Lectures Week 3.

Virasoro Algebra & Quantization :

Conformal Symmetry

$\swarrow$ $\searrow$

Symmetry Algebra $\longleftarrow$ Ward Identity

$\Downarrow$ $\Downarrow$

Spectrum Correlation Function

REVIEW :

[Consider only holomorphic]
(take $\bar{\varepsilon}\bar{V}^*(\bar{z}) = 0$) *

△ WI : (Conformal Ward Identity)

$$\underbrace{(\delta_\varepsilon \mathcal{O}(z',\bar{z}'))}_{}\ \delta(z-z') = \frac{\varepsilon}{2\pi}\oint_{\partial\Omega} dz\ j(z)\ \mathcal{O}(z',\bar{z}')$$

1. $j(z)$ act as Generator
2. Restriction on CF !

Conformal Trans. Generate By. $\delta z = v(z)\varepsilon$

* $\boxed{\text{Symmetry} \Rightarrow \partial_\mu J^\mu = 0}$ $\xrightarrow[\text{Version}]{\text{Quantum}}$ $\boxed{\text{Symmetry} \Rightarrow \delta\langle \mathcal{O}\rangle = \langle J\,\mathcal{O}\rangle}$

(Thm)

* For CFT in 2D : $j(z) = v(z)T(z)$ $\quad v(z)$ is Any holomorphic Function

△ Conformal WI ⇒ OPE coefficient !

假设 : $T(z)\mathcal{O}(0) \sim \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\mathcal{O}^{(n)}(0,0)$

$\Rightarrow$ 代入 CWI : $\underbrace{\delta\langle\mathcal{O}\rangle}_{\text{变换 Generator}} = -\varepsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left[\partial^n v(z)\,\underbrace{\mathcal{O}^{(n)}(0,0)}_{\text{OPE 系数}}\right]$

* **For Quasi - Primary Field**

$\mathcal{O}(z') = \xi^h\,\mathcal{O}(z)$ $\quad$ 其中 h 为 Conformal Dimension
$\quad z' = \xi z$

(由于 $\partial^2 v(z) = 0$ ∴ 不可确定)

$\Downarrow$

$$\Rightarrow T(z)\mathcal{O}(0,0) \sim \cdots + \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0)$$

* For Primary Field

$$\mathcal{O}(z') = (\partial z')^h \, \mathcal{O}(z)$$ 其中 $z'(z)$ 为任意全纯函数 h 为 Conformal Dim

$$\Rightarrow \quad T(z)\,\mathcal{O}(0,0) \sim \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0)$$

* What can we do with WI ?
  - → 1 Restrict Correlation Function ——→ CF
  - → 2. Def: Quantum Generator ⇒ Symmetry Algebra Element
* What can we do with OPE? (and care only 发散)
  - → 1 Calculate Commutator ⇒ Algebra Structure

Symmetry Algebra Element + Algebra Structure → Algebra

RADIAL QUANTIZATION

Conformal Symmetry ——→ WI ——→ Algebra Element

WI ↓ Restrict ↓ OPE ——→ Algebra Relation

Algebra Element + Algebra Relation → Symmetry Algebra ↓ Spectrum

---

o How do we quantize a QFT ? (Canonically)

1. $\mathcal{L}$ Lagrangian
   ↓
2. Legendre trans : H
3. Def Canonical Relation (Equal time Surface) $[x_n(t_1), P_m(t_2)] = \delta(t_1 - t_2)\,\delta_{mn}$
   ↓
4. Mode Expansion → x^+ & x (Annihilation / Creation)
   ↓
5. Spectrum ! $x|0\rangle = 0$ $\quad x^+ \cdots |0\rangle$

- Another Scheme of Quantizing QFT

$H \to$ Generator of Time Transformation ! (from WI)

$\delta \langle O \rangle = \int dz \langle J(z)\, O(z') \rangle$

1. Define a "time Direction"
2. Find the Generator of the "Time" Translation & Understand as H
3. Find Eigen State of Generator

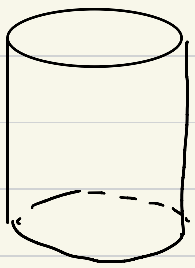
find the Annihlation & Creation

Question : Which Direction can be chosen as "Time" in 2D Euclidean Surface ?

3Q : { 1. Which Direction is time ?
What are the Corresponding Hamiltonian & C/A operators ?
2. What are the Symmetry Algebra ?

