Atiyah-Patodi-Singer index theorem for physicists

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work in progress

Atiyah-Singer index theorem [1968]

$$n_+ - n_- = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}[F^{\mu\nu} F^{\rho\sigma}]$$

Atiyah-Singer index theorem [1968]

is easily understood by Fujikawa-method [1979]:

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{D_{4\mathrm{D}}^2/\Lambda^2}$$

inserting plane wave complete set,

$$\left(= \lim_{\Lambda\to\infty}\int d^4x \int d^4k\; e^{-ikx}\mathrm{tr}\gamma_5 e^{D_{4\mathrm{D}}^2/\Lambda^2} e^{ikx}\right)$$

$$\{D_{4\mathrm{D}}, \gamma_5\} = 0$$

and 1-loop level perturbation,

$$n_+ - n_- = \frac{1}{32\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]$$

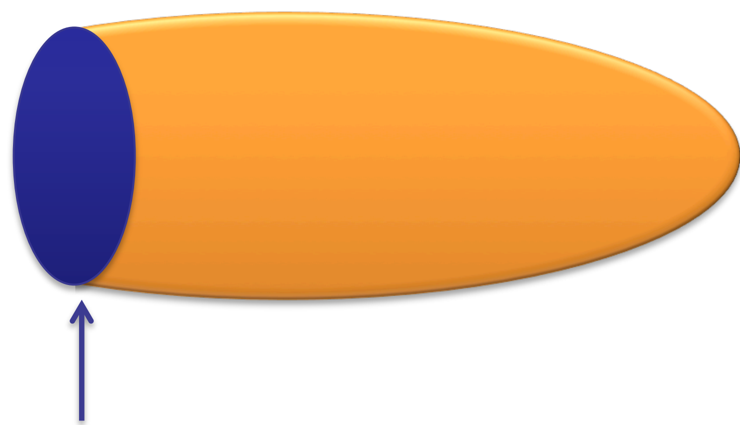
Atiyah-Patodi-Singer index theorem

On a manifold with boundary,

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{D^2_{\mathrm{4D}}/\Lambda^2} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

integer　　　non-integer　　　non-integer

$$\eta(iD^{\mathrm{3D}}) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg} = \sum_{\lambda}^{reg}\mathrm{sgn}\lambda$$



$x_4 = 0$

Atiyah-Patodi-Singer index theorem

OSAKA UNIVERSITY
Live Locally, Grow Globally

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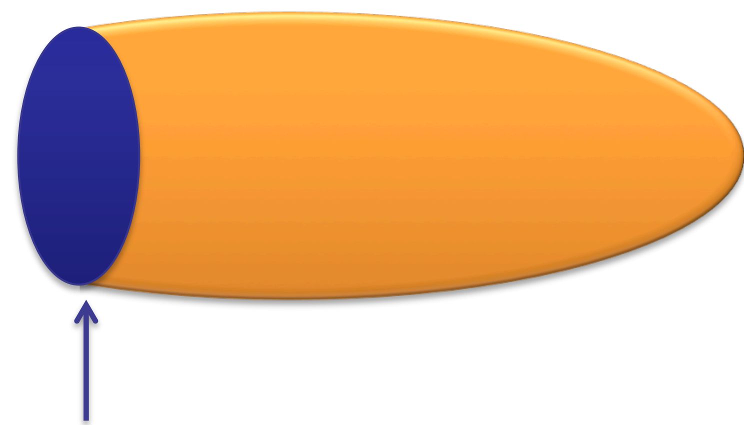
$$\eta(iD^{\rm 3D}) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg} = \sum_{\lambda}^{reg}\mathrm{sgn}\lambda$$

[Atiyah-Patodi-Singer 1975]

The index is obtained for $D_{\rm 4D}$ with APS boundary condition.



