University of California, Los Angeles
Department of Statistics

Statistics C183/C283

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Exam 1
04 May 2009

Name: SOLUTIONS

Problem 1 (20 points)
Answer the following questions:

a. Suppose the average beta for a group of stocks for the years 2004-08 (this is the historical period with the subscript 1) is $\bar{\beta}_1 = 1.0$ and their variance is $\sigma^2_{\bar{\beta}1} = 0.25$. The estimate of beta of stock A obtained from the regression of the returns of stock A on the market for the same period is $\hat{\beta}_A = 1.183$ and its variance is $\sigma^2_{\hat{\beta}_{A1}} = 0.22$. What it your best forecast of beta for stock A using the Vasicek's technique?

$$\beta_V = \frac{0.22}{0.25+0.22}\,1 + \frac{0.25}{0.25+0.22}\,1.183 = 1.097$$

b. You are given that the variance of the return of the market is $\sigma^2_m = 0.2091$ and the covariance between the returns of stock A and the market is $\sigma_{Am} = 0.2474$. Find the beta of stock A.

$$\hat{\beta}_A = \frac{\sigma_{Am}}{\sigma_m^2} = \frac{0.2474}{0.2091} = 1.183$$

c. Portfolios A and B were constructed using the single index model. The beta of portfolio A is $\beta_{PA} = 0.9$ while the beta of portfolio B is $\beta_{PB} = 1.2$. If the variance of the returns of the market is $\sigma^2_m = 0.30$ find the covariance between portfolio A and portfolio B.

$$\sigma_{AB} = (0.9)(1.2)\,0.30 = 0.324$$

$$\mathrm{cov}\left[a_{PA} + b_{PA} R_m + \sum x_i \varepsilon_i,\ \alpha_{PB} + \beta_{PB} R_m + \sum x_i \varepsilon_i\right]$$

d. Assume that the single index model holds. The characteristics of two stocks A and B are the following:

Stock	α	β	σ^2_ϵ
A	-0.0022	1.06	0.01
B	0.0084	0.81	0.05

If $\sigma^2_m = 0.002$ find the correlation coefficient between stocks A and B.

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{(1.06)(0.81)\,0.002}{\sqrt{1.06^2\,0.002 + 0.01}\ \sqrt{0.81^2\,0.002 + 0.05}} = 0.0685$$

Problem 2 **(20 points)**
Answer the following questions:

a. Suppose stocks A and B are chemical stocks and stocks C and D are utility stocks. Let I_C and I_U the indexes for the chemical and utility stocks respectively, and let R_m the market index with variance σ_m^2. The following regression models hold:

$$
\begin{aligned}
R_A &= \alpha_A + \beta_A I_{CH} + \epsilon_A \\
R_B &= \alpha_B + \beta_B I_{CH} + \epsilon_B \\
\hline
R_C &= \alpha_C + \beta_C I_U + \epsilon_C \\
R_D &= \alpha_D + \beta_D I_U + \epsilon_D
\end{aligned}
$$

and

$$
\begin{aligned}
I_{CH} &= \gamma_C + b_{CH} R_m + \delta_{CH} \\
I_U &= \gamma_U + b_U R_m + \delta_U
\end{aligned}
$$

Assumptions: All the error terms are independent, and all the indexes are independent from all the error terms. Also, $var(\epsilon_A) = \sigma_{\epsilon A}^2$, $var(\epsilon_B) = \sigma_{\epsilon B}^2$, $var(\epsilon_C) = \sigma_{\epsilon C}^2$, $var(\epsilon_D) = \sigma_{\epsilon D}^2$, $var(\delta_C) = \sigma_{\delta C}^2$, and $var(\delta_U) = \sigma_{\delta U}^2$.

1. Write down the covariance between stocks A and B.

$$\sigma_{AB} = Cov[\alpha_A + \beta_A I_{CH} + \epsilon_A , \alpha_B + \beta_B I_{CH} + \epsilon_B] = Cov[\beta_A I_{CH}, \beta_B I_{CH}]$$
$$= \beta_A \beta_B \sigma_{CH}^2 = \beta_A \beta_B (b_{CH}^2 \sigma_M^2 + \sigma_{\delta CH}^2)$$

