PORTFOLIO THEORY: A STEP TOWARD ITS PRACTICAL APPLICATION*

MARSHALL E. BLUME†

I. INTRODUCTION

The modern theory of portfolio analysis dates from the pioneering work of Markowitz.[1] Of paramount importance to the practical application of this theory is the ability to assess accurately the future performance of portfolios of risky assets. With accurate assessments, an investor can optimize his choice of a portfolio on the two dimensions of risk and expected return. In spite of the need of obtaining accurate assessments of future performance in applying portfolio analysis to a practical situation, there has been virtually no statistical analysis of the empirical properties of explicit methods of assessing future performance.

This paper proposes two different methods of assessing future performance or, more technically, of assessing predictive or subjective distributions of future returns. These two different methods will be compared by evaluating the accuracy of the predictive distributions derived under these two alternatives, using as a criterion the degree of approximation between these predictive distributions and the underlying distributions generating the future returns.

After reviewing some of the more important theoretical and empirical properties of symmetric, stable distributions in section II, section III develops the first method which assesses predictive distributions of the returns for all the assets individually and then aggregates these distributions into predictive distributions for portfolios. This procedure is implicit in Sharpe's well-known diagonal algorithm for calculating the efficient set.[2] This paper shows that this first method of aggregation requires certain empirical assumptions about the predictive distributions for the individual assets. After sections IV and V describe the sample and the statistical design, section VI uses this first method to assess predictive distributions for a sample of portfolios of common stocks using historical data and compares these predictive distributions to the generating distributions of the future returns. The closeness of the approximation of these predictive distributions to the future generating distributions varies considerably with the particular empirical assumptions used in the process of aggregation.

The second method to be presented in section VII assesses the predictive distributions of future returns for portfolios directly rather than first assessing the distributions for the individual assets. This second or alternative method still requires some of the assumptions needed by the first method. Neverthe-

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† Assistant professor of finance, University of Pennsylvania.

[1] H. Markowitz, "Portfolio Selection," *Journal of Finance* 7 (1952):77–91.

[2] W. F. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science* 9 (1963):277–93.

less, the evidence suggests that these alternatively assessed predictive distributions conform very closely to the future underlying distributions over a wide range of different empirical assumptions.

II. SYMMETRIC, STABLE DISTRIBUTIONS

Mandelbrot, Fama, and Roll[3] have presented substantial empirical and theoretical evidence which suggests that the underlying distributions of returns on common stocks and bonds conform to the so-called non-Gaussian stable family of distributions. Since the methods of assessing predictive distributions proposed below explicitly use symmetric, stable distributions, this section reviews the properties of these distributions which are important for the purposes of this paper.

Symmetric, stable distributions are described by three parameters: a location parameter, μ; a dispersion parameter, γ; and a characteristic exponent, θ.[4] The characteristic exponent $\theta, 0 < \theta \leq 2$, determines the shape of the probability density function. The smaller the value of θ, the greater will be the probability of extreme values or geometrically the greater will be the area contained under the tails of the density function. If $\theta = 2$, the distribution is normal; μ is the expected value; and γ is one-half the variance. If $1 < \theta < 2$, the relevant density function will have more area under the tails than the normal. For such distributions, the variance and all higher-order moments do not exist, but a dispersion parameter γ is defined. The location parameter μ is still the expected value. This paper will only use stable distributions with $\theta > 1$.

Stable distributions possess the important property that they preserve themselves under addition. A formal statement of this property follows: Let the random variables $\tilde{X}_i$, $i = 1, \ldots, n$, be stable and independently distributed with location parameters $\mu(\tilde{X}_i)$, $i = 1, \ldots, n$, with dispersion parameters $\gamma(\tilde{X}_i)$, $i = 1, \ldots, n$, and with the same characteristic exponent θ. Define a new random variable, $\tilde{S}$, as a weighted sum of $\tilde{X}_i$, $i = 1, \ldots, n$, as

$$\tilde{S} = \sum_{i=1}^{n} a_i \tilde{X}_i \,, \tag{1}$$

where a_i, $i = 1, \ldots, n$, are constants. Then $\tilde{S}$ will by symmetric and stable with characteristic exponent θ, location parameter $\mu(\tilde{S})$, given by

$$\mu(\tilde{S}) = \sum_{i=1}^{n} a_i \mu(\tilde{X}_i) \,, \tag{2}$$

and dispersion parameter $\gamma(\tilde{S})$, given by

$$\gamma(\tilde{S}) = \sum_{i=1}^{n} |a_i|^{\theta} \gamma(\tilde{X}_i) \,. \tag{3}$$

If $\theta = 2$ (the normal case), equations (2) and (3) are identical to the corresponding equations for the expected value and variance of a weighted sum of normally distributed random variables except for a constant of proportionality in (3). The reader should note that (3) assumes that the characteristic exponent θ is the same for all the random variables $\tilde{X}_i$, $i = 1, \ldots, n$.

Fama and Roll[5] have presented tech-

[3] B. Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business* 36 (1963): 394–419; E. F. Fama, "The Behavior of Stock-Market Prices," *Journal of Business* 38 (1965):34–105; R. Roll, "The Efficient Market Model Applied to U.S. Treasury Bill Rates" (Ph.D. diss., University of Chicago, 1968).

[4] The characteristic exponent is sometimes symbolized by α instead of θ.

[5] E. F. Fama and R. Roll, "Some Properties of Symmetric Stable Distributions," *Journal of American Statistical Association* 63 (1968):817–36.

niques to assess the values of the location parameter μ and the dispersion parameter γ from a sample drawn from a symmetric, stable process with characteristic exponent θ, $1 < \theta \leq 2$. They argue that the sample mean can be used as an assessment of μ. To assess γ, they first define a variable $s = \gamma^{1/\theta}$. The variable s can then be assessed by

$$\hat{s} = (f_{72} - f_{28})/1.654\,, \tag{4}$$

where f_i is the ith fractile estimated from the sample. They show numerically that $\hat{s}$ has an asymptotic bias of less than .4 percent for all values of the characteristic exponent θ, $1 \leq \theta \leq 2$. An important property of the estimate $\hat{s}$ is that it is not a function of θ. An estimate of the dispersion parameter γ is given by $\hat{s}^{\theta}$, a function of θ.

Methods to assess the value of the characteristic exponent θ are very inadequate. The available procedures require very large samples, such as 2,000 observations. Even then, the sampling errors may be large. In spite of these difficulties, Fama[6] has presented substantial evidence that the characteristic exponent for common stocks is probably between 1.7 and 1.9. What lends credence to this range is that Fama used several different techniques to assess the value of the characteristic exponent; in most cases, the results were similar.

III. ONE METHOD OF ASSESSING PREDICTIVE DISTRIBUTIONS

Implicit in Sharpe's procedure for calculating the efficient set[7] as generalized by Samuelson[8] is a method of assessing predictive distributions of returns for portfolios. This section develops and discusses some of the empirical problems in using it to assess predictive distributions.

Sharpe's diagonal algorithm for tracing the efficient set assumes that the investor's subjective distributions of returns of individual assets are normal. It further assumes the validity of a particular stochastic model which was first suggested by Markowitz.[9] Using a slightly more general form of Sharpe's stochastic model, Fama[10] has proposed a portfolio theory for a stable world in which the variance does not necessarily exist. Samuelson[11] has shown how this more general stochastic model—hence-

[6] Fama, "The Behavior of Stock-Market Prices."

[7] Sharpe, "A Simplified Model for Portfolio Analysis."

[8] P. A. Samuelson, "Efficient Portfolio Selection for Pareto-Levy Investments," *Journal of Financial and Quantitative Analysis* 2 (1967):107–22.

[9] H. Markowitz, *Portfolio Selection: Efficient Diversification of Investment* (New York: John Wiley & Sons, 1959), p. 100, n. 1. This stochastic model or variations of it have been used extensively in the literature. See, e.g., M. E. Blume, "The Assessment of Portfolio Performance: An Application to Portfolio Theory" (Ph.D. diss., University of Chicago, 1968); K. J. Cohen and J. A. Pogue, "An Empirical Evaluation of Alternative Portfolio-Selection Models," *Journal of Business* 40 (1967): 166–93; E. F. Fama, "Portfolio Analysis in a Stable Paretian Market," *Management Science* 11 (1965): 404–19, and "Risk, Return, and Equilibrium in a Stable Paretian Market," Center for Mathematical Studies in Business and Economics Report no. 6831 (Chicago: University of Chicago, 1968); M. Jensen, "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios" (Ph.D. diss., University of Chicago, 1968); B. F. King, "Market and Industry Factors in Stock Price Behavior," *Journal of Business* 39 (suppl.; 1966):139–90; Samuelson, "Efficient Portfolio Selection for Pareto-Levy Investments"; W. F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance* 19 (1964): 425–42; W. F. Sharpe, "A Linear Programming Algorithm for Mutual Fund Portfolio Selection," *Management Science* 13 (1967): 499–510; and Sharpe, "A Simplified Model for Portfolio Analysis" (see n. 2 above).

[10] Fama, "Portfolio Analysis in a Stable Paretian Market," and "Risk, Return, and Equilibrium in a Stable Paretian Market."