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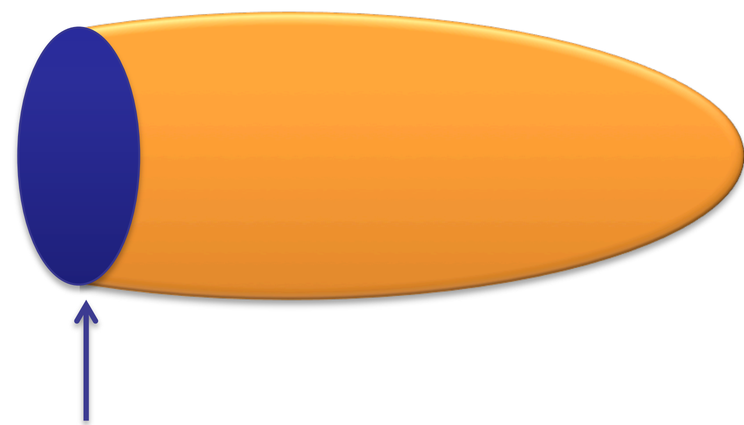
[Atiyah-Patodi-Singer 1975]
The index is obtained for $D_{\rm 4D}$ with APS boundary condition.
But Fujikawa-san didn't do this [little motivation at that time?].



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Atiyah-Patodi-Singer index theorem for topological insulator

Witten 2015 : APS index is the key to understand bulk-edge correspondence in symmetry protected topological insulator:

$$Z_{\text{edge}} \propto \exp(-i\pi\eta(iD^{3\text{D}})/2)$$

T-anomalous

$$Z_{\text{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\text{tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

T-anomalous

$$Z_{\text{edge}}Z_{\text{bulk}} \propto (-1)^{\mathfrak{J}} \Rightarrow$$

T is protected !

$$\mathfrak{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\text{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\text{D}})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16, Freed-Hopkins 16, Witten 16, Yonekura 16]

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$\rightarrow$ We launched a study group reading original APS paper and it took 3 months to translate it into "physics language".

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1. Fujikawa-method is great enough to describe the index theorem with boundary.
2. Original APS boundary condition is mathematically O.K. but unphysical (= unlikely to be realized in nature).
3. Another index can be defined for domain-wall fermion Dirac operator (massive Dirac with local boundary condition) [Callan-Harvey 1985, Kaplan 1992 (Suzuki-san's talk)], which coincides with the APS index.

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✔ 1. Introduction

2. Atiyah-Patodi-Singer index theorem (review)

3. Index from domain-wall Dirac operator

4. Summary and discussion

Dirac operator w/ APS b.c.

We consider a 4D(Euclidean) massless Dirac operator for $x_4 > 0$ (results unchanged for any even-dim.)

$$D^{4\mathrm{D}} = \gamma_4(\partial_4 + H), \quad H = \gamma_4\gamma_i D^i$$

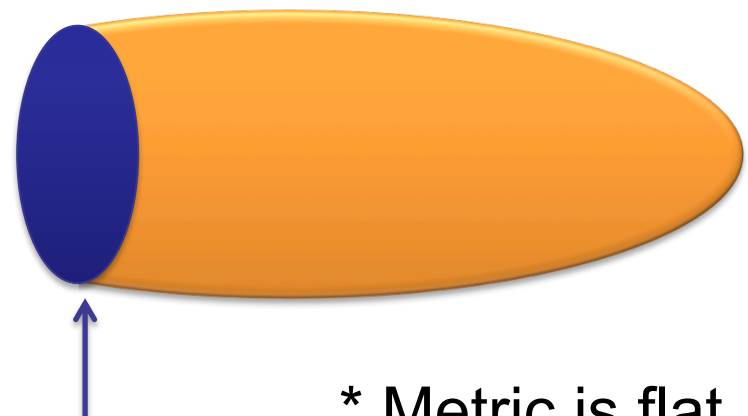
$$A_4 = 0 \ \ \text{gauge}$$

* Gauge group is U(1) or SU(N).

