Lecture 11: Policy Optimization I

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Today's Plan

1. Introduction
2. Policy-based Reinforcement Learning
3. Monte-Carlo Policy Gradient

Value-based RL versus Policy-based RL

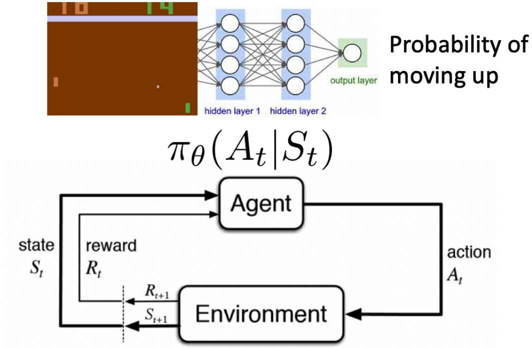
1. In previous lectures, we focused on value-based RL and had value function approximation with parameter θ

$$V_\theta(s) \approx v^\pi(s)$$
$$q_\theta(s, a) \approx q^\pi(s, a)$$

2. A policy is generated directly from the value function using ϵ-greedy or greedy $a_t = \arg\max_a Q(a, s_t)$
3. Instead, we can also parameterize the policy function as $\pi_\theta(a|s)$ where θ is the learnable policy parameter

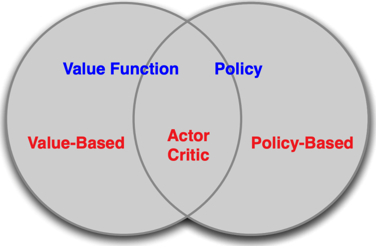
RL Diagram in the View of Policy Optimization

1. Action from the policy is all we need, then let's optimize the policy directly!



Value-based RL versus Policy-based RL

1. Value-based RL
   1. to learn value function
   2. implicit policy based on the value function
2. Policy-based RL
   1. no value function
   2. to learn policy directly
3. Actor-critic
   1. to learn both policy and value function



Advantages of Policy-based RL

1. Advantages:
    1. better convergence properties: we are guaranteed to converge on a local optimum (worst case) or global optimum (best case)
    2. Policy gradient is more effective in high-dimensional action space
    3. Policy gradient can learn stochastic policies, while value function can't
2. Disadvantages:
    1. typically converges to a local optimum
    2. evaluating a policy is inefficient and has high variance

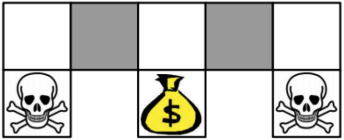
Two Types of Policies

1. Deterministic: given a state, the policy returns the action to take
2. Stochastic: given a state, the policy returns a probability distribution of the actions (e.g., 40% chance to turn left, 60% chance to turn right)

Example: Rock-Paper-Scissors

1. Two-player game
2. What is the best policy?
   1. A deterministic policy is easily beaten
   2. A uniform random policy is optimal (Nash equilibrium)

Example: Aliased Gridworld 1



1. The agent cannot differentiate the two grey states (look the same to the agent)
2. Considers the following features (for all N, E, S, W)

$$\psi(s, a) = [\mathbf{1}(\text{wall to N, a = move E}), \mathbf{1}(\text{wall to S, a = move W}), ...]$$

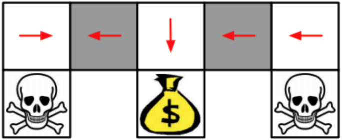
3. If it is value-based RL, value function approximation as:

$$Q_\theta(s, a) = f(\psi(s, a), \theta)$$

4. If it is policy-based RL, policy function approximation as:

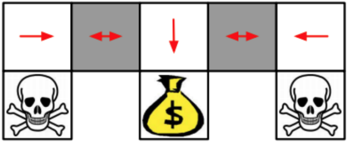
$$\pi_\theta(s, a) = g(\psi(s, a), \theta)$$

Example: Aliased Gridworld 2



1. Value-based RL learns a near-deterministic policy, e.g., greedy or ϵ-greedy
2. Because of the aliasing (the agent cannot differentiate two states), an optimal deterministic policy from value-based RL will either
   1. move W in both grey states (shown by red arrows)
   2. move E in both grey states
3. Either way, 50% chance will get stuck