2. Write down the covariance between stocks A and C.

$$\sigma_{AC} = Cov[\alpha_A + \beta_A I_{CH} + \epsilon_A , \alpha_C + \beta_C I_U + \epsilon_C) = Cov(\beta_A I_{CH}, \beta_C I_U)$$
$$= \beta_A \beta_C Cov(I_{CH}, I_U) = \beta_A \beta_C b_{CH} b_U \sigma_M^2$$

b. Suppose the multi group model holds, and we have data for three industries with 10 stocks in each industry. The 3×3 correlation matrix using the assumption of the multi group model is given below:

$$
\bar{\rho} = \begin{array}{c|ccc}
 & 1 & 2 & 3 \\
1 & 0.4073 & 0.0403 & 0.0722 \\
2 & 0.0403 & 0.4633 & 0.1275 \\
3 & 0.0722 & 0.1275 & 0.0960
\end{array}
$$

1. Suppose the first stock in the first industry has $\sigma_1 = 0.15$ and the last stock in the third industry has $\sigma_{30} = 0.05$. Compute the covariance between these two stocks.

$$\sigma_{1,30} = \sigma_1 \sigma_{30} \bar{\rho}_{13} = (0.15)(0.05)\,0.0722 = 0.0005415$$

2. Suppose short sales are allowed. Using the multi group model the optimum portfolio (point of tangency) can be found by solving $\Phi = A^{-1}C$. Once we obtained the elements of the vector Φ we can compute the z_i's and from there the x_i's. Write down the elements of the matrix A and the vector C.

$$
A = \begin{bmatrix}
1 + \frac{n_1 \rho_{11}}{1-\rho_{11}} & \frac{n_1 \rho_{12}}{1-\rho_{11}} & \frac{n_1 \rho_{13}}{1-\rho_{11}} \\
\frac{n_2 \rho_{21}}{1-\rho_{22}} & 1 + \frac{n_2 \rho_{22}}{1-\rho_{22}} & \frac{n_2 \rho_{23}}{1-\rho_{22}} \\
\frac{n_3 \rho_{31}}{1-\rho_{33}} & \frac{n_3 \rho_{32}}{1-\rho_{33}} & 1 + \frac{n_3 \rho_{33}}{1-\rho_{33}}
\end{bmatrix}
\qquad
C = \begin{bmatrix}
\sum_{n_1} \frac{\bar{R}_i - R_F}{\sigma_i (1-\rho_{11})} \\
\sum_{n_2} \frac{\bar{R}_i - R_F}{\sigma_i (1-\rho_{22})} \\
\sum_{n_3} \frac{\bar{R}_i - R_F}{\sigma_i (1-\rho_{33})}
\end{bmatrix}
$$

Problem 3 (20 points)

Suppose that the single index model holds, $R_f = 0.002$, $\sigma_m^2 = 0.002548013$, and $\bar{R}_m = 0.003602158$. Using the single index model we obtain the following variance covariance matrix for three stocks 1, 2, and 3.

```
> var_covar
            [,1]        [,2]        [,3]
[1,] 0.013422089 0.005415671 0.002142743
[2,] 0.005415671 0.024634485 0.002914967
[3,] 0.002142743 0.002914967 0.010748691
```

Also, the mean returns of the three stocks are:

```
> R_bar
R1 0.005274547
R2 0.001527333
R3 0.010364922
```

The table below show the ranking of the stocks based on the excess return to beta ratio:

```
> table1
     stock         Ratio        col1      col2      col3      col4        col5
[1,]     3  0.0124333171  0.58650977 0.5865098  47.17243  47.17243 0.001334083
[2,]     1  0.0026197367  0.43353212 1.0200419 165.48690 ????????? ???????????
[3,]     2 -0.0002779702 -0.04654724 0.9734946 167.45407 380.11340 0.001260063
```

a. Find the two missing values in the table1 above.

$47.17243 + 165.48690 = 212.662$

$$C_2 = \frac{\sigma_m^2 \text{ col } 2}{1 + \sigma_m^2 \text{ col } 4} = \frac{0.002548013\,(1.0200419)}{1 + 0.002548013\,(212.662)} = 0.001685672$$

b. The last column in table1 are the C_i's. What is the value of C^* when short sales are allowed, and when short sales are not allowed?