APS boundary condition:

$$\left(\frac{H+|H|}{2}\right)\psi|_{x_4=0} = 0,$$

which is non-local.



$$x_4 = 0$$

* Metric is flat.

Recipe by Fujikawa

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In our case, 2 is non-trivial since the complete set needs to respect APS b.c.

$$\phi_+(x_4)|_{x_4=0} = 0, \quad (\partial_4 - \lambda)\phi_-(x_4)|_{x_4=0} = 0, \quad \text{for } \lambda \geq 0,$$
$$\phi_-(x_4)|_{x_4=0} = 0, \quad (\partial_4 + \lambda)\phi_+(x_4)|_{x_4=0} = 0, \quad \text{for } \lambda < 0.$$

$$\gamma_5\phi_\pm = \pm\phi_\pm, \qquad \lambda : \text{eigenvalue of } iD_{3\mathrm{D}} = i\sigma_i D^i.$$

Complete set = plane waves

At LO of adiabatic (small $\partial_{x_4} D^{\mathrm{3D}}$) expansion,

$$\psi(x) = \sum_{\pm}\sum_{\omega}\sum_{\lambda} \alpha^{\pm}_{\omega,\lambda} \phi^{\omega}_{\pm}(x_4)\phi^{\mathrm{3D}}_{\lambda}(\vec{x})$$

$$iD^{\mathrm{3D}}\phi^{\mathrm{3D}}_{\lambda}(\vec{x}) = \lambda\phi^{\mathrm{3D}}_{\lambda}(\vec{x})$$

$$(-\partial^2_{x_4} + \lambda^2)\phi^{\omega}_{\pm} = \lambda^2_{\mathrm{4D}}\phi^{\omega}_{\pm}$$

$$\omega = \sqrt{\lambda^2_{\mathrm{4D}} - \lambda^2}$$

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$$\phi^{\omega}_{+}(x_4) = \frac{1}{\sqrt{2\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

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Surprisingly, NO edge mode appears.

Compute index.

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{-D^\dagger_{\mathrm{4D}} D_{\mathrm{4D}}/\Lambda^2}|_{APSb.c.}$$ ⟵ Inserting the complete set

$$= \lim_{\Lambda\to\infty}\sum_\lambda \int dx_4 \mathrm{sgn}\lambda e^{-\lambda^2/\Lambda^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(-1+\frac{2i|\lambda|}{\omega+i|\lambda|}\right)e^{-\omega^2/\Lambda^2+2i\omega x_4}$$

$$= \sum_\lambda \mathrm{sgn}\lambda \int_0^\infty dx_4 \frac{\partial}{\partial x_4}\left[\frac{1}{2}e^{2|\lambda|x_4}\mathrm{erfc}\left(x_4\Lambda + |\lambda|/\Lambda\right)\right]$$

$$= \sum_\lambda \frac{\mathrm{sgn}\lambda}{2}\lim_{\Lambda\to\infty}\mathrm{erfc}\left(|\lambda|/\Lambda\right)$$

$$= \frac{\eta(iD^{\mathrm{3D}})}{2} \quad \left(+\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

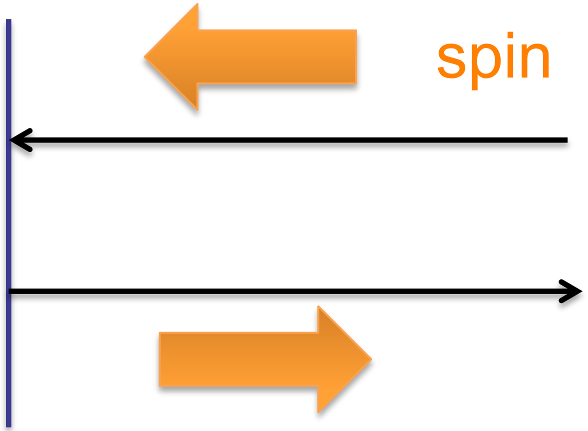
↑ NLO in adiabatic expansion
[Alvarez-Gaume et al. 1985]

Mathematically no problem but…

Unnatural boundary condition put by hand.

