Constructing Three-Way Decision With Fuzzy Granular-Ball Rough Sets Based on Uncertainty Invariance

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***Abstract*—Granular-ball computing (GBC) proposed by Xia adaptively generates a different neighborhood for each object, resulting in greater generality and flexibility. Moreover, GBC greatly improves the efficiency by replacing point input with granular-ball. However, the current GB-based classifiers rigidly assign a specific class label to each data instance and lacks of the necessary strategies to address uncertain instances. These far-fetched certain classification approachs toward uncertain instances may suffer considerable risks. In this article, we introduce three-way decision (3WD) into GBC to construct a novel three-way decision with fuzzy granular-ball rough sets (3WD-FGBRS) from the perspective of uncertainty. This helps to construct reasonable multigranularity spaces for handling complex decision problems with uncertainty. First, 3WD-FGBRS is constructed in a data-driven method based on fuzziness, which avoids the subjective definition of certain risk parameters when calculating the threshold pairs. Based on 3WD-FGBRS, we further propose a sequential three-way decision with fuzzy granular-ball rough sets (S3WD-FGBRS) and analyze the fuzziness loss of multilevel decision result in S3WD-FGBRS. Then, the optimal granular-ball space selection mechanism of S3WD-FGBRS is introduced by combining fuzziness and granular-ball space distance. Finally, extensive comparative experiments are conducted with 3 state-of-the-art GB-based classifiers and 3 classical machine learning classifiers on 12 public benchmark datasets. The results show that our models almost outperform other comparison methods in terms of effectiveness, efficiency and robustness.**

***Index Terms*—Fuzziness, fuzzy granular-ball rough sets (FGBRS), optimal granular-ball space selection, three-way decision (3WD).**

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I. Introduction

GRANULAR computing (GrC) [1], [2], [3], [4] provides solutions to simulate the human cognitive thinking to solve complex problems. From diverse perspectives, rough sets [5], fuzzy sets [6], and quotient spaces [7] are three main models of GrC. Rough sets are widely used to measure the uncertainty and incompleteness of information systems. To process continuous data, neighborhood rough sets (NRS) [8], [9], [10] utilize the neighborhood granulation to convert the equivalence relation into the covering relation in neighborhood space. In addition, numerous extended GrC-based classifiers [11], [12], [13], [14], [15] are designed to employ the information granule as the fundamental unit of computation, thereby greatly enhancing the efficiency of knowledge discovery. Nevertheless, most of these classifiers regard granules as a preliminary feature procession method, without altering the fundamental mathematical model or elevating the core performance of the classifiers themselves.

Based on the idea of GrC, granular-ball computing (GBC) [16], [17], [18], [19] proposed by Xia is a novel method to process data and represent knowledge by replacing traditional information granule input with granular-ball (GB), which follows the rule of "global topology precedence" [20]. After several years of development, GBC is constantly being improved in terms of methods and applications. Chen et al. [21] introduced a GB-based attribute selector, resulting in better classification performance based on the obtained reduction. Xie et al. [22] proposed an efficient minimum spanning tree clustering based on GBC. Furthermore, Xie et al. [23] presented an efficient fuzzy clustering using GB, which improves the feasibility of fuzzy clustering. Cheng et al. [24] proposed a novel GB-based density peaks clustering to obtain much shorter running time without any parameterization. Recently, granular-ball rough sets (GBRS), as a novel rough sets model, is proposed by Xia et al. [19]. Compared with traditional NRS methods, GBRS is a multigranularity learning tool with greater robustness and efficiency by replacing neighborhood granule with GB [25]. The object concepts in GBRS are usually precise and crisp, yet in numerous real-world applications, the object concept may be ambiguous or uncertainty. To solve this problem, this article proposes the concept of fuzzy granular-ball rough sets (FGBRS). Moreover, current works of GBC focus on the traditional two-way classification, which lack strategies to handle instances with uncertainty and may result in misclassification on uncertain cases.

As is well known, three-way decision (3WD) [26], [27], [28] theory proposed by Yao is an emerging approach to address the complex problem with uncertainty. The basic principle of 3WD refers to dividing a universe into three distinct regions and each region corresponds to a decision action. As a generalization of the traditional two-way decision model, 3WD further incorporates a third option, which provides a trisecting-and-acting way for decision-making. Currently, 3WD has been widely applied in different fields. Yao [29] further proposed the sequential three-way decision (S3WD) by introducing the idea of GrC. In essence, S3WD is a progressive computing method with granularity being finer. This means that the same problem can be handled in a multigranularity spaces. Recently, by integrating related theories, i.e., formal concept analysis [30], rough sets [31], and cognitive computing [32], considerable achievements on S3WD are developed and widely applied in a variety of domains, including classification [33], group decision making [34], [35], stock prediction [36], face recognition [37], clustering analysis [38], etc. In terms of the advantage of 3WD [39], [40], we introduce 3WD into GBC to construct the three-way decision with fuzzy granular-ball rough sets (3WD-FGBRS). The delayed decision action reduces decision-making risk to a greater extent by considering the cost or uncertainty of problem-solving. This contributes to establish the reasonable multigranularity structures to tackle the complex problem that involves ambiguous and limited information. The majority of works in 3WD concentrate on computing thresholds by leveraging the provided risk parameters from the viewpoint of misclassification costs. However, in real applications, it is difficult to accurately obtain the risk parameters based on the expert experience. To address the above issue, it is beneficial to introduce fuzziness into 3WD-FGBRS, which provides an objective method to calculate the threshold pairs from perspective of uncertainty. The major contributions of this article are summed up as follows.

1) Three-way decision with fuzzy granular-ball rough sets (3WD-FGBRS) is constructed in a data-driven method based on fuzziness invariance.
2) Sequential three-way decision with fuzzy granular-ball rough sets (S3WD-FGBRS) is further proposed and its fuzziness loss of multilevel decision result is analyzed.
3) The optimal granular-ball space selection mechanism of S3WD-FGBRS is presented by combining fuzziness and granular-ball space distance.

The rest of this article is organized as follows. Section IV-B is a review of related preliminary definitions. In Section V-B, a 3WD-FGBRS is constructed by maintaining the fuzziness invariance. In Section IV, based on 3WD-FGBRS, S3WD-FGBRS is further proposed. Then, the optimal granular-ball space selection mechanism of S3WD-FGBRS is presented. The relevant experiments for the verification of the viability and rationality of our models are shown in Section V. Finally, Section VI concludes this article.

II. Preliminaries

In this section, to facilitate the framework of this article, we review some necessary definitions related to 3WD and GBRS. Let $S = (U, AT \cup D, V, f)$ denote a neighborhood decision system, where $U = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite set of objects; AT and D are the nonempty finite sets of conditional attributes and decision attributes, respectively; V is the set of values of attributes; $f : U \times AT \to V$ denotes a mapping function.

Definition 1: (Indiscernible granular-ball relation) [17] Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, $B \subseteq AT$ and gb denotes a granular-ball. $\forall x_1, x_2 \in U$, the indiscernible granular-ball relation $\mathrm{INDGB}(B)$ of the attribute subset B is defined as

$$\begin{aligned}&\mathrm{INDGB}(B)\\&= \{(x_1, x_2) \in U^2 | \quad f(x_1, a) = f(x_2, a) = gb, \forall a \in B\}.\end{aligned} \tag{1}$$

If $B \subseteq AT$, $U/GB(B) = \{gb_1, gb_2, \ldots, gb_m\}$ denote a granular-ball space induced by $INDGB(B)$.

Definition 2 (See [16]): Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B \subseteq AT$. $U/GB(B) = \{gb_1, gb_2, \ldots, gb_m\}$ is a granular-ball space induced by $INDGB(B)$. $\forall gb \in U/GB(B)$, c and r represent the center and radius of gb, respectively, which are defined as follows:

$$c = \frac{1}{|gb|} \sum_{x \in gb} x \tag{2}$$

$$r = \frac{1}{|gb|} \sum_{x \in gb} \|x - c\|. \tag{3}$$

Here, c is the center of gravity calculated as the average of all objects in gb, and r is the average distance from each point in gb to c.

Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, $B_1, B_2 \subseteq AT$. $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$ and $U/GB(B_2) = \{gb_1^2, gb_2^2, \ldots, gb_l^2\}$ denote two granular-ball spaces induced by $INDGB(B_1)$ and $INDGB(B_2)$, respectively. If $\forall_{gb_k^2 \in U/GB(B_2)} (\exists_{gb_i^1 \in U/GB(B_1)} (gb_k^2 \subseteq gb_i^1))$ then $U/GB(B_1)$ is finer than $U/GB(B_2)$, denoted as $U/GB(B_2) \preceq U/GB(B_1)$. If $\forall_{gb_k^2 \in U/GB(B_2)} (\exists_{gb_i^1 \in U/GB(B_1)} (gb_k^2 \subset gb_i^1))$, then $U/GB(B_1)$ is strictly finer than $U/GB(B_2)$, denoted as $U/GB(B_2) \prec U/GB(B_1)$.

Definition 3. (Granular-ball rough sets) [16]: Let $S = (U, AT \cup D, V, f)$ be a decision system, $B_1 \subseteq AT$. X denotes a target subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$ is a granular-ball space on U. The definition of the upper and lower approximation sets of X with respect to B_1 are, respectively, represented by

$$\begin{aligned}\overline{GBR_{B_1}}(X) &= \bigcup \{gb_i^1 \in U/GB(B_1) \,|\, gb_i^1 \cap X \neq \emptyset\}\\ \underline{GBR_{B_1}}(X) &= \bigcup \{gb_i^1 \in U/GB(B_1) \,|\, gb_i^1 \subseteq X\}.\end{aligned} \tag{4}$$

In numerous real decision-making applications, the states of the target concept often exhibit uncertainty and fuzziness. For instance, when evaluating the pollution level of a river, the results may not cleanly fit into two mutually exclusive categories of pollution-free or polluted. Instead, they may occupy a continuum of states, reflecting varying degrees of contamination. To address this problem, the rough fuzzy sets (RFS) [41] is proposed. In this article, based on the related concepts of rough fuzzy sets, we define the step GB-based fuzzy set as follows:

Definition 4: (Step GB-based fuzzy set) Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. X denotes a target fuzzy subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$ is a granular-ball space on U, where $gb_i^1 = \{x_{i1}, x_{i2}, \ldots, x_{i|gb_i^1|}\}$ $(i = 1, 2, \ldots, m)$. If $\mu_X(x_{i1}) = \mu_X(x_{i2}) = \ldots = \mu_X(x_{i|gb_i^1|}) = \varepsilon_i$ $(0 \leq \varepsilon_i \leq 1$, $i = 1, 2, \ldots, m)$, then X is a step GB-based fuzzy set.

Based on Definition 4, we proposed the definition of GB-based average fuzzy set as follows:

Definition 5. (GB-based average fuzzy set): Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. X denotes a target fuzzy subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$ is a granular-ball space induced by $INDGB(B_1)$, where $gb_i^1 = \{x_{i1}, x_{i2}, \ldots, x_{i|gb_i^1|}\}$ $(i = 1, 2, \ldots, m)$. If $\overline{\mu}(gb_i^1) = \frac{\sum_{x \in gb_i^1} \mu(x)}{|gb_i^1|}$ is called the average membership degree of gb_i^1, we refer to $X_{GB_1}^J$ as the average GB-based fuzzy set of X.

In Definition 5, $\overline{\mu}(gb_i^1)$ can be understood as the probability that gb_i^1 belongs to the target concept X, which is more general than rough membership degree [42], [43].

According to the formula of average fuzziness $F_X = \frac{4}{|U|} \sum_{x \in U} \mu(x)(1 - \mu(x))$ [42], where X denotes a target subset on U, we define the GB-based average fuzziness as follows:

Definition 6. (GB-based average fuzziness): Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. X denotes a target fuzzy subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$ is a granular-ball space induced by $INDGB(B_1)$. The GB-based average fuzziness is defined as follows:

$$F_{X_{GB_1}^J} = \frac{4}{|U|} \sum_{i=1}^{m} h(gb_i^1) \tag{5}$$

where $h(gb_i^1) = \overline{\mu}(gb_i^1)(1 - \overline{\mu}(gb_i^1))$.

The uncertainty of granular-ball rough sets comes from three regions: positive region, negative region, and boundary region as follows:

$$\begin{aligned} F_{X_{GB_1}^J} &= \frac{4}{|U|} \sum_{gb \in \mathrm{POS}(X_{GB_1}^J)} \overline{\mu}(gb)(1 - \overline{\mu}(gb)) \\ &\quad + \frac{4}{|U|} \sum_{gb \in \mathrm{BND}(X_{GB_1}^J)} \overline{\mu}(gb)(1 - \overline{\mu}(gb)) \\ &\quad + \frac{4}{|U|} \sum_{gb \in \mathrm{NEG}(X_{GB_1}^J)} \overline{\mu}(gb)(1 - \overline{\mu}(gb)) \\ &= F(\mathrm{POS}(X_{GB_1}^J)) + F(\mathrm{BND}(X_{GB_1}^J)) + F(\mathrm{NEG}(X_{GB_1}^J)). \end{aligned}$$

Example 1: Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. $X = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5} + \frac{1}{x_6} + \frac{0.8}{x_7} + \frac{0.6}{x_8} + \frac{0.5}{x_9} + \frac{0.3}{x_{10}}$ denotes a target fuzzy subset on U, and $U/GB(B_1) = \{gb_1^1, gb_2^1, gb_3^1\} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6, x_7, x_8\}, \{x_9, x_{10}\}\}$. According to the condition, we have $F_X = 0.708$, and $F_{X_{GB_1}^J} = \frac{4}{10}(0.9 + 0.64 + 0.16) = 0.68$.

III. Three-Way Decision With Fuzzy Granular-Ball Rough Sets Based on Fuzziness Loss

Algorithm S1 [24], as outlined in Section S1 of the Supplementary file, meticulously illustrates the detailed process of generating GBs, which utilizes the distributional characteristics. It further divide a GB by the quantity of objects within it, when the count of objects is sufficiently small, it ensures that the majority of enclosed objects belong to a uniform class. Based on Definiton 4 and Definition 5, we proposed the concept of three-way decision with fuzzy granular-ball rough sets (3WD-FGBRS) as follows:

Definition 7. (3WD-FGBRS): Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. X denotes a target fuzzy subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$, and α, β denotes a pair of thresholds. The three decision regions are represented by

$$\begin{aligned} \mathrm{POS}(X_{GB_1}^J) &= \{x \in U \mid \bar{\mu}(gb_i^1) \geq \alpha\}, \\ \mathrm{BND}(X_{GB_1}^J) &= \{x \in U \mid \beta < \bar{\mu}(gb_i^1) < \alpha\}, \\ \mathrm{NEG}(X_{GB_1}^J) &= \{x \in U \mid \bar{\mu}(gb_i^1) \leq \beta\} \end{aligned} \tag{6}$$

where $\bar{\mu}(gb_i^1)$ denotes the average membership degree of gb_i^1, $i = 1, 2, \ldots, m$.

Within the general 3WD framework [26], the current 3WD can be encompassed in three key aspects: minimizing distance, minimizing cost, and preserving uncertainty invariance. Fuzziness, as an uncertainty measure, is able to describe the uncertain information of granular-ball space. Furthermore, fuzziness loss is objectively to describe the variations of uncertain information. For example, the change from fuzzy sets to GB-based average fuzzy sets may generate the fuzziness loss. Therefore, we introduce fuzziness into 3WD-FGBRS, which offers a novel perspective and approach to enhance its capabilities. In this section, we establish 3WD-FGBRS on the basis of fuzziness loss by constructing the shadowed map and present the pertinent theoretical frameworks. This is easy to obtain clearer decision rules and represent uncertain information, which avoids the need for a priori expert knowledge. More specifically, the granular-balls with average membership degree below α and above β will be assigned to shadowed areas. That is, the range of average membership degree is extended to an uncertain area $[\beta, \alpha]$, which is defined as follows:

Definition 8. (Shadowed map): Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1 \subseteq AT$. X denotes a target fuzzy subset on U, $U/GB(B_1) = \{gb_1^1, gb_2^1, \ldots, gb_m^1\}$. Let a mapping $M : X_{GB_1}^J \to \{0, [\beta, \alpha], 1\}$, which is from $X_{GB_1}^J$ to the set $\{0, [\beta, \alpha], 1\}$, and M is denoted by

$$M(X_{GB_1}^J) = \begin{cases} 0 & \overline{\mu}(gb_i^1) \leq \beta \\ [\beta, \alpha] & \beta < \overline{\mu}(gb_i^1) < \alpha \\ 1 & \overline{\mu}(gb_i^1) \geq \alpha \end{cases} \tag{7}$$

where $i = 1, 2, \ldots, m$.

