Note on Seminar 1

{ # Yellow Book - CFT
CFT on a plane : 1406.4290

1 Symmetry in Field Theory

对于经典和量子场论来说，场论的对称性可以提供很多的信息。对于一个量子理论最重要的两个东西：

- 可观测量（或者称为量子场，关联函数，等等）
- 谱（或者称为量子态，能级，等等）

研究一个量子场论我们就是在获取这个理论的这两个信息。而对称性为我们提供了获取这两个信息极其有力的帮助。而共形对称性更是几乎能够完全帮我们确定这两个信息的基本结构！

可观测量：对于经典场论我们有诺特定理，可以通过对称性给予我们很丰富的守恒的信息。那么对于量子场论我们同样有 Ward Identity。告诉我们对于“量子场”的很多结构。

谱：这个在量子力学之中我们已经可以看到，对于氢原子的量子态，因为氢原子有着 SO（3）对称性，因此可以使用 SO（3）群的表示来分类谱（也就是由角量子数和磁量子数来标记量子态）。对于一般的量子场论来讲，我们依旧可以通过研究对称性，用一些特殊的量子数来标记我们的量子态。

1.1 对称性的表述

两种变换

- 流形的变换

$$x^\mu \to x^{\mu\prime} \quad \phi(x) \to \phi'(x') \tag{1}$$

- 内秉的变换

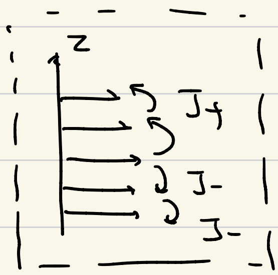
$$\phi(x) \to \phi'(x) \tag{2}$$

How do we use Symmetry To get Spectrum

* How Symmetry Give us the Spectrum ?

Angular Momentum in H Atom :

SO(3) Symmetry → $J_x\ J_y\ J_z$ Generators → $J_+\ J_-\ J_z$
(with a Symmetry Algebra)



1 Rep for Algebra

△ Symmetry gives us algebras
△ Algebra has Representations
△ Element of Representations are Quantum States

→ Virasoro Algebra & Verma Modules

* How Symmetry give us CF ?

Wigner - Eckart Thm in A.M

<u>Thm</u> : For a Tensor Operator T^k_q its E.V can be written as:

$$\langle \gamma; j'm' | T^k_q | \gamma ; jm \rangle = \langle \gamma' j' \| T^k \| \gamma j \rangle \langle j'm' | j k m q \rangle$$

γ means other Quantum #

k label T^k_q 是按 SO(3) 哪阶表示来变换的] : e.g T^2_q $q = -2, -1, 0, 1, 2$

q label T^k 张量分量

Δ ⇒ Abstractly :

$$\langle \mathcal{O} \rangle = \underline{C} \times \underline{\text{Const}}$$

Const: CG coefficient : Defined By Symmetry

$\mathcal{O}$: Covariant Operator e.g. Field Operator

C: Sth. Only Related to Rep & $\mathcal{O}$ But doesn't Dep. on Quantum State

Δ Symmetry + 1 CF of the Rep ⇓ All CF of Elements in the Rep

⇒ Global WI + Primary CF, Local WI ⇒ all CF

Steps in Symmetry Study !

1 Get the Symmetry Transformation ⇒ Conserved Current & Charge (WI)

⇓ ⇓

2. Get the Generators Restriction on CF

⇓ [Central Charge]

3. Algebra of Generators

⇓

4. Rep. of Algebras [Operator Formalism] [WI, OPE ...]

↓ ↓

Spectrums ! CF !

LEFT STEPS

- Generator : $\underbrace{\Phi'(x) - \Phi(x)}_{\text{Same Point}} = \delta_\omega \Phi(x) = -i\,\underbrace{\omega_a}_{\text{十分小的数}}\,\underbrace{G_a}_{\text{Generator (一个算符!)}}\,\Phi(x)$

- Transformation : $x^{\mu'} = x^\mu + \omega_a \boxed{\dfrac{\delta x^\mu}{\delta \omega_a}}$

  $\Phi'(x') = \Phi(x) + \omega_a \boxed{\dfrac{\delta F}{\delta \omega_a}(x)}$

  what the hell Are these ?