Example: Aliased Gridworld 3



1. Policy-based RL can learn the optimal stochastic policy
2. An optimal stochastic policy will randomly move E or W in two grey states

$$\pi_\theta(\text{wall to N and W, move E}) = 0.5$$
$$\pi_\theta(\text{wall to N and W, move W}) = 0.5$$

3. For any starting point, it will reach the goal state in a few steps with high probability

Objective of Optimizing Policy

1. Objective: Given a policy approximator $\pi_\theta(s, a)$ with parameter θ, find the best θ
2. How do we measure the quality of a policy π_θ?
3. In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$$

4. In continuing environments
   1. we can use the average value

   $$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

   2. or the average reward per time-step

   $$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) R(s, a)$$

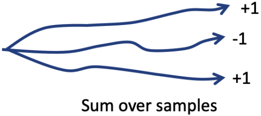
   where d^{π_θ} is stationary distribution of Markov chain for π_θ

Objective of Optimizing Policy

1. The value of the policy is defined as

$$
\begin{aligned}
J(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta}[\sum_t r(s_t, a_t^\tau)] \\
&\approx \frac{1}{m} \sum_m \sum_t r(s_{t,m}, a_{t,m})
\end{aligned}
$$

   It is the same as the value function we defined in the value-based RL



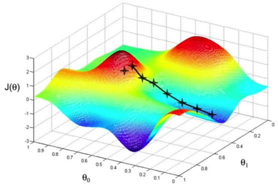
   1. τ is a trajectory sampled from the policy function π_θ
2. The goal of policy-based RL

$$\theta^* = \arg\max_\theta \mathbb{E}_{\tau \sim \pi_\theta}[\sum_t r(s_t, a_t^\tau)]$$

Objective of Optimizing Policy

1. Policy-based RL is an optimization problem that find θ that maximizes $J(\theta)$
2. If $J(\theta)$ is differentiable, we can use gradient-based methods:
   1. gradient ascend
   2. conjugate gradient
   3. quasi-newton
3. If $J(\theta)$ is non-differentiable or hard to compute the derivative, some derivative-free black-box optimization methods:
   1. Cross-entropy method (CEM)
   2. Hill climbing
   3. Evolution algorithm

Policy Optimization using Derivative



1. Consider a function $J(\theta)$ to be any policy objective function
2. Goal is to find parameter θ^* that maximizes $J(\theta)$ by ascending the gradient of the policy,w.r.t parameter θ

$$\Delta\theta = \alpha \nabla_\theta J(\theta)$$

3. Adjust θ in the direction of the gradient, where α is step-size
4. Define the gradient of $J(\mathbf{w})$ to be

$$\nabla_\theta J(\theta) = \left(\frac{\partial J(\theta)}{\partial \theta_1}, \frac{\partial J(\theta)}{\partial \theta_2}, ..., \frac{\partial J(\theta)}{\partial \theta_n}\right)^T$$

Policy Optimization using Derivative-free Methods

1. Sometimes we cannot compute the derivative, i.e., $\nabla_\theta J(\theta)$
2. Derivative free methods:
    1. Cross Entropy Method (CEM)
    2. Finite Difference

Derivative-free Method: Cross-Entropy Method

1. $\theta^* = \arg\max J(\theta)$
2. Treat $J(\theta)$ as a black box score function (not differentiable)

Algorithm 1 CEM for black-box function optimization

for iter $i = 1$ to N **do**
    $\mathcal{C} = \{\}$
    for parameter set $e = 1$ to N **do**
        sample $\theta^{(e)} \sim P_{\mu^{(i)}}(\theta)$
        execute roll-outs under $\theta^{(e)}$ to evaluate $J(\theta^{(e)})$
        store $(\theta^e, J(\theta^{(e)}))$ in $\mathcal{C}$
    end for
    $\mu^{(i+1)} = \arg\max_\mu \sum_{k \in \hat{\mathcal{C}}} \log P_\mu(\theta^{(k)})$
    where $\hat{\mathcal{C}}$ are the top 10% of $\mathcal{C}$ ranked by $J(\theta^{(e)})$
end for

3. Example of CEM for a simple RL problem: `https://github.com/metalbubble/RLexample/blob/master/my_learning_agent.py`

Approximate Gradients by Finite Difference

1. To evaluate policy gradient of $\pi_\theta(s, a)$
2. For each dimension $k \in [1, n]$
   1. estimate *kth* partial derivative of objective function by perturbing θ by a small amount ϵ in *kth* dimension

      $$\frac{\partial J(\theta)}{\partial \theta_k} \approx \frac{J(\theta + \epsilon u_k) - J(\theta)}{\epsilon}$$

      where u_k is unit vector with 1 in *kth* component, 0 else where
3. uses n evaluations to compute policy graident in total n dimensions
4. though noisy and inefficient, but works for arbitrary policies, even if policy is not differentiable.