The three regions are divided according by thresholds to ensure the minimum fuzziness loss of 3WD-FGBRS. Corresponding to any average membership degree of granular-ball, 3WD-FGBRS is established to approximately characterize the target concept as follows.

1) If $\bar{\mu}(gb_i^1)$ is below or equal to β, $\bar{\mu}(gb_i^1)$ is reduced to 0. This indicates that allocating granular-ball gb_i^1 to the negative region will minimize the fuzziness loss.
2) If $\bar{\mu}(gb_i^1)$ is above or equal to α, $\bar{\mu}(gb_i^1)$ is elevated to 1. This indicates that allocating granular-ball gb_i^1 to the positive region will minimize the fuzziness loss.
3) If $\bar{\mu}(gb_i^1)$ is below α and above β, $\bar{\mu}(gb_i^1)$ is transformed into $[\beta, \alpha]$. This indicates that allocating granular-ball gb_i^1 to the boundary region will minimize the fuzziness loss.

From Definition 8, $F(\mathrm{POS}(M(X_{GB_1}^J))) = F(\mathrm{NEG}(M(X_{GB_1}^J))) = 0$, i.e., $F_{M(X_{GB_1}^J)} = F(\mathrm{BND}(M(X_{GB_1}^J)))$. Therefore, the total fuzziness can be calculated as follows:

$$F_{M(X_{GB_1}^J)} = \frac{4}{|U|}\sum_{i=1}^{m}\left|\{x \mid x \in gb_i^1 \wedge \beta < \bar{\mu}(gb_i^1) < \alpha\}\right| \frac{\int_\beta^\alpha t(1-t)\,dt}{\alpha-\beta}. \tag{8}$$

Uncertainty measure [44], [45] is a crucial metric in decision system. As a classical uncertainty measure, the fuzziness does not require expert knowledge, making the calculation of it more objective. Furthermore, fuzziness loss is able to objectively characterize the change of uncertain information without expert experience. To preserve the fuzziness invariance, an objective function is proposed, aimed at determining the optimal threshold pair (α, β).

$$\underset{0\le\beta\le\alpha\le 1}{\operatorname{argmin}} \left|F_{M(X_{GB_1}^J)} - F_{X_{GB_1}^J}\right|$$

where

$$F_{M(X_{GB_1}^J)} - F_{X_{GB_1}^J} = \frac{4}{|U|}\sum_{i=1}^{m}\left(\left|\{x \mid x \in gb_i^1 \wedge \beta < \bar{\mu}(gb_i^1) < \alpha\}\right| \frac{\int_\beta^\alpha t(1-t)\,dt}{\alpha-\beta} - \bar{\mu}(gb_i^1)(1-\bar{\mu}(gb_i^1))\right). \tag{9}$$

Algorithm 1 shows the detail of the process of threshold acquisition. Intuitively, Fig. 1 presents the construction process of 3WD-FGBRS. First, the optimal thresholds are determined by minimizing the fuzziness loss between $X_{GB_1}^J$ and its shadowed map $M(X_{GB_1}^J)$ based on objective function. Then, the granular-balls in FGBRS are divided to establish the three regions to make a decision according to the thresholds. As shown in Fig. 1, the granular-balls with $\bar{\mu}(gb_i^1)$ greater than α are considered to belong to the positive region, which is the blue area. The granular-balls with $\bar{\mu}(gb_i^1)$ less than β are considered to belong to the negative region, which is the red area. The granular-balls with $\bar{\mu}(gb_i^1)$ between β and α are considered to belong to the boundary region, which is the gray area. More specifically, the intuitive construction process of 3WD-FGBRS is shown in Fig. 2, i.e., the actual fuzziness of granular-ball space is characterized by average granular-ball fuzzy sets, then a shadowed map is constructed to provide thresholds for 3WD-FGBRS by minimizing the fuzziness loss.

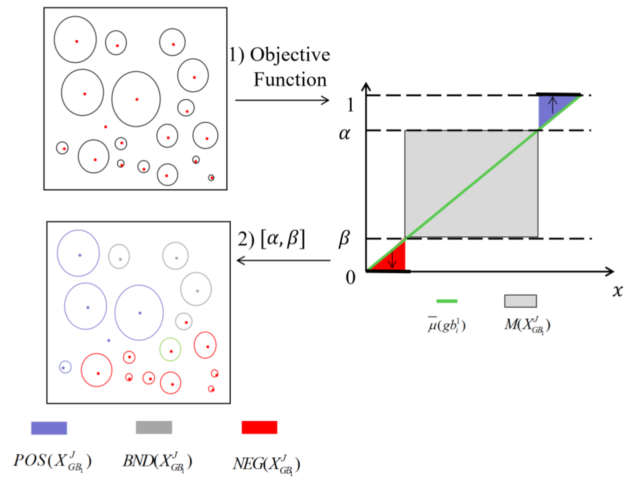

Fig. 1. Interpretation of 3WD-FGBRS based on fuzziness loss.

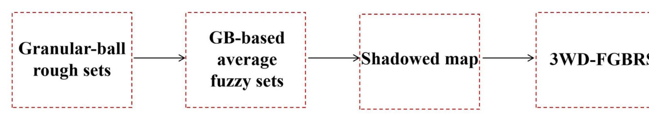

Fig. 2. Construction process of 3WD-FGBRS.

Algorithm 1: The Process of Threshold Acquisition.

Input: $DBGB\_list$, step size L
Output: The decision thresholds α and β

```
for gb ∈ DBGB_list do
    Compute the average membership degree μ̄(gb);
end
Computing F_{X^J_GB} = 1/|U| Σ_{gb∈DBGB_list} h(gb), where
    h(gb) = 4μ̄(gb)(1 − μ̄(gb));
Assuming β = 1 − α and Min = F_{X^J_GB};
for β = 0 to 0.5 do
    α = 1 − β;
    Computing F_{X^J_GB} and |F_{M(X^J_GB)} − F_{X^J_GB}|;
    if |F_{M(X^J_GB)} − F_{X^J_GB}| < Min then
        Min = |F_{M(X^J_GB)} − F_{X^J_GB}|;
        β* = β;
        α* = 1 − β*;
    end
    β = β + L;
end
return α and β;
```

IV. Sequential Three-Way Decision With Fuzzy Granular-Ball Rough Sets Based on Fuzziness Loss

The core to sequential three-way decisions (S3WD) is delayed decision-making, which ensures a more accurate decision in subsequent iterations, as the decision information is transformed with increasingly finer granularity. This decision-making process thus constitutes a progressive approach to continually improves the precision of decision outcomes. Therefore, in order to deal with multigranularity decision problems more efficiently, we further propose the definition of sequential three-way decision with fuzzy granular-ball rough sets (S3WD-FGBRS) based on the fuzziness loss, which utilizes progressive computing to mine useful information. Furthermore, from the perspective of uncertainty, the S3WD-FGBRS with adaptive thresholds is introduced.

Definition 9: Given a neighborhood decision system $S = (U, AT \cup D, V, f)$, $B_1, B_2, \ldots, B_N \subseteq AT$. $U/GB(B_N) \preceq \cdots \preceq U/GB(B_2) \preceq U/GB(B_1)$, and X be a target fuzzy subset on U. The sequential three-way decision with granular-ball rough sets can be denoted by the following series:

$$Seq\_GBS = (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$$

where, $U/GB(B_j) = \{gb_1^j, gb_2^j, \ldots, gb_i^j, \ldots, gb_{\text{num}_j}^j\}$ denotes the jth granular-ball space induced by $INDGB(B_j)$. gb_i^j denotes the ith granular-ball on $U/GB(B_j)$ and num_j denotes the number of granular-balls of $U/GB(B_j)$. According to the threshold pair (α_j, β_j), $U/GB(B_j)$ is divided into $POS(X_{GB_j}^J)$, $BND(X_{GB_j}^J)$ and $NEG(X_{GB_j}^J)$. Herein, (α_j, β_j) denotes the thresholds of the jth granular-ball space.