[11] Samuelson, "Efficient Portfolio Selection for Pareto-Levy Investments."

forth called the market model—can be used to calculate the efficient set.

The market model specifies a relationship between the returns for individual assets and a market factor. The returns for individual assets can be defined in numerous ways; but, for the purposes of this paper, these returns will be measured by what is commonly known as an investment relative. The investment relative, R_{it}, for asset i from time $(t-1)$ to t is the ratio of the value of the asset at time t with dividends of any type immediately reinvested to the value of the asset at time $(t-1)$. Thus, the investment relative can be roughly interpreted as one plus the percentage increase or decrease from time $(t-1)$ to t in the value of an investor's holdings of asset i.

Using this measure of return, the market model asserts that the investment relative $\tilde{R}_{it}$, a random variable, can be expressed as a linear function of some market factor $\tilde{M}_t$:

$$\tilde{R}_{it} = \alpha_i + \beta_i \tilde{M}_t + \tilde{\epsilon}_{it}\,, \qquad (5)$$

where α_i and β_i are constants appropriate to asset i. It further assumes that the disturbances, $\tilde{\epsilon}_{it}$, for all assets are symmetric, stable variates with zero expectations and with the same characteristic exponent $\theta > 1$, that the disturbances for asset i and asset j, $i \neq j$, are independent, and that the distribution of $\tilde{M}_t$ is symmetric and stable with characteristic exponent θ. The distribution of $\tilde{R}_{it}$, which is merely the weighted sum of stable variates, will be stable with characteristic exponent θ and with location and dispersion parameters which are easily derivable from equations (2) and (3).

Sharpe's portfolio theory as generalized by Fama and Samuelson then aggregates these distributions of the investment relatives for the individual assets into a distribution of returns for a portfolio. The measure of the return for a portfolio is analogous to the investment relative—namely, the ratio of the value of the portfolio at time t to the value at time $(t-1)$. To distinguish this ratio from the investment relative, the measure of return for a portfolio will be called the wealth relative.

The wealth relative for a portfolio, $\tilde{W}_t$, will be given by

$$\tilde{W}_t = \sum_i x_{it} \tilde{R}_{it}\,, \qquad (6)$$

where x_{it} is the proportion of total wealth at time $(t-1)$ invested in asset i. The substitution of (5) into (6) yields

$$\begin{aligned} \tilde{W}_t &= \sum_i x_{it}(\alpha_i + \beta_i \tilde{M}_t + \tilde{\epsilon}_{it}) \\ &= \bar{\alpha} + \bar{\beta}\tilde{M}_t + \sum_i x_{it}\tilde{\epsilon}_{it}\,, \end{aligned} \qquad (7)$$

where $\bar{\alpha}$ and $\bar{\beta}$ are weighted averages of the α_i's and β_i's, the weights being the proportion of total wealth invested in each asset at time $(t-1)$.

Since $\tilde{W}_t$ is given as a weighted sum of the market factor, $\tilde{M}_t$, and the disturbances, $\tilde{\epsilon}_{it}$, which were assumed to be independent, symmetric, stable variates with the same characteristic exponent θ, $\tilde{W}_t$ will be stable and symmetric with the characteristic exponent θ. The location and scale parameters of $\tilde{W}_t$ can be derived from (7), using (2) and (3), and are, respectively,

$$\mu(\tilde{W}_t) = \bar{\alpha} + \bar{\beta}\mu(\tilde{M}_t) \qquad (8)$$

and

$$\gamma(\tilde{W}_t) = |\bar{\beta}|^\theta \gamma(\tilde{M}_t) + \sum_i |x_{it}|^\theta \gamma(\tilde{\epsilon}_{it})\,. \qquad (9)$$

If short sales are not allowed and if the returns for each asset are positively related to general market movements, the absolute value signs in (9) can be dropped.

Samuelson[12] uses (8) and (9) to formulate a mathematical program to determine the efficient set. He does not, however, indicate how an investor would in general solve this program. In the particular case in which θ equals 2.0, it can easily be shown that Samuelson's program reduces to Sharpe's quadratic programming formulation,[13] for which numerous algorithms exist.

To summarize, Samuelson's generalized version of Sharpe's procedure for determining the efficient set requires that an investor first assess predictive distributions of the investment relatives for all assets. The investor then must aggregate these distributions for the individual assets into a distribution for a portfolio using equations (8) and (9). This process of aggregation assumes that the market factor, $\tilde{M}_t$, and the disturbances, $\tilde{\epsilon}_{it}$, are independent, symmetric, stable variates with the same characteristic exponent θ. To assume that the disturbances, $\tilde{\epsilon}_{it}$, are independent is tantamount to assuming the absence of industry effects.

This process of aggregation further assumes that the investor's assessed distributions for the market factor and the disturbances are stable. This assumption may be violated even if the corresponding underlying distributions were stable. For instance, if the underlying distribution for the market factor were normal, the investor's assessed distribution might under very reasonable assumptions be a *t* distribution.

The reader should finally note that in developing (8) and (9) it was assumed that the values of α_i and β_i were constants or known with certainty. In practice, an investor would not know these values with certainty but would hold subjective distributions on them. That the values of α_i and β_i are uncertain or random variables means, of course, that (8) and (9) would not follow from (7). Nonetheless, in the context of a large portfolio, an investor might be willing to act as if his assessments of α_i and β_i were certain, so that (8) and (9) would properly characterize his predictive distributions. The reason is that if an investor's assessments of α_i and β_i were unbiased and the errors in these assessments were independent among the different assets, his uncertainty attached to his assessments of $\bar{\alpha}$ and $\bar{\beta}$, merely weighted averages of the α_i's and β_i's, would tend to become smaller, the larger the number of assets in the portfolios and the smaller the proportion in each asset. Intuitively, the errors in the assessments of α_i and β_i would tend to offset each other.[14]

This discussion suggests that the predictive distributions assessed under the assumption that α_i and β_i are constants will better approximate the underlying distributions, the larger the number of securities in the portfolios. Thus, to place Sharpe's generalized method of aggregation on its best footing, the empirical sections will only examine portfolios of twenty or more assets with an equal proportion invested in each. The empirical evidence will suggest that for portfolios of such size the uncertainty attached to $\bar{\alpha}$ and $\bar{\beta}$ in (7) is negligible.

[12] Ibid.

[13] Sharpe, "A Simplified Model for Portfolio Analysis."

[14] That one would tend to be more confident about the value of an average than the values of the components of the average does not hold for all distributions. If the individual components are distributed by stable distributions with characteristic exponent less than one, Fama ("Portfolio Analysis in a Stable Paretian Market") has demonstrated the odd property that one would tend to be more uncertain about the value of the average than the values of the components.

IV. THE SAMPLE

A sample of 251 common stocks[15] which were listed on the New York Stock Exchange continuously from December 30, 1926, through December 30, 1960, was taken from the Price Relative File of the Center for Research in Security Prices at the Graduate School of Business, University of Chicago.[16] Thus, there are 408 consecutive monthly investment relatives for each of the 251 securities—a total of 102,408. These investment relatives are properly adjusted for dividends and changes in capital structure.

The measure of the market factor, M_t, was defined as

$$M_t = n^{-1}\sum_{i=1}^{n} R_{it} ,$$

where n is the number of securities in the measure, and R_{it} is the monthly investment relative of security i from month $(t - 1)$ to t. The securities included for any given month in constructing this measure, henceforth called the market relative, were those securities for which the center's Price Relative File contained twelve monthly investment relatives for the calendar year in which the particular month fell.[17] The number of securities used in constructing M_t varied yearly but ranged from a minimum of 525 in 1927 to a maximum of 1,059 in 1960.

A preliminary analysis of the descriptive accuracy of the market model[18] involving time plots of the residuals from least squares regressions[19] showed that almost without exception the dispersion of the residuals for each security decreased sometime during the late 1930s. Using the sample variance as a descriptive measure of dispersion, 247 of the 251 securities had larger variances of the residuals for the regressions on the first half of the sample, 1926 through 1943, than for the regressions on the second half of the sample, 1944 through 1960. The dispersion of the residuals during this latter period appeared constant for each security.

This reduction in dispersion of the residuals from the pre-World War II period to the post-World War II period presents a complication in evaluating Sharpe's generalized method of assessing a predictive distribution for a portfolio. It was suggested above that his method of aggregation could be examined by measuring the degree of approximation between the predictive distributions and the future generating distributions. If an investor were to use historical data

[15] A complete portfolio analysis should include all assets available to an investor. If the investor restricts himself only to common stocks, he will in general obtain a suboptimal portfolio. Nevertheless, it appears reasonable to restrict the analysis in this paper to common stocks because of the readily available data. If the market model appears useful for common stocks, future work should be done to determine how to include other assets in the analysis.

[16] See L. Fisher and J. H. Lorie, "Rates of Return on Investments in Common Stocks," *Journal of Business* 37 (1964): 1–21. In selecting only securities which were listed continuously from 1926 through 1960, some bias may have been introduced into the sample by the exclusion of the securities which were listed for only part of the period.