ALLOWED : $C^* = 0.001260063$

NOT ALLOWED : $C^* = 0.001685672$

c. When short sales are allowed the composition of the optimum portfolio is:

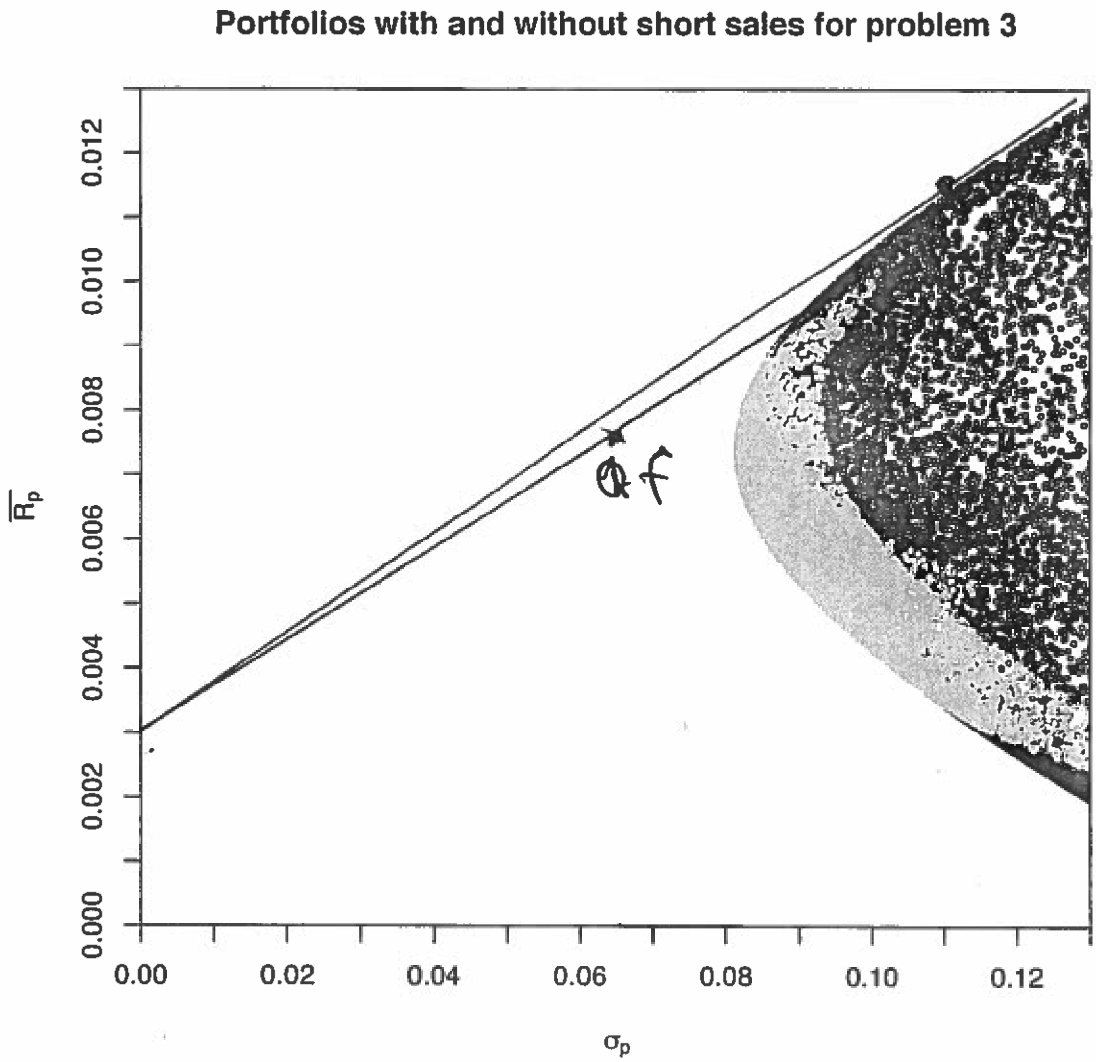
```
> short_sales_composition
     stock    x_short
[1,]     3  0.9648372
[2,]     1  0.2217000
[3,]     2 -0.1865372
```

This portfolio has $\bar{R}_G = 0.01088493$ and $\sigma_G = 0.1046062$. Write the expression that computes these two numbers. No calculations!

$$\bar{R}_G = \sum x_i \bar{R}_i = \underline{x}' \bar{R}$$

$$\sigma_G = \sqrt{\sum x_i^2 \sigma_i^2 + \sum_i^n \sum_{j \neq i}^n x_i x_j \sigma_{ij}} = \sqrt{\underline{x}' \underline{\Sigma}\, \underline{x}}$$

Plot for problem 3:



d. When short sales are not allowed the values of the z_i's are:

```
> no_short_sales
     stock z_no_short
[1,]     3  0.7535748
[2,]     1  0.1236640
```

Find the composition of the optimum portfolio and compute its expected return and standard deviation.