In S3WD-FGBRS, the decision regions of the jth granular-ball space are denoted by

$$POS(X_{GB_j}^J) = \{x \in U | \bar{\mu}(gb_i^j) \geq \alpha_j\},$$
$$BND(X_{GB_j}^J) = \{x \in U | \beta_j < \bar{\mu}(gb_i^j) < \alpha_j\},$$
$$NEG(X_{GB_j}^J) = \{x \in U | \bar{\mu}(gb_i^j) \leq \beta_j\}.$$

Then, a sequence of fuzziness descriptions of 3WD-FGBRS is represented as follows:

$$Seq\_F_{X_{GB}^J} = \left(F_{X_{GB_1}^J}, F_{X_{GB_2}^J}, \ldots, F_{X_{GB_N}^J}\right)$$

where

$$F_{X_{GB_j}^J} = F(\text{POS}(X_{GB_j}^J)) + F(\text{BND}(X_{GB_j}^J)) + F(\text{NEG}(X_{GB_j}^J))$$

$j = 1, 2, \ldots, N$.

According to Definition 8, $F(POS(X_{GB_j}^J)) = F(NEG(X_{GB_j}^J)) = 0$.

$$F_{(X_{GB_j}^J)} = \frac{4}{|U|} \sum_{gb_i^j \in U/GB(B_j)} \left|\{x | x \in gb_i^j \wedge \beta_j < \bar{\mu}(gb_i^j) < \alpha_j\}\right| \frac{\int_{\beta_j}^{\alpha_j} t(1-t)dt}{1 - 2\beta_j}.$$

For simplicity, we further suppose $\alpha_j = 1 - \beta_j$ in this article.

Each granular-ball space of S3WD-FGBRS is considered as a separate 3WD-FGBRS, and the next granular-ball space increases decision information based on the previous granular-ball space. That is, the granular-ball space is continuously refined to update the three decision regions, as a result, the decision result become more precise.

A. Uncertainty of Multilevel Decision Result in S3WD-FGBRS

In this section, to facilitate the theoretical foundation of S3WD-FGBRS, the fuzziness loss analysis of average granular-ball fuzzy set is presented. In S3WD-FGBRS model, the uncertainty typically arises in three regions within each granular-ball space, stemming from the uncertainty associated with the GBs of the positive and negative regions. Specifically, the average membership degree of these GBs may not be strictly equal to 0 or 1. As the granular-ball space becomes finer, the GBs of the positive and negative regions may undergo reclassification, leading to changes in the three disjoint regions. Consequently, the uncertainty loss at each granular-ball space within the S3WD-FGBRS model will vary. Based on Definition 9, we will investigate the changing rules of uncertainty loss of S3WD-FGBRS and present the related theorems (The proofs are provided in Section S2 of Supplementary file) as follows:

Theorem 1: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$, and X be a target fuzzy subset on U. Then, $F_{(X_{GB_{j+1}}^J)} \leq F_{(X_{GB_j}^J)}$ holds.

From Theorem 1, the fuzziness monotonically decreases with the granularity refinement in S3WD-FGBRS. When $U/GB(B_{j+1})$ in Theorem 1 reaches the finest granular-ball space $U/GB(B_N)$, Corollary 1 and Theorem 2 can be obtained as follows:

Corollary 1: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$ and X be a target fuzzy subset on U. Then, $F_{(X_{GB_j}^J)} \geq F_{(X_{GB_N}^J)}$ holds.

Theorem 2: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$ and X be a target fuzzy subset on U. Then, $|F_{X_{GB_{j+1}}^J} - F_{X_{GB_N}^J}| \leq |F_{X_{GB_j}^J} - F_{X_{GB_N}^J}|$ holds.

From Theorem 2, the fuzziness loss between the arbitrary granular-ball space and the finest granular-ball space $U/GB(B_N)$ in S3WD-FGBRS monotonically decreases with the granularity refinement in S3WD-FGBRS.

It is well known that accurately calculating the fuzziness of the three decision regions is crucial for effective decision-making. Nevertheless, it is noteworthy that the fuzziness of each decision region may not necessarily decrease monotonically as the granular-ball space undergoes refinement in S3WD-FGBRS. For simplicity, we solely concentrate on examining the fuzziness in the boundary region by considering the three situations.

Theorem 3: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$ and X be a target fuzzy subset on U. If only the GBs contained in $POS(X^J_{GB_j})$ are subdivided into finer GBs from $U/GB(B_j)$ to $U/GB(B_{j+1})$, then $F(BND(X^J_{GB_{j+1}})) \geq F(BND(X^J_{GB_j}))$ holds.

Theorem 4: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$, and X be a target fuzzy subset on U. If only the GBs contained in $NEG(X^J_{GB_j})$ are subdivided into finer GBs from $U/GB(B_j)$ to $U/GB(B_{j+1})$, then $F(BND(X^J_{GB_{j+1}})) \geq F(BND(X^J_{GB_j}))$ holds.

According to Theorems 3 and 4, when only GBs contained in negative region or positive region are subdivided, the uncertainty of boundary region will increase in S3WD-FGBRS. This outcome is not aligned with human cognition.

Theorem 5: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$, and X be a target fuzzy subset on U. If only the GBs contained in $BND(X^J_{GB_j})$ are subdivided into finer GBs from $U/GB(B_j)$ to $U/GB(B_{j+1})$, then $F(BND(X^J_{GB_{j+1}})) \leq F(BND(X^J_{GB_j}))$ holds.

According to Theorem 5, when only GBs contained in the boundary region are subdivided, the uncertainty of the boundary region will decrease in S3WD-FGBRS. This outcome is aligned with human cognition.

B. Optimal Granular-Ball Space Selection Mechanism

From the above discussion, it is evident that the fuzziness monotonically decreases within the granularity refinement. As well known, uncertainty and granularity reflect decision precision and computational expense at the granular-ball space, respectively. Consequently, it is crucial to address problems by considering the interplay between uncertainty and granularity, which should be considered in granularity optimization in S3WD-FGBRS. For describing a target concept, a higher decision precision will yield in a finer granular-ball space, while the computational expense will become higher in a finer granular-ball space as the number of granular-ball increases. Hence, uncertainty and computational expense are contradictory each other with the granular-ball space being finer. Furthermore, in specific scenarios, it is advisable to select an optimal granular-ball space for decision-making by taking into account user preferences for both decision precision and computational expense.

Definition 10. (Granular-ball space distance): Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, $B_1, B_2 \subseteq AT$ and X is a target fuzzy subset on U. $U/GB(B_1) = \{gb^1_1, gb^1_2, \ldots, gb^1_m\}$ and $U/GB(B_2) = \{gb^2_1, gb^2_2, \ldots, gb^2_l\}$ are two granular-ball space induced by $INDGB(B_1)$ and $INDGB(B_2)$. Then, the granular-ball space distance (GBSD) between $U/GB(B_1)$ and $U/GB(B_2)$ is defined as follows:

$$GBSD(U/GB(B_1), U/GB(B_2)) = \frac{\sum_{j=1}^{m}\sum_{k=1}^{l} gbd_{jk} f_{jk}}{|U|} \tag{10}$$

where $\mathrm{gbd}_{jk} = \frac{\sum_{x\in U}\mu_{\widetilde{\mathrm{gb}}^1_j \cup \widetilde{\mathrm{gb}}^2_k}(x) - \sum_{x\in U}\mu_{\widetilde{\mathrm{gb}}^1_j \cap \widetilde{\mathrm{gb}}^2_k}(x)}{|U|}$, which denotes the granular-ball distance (gbd) between gb^1_j and gb^2_k. f_{jk} represents the amount of intersection objects between gb^1_j and gb^2_k, i.e., $f_{jk} = |gb^1_j \cap gb^2_k|$.

Theorem 6: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, $B_1, B_2, B_3 \subseteq AT$, X be a target fuzzy subset on U. $U/GB(B_1)$, $U/GB(B_2)$, and $U/GB(B_3)$ are three granular-ball spaces in Seq_GBS. If $U/GB(B_3) \preceq U/GB(B_2) \preceq U/GB(B_1)$, then $GBSD(U/GB(B_1), U/GB(B_2)) \leq GBSD(U/GB(B_1), U/GB(B_3))$ holds.

Definition 11. (Granularity measure): Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, and let G be a mapping from the power set of AT to the set of real numbers. Then, G qualifies as a granularity measure provided that it fulfills the ensuing conditions.

1) For any $B \subseteq AT$, $G(B) \geq 0$.
2) For any $B_1, B_2 \subseteq AT$, if $U/B_2 = U/B_1 \subseteq AT$, then $G(B_2) = G(B_1)$.
3) For any $B_1, B_2 \subseteq AT$, if $U/B_2 \prec U/B_1 \subseteq AT$, then $G(B_2) < G(B_1)$.

Theorem 7: Let $S = (U, AT \cup D, V, f)$ be a neighborhood decision system, Seq_GBS $= (U/GB(B_1), U/GB(B_2), \ldots, U/GB(B_N))$, and X be a target fuzzy subset on U. Then, $GBSD(U/GB(B_j), U/GB(B_N))$ is a granularity measure.

