University of California, Los Angeles
Department of Statistics

Statistics C183/C283

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Exam 2
10 May 2006

Name: SOLUTIONS

Problem 1 (20 points)
Answer the following questions:

a. Suppose the single index model holds. Show that the cut-off point C^* can be written as:

$$C^* = (\bar{R}_p - R_f)\beta_p \frac{\sigma_m^2}{\sigma_p^2}.$$

$$C^* = \sigma_m^2 \sum z_j \beta_j = \sigma_m^2 \sum \lambda x_i \beta_j$$

$$= \sigma_m^2 \sum \frac{\bar{R}_p - R_F}{\sigma_p^2} x_j \beta_j = \sigma_m^2 \frac{\bar{R}_p - R_F}{\sigma_p^2} \beta_p$$

$$\Rightarrow C^* = (\bar{R}_p - R_F)\beta_p \frac{\sigma_m^2}{\sigma_p^2}$$

b. Assume the constant correlation model holds and that short sales are allowed. Under what condition the cut-off point C^* is equal to:

$$C^* = \frac{1}{N}\sum_{i=i}^{N} \frac{\bar{R}_i - R_f}{\sigma_i}$$

$$C^* = \frac{\rho}{1-\rho+N\rho} \sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i} = \frac{\rho/\rho}{\frac{1}{\rho} - \frac{\rho}{\rho} + \frac{N\rho}{\rho}} \sum \frac{\bar{R}_i - R_F}{\sigma_i}$$

$$= \frac{1}{\frac{1}{\rho} - 1 + N} \sum \frac{\bar{R}_i - R_F}{\sigma_i} \quad \text{IF } \frac{1}{\rho} \approx 1 \Rightarrow C^* = \frac{1}{N} \sum \frac{\bar{R}_i - R_F}{\sigma_i}.$$

Problem 2 **(25 points)**
You are given the following data:

Stock i	R_i	σ_i
1	0.29	0.03
2	0.19	0.02
3	0.08	0.15

a. Assume short sales are allowed, $R_f = 0.05$, and $\bar{\rho} = 0.5$. Rank the stocks based on the excess return to standard deviation ratio, find the cut-off point C^*, and find the optimum portfolio.

$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum \frac{\bar{R}_i - R_F}{\sigma_i}$	C_i
$\frac{0.29-0.05}{0.03} = 8$	$\frac{0.5}{1-0.5+0.5} = 0.5$	8	4
$\frac{0.19-0.05}{0.02} = 7$	$\frac{0.5}{1-0.5+2(0.5)} = \frac{1}{3}$	15	5
$\frac{0.08-0.05}{0.15} = 0.2$	$\frac{0.5}{1-0.5+3(0.5)} = \frac{1}{6}$	15.2	3.8

$$Z_1 = \frac{1}{(1-0.5)0.03}\left[8-3.8\right] = 280$$

$$Z_2 = \frac{1}{(1-0.5)0.02}\left[7-3.8\right] = 320$$

$$Z_3 = \frac{1}{(1-0.5)0.15}\left[0.2-3.8\right] = -48$$

$$\sum Z_i = 552$$

$$\therefore X_1 = \frac{280}{552} = 0.507, \quad X_2 = \frac{320}{552} = 0.580$$

$$X_3 = -\frac{48}{552} = -0.087.$$

b. The above solution could have been found using the techniques that discussed earlier in class through the following:

$$Z = \Sigma^{-1}R = \begin{pmatrix} 0.00090 & 0.00030 & 0.00225 \\ 0.00030 & 0.00040 & 0.00150 \\ 0.00225 & 0.00150 & 0.02250 \end{pmatrix}^{-1} \begin{pmatrix} 0.29-0.05 \\ 0.19-0.05 \\ 0.08-0.05 \end{pmatrix} = \begin{pmatrix} 280.00 \\ 320.00 \\ -48.00 \end{pmatrix}.$$

Explain what you see here and verify that the solution is the same as with part (a) (~~there maybe some rounding differences~~).