APS b.c. does not preserve angular momentum but keeps helicity to keep it massless.

When the fermion reflects at the boundary,



$x_4 = 0$

spin flips.

What can (non-locally) flip the spin in physics?

Only God (= fine-tuning) can do it.

In Mathematics, we can put ANY boundary condition by hand to 1st order differential equations.

But in physics, this is not true: since

$$\partial_x \phi(x) \to \frac{\phi(x+a) - \phi(x)}{a},$$

only “natural” boundary conditions survive the continuum limit, otherwise, fine-tuning is needed [Luescher 2006].

More details

The original APS condition would be contaminated by quantum correction,

$$\left(\frac{H+|H|}{2}+\frac{c}{a}\right)\psi|_{x_4=0}=0,$$

APS condition correction

which end up with Dirichlet b.c.

in the continuum limit $a \to 0$

unless we fine-tune c to vanish.

Another complaint

Physical meaning of boundary effect is unclear.

There exists no edge-localized mode under APS boundary condition.

Non-trivial bulk plane-wave integral gives $\dfrac{\eta(iD^{\mathrm{3D}})}{2}$

This is physically very different from Witten's scenario in topological materials.

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   Mathematically no problem but unphysical.
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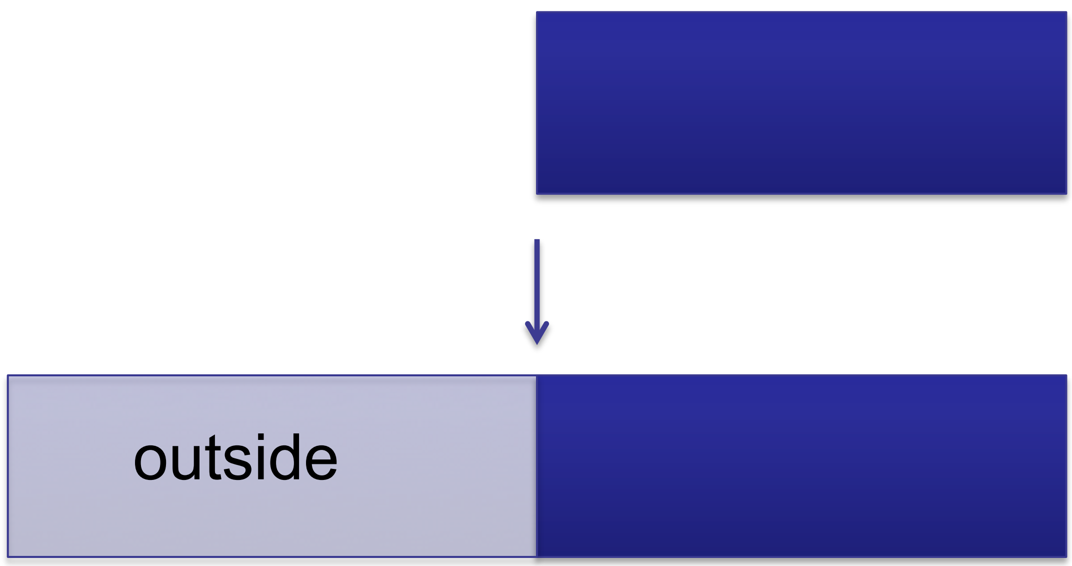