From Theorem 7, $GBSD(U/GB(B_j), U/GB(B_N))$ can be used to reflect the granularity content of a granule-ball space in Seq_GBS as a granularity measure. Namely, a lower $GBSD(U/GB(B_j), U/GB(B_N))$ results in a higher computational expense, which means handling more GBs. For convenience, we use $G_{X^J_{GB_j}}$ to denote $GBSD(U/GB(B_j), U/GB(B_N))$. That is, $G_{X^J_{GB_j}} = GBSD(U/GB(B_j), U/GB(B_N))$.

If decision precision and computational expense desired by the users are denoted as F_{user} and G_{user}, respectively, the objective of granular-ball space selection is to identify a granular-ball space $U/GB(B_j)$ to fulfills the constraints $F_{X^J_{GB_j}} \leq F_{\text{user}}$ and $G_{X^J_{GB_j}} \geq G_{\text{user}}$. Subsequently, related decision actions are executed on this granular-ball space. The granular-ball space optimization mechanism is illustrated in Fig. 3. Herein, the granular-ball space $U/GB(B_s)$ satisfies the computational expense demand, but fails short of meeting the decision precision criterion; the granular-ball space $U/GB(B_t)$ fulfills the decision precision criterion, but fails short of meeting the computational expense demand; the granular-ball space $U/GB(B_r)$ satisfies both the computational expense and decision precision constraints, and efficient decision is achieved based on granular-ball space optimization. This computation is formalized as the following optimization problem:

$$\underset{U/GB(B_j)}{\operatorname{argmin}}\ F_{X^J_{GB_j}} \tag{11}$$

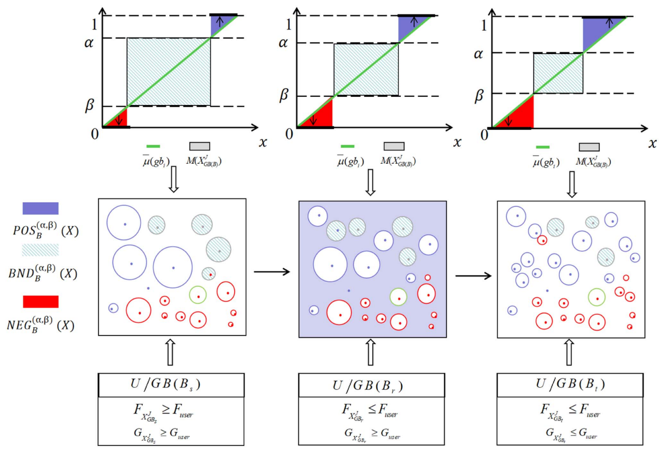

Fig. 3. Optimal granular-ball space that fulfills the user preferences.

TABLE I
INFORMATION OF EXPERIMENTAL DATASETS

	Granular-ball spaces	User requirements
1	$U/GB(B_1)$	$F_{X^J_{GB_1}} = 0.976$, $G_{X^J_{GB_1}} = 0.54$
2	$U/GB(B_2)$	$F_{X^J_{GB_2}} = 0.829$, $G_{X^J_{GB_2}} = 0.35$
3	$U/GB(B_3)$	$F_{X^J_{GB_3}} = 0.756$, $G_{X^J_{GB_3}} = 0.29$
4	$U/GB(B_4)$	$F_{X^J_{GB_4}} = 0.703$, $G_{X^J_{GB_4}} = 0.176$

s.t.

$$F_{X^J_{GB_j}} \leq F_{\text{user}}$$
$$G_{X^J_{GB_j}} \geq G_{\text{user}}.$$

Example 2: Suppose that Seq_GBS $= (U/GB(B_1), U/GB(B_2), U/GB(B_3), U/GB(B_4))$, X denotes a fuzzy target subset on U. $F_{\text{user}} = 0.9$ and $G_{\text{user}} = 0.2$. The details of granular-ball spaces in Seq_GBS are shown in Table I.

According to formula (11), because $F_{\text{user}} = 0.9$ and $G_{\text{user}} = 0.2$, then, $U/GB(B_2)$ and $U/GB(B_3)$ meet the user requirements for decision precision and computational expense, while $U/GB(B_3)$ has less uncertainty. Therefore, $U/GB(B_3)$ is the optimal granular-ball space in Seq_GBS.

V. Experiment

In this section, we meticulously outline the experimental setup, encompassing validation procedures, evaluation metrics and comparison methods, and hyperparameter configurations. Our experiments primarily concentrate on two pivotal research aspects: (1) The trends of uncertainty and decision quality with the changing granularity are exhibited to verify the related properties of S3WD-FGBRS; (2) 3WD-FGBRS outperforms the GB-based classifiers and conventional classifier algorithms, including CART, KNN, GBKNN, GBKNN++, ACCGBKNN, and 3WD-NRS, in terms of comprehensive performance. This conclusion is derived from measurement using specific evaluation criteria in experiments. The experimental environments are Windows 10, 64-bit OS, Intel i5-4210 M CPU, 8.0 GB of DDR3 running memory, and the programming software is Python 3.9.7. Experiments are performed based on twelve datasets from UCI and KEEl, which are shown in Table II, which are from https://archive.ics.uci.edu/ml/datasets.php and https://sci2s.ugr.es/keel/datasets.php . Moreover, for every test in the experiment, we used tenfold cross-validation to ensure the reliability of the results.

TABLE II
INFORMATION OF EXPERIMENTAL DATASETS

	Datasets	Characteristics	Instances	Attributes
1	Raisin	Integer, Real	900	7
2	Steel Plates Faults	Integer, Real	1941	27
3	Wifi_Loc	Real	2000	7
4	Segment	Real	2310	19
5	Iranian Churn	Integer	3150	13
6	Rice	Real	3810	7
7	Banana	Real	5300	2
8	Satimage	Integer	6435	36
9	Twonorm	Integer, Real	7400	20
10	Mushroom	Categorical	8124	22
11	Dry Bean	Integer, Real	13611	16
12	HTRU	Real	17000	8

A. Analysis of Uncertainty and Decision Quality for S3WD-FGBRS

For the experiment, in addition to using fuzziness to measure uncertainty, we employ three measures [46] including CAR, NPE, and CRR to evaluate the decision quality of S3WD-FGBRS, where, $CAR = \frac{|POS(X^J_{GB_j}) \cap X|}{|POS(X^J_{GB_j})|}$ denotes correct acceptance rate, $NPE = \frac{|BND(X^J_{GB_j}) \cap X|}{|BND(X^J_{GB_j})|}$ denotes noncommitment of positive error and $CRR = \frac{|NEG(X^J_{GB_j}) \cap X|}{|NEG(X^J_{GB_j})|}$ denotes correct-rejection rate. The experiments with respect to the uncertainty and decision quality on twelve datasets are presented in Figs. 4 and 5, respectively. As depicted in Fig. 4, the x-coordinate represents four distinct granularity levels, while the y-coordinate corresponds to each fuzziness value. From these results, the following observations can be made.

1) The uncertainty primarily originates from the boundary region at each granularity level within the S3WD-FGBRS framework.
2) As the granularity levels transition from coarser to finer in S3WD-FGBRS, the total uncertainty $F_{X^J_{GB_1}}$ monotonously decrease, while the uncertainty with respect to three regions exhibit a nonmonotonicity.
3) The difference between $F_{M(X^J_{GB_1})}$ and $F_{X^J_{GB_1}}$ is very small on each granularity.

