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Giuseppe Morandi Pasquale Sodano
Arturo Tagliacozzo Valerio Tognetti
(Eds.)

Field Theories for Low-Dimensional Condensed Matter Systems

Spin Systems
and Strongly Correlated Electrons

With 55 Figures



Springer

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Preface

This book is especially addressed to young researchers in theoretical physics with a basic background in Field Theory and Condensed Matter Physics. The topics were chosen so as to offer the largest possible overlap between the two expertises, selecting a few key problems in Condensed Matter Theory which have been recently revisited within a field-theoretic approach.

The presentation of the material is aimed not only at providing the reader with an overview of this exciting frontier area of modern theoretical physics, but also at elucidating most of the tools needed for a technical comprehension of the many papers appearing in current issues of physics journals and, hopefully, to enable the reader to tackle research problems in this area of physics. This makes the material a live creature: while not pretending it to be exhaustive, it is tutorial enough to be useful to young researchers as a starting point in anyone of the topics covered in the book.

Each chapter in this book has its own standpoint as guideline and introduces the reader to the different techniques used in dealing with the many theoretical aspects of today's Condensed Matter Physics. The material of each chapter is rather self-contained, except for Chap. 3, where it is required some familiarity with Conformal Field Theory. An introduction to this subject, as well as to most of the material presented in this book, may be found in the books quoted in the Introduction. Chapter 6 is mostly sketchy and well reflects the outburst of creative activity in the young field of Mesoscopic Physics.

After a short introduction, the topics are organized into six chapters, each inspired by the mini-courses delivered at the "Workshop with Learning" on "*Field Theories for Low-Dimensional Condensed Matter Systems: Spin Systems and Strongly Correlated Electrons*" held in Chia - Laguna (Italy) in the fall of 1997. Lecturers were (in the order of the chapters appearing in the book) H.J. Schulz, R.B. Laughlin, I. Affleck, A. Auerbach, G. Semenoff and B.L. Altshuler. This stimulating workshop was promoted by the Laboratory "Forum of Condensed Matter Theory" of the Italian Istituto Nazionale di Fisica della Materia (I.N.F.M.) and greatly benefited from the close interplay with the Fifth Chia Meeting on "*Common Trends in Condensed Matter and High Energy Physics*" organized by Profs. A. Barone of the University of Naples and A. Devoto of the University of Cagliari.

This book would not be out without the moral support of many friends and the financial support of the Laboratory “Forum of Condensed Matter Theory” of the Istituto Nazionale di Fisica della Materia (I.N.F.M.), of the Gruppo Nazionale di Struttura della Materia (G.N.S.M.) and the Istituto Nazionale di Fisica Nucleare (I.N.F.N.).

We greatly thank all the lecturers, senior researchers and students who attended the Chia Workshop; their enthusiasm, interest and skills were determining at all the stages of this enterprise; in particular, we are grateful to R.B. Laughlin who advised us in the early stages of the project and strongly encouraged us to converge towards the final form of this book. We warmly thank A. Devoto for his invaluable help and, most importantly, for keeping alive in Chia - Laguna the charming tradition of periodical international meetings on the interplay between Field Theory and Condensed Matter Physics; we feel that, without this long standing tradition, this project for a book could not have taken its shape.

Special thanks are due to Giovanna Savoldi of I.N.F.M. (Unità di Genova) and Isabella Panico of I.N.F.N. (Sezione di Perugia), who collaborated with great skill and enthusiasm in gathering the first materials for the book. We thank also Guido Celentano who greatly helped in preparing the final version of the book.

Most sadly, shortly after completing his contribution, Prof. H.J. Schulz passed away. We mourn him as a great person and an outstanding scientist and we feel that his death is an invaluable loss for the whole scientific community. We like to dedicate this volume to his memory.

Naples, March 2000

*Giuseppe Morandi
Pasquale Sodano
Arturo Tagliacozzo
Valerio Tognetti*

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1 Introduction

Giuseppe Morandi, Pasquale Sodano, Arturo Tagliacozzo,
and Valerio Tognetti

The interplay between Quantum Field Theory (QFT) and Condensed Matter Physics (CMP) is not at all surprising since condensed matter systems usually involve a large number of degrees of freedom interacting among themselves. In fact, the description of a condensed-matter system in terms of interacting quantum fields dates back at least to the late Fifties [1]. However, and until quite recently, the use of field-theoretic techniques has not gone much beyond the framework of perturbative expansions, mean-field approaches and/or, at most, selective resummation of some classes of Feynman diagrams.

The last two decades witnessed two important developments: the approach to critical phenomena in CMP by means of the Renormalization Group (RG) [2] and the progress of non-perturbative methods in QFT [3]. Field theoretical investigations of low dimensional condensed matter systems [4] evidenced that non-perturbative methods such as Conformal Field Theory [5] and Topological Chern–Simons Field Theories [6] may turn out very useful for the description of real systems.

The discovery of the quantization of the Hall conductance in heterostructures (QHE) [7] and the ability to produce samples displaying properties of low dimensional quantum antiferromagnets (QAFM) set the stage for the appreciation of interesting relationships between condensed matter systems and lower dimensional gauge theories. For example, by studying spin chains, spin ladders and planar spin systems, an intuitive and non-perturbative picture can be gained of the ground state of a corresponding lattice gauge theory. Finally, recent advances in mesoscopic physics [8], especially in connection with analyses of the quantum transport in quantum wires and dots and of the conductance in small size systems, provide an ideal testing ground for Random Matrix Theory (RMT) [9], which has also been successfully used in the study of chiral symmetry breaking in Quantum Chromodynamics and Quantum Gravity in two dimensions [10].

There is a wealth of classical and quantum models for which exact solutions are available, which are essentially one-dimensional (1d) [11]. Relevant to the subject matter of this book are quantum models in 1d such as the Tomonaga–Luttinger model, the $S = 1/2$ AFM Heisenberg chain, the 1d Hubbard model. These models are exactly solvable within the Bethe Ansatz method [12], which provides exact results for the energy spectrum and some thermodynamic quantities. A similar approach applies to the Haldane–

Shastry model, that proves very useful for the investigation of spinon excitations relevant both for the description of the fractional QHE (see Chap. 2) and for the low energy sector of some $1 + 1d$ lattice gauge theories (see Chap. 5). However, the very fact that an exact solution is attainable and that all these models can be variously connected among themselves [11] supports the idea that they constitute a very special class of statistical models; only in some instances one may hope to extend these techniques to higher space dimensions. In addition, the bosonization method [13] allows for a transparent description of the low energy properties like the long distance behaviour of the correlation functions and paves the way to the analysis of non Fermi-Liquid (Luttinger-Liquid-type in particular) behaviours and to the study of spin-charge separation in one-dimensional itinerant AFM models. The Bethe Ansatz and Bosonization methods, although not explained in detail in this book, are recurrent in the first three chapters. For a review of these methods the interested reader is referred to [12,13].

The excitement for the study of low-dimensional quantum systems in CMP was triggered by the recent discovery of the QHE and of high- T_c superconductors (HTS). In most cases these phenomena are characterized by strong electron correlations, which prevent a naive use of perturbation techniques; furthermore, the low dimensionality enhances the role of quantum fluctuations. Theoretical studies of strongly correlated electron systems greatly benefited from modern advances in non-perturbative QTF in lower space-time dimensions. Infrared divergences in perturbation theory make its breakdown in low dimensions almost unavoidable. The Tomonaga–Luttinger model is the remarkable example of a QFT, in which a complete summation of diagrams in perturbation theory can be achieved [14]. However, it is well known that this is not feasible in general and, usually, one has to resort only to a partial sum of the perturbative expansion to get rid of the ubiquitous logarithmic singularities in the vertex corrections. A remarkable example of the latter situation is provided by the Kondo problem, which can be regarded as an effectively one-dimensional problem, due to the linear dispersion of band electrons screening the magnetic impurity.

Both in interacting electron systems, as well as in spin systems, the use of an effective action describing the low energy sector of the theory allows for the recognition of the role of quantum fluctuations in destroying ordered phases in low dimensions (see Chap. 4). Recently [2], the RG approach has been applied to the action for interacting electrons. In fermionic systems, the scaling transformation was generated by integrating out the modes of the system high in energy with respect to the Fermi surface to give rise to an effective action for the low-energy dynamics. A milestone in this procedure has been the recognition that, in 3-space dimensions, the Fermi Liquid picture is the fixed point to which well behaved systems universally flow. The superconducting ground state in metals may be cast in the same formalism; the Meissner effect immediately follows from the resulting Abelian gauge theory

with minimal coupling to the e-m fields and its appearance is a manifestation of the Anderson–Higgs–Kibble mechanism for symmetry breaking of gauge theories.

The consequences of non-perturbative approaches to interacting fermion systems in two space dimensions is still widely unexplored [15]. For the Kondo problem a RG analysis allows to go beyond perturbation theory towards the strong coupling fixed point, in which the impurity spin is fully screened and the system exhibits local Fermi Liquid behaviour. The scaling is, however, unable to describe the properties of the strongly correlated phase [16]. Close to the strong coupling fixed point, a new non-perturbative method based on conformal field theory with boundaries can be successfully adopted (see Chap. 3). Conformal symmetry then allows one to compute the QFT correlation functions non-perturbatively (i.e., without explicitly solving the interacting problem). When supplemented by bosonization, the method has proved to be most useful in the analysis of the Multichannel Kondo problem.

To compute effective actions for spin and strongly correlated electron systems, functional methods - mainly (but not only) in the coherent state basis [17] - reveal themselves very useful. A variety of saddle points can be tested to characterize different phases in which spin systems may exist (see Chap. 4). If a semiclassical approach is justified most of these Hamiltonians can be mapped onto a Nonlinear Sigma Model ($\text{NL}\sigma\text{M}$). In $2 + 1d$ the RG analysis of the $\text{NL}\sigma\text{M}$ leads to the understanding of its phase diagram: the ground state is disordered or ordered, depending on the relative magnitude of the parameters of the Hamiltonian and, in analogy with QFT models of particle physics [18], the switching from an ordered to a disordered phase is signaled by the dynamical generation of a mass for some relevant branch of the excitation spectrum. A definitely non-perturbative result, anticipated by D.F.M. Haldane in the early Eighties, namely the presence of a topological term in the effective action for half-odd-integer quantum spin chains, has proved to have far-reaching consequences in the physics of one-dimensional QAFM's and spin ladders, as it implies that half-odd-integer-spin chains (as well as similar spin ladders with an odd number of legs) have a quantum-critical ground state with algebraic decay of correlations while integer-spin chains and ladders (or half-odd-integer ladders with an even number of legs) are in a quantum disordered but strongly correlated phase known as the Spin Liquid. The Haldane's argument by now has been convincingly tested experimentally and provides us with a beautiful example of the fact that, in some instances, a topological term in the effective action plays a leading role in understanding the physics of a lower dimensional system. In other circumstances, gauge interactions of non electromagnetic origin [19] among non relativistic electrons can be generated effectively in strongly correlated metals, leading to non-Fermi Liquid behaviour. Resonant Valence Bond states or half filled Landau levels in the fractional QHE may be approached this

way and the resulting picture of composite fermions [20] is now obtaining experimental support, thus becoming as real as the quasiparticle concept.

There is an interesting relation between lattice gauge theories [21] in the limit of strong coupling and QAFM, which is valid in any space dimension (see Chap. 5). Using the Hamiltonian approach to lattice gauge theories with staggered fermions, one is able to evidence that - at strong coupling - both the mass spectrum and the order parameter for chiral symmetry breaking are known once the spin correlators of the QAFM are given. For example, the QAFM chain with spin 1/2 is related to both the strongly-coupled lattice two-flavor Schwinger model and to the two-flavor QCD in 1 + 1 dimensions in the limit of large number of colors: the spectrum of these gauge theories is determined from the knowledge of the spin correlation functions of the QAFM chain and the destruction of Neel order in QAFM amounts to chiral symmetry breaking. Thus, there is an example in which exact results of a statistical model allow for detailed descriptions of the low-lying excitation spectrum of a gauge model. The relationship between gauge theories and QAFM may survive also in the weak coupling limit: an interesting proposal in this direction has been made recently using a RG method [22]. Also at finite temperatures, gauge theories may be usefully mapped onto spin systems. In fact, confinement-deconfinement transitions in finite-temperature gauge theories can be regarded as an order-disorder transitions for a pertinent spin model. This is obtained explicitly from the computation of the effective action of the Polyakov loop operators [23].

Disordered materials also provide a remarkable testing ground for non-perturbative approaches. Besides their importance for technical applications, they manifest new physical properties which are absent in clean crystalline materials. Diagrammatic perturbative expansions, although very powerful and useful in the study of weak localization [24] provide accurate results only when the quantum effects are weak. Disordered systems are best treated by the so-called replica method [25] or by the supersymmetry method [26]. Using these approaches, one can reduce the problem of localization (weak or strong) to matrix NL σ M's. In the case of a small disordered metal particle the 0d NL σ M can be solved and the level-level correlation function found. Quite remarkably, this correlation function turns out to coincide with the corresponding function emerging from RMT approaches to complex nuclei and quantum chaos [27,9]. In general, a relation exists between the universal properties of the two-point correlation function of the eigenvalues of large matrices and universal conductance fluctuations in disordered conductors [8,10].

RMT is a remarkable example of the concept of universality introduced in the RG approach to Statistical Physics. In RMT the statistical properties of the spectrum of a complex system are characterized by using random matrices: i.e., the Hamiltonian describing the system is modeled as a large matrix with random elements. The matrices are defined so as to possess symmetries and constraints corresponding to the ones of the Hamiltonian.

Small metallic grains down to the limit of nanosize and quantum dots offer a unique testing ground for the statistical study of the energy spectra [28] (see also Chap. 6). The sample-dependent features of the spectrum due to disorder should disappear in the limit in which the dimensionless conductance goes to infinity. This occurs because at times longer than the inverse Thouless energy, the only energy scale left is the mean level spacing.

Two main features are responsible for moving away from the universal regime limit in nanophysics, namely the decrease of the conductance occurring when the system is more and more closed with respect to the leads and the increase of the role of e-e correlations. In Chap. 6, the study of the transition from the Poisson spectral statistics (typical of classically integrable systems) to the Wigner spectral statistics (typical of classically chaotic systems) is addressed, showing that the features of the spectrum of an interacting 0d system can be obtained non-perturbatively using a mapping onto the Anderson–Cayley model. It should be stressed that, while the physics of disordered non interacting systems is by now well understood, the interplay between disorder and interactions is largely unexplored [29], with solutions existing only in 1d systems [30]. Recently, the analysis of planar systems has been the source of big surprises, mainly because of the possibility of a 2d Metal-Insulator transition; this would make the one parameter scaling in localization inapplicable.

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2 Fermi Liquids and Luttinger Liquids

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2.1 Introduction

In these lecture notes, corresponding roughly to lectures given at the summer school in Chia Laguna, Italy, in September 1997, an attempt is made to present the physics of three-dimensional interacting fermion systems (very roughly) and that of their one-dimensional counterparts, the so-called Luttinger liquids (in some more detail). These subjects play a crucial role in a number of currently highly active areas of research: high temperature and organic superconductors, quantum phase transitions, correlated fermion systems, quantum wires, the quantum Hall effect, low-dimensional magnetism, and probably some others. Some understanding of this physics thus certainly should be useful in a variety of areas, and it is hoped that these notes will be helpful in this.

As the subject of these lectures was quite similar to those delivered at Les Houches, some overlap in the notes [1] was unavoidable. However, a number of improvements have been made, for example a discussion of the “Klein factors” occurring in the bosonization of one-dimensional fermions, and new material added, mainly concerning spin chains and coupled Luttinger liquids. Some attempt has been made to keep references up to date, but this certainly has not always been successful, so we apologize in advance for any omissions (but then, these are lecture notes, not a review article).

2.2 Fermi Liquids

Landau’s Fermi liquid theory [2–4] is concerned with the properties of a many-fermion system at low temperatures (much lower than the Fermi energy) in *the normal state*, i.e. in the absence or at least at temperatures above any symmetry breaking phase transition (superconducting, magnetic, or otherwise). The ideal example for Landau’s theory is liquid helium 3, above its superfluid phase transition, however, the conceptual basis of Landau’s theory is equally applicable to a variety of other systems, in particular electrons in metals. Quantitative applications are however more difficult because of a variety of complications which appear in real systems, in particular the absence of translational and rotational invariance and the presence of electron-phonon interactions, which are not directly taken into account in Landau’s theory.

Subsequently, I will first briefly discuss the case of a noninteracting many-fermion system (the *Fermi gas*), and then turn to Landau's theory of the interacting case (the *liquid*), first from a phenomenological point of view, and then microscopically. A much more detailed and complete exposition of these subjects can be found in the literature [5–9].