Answer 1 : Radial Direction is time :

- CFT on a Cylinder is Extremely Interesting

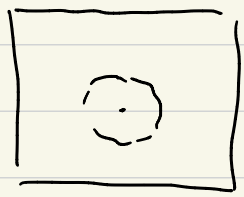


t

(x, t)

$x \sim x + L$

Conformal Transformation



$z = e^{\frac{2\pi}{L}(t + ix)}$

$e^{2\pi/L\, t}$ is Radial Direction

* Coordinate :	(x,t)	$(z, \bar{z})$	$\to$ physical when $z = \overline{(\bar{z})}$
* CF :	$\langle O O \rangle$	$= \langle F(O) F(O) \rangle$	
* Time :	t	$r \sim e^{2\pi/L\, t}$	
* Hamiltonian :	$\hat{H}$	$\hat{D}$	
* EM Tensor	$T^{\mu\nu}$	$\longleftrightarrow$ $T(z)\ \bar{T}(\bar{z})$	

Conformal Trans.

Relation Between $\hat{H}$ & $\hat{D}$ $\quad T_C^{\mu\nu} \to$ Cylinder $\quad T_P^{\mu\nu} \to$ plane

$$\hat{D} = \int_C \underbrace{r T_{rr}}_{\downarrow} (r d\theta) \quad \longrightarrow \text{换到 Holomorphic Co.} \quad z = re^{i\theta} \quad \bar{z} = re^{-i\theta}$$

Dialation Conserved Charge

$$\hat{H} = \int_0^L T'_{tt}\, dx \quad \longrightarrow \text{换 Holomorphic Co.}$$

T' 与 T 之间有 Conformal Trans 联系 $\quad z = e^{\frac{2\pi}{L} w}$

$$\Rightarrow \quad \hat{D} = \hat{H} + \frac{\pi c}{6} \quad !!$$

Central Charge shift the 0-point of Energy!
($\hat{H}$ 为已知 Cylinder 上理论的 Hamiltonian)

Remark: Why we have to Specify "time Direction" before talking about Symmetry Algebra?

-> We talk about Algebra only on "Equal time Slice" * !!

Answer 2: What are the Symmetry Algebra

Elements of Symmetry Algebra: Generators

* Classically: $\delta \mathcal{O} = -i\varepsilon G_a \mathcal{O} = \mathcal{O}'(z) - \mathcal{O}(z)$

* Quantum: $\delta \langle \mathcal{O}(0) \rangle = \frac{\varepsilon}{2\pi} \oint_C \langle v(z) T(z) \mathcal{O}(0) \cdots \rangle dz$

C: 包含且仅包含 0

$$= i\varepsilon \underbrace{\operatorname*{Res}_{z\to 0} \langle v(z) T(z) \mathcal{O}(0) \rangle}_{\text{Serve as Generator}}$$

$\operatorname*{Res}_{z\to 0} v(z) T(z) \quad$ 展开 $v(z) = \sum_{n\in\mathbb{Z}} z^{n+1} v_n$

$$\Rightarrow \delta_v \langle \mathcal{O}(0) \rangle = \sum_{n\in\mathbb{Z}} i\varepsilon v_n \left\langle \frac{1}{2\pi i} \oint_0 dz\, z^{n+1} T(z) \mathcal{O}(0) \cdots \right\rangle$$

<u>Def</u> $L_n^{(0)} = \frac{1}{2\pi i}\oint_0 dz\, z^{n+1} T(z)$ (0点附近)

$$\Rightarrow \delta\langle \mathcal{O}(0)\cdots\rangle = \sum_{n\in\mathbb{Z}} i\underset{\sim}{\varepsilon_n} \left\langle \left(L_n^{(0)}\mathcal{O}(0)\right)\cdots\cdots \right\rangle$$

$$\delta\mathcal{O} = \sum_a i\omega_a G_a \mathcal{O}$$

] 类比!

$$\boxed{\text{Conclusion: Generator of Quantum Conformal Field theory is } L_n^{(z)}}$$

* Remark: How to understand $L_n^{(z)}$

this is an operator 作用于 z点的场上面

$$L_n^{(z)}: \mathcal{O}(z) \to \frac{1}{2\pi i}\oint_z T(z')\,\mathcal{O}(z)\,dz'$$

* 一般写 L_n 指 $L_n^{(0)}$ 即作用于 0点的 L_n !!

