$$\boldsymbol{F} = m\boldsymbol{a} \Leftrightarrow \delta \int_0^t L \ \mathrm{d}t = 0$$

本文采用 *Einstein* 求和约定.

$$\Rightarrow$$

考虑一处于平衡、有约束的体系. 考虑某时刻 t 系统坐标一个微小的, 与运动方程和约束条件均兼容的虚拟位移 (虚位移). 记作用于质点 i 上的力为 F . 由于每一质点处于平衡, 有 $\boldsymbol{F}_i = 0$. 因此

$$\boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0$$

现将作用于质点 i 上的力 $\boldsymbol{F}_i$ 分为两部分: $\boldsymbol{F}_i = \boldsymbol{F}_i^a + \boldsymbol{F}_i^c$, 其中 $\boldsymbol{F}_i^a$ 称主动力, 是除了由约束引起的所有力. $\boldsymbol{F}_i^c$ 称约束力, 假设其恒满足 $\boldsymbol{F}_i^c \cdot \delta \boldsymbol{r}_i = 0$. 于是

$$\boldsymbol{F}_i^a \cdot \delta \boldsymbol{r}_i = 0$$

a.k.a. 虚功原理.

系统并不处于力学平衡, 可以将 $\boldsymbol{F}_i = 0$ 替换为 $(\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i \cdot \delta \boldsymbol{r}_i) \cdot = 0$, 于是虚功原理为

$$(\boldsymbol{F}_i^a - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = 0$$

a.k.a. *d'Alembert* 原理. 现考虑完整约束的系统.

$$\delta \boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

于是主动力的虚功为

$$\boldsymbol{F}_i^a \cdot \delta \boldsymbol{r}_i = \boldsymbol{F}_i^a \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = Q_j \delta q_j$$

d'Alembert 原理中另一项

$$\dot{\boldsymbol{p}}_i \cdot \delta \boldsymbol{r}_i = m_i \ddot{r}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j = \frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \dot{r}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) - m_i \dot{r}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right)$$

另一方面,

$$\boldsymbol{v}_i = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}$$

从而

$$\frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

我们可以重新表述 *d'Alembert* 原理:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0$$

a.k.a. *Euler-Lagrange* 方程.
对于完整约束, 各 q_j 独立, 于是 *d'Alembert* 原理要求

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

若主动力由仅由坐标决定的势能给出

$$\boldsymbol{F}_i = -\frac{\partial V}{\partial \boldsymbol{r}_i}$$

则广义力可写为

$$Q_j = -\frac{\partial V}{\partial q_j}$$

定义系统的 *Lagrangian*

$$L = T - V$$

则 *Euler-Lagrange* 方程可以写为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

记

$$q(t) = (q_1(t),\ q_2(t),\ ...,\ q_s(t))$$

再来考虑 *d'Alembert* 原理

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\delta q = 0$$

系统的真实运动要求

$$\delta q(t) = \delta q(0) = 0, \quad \delta t = 0$$

故

$$\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\delta q\ \mathrm{d}t + \left[\left(\frac{\partial L}{\partial \dot{q}}\ \delta q\right)\right]_0^t = 0$$

$$\Rightarrow \int_0^t \delta L\left(q,\ \dot{q},\ t\right)\ \mathrm{d}t$$

i.e.

$$\delta \int_0^t L\left(q,\ \dot{q},\ t\right)\ \mathrm{d}t = 0$$

$$\Leftarrow$$

考虑前述系统的 *Lagrangian*

$$L\left(q,\ \dot{q},\ t\right)$$

系统的真实运动要求

$$\delta q(t) = \delta q(0) = 0, \quad \delta t = 0$$

最小作用量原理给出

$$\begin{aligned}
0 &= \delta S \\
&= \delta \int_0^t L\left(q,\ \dot{q},\ t\right)\ \mathrm{d}t \\
&= \int_0^t \delta L\left(q,\ \dot{q},\ t\right)\ \mathrm{d}t \\
&= \int_0^t \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right)\ \mathrm{d}t \\
&= \int_0^t \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)\delta q\ \mathrm{d}t + \left[\left(\frac{\partial L}{\partial \dot{q}}\ \delta q\right)\right]_0^t = 0 \\
&= \int_0^t \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)\delta q\ \mathrm{d}t
\end{aligned}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

代入 $L = T - V$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

$$\Rightarrow m_i \ddot{q}_i = Q_i$$

写成矢量形式

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$$