[17] Measures of the market factor were also constructed for relatives of two, three, four, six, and twelve months in an analogous way to the one-month index. In fact, this seemingly strange definition of n in terms of securities available for an entire calendar year is an attempt to keep these measures on the same footing. These other measures will not be used in this paper.

[18] For a more complete discussion of the specification of the market model, the reader is referred to Blume ("The Assessment of Portfolio Performance" [see n. 9 above]).

[19] J. Wise ("Linear Estimators for Linear Regression Systems having Infinite Residual Variances" [Berkeley-Stanford Mathematic Economics Seminar, October 1963]) gives some justification for using least squares if the disturbances are stably distributed.

through 1943 to assess a predictive distribution for a portfolio, he would more than likely find that the dispersion parameter of his predictive distribution would overstate the dispersion parameter of the generating process after 1943. It might be possible to devise statistical tests which would measure how much of the differences between the predictive distributions and the future generating processes were attributable, on the one hand, to bad judgement about the values of the dispersion parameters and, on the other hand, to violations of the assumptions required by Sharpe's generalized aggregation process. This paper will avoid this statistical problem by restricting the empirical analysis to the post-World War II period, 1944 through 1960. Though this restriction will limit the generality of the study, it will not materially affect the conclusions about the properties of Sharpe's method of assessing predictive distributions for portfolios.

V. THE STATISTICAL DESIGN

The data consisting of 204 monthly investment relatives for each of 251 securities from 1944 through 1960 will be used in part to assess predictive distributions for the wealth relatives of various portfolios and in part to determine how closely these predictive distributions approximate the future generating distributions. It was arbitrarily decided to use 102 months of data to assess these predictive distributions. The reader should note that the use of 102 months of data to assess these distributions means that July 1951 is the first month for which predictive distributions can be obtained.

To assess the predictive distributions for July 1951, the 102 monthly investment relatives from January 1944 through June 1951, were regressed on the corresponding market relatives by simple least squares. This process yielded estimates of the constant, α_i, and the slope, β_i, for each security. An estimate of the dispersion parameter of the disturbances, $\hat{s}(\tilde{\epsilon}_{it})$, was obtained for each security by Fama and Roll's formula, equation (4).

With these assessments, an investor need only assess a predictive distribution for the market relative to obtain a predictive distribution for the wealth relative of any portfolio for July 1951. This paper will avoid the interesting problem of assessing a predictive distribution for the market relative by comparing the predictive distributions of the wealth relatives conditional on the market relative, $P(\tilde{W}_t|\tilde{M}_t)$, to the conditional generating distributions, rather than comparing the unconditional distributions. In fact, the comparison of the conditional distributions will yield the same, if not more, insight into Sharpe's aggregation process than a comparison of the unconditional distributions. The rationale for this assertion follows directly from the form of the conditional predictive distributions implied by the market model. From (7), $P(\tilde{W}_t|\tilde{M}_t)$ will be stable and symmetric, with characteristic exponent θ, location parameter

$$\mu(\tilde{W}_t|\tilde{M}_t) = \bar{\alpha} + \bar{\beta}\tilde{M}_t\,, \qquad (10)$$

and dispersion parameter

$$\gamma(\tilde{W}_t|\tilde{M}_t) = \sum_i x_{it}{}^{\theta}\gamma(\tilde{\epsilon}_{it})\,, \qquad x_{it} \geq 0\,. \qquad (11)$$

The assumptions of the market model are used in the same way in obtaining the conditional predictive distributions as in obtaining the unconditional predictive distributions. Thus, violations of the assumptions of the market model would tend to cause both the conditional and unconditional predictive distributions to deviate in the same way from the corresponding generating distributions. Further, in examining the conditional predictive distributions rather than the unconditional distributions, it can be stated that the differences between the assessed and underlying distributions are not caused by differences between the predictive distribution of the market relatives and the corresponding underlying distribution.

For the above reasons, predictive distributions conditional on the value of the market relative in July 1951, rather than the unconditional predictive distributions, were assessed for various portfolios to be described below. Conditional predictive distributions were also assessed for each of these various portfolios for each of the remaining 101 months, August 1951 through December 1960. It was desired to use the 102 months of historical data immediately preceding the month under consideration to obtain the required assessments. However, to reduce the computations, the assessments were revised only every six months rather than every month.

To compare these conditional predictive distributions with the conditional generating distributions, the following procedure was used: For each portfolio, the actual wealth relative for July 1951 can be considered a drawing from the generating or underlying conditional distribution. If the conditional predictive distribution were identical to the conditional underlying distribution, the actual wealth relative can in turn be thought of as a drawing from the predictive distribution; the value of the cumulative density function of the predictive distribution for this drawing was calculated. Henceforth, the phrase "the value of the cumulative density function" will be shortened to "the cumulative frequency." This process was repeated for each of the 101 remaining months, August 1951 through December 1960, to yield 102 cumulative frequencies for each portfolio. Mood[20] shows that if these conditional predictive distributions were identical to the underlying distributions, these 102 cumulative frequencies would behave as if they were drawn from a uniform distribution over the unit interval. Thus, the degree of approximation between the conditional predictive distributions and the underlying distributions can be examined by determining how closely these calculated cumulative frequencies conform to a uniform distribution.[21]

The portfolios for which these conditional predictive distributions were assessed were chosen with several goals in mind: First, the portfolios contained twenty or more securities with an equal proportion invested in each. Section III discussed the reason for this restriction. Second, it was desired to determine whether there was any interaction between Sharpe's aggregation process and the number of securities in portfolios. To this end, portfolios of twenty, thirty-

[20] A. M. Mood, *Introduction to the Theory of Statistics* (New York: McGraw-Hill Book Co., 1950), p. 107.

[21] There are, of course, other ways to compare the conditional predictive distributions to the underlying generating distributions. For example, one might compare the values of the mean and dispersion parameter for each predictive distribution with estimates of these parameters from some later period, essentially the technique of I. Friend and D. Vickers ("Portfolio Selection and Investment Performance," *Journal of Finance* 20 [1965]:391–415). This procedure seems less powerful than the one in this paper because it used only two parameters rather than the whole distribution. It is easy to conceive of two different distributions with the same mean and dispersion parameter. The procedure to be used in this paper is sensitive to any and all differences between the predictive and generating distributions.

five, and fifty securities were selected. Third, some of the portfolios were selected so that each security represented a different two-digit SEC industry. In such portfolios, King's results[22] suggest that industry effects are likely to be minimal. It will be recalled that the market model assumes that there are no industry effects. Finally, the portfolios in the efficient set will have widely different predictive distributions. Some portfolios will have small dispersion parameters; others will have large dispersion parameters. In a large portfolio, the market effect will tend to be the dominant effect. Therefore, the portfolios were selected to have widely diverse relationships to the market by maximizing the diversity of the averages of the estimates of β_i.[23]

In line with these criteria, five sets of portfolios were selected. Within each set, the portfolios contain no securities in common, and the diversity of the averages of the estimates of β_i is about as great as possible. Among sets, the portfolios may contain common securities. The first and second sets consist respectively of six portfolios of twenty securities[24] and two portfolios of thirty-five securities,[25] in which each security represents a different two-digit SEC industry. The third, fourth, and fifth sets consist respectively of twelve portfolios of twenty securities, seven of thirty-five securities, and five of fifty securities.[26] These portfolios were not controlled on industry

VI. THE EMPIRICAL RESULTS

Predictive distributions conditional on the appropriate market relative, M_o, were assessed for four assumed values of the characteristic exponent, $\theta = 1.7, 1.8, 1.9$, and 2.0, for each of these thirty-two portfolios for each of the 102 months from July 1951 through December 1960.

[22] King, "Market and Industry Factors in Stock Price Behavior."

[23] Selecting portfolios by maximizing the diversity of the estimates of $\bar{\beta}$ will limit the conclusions which can be made about the adequacy of any assessment procedure for determining the efficient set. The reason is that the equation for the expected wealth relative (8) involves both the estimates for $\bar{\alpha}$ and $\bar{\beta}$. To maximize the diversity of the values of the expected wealth relative, one would need to consider the estimates of $\bar{\alpha}$ and $\bar{\beta}$ simultaneously, which is not done in selecting the portfolios in this paper. Nevertheless, selecting the portfolios by maximizing the diversity of the estimates of $\bar{\beta}$ has the effect of maximizing the values of the dispersion parameters since equation (9) does not involve the estimates of $\bar{\alpha}$. The primary conclusions in this paper will be about properties of the dispersion estimates for portfolios and not about the expected wealth relatives, so that the method of selecting portfolios used in the text should be adequate.

[24] The estimates of β_i from the period January 1944 through June 1951 were ranked in ascending order. A portfolio of twenty securities was selected by including that security with the smallest estimate of β_i. The security with the next smallest estimate of β_i from a different two-digit SEC industry was added to the portfolio. This process was repeated until twenty securities were included. This portfolio has the smallest average of the estimates of β_i of all possible portfolios containing twenty securities from different industries. Another portfolio was selected by including the security with the largest estimate of β_i. The security from a different two-digit SEC industry with the next largest estimate of β_i was added. This process was repeated until the portfolio contained twenty securities. These forty securities were then removed from the list of available securities. The process was repeated two more times to obtain six portfolios of twenty securities. These six were all that could be obtained by controlling on industry in this manner.