2.2.1 The Fermi Gas

In a noninteracting translationally invariant systems, the single-particle eigenstates are plane waves

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} \quad (2.1)$$

with energy

$$\varepsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}, \quad (2.2)$$

where Ω is the volume of the system, and we will always use units so that $\hbar = 1$. The ground state of an N -particle system is the well-known Fermi sea: all states up to the *Fermi wavevector* k_F are filled, all the other states are empty. For spin-1/2 fermions the relation between particle number and k_F is

$$N = \Omega \frac{k_F^3}{3\pi^2}. \quad (2.3)$$

The energy of the last occupied state is the so-called *Fermi energy* $E_F = k_F^2/(2m)$, and one easily verifies that

$$E_F = \frac{\partial E_0(N)}{\partial N} = \mu(T=0) \quad (2.4)$$

i.e. E_F is the zero-temperature limit of the chemical potential ($E_0(N)$ in the formula above is the ground state energy).

It is usually convenient to define the Hamiltonian in a way so that the absolute ground state has a well-defined fixed particle number. This is achieved simply by including the chemical potential μ in the definition of the Hamiltonian, i.e. by writing

$$H = \sum_{\mathbf{k}} \xi_{\mathbf{k}} n_{\mathbf{k}}, \quad (2.5)$$

where $n_{\mathbf{k}}$ is the usual number operator, $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$, and the spin summation is not written explicitly (at finite temperature this of course brings one to the usual grand canonical description where small fluctuations of the particle number occur). With this definition of the Hamiltonian, the elementary excitations of the Fermi gas are

- addition of a particle at wavevector \mathbf{k} ($\delta n_{\mathbf{k}} = 1$). This requires $|\mathbf{k}| > k_F$, and thus the energy of this excitation is $\epsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu > 0$,
- destruction of a particle at wavevector \mathbf{k} ($\delta n_{\mathbf{k}} = -1$), i.e. creation of a hole. This requires $|\mathbf{k}| < k_F$, and thus the energy is $\epsilon_{\mathbf{k}} = \mu - \epsilon_{\mathbf{k}} > 0$.

The dispersion relation of the elementary particle and hole excitation is shown in Fig. 2.1a.

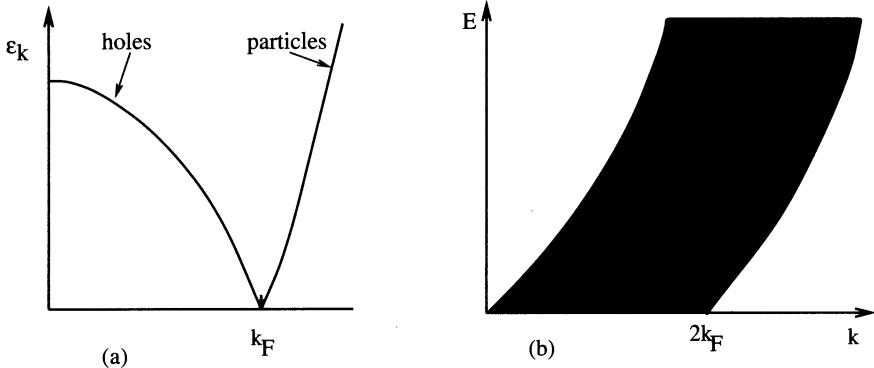


Fig. 2.1. (a) the energy-momentum relation for the elementary particle ($k > k_F$) and hole ($k < k_F$) excitations; and (b) the particle-hole continuum

These excitations change the total number of particles. Construction of states at constant particle number is of course straightforward: one takes one particle from some state \mathbf{k} , with $|\mathbf{k}| < k_F$, and puts it into a state \mathbf{k}' , with $|\mathbf{k}'| > k_F$. These *particle-hole excitations* are parameterized by the *two* quantum numbers \mathbf{k}, \mathbf{k}' and thus form a continuum, as shown in Fig. 2.1b. The restriction on the allowed values of \mathbf{k}, \mathbf{k}' insures that all particle-hole states have positive energy. Higher excited states, i.e. states with many particles and many holes, are straightforwardly constructed, the only restriction being imposed by the Pauli principle.

Thermodynamic quantities are easily obtained and are all determined by the density of states at the Fermi energy. For example, the specific heat obeys the well known linear law $C(T) = \gamma T$, with

$$\gamma = \frac{2\pi^2}{3} N(E_F) k_B^2 \quad (2.6)$$

and similarly the (Pauli) spin susceptibility χ and the compressibility κ are given by

$$\chi = 2N(E_F)\mu_B^2 \quad (2.7)$$

$$\kappa = 2N(E_F)/\rho^2. \quad (2.8)$$

Here for the quadratic dispersion relation (2.2) the density of states (per spin) at the Fermi energy is given by $N(E_F) = mk_F/(2\pi^2)$, but it should be emphasized that eqs. (2.6) to (2.8) are valid for an arbitrary density of states, in particular in solids where bandstructure effects can change the electronic dispersion relation quite drastically. Thus, for noninteracting electrons one expects the so-called “Wilson ratio”

$$R_W = \frac{\pi^2 k_B^2}{3\mu_B^2} \frac{\chi}{\gamma} \quad (2.9)$$

to be unity, independently of details of the bandstructure. Any deviation from unity is necessarily an indication of some form of interaction effect.

2.2.2 Landau’s Theory of Fermi Liquids

Basic hypothesis. Landau’s theory is to a large extent based on the idea of a continuous and one-to-one correspondence between the eigenstates (ground state *and* excited states) of the noninteracting and the interacting system. For this to be an acceptable hypothesis it is crucial that the interactions do not lead to any form of phase transition or symmetry-broken ground state.

In particular one can consider a state obtained by adding a particle (with momentum $|\mathbf{p}| > k_F$) to the noninteracting ground state:

$$|\mathbf{p}, N+1\rangle = a_{\mathbf{p}}^+ |0, N\rangle. \quad (2.10)$$

Here $a_{\mathbf{p}}^+$ is a fermion creation operator for momentum state \mathbf{p} , and $|0, N\rangle$ is the N -particle ground state of the noninteracting system. Now we add some form of particle-particle interaction. In a translationally invariant system, interactions conserve total momentum, and thus even after switching on the interaction the state still has total momentum \mathbf{p} . However, the interaction of the added particle with the filled Fermi sea, as well as the interaction of the particles in the sea amongst themselves, will change the distribution of particles in \mathbf{k} -space, and of course also modify the energy of our state. The complex formed by the particle added at \mathbf{p} and the perturbed distribution of the other particles is called a *Landau quasiparticle*. The Pauli principle implied $|\mathbf{p}| > k_F$ in the absence of interactions, and by the continuity hypothesis the same restriction remains valid in the interacting case. In particular, the value of k_F , which imposes a lower limit on the allowed momentum of the quasiparticle, is unchanged by the interactions.

Analogous considerations can be performed for a state obtained by destruction of a particle (e.g. creation of a hole):

$$|\mathbf{p}, N-1\rangle = a_{-\mathbf{p}}^- |0, N\rangle. \quad (2.11)$$

Note that due to the momentum $-\mathbf{p}$ the total momentum of this state is indeed \mathbf{p} .

The quasi-particle concept has a certain number of limitations, mainly due to the fact that, as will be discussed below, the lifetime of a quasi-particle is finite. However, for excitations close to k_F one has $1/\tau \propto (\varepsilon - E_F)^2$, i.e. the lifetime becomes much longer than the inverse excitation energy, and the quasi-particles therefore are reasonably well defined. In practice, this means that Landau's theory is useful for phenomena at energy scales much smaller than the Fermi energy, but inapplicable otherwise. In metals, where $E_F \approx 3 \dots 5 \text{ eV}$, this restriction is not too serious when one is concerned with thermodynamic or transport properties. One should also note that the ground state energy itself has important contributions from states well below E_F , and therefore is not accessible to Landau's theory.

Equilibrium properties. In order to derive physical quantities from the picture of the low-energy excitations, we need some information about the energetics of the quasiparticles and of their interactions. To be specific, starting from the ground state quasiparticle distribution

$$\begin{aligned} n_0(\mathbf{k}) &= 1 \text{ if } |\mathbf{k}| < k_F \\ &= 0 \text{ if } |\mathbf{k}| > k_F \end{aligned} \quad (2.12)$$

one considers changes in quasiparticle occupation number of the form $n_0(k) \rightarrow n_0(k) + \delta n(k)$, i.e. $\delta n(k) = 1$ represents an excited quasi-particle, $\delta n(k) = -1$ an excited quasi-hole (with the notation $k = (\mathbf{k}, \sigma)$, and $\sigma = \uparrow, \downarrow$ the spin index). The corresponding change in energy is

$$\delta E = \sum_k \varepsilon_{\mathbf{k}}^0 \delta n(k) + \frac{1}{2\Omega} \sum_{kk'} f(k, k') \delta n(k) \delta n(k') , \quad (2.13)$$

where the first and second term represent the energy of a single quasi-particle and the interaction between quasiparticles, respectively. To be more precise, we assume that the chemical potential is included in the Hamiltonian, as in (2.5). Consequently, $\varepsilon_{\mathbf{k}}^0$ vanishes on the Fermi surface, and, given that we are mainly interested in phenomena in the vicinity of k_F , it is sufficient to retain the lowest order term in an expansion around $|\mathbf{k}| = k_F$. One thus writes

$$\varepsilon_{\mathbf{k}}^0 = \frac{k_F}{m^*} (|\mathbf{k}| - k_F) , \quad (2.14)$$

thus defining the *effective mass* m^* which is different from the “bare” mass m due to interaction effects that could in principle be calculated from a microscopic theory of the system.

The energy of a quasi-particle added to the system is easily obtained from (2.13) by calculating the difference in δE between a state with $\delta n(k) = 1$ and a state with $\delta n(k) = 0$. One finds

$$\varepsilon_k = \varepsilon_{\mathbf{k}}^0 + \frac{1}{\Omega} \sum_{k'} f(k, k') \delta n(k') , \quad (2.15)$$

i.e. the energy of an added quasi-particle is not just the “bare” quasi-particle energy $\varepsilon_{\mathbf{k}}^0$ but also depends, via the interaction term, on the presence of the other quasi-particles. Given that the non-interacting particles obey Fermi–Dirac statistics, the quasi-particles do so too, and consequently, the occupation probability of a quasi-particle state is given by

$$n(\mathbf{k}) = \frac{1}{e^{\beta\varepsilon_{\mathbf{k}}} + 1}. \quad (2.16)$$

Note that the full and not the bare quasi-particle energy enters this expression. In principle, $n(\mathbf{k})$ thus has to be determined self-consistently from (2.15) and (2.16).

For the subsequent calculations, it is convenient to transform the quasiparticle interaction $f(\mathbf{k}, \mathbf{k}')$. First, spin symmetric and antisymmetric f -functions are defined via

$$\begin{aligned} f(\mathbf{k} \uparrow, \mathbf{k}' \uparrow) &= f^s(\mathbf{k}, \mathbf{k}') + f^a(\mathbf{k}, \mathbf{k}') \\ f(\mathbf{k} \uparrow, \mathbf{k}' \downarrow) &= f^s(\mathbf{k}, \mathbf{k}') - f^a(\mathbf{k}, \mathbf{k}'). \end{aligned} \quad (2.17)$$

Moreover, given the implicit restrictions of the theory, one is only interested in processes where all involved particles are very close to the Fermi surface. Under the assumption that the interaction functions are slowly varying as a function of \mathbf{k} , one then can set $|\mathbf{k}| = |\mathbf{k}'| = k_F$. Because of rotational symmetry, the f -functions then can only depend on the angle between \mathbf{k} and \mathbf{k}' , called θ . One can then expand the f -function in a Legendre series as

$$f^{a,s}(\mathbf{k}, \mathbf{k}') = \sum_{L=0}^{\infty} f_L^{a,s} P_L(\cos \theta), \cos \theta = \frac{\mathbf{k} \cdot \mathbf{k}'}{k_F^2}, \quad (2.18)$$

where the P_L are the Legendre polynomials. Finally, one usually puts these coefficients into dimensionless form by introducing

$$F_L^{a,s} = \frac{k_F m^*}{\pi^2} f_L^{a,s}. \quad (2.19)$$

We are now in a position to calculate some equilibrium properties. The first one will be the specific heat at constant volume

$$C_V = \frac{1}{V} \frac{\partial U}{\partial T}, \quad (2.20)$$

where U is the internal energy. The temperature-dependent part of U comes from thermally excited quasi-particles, as determined by the distribution (2.16). In principle, in this expression $\varepsilon_{\mathbf{k}}$ is itself temperature-dependent, because of the temperature dependent second term in (2.15). However, one can easily see that this term only gives contributions of order T^2 , and therefore can be neglected in the low-temperature limit. Consequently, one can

indeed replace ε_k by ε_k^0 , and then one only has to replace the bare mass by m^* in the result for a non-interacting system to obtain

$$C_\Omega = \frac{m^* k_F}{3} k_B^2 T . \quad (2.21)$$

The spin susceptibility (at $T = 0$) is related to the second derivative of the ground state energy with respect to the (spin) magnetization M :

$$\chi = \left[\Omega \frac{\partial^2 E_0}{\partial M^2} \right]^{-1} . \quad (2.22)$$

Spin magnetization is created by increasing the number of \uparrow spin particles and decreasing the number of \downarrow spins ($M = \mu_B(N_\uparrow - N_\downarrow)$), i.e. by changing the Fermi wavevectors for up and down spins: $k_F \rightarrow k_F + \delta k_F$ for $\sigma = \uparrow$ and $k_F \rightarrow k_F - \delta k_F$ for $\sigma = \downarrow$.

By calculating with (2.13) the corresponding change of the ground state energy, we obtain from (2.22):

$$\chi = \frac{1}{1 + F_0^a} \frac{\mu_B^2 k_F m^*}{\pi^2} . \quad (2.23)$$

Note that here, and contrary to the specific heat, interactions enter not only via m^* but also explicitly via the coefficient F_0^a , which is the only coefficient that appears here because the distortion of the Fermi distribution function is antisymmetric in the spin index and has spherical symmetry ($L = 0$). The Wilson ratio is then

$$R_W = \frac{1}{1 + F_0^a} . \quad (2.24)$$

Following a similar reasoning, one can calculate the compressibility κ of a Fermi liquid:

$$\kappa = -\frac{1}{\Omega} \frac{\partial \Omega}{\partial P} = \left[\Omega \frac{\partial^2 E_0}{\partial \Omega^2} \right]^{-1} = \frac{m^* k_F}{\pi^2 \rho^2 (1 + F_0^s)} . \quad (2.25)$$

It is also interesting that in a translationally invariant system as we have considered here, the effective mass is not independent of the interaction coefficients. One can show indeed, by exploiting the Galilean invariance of the system, that

$$\frac{m^*}{m} = 1 + F_1^s / 3 . \quad (2.26)$$

Nonequilibrium properties. As far as equilibrium properties are concerned, Landau's theory is phenomenological and makes some important

qualitative predictions, the most prominent being that even in the presence of interactions the low-temperature specific heat remains linear in temperature and that the spin susceptibility tends to a constant as $T \rightarrow 0$. However, Landau's theory has little quantitative predictive power because the crucial Landau parameters have actually to be determined from experiment. The situation is different for non-equilibrium situations, where the existence of new phenomena, in particular collective modes, is predicted. These modes are another kind of elementary excitations which, contrary to quasiparticles, involve a coherent motion of the whole system. We shall not enter into the details of the treatment of non-equilibrium properties of Fermi liquids (see [5,8,9]) and just briefly sketch the general conceptual framework and some of the more important results. To describe non-equilibrium situations, one makes two basic assumptions:

- Deviations from equilibrium are described by a *Boltzmann equation* for a space- and time- dependent *quasiparticle distribution function* $n(\mathbf{k}, \mathbf{r}, t)$, which describes the density of quasi-particles of momentum and spin (\mathbf{k}, σ) at point \mathbf{r} and time t . At equilibrium, n is of course given by (2.12). The fact that in the distribution function one specifies simultaneously momentum and position of course imposes certain restrictions, due to the quantum-mechanical nature of the underlying problem. More precisely, spatial and temporal variations of the distribution function have to be slow compared to the typical wavelength and frequency of the quasi-particles. We have then the conditions $v_F|\mathbf{q}|, |\omega| < E_F$, where \mathbf{q} and $|\omega|$ set the scale of the spatial and temporal variations of $n(\mathbf{r}, t)$.
- Because of the \mathbf{r} -dependent n , the quasiparticle energy is itself, via (2.15), \mathbf{r} -dependent. One then assumes the following quasi-classical equations of motion

$$\begin{aligned}\dot{\mathbf{r}} &= \nabla_{\mathbf{k}} \varepsilon_{\mathbf{k}}(\mathbf{r}) \\ \dot{\mathbf{k}} &= -\nabla_{\mathbf{r}} \varepsilon_{\mathbf{k}}(\mathbf{r}).\end{aligned}\tag{2.27}$$

Note in particular that a space-dependent distribution function gives rise, via the $f(k, k')$ function, to a force acting on a quasi-particle.