$X_3 = \frac{z_3}{\sum z_i} = 0.8590304$

$X_1 = \frac{z_1}{\sum z_i} = 0.1409696$

$\bar{R}_G = 0.859(0.0103649) + 0.14097(0.00527),$

$\boxed{\bar{R}_G = 0.009647}$

$\sigma_G = \sqrt{0.859^2(0.010749) + 0.14097^2(0.013422) + 2(0.859)(0.14097)(0.0021427)}$

$\Rightarrow \sigma_G = 0.0934$

e. On the previous page you see the plot of the expected return against the standard deviation of many portfolios of these three stocks when short sales are allowed and when short sales are not allowed. Indicate the position of the optimum portfolio on this graph when short sales are allowed and when short sales are not allowed.

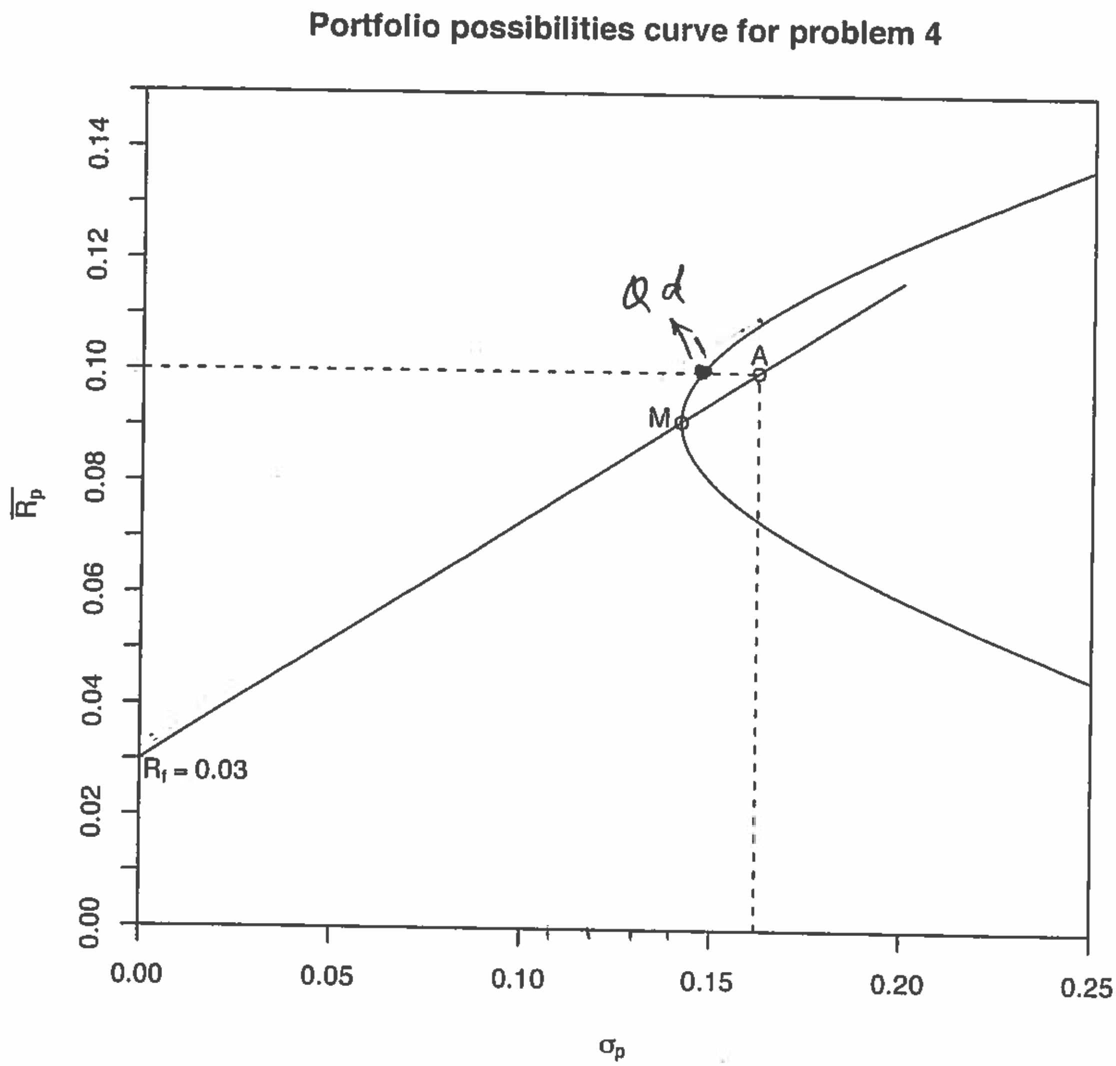
POINT OF TANGENCY

f. Consider the case when short sales are not allowed. Suppose a new portfolio is constructed as follows: 70% in the optimum portfolio and 30% in the risk free asset. Find the expected return and standard deviation of this new portfolio and place it on the graph on the previous page.