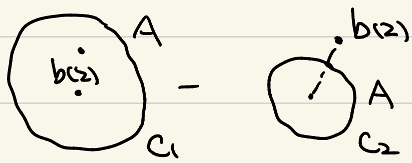
$$L_n(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\cdots) = \sum_i\left(\mathcal{O}_1\cdots\left(L_n^{(z_i)}\mathcal{O}_i\right)\cdots\right)$$

] <u>Def</u>

Relation of Symmetry Algebra: Commutators (in Radial Quantization)

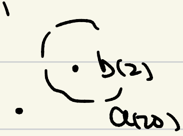
OPE: Help us calculate Commutators: (in Radial Quantization)

Take $A = \oint_C a(w)\,dw$ Then $[A, b(z)] =$



$$= \oint_{C_1} a(w)b(z)\,dw - \oint_{C_2} a(w)b(z)\,dw$$

=



$$= \oint_z a(w)b(z)\,dw$$

Take $B = \int_C b(z)\,dz$

$$\underline{[A, B]} = \oint_C [A, b(z)]\,dz = \oint_C dz \oint_z dw\, \underline{(a(w)b(z))}$$

Commutator (in Radial Quantization)

OPE 于 w点 !!

$[L_n, L_m]$ (认为作用0点の场)

OPE于 w

$$[L_n, L_m] = \oint_0 dz \oint_z dw \left(T(z)\, T(w) \right) z^{n+1} w^{n+1}$$

$$= \oint_0 dz \oint_z dw\, z^{n+1} w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \right]$$

$\cdots$

$$= (n-m) L_{m+n} + \frac{n(n^2-1)}{12} c\, \delta_{n+m,0}$$

→ Virasoro Algebra

$$\boxed{[L_n, L_m] = (n-m) L_{m+n} + \frac{n(n^2-1)}{12} c\, \delta_{n+m,0}}$$

→ $\hat{D} = \int_0^{2\pi} r T_{rr}\, r d\theta = $ 代入 $\cdots = L_0 + \bar{L}_0$

shift in 0-point

* L_0 act like Hamiltonian (of corse $\hat{D} = \hat{H} + \frac{\pi c}{6}$) on the z plane

* Spectrum of L_0 is the Energy Level

* $[L_0, L_n] = -n L_n \quad (n \neq 0)$: $\begin{cases} L_n \text{ is Annihilation Operator} \\ L_{-n} \text{ is Creation Operator} \end{cases}$

* Def : Primary State (Generalization of Vacuum)

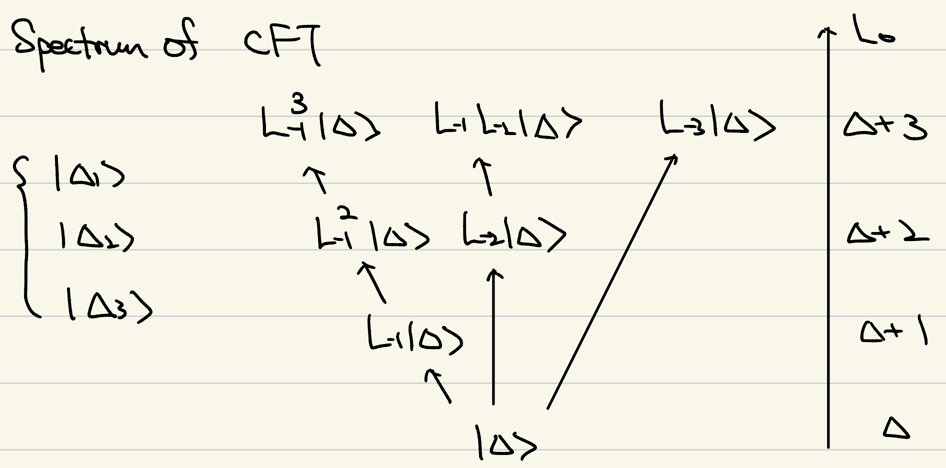
$\forall L_n\ n>0 \quad L_n|\Delta\rangle = 0$

其中 $L_0|\Delta\rangle = \Delta|\Delta\rangle$

then $|\Delta\rangle$ is a primary State

$\Delta=0$ 时为真空态!

* Spectrum of CFT



Conclusion :

1. Symmetry Algebra :

   Virasoro Algebra

$$[L_n, L_m] = (n-m) L_{m+n} + \frac{n(n^2-1)}{12} c\, \delta_{n+m,0}$$

2. Representation

$\lvert h\rangle$ Primary
$\swarrow \downarrow \searrow$
Descendants

$\Rightarrow$ 1 Highest Weight Rep : Verma Module !