By linearizing the Boltzmann equation and studying the collisionless regime, where the collision term in the Boltzmann equation can be neglected, one finds collective mode solutions which correspond to oscillations of the Fermi surface. The most important one is the longitudinal symmetric mode which, like ordinary sound, involves fluctuations of the particle density. This kind of sound appears, however, in a regime where ordinary sound cannot exist (the existence of collisions is indeed crucial for the propagation of ordinary sound waves) and is a purely quantum effect. Since collisions can always be neglected at very low temperatures, this new kind of sound has been called by Landau *zero sound*. The collision term of the Boltzmann equation is on the contrary essential to calculate the quasi-particle lifetime τ . One can find

indeed, for a quasiparticle of energy $\varepsilon_{\mathbf{p}}$

$$\tau^{-1} \propto m^{*3} \frac{(\pi T)^2 + \varepsilon_{\mathbf{p}}^2}{1 + e^{-\beta \varepsilon_{\mathbf{p}}}}. \quad (2.28)$$

The most important result here is the divergence of the lifetime for low energies and temperatures as $\tau \propto \max(\varepsilon_{\mathbf{p}}, T)^{-2}$, so that the product $\varepsilon_{\mathbf{p}}\tau$ in fact diverges as the Fermi surface is approached. This shows that the quasiparticle becomes a well-defined (nearly-) eigenstate at low excitation energies, i.e. in the region where Landau's theory is applicable. On the other hand, at higher energies the quasiparticle becomes less and less well-defined. One may note that initially we had assumed that a quasiparticle is an *exact eigenstate* of the interacting system, which was obtained from a noninteracting eigenstate by switching on the interaction, and therefore should have infinite lifetime. We now arrive at the conclusion that the lifetime is not strictly infinite, but only very long at low energies. In the following section we will try to clarify this from a microscopical point of view.

2.2.3 Microscopic Basis of Landau's Theory

At our current knowledge, it does not seem generally possible to *derive* Landau's theory starting from some microscopic Hamiltonian, apart possibly in perturbation theory for small interactions. It is however possible to formulate the basic hypotheses in terms of microscopic quantities, in particular one- and two-particle Green functions. This will be outlined below.

Quasiparticles. As far as single particle properties are concerned it is sufficient to consider the one-particle Green function

$$\mathcal{G}(k, \tau) = -\langle T_{\tau} a_k(\tau) a_k^+(0) \rangle, \quad (2.29)$$

where τ is the usual (Matsubara) imaginary time. In this quantity, interaction effects appear via self-energy corrections Σ in the Fourier transformed function

$$\mathcal{G}(k, \omega) = \frac{1}{i\omega - \varepsilon_k^{00} - \Sigma(k, \omega)}. \quad (2.30)$$

Here ε_k^{00} is the bare particle energy, without any effective mass effects. Excitation energies of the system then are given by the poles of $\mathcal{G}(k, \omega)$. In these terms, Landau's assumption about the existence of quasiparticles is equivalent to assuming that $\Sigma(k, \omega)$ is sufficiently regular close to the Fermi surface as to allow an expansion for small parameters. Regularity in (k, ω) space implies that in real space the self-energy has no contributions that decay slowly in time and/or space. Given that the self-energy can be calculated in terms

of the effective interaction between particles this is certainly a reasonable assumption when the particle-particle interaction is short-range (though there is no formal prove of this). For Coulomb interactions, screening has to be invoked to make the effective interaction short ranged.

One can further notice that $\Sigma(k_F, 0)$ just renormalizes the chemical potential. Given that we want to work at fixed particle number we can absorb this term in the effective μ . Expanding then to first order around the Fermi surface, the Green function takes the form

$$\mathcal{G}(k, \omega) = \frac{z}{i\omega - \varepsilon_k^0}, \quad (2.31)$$

where ε_k^0 has the form (2.14) of the phenomenological approach, with

$$m^* = m \left(1 - \frac{\partial \Sigma}{\partial \omega} \right) \left(1 + \frac{m}{k_F} \frac{\partial \Sigma}{\partial k} \right)^{-1}, \quad (2.32)$$

and the *quasiparticle renormalization factor* is

$$z = \left(1 - \frac{\partial \Sigma}{\partial \omega} \right)^{-1}. \quad (2.33)$$

All derivatives are to be taken at the Fermi surface and at $\omega = 0$. One should notice that a sum rule imposes that the frequency-integrated spectral density

$$A(k, \omega) = -\frac{1}{\pi} \text{Im } \mathcal{G}(k, i\omega \rightarrow \omega + i\delta) \quad (2.34)$$

equals unity. Consequently, in order to fulfill the sum rule, if $z < 1$ there has to be a contribution in addition to the *quasiparticle pole* in (2.31). This is the so-called “incoherent background” from single and multiple particle-hole pair excitations which can extend to rather high energies but becomes small close to the Fermi surface.

The form (2.31) gives rise to a jump in the momentum distribution function at k_F of height z , instead of unity in the noninteracting case (Fig. 2.2). In addition to the jump, the incoherent background gives rise to a contribution which is continuous through k_F .

A finite quasiparticle lifetime arises if the expansion of $\Sigma(k, \omega)$ is carried to second order. Then (2.31) generalizes to

$$\mathcal{G}(k, \omega) = \frac{z}{i\omega - \varepsilon_k^0 + i\text{sign}(\omega)\tau(\omega)^{-1}}, \quad (2.35)$$

where $\tau(\omega)$ is typically given by an expression like (2.28).

Quasiparticle interaction. The quasiparticle interaction parameters $f(k, k')$ are expected to be connected to the two-particle vertex function.

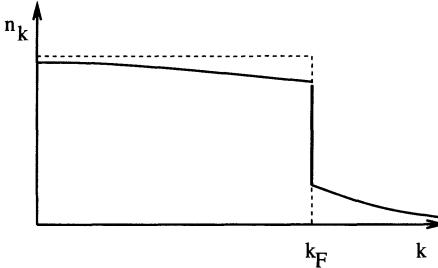


Fig. 2.2. The momentum distribution function $n_{\mathbf{k}} = \langle a_{\mathbf{k}}^+ a_{\mathbf{k}} \rangle$ in the interacting (full line) and noninteracting (dashed line) cases

This function, which we will denote $\Gamma^{(2)}(P_1, P_2; K)$ describes the scattering of two particles from initial state P_1, P_2 to the final state $P_1 - K, P_2 + K$, and the notation is $P_i = (\mathbf{p}_i, \omega_i, \sigma_i)$. The contribution of first and second order in the interaction potential $V(\mathbf{k})$ are shown in Fig. 2.3. Let us now study the case of small transfer K , but arbitrary $P_{1,2}$, only restricted to be close to the Fermi surface. One then notices that diagram 2.3b (part c) gives rise to singularities, because the poles of the two intervening Green functions coalesce. On the other hand diagrams 2.3b (part a) and 2.3b (part b) remain non-singular for small K . This motivates one to introduce a *two-particle irreducible function* $\tilde{\Gamma}^{(2)}$ which is the sum of all contributions which do not contain a single product $\mathcal{G}(Q)\mathcal{G}(K+Q)$. This function then is non-singular for small K , and consequently the total vertex function is determined by the integral equation

$$\begin{aligned} \Gamma(P_1, P_2; K) &= \tilde{\Gamma}(P_1, P_2) + \frac{1}{(2\pi)^4} \\ &\quad \times \int d\omega d^3q \tilde{\Gamma}(P_1, Q) \mathcal{G}(Q) \mathcal{G}(K+Q) \Gamma(Q, P_2, K). \end{aligned} \quad (2.36)$$

For simplicity, the spin summation is omitted here. The singular contribution now comes from small K and Q in the vicinity of the Fermi surface. In this area the Q -dependence of the Γ 's in (2.36) is non-singular and can be neglected. The energy and radial momentum integral over Q can then be done, leading to

$$\begin{aligned} \Gamma(P_1, P_2; K) &= \tilde{\Gamma}(P_1, P_2) + \frac{z^2 k_F^2}{(2\pi)^3} \\ &\quad \times \int d^2\Omega_{\mathbf{q}} \tilde{\Gamma}(P_1, Q) \frac{\hat{\mathbf{q}} \cdot \mathbf{k}}{\omega - v_F \hat{\mathbf{q}} \cdot \mathbf{k}} \Gamma(Q, P_2, K), \end{aligned} \quad (2.37)$$

where $\hat{\mathbf{q}}$ is a vector on the Fermi surface, and $d^2\Omega_{\mathbf{q}}$ is the corresponding angular integration. Here only the quasiparticle pole in \mathcal{G} has been taken into account. The contribution from the incoherent parts can in principle be absorbed into the definition of $\tilde{\Gamma}$.

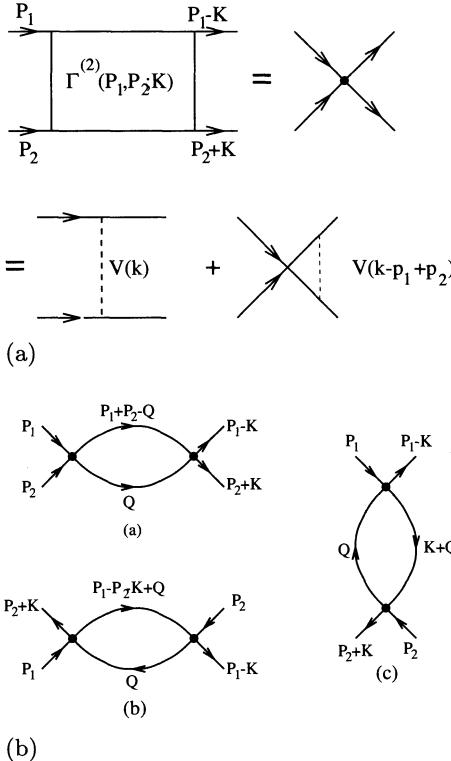


Fig. 2.3. The lowest (a) and second order (b) contributions to the two-particle vertex $\Gamma^{(2)}$. Note that the external lines do not represent actual Green functions but only indicate the external “connections”

The expression (2.37) is clearly singular because it has radically different behavior according to whether one first sends \mathbf{k} or ω to zero. In the first case, the limit of Γ can be related to the Landau f -function, while the second one is relevant for the calculation of the transition probabilities determining the lifetime of quasiparticles. Here we will consider only the former case. Sending \mathbf{k} to zero first one finds straightforwardly

$$\lim_{\omega \rightarrow 0} (\lim_{\mathbf{k} \rightarrow 0} \Gamma) \equiv \Gamma^\omega(P_1, P_2) = \tilde{\Gamma}(P_1, P_2). \quad (2.38)$$

Closer inspection then shows that in this case the poles of the two Green functions in (2.36) are always on the same side of the real axis and consequently the singular term in (2.36) vanishes. To make the identification between $\Gamma^\omega(P_1, P_2)$ and the Landau f -function we notice that the density response function at energy-momentum K , whose poles give the collective (zero-sound) modes, contains the interactions via $\Gamma(P_1, P_2, K)$. In particular, the existence of a pole in the response function implies a pole in Γ . The comparison between the equations for this pole and those which can be ob-

tained through the Boltzmann equation within Landau's theory allows then the identification

$$f(k, k') = z^2 \Gamma^\omega(k, k') . \quad (2.39)$$

2.2.4 Summary

The basic assumption of Landau's theory is the existence of low-energy quasiparticles with a very long lifetime, and their description in terms of a rather simple energy functional, (2.13). From this a number of results for thermodynamic properties is obtained. At this level, the theory is of little quantitative power because the Landau parameters are not determined. Qualitatively, however, the predictions are important: the low-temperature thermodynamic properties of an interacting fermion system are very similar to those of a noninteracting system, the interactions only lead to quantitative renormalizations. Actual quantitative predictions are obtained when one extends the theory to nonequilibrium properties, using the Boltzmann equation [5]. A new phenomenon predicted (and actually observed in ${}^3\text{He}$ [10]) is the existence of collective excitations, called "zero sound". This approach also allows the calculation of the quasiparticle lifetime and its divergence as the Fermi energy is approached, as well as the treatment of a number of transport phenomena.

As already mentioned, the ideal system for the application of Landau's theory is ${}^3\text{He}$, which has both short-range interaction and is isotropic. The application to electrons in metals is more problematic. First, the interactions are long-ranged (Coulombic). This can however be accommodated by properly including screening effects. More difficulties, at least at the quantitative level, arise because metals are naturally anisotropic. This problem is not of fundamental nature: even when the Fermi surface is highly anisotropic, an expansion like (2.13) can still be written down and thus interaction parameters can be defined. However, a simple Legendre expansion like (2.18) is not in general possible and the description of the quasiparticle interaction in terms of a few parameters becomes impossible. An exception case, with a very nearly spherical Fermi surface, are the alkali metals, where a determination of Landau parameters can indeed be attempted [5]. It should be noticed that the difficulties with the Landau description of metals are not of conceptual nature and in particular do not invalidate the quasiparticle concept but are rather limitations on the usefulness of the theory for quantitative purposes.

Landau's theory can be interpreted in terms of microscopic quantities like Green functions (the *quasiparticle pole*) and interaction vertices, as discussed above. It should however be emphasized that these arguments do provide a microscopic interpretation of Landau's picture, rather than proving its correctness. Similar remarks apply to the calculated diverging quasiparticle lifetime: this at best show that Landau's picture is internally consistent. Considerable progress towards a deeper formal understanding of Fermi liquid theory has been made in recent years [11,12].

2.3 Renormalization Group for Interacting Fermions

In this chapter, we will consider properties of interacting fermions in the framework of renormalization group theory. This will serve two purposes: first, the treatment of one-dimensional interacting fermions, which will be considered in considerable detail in the following chapters, gives rise to divergences which can only be handled by this approach. Results obtained in this way will be an essential ingredient in the subsequent discussion of “Luttinger liquids”. More generally, the renormalization group method will clarify the status of both Landau’s Fermi liquid theory and the Luttinger liquid picture as renormalization group fixed points, thus establishing a link with a number of other phenomena in condensed matter physics. We will formulate the problem in terms of fermion functional integrals, as done by Bourbonnais in the one-dimensional case [13] and more recently for two and three dimensions by Shankar [14]. For the most part, I will closely follow Shankar’s notation.

Before considering the interacting fermion problem in detail, let us briefly recall the general idea behind the renormalization group, as formulated by Kadanoff and Wilson: one is interested in the statistical mechanics of a system described by some Hamiltonian H . Equilibrium properties then are determined by the partition function

$$Z = \sum_{\text{configurations}} e^{-\beta H} = \sum_{\text{configurations}} e^{-S}, \quad (2.40)$$

where the second equality defines the *action* $S = \beta H$. Typically, the action contains degrees of freedom at wavevectors up to some *cutoff* Λ , which is of the order of the dimensions of the Brillouin zone. One wishes to obtain an “effective action” containing only the physically most interesting degrees of freedom. In standard phase transition problems this is the vicinity of the point $\mathbf{k} = 0$, however, for the fermion problem at hand the surface $|\mathbf{k}| = k_F$ is relevant, and the cutoff has to be defined with respect to this surface. In order to achieve this one proceeds as follows:

1. Starting from a cutoff-dependent action $S(\Lambda)$ one eliminates all degrees of freedom between Λ and Λ/s , where s is a factor larger than unity. This gives rise to a new action $S'(\Lambda' = \Lambda/s)$.
2. One performs a “scale change” $\mathbf{k} \rightarrow s\mathbf{k}$. This brings the cutoff back to its original value and a new action $S'(\Lambda)$ is obtained. Because of the degrees of freedom integrated out, coupling constants (or functions) are changed.
3. One chooses a value of s infinitesimally close to unity: $s = 1 + \varepsilon$, and performs the first two steps iteratively. This then gives rise to differential equations for the couplings, which (in favourable circumstances) can be integrated until all non-interesting degrees of freedom have been eliminated.

