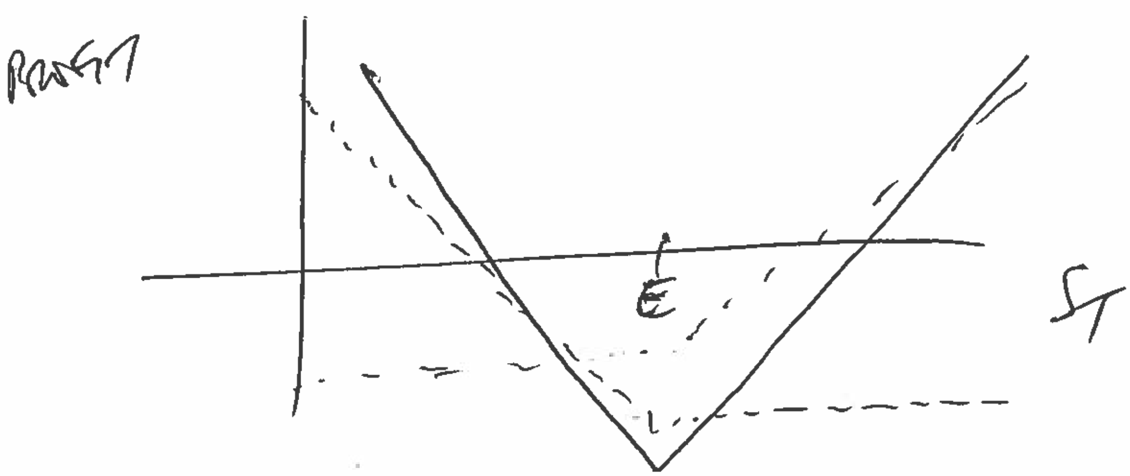


Part B:
A *straddle* is an option trading strategy where the investor buys a put and a call with the same expiration date and exercise price.

a. Construct a table that shows the *payoff* of the put, the call, and the total. Please do not use numbers. Use E, S_T, etc.

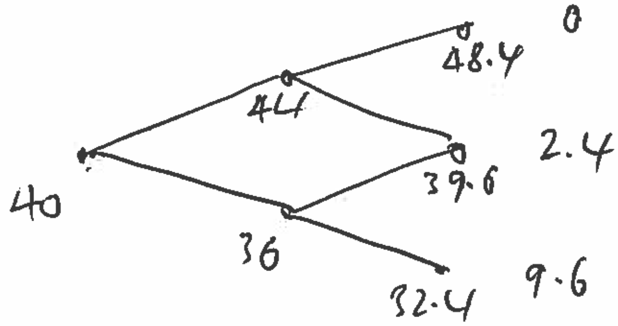
S_T	LONG CALL	LONG PUT	TOTAL
$S < E$	0	$E - S$	$E - S$
$S > E$	$S - E$	0	$S - E$

b. Draw the diagram that shows the *profit* of the put, the call, and the total. Again, no numbers!



Problem 3 (15 points)
Answer the following questions:

a. The price of a stock at time $t = 0$ is \$40. Over each of the next two 3-month periods it is expected to increase by 10% or decrease by 10%. The risk-free continuous interest rate is 12% per year. What is the value of a 6-month European put option with exercise price of \$42? Show all your work and place all the values on a 2-step binomial tree.



$u = 1.1 \quad d = 0.9$

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.12 \frac{1}{4}} - 0.9}{1.1 - 0.9} = 0.65227$$

$$1 - p = 0.34773$$

$$\frac{0 \cdot p + 2.4(1-p)}{e^{0.12 \frac{1}{4}}} = 0.80988 \qquad \frac{2.4p + 9.6(1-p)}{e^{0.12 \frac{1}{4}}} = 4.75873$$

FINALLY, $P = \dfrac{0.80988\, p + 4.75873(1-p)}{e^{0.12 \frac{1}{4}}} \Rightarrow \underline{P = 2.11850}$

b. Suppose the return of the underlying stock of a European call is equal to the risk-free interest rate. Show that the probability that a European call option will be exercised at time T is equal to $\Phi(d_2)$. Assume lognormal property of stock prices. Also, time now is 0, therefore $\Delta t = T$.

$\ln S_T \sim N\left(\ln S_0 + (r - \tfrac{1}{2}\sigma^2)T,\ \sigma\sqrt{T}\right)$ because $\mu = r$

$$P(S_T > E) = P(\ln S_T > \ln E) = P\left(Z > \frac{\ln E - \ln S_0 - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z > \frac{\ln \frac{E}{S_0} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = P\left(Z < \frac{\ln S/E + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = \Phi(d_2)$$

c. Refer to part (b): Again, the underlying stock earns the risk-free interest rate. Give an expression of the value of the European call that pays off \$100 if the price of the stock at time T is greater than E.