  - 一个记号：标记表征变换的样子与场的协变方式

Given Transformation	$\longleftrightarrow$	$\dfrac{\delta x^\mu}{\delta \omega_a} \Big/ \dfrac{\delta F}{\delta \omega_a}(x)$ 数据	$\longleftrightarrow$	Generator
Finite Trans.	①	Infinitesimal Trans.	②	Generator

* Relation ① : 求导 : $\begin{cases} x \to x + a = x' & \dfrac{dx'}{da} = 1 \\ \Phi(x) \to \Phi'(x') & \dfrac{dF}{d\omega} = 0 \end{cases}$ (不再认为 $\Phi(x)$ 是 x 函数而是 ω_a)

x^μ 对应 : $x^\mu \to x^{\mu'}$ 的映射

F 对应 : $\Phi(x) \to \Phi'(x')$ 的映射 !

* Relation ② : $\Phi'(x') - \Phi(x) = \omega_a \dfrac{\delta F}{\delta \omega_a}(x)$

$$\Phi(x) = \Phi(x') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi \Big|_{x'} \quad \to \text{Taylor Expansion}$$

$$\Phi'(x') - \Phi(x') = -i\omega_a G_a \Phi(x')$$

$$\Rightarrow \quad iG_a \Phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi - \frac{\delta F}{\delta \omega_a}$$

<u>Ref</u> : CFT P39 (Any QFT Textbook)

* $G_a \Rightarrow$ Operator $\Rightarrow$ Algebra $\Rightarrow$ Spectrum !

Conformal Symmetry

$j(z)T(z)$ → 对应：

* Conformal Transformation : * Transformation of Manifold
  * Metric Inv. $g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x)$

* Infinitesimal T. : $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ → Transformation of Manifold / Coordinate Change

$\because$ $g_{\mu\nu}$ is a Tensor

$$\therefore\ g'_{\mu\nu}(x') = \underbrace{\frac{\partial x^a}{\partial x'^{\mu}} \frac{\partial x^b}{\partial x'^{\nu}}}_{\text{Matricies}} g_{ab}(x) \qquad \text{其中：} \left.\frac{\partial x^a}{\partial x'^{\mu}}\right|_x = \delta^a_{\mu} - \partial_{\mu}\varepsilon^a(x)$$

$$\Rightarrow\ g'_{\mu\nu}(x') = g_{\mu\nu}(x) - (\partial_{\mu}\varepsilon_{\nu}(x) + \partial_{\nu}\varepsilon_{\mu}(x))$$

* Restriction of CT. $g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x)$

$$\Rightarrow\ (1-\Lambda(x))\, g_{\mu\nu}(x) = (\partial_{\mu}\varepsilon_{\nu}(x) + \partial_{\nu}\varepsilon_{\mu}(x))$$

$$\Rightarrow\ \boxed{\text{C.T. Relation: } (\partial_{\mu}\varepsilon_{\nu}(x) + \partial_{\nu}\varepsilon_{\mu}(x)) = f(x)\, g_{\mu\nu}(x)}$$

△ Now consider flat Euclidean S.T. $g_{\mu\nu}(x) = \mathrm{Id}_d$ Ref: CFT P4b

- 双边 Tr : $f(x) = \frac{2}{d}\partial_{\rho}\varepsilon^{\rho}(x)$
- 一阶导 $2\partial^2\varepsilon_{\rho} = (2-d)\,\partial_{\rho} f$ $\quad (2\partial_{\mu}\partial_{\nu}\varepsilon_{\rho} = \eta_{\mu\rho}\partial_{\nu} f + \eta_{\nu\rho}\partial_{\mu} f - \eta_{\mu\nu}\partial_{\rho} f)$
- 一阶导 $(d-1)\,\partial^2 f = 0$ $\quad ((2-d)\,\partial_{\mu}\partial_{\nu} f = \eta_{\mu\nu}\partial^2 f)$

Contraction

$$\Rightarrow \begin{cases} d=2 \text{ Is Unique: } d=2 \text{ 时 } \partial^2\varepsilon_{\rho}(x) = 0 \quad (\text{全纯函数!}) \\ d\geq 3 \text{ 时 } \quad \partial_{\mu}\partial_{\nu} f = 0, \quad \partial^2 f(x) = 0 \end{cases}$$

Global C. T.