2.3.1 One Dimension

The one-dimensional case, which has interesting physical applications, will here be mainly used to clarify the procedure. Let us first consider a noninteracting problem, e.g. a one-dimensional tight-binding model defined by

$$H = \sum_k \xi_k a_k^\dagger a_k , \quad \xi_k = -2t \cos k - \mu , \quad (2.41)$$

where t is the nearest-neighbour hopping integral. We will consider the metallic case, i.e. the chemical potential is somewhere in the middle of the band. Concentrating on low-energy properties, only states close to the “Fermi points” $\pm k_F$ are important, and one can then linearize the dispersion relation to obtain

$$H = \sum_{k,r=\pm} v_F (rk - k_F) a_{kr}^\dagger a_{kr} , \quad (2.42)$$

where $v_F = 2t \sin k_F$ is the Fermi velocity, and the index r differentiates between right- and left-going particles, i.e. particles close to k_F and $-k_F$. To simplify subsequent notation, we (i) choose energy units so that $v_F = 1$, (ii) translate k -space so that zero energy is at $k = 0$, and (iii) replace the k -sum by an integral. Then

$$H = \sum_{r=\pm} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} r k a_r^\dagger(k) a_r(k) . \quad (2.43)$$

For the subsequent renormalization group treatment we have to use a functional integral formulation of the problem in terms of Grassmann variables (a detailed explanation of this formalism is given by Negele and Orland [15]). The partition function becomes

$$Z(\Lambda) = \int \mathcal{D}\phi e^{-S(\Lambda)} , \quad (2.44)$$

where $\mathcal{D}\phi$ indicates functional integration over a set of Grassmann variables. The action is

$$S(\Lambda) = \int_0^\beta d\tau \left\{ \sum_{r=\pm} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \phi_r^*(k, \tau) \partial_\tau \phi_r(k, \tau) + H(\phi^*, \phi) \right\} , \quad (2.45)$$

where the zero-temperature limit $\beta \rightarrow \infty$ has to be taken, and $H(\phi^*, \phi)$ indicates the Hamiltonian, with each a^\dagger replaced by a ϕ^* , and each a replaced by a ϕ . Fourier transforming with respect to the imaginary time variable

$$\phi_r(k, \tau) = T \sum_{\omega_n} \phi_r(k, \omega_n) e^{-i\omega_n \tau} \quad (\omega_n = 2\pi(n + 1/2)T) \quad (2.46)$$

and passing to the limit $T \rightarrow 0$ one obtains the noninteracting action

$$S_0(\Lambda) = \sum_{r=\pm} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \phi_r^*(k, \omega) [-i\omega + rk] \phi_r(k, \omega) . \quad (2.47)$$

We notice that this is diagonal in k and ω which will greatly simplify the subsequent treatment. Because of the units chosen, ω has units of $(\text{length})^{-1}$ (which we will abbreviate as L^{-1}), and then $\phi_r(k, \omega)$ has units $L^{3/2}$.

We now integrate out degrees of freedom. More precisely, we will integrate over the strip $\Lambda/s < |k| < \Lambda$, $-\infty < \omega < \infty$. The integration over all ω keeps the action local in time. One then has

$$Z(\Lambda) = Z(\Lambda, \Lambda/s) Z(\Lambda/s) , \quad (2.48)$$

where $Z(\Lambda, \Lambda/s)$ contains the contributions from the integrated degrees of freedom, and $Z(\Lambda/s)$ has the same form of (2.44). The new action is then $S'_0(\Lambda/s) = S_0(\Lambda/s)$. Introducing the scale change

$$k' = ks , \quad \omega' = \omega s , \quad \phi' = \phi s^{-3/2} \quad (2.49)$$

one easily finds that $S'_0(\Lambda) = S_0(\Lambda)$. The action does not change therefore under scale change (or *renormalization*): we are at a *fixed point*. One should notice that the scale change of k implies that k' is quantized in units of $\Delta k' = 2\pi s/L$, i.e. eliminating degrees of freedom actually implies that we are considering a shorter system, with correspondingly less degrees of freedom. This means that even though the action is unchanged the new $Z(\Lambda)$ is the partition function of a shorter system. To derive this in detail, one has to take into account the change in the functional integration measure due to the scale change on ϕ .

Before turning to the problem of interactions, it is instructive to consider a quadratic but diagonal perturbation of the form

$$\delta S_2 = \sum_{r=\pm} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \mu(k, \omega) \phi_r^*(k, \omega) \phi_r(k, \omega) . \quad (2.50)$$

We assume that $\mu(k, \omega)$ can be expanded in a power series

$$\mu(k, \omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \dots \quad (2.51)$$

Under the scale change (2.49) one then has

$$\mu_{nm} \rightarrow s^{1-n-m} \mu_{nm} . \quad (2.52)$$

There now are three cases:

1. a parameter μ_{nm} grows with increasing s . Such a parameter is called *relevant*. This is the case for μ_{00} ;

2. a parameter remains unchanged (μ_{10}, μ_{01}). Such a parameter is *marginal*;
3. finally, all other parameters decrease with increasing s . These are called *irrelevant*.

Generally, one expects relevant parameters, which grow after elimination of high-energy degrees of freedom, to strongly modify the physics of the model. In the present case, the relevant parameter is simply a change in chemical potential, which doesn't change the physics much (the same is true for the marginal parameters). One can easily see that another relevant perturbation is a term coupling right- and left-going particles of the form $m(\phi_1^* \phi_2 + \phi_2^* \phi_1)$. This term in fact does lead to a basic change: it leads to the appearance of a gap in the spectrum.

Let us now introduce fermion–fermion interactions. The general form of the interaction term in the action is

$$S_I = \int_{k\omega} u(1234) \phi^*(1)\phi^*(2)\phi(3)\phi(4) . \quad (2.53)$$

Here $\phi(3)$ is an abbreviation for $\phi_{r3}(k_3, \omega_3)$, and similarly for the other factors, while u is an interaction function to be specified. The integration measure is

$$\begin{aligned} \int_{k\omega} = & \left(\prod_{i=1}^4 \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \right) \delta(k_1 + k_2 - k_3 - k_4) \\ & \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) . \end{aligned} \quad (2.54)$$

We now note that the dimension of the integration measure is L^{-6} , and the dimension of the product of fields is L^6 . This in particular means that if we perform a series expansion of u in analogy to (2.51) the constant term will be s -independent, i.e. marginal, and all other terms are irrelevant. In the following we will thus only consider the case of a constant (k - and ω -independent) u .

These considerations are actually only the first step in the analysis: in fact it is quite clear that (unlike in the noninteracting case above) integrating out degrees of freedom will not in general leave the remaining action invariant. To investigate this effect, we use a more precise form of the interaction term:

$$\begin{aligned} S_I = & \int_{k\omega} \sum_{ss'} \{ g_1 \phi_{s+}^*(1) \phi_{s'-}^*(2) \phi_{s'+}^*(3) \phi_{s-}^*(4) \\ & + g_2 \phi_{s+}^*(1) \phi_{s'-}^*(2) \phi_{s'-}^*(3) \phi_{s+}^*(4) \} . \end{aligned} \quad (2.55)$$

Here we have reintroduced spin, and the two coupling constants g_1 and g_2 denote, in the original language of (2.42), backward ($(k_F, -k_F) \rightarrow (-k_F, k_F)$) and forward ($(k_F, -k_F) \rightarrow (k_F, -k_F)$) scattering. Note that in the absence of spin the two processes are actually identical.

Now, the Kadanoff–Wilson type mode elimination can be performed via

$$e^{-S'} = \int \mathcal{D}\bar{\phi} e^{-S} , \quad (2.56)$$

where $\mathcal{D}\bar{\phi}$ denotes integration only over degrees of freedom in the strip $\Lambda/s < |k| < \Lambda$. Dividing the field ϕ into $\bar{\phi}$ (to be eliminated) and ϕ' (to be kept), one easily sees that the noninteracting action can be written as $S_0 = S_0(\phi') + S_0(\bar{\phi})$. For the interaction part, things are a bit more involved:

$$S_I = \sum_{i=0}^4 S_{I,i} = S_{I,0} + \bar{S}_I . \quad (2.57)$$

Here $S_{I,i}$ contains i factors $\bar{\phi}$. We then obtain

$$e^{-S'} = e^{-S_0(\phi') - S_{I,0}} \int \mathcal{D}\bar{\phi} e^{-S_0(\bar{\phi}) - \bar{S}_I} . \quad (2.58)$$

Because \bar{S}_I contains up to four factors $\bar{\phi}$, the integration is not straightforward, and has to be done via a perturbative expansion, giving

$$\int \mathcal{D}\bar{\phi} e^{-S_0(\bar{\phi}) - \bar{S}_I} = Z_0(\Lambda, \Lambda/s) \exp \left[- \sum_{i=1}^{\infty} \frac{1}{n!} \langle \bar{S}_I^n \rangle_{\bar{0},\text{con}} \right] , \quad (2.59)$$

where the notation $\langle \dots \rangle_{\bar{0},\text{con}}$ indicates averaging over $\bar{\phi}$ and only the connected diagrams are to be counted. It can be easily seen, moreover, that because of the $U(1)$ invariance of the original action associated to the particle number conservation, terms which involve an odd number of ϕ' or $\bar{\phi}$ fields are identically zero. The first order cumulants give corrections to the energy and the chemical potential and are thus of minor importance. The important contributions come from the second order term $\langle S_{I,2}^2 \rangle_{\bar{0},\text{con}}$ which after averaging leads to terms of the form $\phi'^* \phi'^* \phi' \phi'$, i.e. to corrections of the interaction constants $g_{1,2}$. The calculation is best done diagrammatically, and the four intervening diagram are shown in Fig. 2.4.

One can easily see that not all of these diagrams contribute corrections to g_1 or g_2 . Specifically, one has

$$\begin{aligned} \delta g_1 &\propto g_1 g_2 [(a) + (c)] + 2g_1^2(d) \\ \delta g_2 &\propto (g_1^2 + g_2^2)(a) + g_2^2(b) , \end{aligned} \quad (2.60)$$

where the factor 2 for diagram (d) comes from the spin summation over the closed loop. Because the only marginal term is the constant in $u(1234)$, one can set all external energies and momenta to zero. The integration over the internal lines in diagram (a) then gives

$$\begin{aligned} (a) &= \int_s \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{1}{i\omega - k} \frac{1}{-i\omega - k} \\ &= \int_{\Lambda/s}^{\Lambda} \frac{dk}{2\pi} \frac{1}{k} = \frac{1}{2\pi} d\ell , \end{aligned} \quad (2.61)$$

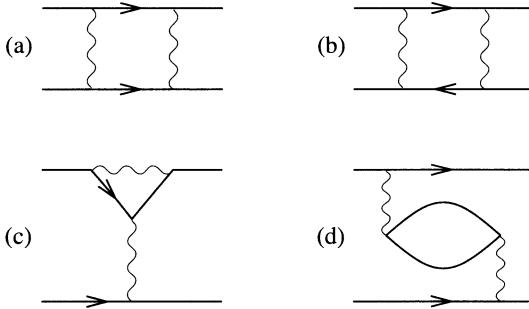


Fig. 2.4a–d. The diagrams intervening in the renormalization of the coupling constants g_1 and g_2 . Note that in (b) the direction of one arrow is reversed with respect to (a), i.e. this is a particle-hole diagram

where $s = 1 + d\ell$, and similarly the particle-hole diagrams (b) to (d) give a contribution $-d\ell/(2\pi)$. Performing this procedure recursively, using at each step the renormalized couplings of the previous step, one obtains the renormalization group equations

$$\frac{dg_1}{d\ell} = -\frac{1}{\pi} g_1^2(\ell), \quad \frac{dg_2}{d\ell} = -\frac{1}{2\pi} g_1^2(\ell), \quad (2.62)$$

where $s = e^\ell$. These equations describe the effective coupling constants to be used after degrees of freedom between Λ and Λe^ℓ have been integrated out. As initial conditions one of course uses the bare coupling constants appearing in (2.55). Equations (2.62) are easily solved. The combination $g_1 - 2g_2$ is ℓ -independent, and one has further

$$g_1(\ell) = \frac{g_1}{1 + g_1 \ell}. \quad (2.63)$$

There then are two cases:

- Initially, $g_1 \geq 0$. One then renormalizes to the *fixed line* $g_1^* = g_1(\ell \rightarrow \infty) = 0$, $g_2^* = g_2 - g_1/2$, i.e. one of the couplings has actually vanished from the problem, but there is still the free parameter g_2^* . A case like this, where perturbative corrections lead to irrelevancy, is called “marginally irrelevant”.
- Initially, $g_1 < 0$. Then g_1 diverges at some finite value of ℓ . We should however notice that, well before the divergence, we have left the weak-coupling regime where the perturbative calculation leading to (2.62) is valid. We should thus not overinterpret the divergence and just remember the renormalization towards strong coupling. This type of behavior is called “marginally relevant”.

We will discuss the physics of both cases in the next section.

Two remarks are in order here: first, had we done a straightforward order-by-order perturbative calculation, integrals like (2.61) would have been logarithmically divergent, both for particle-particle and particle-hole diagrams. This would have lead to inextricably complicated problem already at the next order. Secondly, for a spinless problem, the factor 2 in the equation for $g_1(\ell)$ is replaced by unity. Moreover, in this case only the combination $g_1 - g_2$ is physically meaningful. This combination then remains unrenormalized.

2.3.2 Two and Three Dimensions

We will now follow a similar logic as above to consider two and more dimensions. Most arguments will be made for the two-dimensional case, but the generalization to three dimensions is straightforward. The argument is again perturbative, and we thus start with free fermions with energy

$$\xi_{\mathbf{K}} = \frac{\mathbf{K}^2}{2m} - \mu = v_F k + O(k^2) \quad (v_F = k_F/m). \quad (2.64)$$

We use upper case momenta \mathbf{K} to denote momenta measured from zero, and lower case to denote momenta measured from the Fermi surface: $k = |\mathbf{K}| - k_F$. The Fermi surface geometry now is that of a circle as shown in Fig. 2.5. One notices in particular that states are now labeled by two quantum numbers which one can take as radial (k) and angular (θ). Note that the cutoff is applied around the low-energy excitations at $|\mathbf{K}| - k_F$, not around $\mathbf{K} = 0$. The noninteracting action then takes the form

$$S_0 = k_F \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} [\phi^*(k\theta\omega)(-\mathrm{i}\omega - k)\phi(k\theta\omega)]. \quad (2.65)$$

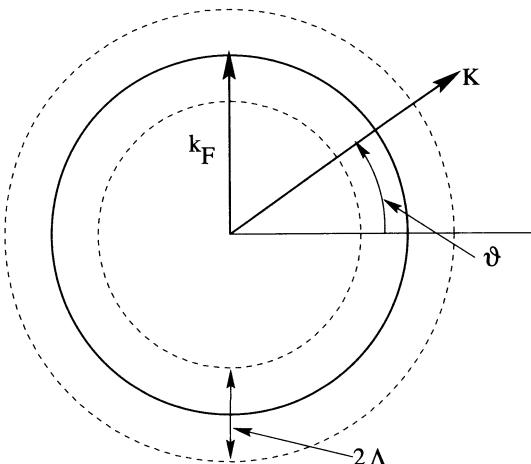


Fig. 2.5. Fermi surface geometry in two dimensions

One notices that this is just a (continuous) collection of one-dimensional action functional, parameterized by the variable θ . The prefactor k_F comes from the two-dimensional integration measure $d^2K = (k_F + k)dkd\theta$, where the extra factor k has been neglected because it is irrelevant, as discussed in the previous section.

The general form of the interaction term is the same as in the one-dimensional case

$$S_I = \int_{\mathbf{K}\omega} u(1234)\phi^*(1)\phi^*(2)\phi(3)\phi(4) , \quad (2.66)$$

however, the integration measure is quite different because of two-dimensional \mathbf{K} -space. Performing the integration over \mathbf{K}_4 and ω_4 in the two-dimensional analogue of (2.54), the measure becomes

$$\int_{\mathbf{K}\omega} = \left(\frac{k_F}{2\pi}\right)^3 \left(\prod_{i=1}^3 \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi}\right) \Theta(\Lambda - |k_4|) . \quad (2.67)$$

Here $\mathbf{K}_4 = \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3$. Now the step function poses a problem because one easily convinces oneself that even when $\mathbf{K}_{1,2,3}$ are on the Fermi surface, in general \mathbf{K}_4 can be far away from it. This is quite different from the one-dimensional case, where everything could be (after a trivial transformation) brought back into the vicinity of $k = 0$.

