

University of California, Los Angeles  
Department of Statistics

Statistics C183/C283

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Homework 1 - Solutions

**Exercise 1**

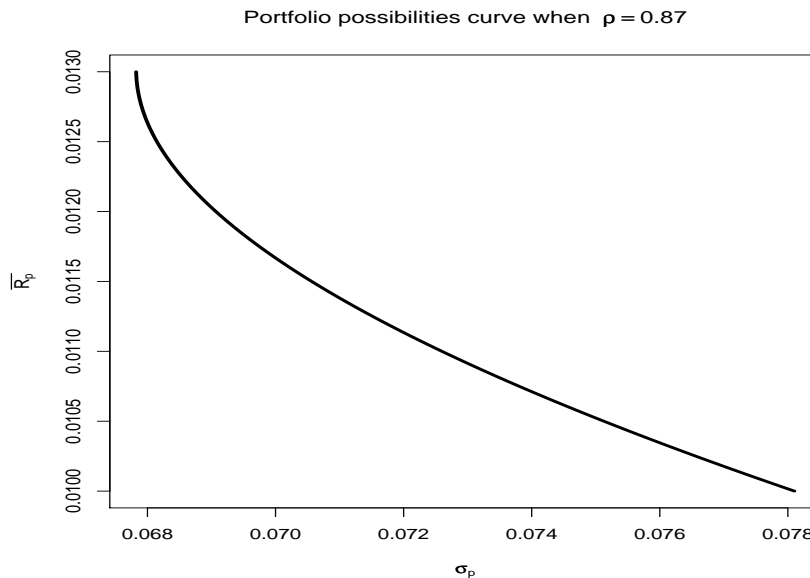
In this situation all the available funds will be invested in the least risky asset, therefore  $x_2 = 1, x_1 = 0$ . We know that

$$x_2 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

We set  $x_2 = 1$  and solve for  $\rho$  to get  $\rho = \frac{\sigma_2}{\sigma_1}$ . Therefore if the correlation coefficient is larger than the ratio of the smaller to larger standard deviation diversification is not useful in reducing the risk. In our example if  $\rho$  is larger than  $\frac{\sqrt{0.0046}}{\sqrt{0.0061}} = 0.87$  then all funds should be invested in TEXACO (least risky stock).

**Exercise 2**

Using  $\rho = 0.87$  we construct the portfolio possibilities curve shown below.



**Exercise 3**

Again we use the expressions for  $x_1$  and  $x_2$  that we found in class:

$$x_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \frac{0.0046 - (-1)\sqrt{0.0061}\sqrt{0.0046}}{0.0061 + 0.0046 - 2(-1)\sqrt{0.0061}\sqrt{0.0046}} = 0.465.$$

And,  $x_2 = 1 - x_1 = 1 - 0.465 = 0.535$ . For this combination (46.5% IBM, 53.5% TEXACO) the risk will be zero. This is because  $\rho = -1$ .

**Exercise 4**

The variance of an equally weighted portfolio is equal to

$$\sigma_p^2 = \frac{1}{N}(\bar{\sigma}_i^2 - \bar{\sigma}_{ij}) + \bar{\sigma}_{ij},$$

and when  $\bar{\sigma}_i^2 = 50$ ,  $\bar{\sigma}_{ij} = 10$  we get

$N$	$\sigma_p^2$
5	18
10	14
20	12
50	10.8
100	10.4

**Exercise 5**

Let's choose  $\rho = 0.95$ . To find  $x_1, x_2$  we use the expression from exercise 3:

$$x_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \frac{0.0046 - (0.95)\sqrt{0.0061}\sqrt{0.0046}}{0.0061 + 0.0046 - 2(0.95)\sqrt{0.0061}\sqrt{0.0046}} = -0.68.$$