[25] The same procedure as used for portfolios of twenty securities controlled on industry was used. After two portfolios were selected, there were less than thirty-five industries remaining.

[26] The method of selecting portfolios of twenty, thirty-five, and fifty securities for which there was no control on industry was as follows: Portfolio one of n securities consisted of those n securities whose estimates of β_i in the period from January 1944 through June 1951 were the smallest. Portfolio two of n securities consisted of those n securities with the next smallest estimates of β_i, and so on until the number of the remaining securities was less than n. This process was repeated for each value of n.

These particular values were chosen because Fama's research, cited earlier in the paper, suggested that the true value of θ is between 1.7 and 1.9 and because it was desired to examine the effect of the assumption of normality ($\theta = 2$). In accordance with the discussion above, these conditional predictive distributions were assumed to be symmetric and stable with location parameters,

$$\hat{\mu} = \hat{\hat{\alpha}} + \hat{\hat{\beta}} M_o ,$$

and dispersion parameters, $\hat{\gamma} = [n^{-\theta} \Sigma \hat{s}_i^{\theta}]$. The actual wealth relative, W_o, was determined and the standardized variate $t(\theta)$ was calculated as

$$t(\theta) = \frac{W_0 - (\hat{\hat{\alpha}} + \hat{\hat{\beta}} M_o)}{\hat{\gamma}^{1/\theta}} . \qquad (12)$$

Using a program published by Fama and Roll,[27] the value of the cumulative density function or in short the cumulative frequency corresponding to $t(\theta)$ was calculated for each value of θ for each month for each portfolio.

It was just argued that if these conditional predictive distributions were identical to the underlying distributions, these cumulative frequencies would behave as if drawn from a uniform distribution. This paper will measure the degree of approximation between the predictive distributions and the underlying distributions by determining how closely these cumulative frequencies approximate drawings from a uniform distribution.

These cumulative frequencies were first examined for each portfolio and for each value of θ. The unit interval was partitioned into ten equal subintervals, and the number of cumulative frequencies in each subinterval was counted for each portfolio and for each value of θ.[28] For reasons of space, only the results for the six portfolios of twenty securities controlled on industry are presented in table 1. These results are typical of those for the other twenty-six portfolios. If the conditional predictive distributions were identical to the underlying distributions, one would expect to observe 10.2 observations in each cell.

These results are quite interesting. For the predictive distributions assessed under the normality assumption ($\theta = 2$), nine of the twelve extreme subintervals, the 0.0–0.1 and the 0.9–1.0 subintervals, contain more cumulative frequencies than expected.[29] The remaining three extreme subintervals contain ten cumulative frequencies each, which is only .2 less than the expected number. Thus, the cumulative frequencies under the assumption of normality appear to deviate from uniformly distributed variates in that there are too many extreme values. This means that the number of extreme observations of the wealth relatives is larger than that anticipated on

[27] Fama and Roll, "Some Properties of Symmetric Stable Distributions."

[28] The choice of ten equal subintervals is of course arbitrary. Later, the cumulative frequencies will be presented differently to avoid this arbitrary choice.

[29] Strictly speaking, one would only expect half of the extreme subintervals to contain more cumulative frequencies than expected if the number of cumulative frequencies in, for instance, the 0.0–0.1 subinterval were symmetrically distributed under the hypothesis that the cumulative frequencies were uniformly distributed over the ten cells. In this case, the mean would be the same as the median. In fact, the distribution of the number of cumulative frequencies in the 0.0–0.1 subinterval is skewed to the right; and more particularly, W. Feller (*An Introduction to Probability Theory and Its Applications* [New York: John Wiley & Sons, 1957], 1:34) shows that the distribution will be binomial with $n = 102$, and $p = .10$. For such a binomial distribution, the median is not explicitly defined but is some value between 10 and 11, so that the mean of 10.2 approximates the median. Throughout this study, the mean will be used where, to be precise, the median should be used. This approximation will not change any of the conclusions.

the basis of the predictive distributions, so that the predictive distributions tend to understate the probability of extreme values.

As the characteristic exponent used in assessing the predictive distributions is reduced from 2.0, the tendency for the predictive distributions to understate the probability of extreme values becomes less. This is evidenced, for instance, in the 0.00–.1 subinterval of the first portfolio. For $\theta = 2.0$, there are twenty-two cumulative frequencies; but for $\theta = 1.9$, there are only nineteen cumulative frequencies. Finally, when the characteristic exponent assumes the value 1.7, six of the twelve extreme subintervals contain more than expected and six less than expected. These results suggest that the predictive distributions assessed using a value of θ of 1.7 do not systematically tend to understate or overstate the probability of extreme values.

TABLE 1

PREDICTIVE RESULTS USING $[n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta}$ AS ASSESSMENT OF γ FOR PORTFOLIOS OF 20 SECURITIES CONTROLLED ON INDUSTRY

β	No.	θ	SUBINTERVALS										χ^2
			0.0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0	
0.433....	1	1.7	17	9	9	14	8	7	8	10	14	6	11.33
		1.8	17	10	12	12	6	6	7	11	10	11	9.76
		1.9	19	9	12	12	5	6	7	8	9	15	16.63
		2.0	22	8	12	10	5	6	6	7	11	15	23.88
0.583....	2	1.7	4	11	9	9	11	12	11	9	15	11	7.02
		1.8	6	11	8	8	11	11	10	9	16	12	6.63
		1.9	9	9	9	7	10	10	9	10	14	15	5.25
		2.0	10	8	10	7	9	9	10	9	12	18	8.20
0.716....	3	1.7	7	4	12	16	11	9	13	10	9	11	9.57
		1.8	8	7	10	16	9	9	11	9	8	15	8.00
		1.9	9	6	11	17	7	9	11	7	8	17	13.69
		2.0	10	9	9	15	7	8	11	5	11	17	11.33
1.237....	4	1.7	8	13	12	10	9	14	12	11	8	5	6.63
		1.8	11	11	15	7	8	13	12	10	7	8	6.43
		1.9	15	10	14	5	8	13	11	10	7	9	8.78
		2.0	16	13	11	4	8	11	13	9	7	10	10.35
1.328....	5	1.7	9	15	13	11	7	11	9	8	8	11	5.45
		1.8	15	13	10	13	4	11	8	5	9	14	12.31
		1.9	19	11	10	12	3	11	6	7	8	15	18.59
		2.0	21	10	11	10	3	11	6	7	8	15	22.12
1.560....	6	1.7	12	10	9	8	13	11	10	4	14	11	7.02
		1.8	12	11	10	7	12	10	11	3	9	17	11.53
		1.9	14	10	11	6	11	9	11	4	6	20	18.39
		2.0	17	9	9	6	11	9	11	3	6	21	25.06

The values of the χ^2 statistic assuming an expected number of 10.2 in each cell are given in table 1 as a summary measure of the degree of approximation of the cumulative frequencies to uniform variates. The distribution of the χ^2 statistic will have nine degrees of freedom, an expected value of nine and a cumulative density function tabulated in table 2.[30] The value of the χ^2 statistic will tend to be larger, the poorer the approximation of the cumulative frequencies to uniform variates. The values of the χ^2 statistic suggest that the predictive distributions assessed under the normality assumption ($\theta = 2$) are generally poorer approximations of the underlying generating processes than those assessed with smaller values of the characteristic exponent. This conclusion is consistent with the conclusions reached in analyzing the number of cumulative frequencies in the extreme subintervals.

What is perhaps more important for an investor who wishes to use Sharpe's method for calculating the efficient set as generalized by Samuelson is that the predictive distributions assessed with a value of the characteristic exponent of 1.7 or 1.8 seem to conform very closely to the future generating distributions. The .95 fractile of the χ^2 statistic with nine degrees of freedom is 16.92. The values of the χ^2 statistic in table 1 are all less than 16.92 for θ equals 1.7 or 1.8.