Computing the Policy Gradient Analytically

1. Assume policy π_θ is differentiable whenever it is no-zero
2. and we can compute the gradient $\nabla_\theta \pi_\theta(s, a)$
3. Likelihood ratios exploit the following tricks

$$\begin{aligned}\nabla_\theta \pi_\theta(s, a) &= \pi_\theta(s, a) \frac{\nabla_\theta \pi_\theta(s, a)}{\pi_\theta(s, a)} \\ &= \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)\end{aligned}$$

4. The score function is $\nabla_\theta \log \pi_\theta(s, a)$

Policy Gradient for One-Step MDPs

1. Consider a simple class of one-step MDPs
   1. Starting in state $s \sim d(s)$
   2. Terminating after one time-step with reward $r = R(s, a)$
2. Use likelihood ratios to compute the policy gradient

$$
\begin{aligned}
J(\theta) =& \mathbb{E}_{\pi_\theta}[r] \\
=& \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s, a) r
\end{aligned}
$$

3. The gradient is as

$$
\begin{aligned}
\nabla_\theta J(\theta) =& \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a) r \\
=& \mathbb{E}_{\pi_\theta}[r \nabla_\theta \log \pi_\theta(s, a)]
\end{aligned}
$$

Policy Example: Softmax Policy

1. Simple policy model: weight actions using linear combination of features $\phi(s, a)^T\theta$
2. Probability of action is proportional to the exponetiated weight

$$\pi_\theta(s, a) = \frac{\exp^{\phi(s,a)^T\theta}}{\sum_{a'} \exp^{\phi(s,a')^T\theta}}$$

3. The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \phi(s, a) - \mathbb{E}_{\pi_\theta}[\phi(s, .)]$$

Policy Example: Gaussian Policy

1. In continuous action spaces, a Gaussian policy can be naturally defined
2. Mean is a linear combination of state features $\mu(s) = \phi(s)^T\theta$
3. Variance may be fixed σ^2 or can also be parameterized
4. Policy is Gaussian, the continuous $a \sim \mathcal{N}(\mu(s), \sigma^2)$
5. The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \frac{(a - \mu(s))\phi(s)}{\sigma^2}$$

Policy Gradient for Multi-step MDPs

1. Denote a state-action trajectory from one episode as $\tau = (s_0, a_0, r_1, ...s_{T-1}, a_{T-1}, r_T, s_T) \sim (\pi_\theta, P(s_{t+1}|s_t, a_t))$
2. Use $R(\tau) = \sum_{t=0}^{T} R(s_t, a_t)$ to be the sum of rewards for a trajectory τ
3. The policy objective is

$$J(\theta) = \mathbb{E}_{\pi_\theta}\Big[\sum_{t=0}^{T} R(s_t, a_t)\Big] = \sum_{\tau} P(\tau;\theta)R(\tau)$$

   where $P(\tau;\theta)$ denotes the probability over trajectories when executing the policy π_θ
4. Then our goal is to find the policy parameter θ

$$\theta^* = \arg\max_{\theta} J(\theta) = \arg\max_{\theta} \sum_{\tau} P(\tau;\theta)R(\tau)$$

Policy Gradient for Multi-step MDPs

1. Our goal is to find the policy parameter θ

$$\theta^* = \arg\max_\theta J(\theta) = \arg\max_\theta \sum_\tau P(\tau;\theta)R(\tau)$$

2. Take the gradient with respect to θ:

$$\begin{aligned}
\nabla_\theta J(\theta) =& \nabla_\theta \sum_\tau P(\tau;\theta)R(\tau) \\
=& \sum_\tau \nabla_\theta P(\tau;\theta)R(\tau) \\
=& \sum_\tau \frac{P(\tau;\theta)}{P(\tau;\theta)} \nabla_\theta P(\tau;\theta)R(\tau) \\
=& \sum_\tau P(\tau;\theta)R(\tau)\frac{\nabla_\theta P(\tau;\theta)}{P(\tau;\theta)} \\
=& \sum_\tau P(\tau;\theta)R(\tau)\nabla_\theta \log P(\tau;\theta)
\end{aligned}$$