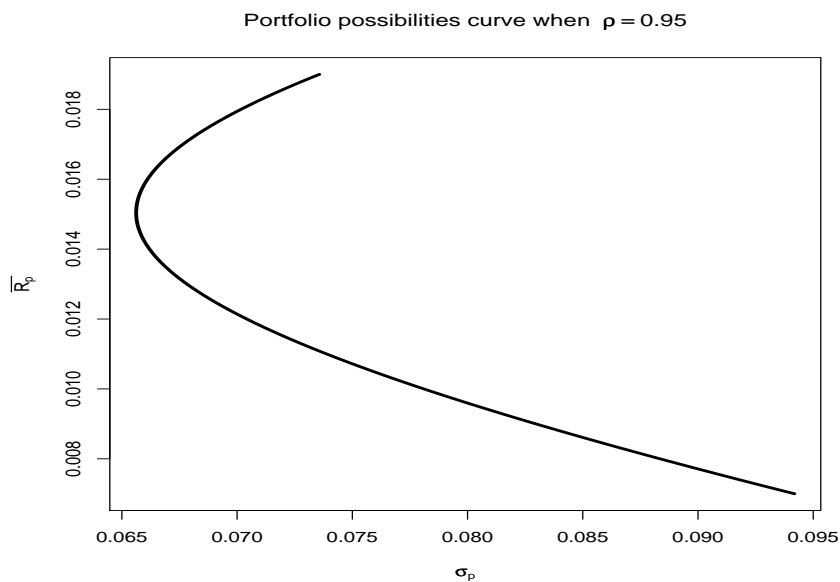
And,  $x_2 = 1 - x_1 = 1 - (-0.68) = 1.68$ .

This means that IBM should be held short. This combination  $(-68\%, 168\%)$  will give us risk less than the least risky stock. The risk will be equal to

$$\sigma_p^2 = (-0.68)^2(0.0061) + 1.68^2(0.0046) + 2(-0.68)(1.68)(0.95)\sqrt{0.0061}\sqrt{0.0046} = 0.0043.$$

We observe that the variance of this combination is less than the variance of the two stocks.

Using  $\rho = 0.95$  we construct the portfolio possibilities curve shown below.

**Exercise 6**

See solutions of lab 1 on the course website.

## Exercise 7

- a. We need the elements of the inverse of the variance covariance matrix. Then we sum each row of the inverse and divide by the sum of all the elements of the inverse. Here are the R commands:

```
xx <- diag(c(0.010025, 0.002123, 0.005775), 3,3)
xx
      [,1] [,2] [,3]
[1,] 0.010025 0.000000 0.000000
[2,] 0.000000 0.002123 0.000000
[3,] 0.000000 0.000000 0.005775

xxinv <- solve(xx)
xxinv
      [,1] [,2] [,3]
[1,] 99.75062 0.0000 0.0000
[2,] 0.00000 471.0316 0.0000
[3,] 0.00000 0.0000 173.1602

x1 <- sum(xxinv[1,]) / sum(xxinv)
x2 <- sum(xxinv[2,]) / sum(xxinv)
x3 <- sum(xxinv[3,]) / sum(xxinv)

x1; x2; x3
[1] 0.1340838
[1] 0.633156
[1] 0.2327602
```

- b. We first find the vector  $\mathbf{Z}$ :

$$\mathbf{Z} = \Sigma^{-1} \mathbf{R}_1 = \begin{pmatrix} 0.010025 & 0.000000 & 0.000000 \\ 0.000000 & 0.002123 & 0.000000 \\ 0.000000 & 0.000000 & 0.005775 \end{pmatrix}^{-1} \begin{pmatrix} 0.005174 - 0.001 \\ 0.010617 - 0.001 \\ 0.016947 - 0.001 \end{pmatrix} = \begin{pmatrix} 0.4163591 \\ 4.5299105 \\ 2.7613853 \end{pmatrix}.$$

The sum of the  $z_i$ 's is  $\sum_{i=1}^3 z_i = 7.707655$ .

Therefore  $x_1 = \frac{0.4163591}{7.707655} = 0.05401891$ ,  $x_2 = \frac{4.5299105}{7.707655} = 0.58771579$ ,  $x_3 = \frac{2.7613853}{7.707655} = 0.35826530$ .