To see the implications of this point, it is convenient to replace the sharp cutoff in (2.67) by a soft cutoff, imposed by an exponential:

$$\Theta(\Lambda - |k_4|) \rightarrow \exp(-|k_4|/\Lambda) . \quad (2.68)$$

Introducing now unit vectors $\boldsymbol{\Omega}_i$ in the direction of \mathbf{K}_i via $\mathbf{K}_i = (k_F + k_i)\boldsymbol{\Omega}_i$ one obtains

$$\begin{aligned} k_4 &= |k_F(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_3) + k_1\boldsymbol{\Omega}_1 + \dots| - k_F \approx k_F(|\Delta| - 1) \\ \Delta &= \boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_3 . \end{aligned} \quad (2.69)$$

Now, integrating out variables leaves us with $\Lambda \rightarrow \Lambda/s$ in (2.67) everywhere, including the exponential cutoff factor for k_4 . After the scale change (2.49) the same form of the action as before is recovered, with

$$u'(k'_i, \omega'_i, \theta'_i) = e^{-(s-1)(k_F/\Lambda)|\Delta|-1} u(k_i/s, \omega_i/s, \theta_i) . \quad (2.70)$$

We notice first that nothing has happened to the angular variable, as expected as it parameterizes the Fermi surface which is not affected. Secondly, as in the one-dimensional case, the k and ω dependence of u is scaled out, i.e. *only the values $u(0, 0, \theta_i)$ on the Fermi surface are of potential interest (i.e. marginal)*. Thirdly, the exponential prefactor in (2.70) suppresses couplings for which $|\Delta| \neq 1$. This is the most important difference with the one-dimensional case.

A first type of solution to $|\Delta| = 1$ is

$$\begin{aligned}\Omega_1 &= \Omega_3 \Rightarrow \Omega_2 = \Omega_4, \text{ or} \\ \Omega_1 &= \Omega_4 \Rightarrow \Omega_2 = \Omega_3.\end{aligned}\tag{2.71}$$

These two cases only differ by an exchange of the two outgoing particles, and consequently there is a minus sign in the respective matrix element. Both processes depend only on the angle θ_{12} between Ω_1 and Ω_2 , and we will write

$$u(0, 0, \theta_1, \theta_2, \theta_1, \theta_2) = -u(0, 0, \theta_1, \theta_2, \theta_2, \theta_1) = F(\theta_1 - \theta_2).\tag{2.72}$$

We can now consider the perturbative contributions to the renormalization of F . To lowest nontrivial (second) order the relevant diagrams are those of Fermi liquid theory and are reproduced in Fig. 2.6a.

Consider diagram (a). To obtain a contribution to the renormalization of F , both \mathbf{Q} and $\mathbf{P} - \mathbf{Q}$ have to lie in the annuli to be integrated out. As can be seen from Fig. 2.6b, this will give a contribution of order $d\ell^2$ and therefore does not contribute to a renormalization of F . The same is true (if we consider the first case in (2.71)) for diagram (b). Finally, for diagram (c), because \mathbf{K} is small, the poles of both intervening Green functions are on the same side of the real axis, and here then the frequency integration gives a zero result. For the second process in (2.71) the same considerations apply, with the roles of diagrams (b) and (c) interchanged. The conclusion then is that F is not renormalized and remains marginal:

$$\frac{dF}{d\ell} = 0.\tag{2.73}$$

The third possibility is to have $\Omega_1 = -\Omega_2$, $\Omega_3 = -\Omega_4$. Then the angle between Ω_1 and Ω_3 can be used to parameterize u :

$$u(0, 0, \theta_1, -\theta_1, \theta_3, -\theta_3) = V(\theta_1 - \theta_3).\tag{2.74}$$

In this case $\mathbf{P} = 0$, and therefore in diagram (a) if \mathbf{Q} has to be eliminated, so is $-\mathbf{Q}$. Consequently, one has a contribution of order $d\ell$. For the other two diagrams, one finds again negligible contributions of order $d\ell^2$. Thus, one obtains

$$\frac{dV(\theta_1 - \theta_3)}{d\ell} = -\frac{1}{8\pi^2} \int_0^{2\pi} \frac{d\theta}{2\pi} V(\theta_1 - \theta) V(\theta - \theta_3).\tag{2.75}$$

This is a renormalization equation for a function, rather than for a constant, i.e. one here has an example of a “functional renormalization group”. Nevertheless, a Fourier transform

$$V_\lambda = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\lambda\theta} V(\theta)\tag{2.76}$$

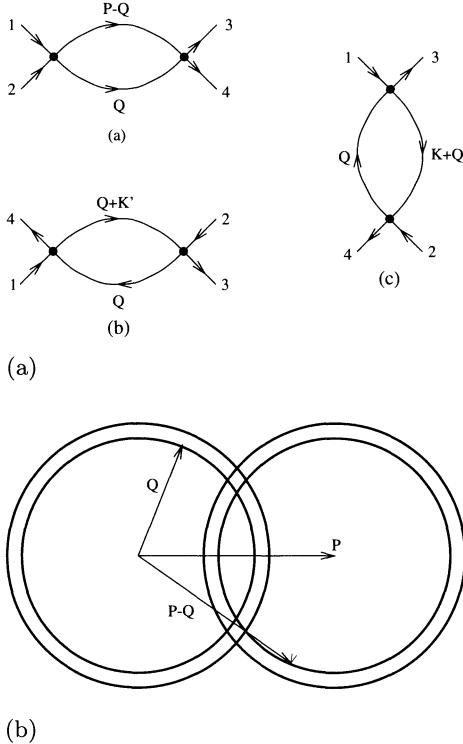


Fig. 2.6. (a) Second order diagrams renormalizing the coupling function. Here $P = \mathbf{K}_1 + \mathbf{K}_2$, $\mathbf{K}_3 = \mathbf{K}_1 - \mathbf{K}$, and $\mathbf{K}_4 = \mathbf{K}_1 - \mathbf{K}'$. \mathbf{Q} is the loop integration variable. Note that these diagrams are actually identical to those occurring in Landau's theory (Fig. 2.3b). (b) Phase space for diagram (a). The rings are the degrees of freedom to be integrated out between Λ and Λ/s . Note that *only* if $P = 0$ are Q and $P - Q$ simultaneously in the area to be integrated, giving a contribution of order $d\ell$

brings this into a more standard form:

$$\frac{dV_\lambda}{d\ell} = -\frac{V_\lambda^2}{4\pi}. \quad (2.77)$$

This has the straightforward solution

$$V_\lambda = \frac{V_\lambda}{1 + V_\lambda \ell / (4\pi)}. \quad (2.78)$$

From (2.73) and (2.78) there are now two possibilities:

1. At least one of the V_λ is negative. Then one has a divergence of $V_\lambda(\ell)$ at some finite energy scale. Given that this equation only receives contributions from BCS-like particle-particle diagrams, the interpretation of this

as a superconducting pairing instability is straightforward. The index λ determines the relative angular momentum of the particles involved.

2. All $V_\lambda > 0$. Then one has the fixed point $V_\lambda = 0$, $F(\theta_1 - \theta_2)$ arbitrary. What is the underlying physics of this fixed point? One notices that here $\theta_3 = \theta_1$, $\theta_4 = \theta_2$, i.e. the marginal term in the action is $\phi_{\theta_1}^* \phi_{\theta_2}^* \phi_{\theta_1} \phi_{\theta_2}$. In the operator language, this translates into

$$H_{\text{int}} \approx \int d\theta_1 d\theta_2 n_{\theta_1} n_{\theta_2}. \quad (2.79)$$

We now can recognize this as an operator version of Landau's energy functional, (2.13). The fixed point theory is thus identified as *Landau's Fermi liquid theory*.

The generalization of the above to three dimensions is rather straightforward. In addition to the forward scattering amplitudes F , scattering where there is an angle $\phi_{12;34}$ spanned by the planes (Ω_1, Ω_2) and (Ω_3, Ω_4) is also marginal. For $\phi_{12;34} \neq 0$ these processes are the ones contributing to the quasiparticle lifetime, as discussed in Sect. 2.2.2, however they do not affect equilibrium properties. The (zero temperature) fixed point properties thus still only depend on amplitudes for $\phi_{12;34} = 0$, i.e. the Landau f -function.

2.4 Bosonization and the Luttinger Liquid

The Fermi liquid picture described in the preceding two sections is believed to be relevant for most three-dimensional itinerant electron systems, ranging from simple metals like sodium to heavy-electron materials. The best understood example of non-Fermi liquid properties is that of interacting fermions in one dimension. This subject will be discussed in the remainder of these lecture notes. We have already started this discussion in Sect. 2.3.1, where we used a perturbative renormalization group to find the existence of one marginal coupling, the combination $g_1 - 2g_2$. This approach, pioneered by Sólyom and collaborators in the early 70's [16], can be extended to stronger coupling by going to second or even third order [17] in perturbation theory. A principal limitation remains however the reliance on perturbation theory, which excludes the treatment of strong-coupling problems. An alternative method, which allows one to handle, to a certain extent, strong-interaction problems as well, is provided by the bosonization approach, which will be discussed now and which forms the basis of the so-called Luttinger liquid description. It should be pointed out, however, that entirely equivalent results can be obtained by many-body techniques, at least for the already highly nontrivial case of pure forward scattering [18,19].

2.4.1 Spinless Model: Representation of Excitations

The bosonization procedure can be formulated precisely, in the form of operator identities, for fermions with a linear energy-momentum relation, as

discussed in Sect. 2.3.1. To clarify notation, we will use a_+ -(a_-) operators for right-(left-)moving fermions. The linearized noninteracting Hamiltonian (2.42) then becomes

$$H_0 = v_F \sum_k \left((k - k_F) a_{+,k}^\dagger a_{+,k} + (-k - k_F) a_{-,k}^\dagger a_{-,k} \right), \quad (2.80)$$

and the density of states is $N(E_F) = 1/(\pi v_F)$. In the *Luttinger model* [20–22], one generalizes this kinetic energy by letting the momentum cutoff Λ tend to infinity. There then are two branches of particles, “right movers” and left movers”, both with unconstrained momentum and energy, as shown in Fig. 2.7. At least for weak interaction, this addition of extra states far from the Fermi energy is not expected to change the physics much. However, this modification makes the model exactly solvable even in the presence of nontrivial and possibly strong interactions. Moreover, and most importantly, many of the features of this model carry over even to strongly interacting fermions on a lattice.

We now introduce the Fourier components of the particle density operator for right and left movers:

$$\rho_\pm(q) = \sum_k a_{\pm,k+q}^\dagger a_{\pm,k}. \quad (2.81)$$

The noninteracting Hamiltonian (and a more general model including interactions, see below) can be written in terms of these operators in a rather simple form and then be solved exactly. This is based on the following facts:

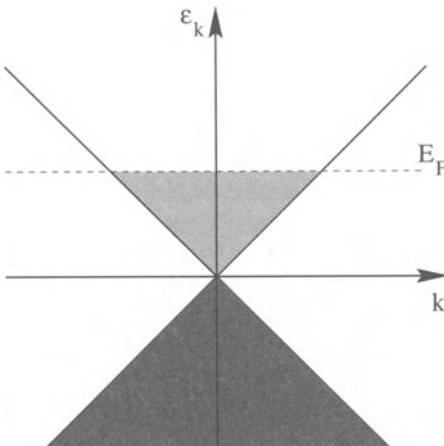


Fig. 2.7. Single-particle energy spectrum of the Luttinger model. Occupied states are shown in grey, the dark grey area represents the states added to make the model solvable

- the density fluctuation operators ρ_α , with $\alpha = \pm$, obey Bose type commutation relations:

$$[\rho_\alpha(-q), \rho_{\alpha'}(q')] = \delta_{\alpha\alpha'} \delta_{qq'} \frac{\alpha q L}{2\pi}. \quad (2.82)$$

The relation (2.82) for $q \neq q'$ or $\alpha \neq \alpha'$ can be derived by straightforward operator algebra. The slightly delicate part is (2.82) for $q = q'$. One easily finds

$$[\rho_+(-q), \rho_+(q)] = \sum_k (\hat{n}_{k-q} - \hat{n}_k), \quad (2.83)$$

where \hat{n}_k is an occupation number *operator*. In a usual system with a finite interval of states between $-k_F$ and k_F occupied, the summation index of one of the \hat{n} operators could be shifted, giving a zero answer in (2.83). In the present situation, with an infinity of states occupied below k_F , this is not so. Consider for example the ground state and $q > 0$. Then each term in (2.83) with $k_F < k < k_F + q$ contributes unity to the sum, all other terms vanish, thus establishing the result (2.82). More generally, consider a state with all levels below a certain value k_0 ($< k_F$) occupied, but an arbitrary number of particle hole pairs excited otherwise. One then has, assuming again $q > 0$,

$$\begin{aligned} \sum_k (\hat{n}_{k-q} - \hat{n}_k) &= \left(\sum_{k \geq k_0} + \sum_{k < k_0} \right) (\hat{n}_{k-q} - \hat{n}_k) \\ &= \sum_{k \geq k_0} (\hat{n}_{k-q} - \hat{n}_k) \\ &= \sum_{k \geq k_0-q} \hat{n}_k - \sum_{k \geq k_0} \hat{n}_k \\ &= \sum_{k_0-q \leq k < k_0} \hat{n}_k = \frac{Lq}{2\pi}. \end{aligned} \quad (2.84)$$

The result is independent of k_0 , and one thus can take the limit $k_0 \rightarrow -\infty$. Together with an entirely parallel argument for ρ_- , this then proves (2.82). Moreover, for $q > 0$ both $\rho_+(-q)$ and $\rho_-(q)$ annihilate the noninteracting groundstate. One can easily recover canonical Bose commutation relations by introducing normalized operators, for example

$$\tilde{\rho}_+(q) = \sqrt{2\pi/(qL)} \rho_+(q) \quad (2.85)$$

would be a canonical creation operator, but we won't use this type of operators in the following.

- The noninteracting Hamiltonian obeys a simple commutation relation with the density operators. For example

$$[H_0, \rho_+(q)] = v_F q \rho_+(q), [H_0, \rho_\alpha(q)] = v_F \alpha q \rho_\alpha(q), \quad (2.86)$$

i.e. states created by $\rho_+(q)$ are eigenstates of H_0 , with energy $v_F q$. Consequently, the kinetic part of the Hamiltonian can be re-written as a term bilinear in boson operators, i.e. quartic in fermion operators:

$$H_0 = \frac{2\pi v_F}{L} \sum_{q>0, \alpha=\pm} \rho_\alpha(q) \rho_\alpha(-q) . \quad (2.87)$$

This equivalence may be made more apparent noting that $\rho_+(q)$ creates particle-hole pairs that all have total momentum q . Their energy is $\varepsilon_{k+q} - \varepsilon_k$, which, because of the linearity of the spectrum, equals $v_F q$, *independently of k*. Thus, states created by $\rho_+(q)$ are linear combinations of individual electron-hole excitations all with the same energy, and therefore are also eigenstates of (2.80).

3. The above point shows that the spectra of the bosonic and fermionic representations of H_0 are the same. To show complete equivalence, one also has to show that the degeneracies of all the levels are identical. This can be achieved calculating the partition function in the two representations and demonstrating that they are equal. This then shows that the states created by repeated application of ρ_\pm on the ground state form a complete set of basis states [23,24].

We now introduce interactions between the fermions. As long as only forward scattering of the type $(k_F; -k_F) \rightarrow (k_F; -k_F)$ or $(k_F; k_F) \rightarrow (k_F; k_F)$ is introduced, the model remains exactly solvable. The interaction Hamiltonian describing these processes takes the form

$$H_{\text{int}} = \frac{1}{2L} \sum_{q, \alpha=\pm} (g_2(q) \rho_\alpha(q) \rho_{-\alpha}(-q) + g_4(q) \rho_\alpha(q) \rho_\alpha(-q)) . \quad (2.88)$$

Here, $g_2(q)$ and $g_4(q)$ are the Fourier transforms of a real space interaction potential, and in a realistic case one would of course have $g_2(q) = g_4(q) = g(q)$, but it is useful to allow for differences between g_2 and g_4 . For Coulomb interactions one expects $g_2, g_4 > 0$. In principle, the long-range part of the Coulomb repulsion leads to a singular q -dependence. Such singularities in the g_i can be handled rather straightforwardly and can lead to interesting physical effects as will be discussed below. Here I shall limit myself to nonsingular g_2, g_4 . Electron-phonon interactions can lead to effectively attractive interactions between electrons, and therefore in the following I will not make any restrictive assumptions about the sign of the constants. One should however notice that a proper treatment of the phonon dynamics and of the resulting retardation effects requires more care [25].