Policy Gradient for Multi-step MDPs

1. Our goal is to find the policy parameter θ

$$\theta^* = \arg\max_\theta J(\theta) = \arg\max_\theta \sum_\tau P(\tau;\theta)R(\tau)$$

2. Take the gradient with respect to θ:

$$\nabla_\theta J(\theta) = \sum_\tau P(\tau;\theta)R(\tau)\nabla_\theta \log P(\tau;\theta)$$

3. Approximate with empirical estimate for m sample paths under policy π_θ:

$$\nabla_\theta J(\theta) \approx \frac{1}{m}\sum_{i=1}^{m} R(\tau_i)\nabla_\theta \log P(\tau_i;\theta)$$

Decomposing the Trajectories into States and Actions

1. Approximate with empirical estimate for m sample paths under policy π_θ:

$$\nabla_\theta J(\theta) \approx \frac{1}{m}\sum_{i=1}^{m} R(\tau_i)\nabla_\theta \log P(\tau_i;\theta)$$

2. Decompose $\nabla_\theta \log P(\tau;\theta)$

$$\begin{aligned}
\nabla_\theta \log P(\tau;\theta) =& \nabla_\theta \log \left[\mu(s_0)\prod_{t=0}^{T-1}\pi_\theta(a_t|s_t)p(s_{t+1}|s_t,a_t)\right] \\
=& \nabla_\theta \Big[\log \mu(s_0) + \sum_{t=0}^{T-1}\log \pi_\theta(a_t|s_t) + \log p(s_{t+1}|s_t,a_t)\Big] \\
=& \sum_{t=0}^{T-1}\nabla_\theta \log \pi_\theta(a_t|s_t)
\end{aligned}$$

Likelihood Ratio Policy Gradient

1. Our goal is to find the policy parameter θ

$$\theta^* = \arg\max_\theta V(\theta) = \arg\max_\theta \sum_\tau P(\tau;\theta)R(\tau)$$

2. Approximate with empirical estimate for m sample paths under policy π_θ:

$$\nabla_\theta V(\theta) \approx \frac{1}{m}\sum_{i=1}^{m} R(\tau_i)\nabla_\theta \log P(\tau_i;\theta)$$

3. And we have $\nabla_\theta \log P(\tau_i;\theta) = \sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t|s_t)$

$$\nabla_\theta J(\theta) \approx \frac{1}{m}\sum_{i=1}^{m} R(\tau_i) \sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t^i|s_t^i)$$

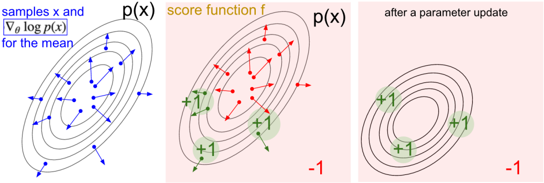
   1. It do not need to know the dynamics model!

Understanding Score Function Gradient Estimator

1. Consider the generic form of $E_{\tau\sim\pi_\theta}[R(\tau)]$ as

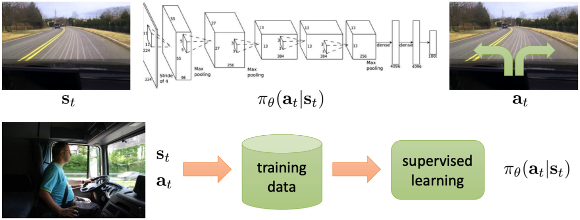
$$\nabla_\theta \mathbb{E}_{x\sim p(x;\theta)}[f(x)] = \mathbb{E}_x[f(x)\nabla_\theta \log p(x;\theta)]$$

2. The above gradient can be understood as:
   1. Shift the distribution p through its parameter θ to let its future samples x achieve higher scores as judged by $f(x)$
   2. The direction of $f(x)\nabla_\theta \log p(x;\theta)$ pushes up the log probability of the sample, in proportion to how good it is