As a further analysis of the conditional predictive distributions, the number of the thirty-two portfolios for which the number of cumulative frequencies in the 0.0–0.1 subinterval exceeded the expected number 10.2 was counted for each value of θ. This process was repeated for the 0.9–1.0 subinterval. Table 3 contains these figures. If the conditional predictive distributions were identical to the underlying distributions, one would expect that approximately half of the extreme subintervals would contain more cumulative frequencies than expected. Since there are sixty-four extreme subintervals, one would expect that approximately thirty-two would contain more than expected. The results in table 3 suggest that the predictive distributions in terms of the probabilities of extreme values best approximate the underlying distributions for some value

TABLE 2

CUMULATIVE χ^2 DISTRIBUTION FOR 9 DEGREES OF FREEDOM

$F(\chi)^2$....	0.01	0.02	0.05	0.10	0.20	0.30	0.50	0.70	0.80	0.90	0.95	0.98	0.99
χ^2.....	2.09	2.53	3.33	4.17	5.38	6.39	8.34	10.66	12.24	14.68	16.92	19.68	21.67

TABLE 3

NUMBER OF CUMULATIVE FREQUENCIES IN EXTREME SUBINTERVALS WHICH EXCEED 10.2

θ	Subintervals		Total
	0.0–0.1	0.9–1.0	
1.7..........	12	12	24
1.8..........	21	21	42
1.9..........	27	27	54
2.0..........	29	30	59

[30] The χ^2 statistic is calculated by

$$\chi^2 = \sum_{i=1}^{10} \frac{(D_i - E_i)^2}{E_i},$$

where D_i is the observed number of cumulative frequencies and E_i is the expected number in each cell i. The expected number E_i is of course 10.2 for this calculation. Since there are ten cells, the χ^2 statistic will have $10 - 1 = 9$ degrees of freedom. Some might argue that there are only 7 degrees of freedom since the mean and dispersion parameter are estimated from historical data. This would be true if the hypothesis were that the underlying process were identical, but in this paper the hypothesis is that the predictive distributions are identical to the future distributions. The way in which the predictive distributions are assessed is irrelevant to this hypothesis.

of θ between 1.7 and 1.8 since for θ equals 1.7 only twenty-four extreme subintervals contain more than 10.2 cumulative frequencies and for θ equals 1.8, forty-two extreme subintervals.

Table 3 also yields insight into the assumption that the predictive distributions are symmetric. If the conditional generating distributions were asymmetric, one would expect that the number of 0.0–0.1 subintervals for which the number of cumulative frequencies exceeded 10.2 would differ from the corresponding number for the 0.9–1.0 subintervals. An examination of table 3 indicates that there is no substantial difference between the numbers for the 0.0–0.1 and the 0.9–1.0 subintervals. As a matter of fact, the numbers are the same for three of the four values of θ.[31]

Although it is impractical to present the results for each of the thirty-two portfolios, the results can be summarized in a compact form. To do this, the cumulative frequencies for each of the five major classes of portfolios were pooled.[32] Table 4(A) contains these pooled cumulative frequencies, the expected number in each cell if the cumulative frequencies were uniformly distributed, and the corresponding χ^2 statistic. With this table, it is difficult to compare the results among groups of portfolios because the total number of cumulative frequencies differ. To facilitate comparison, table 4(B) was constructed by dividing the number of cumulative frequencies in each cell by the number of portfolios in the group. If the cumulative frequencies were uniform, one would expect 10.2 in each cell.

The figures in table 4 again support the conclusion that the predictive distributions assessed under the normality assumption tend to understate the probability of extreme values and that those distributions assessed with θ between 1.7 and 1.8 tend to conform very closely to the underlying distributions. For θ equals 2.0, the numbers of cumulative frequencies in the 0.0–0.1 and 0.9–1.0 subintervals are without exception greater than expected. As the value of θ used in assessing the predictive distributions decreases, the understatement becomes less. In fact, an analysis of the extreme subintervals suggests that the predictive distributions assessed with some value of θ between 1.7 and 1.8 would neither consistently overstate nor understate the

[31] This result conflicts with that of F. D. Arditti ("Risk and the Required Return on Equity," *Journal of Finance* 2 [1967]:19–36). He found that the distributions of returns of common stocks were sometimes skewed to the right. There are several possible reasons for the different results. First, he examined yearly returns rather than monthly returns. It is possible that the distributions of yearly returns may be more skewed than the distributions of monthly returns. Second, this paper examines the distributions of returns of portfolios rather than of individual stocks. Since the return of a portfolio is an average of the returns of individual stocks, the distributions of returns of portfolios might tend to be less skewed than the distributions of the individual stocks. If the variances of the returns exist, the central limit theorem is applicable, so that the above statement is certainly true. If, however, the variances do not exist, there is no guarantee that the distributions of returns of portfolios will approach symmetric distributions. For a further discussion, the reader is referred to W. Feller, *An Introduction to Probability Theory and Its Applications* (New York: John Wiley & Sons, 1966), 2: esp. pp. 302–6. Third, if the returns of individual securities are really distributed by stable distributions for which the variance and higher moments do not exist, the use of the sample third moment about the mean to measure skewness is questionable.

[32] In pooling data, it must be assumed that all the data are being generated by the same process. This assumption appears reasonable for the cumulative frequencies in table 1. If the generating processes were different among portfolios they would most likely differ with the value of $\bar{\beta}$, but there appears to be little or no relationship between the distributions of the cumulative frequencies and the values of $\bar{\beta}$. Since this is the most likely relationship, it seems safe to assume that all the data are generated by the same process.

TABLE 4

SUMMARY OF PREDICTIVE RESULTS USING $[n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta}$ AS ASSESSMENT OF $\gamma^{1/\theta}$

Securities per Portfolio	Controlled on Industry	θ	Expected No.	A: Subintervals 0.0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0	χ^2
20......	Yes	1.7	61.2	57	62	64	68	59	64	63	52	68	55	4.21
		1.8	61.2	69	63	65	63	50	60	59	47	59	77	10.94
		1.9	61.2	85	55	67	59	44	58	55	46	52	91	35.81
		2.0	61.2	96	57	62	52	43	54	57	40	55	96	55.78
35......	Yes	1.7	20.4	19	22	19	23	23	22	15	23	20	18	3.16
		1.8	20.4	23	21	23	19	20	21	13	20	21	23	3.84
		1.9	20.4	29	22	20	17	18	19	12	17	24	26	10.90
		2.0	20.4	36	20	22	12	16	16	14	13	23	32	29.04
20......	No	1.7	122.4	110	152	120	129	130	133	106	126	111	107	15.51
		1.8	122.4	147	141	113	126	114	123	95	112	115	138	18.63
		1.9	122.4	190	121	107	117	106	116	87	110	111	159	65.56
		2.0	122.4	226	111	93	110	101	103	85	102	108	185	152.42
35......	No	1.7	71.4	72	69	71	99	79	69	67	62	59	67	15.58
		1.8	71.4	92	74	61	91	72	60	63	61	59	81	20.71
		1.9	71.4	114	63	66	80	67	59	56	56	51	102	55.86
		2.0	71.4	126	68	70	65	61	49	58	54	52	111	85.05
50......	No	1.7	51.0	54	60	57	52	53	51	45	48	43	47	5.02
		1.8	51.0	68	62	54	44	48	41	48	36	51	58	16.68
		1.9	51.0	89	53	53	42	39	38	46	27	53	70	55.14
		2.0	51.0	108	48	51	34	35	35	41	25	48	85	117.65

B

20......	Yes	1.7	10.2	9.5	10.3	10.7	11.3	9.8	10.7	10.5	8.7	11.3	9.2
		1.8	10.2	11.5	10.5	10.8	10.5	8.3	10.0	9.8	7.8	9.8	12.8
		1.9	10.2	14.2	9.2	11.2	9.8	7.3	9.7	9.2	7.7	8.7	15.2
		2.0	10.2	16.0	9.5	10.3	8.7	7.2	9.0	9.5	6.7	9.2	16.0
35......	Yes	1.7	10.2	9.5	11.0	9.5	11.5	11.5	11.0	7.5	11.5	10.0	9.0
		1.8	10.2	11.5	10.5	11.5	9.5	10.0	10.5	6.5	10.0	10.5	11.5
		1.9	10.2	14.5	11.0	10.0	8.5	9.0	9.5	6.0	8.5	12.0	13.0
		2.0	10.2	18.0	10.0	11.0	6.0	8.0	8.0	7.0	6.5	11.5	16.0
20......	No	1.7	10.2	9.2	12.7	10.0	10.7	10.8	11.1	8.8	10.5	9.2	8.9
		1.8	10.2	12.2	11.7	9.4	10.5	9.5	10.2	7.9	9.3	9.6	11.5
		1.9	10.2	15.8	10.1	8.9	9.7	8.8	9.7	7.2	9.2	9.2	13.2
		2.0	10.2	18.8	9.2	7.7	9.2	8.4	8.6	7.1	8.5	9.0	15.4
35......	No	1.7	10.2	10.3	9.9	10.1	14.1	11.3	9.9	9.6	8.9	8.4	9.6
		1.8	10.2	13.1	10.6	8.7	13.0	10.3	8.6	9.0	8.7	8.4	11.6
		1.9	10.2	16.3	9.0	9.4	11.4	9.6	8.4	8.0	8.0	7.3	14.6
		2.0	10.2	18.0	9.7	10.0	9.3	8.7	7.0	8.3	7.7	7.4	15.9
50......	No	1.7	10.2	10.8	12.0	11.4	10.4	10.6	10.2	9.0	9.6	8.6	9.4
		1.8	10.2	13.6	12.4	10.8	8.8	9.6	8.2	9.6	7.2	10.2	11.6
		1.9	10.2	17.8	10.6	10.6	8.4	7.8	7.6	9.2	5.4	10.6	14.0
		2.0	10.2	21.6	9.6	10.2	6.8	7.0	7.0	8.2	5.0	9.6	17.0

probability of extreme values. For θ equals 1.8, all ten of the extreme subintervals contain more cumulative frequencies than expected. For θ equals 1.7, only two of the ten extreme subintervals contain more than expected. It therefore appears that for θ equals 1.8 the predictive distributions slightly understate the probability of extreme values, and for θ equals 1.7, slightly overstate the probability of the extreme values. The values of the χ^2 statistic for the predictive distribitions assessed with θ equals 1.7 are all less than 16.92, the .95 fractile of the χ^2 distribution. This result suggests that the predictive distributions using a value of θ of 1.7 conform very closely to the underlying distributions. The values of the χ^2 statistic for θ equals 1.8 are all larger than the corresponding values for θ equals 1.7 and in two cases are larger than 16.92. This implies that the predictive distributions assessed with θ equals 1.8 conform less closely to the underlying distributions than those with θ equals 1.7. Yet, an analysis of the extreme subintervals suggested that the predictive distributions would conform most closely to the underlying distributions for some value of the characteristic exponent between 1.7 and 1.8.