△ What is A Quantum State ?

$\rightarrow$ 一种 Denote 能级的方式 (抽象的)

$\rightarrow$ Quantum State of Field Theory & Field (具体存在的"东西")

○ $H \underbrace{a_k^\dagger \lvert 0\rangle}_{\lvert\psi_k\rangle} = (E_0 + \omega_k) \underbrace{a_k^\dagger \lvert 0\rangle}_{\lvert\psi_k\rangle}$	○ $\hat{H} \lvert h\rangle = h \lvert h\rangle$
○ For Free Scalar Field : $a_k^\dagger = \frac{1}{\sqrt{2a\omega_k}} (a\omega_k \tilde{\varphi}_k^\dagger - i \tilde{\pi}_k^\dagger)$ 其中 φ_k π_k 为 φ_n 场形式的叠加!!!	○ CFT $\lvert h\rangle = \phi(0)\lvert 0\rangle$ $\phi(0)$ 即 Primary Field
○ Scalar Field State = Superposition of Many Fields	○ Primary State = P.. Field

STATE - OPERATOR CORRESPONDENCE

- What is the Actual Realization of "Primary State" & Verma Module

For a primary Field $\phi(z)$ (Ignore $\bar{z}$)

$$L_n^{(z)}\phi(z) = \frac{1}{2\pi i}\oint_z dw\, w^{n+1}\, \underline{T(w)\phi(z)} \qquad \text{Primary OPE}$$

$$= \frac{1}{2\pi i}\oint_z dw\, w^{n+1}\left[\frac{h\phi(z)}{(z-w)^2} + \frac{\partial}{z-w}\phi(z) + \cdots\right]$$

$$\begin{cases} * \ L_0^{(z)}\phi(z) = h\,\phi(z) \\ * \ L_n^{(z)}\phi(z) = 0 \quad \text{if } n \geq 1 \end{cases} \qquad \begin{cases} L_0|\Delta\rangle = \Delta|\Delta\rangle \\ L_n|\Delta\rangle = 0 \quad n \geq 1 \end{cases}$$

- Primary Field have Same Math Structure as Primary State!

- Every Primary Field gives us a Rep of the Vir Algebra

  Every Primary Field gives us a Existence of an Energy Level!!

THE WHOLE STRUCTURE OF CFT

- Primary Field $\quad V_\Delta(z) \sim |\Delta\rangle$
- Descendants $\quad L_{-n}^{(z)} V_\Delta(z) \sim L_{-n}|\Delta\rangle$

Abstract Notion but Should be understand as Left!

- $\mathcal{V}_\Delta : \{ V_\Delta(z), \{L_{-n}^{(z)}\} V_\Delta(z) \}$ or $\{ |\Delta\rangle, \{L_{-n}\}|\Delta\rangle \}$

* Is that Enough? X → Degenerate Fields !!
(States)

* What can we do with WI ?

→ 1 Restrict Correlation Function → CF

→ 2. Def: Quantum Generator ⇒ Symmetry Algebra Element

* What can we do with OPE? (and care only 发散)

→ 1 Calculate Commutator ⇒ Algebra Structure

Algebra

CORRELATION FUNCTION OF CFT

o Get Primary CF :

- Conformal WI → OPE → 神奇Relation (类似 Primary Conformal WI)

$$\underbrace{\left\langle T(z) V_{\Delta_1}(z_1) V_{\Delta_2}(z_2) \cdots \right\rangle}_{\text{Primary \& Descendant CF}} = \sum_{i=1}^{N} \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i)} \partial_i \right) \underbrace{\left\langle V_{\Delta_1}(z_1) \cdots \right\rangle}_{\text{Primary CF}}$$

- 双边进行 : $\oint_{\underbrace{\Omega}_{\text{包含 } z_1 \cdots z_n \text{ 区域}}} dz\, \varepsilon(z) \left\langle T(z) \prod_i V_{\Delta_i}(z_i) \right\rangle = \oint_{\Omega} dz\, \varepsilon(z) \left(\cdots \right)$

△ $\varepsilon(z) = z^2$ 时 (右边积分 Res = 0)

$$\Rightarrow \left\langle L_{-1} \prod_{i=1}^{N} V_{\Delta_i}(z_i) \right\rangle = 0$$