Comparison to Maximum Likelihood

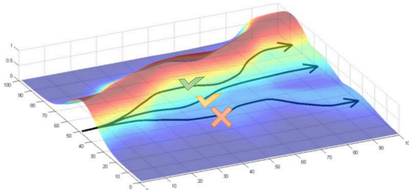
1. Policy gradient estimator: $\nabla_\theta J(\theta) \approx \frac{1}{M}\sum_{m=1}^{M}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(a_{t,m}|s_{t,m})\right)\left(\sum_{t=1}^{T} r(s_{t,m}, a_{t,m})\right)$
2. Maximum likelihood estimator: $\nabla_\theta J_{ML}(\theta) \approx \frac{1}{M}\sum_{m=1}^{M}\left(\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(a_{t,m}|s_{t,m})\right)$
3. Interpretation: good action is made more likely, bad action is made less likely



Comparison to Maximum Likelihood

$$\nabla_\theta J(\theta) \approx \frac{1}{m} \sum_{i=1}^{m} R(\tau_i) \nabla_\theta \log P(\tau_i; \theta)$$

1. If going up the hill leads to higher reward, change the policy parameters to increase the likelihood of trajectories that move higher



Large Variance of Policy Gradient

1. We have the following approximate update

$$\nabla_\theta J(\theta) \approx \frac{1}{m} \sum_{i=1}^{m} R(\tau_i) \sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i)$$

2. Unbiased but very noisy
3. Two fixes:
    1. Use temporal causality
    2. Include a baseline

Reduce Variance of Policy Gradient using Causality

1. Previously $\nabla_\theta \mathbb{E}_\tau[R] = \mathbb{E}_\tau\Big[\Big(\sum_{t=0}^{T-1} r_t\Big)\Big(\sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t|s_t)\Big)\Big]$
2. We can derive the gradient estimator for a single reward term $r_{t'}$ as

$$\nabla_\theta \mathbb{E}_\tau[r_{t'}] = \mathbb{E}_\tau\Big[r_{t'} \sum_{t=0}^{t'} \nabla_\theta \log \pi_\theta(a_t|s_t)\Big]$$

3. Summing this formula over t, we obtain

$$\begin{aligned}
\nabla_\theta J(\theta) = \nabla_\theta \mathbb{E}_{\tau\sim\pi_\theta}[R] &= \mathbb{E}_\tau\Big[\sum_{t'=0}^{T-1} r_{t'} \sum_{t=0}^{t'} \nabla_\theta \log \pi_\theta(a_t|s_t)\Big] \\
&= \mathbb{E}_\tau\Big[\sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t|s_t) \sum_{t'=t}^{T-1} r_{t'}\Big] \\
&= \mathbb{E}_\tau\Big[\sum_{t=0}^{T-1} G_t \cdot \nabla_\theta \log \pi_\theta(a_t|s_t)\Big]
\end{aligned}$$

Reduce Variance of Policy Gradient using Causality

1. Therefore we have

$$\nabla_\theta \mathbb{E}_{\tau \sim \pi_\theta}[R] = \mathbb{E}_\tau \Big[\sum_{t=0}^{T-1} G_t \cdot \nabla_\theta \log \pi_\theta(a_t | s_t) \Big]$$

2. $G_t = \sum_{t'=t}^{T-1} r_{t'}$ is the return for a trajectory at step t
3. Causality: policy at time t' cannot affect reward at time t when $t < t'$
4. Then we can have the following estimated update

$$\nabla_\theta \mathbb{E}[R] \approx \frac{1}{m} \sum_{i=1}^{m} \sum_{t=0}^{T-1} G_t^{(i)} \cdot \nabla_\theta \log \pi_\theta(a_t^i | s_t^i)$$

REINFORCE: A Monte-Carlo policy gradient algorithm

1. The algorithm simply samples multiple trajectories following the policy π_θ while updating θ using the estimated gradient

REINFORCE, A Monte-Carlo Policy-Gradient Method (episodic)

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$
Repeat forever:
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$
  For each step of the episode $t = 0, \ldots, T-1$:
    $G \leftarrow$ return from step t
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \nabla_{\boldsymbol{\theta}} \ln \pi(A_t|S_t, \boldsymbol{\theta})$

2. Classic paper: Williams (1992). Simple statistical gradient-following algorithms for connectionist reinforcement learning: introduces REINFORCE algorithm