Putting together (2.87) and (2.88), the complete interacting Hamiltonian, the *Tomonaga-Luttinger model*, then becomes a bilinear form in boson operators that is easily diagonalized by a Bogolyubov transformation. A first

consequence is the expression for the excitation spectrum

$$\omega(q) = |q| \sqrt{\left(v_F + \frac{g_4(q)}{2\pi} \right)^2 - \left(\frac{g_2(q)}{2\pi} \right)^2}. \quad (2.89)$$

The diagonal boson operators are linear combinations of the original ρ operators, and consequently, these elementary excitations are collective density oscillations, their energy being determined both by the kinetic energy term and the interactions.

We note here that in order for the Bogolyubov transformation to be a well-defined unitary transformation, $g_2(q)$ has to decrease at large q at least as $|q|^{-1/2}$ [23]. On the other hand, the large- q behavior of g_2 is unimportant for the low-energy properties of the model. We therefore in the following will almost always use a q -independent g_2 and g_4 . An approximate and frequently used way to cure the divergences arising due to this procedure is to keep a parameter α in subsequent formulae as a finite short-distance cutoff, of the order of a lattice spacing. One can then also include the “backward scattering” $(k_F; -k_F) \rightarrow (-k_F; k_F)$, because for spinless electron this is just the exchange analogue of forward scattering and does not constitute a new type of interaction. It is worthwhile emphasizing here that the solution is valid for arbitrarily strong interactions, no perturbative expansion is needed!

Up to this point, the construction does not allow for a direct calculation of correlation functions like the one-particle Green function or more generally any function involving individual creation or destruction operators. This type of correlation function becomes tractable by representing single particle operators in terms of the boson operators. To this end, we introduce the field operators

$$\phi(x) = -\frac{i\pi}{L} \sum_{q \neq 0} \frac{1}{q} e^{-\alpha|q|/2-iqx} (\rho_+(q) + \rho_-(q)) - N \frac{\pi x}{L}, \quad (2.90)$$

$$\Pi(x) = \frac{1}{L} \sum_{q \neq 0} e^{-\alpha|q|/2-iqx} (\rho_+(q) - \rho_-(q)) + \frac{J}{L}, \quad (2.91)$$

$$N = N_+ + N_-, \quad J = N_+ - N_-. \quad (2.92)$$

Here N_{\pm} are the numbers of particles added to the ground state on the right- and left-moving branch i.e.

$$N_{\pm} = \rho_{\pm}(q=0) = \sum_k (\hat{n}_{\pm,k} - \langle \hat{n}_{\pm,k} \rangle_0). \quad (2.93)$$

Because addition of a particle changes both N and J , one has the “selection rule” $(-1)^N = (-1)^J$; α is a cutoff parameter which (at least in principle, see the discussion above) has to be set to zero in the end of any calculation. The fields ϕ and Π then obey canonical boson commutation relations:

$$[\phi(x), \Pi(y)] = i\delta(x-y), \quad (2.94)$$

and ϕ is related to the local particle density via

$$\partial_x \phi = -\pi(\rho(x) - \rho_0), \quad (2.95)$$

where ρ_0 is the average particle density in the ground state. More precisely, in a lattice model this would represent the slowly varying components ($q \approx 0$) of the density, whereas components with $q \approx 2k_F$ would correspond to crossproducts between ψ_{\pm} .

The expression for the single fermion operators then is

$$\psi_{\pm}(x) = \lim_{\alpha \rightarrow 0} \frac{1}{\sqrt{2\pi\alpha}} U_{\pm} \exp [\pm ik_F x \mp i\phi(x) + i\theta(x)], \quad (2.96)$$

where the upper and lower sign refer to right- and left-moving electrons respectively, and

$$\begin{aligned} \theta(x) &= \pi \int^x \Pi(x') dx' \\ &= \frac{i\pi}{L} \sum_{q \neq 0} \frac{1}{q} e^{-\alpha|q|/2 - iqx} [\rho_+(q) - \rho_-(q)] + J \frac{\pi x}{L}. \end{aligned} \quad (2.97)$$

The U -operators (sometimes referred to as “Klein factors”; they are non-Hermitian: $U_{\alpha}^\dagger \neq U_{\alpha}$) decrease the total particle number on one of the branches by unity and are necessary because the boson fields all conserve the total particle number. These operators also insure proper anticommutation between right- and left-going operators: one easily checks that the “chiral components” $\phi_+ = \theta - \phi$ and $\phi_- = \theta + \phi$ commute with each other, one therefore needs the anticommutation relations $[U_{\alpha}, U_{\beta}]_+ = 0$, etc. In the thermodynamic limit $L \rightarrow \infty$, the fact that the U 's change the particle number is of minor importance because this represents a shift of k_F by a quantity of order $1/L$, and one then can replace the U_{α} 's by Majorana (Hermitian) fermion operators obeying $[\eta_{\alpha}, \eta_{\beta}]_+ = 0$, [26] as discussed in more detail in the appendix. In single-chain problems these effects play a minor role, however in the many chain systems to be discussed at the end of these notes, proper account of anticommutation is crucial.

A detailed derivation of the important (2.96) as an operator identity is given in the literature [23,24]. However, a simple plausibility argument can be given: creating a particle at site x requires introducing a kink of height π in ϕ , i.e. ϕ has to be shifted by π at points on the left of x . Displacement operators are exponentials of momentum operators, and therefore a first guess would be $\psi(x) \approx \exp(i\pi \int_{-\infty}^x \Pi(x') dx')$. However, this operator commutes with itself, instead of satisfying canonical anticommutation relations. Anticommutation is achieved by multiplying with an operator, acting at site x , that changes sign each time a particle passes through x . Such an operator is $\exp(\pm i\phi(x))$. The product of these two factors then produces (2.96).

The full Hamiltonian can also be simply expressed in terms of ϕ and Π . In the long-wavelength limit, neglecting the momentum dependence of the g_i , one can express the total Hamiltonian in the field phase formalism

$$H = H_0 + H_{\text{int}} = \int dx \left(\frac{\pi u K}{2} \Pi(x)^2 + \frac{u}{2\pi K} (\partial_x \phi)^2 \right). \quad (2.98)$$

This is obviously just the Hamiltonian of an elastic string, with the eigenmodes corresponding to the collective density fluctuations of the fermion liquid. It is important to notice that these collective modes are the only (low-energy) excited states, and that in particular *there are no well-defined single particle excitations*, nor are there the incoherent particle-hole pair excitations typical of a Fermi gas. The parameters in (2.98) are given by

$$u = \sqrt{\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2}, \quad K = \sqrt{\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}}. \quad (2.99)$$

The energies of the eigenstates are $\omega(q) = u|q|$, in agreement with (2.89). From the continuity equation, the expression (2.95) for the local particle density and the equation of motion of ϕ the (number) current density is

$$j(x) = uK\Pi(x). \quad (2.100)$$

Note in particular that for $g_2 = g_4$ one has $uK = v_F$, i.e. the expression for the current density is interaction-independent. The relation $uK = v_F$ holds in particular for systems with full (Galilean) translational invariance. On the other hand, in the continuum limit of lattice systems this relation is in general not true.

The most remarkable result here is the “collectivization” of the dynamics: there are no quasiparticle-like excitations. In fact there is a rather simple physical picture explaining this: imagine accelerating one particle a little bit in one direction. Very soon it will hit its neighbour and transmit its momentum to it, and the neighbour will in turn transmit its momentum to a further neighbour, and so on. Quite quickly, the initial localized motion will have spread coherently through the whole system. This picture can be formalized noting that in one dimension the difference between a solid and a fluid is not well-defined: whereas in higher dimensions solids and fluids are differentiated by the presence or absence of long-wavelength transverse modes, no transverse modes can exist in a system with movement along only one direction. The long-wavelength modes thus can equally well be considered as the phonons of a one-dimensional crystal [27,28]. Note that on the contrary in dimensions larger than one the neighbours of any given particle can be pushed aside, giving rise to a backflow that allows the particle to move through the system more or less freely.

The exclusive existence of collective excitations, at least at low energies, is one of the typical properties of the *Luttinger liquid*. Rather than discussing the physics of the spinless case in detail, we will turn now to the more interesting case of fermions with spin.

2.4.2 Model with Spin: the Concept of the Luttinger Liquid

In the case of spin-1/2 fermions, all the fermion operators acquire an additional spin index s . Following the same logic as above, the kinetic energy then takes the form

$$\begin{aligned} H_0 &= v_F \sum_{k,s} \left((k - k_F) a_{+,k,s}^\dagger a_{+,k,s} + (-k - k_F) a_{-,k,s}^\dagger a_{-,k,s} \right) \\ &= \frac{2\pi v_F}{L} \sum_{q>0, \alpha=\pm, s} \rho_{\alpha,s}(q) \rho_{\alpha,s}(-q) , \end{aligned} \quad (2.101)$$

where density operators for spin projections $s = \uparrow, \downarrow$ have been introduced:

$$\rho_{\pm,s}(q) = \sum_k a_{\pm,k+q,s}^\dagger a_{\pm,k,s} . \quad (2.102)$$

There are now two types of interaction. First, the “backward scattering” $(k_F, s; -k_F, t) \rightarrow (-k_F, s; k_F, t)$ which for $s \neq t$ cannot be re-written as an effective forward scattering (contrary to the spinless case). The corresponding Hamiltonian is

$$H_{\text{int},1} = \frac{1}{L} \sum_{k,p,q,s,t} g_1 a_{+,k,s}^\dagger a_{-,p,t}^\dagger a_{+,p+2k_F+q,t} a_{-,k-2k_F-q,s} . \quad (2.103)$$

And, of course, there is also the forward scattering, of a form similar to the spinless case

$$H_{\text{int},2} = \frac{1}{2L} \sum_{q,\alpha,s,t} \left(g_2(q) \rho_{\alpha,s}(q) \rho_{-\alpha,t}(-q) + g_4(q) \rho_{\alpha,s}(q) \rho_{\alpha,t}(-q) \right) . \quad (2.104)$$

To go to the bosonic description, one introduces ϕ and Π fields for the two spin projections separately, and then transforms to charge and spin bosons via $\phi_{\rho,\sigma} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$, $\Pi_{\rho,\sigma} = (\Pi_\uparrow \pm \Pi_\downarrow)/\sqrt{2}$. The operators ϕ_ν and Π_ν obey Bose-like commutation relations:

$$[\phi_\nu(x), \Pi_\mu(y)] = i\delta_{\nu\mu}\delta(x-y) , \quad (2.105)$$

and single fermion operators can be written in a form analogous to (2.96):

$$\psi_{\pm,s}(x) = \lim_{\alpha \rightarrow 0} \frac{\eta_{\pm,s}}{\sqrt{2\pi\alpha}} \exp \left(\pm ik_F x - i(\pm(\phi_\rho + s\phi_\sigma) + (\theta_\rho + s\theta_\sigma))/\sqrt{2} \right) , \quad (2.106)$$

where $\theta_\nu(x) = \pi \int^x \Pi_\nu(x') dx'$.

The full Hamiltonian $H = H_0 + H_{\text{int},1} + H_{\text{int},2}$ then takes the form

$$H = H_\rho + H_\sigma + \frac{2g_1}{(2\pi\alpha)^2} \int dx \cos(\sqrt{8}\phi_\sigma) . \quad (2.107)$$

Here α is a short-distance cutoff, and for $\nu = \rho, \sigma$

$$H_\nu = \int dx \left(\frac{\pi u_\nu K_\nu}{2} \Pi_\nu^2 + \frac{u_\nu}{2\pi K_\nu} (\partial_x \phi_\nu)^2 \right), \quad (2.108)$$

with

$$u_\nu = \sqrt{\left(v_F + \frac{g_{4,\nu}}{\pi}\right)^2 - \left(\frac{g_\nu}{2\pi}\right)^2}, \quad K_\nu = \sqrt{\frac{2\pi v_F + 2g_{4,\nu} + g_\nu}{2\pi v_F + 2g_{4,\nu} - g_\nu}}, \quad (2.109)$$

and $g_\rho = g_1 - 2g_2$, $g_\sigma = g_1$, $g_{4,\rho} = g_4$, $g_{4,\sigma} = 0$. The choice of sign (which is the conventional one) for the cosine-term in (2.107) corresponds to a particular “gauge choice”, as discussed in the appendix.

For a noninteracting system one thus has $u_\nu = v_F$ (charge and spin velocities equal!) and $K_\nu = 1$. For $g_1 = 0$, (2.107) describes independent long-wavelength oscillations of the charge and spin density, with linear dispersion relation $\omega_\nu(k) = u_\nu |k|$, and the system is conducting. As in the spinless case, there are no single-particle or single particle-hole pair excited states. This model (no backscattering), usually called the Tomonaga-Luttinger model, is the one to which the bosonization method was originally applied [20,21,29].

For $g_1 \neq 0$ the cosine term has to be treated perturbatively. We have already obtained the corresponding renormalization group equations in the previous Section (2.62). In particular, for repulsive interactions ($g_1 > 0$), g_1 is renormalized to zero in the long-wavelength limit, and at the fixed point one has $K_\sigma^* = 1$. The three remaining parameters in (2.107) then completely determine the long-distance and low-energy properties of the system.

It should be emphasized that (2.107) has been derived here for fermions with linear energy-momentum relation. For more general (e.g. lattice) models, there are additional operators arising from band curvature and the absence of high-energy single-particle states [23]. One can however show that all these effects are, at least for not very strong interaction, irrelevant in the renormalization group sense, i.e. they do not affect the low-energy physics. Thus, (2.107) is still the correct effective Hamiltonian for low-energy excitations. The lattice effects however intervene to give rise to “higher harmonics” in the expression for the single-fermion operators, i.e. there are low energy contributions at wavenumbers $q \approx (2m+1)k_F$ for arbitrary integer m [27].

The Hamiltonian (2.107) also provides an explanation for the physics of the case of negative g_1 , where the renormalization group scales to strong coupling (2.62). In fact, if $|g_1|$ is large in (2.107), it is quite clear that the elementary excitations of ϕ_σ will be small oscillations around one of the minima of the cosine term, or possibly soliton-like objects where ϕ_σ goes from one of the minima to the other. Both types of excitations have a gap, i.e. for $g_1 < 0$ one has a *gap in the spin excitation spectrum*, whereas the charge excitations remain massless. This can actually investigated in more detail in an exactly solvable case [30]. We will subsequently concentrate on the case $g_1 > 0$, so that there is no spin gap, however investigations of spectral

functions as described below have also been recently performed for the case with a spin gap [31].

Spin-charge separation. One of the more spectacular consequences of the Hamiltonian (2.107) is the complete separation of the dynamics of the spin and charge degrees of freedom. For example, in general one has $u_\sigma \neq u_\rho$, i.e. the charge and spin oscillations propagate with different velocities. Only in a noninteracting system or if some accidental degeneracy occurs one does have $u_\sigma = u_\rho = v_F$. To make the meaning of this fact more transparent, let us create an extra particle in the ground state, at $t = 0$ and spatial coordinate x_0 . The charge and spin densities then are easily found, using $\rho(x) = -(\sqrt{2}/\pi)\partial_x\phi_\rho$ (note that $\rho(x)$ is the deviation of the density from its average value) and $\sigma_z(x) = -(\sqrt{2}/\pi)\partial_x\phi_\sigma$:

$$\begin{aligned} \langle 0 | \psi_+(x_0) \rho(x) \psi_+^\dagger(x_0) | 0 \rangle &= \delta(x - x_0), \\ \langle 0 | \psi_+(x_0) \sigma_z(x) \psi_+^\dagger(x_0) | 0 \rangle &= \delta(x - x_0). \end{aligned} \quad (2.110)$$

Now, consider the time development of the charge and spin distributions. The time-dependence of the charge and spin density operators is easily obtained from (2.107) (using the fixed point value $g_1 = 0$), and one obtains

$$\begin{aligned} \langle 0 | \psi_+(x_0) \rho(x, t) \psi_+^\dagger(x_0) | 0 \rangle &= \delta(x - x_0 - u_\rho t), \\ \langle 0 | \psi_+(x_0) \sigma_z(x, t) \psi_+^\dagger(x_0) | 0 \rangle &= \delta(x - x_0 - u_\sigma t). \end{aligned} \quad (2.111)$$

Because in general $u_\sigma \neq u_\rho$, after some time charge and spin will be localized at completely different points in space, i.e. *charge and spin have separated completely*.

Here a linear energy-momentum relation has been assumed for the electrons, and consequently the shape of the charge and spin distributions is time-independent. If the energy-momentum relation has some curvature (as is necessarily the case in lattice systems) the distributions will widen with time. However this widening is proportional to \sqrt{t} , and therefore much smaller than the distance between charge and spin. Thus, the qualitative picture of spin-charge separation is unchanged.