The above analysis was based upon an arbitrary partitioning of the unit interval into ten equal subintervals. To determine the sensitivity of the conclusions to this choice, the cumulative frequencies were graphed on what will be called uniform probability plots. These plots avoid this arbitrary partitioning. First, the cumulative frequencies for the portfolios within each group and for each value of θ were ranked in ascending order. Then, the ith ordered cumulative frequency was plotted on the horizontal axis against the fraction $i/(T+1)$, where T is the total number of cumulative frequencies in the pooled data for a given group. Since the fractions, $i/(T+1)$, $i = 1, 2, \ldots, T$, are unbiased estimates of the cumulative frequencies of the uniform distribution over the unit interval, the plot of the cumulative frequencies against these fractions would tend to fall on a straight line from the lower left hand corner to the upper right hand corner if the cumulative frequencies were uniformly distributed. If there were too many extreme cumulative frequencies, the plots would tend to take the shape of the reverse of an elongated S, whereas if there were not enough, the plots would tend to take the shape of an elongated S.

Although these plots were prepared for each of the five groups of portfolios, figure 1, because of space, presents only the plots for the portfolios of twenty securities controlled on industry. These plots are representative of the other four groups. The shape of these plots for θ equals 1.9 and 2.0 suggests that there are more extreme cumulative frequencies than expected. This implies that the predictive distributions tend to understate the probability of extreme values. The plots for θ equals 1.7 and 1.8 suggest that the corresponding conditional predictive distributions conform very closely to the underlying generating distributions. Therefore, the conclusions of this section do not seem to rest upon the arbitrary partitioning of the unit interval into ten equal intervals.

VII. AN EXPLANATION AND AN ALTERNATIVE

The last section presented empirical evidence which suggested that the predictive distributions assessed using a value of the characteristic exponent θ between 1.7 and 1.8 conformed very closely to the underlying distributions. The predictive distributions assessed

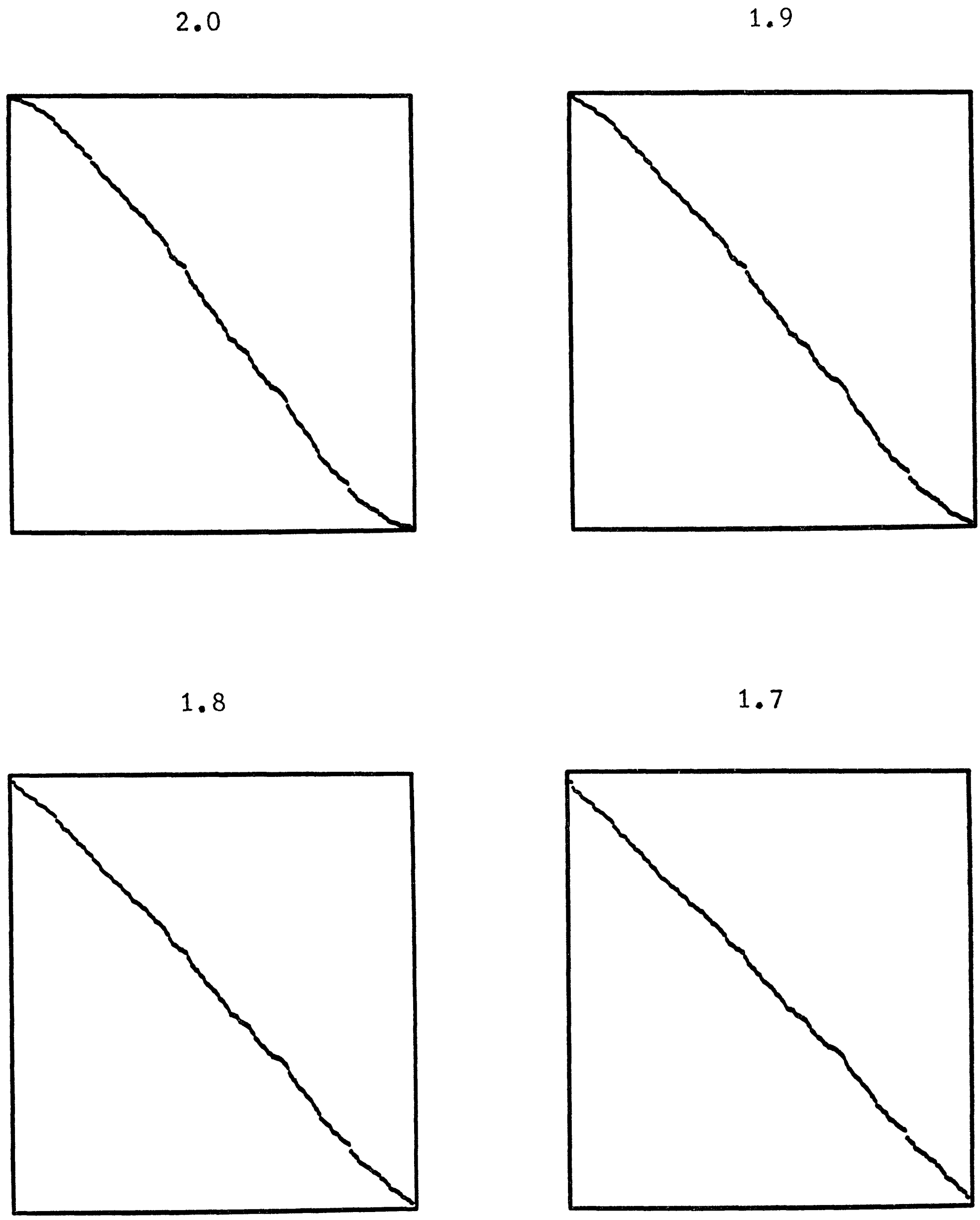

Fig. 1.—Uniform probability plots for predictive results using $[n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta}$ as assessment of $\gamma^{1/\theta}$ for portfolios of twenty securities controlled on industry.

using a value of θ greater than 1.8 tended to understate the probability of extreme values. Mandelbrot and later Fama[33] have publicized one reason for this understatement. For any given location and dispersion parameters, the probability of extreme values will be inversely related to the value of the characteristic exponent.[34] Thus, if the true value of θ were really between 1.7 and 1.8, the effect of assuming a value greater than 1.8 would be to understate the probability of extreme values. Although this effect, henceforth called the "distribution" effect, undoubtedly accounts for some of the understatement, the empirical evidence will suggest that its numerical importance is small compared to other effects.

Another reason, which is not well known, will be called the "aggregation" effect. The "aggregation" effect arises because of the way in which the dispersion parameters of the individual securities are aggregated into a dispersion parameter for a portfolio. An estimate of the dispersion parameter for the nonmarket effect of a portfolio with an equal investment in each security, according to Sharpe's procedure, is given by

$$\hat{\gamma}^{1/\theta} = [n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta} . \tag{13}$$

By partially differentiating (13) with respect to θ, it can be shown that $\hat{\gamma}^{1/\theta}$ is a decreasing function of θ if $1 \leq \theta \leq 2$ and $\hat{s}_i > 0$ for all i. Thus, if the true value of θ were between 1.7 and 1.8, assuming a value of θ greater than 1.8 would cause the estimate $\hat{\gamma}^{1/\theta}$ to be smaller than if one assumed the correct value. This understatement of the dispersion parameter would tend to cause an understatement of the probability of extreme values.

A third reason for this understatement is that (13) assumes that there are no industry effects. If the returns for individual securities were, for instance, normal ($\theta = 2$) and if the covariances between different securities were on balance positive, the value of $\hat{\gamma}^{1/2}$, given by (13), would tend to understate the true value of the dispersion parameter. This failure to account for industry effects would lead to an understatement of the probability of extreme values for the predictive distributions assessed using a value of θ of 2.0. Using a value of the characteristic exponent smaller than 2.0 in assessing the predictive distributions will cause the estimate of the dispersion parameter to be larger since $\hat{\gamma}^{1/\theta}$ is a decreasing function of θ. Thus, the understatement of the probability of extreme values will tend to be less than that observed for the predictive distributions assessed under the assumption of normality. For some value of θ less than 2.0, the probability of extreme values would be correctly stated. In short, the use of a value of the characteristic exponent less than the true value will tend to compensate for failure to account for industry effects if they are on balance positive.