2 $d \geq 3$ 时

$\partial^2 f(x) = 0 \quad ((2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f) \Rightarrow f(x) = A + B_\mu x^\mu$

$2\partial_\mu\partial_\nu \varepsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \Rightarrow \varepsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$

$$\Updownarrow$$

$$2\partial_\mu\partial_\nu \varepsilon_\rho = C$$

$\Rightarrow$ C.T.

$$\begin{cases} x'^\mu = x^\mu + a^\mu & \longrightarrow \quad P_\mu = \partial_\mu \\ x'^\mu = \alpha x^\mu & \longrightarrow \quad D = -i x^\mu \partial_\mu \\ x'^\mu = M^\mu{}_\nu x^\nu & \longrightarrow \quad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ x^{\mu\prime} = \dfrac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2} \iff \dfrac{x^\mu}{x^2} \text{ 平移 } b^\mu \; \dfrac{x'^\mu}{x'^2} & \longrightarrow \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{cases}$$

$\Rightarrow$ Global C.T. Algebra: J

Take: $J_{\mu\nu} = L_{\mu\nu}$ $\quad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu)$

$J_{-10} = D$ $\quad J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu)$

$J_{ab} = -J_{ba}$

$$\eta = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & \end{matrix} \\ \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \end{array}$$

J 满足: $[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac})$

Lorentz Group Algebra !! $\quad SO(d+1,1)$ Lorentz Group

Note: We consider Flat Euclidean Manifold. But get a Lorentzian Symmetry!

2. $d = 2$ 时

1) 上述约束依旧成立 → Global Conformal Symmetry:

2) 其他约束: $2\partial^2 \varepsilon_\rho = (2-d)\partial_\rho f = 0$ $\quad \therefore$ $\varepsilon_\rho(x)$ 是全纯函数

$$x^{\mu\prime} = x^\mu + \varepsilon^\mu(x)$$

$$\boxed{\text{CT in 2D} = \text{Holomorphic Transformation}}$$

Consequences:
- Conformal Symmetry is a local Symmetry 因为 $\varepsilon^\mu(x)$ Dep. on x
  → Infinite Generators
- Better use Language of Complex Analysis to write Theory

Change into Another Coordinate : Holomorphic Coordinate

(Notation from Polchinski String)

Def $z = \sigma^1 + i\sigma^2 \qquad \bar{z} = \sigma^1 - i\sigma^2$

$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \qquad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$

$$\begin{cases} \sigma^1 = \frac{1}{2}(z + \bar{z}) \\ \sigma^2 = \frac{1}{2i}(z - \bar{z}) \end{cases}$$

- Explanation: View z & $\bar{z}$ as seperate (理论定义在两个复平面上)
  但只有 $\bar{z} = \overline{(z)}$ 时为 Physical!!!

Def: $v^z = v^1 + iv^2 \qquad v^{\bar{z}} = v^1 - iv^2$

(Rank 1 Tensor) $v_z = \frac{1}{2}(v^1 - iv^2) \qquad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2)$

Remind Tensor have same Dim as Manifold

- Remark: △ Redefine Tensor in New coordinate

  △ $V_\mu \sim \partial \qquad V^\mu \sim dx^\mu$

  因为 $V \sim V_\mu \underline{dx^\mu} \sim V^\mu \underline{\partial_\mu}$

  十分合理 in GR sense

  △ 虽然用 flat Spacetime: $M = \mathbb{R}^2$ 但上下指标有区分!!

  △ Metric 十分合理

$$ds^2 = \underset{\text{flat } \mathbb{R}^2}{\underline{g_{\mu\nu}\, d\sigma^\mu d\sigma^\nu}} = g_{zz}\, dz\, dz + g_{z\bar{z}}\, dz\, d\bar{z} + g_{\bar{z}z}\, d\bar{z}\, dz + g_{\bar{z}\bar{z}}\, d\bar{z}\, d\bar{z}$$

$$= d\sigma^1 d\sigma^1 + d\sigma^2 d\sigma^2$$

$$= \frac{1}{2}(dz + d\bar{z})\frac{1}{2}(dz + d\bar{z}) + \frac{-1}{4}(dz - d\bar{z})(dz - d\bar{z})$$

$$= \frac{1}{4}(dz\,dz + 2\,dz\,d\bar{z} + d\bar{z}\,d\bar{z})$$

$$- \frac{1}{4}(dz\,dz - 2\,dz\,d\bar{z} + d\bar{z}\,d\bar{z})$$

$$= \frac{1}{2}\, dz\, d\bar{z} + \frac{1}{2}\, d\bar{z}\, dz$$

$$g_{\mu\nu} = \begin{array}{c} \begin{matrix} zz & z\bar{z} \end{matrix} \\ \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \end{array} \longrightarrow \text{Verify } v^z \text{ 与 } v_z \text{ 关系!!}$$

