From *Options Futures and Other Derivatives* by John Hull, Prentice Hall 6th Edition, 2006.

Exercise 1

The Black-Scholes formula for the value C of a European call option at time t and expiration time at time T is:

$$C = S_0\Phi(d_1) - \frac{E}{e^{r(T-t)}}\Phi(d_2)$$

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

Answer the following questions:

1. Find $\Phi'(d_1)$.
2. Show that $S_0\Phi'(d_1) = \frac{E}{e^{r(T-t)}}\Phi'(d_2)$.
3. Find $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$.
4. Show that

$$\frac{\partial C}{\partial t} = -rEe^{-r(T-t)}\Phi(d_2) - S_0\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}}.$$

5. Show that $\frac{\partial C}{\partial S} = \Phi(d_1)$.
6. Show that C satisfies the Black-Scholes differential equation.
7. Show that C satisfies the boundary conditions for a European call option, $C = max[S-E, 0]$ as $t \to T$.

Exercise 2

Assume that a non-dividend-paying stock has an expected return of μ and volatility of σ. A financial institution has just announced that it will trade a security that pays off a dollar amount equal to $ln(S_T)$ at time T, where S_T denotes the value of the stock price at time T. Answer the following questions:

a. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price at time T.

b. Confirm that your price satisfies the Black-Scholes-Merton differential equation.