$\bar{R}_G = 0.70(0.009647) + 0.30(0.002) = 0.007353$

$\sigma_G = 0.70(0.0934) = 0.06538$

Plot for problem 4:



2 3 4 5 6 7

Problem 4 (20 points)
You are constructing a portfolio from three assets. The first two assets are stocks 1 and 2. The third asset is the risk free T-bill which has return 3%. The characteristics of the two stocks are as follows:

Stock	$\bar{R}$	σ
1	0.15	0.30
2	0.08	0.15

The correlation coefficient between stocks 1 and 2 is $\rho_{12} = 0.15$.

a. Find the composition of the minimum risk portfolio (point M on the graph on the previous page).

$$X_1 = \frac{0.15^2 - 0.15(0.30)(0.15)}{0.30^2 + 0.15^2 - 2(0.15)(0.30)(0.15)} = 0.1590909$$

$$X_2 = 1 - X_1 = 0.8409091$$

b. Suppose you want to form a portfolio by combining the two stocks and the risk free asset that will give you expected return of 10% (point A on the graph). Find the composition of portfolio A in stock 1, stock 2, and the risk free asset. Note: For your convenience you are given that $\bar{R}_M = 0.09113636$.

LET X = ~~FRACTION~~ IN M, $1-X$ IN RF

$$X(0.09113636) + (1-X)0.03 = 0.10 \Rightarrow X = \frac{0.10 - 0.03}{0.09113636 - 0.03} = 1.144981$$

$\therefore \boxed{1 - X = -0.1449815}$ IN RF

STOCK 1: $1.144981(0.1590909) = \boxed{0.182156} \rightarrow 1$

STOCK 2: $1.14498(0.8409091) = \boxed{0.962825} \rightarrow 2$

c. Calculate the standard deviation of portfolio A. Note: For your convenience you are given that $\sigma_M = 0.1414013$

$$\sigma_A = 1.144981\,\sigma_M = 1.144981(0.1414013) =$$

$$\Rightarrow \boxed{\sigma_A = 0.1619}$$

d. Suppose you still want a return of 10% but you are only allowed to combine the two stocks. Find the composition of this portfolio and show it on the graph.

$\boxed{X_1 + X_2 = 1}$

$$\bar{R}_d = X_1(0.15) + (1 - X_1)0.08 = 0.10$$

$$X_1 = \frac{0.10 - 0.08}{0.15 - 0.08} = 0.2857$$

$$X_2 = 0.7143$$

Problem 5 (20 points)
Using the constant correlation model we completed the table below on 6 stocks. Assume $R_f = 0.001$ and average correlation $\rho = 0.2530345$.

> table1

	Rbar	Rbar_Rf	sigma	Ratio	col1	col2	col3
R4	0.015036250	0.014036250	0.1181161	0.118834339	0.2530345	0.1188343	0.03006919
R3	0.010364922	0.009364922	0.1029387	0.090975688	0.2019374	0.2098100	0.04236849
R5	0.009990582	0.008990582	0.1360799	0.066068411	0.1680099	?????????	??????????
R1	0.005274547	0.004274547	0.1152052	0.037103779	0.1438429	0.3129822	0.04502026
R6	0.003806880	0.002806880	0.1385122	0.020264496	0.1257541	0.3332467	0.04190712
R2	0.001527333	0.000527333	0.1560782	0.003378646	0.1117065	0.3366254	0.03760324

a. Find the two missing numbers in the table above.

$\boxed{0.2758784}$ $\boxed{0.0463503}$

$0.2098100 + 0.0660 \quad \left(\sum_{j=1} \frac{\bar{R}_j - R_f}{\sigma_j}\right)$

$(0.1680099)(0.2758784)$

col1 × col2

b. Find the cut-off point C^* if short sales are not allowed.

$C^* = 0.0463503$

c. Find the cut-off point C^* if short sales are allowed.

$C^* = 0.0376032$

d. Write down the expression in matrix form that computes the variance of the portfolio when short sales are allowed. No calculations.

$$\sigma_p^2 = X' \Sigma X$$

e. You are given a new stock with $\bar{R} = 0.005$, and $\sigma = 0.15$. Will anything change when short sales are allowed and when short sales are not allowed in terms of the portfolio allocation. Briefly explain your answer without doing all the calculations.

$(0.005 - 0.001)/0.15 = 0.0267$

NO SHORT SALES : NOTHING CHANGE

SHORT SALES : WE HAVE DO CALCULATIONS