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→ Lets consider domain-wall fermion operator:

$$D_{4\mathrm{D}} + M\epsilon(x_4), \quad \epsilon(x_4) = \mathrm{sgn} x_4$$

Callan-Harvery 1985,
Kaplan 1992

Our goal : we will show

domain-wall fermion determinant with a PV field defines an index

$$\det \frac{D_{\rm 4D} + M\epsilon(x_4)}{D_{\rm 4D} - M} \qquad \epsilon(x_4) = \mathrm{sgn} x_4$$

$$\propto (-1)^{\mathfrak{I}}$$

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* $\gamma 5$ Hermiticity:

$$= \det \frac{\gamma_5 (D_{\mathrm{4D}} + M\epsilon(x_4))\gamma_5}{\gamma_5 (D_{\mathrm{4D}} - M)\gamma_5} = \left(\det \frac{D_{\mathrm{4D}} + M\epsilon(x_4)}{D_{\mathrm{4D}} - M} \right)^*$$

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which coincides with the APS index,

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

Two remarks

1. $$\det \frac{D_{4\mathrm{D}} + M\epsilon(x_4)}{D_{4\mathrm{D}} - M} \to 1 \ (\text{for } x_4 < 0)$$

$$\to \det \frac{D_{4\mathrm{D}} + M}{D_{4\mathrm{D}} - M} \ (\text{for } x_4 > 0)$$

($\theta = \pi$ physics.)

is a good model for topological insulator.

2. Dimension is different from Suzuki-san's talk:

$$\det \frac{D_{5\mathrm{D}} + M\epsilon(x_5)}{D_{5\mathrm{D}} - M}$$

Index in terms of eta-invariants

First, let's rewrite

$$\det\frac{D_{4\mathrm{D}}+M\epsilon(x_4)}{D_{4\mathrm{D}}-M}=\det\frac{i\gamma_5(D_{4\mathrm{D}}+M\epsilon(x_4))}{i\gamma_5(D_{4\mathrm{D}}-M)}=\det\frac{iH_{DW}}{iH_{PV}}=\frac{\prod i\lambda_{DW}}{\prod i\lambda_{PV}}$$

$$\propto\exp\left[i\pi\left(\sum_{\lambda_{DW}}\frac{\mathrm{sgn}\lambda_{DW}}{2}-\sum_{\lambda_{DW}}\frac{\mathrm{sgn}\lambda_{PV}}{2}\right)\right]$$

where

$$H_{DW}=\gamma_5(D_{4\mathrm{D}}+M\epsilon(x_4))$$

$$H_{PV}=\gamma_5(D_{4\mathrm{D}}-M)$$

Therefore, we have

$$\mathfrak{I}=\frac{\eta(H_{DW})}{2}-\frac{\eta(H_{PV})}{2}. \qquad \eta(H)\equiv\sum_\lambda\mathrm{sgn}\lambda$$

PV part = Atiyah-Singer index

$$\eta(H_{PV}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{PV}}{(\sqrt{H_{PV}^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H_{PV} e^{-tH_{PV}^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(-1+\frac{D}{M}\right) e^{-t'D^\dagger D/M^2} e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

$$= -\frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

Note: Atiyah-Singer index can be defined by massive Dirac operator,

$$H_{PV} = \gamma_5(D_{4\mathrm{D}} - M)$$

Domain-wall fermion part

Now let's compute

$$\eta(H_{DW}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{DW}}{(\sqrt{H_{PV}^2})^{1+s}} = \lim_{s\to 0}\frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt t^{(s-1)/2}\mathrm{Tr} H_{DW} e^{-tH_{DW}^2}$$

$$H_{DW} = \gamma_5(D_{\mathrm{4D}} + M\epsilon(x_4))$$

In the free fermion case,

$$H_{DW}^2 = -\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4).$$

→ eigenvalue problem = Schrodinger equation with δ-function-like potential.

Complete set in the free case

Solutions to $(-\partial_{x_4}^2 + \omega^2 - 2M\gamma_4\delta(x_4))\varphi = 0$ are

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\mathrm{edge}}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$

→ Edge mode appears !

where $\omega = \sqrt{\boldsymbol{p}^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,\mathrm{edge}}_{\pm,e/o} = \pm\varphi^{\omega,\mathrm{edge}}_{\pm,e/o}$

3D direction = conventional plane waves.