Inverse !!

$$g^{\mu\nu} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Warm up : Get used to holomorphic Coordinate

* Derivative : $\partial z = \frac{1}{2}(\partial_1 - i\partial_2)(\sigma^1 + i\sigma^2) = \frac{1}{2}(1+1) = 1$

$$\partial \bar{z} = \frac{1}{2}(\partial_1 - i\partial_2)(\sigma^1 - i\sigma^2) = \frac{1}{2}(1-1) = 0$$

$$\bar{\partial} z = 0$$

$$\bar{\partial} \bar{z} = 1$$

* Intergral Unit : $\sqrt{g_\sigma}\, d\sigma^1 d\sigma^2 = \sqrt{-g_z}\, d^2 z$ $\qquad -g_z = \frac{1}{4} \qquad g_\sigma = 1$

* Delta Function : Defi : $\int d^2 z\, \delta^2(z,\bar{z}) = 1$

由于 $\int d\sigma^1 \sigma^2\, \delta(\sigma^1)\delta(\sigma^2) = 1$ 由 $d\sigma^1 d\sigma^2 = \frac{1}{2} d^2 z$

$$\therefore \int d^2 z \left[\frac{1}{2} \delta(\sigma^1)\delta(\sigma^2) \right] = 1$$

$$\Rightarrow \delta^2(z,\bar{z}) = \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2)$$

* Stokes . Thm :

$$\int_R d^2 z\, (\partial v^z + \bar{\partial} v^{\bar{z}}) = i \oint_{\partial R} (v^z d\bar{z} - v^{\bar{z}} dz)$$

* 全反对称 Tensor : $\varepsilon_{\mu\nu} = \sqrt{|g|}\, \hat{\varepsilon}_{\mu\nu}$ 全反对称算子 !!

$$\hat{\varepsilon}_{\mu\nu} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad g = 1 \qquad \therefore \varepsilon_{\mu\nu} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\varepsilon_{z\bar{z}} = \frac{\partial \sigma^1}{\partial z}\frac{\partial \sigma^2}{\partial \bar{z}} \varepsilon_{12} + \frac{\partial \sigma^2}{\partial z}\frac{\partial \sigma^1}{\partial \bar{z}} \varepsilon_{21}$$

$$= \frac{1}{2} \times \frac{i}{2} \times 1 + \frac{1}{2i} \times \frac{1}{2} \times -1$$

$$= \frac{i}{2}$$

$$\varepsilon_{\mu\nu} = \begin{bmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix} \qquad \varepsilon^{\mu\nu} = \begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix}$$

* Conformal Transformation : $w(z)$ => 全纯函数 !!

C.T. in 2D

$z' = z + \varepsilon(z)$ => Infinitesimal Trans !

由于 $\varepsilon(z)$ 为全纯函数 $\begin{cases} \varepsilon(z) = \sum_{-\infty}^{+\infty} c_n z^{n+1} \\ \bar{\varepsilon}(\bar{z}) = \sum_{-\infty}^{+\infty} \bar{c}_n \bar{z}^{n+1} \end{cases}$ Laurent Expansion

o Consider a Scalar field

$$\phi'(z',\bar{z}') = \phi(z,\bar{z}) = \phi(z',\bar{z}') - \varepsilon(z')\partial'\phi(z',\bar{z}') - \bar{\varepsilon}(\bar{z}')\bar{\partial}'\phi(z',\bar{z}')$$

$$\delta\phi = \phi'(z',\bar{z}') - \phi(z',\bar{z}')$$

$$= -\varepsilon(z)\partial\phi - \bar{\varepsilon}(\bar{z})\bar{\partial}\phi$$ Laurent Expansion

$$= -\sum_n \left[c_n l_n \phi + \bar{c}_n \bar{l}_n \phi \right]$$

* Generators are : l_n $\bar{l}_n$ $n = -\infty, \cdots, +\infty$

$$\begin{cases} l_n = -z^{n+1}\partial \\ \bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \end{cases}$$

* Algebra of C.T. : (Classic)

$$\left.\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n) \bar{l}_{m+n} \end{aligned}\right] \text{2套 Algebra}$$

$$[l_m, \bar{l}_n] = 0$$

-> Witt Algebra !!

