比耐方程

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考虑一质点在有心势场 $V(r)$ 中运动, 它的 *Lagrangian*

$$L(r,\dot{r},\theta)=\frac{\mu}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r) \tag{1}$$

式中各量意义显然. 循环坐标 θ 对应的广义动量守恒:

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=\mu r^2\dot{\theta}=J \tag{2}$$

i.e, 熟知的角动量. 又 *Lagrangian* 不显含时间, 有能量守恒

$$E=\frac{\mu}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)+V(r) \tag{3}$$

下面开始计算:

$$\dot{r}=\dot{\theta}\frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{4}$$

代入 (4),

$$\begin{aligned}\frac{2\left(E-V(r)\right)}{\mu}&=\dot{\theta}^2\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2+r^2\right]\\&=\frac{h^2}{r^4}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2+r^2\right]\\&=h^2\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+u^2\right]\end{aligned} \tag{5}$$

其中 $h:=r^2\dot{\theta}$、$u=\dfrac{1}{r}$. 两端对 r 求导:

$$\begin{aligned}h^2\left(2\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}u}{\mathrm{d}\theta}+2u\frac{\mathrm{d}u}{\mathrm{d}r}\right)&=\frac{2F(\frac{1}{\mu})}{\mu}\\h^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\frac{\mathrm{d}u}{\mathrm{d}r}\right)&=\frac{F(\frac{1}{\mu})}{\mu}\\h^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)&=-\frac{F(\frac{1}{\mu})}{\mu}\end{aligned} \tag{6}$$

这就是我们想要的结果.