"Automatic" boundary condition

We didn't put any boundary condition by hand. But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\bigg|_{x_4=0} = 0, \qquad \varphi^{\omega}_{\pm,o}(x_4=0)=0.$$

is automatically satisfied due to the δ-function-like potential.

This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep helicity.

Fujikawa-method

$$\eta(H_{DW}) = \frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt' t'^{\frac{s-1}{2}} \mathrm{Tr}\gamma_5 \left(\epsilon(x_4) + \frac{D}{M}\right) e^{-t' H_{DW}^2/M^2} e^{-t'},$$

Perturbative expansion

We insert our complete set $\{\varphi^{\omega,\mathrm{edge}}_{\pm,e/o}(x_4) \times e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\}$

(Secret computation until we submit a paper.)

100% edge-mode effect

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{3\mathrm{D}})$$

$$\parallel$$

$(CS \bmod \text{integer})$

$\epsilon(x_4) = \mathrm{sgn} x_4$

Total index

$$\mathfrak{I} = \frac{\eta(H_{DW}))}{2} - \frac{\eta(H_{PV})}{2}$$

$$= \frac{1}{2}\left[\frac{1}{32\pi^2}\int d^4x\ \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{\mathrm{3D}})\right.$$

$$\left.+\frac{1}{32\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}\right]$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})$$

which coincides with APS index !

Physical meaning of 1/2

$$\eta(H_{DW}) = \frac{1}{32\pi^2}\int d^4x\ \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{\mathrm{3D}})$$

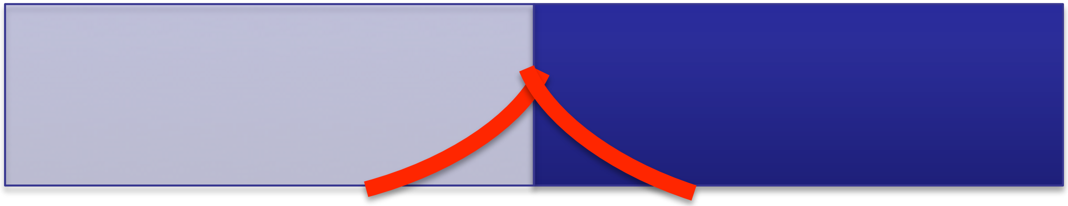
$$= -\left[\int d^4x_{x_4<0}\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \frac{1}{2}\eta(iD^{\mathrm{3D}})\right]$$

$$+\left[\int d^4x_{x_4>0}\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})\right]$$

The edge mode contributes its half to topological phase and another half to normal phase (and PV regulator determines which is which).

$$x_4 = 0$$

$$\varphi^{\mathrm{edge}}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$



We also confirm

$$\mathfrak{J} = \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}.$$

is independent of mass,
$$\frac{\partial \mathfrak{J}}{\partial M} = 0.$$

and stable against change of gauge fields
$$\frac{\delta \mathfrak{J}}{\delta A_\mu(x)} = 0.$$

Contents

✔ 1. Introduction

✔ 2. Atiyah-Patodi-Singer index theorem (review)

Mathematically no problem but unphysical.

✔ 3. Index from domain-wall Dirac operator

More realistic set-up, edge modes give eta-invariant, meaning of ½ is clear, coincides with APS index.

4. Summary and discussion

Summary

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1. Fujikawa-method is great enough to describe the index theorem with boundary.

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Summary

1. Fujikawa-method is great enough to describe the index theorem with boundary.
2. Original APS is mathematically O.K. but unphysical (no “outside”, massless, non-local b.c. by hand, breaking rotational sym., no edge mode, no physical meaning of ½)
3. The index can be defined by Domain-wall fermion Dirac operator (having “outside”, massive, local b.c. automatically given, edge mode explain eta-inv. and 1/2), which coincides with the APS index.