In Fig. 5, we assess the performance of S3WD-FGBRS using the metrics of CAR, NPE, and CRR. Evidently, there is no significant change in CAR and CRR. This stability can be attributed to our approach of postponing decisions to subsequent levels when insufficient valuable information is available in S3WD-FGBRS. This strategy is able to make decisions of acceptance and rejection with remarkable accuracy. Consequently, the significant

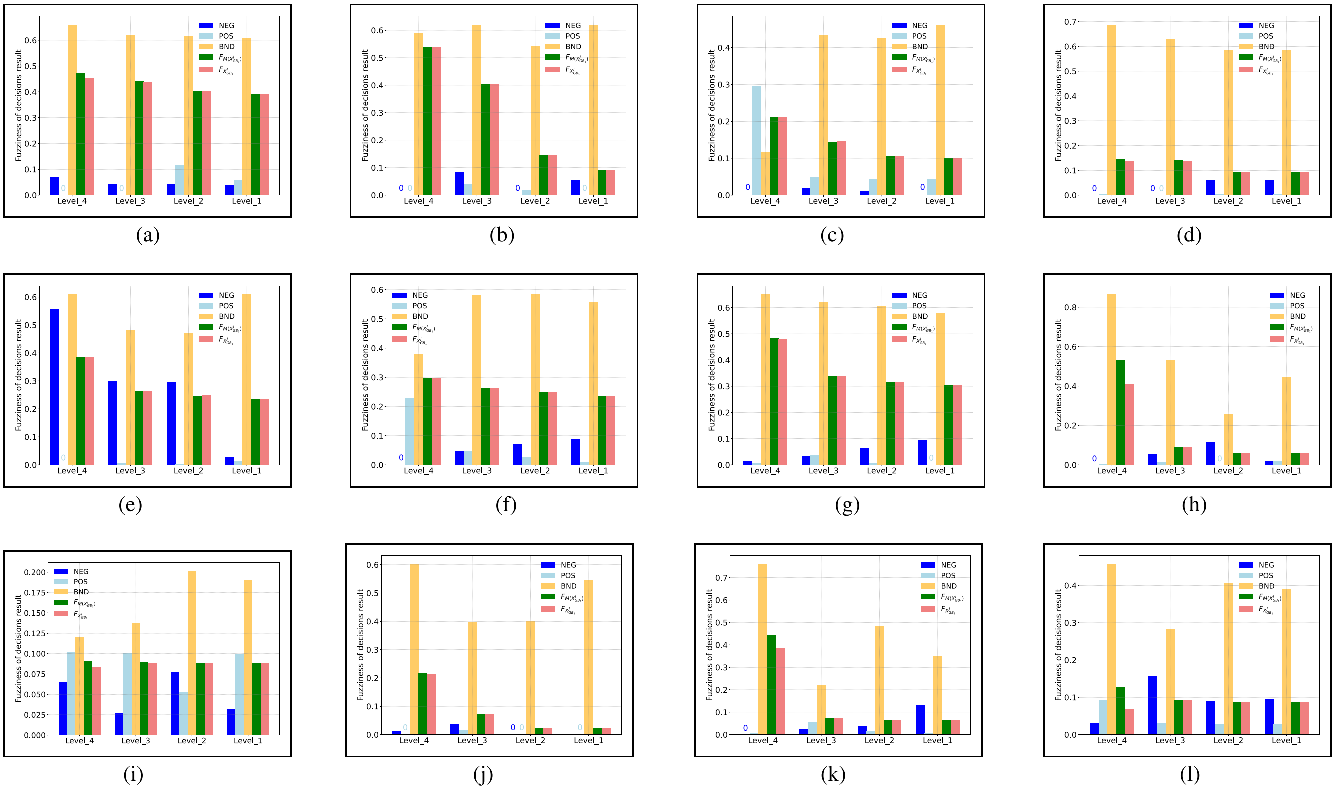

Fig. 4. Change of uncertainty for S3WD-FGBRS. (a) Raisin. (b) SPF. (c) Wifi_Loc. (d) Segment. (e) Iranian churn. (f) Rice. (g) Banana. (h) Satimage. (i) Twonorm. (j) Mushroom. (k) Dry bean. (l) HTRU.

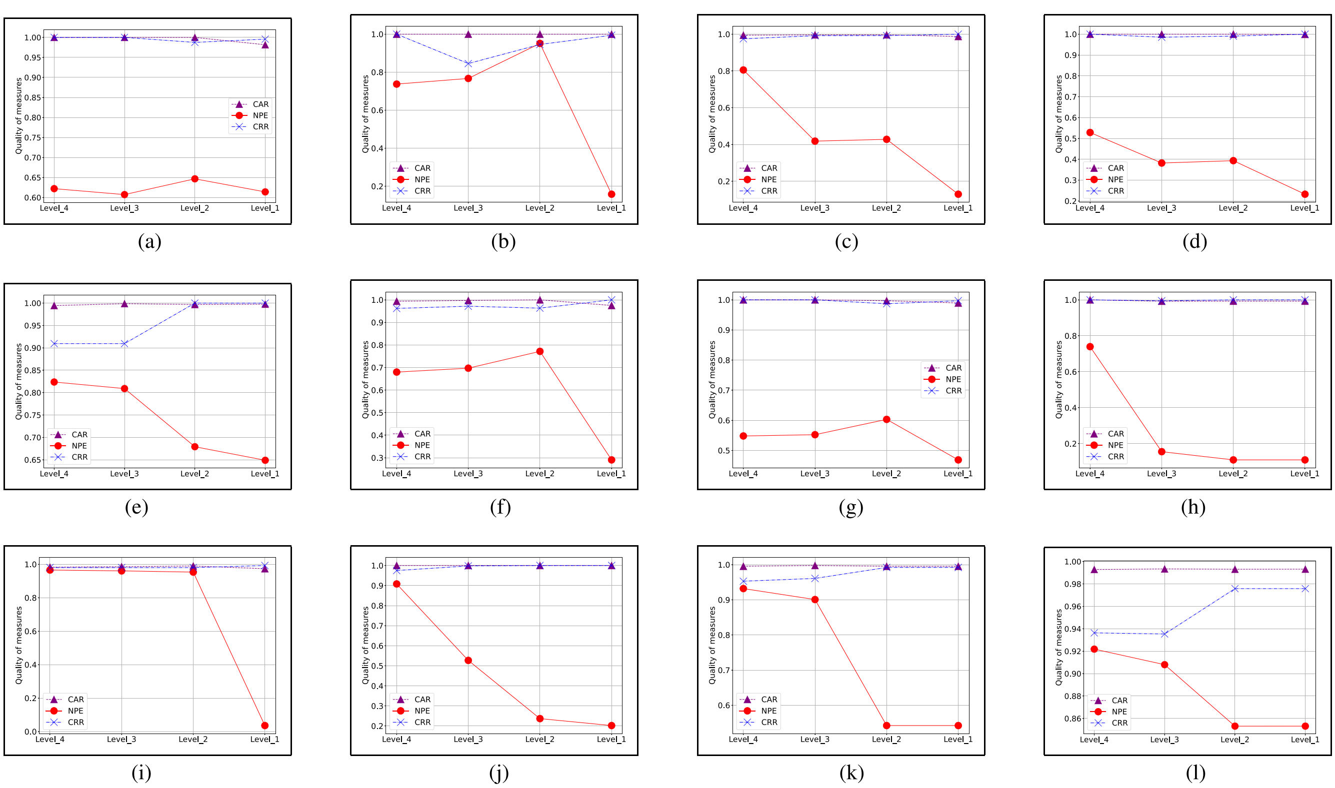

Fig. 5. Change of decision quality in S3WD-FGBRS. (a) Raisin. (b) SPF. (c) Wifi_Loc. (d) Segment. (e) Iranian churn. (f) Rice. (g) Banana. (h) Satimage. (i) Twonorm. (j) Mushroom. (k) Dry bean. (l) HTRU.

CAR and CRR values are clearly demonstrated in Fig. 5. At each granularity, both CAR and CRR approach one. Furthermore, we enhance the information content by dividing the boundary region into three distinct regions. Significantly, it is worth observing that the NPE reaches zero at level-1, indicating a substantial enhancement in decision quality.

B. 3WD-FGBRS vs Other Classifiers

The performance of 3WD-FGBRS has been thoroughly validated in the conducted experiment, encompassing crucial aspects such as accuracy, time cost, robustness, and stability. To further evaluate its effectiveness, four widely recognized machine learning metrics–namely, $Accuracy$, $Precision$, $Recall$, and F1 score ($F1$)–have been calculated. These metrics provide a comprehensive assessment of classification performance, with higher values indicating superior outcomes. Notably, $F1$ serves as the harmonic mean between $Precision$ and $Recall$, providing a balanced evaluation of both. $Accuracy$ is determined by R_{test} and U_{test}, which represent the ratio of correctly predicted test samples to the overall number of test samples. $Precision$ quantifies the proportion of elements accurately labeled as positive, while $Recall$ reflects the proportion of actual positive elements that are correctly identified. Specifically, tp denotes the total number of true positives, tn denotes the total number of true negatives, fp denotes the total number of false positives, and fn denotes the total number of false negatives, where, Accuracy $= \frac{R_{\text{test}}}{U_{\text{test}}}$, Precision $= \frac{tp}{tp+fp}$, and F1 $= \frac{2*Precision*Recall}{Precision+Recall}$.