To measure the magnitude of these three effects, the predictive tests were replicated using an alternative method of assessing the conditional predictive distributions. This alternative used an estimate of $\gamma^{1/\theta}$ which is independent of θ and which properly accounts for any industry

[33] Mandelbrot, "The Variation of Certain Speculative Prices"; Fama, "The Behavior of Stock-Market Prices."

[34] Fama and Roll ("Some Properties of Symmetric Stable Distributions") have published tables of symmetric, stable distributions. These tables indicate that the probability of extreme values will be inversely related to the value of the characteristic exponent for values of the standardized stable variates greater than 0.85, which is approximately the .72 fractile for any symmetric, stable distribution with characteristic exponent between 1.0 and 2.0. Because of symmetry, the statement would also be correct for variates less than -0.85.

effects. To obtain this estimate, the residuals of the individual securities in a portfolio were averaged to obtain a portfolio residual for each of the same 102 months used in obtaining the assessments of α_i and β_i. The parameter $\gamma^{1/\theta}$ for the portfolio was estimated from these portfolio residuals using (4). This estimate, designated $\hat{s}(n^{-1}\Sigma\epsilon_{it})$, is independent of θ, so that the "aggregation" effect is absent. Moreover, this estimate properly accounts for any cross-sectional dependence among the disturbances.[35]

Tables 5 and 6 and figure 2 present the results of this replication. Table 5 contains the results for the six portfolios of twenty securities controlled on industry in exactly the same format as table 1, which was discussed in the last section.

[35] This technique is mathematically identical to the following: Calculate the wealth relatives for the portfolio, $\Sigma_i n^{-1} R_{it}$. Regress these wealth relatives on the market relatives to obtain estimates of $\bar{\alpha}$, $\bar{\beta}$, and $\gamma^{1/\theta}$. With these estimates, assess predictive distributions and perform the predictive tests.

The estimates of $\bar{\alpha}$ and $\bar{\beta}$ will be the same whether they are calculated by regressing the wealth relatives on the market relatives or by averaging the estimates of α_i and β_i. To show this, note that the estimates of $\bar{\beta}$ obtained by regressing the wealth relatives on the market are

$$\hat{\bar{\beta}} = \left[\sum_t \sum_i \left(\frac{1}{n} R_{it}\right)(M_t - \bar{M})\right] \Big/ \sum_t M_t(M_t - \bar{M}) \ .$$

Factoring out $1/n$ and rearranging the order of summation,

$$\hat{\bar{\beta}} = \frac{1}{n}\sum_i \left[\frac{\sum_t R_{it}(M_t - \bar{M})}{\sum_t M_t(M_t - \bar{M})}\right] = \frac{1}{n}\sum_i \hat{\beta}_i \ .$$

It is easily shown that $\bar{\alpha}$ will be the same whether estimated by regressing the wealth relatives on the market relatives or by averaging the estimates $\hat{\alpha}_i$. Thus, the portfolio residuals will be the same whether given by the regressions of the wealth relatives or from the averages of the residuals of the individual securities.

Table 6 contains the pooled results for the five different groups of portfolios. The format of this table is identical to that of table 4. Figure 2 contains the uniform probability plots for the same portfolios as in figure 1.

The differences between the predictive results using $[n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta}$ as an assessment of $\gamma^{1/\theta}$ and the results using $\hat{s}(n^{-1}\Sigma\epsilon_{it})$ can be attributed to the "aggregation" effect and failure to account for industry effects. The results in the last section which used $[n^{-\theta}\Sigma\hat{s}_i^{\theta}]^{1/\theta}$ as an assessment of $\gamma^{1/\theta}$ suggested that the shape of the predictive distributions was quite sensitive to the choice of θ and also that the predictive distributions tended to understate the probability of extreme values when a characteristic exponent greater than 1.8 was assumed.

A first examination of the results using $\hat{s}(n^{-1}\Sigma\epsilon_{it})$ as an assessment of $\gamma^{1/\theta}$ suggests that the shape of the predictive distributions are considerably less sensitive to the choice of θ. For example, there appears to be no consistent tendency for the predictive distributions for the six portfolios controlled on industry (table 5) to understate or overstate the probability of extreme values for any value of θ. For θ equals 2.0, six of the twelve extreme subintervals contain more cumulative frequencies than expected; for θ equals 1.7, five contain more than expected. In addition, the values of the χ^2 statistic in table 5 suggest that there is no relationship between the value of θ and the degree of approximation. Two of the six portfolios have the smallest value of the χ^2 statistic for θ equals 2.0; two for θ equals 1.8; and two for θ equals 1.7. The pooled results (table 6) and the uniform probability plots (fig. 2) yield roughly the same conclusions as the results for the individual portfolios. On the basis of this evidence, it seems safe to conclude that the "aggregation" effect and failure to account for industry effects are jointly much more important numerically than the "distribution" effect. For the portfolios controlled on industry, industry effects are likely to be minimal, so that the differences between the two methods of assessing predictive distributions can be attributed solely to the "aggregation" effect.

Besides being useful in analyzing the importance of the "distribution" and the "aggregation" effects, these alternatively assessed predictive distributions appear to conform very closely to the underlying distributions for any value of the characteristic exponent between 1.7 and 2.0. This is reflected in table 5 in that the values of the χ^2 statistic range from 5.45 to 14.27 all less than 16.92, the .95 fractile of the χ^2 distribution under the hypothesis that the predictive distributions are identical to the underlying distributions. It is also reflected in table 6 for the pooled sample in that the values of the χ^2 statistic are all less than 16.92 with the exception of three of the four χ^2 values for the portfolios of thirty-five securities not controlled on industry. That three of the four χ^2 values are larger than the .95 fractile for this group may be an artifact of the arbitrary subdivision of the unit interval. If one examines in table 6 the 0.0–0.1 and the

TABLE 5

PREDICTIVE RESULTS USING $\hat{s}(n^{-1}\Sigma\epsilon_{it})$ AS ASSESSMENT OF $\gamma^{1/\theta}$ FOR PORTFOLIOS OF 20 SECURITIES CONTROLLED ON INDUSTRY

β	No.	θ	Subintervals										χ^2
			0.0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0	
0.433....	1	1.7	12	9	11	15	10	8	9	12	12	4	7.81
		1.8	13	9	10	15	10	8	9	11	11	6	5.65
		1.9	16	6	10	15	10	8	9	11	11	6	9.76
		2.0	16	6	10	15	10	8	9	11	11	6	9.76
0.583....	2	1.7	5	12	9	7	11	11	12	7	15	13	8.59
		1.8	5	12	9	7	11	11	12	7	15	13	8.59
		1.9	6	11	9	7	11	11	12	7	15	13	7.41
		2.0	6	11	9	7	11	11	12	7	14	14	7.22
0.716....	3	1.7	7	7	10	16	10	9	13	7	11	12	7.61
		1.8	7	7	10	16	10	9	13	7	9	14	8.78
		1.9	8	6	10	16	10	9	13	6	10	14	9.57
		2.0	8	6	10	16	10	9	13	6	9	15	10.55
1.237....	4	1.7	12	13	11	8	8	14	11	11	5	9	6.43
		1.8	12	13	11	8	8	14	11	10	6	9	5.45
		1.9	13	12	12	7	8	14	11	10	6	9	6.24
		2.0	13	12	12	7	8	14	11	10	6	9	6.24
1.328....	5	1.7	9	18	9	14	5	11	10	4	11	11	14.27
		1.8	9	18	9	14	5	11	10	4	11	11	14.27
		1.9	10	17	9	14	5	11	10	3	11	12	14.27
		2.0	13	14	9	14	5	11	10	3	11	12	11.92
1.560....	6	1.7	8	10	12	8	14	13	10	5	14	8	8.00
		1.8	8	10	12	8	14	13	10	4	15	8	9.96
		1.9	8	10	12	8	14	13	10	4	14	9	8.78
		2.0	10	9	11	8	14	13	10	4	14	9	8.20

TABLE 6

SUMMARY OF PREDICTIVE RESULTS USING $\hat{s}(n^{-1}\Sigma\epsilon_{it})$ AS ASSESSMENT OF $\gamma^{1/\theta}$

Securities per Portfolio	Controlled on Industry	θ	Expected No.	A: Subintervals 0.0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0	χ^2
20.......	Yes	1.7	61.2	53	69	62	68	58	66	65	46	68	57	8.46
		1.8	61.2	54	69	61	68	58	66	65	43	67	61	9.34
		1.9	61.2	61	62	62	67	58	66	65	41	67	63	8.62
		2.0	61.2	66	58	61	67	58	66	65	41	65	65	9.01
35.......	Yes	1.7	20.4	18	19	23	23	23	22	18	25	17	16	4.33
		1.8	20.4	19	18	23	23	23	22	18	25	16	17	4.33
		1.9	20.4	19	18	23	23	23	22	18	24	16	18	3.65
		2.0	20.4	19	18	23	23	23	22	18	24	15	19	3.94
20.......	No	1.7	122.4	127	143	113	140	118	130	110	107	118	118	11.03
		1.8	122.4	135	136	112	139	119	130	109	106	117	121	10.43
		1.9	122.4	140	132	111	139	119	130	109	104	113	127	12.29
		2.0	122.4	142	133	108	139	119	130	108	104	112	129	14.27
35.......	No	1.7	71.4	82	68	73	91	76	69	71	55	59	70	13.48
		1.8	71.4	90	62	71	90	77	69	71	53	59	72	18.35
		1.9	71.4	90	63	70	90	77	69	71	52	58	74	19.11
		2.0	71.4	94	59	70	90	77	69	71	52	57	75	23.06
50.......	No	1.7	51.0	54	63	53	53	53	49	41	56	42	46	7.84
		1.8	51.0	59	59	52	53	53	49	41	53	43	48	6.24
		1.9	51.0	63	55	53	52	53	50	40	51	43	50	6.98
		2.0	51.0	66	52	55	50	53	50	40	51	43	50	8.51