$\sum Z_i = 552,$

$$Z_1 = 280/552 = 0.507$$

$$Z_2 = 320/552 = 0.580$$

$$Z_3 = -48/552 = -0.087$$

Problem 3 (30 points)

Suppose the single index model holds. Also short sales are allowed and there is a risk free rate $R_f = 0.002$. For 3 stocks the following were obtained based on monthly returns for a period of 5 years:

Stock i	α	β	σ_ϵ^2
1	0.01	1.08	0.003
2	0.04	0.80	0.006
3	0.08	1.22	0.001

The expected return and variance of the market are $\bar{R}_m = 0.10$ and $\sigma_m^2 = 0.002$ for the same period.

a. Suppose that the optimum portfolio consists of 30% of stock 1, 50% of stock 2, and 20% of stock 3. What is the β of this portfolio.

$$\beta_p = \sum x_i \beta_i = 0.30(1.08) + 0.5(0.80) + 0.20(1.22) \Rightarrow \underline{\beta_p = 0.968}$$

$$\alpha_p = \sum x_i \alpha_i = 0.3(0.01) + 0.5(0.04) + 0.2(0.08) = 0.039$$

$$\bar{R}_1 = 0.01 + 1.08(0.10) = 0.118 \qquad \bar{R}_3 = 0.08 + 1.22(0.10) = 0.202$$

$$\bar{R}_2 = 0.04 + 0.80(0.10) = 0.12 \qquad \boxed{R_p = 0.1358} \qquad \bar{R}_p = \alpha_p + \beta_p \bar{R}_m = 0.1358$$

b. Suppose that you are a portfolio manager and you have \$500000 to invest in this optimum portfolio on behalf of a client. In addition this client wants to invest another \$300000 by borrowing this amount at the risk free rate $R_f = 0.002$. What is the expected return and standard deviation of this portfolio. Show it on the expected return standard deviation space.

$$\bar{R}_c = (1-x)R_F + x\bar{R}_p = -0.6(0.002) + 1.6\left[0.30(0.118) + 0.50(0.12) + 0.20(0.202)\right]$$

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum x_i^2 \sigma_{\epsilon i}^2 = 0.968^2(0.002) = 0.00187 + 0.00181 = 0.003684 \Rightarrow \underline{\bar{R}_c = 0.21608}$$

$$\sigma_c = x\sigma_p = 1.6\left[(1.08^2 \cdot 0.3^2 + 0.80^2 \cdot 0.5^2 + 1.22^2 \cdot 0.2^2)0.002 + 0.3^2 \cdot 0.003 + 0.5^2 \cdot 0.006 + 0.2^2 \cdot 0.001 + (1.08 \times 0.80 \cdot 0.3 \cdot 0.5 + 1.08 \times 1.22 \cdot 0.3 \cdot 0.2 + 0.80 \times 1.22 \cdot 0.5 \cdot 0.2)0.002 \times 2\right]^{1/2} \Rightarrow \sigma_c = 0.09711$$

$= 1.6\sqrt{\cdot}$

$\underline{\sigma_p = 0.060696}$

c. What is the covariance between the portfolio of part (a) and the market?

$$\text{Cov}\left[0.30R_1 + 0.50R_2 + 0.20R_3,\ R_m\right] =$$

$$0.30\,\text{Cov}(R_1, R_m) + 0.50\,\text{Cov}(R_2, R_m) + 0.20\,\text{Cov}(R_3, R_m)$$

$$= 0.30[1.08 \times 0.002] + 0.50[0.80 \times 0.002] + 0.20[1.22 \times 0.002] \Rightarrow \text{Cov}(R_p, R_m) = \beta_p \sigma_m^2 = 0.001936$$

d. If the client wants to allocate 60% of his initial funds in the optimum portfolio and the remaining 40% in the risk free asset, what would be the expected return and standard deviation of this position?