Moreover, we utilize the Wilcoxon rank-sum test to conclusively determine the presence of any substantial differences between the compared classifiers. In addition, the efficacy of the method is further validated through the time efficiency, as well as the anti-noise capability and stability. Relevant evaluation criteria will be mentioned in the corresponding experimental sections. 3WD-FGBRS is contrasted with three state-of-the-art GB-based classifiers and three classic machine learning classifiers. The descriptions of these classifiers and their hyperparameter settings are as follows.

1) *CART:* All hyperparameters of CART are aligned with the default parameters in scikit-learn.
2) *KNN:* Set k to 1, 3, 5, 7, 9, 11, 13, and 15 and select the appropriate k value as the final result.
3) *3WD-NRS [47]:* A classic three-way classifier based on neighborhood rough sets.
4) GBKNN [16]: An original GB-based KNN algorithm.
5) *GBKNN++ [48]:* An improved GB-based KNN algorithm based on GBG++ method.
6) *ACC-GBKNN [18]:* In this classifier, the K-means method in ACC-GBCKNN classifier is replaced by k -division.

Effectiveness: The comparison experiment includes six different classification algorithms, including KNN, CART, 3WD-NRS, GBKNN, GBKNN++, and ACC-GBKNN. Table III presents four evaluation metrics, namely $Accuracy$, $Precision$, $Recall$, and $F1$. Subsequently, statistical analysis is performed on the results, including Friedman test, Wilkerson rank-sum test, and mean ranking analysis. $Win/Loss$ represents the win-loss ratio following pairwise comparisons between 3WD-FGBRS and the comparison algorithms. $p-value$ reflects the difference between 3WD-FGBRS and the comparison algorithms. If $p-value < 0.05$, it indicates a significant difference between 3WD-FGBRS and the comparison algorithms. Otherwise, there is no statistical difference. Rank represents the average ranking. A higher rank value indicates a more effective algorithm. In terms of $Win/Loss$, 3WD-FGBRS wins 238 times out of 288 total comparison times against other algorithms. In terms of $p-value$, 3WD-FGBRS shows significant differences with the comparison algorithms. Regarding $rank$, 3WD-FGBRS obtains the highest overall score, followed by GBKNN and GBKNN++. The superior performance of 3WD-FGBRS in terms of effectiveness can be attributed to the key factors as follows: 1) in comparison to the traditional classifiers such as CART, KNN and 3WD-NRS, 3WD-FGBRS inherits the strengths of GB-based classifiers. Specifically, GBs are well-suited for representing datasets with spherical distributions and exhibit resilience against noisy data; 2) compared to the GB-based classifiers including GBKNN, GBKNN++, and ACC-GBKNN, the innovative 3WD concept enhances uncertainty handling, thereby significantly improving classification performance. Moreover, the ROC curves and AUC values results are shown in Section S4 of supplementary file.

Efficiency: Table IV presents the details of the time required for each methods on datasets. With the increase in dataset size, the time required for classification by KNN and 3WD-NRS significantly increases. From the perspective of rankings, the time consumed by 3WD-FGBRS ranks 3th among all the algorithms, while the other GB-based methods, namely GBKNN++, GBKNN, ACC-GBKNN, rank 2th, 4th, and 5th. This implies that the time required by 3WD-FGBRS is not significantly different from that of other GB-based methods. Although the time consumption of 3WD-FGBRS exceeds that of CART (the 1st ranked), much lower than that of 3WD-NRS and KNN in all datasets. Due to the acquirement of thresholds in 3WD introduced, 3WD-FGBRS incurs slightly more time compared to the other GB-based methods. However, this increase of time cost is entirely acceptable. Overall, 3WD-GBRS provides a classification solution with relatively low time complexity.

Robustness: To verify the robustness of 3WD-FGBRS, we manually introduced noise proportions of 10%, 20%, 30%, and 40% on each dataset. Literatures [16], [48] has clearly demonstrated that GB-based classifiers exhibit superior robustness compared to traditional classifiers. Therefore, for this evaluation, we solely selected GB-based classifiers—specifically, GBKNN, ACC-GBKNN and GBKNN++, as the comparison methods. The evaluation process involves the following steps: 1) we randomly chose samples from each dataset and altered their labels to introduce noise; 2) we performed ten-fold cross-validation on the datasets with varying noise ratios to assess the robustness of these classifiers based on test accuracy. Fig. 6 visually presents the results in a heat map, clearly indicating the ranking of test accuracy for each classifier across different datasets and noise ratios. For instance, on the SPF dataset with a label noise ratio of 10%, 3WD-FGBRS achieves the highest test accuracy, marked as 1 in Fig. 6(a). 3WD-FGBRS rankes first on most datasets, outperforming GBKNN, ACC-GBKNN, and GBKNN++ in

TABLE III
STATISTICAL ANALYSIS OF VARIOUS ALGORITHMS

No.	Metrics	CART	KNN	GBKNN	GBKNN++	ACC-GBKNN	3WD-NRS	3WD-FGBRS
1	$Accuracy$	0.8056	0.6922	0.8489	0.8522	0.8033	0.5114	**0.8790**
	$F1$	0.8046	0.6567	0.8438	0.8513	0.8025	0.6767	0.8693
	$Recall$	0.8081	0.7415	0.8659	0.8156	0.8333	1.0000	0.9048
	$Precision$	0.8044	0.6978	0.8267	0.8811	0.7889	0.5114	**0.8406**
2	$Accuracy$	0.8669	0.9763	0.9043	0.8643	0.8663	0.4818	**0.9829**
	$F1$	0.8104	0.9505	0.8458	0.8513	0.8524	0.6503	0.9859
	$Recall$	0.7865	0.9171	0.8048	0.9777	0.9677	1.0000	0.9863
	$Precision$	0.9029	0.9951	0.9455	0.7794	0.7797	0.4818	**0.9871**
3	$Accuracy$	0.9580	0.9720	0.9745	0.9660	0.9640	0.6408	**0.9806**
	$F1$	0.9567	0.9747	0.9743	0.9660	0.9639	0.7811	0.9791
	$Recall$	0.9665	0.9647	0.9769	0.9690	0.9670	1.0000	0.9853
	$Precision$	0.9510	0.9890	0.9730	0.9647	0.9631	0.6408	**0.9744**
4	$Accuracy$	**0.9926**	0.9919	0.9879	0.9870	0.9837	0.3542	0.9917
	$F1$	0.9742	0.9719	0.9558	**0.9745**	0.9672	0.5231	0.9628
	$Recall$	0.9761	0.9681	0.9869	0.9800	0.9532	1.0000	0.9860
	$Precision$	0.9727	**0.9764**	0.9273	0.9358	0.9392	0.3542	0.9444
5	$Accuracy$	0.9419	0.9463	0.9454	0.9436	**0.9485**	0.1730	0.9481
	$F1$	0.8124	0.8243	0.8119	0.8910	0.9035	0.2950	0.7106
	$Recall$	0.8326	0.8510	0.8878	0.8051	0.8525	1.0000	0.8663
	$Precision$	0.7981	0.8095	0.7515	**0.8289**	0.8255	0.1730	0.6071
6	$Accuracy$	0.8837	0.7937	0.9202	0.9050	0.8858	0.8040	**0.9419**
	$F1$	0.8647	0.7232	0.9061	0.9028	0.8838	0.8914	0.9314
	$Recall$	0.8621	0.8210	0.9092	0.8859	0.8840	1.0000	0.9344
	$Precision$	0.8681	0.7571	0.9043	0.8924	0.8557	0.8040	**0.9293**
7	$Accuracy$	0.8764	0.7398	0.8911	0.8826	0.8753	0.5850	**0.9188**
	$F1$	0.8621	0.7288	0.8763	0.8810	0.8742	0.7382	0.9030
	$Recall$	0.8627	0.7159	0.8929	0.8542	0.8740	1.0000	0.9322
	$Precision$	0.8620	0.7673	0.8607	**0.8805**	0.8520	0.5850	0.8766
8	$Accuracy$	0.9692	0.9826	0.9767	0.9865	0.9849	0.3402	**0.9905**
	$F1$	0.9361	0.9609	0.9507	0.9818	0.9796	0.5077	0.9744
	$Recall$	0.9354	0.9718	0.9641	0.9843	0.9791	1.0000	0.9795
	$Precision$	0.9374	0.9562	0.9394	0.9624	0.9604	0.3402	**0.9698**
9	$Accuracy$	0.8431	0.9447	0.9708	0.9561	0.9473	0.7857	**0.9827**
	$F1$	0.8424	0.9446	0.9708	0.9561	0.9473	0.8800	0.9825
	$Recall$	0.8456	0.9466	0.9709	0.9613	0.9527	1.0000	0.9862
	$Precision$	0.8399	0.9445	0.9708	0.9513	0.9426	0.7857	**0.9789**
10	$Accuracy$	0.9685	0.9936	0.9852	0.9765	0.9849	0.9562	**0.9983**
	$F1$	0.9767	0.9942	0.9858	0.9762	0.9845	0.9776	0.9985
	$Recall$	0.9622	0.9890	0.9857	0.9774	0.9988	1.0000	0.9977
	$Precision$	**1.0000**	1.0000	0.9872	0.9808	0.9774	0.9562	0.9992
11	$Accuracy$	0.9387	0.9589	0.9597	0.9439	0.9536	0.8446	**0.9843**
	$F1$	0.7980	0.8704	0.8749	0.9066	0.9168	0.9158	0.9324
	$Recall$	0.7853	0.8647	0.8612	0.9032	0.9096	1.0000	0.9717
	$Precision$	0.8236	0.9004	0.8958	0.8175	0.8284	0.8446	**0.9077**
12	$Accuracy$	0.9677	0.9620	0.9776	0.9752	0.9727	0.0793	**0.9795**
	$F1$	0.8246	0.7588	0.8710	0.9227	0.9169	0.1470	0.8607
	$Recall$	0.8212	0.7377	0.9188	0.8279	0.8389	1.0000	0.9555
	$Precision$	0.8310	0.7811	0.8310	**0.8951**	0.8613	0.0793	0.7850
Statistics	$win/loss$	41/7	40/8	42/6	39/9	40/8	36/12	**238/50**
	$p-vlaue$	**3.5272E-05**	0.0162	0.0133	0.0090	0.0047	0.0011	
	$rank$	2.8333	3.4900	4.5104(2)	4.3958(3)	3.9583	2.8542	**5.9583(1)**