Physical properties. The simple form of the Hamiltonian (2.107) at the fixed point $g_1^* = 0$ makes the calculation of physical properties rather straightforward. The specific heat now is determined both by the charge and spin modes, and consequently the specific heat coefficient γ is given by

$$\gamma/\gamma_0 = \frac{1}{2}(v_F/u_\rho + v_F/u_\sigma). \quad (2.112)$$

Here γ_0 is the specific heat coefficient of noninteracting electrons of Fermi velocity v_F .

The spin susceptibility and the compressibility are equally easy to obtain. Note that in (2.107) the coefficient u_σ/K_σ determines the energy necessary to create a nonzero spin polarization, and, as in the spinless case, u_ρ/K_ρ fixes the energy needed to change the particle density. Given the fixed point value $K_\sigma^* = 1$, one finds

$$\chi/\chi_0 = v_F/u_\sigma, \quad \kappa/\kappa_0 = v_F K_\rho/u_\rho, \quad (2.113)$$

where χ_0 and κ_0 are the susceptibility and compressibility of the noninteracting case. From (2.112) and (2.113) the Wilson ratio is

$$R_W = \frac{\chi}{\gamma} \frac{\gamma_0}{\chi_0} = \frac{2u_\rho}{u_\rho + u_\sigma}. \quad (2.114)$$

The quantity $\Pi_\rho(x)$ is proportional to the current density. As before, the Hamiltonian commutes with the total current, one thus has

$$\sigma(\omega) = 2K_\rho u_\rho \delta(\omega) + \sigma_{\text{reg}}(\omega), \quad (2.115)$$

i.e. the product $K_\rho u_\rho$ determines the weight of the DC peak in the conductivity. If the total current commutes with the Hamiltonian σ_{reg} vanishes, however more generally this part of the conductivity varies as ω^3 at low frequencies [32].

The above properties, linear specific heat, finite spin susceptibility, and DC conductivity are those of an ordinary Fermi liquid, the coefficients u_ρ , u_σ , and K_ρ determining renormalizations with respect to noninteracting quantities. However, the present system is *not a Fermi liquid*. This is in fact already obvious from the preceding discussion on charge-spin separation, and can be made more precise considering the single-particle Green function. Using the representation (2.106) of fermion operators one finds (at the fixed point $g_1 = 0$)

$$\begin{aligned} G^R(x, t) &= -i\Theta(t) \left\langle \left[\psi_{+,s}(x, t), \psi_{+,s}^\dagger(0, 0) \right]_+ \right\rangle \\ &= -\frac{\Theta(t)}{\pi} e^{ik_F x} \text{Re} \left(\frac{1}{\sqrt{(u_\rho t - x)(u_\sigma t - x)}} \left[\frac{\alpha^2}{(\alpha + iu_\rho t)^2 + x^2} \right]^{\delta/2} \right) \end{aligned} \quad (2.116)$$

where the presence of the Heaviside function ensures the retarded nature of G^R and $[,]_+$ denotes the anticommutation of fermion operators. Note that this expression factorizes into a spin and a charge contribution which propagate with different velocities. Fourier transforming (2.116) gives the momentum distribution function in the vicinity of k_F :

$$n_k \approx n_{k_F} - \text{const.} \times \text{sign}(k - k_F) |k - k_F|^\delta, \quad (2.117)$$

and for the single-particle density of states (i.e. the momentum-integrated spectral density) one finds:

$$N(\omega) \approx |\omega|^\delta. \quad (2.118)$$

In both cases $\delta = (K_\rho + 1/K_\rho - 2)/4$. Note that for any $K_\rho \neq 1$, i.e. *for any nonvanishing interaction*, the momentum distribution function and the density of states have power-law singularities at the Fermi level, with a vanishing single particle density of states at E_F . This behavior is obviously quite different from a standard Fermi liquid which would have a finite density of states and a step-like singularity in n_k . The absence of a step at k_F in the momentum distribution function implies the *absence of a quasiparticle pole* in the one-particle Green function. In fact, a direct calculation of the spectral function $A(k, \omega)$ from (2.116) [33,34] shows that the usual quasiparticle pole is replaced by a continuum, with a lower threshold at $\min_\nu(u_\nu(k - k_F))$ and branch cut singularities at $\omega = u_\rho p$ and $\omega = u_\sigma p$:

$$A(k, \omega) \approx (\omega - u_\sigma(k - k_F))^{\delta-1/2} \left| \omega - u_\rho(k - k_F) \right|^{(\delta-1)/2} \quad (u_\rho > u_\sigma), \quad (2.119)$$

$$A(k, \omega) \approx (\omega - u_\rho(k - k_F))^{(\delta-1)/2} \left| \omega - u_\sigma(k - k_F) \right|^{\delta-1/2} \quad (u_\rho < u_\sigma). \quad (2.120)$$

The coefficient K_ρ also determines the long-distance decay of all other correlation functions of the system: Using the representation (2.106) the charge and spin density operators at $2k_F$ are

$$\begin{aligned} O_{\text{CDW}}(x) &= \sum_s \psi_{-,s}^\dagger(x) \psi_{+,s}(x) \\ &= \lim_{\alpha \rightarrow 0} \frac{e^{2ik_F x}}{\pi \alpha} e^{-i\sqrt{2}\phi_\rho(x)} \cos[\sqrt{2}\phi_\sigma(x)], \end{aligned} \quad (2.121)$$

$$\begin{aligned} O_{\text{SDW}_x}(x) &= \sum_s \psi_{-,s}^\dagger(x) \psi_{+,-s}(x) \\ &= \lim_{\alpha \rightarrow 0} \frac{e^{2ik_F x}}{\pi \alpha} e^{-i\sqrt{2}\phi_\rho(x)} \cos[\sqrt{2}\theta_\sigma(x)]. \end{aligned} \quad (2.122)$$

Similar relations are also found for other operators. It is important to note here that all these operators decompose into a product of one factor depending on the charge variable only by another factor depending only on the spin field. Using the Hamiltonian (2.107) at the fixed point $g_1^* = 0$ one finds for example for the charge and spin correlation functions¹

$$\begin{aligned} \langle n(x)n(0) \rangle &= K_\rho/(\pi x)^2 + A_1 \cos(2k_F x)x^{-1-K_\rho} \ln^{-3/2}(x) \\ &\quad + A_2 \cos(4k_F x)x^{-4K_\rho} + \dots, \end{aligned} \quad (2.123)$$

$$\langle \mathbf{S}(x) \cdot \mathbf{S}(0) \rangle = 1/(\pi x)^2 + B_1 \cos(2k_F x)x^{-1-K_\rho} \ln^{1/2}(x) + \dots, \quad (2.124)$$

¹ The time- and temperature dependence is also easily obtained, see [35].

with model dependent constants A_i, B_i . The ellipses in (2.123) and (2.124) indicate higher harmonics of $\cos(2k_F x)$ which are present but decay faster than the terms shown here. Similarly, correlation functions for singlet (SS) and triplet (TS) superconducting pairing are

$$\begin{aligned}\langle O_{\text{SS}}^\dagger(x)O_{\text{SS}}(0)\rangle &= Cx^{-1-1/K_\rho} \ln^{-3/2}(x) + \dots, \\ \langle O_{\text{TS}_\alpha}^\dagger(x)O_{\text{TS}_\alpha}(0)\rangle &= Dx^{-1-1/K_\rho} \ln^{1/2}(x) + \dots.\end{aligned}\quad (2.125)$$

The logarithmic corrections in these functions [36] have been studied in detail recently [37–40]. The corresponding susceptibilities (i.e. the Fourier transforms of the above correlation functions) behave at low temperatures as

$$\chi_{\text{CDW}}(T) \approx T^{K_\rho-1} |\ln(T)|^{-3/2}, \quad \chi_{\text{SDW}}(T) \approx T^{K_\rho-1} |\ln(T)|^{1/2}, \quad (2.126)$$

$$\chi_{\text{SS}}(T) \approx T^{1/K_\rho-1} |\ln(T)|^{-3/2}, \quad \chi_{\text{TS}}(T) \approx T^{1/K_\rho-1} |\ln(T)|^{1/2}, \quad (2.127)$$

i.e. for $K_\rho < 1$ (spin or charge) density fluctuations at $2k_F$ are enhanced and diverge at low temperatures, whereas for $K_\rho > 1$ pairing fluctuations dominate. The “phase diagram”, showing in which part of parameter space which type of correlation diverges for $T \rightarrow 0$ is shown in Fig. 2.8.

These correlation functions with their power law variations actually determine experimentally accessible quantities: the $2k_F$ and $4k_F$ charge correlations lead to X-ray scattering intensities $I_{2k_F} \approx T^{K_\rho}$, $I_{4k_F} \approx T^{4K_\rho-1}$, and similarly the NMR relaxation rate due to $2k_F$ spin fluctuations varies as $1/T_1 \approx T^{K_\rho}$. The remarkable fact in all the above results is that there is only one coefficient, K_ρ , which determines all the asymptotic power laws.

We here re-emphasize the two important properties of spin-1/2 interacting fermions in one dimension: (i) correlation functions show power-law decay, with interaction-dependent powers determined by one coefficient, K_ρ ; and (ii) “spin-charge separation”: spin and charge degrees of freedom propagate with different velocities. Both these properties are typical of the Luttinger liquid and invalidate the Landau quasiparticle concept in one dimension.

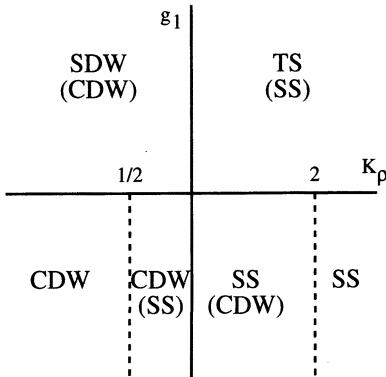


Fig. 2.8. Phase diagram for interacting spin-1/2 fermions

A nice experimental example of different spin-charge separation (different velocities for the spin and charge modes) is provided by Raman scattering experiments on single-channel quantum wires [41–43]. On the other hand, the situation in quasi-onedimensional conductors [44] is less clear: in compounds of the TTF–TCNQ series, the observation of strong $4k_F$ CDW fluctuations [45] seems only to be explainable in the Luttinger-liquid picture. On the other hand, in the much studied $(\text{TMTSF})_2\text{X}$ family of compounds, the NMR data can be interpreted in the Luttinger liquid framework [46], but much of the magnetotransport work of Chaikin and collaborators is explained rather satisfactorily using a Fermi liquid like picture.

Long-range interactions: Wigner crystallization. The above calculations can be straightforwardly generalized to the case of long-range interactions. Of interest is the case of unscreened Coulomb interactions ($V(r) = e^2/r$) which for example is of relevance for the physics of an isolated quantum wire. The short-distance singularity has to be cut off, and for example in a wire of diameter d an approximate form would be $V(r) = e^2/\sqrt{x^2 + d^2}$, leading to a Fourier transform $V(q) = 2e^2 K_0(qd)$. The long-range nature of the interaction is only of importance for the forward-scattering processes, and these appear only in the charge part of the Hamiltonian which is now given by

$$H_\rho = \frac{v_F}{2\pi} \int dx (\pi^2(1 + \tilde{g}_1)\Pi_\rho^2 + (1 - \tilde{g}_1)(\partial_x \phi_\rho)^2) + \frac{1}{\pi^2} \int dx dx' V(x - x') \partial_x \phi_\rho \partial_{x'} \phi_\rho , \quad (2.128)$$

where $\tilde{g}_1 = g_1/(2\pi v_F)$. The elementary excitations then are found to be charge oscillations (plasmons), with energy-momentum relation

$$\omega_\rho(q) = v_F |q| \sqrt{(1 + \tilde{g}_1)(1 - \tilde{g}_1 + 2\tilde{V}(q))} \quad (2.129)$$

where $\tilde{V}(q) = V(q)/(\pi v_F)$. The long-wavelength form ($q \rightarrow 0$), $\omega_\rho(q) \approx |q^2 \ln q|^{1/2}$, agrees with RPA calculations [47,48], however, the effect of g_1 , which is a short-range exchange contribution, are usually neglected in those calculations. The spin modes are still given by $\omega_\sigma(q) = u_\sigma |q|$, with $u_\sigma = v_F \sqrt{1 - \tilde{g}_1^2}$.

In the evaluation of correlation functions, the charge averages lead to

$$\langle (\phi_\rho(x) - \phi_\rho(0))^2 \rangle = \int_0^\infty dq \frac{1 - \cos qx}{q} \sqrt{\frac{1 + \tilde{g}_1}{1 - \tilde{g}_1 + 2\tilde{V}(q)}} \approx c_2 \sqrt{\ln x} , \quad (2.130)$$

with $c_2 = \sqrt{(1 + \tilde{g}_1)\pi v_F/e^2}$. One thus obtains for example

$$\begin{aligned} \langle \rho(x) \rho(0) \rangle &= A_1 \cos(2k_F x) \exp(-c_2 \sqrt{\ln x})/x \\ &\quad + A_2 \cos(4k_F x) \exp(-4c_2 \sqrt{\ln x}) + \dots , \end{aligned} \quad (2.131)$$

where $A_{1,2}$ are interaction dependent constants, and only the most slowly decaying Fourier components are exhibited. The most interesting point here is the extremely slow decay (much slower than any power law!) of the $4k_F$ component, showing an incipient charge density wave at wavevector $4k_F$. This slow decay should be compared with the power-law decay found for short-range interactions (2.123). The $4k_F$ oscillation period is exactly the average interparticle spacing, i.e. the structure is that expected for a one-dimensional *Wigner crystal*. Of course, because of the one-dimensional nature of the model, there is no true long-range order, however, the extremely slow decay of the $4k_F$ oscillation would produce strong quasi-Bragg peaks in a scattering experiment. It is worthwhile to point out that this $4k_F$ contribution arises even if the Coulomb interaction is extremely weak and depends only on the long-range character of the interaction. On the other hand, any $2k_F$ scattering is considerably weaker, due to the $1/x$ prefactor in (2.131) which has its origin in the contribution of spin fluctuations. The $2k_F$ spin correlations equally contain a $1/x$ factor and thus have the same asymptotic decay as the $2k_F$ charge correlations, (2.131). On the other hand, correlation functions that involve operators changing the total number of particles (e.g. the single particle Green function and pairing correlation functions) decay like $\exp[-\text{cst.} \times (\ln x)^{-3/2}]$, i.e. *faster* than any power law. This in particular means that the momentum distribution function n_k and all its derivatives are continuous at k_F , and there is only an essential singularity at k_F .

It is instructive to compare the above result (2.131), obtained in the limit of weak Coulomb interactions, with the case of strong repulsion (or, equivalently, heavy particles). The configuration of minimum potential energy is one of a chain of equidistant particles with lattice constant a , and quantum effects are expected to lead only to small oscillations in the distances between particles. The Hamiltonian then is

$$H = \sum_l \frac{p_l^2}{2m} + \frac{1}{4} \sum_{l \neq m} V''(ma)(u_l - u_{l+m})^2 , \quad (2.132)$$

where u_l is the deviation of particle l from its equilibrium position. In the long-wavelength limit, the oscillation of this lattice have energy $\omega(q) = \sqrt{2/(ma)eq|\ln(qa)|^{1/2}}$. The most slowly decaying part of the density-density correlation function then is

$$\langle \rho(x) \rho(0) \rangle \approx \cos(2\pi x/a) \exp\left(-\frac{4\pi}{(2me^2a)^{1/2}} \sqrt{\ln x}\right) . \quad (2.133)$$

Noticing that $k_F = \pi/(2a)$, one observes that the results (2.131) and (2.133) are (for $g_1 = 0$) identical as far as the long-distance asymptotics are con-

cerned, *including the constants in the exponentials*. Equation (2.131) was obtained in the weak interaction limit, whereas (2.133) applies for strong Coulomb forces. Similarly, the small- q limit of the charge excitation energies is identical.

2.4.3 Summary

In this section we have developed the basic bosonization formalism for one-dimensional interacting fermions and seen some elementary and direct applications to the calculation of some physical properties, in particular correlation functions. We have seen that the properties of the one-dimensional interacting system, the *Luttinger liquid*, are fundamentally different from two- or three-dimensional Fermi liquids. In particular the elementary excitations are *not quasiparticles* but rather collective oscillations of the charge and spin density, propagating coherently, but in general at different velocities. This gives rise to the interesting phenomenon of *spin-charge separation*. Finally, and again contrary to the Fermi-liquid case, most correlation functions show *non-universal powerlaws, with interaction-dependent exponents*. However, all these exponents depend only on one parameter, K_ρ , the spin analogue of which, K_σ , being fixed to unity by spin-rotation invariance ($K_\sigma \neq 1$ is possible if spin rotation invariance is broken). Beyond K_ρ , the only parameters that intervene in the low-energy physics of a Luttinger liquid are the velocities of the spin and charge modes, $u_{\rho,\sigma}$. In the spinless case only two parameters, K and u are involved. Finally, we have seen that long-range (Coulomb) interactions can profoundly modify these properties.