Compute the mean and variance of the point of tangency  $G$ :

$$\bar{R}_G = \mathbf{x}'_G \bar{\mathbf{R}} = 0.01259079.$$

$$\sigma_G^2 = \mathbf{x}'_G \Sigma \mathbf{x}_G = 0.001503803.$$

## Exercise 8

1. The investor will move up from point  $A$  until the tangent, or move to the left of point  $A$ , again until the tangent.
2. Portfolio  $Z$  cannot be on the efficient frontier.
3.  $\bar{R}_p = \sum_{i=1}^n x_i \bar{R}_i = \frac{1}{25} \sum_{i=1}^{25} \bar{R}_i = \frac{25(0.15)}{25} = 0.15$ .

$$\sigma_p^2 = \sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j \sigma_{ij} = \frac{1}{25^2} 25(0.60^2) + 25(24) \frac{1}{25} \frac{1}{25} 0.5(0.60)(0.60) = 0.1872.$$

Therefore,  $\sigma_p = \sqrt{0.1872} = 0.4327$ .

In general,  $\sigma_p^2 = \frac{\sigma^2}{n} + \frac{n-1}{n} \rho \sigma^2$ .

4. We want  $\sigma_p^2 = \frac{\sigma^2}{n} + \frac{n-1}{n}\rho\sigma^2 < 0.43^2$ . Solve for  $n$  to get  $n \approx 37$ .

### Exercise 9

The solution is based on  $\mathbf{Z} = \mathbf{\Sigma}^{-1}\mathbf{R}$ , where,  $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ , and  $\mathbf{R} = \begin{pmatrix} \bar{R}_A = \bar{R}_1 - R_f \\ \bar{R}_B = \bar{R}_2 - R_f \end{pmatrix}$ .

The inverse of  $\mathbf{\Sigma}$  is  $\mathbf{\Sigma}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$ .

Therefore,  $\mathbf{Z} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \bar{R}_A \\ \bar{R}_B \end{pmatrix}$ .

It follows that  $Z_1 = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$  and  $Z_2 = \frac{-\bar{R}_A\rho\sigma_1\sigma_2 + \bar{R}_B\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$ .

Finally,  $X_1 = \frac{Z_1}{Z_1 + Z_2} = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\rho\sigma_1\sigma_2}{\bar{R}_A\sigma_1^2 + \bar{R}_B\sigma_2^2 - (\bar{R}_A + \bar{R}_B)\rho\sigma_1\sigma_2}$ , or  $X_1 = \frac{\bar{R}_A\sigma_2^2 - \bar{R}_B\sigma_{12}}{\bar{R}_A\sigma_1^2 + \bar{R}_B\sigma_2^2 - (\bar{R}_A + \bar{R}_B)\sigma_{12}}$ , and  $X_2 = 1 - X_1$ .

### Exercise 10

a. We want to minimize

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\ \text{subject to} \quad & E = R_f + (\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{x} \end{aligned}$$

Using the method of Lagrange multipliers we minimize:

$$\min S = \frac{1}{2}\mathbf{x}'\mathbf{\Sigma}\mathbf{x} - \lambda(R_f + (\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{x} - E)$$

We take derivatives w.r.t.  $\mathbf{x}$  and  $\lambda$ .

$$\begin{aligned} \frac{\partial S}{\partial \mathbf{x}} &= \mathbf{\Sigma}\mathbf{x} - \lambda(\bar{\mathbf{R}} - R_f\mathbf{1}) = \mathbf{0} \Rightarrow \mathbf{x} = \lambda\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1}) \\ \frac{\partial S}{\partial \lambda} &= R_f + (\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{x} - E = 0 \Rightarrow (\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{x} = E - R_f \end{aligned}$$

Therefore,

$$(\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{x} = E - R_f = (\bar{\mathbf{R}} - R_f\mathbf{1})'\lambda\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1}) \Rightarrow \lambda = \frac{E - R_f}{(\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1})}.$$

And finally,

$$\mathbf{x} = \frac{E - R_f}{(\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1})}\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1}).$$

b. The minimized variance is  $\sigma^2 = \mathbf{x}'\mathbf{\Sigma}\mathbf{x}$ , where  $\mathbf{x}$  is the result from (a). Substitute  $\mathbf{x}$  into  $\sigma^2$  to get the result. The result shows that  $E$  is linear in  $\sigma$ . All the combination between  $R_f$  and  $\mathbf{x}$  will be on a straight line with slope  $\sqrt{(\bar{\mathbf{R}} - R_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\bar{\mathbf{R}} - R_f\mathbf{1})}$ .