PAYOFF AT EXPIRATION IS: $100\, P(S_T > E) = 100\, \Phi(d_2)$

SINCE THIS IS RISKLESS THE P.V. WILL BE THE PRICE OF THE CALL DISCOUNTED USING CONTINUOUS RISK-FREE INTEREST RATE.

$$C = \frac{100\, \Phi(d_2)}{e^{rt}}$$

Problem 4 **(20 points)**
Answer the following questions:

a. Assume that the price S of stock A follows the lognormal distribution. Its current value is $50, with expected return and volatility 12% and 30% respectively per year. What is the probability that the stock price will be larger than $80 in two years?

$$P(S_T > 80) = P(\ln S_T > \ln 80) = P\left(Z > \frac{\ln 80 - \ln 50 - \left(0.12 - \frac{0.3^2}{2}\right)2}{0.3\sqrt{2}}\right)$$

$$= P(Z > 0.75) = 1 - 0.7734 = \underline{0.2266}.$$

b. Refer to question (a). A European put is written on stock A with expiration date 6 months from now and with exercise price $60. What is the probability that this put option will not be exercised?

$$P(S_T > 60) = P(\ln S_T > \ln 60)$$

$$= P\left[Z > \frac{\ln 60 - \ln 50 - \left(0.12 - \frac{0.3^2}{2}\right)0.5}{0.3\sqrt{0.5}}\right)$$

$$= P(Z > 0.68) = 1 - 0.7517 = \underline{0.2483}$$

c. Suppose a call option is currently prices at $110. You want to estimate volatility by trial and error using the Black-Scholes formula for c. You start with an initial guess of $\sigma = 0.30$ that gives $c = \$115$. What should be your next guess for σ? Explain!

$C = 110$ WITH $\sigma_1 = 0.30$ WE GET $C_1 = 115$

$\therefore$ OUR NEXT GUESS SHOULD BE $\sigma < 0.30$.

d. Consider the binomial option pricing model for a European put, with exercise price $52, current stock price $50, $u = 1.2$, $d = 0.8$ for a 30-period binomial tree. Find the maximum number of up movements so that the put will be in the money at expiration.

WE WANT $S_0 u^k d^{30-k} < 52$

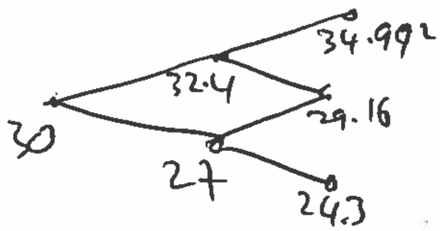
$$\left(\frac{u}{d}\right)^k < \frac{52}{S_0 d^{30}} \Rightarrow k < \frac{\log\left(\frac{52}{50(0.8)^{30}}\right)}{\log \frac{1.2}{0.8}}$$

$$\underline{k \le 16}.$$

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.05 \frac{2}{12}} - 0.9}{1.08 - 0.9} = 0.6020$$

Problem 5 (25 points)
Answer the following questions:

a. A stock price is currently \$30. During each 2-month period for the next 4 months the stock will increase by 8% or decrease by 10%. The risk-free continuous interest rate is 5% per year. Use a two-step binomial tree to calculate the value of an option that pays off at expiration amount equal to $max[(30 - S_T), 0]^2$, where S_T is the price of the stock in 4 months.



0

$0.84^2 = 0.7056$

$5.7^2 = 32.49$

VALUE of THE OPTION :

$$\frac{0.7056 \times 2 \times p(1-p) + 32.49\,(1-p)^2}{e^{0.05 \frac{4}{12}}}$$

$\Rightarrow \quad c = 5.394$

b. Assume the Black-Scholes model applies. Consider an option on a non-dividend paying stock when the stock price is \$30, the exercise price is \$29, the continuously risk-free interest rate 5%, the volatility is 25% per year, and the time to expiration is 4 months.

1. What is the price of the option if it is a European call?

$$c = S_0\,\phi(d_1) - E e^{-rt}\phi(d_2) = 30\,\phi(0.4225) - 29\,e^{-0.05 \frac{4}{12}}\,\phi(0.2782)$$

$$\boxed{c = 2.4778}$$

$$d_1 = \frac{\ln\frac{30}{29} + \left(0.05 + \frac{0.25^2}{2}\right)\frac{1}{3}}{0.25\sqrt{\frac{1}{3}}} = 0.4225$$

$$\phi(d_1) = 0.6628$$

$$d_2 = d_1 - 0.25\sqrt{\frac{1}{3}} = 0.2782$$

$$\phi(d_2) = 0.6103$$

2. What is the price of the option if it is an American call?

SAME!

3. What is the price of the option if it is a European put?

$$p + S_0 = c + \frac{E}{e^{rt}}$$

$$p = 2.4778 + \frac{29}{e^{0.05 \cdot 4/12}} - 30 \quad \Rightarrow \quad p = 0.9989$$

c. A stock price is observed weekly with S_i being the ith observation. Define $u_i = ln(S_i/S_{i-1})$. Suppose that there are 40 observations on u_i and $\sum_{i=1}^{40} u_i = 0.18$ while $\sum_{i=1}^{40} u_i^2 = 0.06$. Estimate the stock price volatility per year.

WEEKLY:

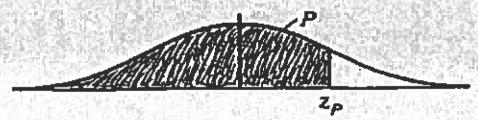
$$s = \left(\frac{1}{40-1}\left[0.06 - \frac{0.18^2}{40}\right]\right)^{1/2} = 0.03896$$

$$\sigma_{\text{ANNUAL}} = 0.03896\sqrt{52} \Rightarrow \sigma = 0.2809$$

The standard normal distribution table. Note: $P(Z \leq 1.13) = 0.8708$.

TABLE 2

Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998