B

Securities per Portfolio	Controlled on Industry	θ	Expected No.	0.0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0
20.......	Yes	1.7	10.2	8.8	11.5	10.3	11.3	9.7	11.0	10.8	7.7	11.3	9.5
		1.8	10.2	9.0	11.5	10.2	11.3	9.7	11.0	10.8	7.2	11.2	10.2
		1.9	10.2	10.2	10.3	10.3	11.2	9.7	11.0	10.8	6.8	10.8	10.5
		2.0	10.2	11.0	9.7	10.2	11.2	9.7	11.0	10.8	6.8	10.8	10.8
35.......	Yes	1.7	10.2	9.0	9.5	11.5	11.5	11.5	11.0	9.0	12.5	8.5	8.0
		1.8	10.2	9.5	9.0	11.5	11.5	11.5	11.0	9.0	12.5	8.0	8.5
		1.9	10.2	9.5	9.0	11.5	11.5	11.5	11.0	9.0	12.0	8.0	9.0
		2.0	10.2	9.5	9.0	11.5	11.5	11.5	11.0	9.0	12.0	7.5	9.5
20.......	No	1.7	10.2	10.6	11.9	9.4	11.7	9.8	10.8	9.2	8.9	9.8	9.8
		1.8	10.2	11.2	11.3	9.3	11.6	9.9	10.8	9.1	8.8	9.7	10.1
		1.9	10.2	11.7	11.0	9.2	11.6	9.9	10.8	9.1	8.7	9.4	10.6
		2.0	10.2	11.8	11.1	9.2	11.6	9.9	10.8	9.0	8.7	9.3	10.7
35.......	No	1.7	10.2	11.7	9.7	10.4	13.0	10.9	9.9	10.1	7.9	8.4	10.0
		1.8	10.2	12.9	8.9	10.1	12.9	11.0	9.9	10.1	7.6	8.4	10.3
		1.9	10.2	12.9	9.0	10.0	12.9	11.0	9.9	10.1	7.4	8.3	10.6
		2.0	10.2	13.4	8.4	10.0	12.9	11.0	9.9	10.1	7.4	8.1	10.7
50.......	No	1.7	10.2	10.8	12.6	10.6	10.6	10.6	9.8	8.2	11.2	8.4	9.2
		1.8	10.2	11.8	11.8	10.4	10.6	10.6	9.8	8.2	10.6	8.6	9.6
		1.9	10.2	12.6	11.0	10.6	10.4	10.6	10.0	8.0	10.2	8.6	10.0
		2.0	10.2	13.2	10.4	11.0	10.0	10.6	10.0	8.0	10.2	8.6	10.0

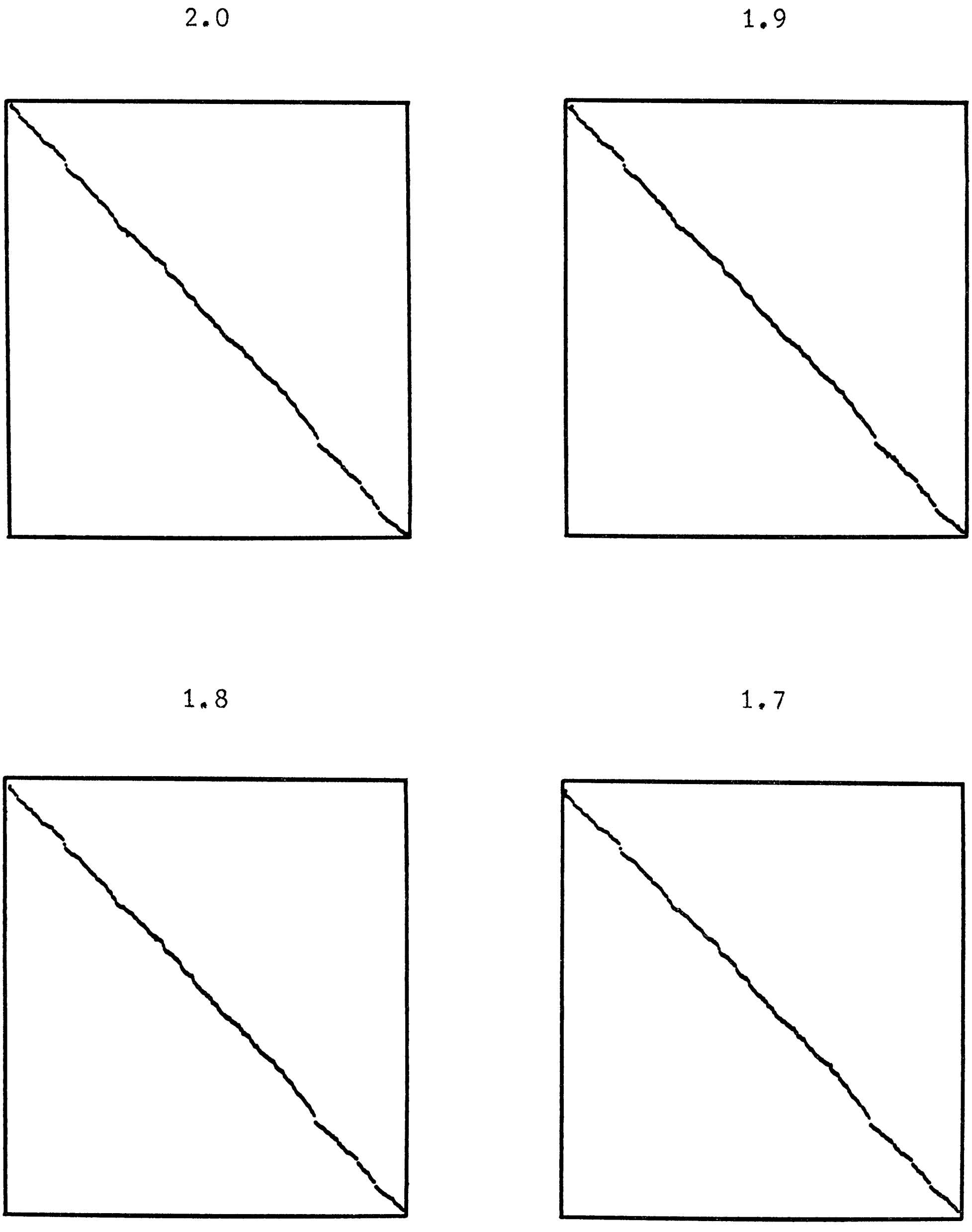

FIG. 2.—Uniform probability plots for predictive results using $\hat{s}(n^{-1}\,\Sigma\epsilon_{it})$ as assessment of $\gamma^{1/\theta}$ for portfolios of twenty securities controlled on industry.

0.1–0.2 subintervals for this group, one finds that there are more cumulative frequencies than expected in the 0.0–0.1 subinterval and less in the 0.1–0.2 subinterval for all values of θ. Thus, the large values of the χ^2 statistic for these portfolios may well result only from the arbitrary partitioning of the unit interval. The uniform probability plots in figure 2 appear to conform very closely to a straight line from the lower left hand corner to the upper right hand corner for all values of θ.

Before concluding this section, the assumption of linearity and the adequacy of the assessments in the context of portfolios for the period from 1944 through 1960 can be examined jointly. Strong empirical evidence indicated that these alternatively assessed conditional predictive distributions are almost insensitive to the assumed value of θ, $1.7 \leq \theta \leq 2.0$. Moreover, these distributions properly account for any industry effects. Therefore, the degree of correspondence between these predictive distributions and the underlying distributions will be determined jointly by the correctness of the assumption of linearity and the adequacy of the assessments. The evidence just presented, however, indicated that these conditional predictive distributions conform very closely to the underlying distributions, so that it can be concluded that the assumption of linearity and the assessments are jointly adequate.

VIII. CONCLUSION

The empirical results of this paper suggest that the choice of a particular method of assessing future performance or predictive distributions of future returns of portfolios using historical data is of crucial importance in obtaining accurate assessments. For example, the method inherent in Sharpe's generalized procedure for the calculation of the efficient set yields predictive distributions whose accuracy is extremely sensitive to the form of the distributions assumed for the individual securities. The reasons for this sensitivity were discussed.

An alternative method proposed in this paper yields predictive distributions which appeared extremely accurate over a very wide range of assumptions. In short, this method is considerably more robust to the assumptions than the first method and therefore of the two methods examined would seem to be the preferred one in the practical application of portfolio theory.