$$x = 0.60, \quad 1 - x = 0.4$$

$$\bar{R}_c = 0.4(0.002) + 0.6(0.1358) \Rightarrow \bar{R}_c = 0.08228.$$

$$\sigma_c = 0.6\,\sigma_p = 0.6(0.060696) \Rightarrow \sigma_c = 0.03642$$

e. What is the covariance between stock 1 and the market?

$$\text{Cov}(R_1, R_m) = \text{Cov}(\alpha + \beta R_m + \epsilon,\ R_m)$$

$$= \beta\sigma_m^2 = 1.08(0.002) \Rightarrow \underline{\text{Cov}(R_1, R_m) = 0.00216}$$

Problem 4 (25 points)
Assume that $\sigma_m^2 = 10, R_f = 0.05$. You are also given $\beta_1 = 1, \beta_2 = 1.5, \beta_3 = 1, \beta_4 = 2, \beta_5 = 1, \beta_6 = 1.5, \beta_7 = 2, \beta_8 = 0.8, \beta_9 = 1, \beta_{10} = 0.6$. The table below shoes the procedure for finding the cut-off point C^*.

Stock i	$\frac{\bar{R}_i - R_f}{\beta_i}$	$\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{\epsilon j}^2}$	$\frac{\beta_i^2}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{\epsilon j}^2}$	C_i
1	10.0	0.20	??? ~~0.20~~	0.02000	0.02000	1.67
2	8.0	0.45	0.65	0.05625	0.07625	3.69
3	7.0	0.35	1.00	0.05000	0.12625	4.42
4	6.0	2.40	A ??? 3.40	0.40000	0.52625	5.43
5	6.0	0.15	3.55	0.02500	0.55125	??? C
6	4.0	0.30	3.85	0.07500	0.62625	5.30
7	3.0	0.30	4.15	0.10000	B ???	5.02
8	2.5	0.10	4.25	0.04000	0.76625	4.91
9	2.0	0.10	4.35	0.05000	0.81625	4.75
10	1.0	0.06	4.41	0.06000	0.87625	4.52

a. Find the three missing values. A: 0.20

B: 0.72625

$$C_5 = \frac{\sigma_m^2 \sum \frac{\bar{R}_i - R_f}{\sigma_{\epsilon i}^2}\beta_i}{1 + \sigma_m^2 \sum \frac{\beta_i^2}{\sigma_{\epsilon i}^2}} = \frac{0.10 \cdot 3.55}{1 + 0.10\,(0.55125)} \Rightarrow C_5 = 5.451$$

b. If short sales are not allowed find the cut-off point C^* and the ~~composition of the optimum portfolio~~ value of Z_1.

$$C^* = 5.45 \qquad Z_1 = \frac{\beta_1}{\sigma_{\epsilon_1}^2}(10 - 5.45) = \frac{1}{50}(10 - 5.45)$$

$$\frac{\beta_1^2}{\sigma_{\epsilon_1}^2} = 0.02 \Rightarrow \frac{1}{\sigma_{\epsilon_1}^2} = 0.02 \Rightarrow \sigma_{\epsilon_1}^2 = 50 \qquad \Rightarrow Z_1 = 0.091$$

c. If short sales are allowed find the cut-off point C^* and the ~~composition of the optimum portfolio~~ value of Z_1.

$$C^* = 4.52$$

$$Z_1 = \frac{1}{50}[10 - 4.52] \Rightarrow Z_1 = 0.1096$$

d. Find the correlation coefficient between stock 1 and the market.

$$\rho = \frac{Cov(R_1, R_m)}{\sigma_1 \sigma_m} = \frac{1 \times 10}{\sqrt{1 \times 10 + 50}\,\sqrt{10}} \Rightarrow \rho = \cancel{0.258}\ 0.408$$

$$\sigma_1 = \sqrt{\beta_1^2 \sigma_m^2 + \sigma_{\epsilon_1}^2}$$