The bold values represents the best one among all the results.

TABLE IV
REQUIRED TIME OF VARIOUS ALGORITHMS

No.	CART	KNN	GBKNN	GBKNN++	ACC-GBKNN	3WD-NRS	3WD-FGBRS
1	**0.0543**	1.1230	0.1726	0.1365	0.4316	1.1139	0.0665
2	**0.0957**	5.5952	0.1481	0.1564	0.2648	5.7592	0.2238
3	**0.0591**	5.1642	0.2132	0.1093	0.3964	2.226	0.1723
5	**0.0845**	11.7397	0.9892	0.2143	0.8500	13.6651	0.4824
6	**0.1733**	18.5258	1.7407	0.3769	1.9210	22.5035	0.4781
7	**0.1006**	36.4797	3.3159	0.6246	2.8285	260.445	0.9364
8	**0.1344**	60.1761	1.5582	0.5121	1.0440	122.0209	1.3373
9	**0.6507**	77.7871	5.6333	0.9272	4.9394	229.2911	1.4386
10	**0.0417**	93.3932	0.5402	0.2434	0.8174	320.3618	1.6648
11	**0.6590**	287.0782	7.979	1.7393	5.8750	1466.1927	3.8530
12	**0.5081**	373.8240	20.6019	2.4093	9.8970	1990.3189	4.7218
$rank$	**6.8333(1)**	1.8333	3.9167	5.9167(2)	3.7500	1.1667	4.5833(3)

The bold values represents the best one among all the results.

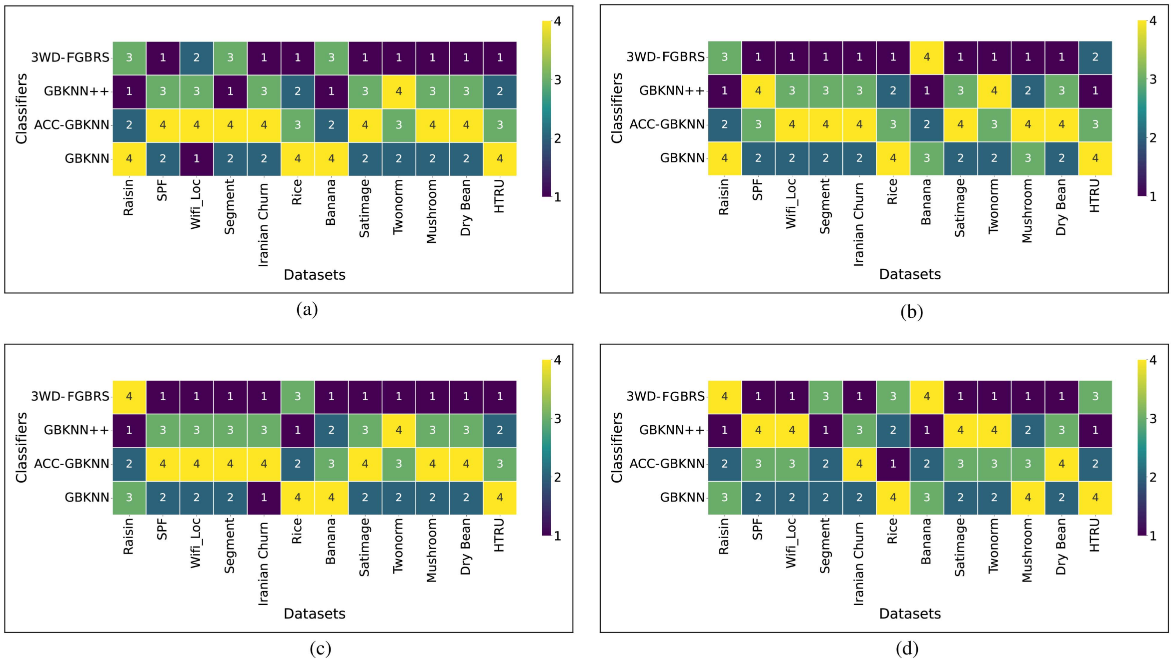

Fig. 6. Comparison on ranking of test $Accuracy$ under each noise rate. (a) Noise rate: 10%. (b) Noise rate: 20%. (c) Noise rate: 30%. (d) Noise rate: 40%.

terms of test accuracy for the majority of datasets with label noise ratios of 10%, 20%, 30%, and 40%. The main reason for the outstanding noise robustness of 3WD-FGBRS is that it configured with boundary region, which naturally mitigates the impact of noisy samples. In addition, the label assignment for a GB is determined by the majority of its constituent samples, further enhancing its resilience against noise.

Furthermore, to provide a more intuitive explanation of the advantages of the model proposed in this article, a real-time application case of 3WD-FGBRS are shown in Section S3 of supplementary file.

VI. Conclusion

Considering the shortcomings of the existing GB-based classifier, this article introduces the three-way decision theory into GBC to construct a 3WD-FGBRS in terms of uncertainty. Based on 3WD-FGBRS, S3WD-FGBRS is further presented. The first experimental results present that the trends of uncertainty and decision quality with the changing granularity of S3WD-FGBRS. For the second experiment, the performance of 3WD-FGBRS is validated, i.e., 3WD-FGBRS almost outperforms other comparison methods in term of effecitveness, efficiency and robustness. This comprehensive comparison encompassed 3 state-of-the-art GB-based classifiers and 3 classical machine learning classifiers, on 12 public benchmark datasets. Consequently, our work establishes a significant foundation for future endeavors in GBC, aiming for enhanced robustness and generality. Notably, our efforts are exploratory in nature and inevitably carry certain limitations, e.g., our current work primarily focuses on addressing the classification of labeled datasets using 3WC-SGB approach, but we recognize the importance of extending this work to handle unlabeled real-time datasets. To tackle this challenge, we propose the following potential directions for future research.

1) For real-time unlabeled data, we will incorporate an incremental learning framework into 3WC-SGB. This would involve dynamically updating the granular-ball spaces and thresholds as new data points arrive. Specifically, we can design an adaptive algorithm that leverages the existing labeled data to initialize the model, and then continuously refines it as new unlabeled data is processed, potentially using semisupervised learning techniques to exploit the unlabeled data.
2) *Sliding Window Mechanism:* When handling real-time data, a sliding window technique will be employed to dynamically capture the temporal characteristics of the data, improving the classification performance.
3) *Real-Time Data Preprocessing:* To boost classifier performance, we plan to introduce a real-time data preprocessing module that includes noise reduction and feature extraction, ensuring data quality before classification.

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