What's next ?

1. Generalization to odd dimensions
2. Generalization to Shamir-type domain-wall fermions
3. Generalization to other domain-walls,
4. Non-perturbative formulation of APS index theorem on a lattice
5. Application to 6D formulation of lattice chiral gauge theory

[F-Onogi-Yamamoto-Yamamura 2016]

Back-up slides

Atiyah-Singer index from massive Dirac operator (details)

$$H = \gamma_5 (D_{\rm 4D} + M)$$

anti-commutes with $D_{\rm 4D}$

$\rightarrow$ any eigenmode ϕ_λ and $D_{\rm 4D}\phi_\lambda$

make a pair with eigenvalues $\pm\lambda$

unless $D_{\rm 4D}\phi_\lambda = 0.$

For zero-modes of $D_{\rm 4D}$, $H = \pm M$

for $\pm$ chirality. $\Rightarrow$ $\mathfrak{I} = \eta(H).$

Massless lattice Dirac operator naturally includes eta-invariant.

Massless overlap Dirac operator[Neuberger 98]

$$D^{ov} = \frac{1}{a}\left[1+\gamma_5\frac{H_W}{\sqrt{H_W^2}}\right], \quad H_W = \gamma_5(D_W - 1/a)$$

has an index, which end up with eta-inv. of massive Wilson Dirac operator:

$$\mathrm{Tr}\gamma_5\left(1-\frac{D_{ov}a}{2}\right) = -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(H_W),$$

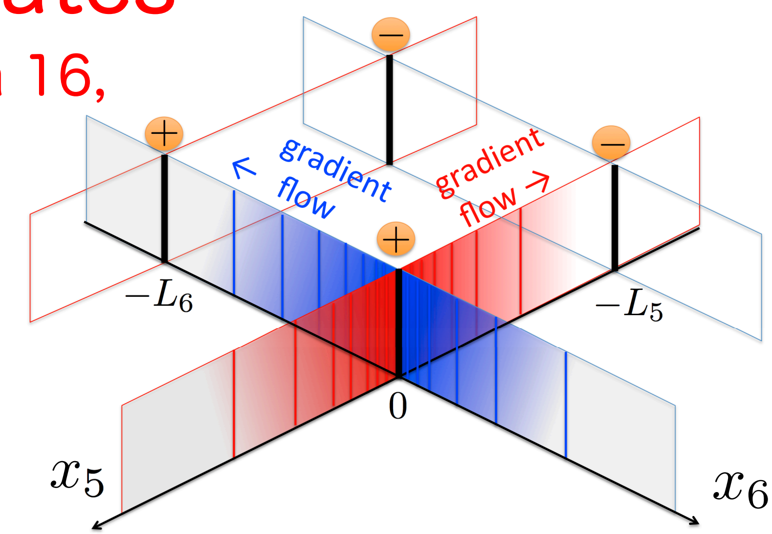
Domain-wall is more interesting than simple boundary.

Homology : boundary of boundary = empty.

Junction of domain-walls are non-trivial if they carry different quantum numbers

→ edge of edge states

[F-Onogi-Yamamoto-Yamamura 16, Hashimoto-Wu-Kimura 16]



Why APS boundary needed?

Because Anti-Hermiticity of $D^{\rm 4D}$ should be unchanged.

$$0 = \int d^4x \phi_2^\dagger(x) D^{\rm 4D} \phi_1(x) + \int d^4x (D^{\rm 4D}\phi_2(x))^\dagger \phi_1(x)$$

$$= \int_{x_4=0} d^3x \phi_2^\dagger(x) \gamma_4 \phi_1(x) \qquad D^{\rm 4D} = \gamma_4(\partial_4 + H), \quad H = \gamma_4 \gamma_i D^i$$

This holds for APS boundary condition since

$$H\gamma_4 \phi_i|_{x_4=0} = -\gamma_4 H \phi_i|_{x_4=0}$$