Reducing Variance Using a Baseline

1. The original update

$$\nabla_\theta \mathbb{E}_{\tau\sim\pi_\theta}[R] = \mathbb{E}_\tau\left[\sum_{t=0}^{T-1} G_t \cdot \nabla_\theta \log \pi_\theta(a_t|s_t)\right]$$

   1. $G_t = \sum_{t'=t}^{T-1} r_{t'}$ is the return for a trajectory which might have high variance

2. We subtract a baseline $b(s)$ from the policy gradient to reduce variance

$$\nabla_\theta \mathbb{E}_{\tau\sim\pi_\theta}[R] = \mathbb{E}_\tau\left[\sum_{t=0}^{T-1} \left(G_t - b(s_t)\right) \cdot \nabla_\theta \log \pi_\theta(a_t|s_t)\right]$$

3. A good baseline is the expected return

$$b(s_t) = \mathbb{E}[r_t + r_{t+1} + \ldots + r_{T-1}]$$

Reducing Variance Using a Baseline

$$\nabla_\theta \mathbb{E}_{\tau\sim\pi_\theta}[R] = \mathbb{E}_\tau\Big[\sum_{t=0}^{T-1} \big(G_t - b(s_t)\big) \cdot \nabla_\theta \log \pi_\theta(a_t|s_t)\Big]$$

1. Interpretation: increase the logprob of action a_t proportional to how much returns G_t are better than the expected return
2. We can **prove** that baseline $b(s)$ can reduce variance, without changing the expectation:

$$\mathbb{E}_\tau\Big[\nabla_\theta \log \pi_\theta(a_t|s_t) b(s_t)\Big] = 0, \tag{1}$$

$$E_\tau\Big[\nabla_\theta \log \pi_\theta(a_t|s_t)(G_t - b(s_t))\Big] = E_\tau\Big[\nabla_\theta \log \pi_\theta(a_t|s_t) G_t\Big] \tag{2}$$

$$Var_\tau\Big[\nabla_\theta \log \pi_\theta(a_t|s_t)(G_t - b(s_t))\Big] < Var_\tau\Big[\nabla_\theta \log \pi_\theta(a_t|s_t) G_t\Big] \tag{3}$$

3. Thus subtracting a baseline is unbiased in expectation but reduces variance

Vanilla Policy Gradient Algorithm with Baseline

procedure Policy Gradient(α)

Initialize policy parameters θ and baseline values $b(s)$ for all s, e.g. to 0

for iteration $= 1, 2, \dots$ **do**

Collect a set of m trajectories by executing the current policy π_θ

for each time step t of each trajectory $\tau^{(i)}$ **do**

Compute the *return* $G_t^{(i)} = \sum_{t'=t}^{T-1} r_{t'}$

Compute the *advantage estimate* $\hat{A}_t^{(i)} = G_t^{(i)} - b(s_t)$

Re-fit the baseline to the empirical returns by updating $\mathbf{w}$ to minimize

$$\sum_{i=1}^{m} \sum_{t=0}^{T-1} \|b(s_t) - G_t^{(i)}\|^2$$

Update policy parameters θ using the policy gradient estimate $\hat{g}$

$$\hat{g} = \sum_{i=1}^{m} \sum_{t=0}^{T-1} \hat{A}_t^{(i)} \nabla_\theta \log \pi_\theta(a_t^{(i)} | s_t^{(i)})$$

with an optimizer like SGD ($\theta \leftarrow \theta + \alpha \cdot \hat{g}$) or Adam

return θ and baseline values $b(s)$

Practical Implementation of the Algorithm

1. In practice we usually do not compute the gradients $\sum_t \hat{A}_t \nabla_\theta \log \pi_\theta(a_t|s_t)$ individually
2. Instead, we accumulate data from current batch as

$$L(\theta) = \sum_t \hat{A}_t \log \pi_\theta(a_t|s_t;\theta)$$

3. Then the policy gradient estimator $\hat{g} = \nabla_\theta L(\theta)$
4. We also could have a joint loss with value function approximation as

$$L(\theta, \mathbf{w}) = \sum_t \left(\hat{A}_t \log \pi_\theta(a_t|s_t;\theta) - ||b(s_t) - \hat{R}_t||^2 \right)$$

5. Then solve this using auto diff

Concluding Remark

1. Derive the policy gradient by yourself to get a deeper understanding!!!
2. Next lecture will be on actor-critic policy gradient and some variants