Properties : * l_0 l_{-1} l_1 : $\begin{cases} l_0 = -z\partial \longrightarrow \text{Scale D} \\ l_1 = -z^2\partial \longrightarrow \text{Special C.T.} \\ l_{-1} = -\partial \longrightarrow \text{Translation} \end{cases}$

Is Witt Algebra the Conformal Symmetry Algebra in 2D CFT ?

No !

$$\boxed{\text{Virasoro Algebra} : \quad [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{n+m,0}}$$

* Central Extension of Witt Algebra !!
* $m = 0, \pm 1$ 일 때 $\Rightarrow$ Same as Witt Algebra

What Can We get from Global Conformal Symmetry ?

$\Big\langle$ Quantum & Classic

2D & $\gtrsim$ 3D

$\Rightarrow$ CF of Primary Fields ! (Primary Field is a kind of Quasi - Primary)

> **Def** : Quasi-Primary Field has Δ : conformal Dimension in d - dim
>
> ○ $x \to x'$ (Global C.T.)
>
> ○ $\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x)$

Why Primary ? ○ 모든 State 는 Primary State : Label Vir. Algebra 표현!

○ Simplist Field :

We study the Simplist !!

Remember : $\underset{\sim}{1}$ CF $\xrightarrow[\text{Wigner-Eckart}]{}$ all CF

$\left|\frac{\partial x'}{\partial x}\right|(x)$ = Jacobi = $\frac{1}{\sqrt{|g(x)|}} = \Lambda(x)^{-d/2}$ d = dimension !

What's the Correlation Function of Primary Fields ?

2- point Function : $\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\Delta/d_1} \left|\frac{\partial x'}{\partial x}\right|_{x_2}^{\Delta/d_2} \langle \phi_1(x_1')\, \phi_2(x_2') \rangle$

Note: Assume Jacobi 不变 (Anomaly free)

则 $\langle \phi_1'(x_1')\phi_2'(x_2')\rangle = \langle F(\phi_1(x_1))\, F(\phi_2(x_2))\rangle$

<u>Thm</u>: Jacobi 不变 $\Rightarrow$ CF 与函数形式满足同一变换关系!

- Under Scale Trans: $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$
- Translation Symmetry:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \langle f(|x_1-x_2|)\rangle$$

$$\Rightarrow f(|x_1-x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1-x_2|)$$

$$\therefore f(|x_1-x_2|) = \frac{C_\Delta}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

- Under SCT: $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2b\cdot x+b^2x^2)^d} = \frac{1}{\gamma^d}$

$$|x_i'-x_j'| = \frac{|x_i-x_j|}{(1-2b\cdot x_i+b^2x_i^2)^{1/2}\,(1-2b\cdot x_j+b^2x_j^2)^{1/2}}$$

$$\Rightarrow \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \quad \therefore \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} = 1$$

"=" 仅于 $\Delta_1=\Delta_2$ 时成立

<u>Conclusion</u>:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \Delta_1=\Delta_2 \\ 0 & \Delta_1\neq\Delta_2 \end{cases}$$

3-Point Function:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}\, x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

What CF Teaches us ?

* 2-Point Function : only $\Delta_1 = \Delta_2$ can be in a "$\langle\ \rangle$"

  3-Point Function Then $\langle \phi_1 \phi_2 \phi_3 \rangle \neq 0 \Rightarrow \underline{\phi_2 \phi_3 \sim \phi_1}$

$$\boxed{\begin{matrix} \phi_2 & & \\ & x \longrightarrow & \phi_1 \\ \phi_3 & & \end{matrix}}$$ OPE structure.

* Correlation Function can be built up with 2 point Function + OPE

OPE : $V_{G_1}(x_1) V_{G_2}(x_2) = \sum_{G} C^{G}_{G_1 G_2}(x_1, x_2) V_{G_3}(x_3)$

$$\langle V V V V \rangle = \sum \sum C C \langle V V \rangle$$