For lack of time and space, we have not touched here upon a number of interesting uses and generalizations of the bosonization method, in particular the Kondo effect [49] and applications to Fermi systems in more than one dimension [50,51]. An extensive recent review on Luttinger liquids and bosonization has been given by Voit [52].

2.5 Applications

We will subsequently discuss some results from applications of the formalism developed to interesting physical questions: transport and the effect of disorder, and the physics of antiferromagnetic spin chains. Here, we will come across the two cases where Luttinger liquid behavior is probably best established experimentally: the physics of quantum Hall edge states, upon which we will only touch briefly, and quantum spin systems, in particular the spin-1/2 antiferromagnet, which we will discuss in some detail, with some reference to experiment.

2.5.1 Transport

Conductivity and conductance. In the previous section we were concerned with equilibrium properties and correlation functions, in order to characterize the different phases possible in a one-dimensional system of interacting fermions. Here, we will investigate transport, in particular the DC conductivity. Finite-frequency effects have also been investigated, and the reader is referred to the literature [32,53].

To clarify some of the basic notions, let us first consider a Luttinger model in the presence of a weak space- and time-dependent external potential φ . The interaction of the fermions with φ is described by the extra term

$$H_{\text{ext}} = -e \int dx \hat{\rho}(x) \varphi(x, t) \quad (2.134)$$

in the total Hamiltonian. We will assume that the external field is slowly varying in space, so that in the particle-density operator $\hat{\rho}$ only products of either two right- or two left-going fermion operators appear but no cross terms. Standard linear response theory tells us that the current induced by the potential is given by

$$j(x, t) = -\frac{e^2}{\hbar} \int_{-\infty}^t dt' \int dx' D_{j\rho}(x - x', t - t') \varphi(x', t') , \quad (2.135)$$

where the *retarded current-density correlation function* is given by

$$\begin{aligned} D_{j\rho}(x, t) &= -i\Theta(t) \langle [j(x, t), \rho(0, 0)] \rangle \\ &= -\frac{u_\rho K_\rho}{\pi} \Theta(t) (\delta'(x - u_\rho t) + \delta'(x + u_\rho t)) . \end{aligned} \quad (2.136)$$

The second line is the result for spin-1/2 electrons. For spinless fermions one has to make the replacement $u_\rho K_\rho \rightarrow uK/2$.

Let us now first consider the situation where we adiabatically switch on a potential of frequency ω and wavenumber q along the whole length of the system. From (2.136) one then straightforwardly obtains the q - and ω -dependent conductivity as

$$\sigma(q, \omega) = \frac{4e^2}{\hbar} u_\rho K_\rho \frac{i(\omega + i0^+)}{(\omega + i0^+)^2 - u_\rho^2 q^2} . \quad (2.137)$$

In particular, the real part of the conductivity for constant applied field is

$$\text{Re } \sigma(0, \omega) = \frac{2e^2}{\hbar} u_\rho K_\rho \delta(\omega) , \quad (2.138)$$

in agreement with (2.115) (where units with $e^2 = \hbar = 1$ were used).

Applying on the other hand a static field over a finite part of the sample, one obtains a current $j = 2e^2 K_\rho U/h$, where U is the applied tension. The conductance thus is

$$G = \frac{2e^2}{h} K_\rho , \quad (2.139)$$

and depends on K_ρ only, not on u_ρ . For the noninteracting case $K_\rho = 1$ this is Landauer's well-known result [54]. Note that interactions affect the value of the conductance. The conductance here is independent on the length over which the field is applied. Noting that in dimension d the conductance is related to the DC conductivity via $G = L^{d-2}\sigma$, a length-independent conductance implies an infinite conductivity in one dimension, in agreement with (2.136). The fact that u_ρ does not appear in the expression for G can be understood noting that applying a static field over a finite (but large) part of the sample, one is essentially studying the wavenumber-dependent conductivity at strictly zero frequency, which from (2.137) is given by $\sigma(q, 0) = 2e^2 K_\rho \delta(q)/\hbar$, indeed independent on u_ρ . On the other hand, applying a field of finite frequency over a finite length ℓ , one can see that one measures the conductivity $\sigma(q \rightarrow 0, \omega)$ if $\omega > u_\rho/\ell$.

The situation of a finite static potential drop over only a finite part of a wire is clearly difficult to realize experimentally. However, the result (2.139), or more precisely its spinless analogue $G = Ke^2/h$, applies to the chiral Luttinger liquid as realized on the edge of quantum Hall systems [55–57]. On the other hand, for a quantum wire connected to measuring contacts which impose a potential difference over the whole length of the wire, one obtains [58,59,56,60]

$$G = \frac{2e^2}{h} , \quad (2.140)$$

independent of K_ρ , i.e. the momentum-conserving interactions play no role in the DC conductance, as to be expected. On the other hand, the value of K_ρ plays an important role for the effects of random and contact potentials [61,62]. These results permit a consistent explanation of some experiments on quantum wires [63], but leave open questions in other cases [64].

Persistent current. The Luttinger model description can be used straightforwardly to obtain the current induced in a strictly one-dimensional ring threaded by a magnetic flux Φ [65]. The argument can in fact be made very simply: in the one-dimensional geometry, the vector field can be removed entirely from the Hamiltonian via a gauge transformation, which then leads to the boundary condition $\psi(L) = \exp(2\pi i \Phi / \Phi_0) \psi(0)$ for the fermion field operator. Here L is the perimeter of the ring, and $\Phi_0 = hc/e$. For spinless fermions, this is achieved by replacing

$$\Pi(x) \rightarrow \Pi(x) + \frac{2\Phi}{L\Phi_0} \quad (2.141)$$

in the bosonization formula, (2.96). The total J -dependent part of the Hamiltonian then becomes

$$H_J = \frac{\pi u K}{2L} (J + 2\Phi/\Phi_0)^2 , \quad (2.142)$$

giving rise to a number current

$$j = \frac{\Phi_0}{2\pi} \frac{\partial E}{\partial \Phi} = \frac{uK}{L} \left(J + \frac{2\Phi}{\Phi_0} \right) . \quad (2.143)$$

At equilibrium, J is chosen so as to minimize the energy. Given that at constant total particle number J can only change by two units, one easily sees that the equilibrium (persistent) current has periodicity Φ_0 , and reaches its maximum value uK/L at $\Phi = \Phi_0/2$, giving rise to the familiar sawtooth curve for the current as a function of flux.

For fermions with spin, as long as there is no spin gap ($g_1 > 0$), the above results can be taken over, with the simple replacement $uK \rightarrow 2u_\rho K_\rho$, the factor 2 coming from the spin degeneracy. Note in particular that the persistent current, an equilibrium property, is given by the same combination of parameters as the Drude weight in the conductivity. This is an illustration of Kohn's result [66] relating the Drude weight to the effect of a magnetic flux through a ring.

In the case of negative g_1 , electrons can be transferred from the right to the left-going branch only by pairs. Consequently, the periodicity of the current and the ground state energy is doubled to $2\Phi_0$, and the maximum current is equally doubled. This behavior has actually been found in numerical calculations [67,68].

Quantum Hall edge states. The Luttinger liquid picture has an interesting application to the physics of the fractional quantum Hall effect, as discovered and discussed by Wen [69,70]. To see how this comes about, consider the states available in the different Landau levels in the vicinity of the edge of the quantum Hall device, as shown in Fig. 2.9 [71]. It is clear that low-energy states only exist at the edge (the bulk quantum Hall state is well-known to be characterized by a finite excitation gap), and close to the Fermi energy (i.e. the edge) the states have a linear dispersion relation. This can be made particularly clear if one assumes a disk-shaped sample: the states have a wavefunction

$$\approx z^k \approx (e^{i\theta})^k , \quad (2.144)$$

with k increasing linearly with radial position. The angular momentum quantum number k thus plays a role very similar to linear momentum in the linear geometry we have assumed up to now. One thus can linearize the dispersion in Fig. 2.9 and obtains essentially the spinless model discussed in Sect. 2.4.1,

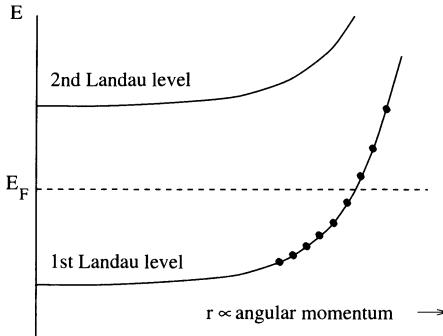


Fig. 2.9. Quantum states in the lowest Landau level in the vicinity of the edge of a quantum hall device. The spatial variation of the confining potential is assumed to be slow on the scale of the magnetic length, so that the energies of the different quantum states are determined by the local value of the confining potential. For a circular device the angular momentum of a state increases proportionally to its distance r from the origin

the only difference being that here only right-going particles exist. This difference is the origin of the term *chiral* Luttinger liquid (in fact, the left going branch is to be found on the opposite edge of the device). Because there are no left-going particles (or at least they can be thought of as being at a macroscopic distance), there also is no right-left interaction, and consequently one expects the noninteracting value $K = 1$. Moreover all the left-going components of the fields have to be projected out, for example one has to replace $\phi \rightarrow \phi_+ = (\phi - \theta)/2$.

However, straightforward adoption of this scheme leads to trouble: from the preceding subsection we know that $K = 1$ leads to a conductance (which in the present case is the Hall conductance) of $G = e^2/h$, *different* from the well-known

$$G = \nu \frac{e^2}{h} \quad (2.145)$$

valid for a fractional quantum Hall state ($\nu = 1/m$ is the filling factor). To repair this problem one makes the *hypothesis* that instead of (2.95) and (2.100) one has

$$\rho(x) = -\frac{\sqrt{\nu}}{\pi} \frac{\partial \phi_+}{\partial x} \quad , \quad j(x) = u\sqrt{\nu} \Pi_+(x) , \quad (2.146)$$

where the subscripts indicate projection on right-going states. With these definitions following the calculations of the previous subsection one now straightforwardly reproduces the correct result, (2.145). The appearance of the factors $\sqrt{\nu}$ in (2.146) indicates that the objects occupying the states in Fig. 2.9

are not free electrons but rather strongly affected by the physics of the bulk of the samples. A more detailed derivation, starting from a Chern–Simons field theory for the bulk physics, has also been given by Wen [70].

Beyond reproducing the correct value of the Hall conductance, the above hypothesis leads to a number of interesting conclusions. Consider first the creation operator for a real electron (charge e) on the edge. Following the arguments of Sect. 2.4.1, because of (2.146), the bosonized version of the electron operator now must create a jump of ϕ of height $\pi/\sqrt{\nu}$, rather than of height π . This leads to

$$\psi_+(x) \approx e^{-i\phi_+(x)/\sqrt{\nu}}, \quad (2.147)$$

x being the coordinate along the perimeter of the sample. Now, these operators obey the relation

$$\psi_+(x')\psi_+(x) = e^{\pm i\pi/\nu}\psi_+(x)\psi_+(x'). \quad (2.148)$$

But the real electron is still a fermion, i.e. ψ_+ must obey anticommutation relations. Thus $m = 1/\nu$ *has to be an odd integer*. One thus reproduces one of the fundamental facts of the fractional quantum Hall effect. From (2.147) one also finds a decay of the single-electron Green function as

$$G(x, t) \propto \frac{1}{(x - ut)^{1/\nu}}. \quad (2.149)$$

Another fundamental property of the quantum Hall state appears when one considers the fractionally charged elementary excitation of charge $e\nu$ at the edge. A charge- $e\nu$ object is created by

$$\psi_{+\nu}(x) \approx e^{-i\sqrt{\nu}\phi_+(x)}, \quad (2.150)$$

leading to a slow decay of the corresponding Green function, with exponent ν , instead of $1/\nu$ in (2.149). One now has the relation

$$\psi_{+\nu}(x')\psi_{+\nu}(x) = e^{\pm i\pi\nu}\psi_{+\nu}(x)\psi_{+\nu}(x'), \quad (2.151)$$

i.e. exchanging the fractionally charged objects one obtains nontrivial ($\neq \pm 1$) phase factors and these quasiparticles thus also obey fractional statistics.

A single hypothesis, the insertion of the factors $\sqrt{\nu}$ in (2.146), thus reproduces two of the fundamental facts about the fractional quantum Hall effect! In addition one obtains a basis for the treatment of transport phenomena mentioned at the end of Sect. 2.5.2 and results for the asymptotics of Green functions.

2.5.2 Disorder

Effects of isolated impurities. The infinite conductivity in the ideally pure systems considered up to here is a natural but hardly realistic result: any realistic system will contain some form of inhomogeneity. This in general leads to a finite conductivity, and in one dimension one can anticipate even more dramatic effects: in a noninteracting system any form of disorder leads to localization of the single-particle eigenstates [72,73]. How this phenomenon occurs in interacting systems will be discussed in this and the following section.

Following Kane and Fisher [74], consider first the case of a single inhomogeneity in an otherwise perfect one-dimensional system. The extra term in the Hamiltonian introduced by a localized potential $v(x)$ is (for spinless fermions)

$$H_{\text{barrier}} \propto \int dx \psi^\dagger(x) \psi(x) . \quad (2.152)$$

Decomposing the product of fermion operators into right- and left-going parts, one has

$$\psi^\dagger \psi = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ . \quad (2.153)$$

In the bosonic representation, the first two terms are proportional to $\partial_x \phi$ (see (2.95)), and therefore the corresponding contribution in (2.152) can in fact be eliminated by a simple unitary transformation of ϕ . These terms represent scattering with momentum transfer $q \ll 2k_F$, i.e. they do not transfer particles between k_F and $-k_F$ and therefore do not affect the conductance in any noticeable way. On the other hand, the last two terms in (2.153) represent scattering with $|q| \approx 2k_F$, i.e. from the $+$ to the $-$ branch and *vice versa*. These terms certainly are expected to affect the conductance, because they change the direction of propagation of the particles. The bosonic representation of these terms is

$$H_{\text{barrier}} = \frac{V(2k_F)}{\pi\alpha} \cos 2\phi(0) , \quad (2.154)$$

where the potential $V(x)$ is assumed to be centered at $x = 0$. For this reason, only the value of the ϕ at $x = 0$ intervenes in (2.154).

One now can integrate out all the degrees of freedom away from $x = 0$, to obtain an effective action implying only the time-dependence of $\phi(0)$. Then a renormalization group equation for $V \equiv V(2k_F)$ can be found as

$$\frac{dV}{d\ell} = (1 - K)V , \quad (2.155)$$

where $E = E_0 e^{-\ell}$, E_0 is the original cutoff, and E is the renormalized cutoff.

From (2.155) it is clear that there are three regimes:

1. For $K > 1$ one has $V(\ell \rightarrow \infty) = 0$, i.e. as far as the low-energy physics is concerned, the system behaves like one without the barrier. In particular, the low-temperature conductance takes the “pure” value $G = e^2 K / h$, with corrections of order $T^{2(K-1)}$. We note that in this case superconducting fluctuations dominate, and the perfect transmission through the barrier can be taken as a manifestation of superconductivity in the one-dimensional system.
2. For the noninteracting case $K = 1$, V is invariant, and one thus has partial transmission and a non-universal conductance depending on V .
3. For $K < 1$, $V(\ell)$ scales to infinity. Though the perturbative calculation does not provide any direct way to treat this case, it is physically clear that the transmission and therefore the conductance should vanish.

Note that the non-interacting case is marginal, separating the regions of perfect and zero transmission. These results are very similar to earlier ones by Mattis [75] and by Luther and Peschel [76] who treat disorder in lowest-order perturbation theory.

The case of $K < 1$ can be further analyzed considering the case of two finite Luttinger liquids coupled by a weak tunneling barrier, as would be appropriate for a strong local potential. The barrier Hamiltonian then is

$$H_{\text{barrier}} = t \left(\psi_1^\dagger(0)\psi_2(0) + \psi_2^\dagger(0)\psi_1(0) \right) \approx \frac{t}{\pi\alpha} \cos 2\theta(0). \quad (2.156)$$

Here $\psi_{1,2}$ are the field operators to the left and to the right of the barrier. The operators have to satisfy the *fixed boundary condition* $\psi_i(x = 0) = 0$, different from the periodic boundary conditions we have used so far. Noting that the ψ_i can be decomposed into left- and right-going parts as $\psi_i = \psi_{i+} + \psi_{i-}$, this can be achieved, using (2.96), by imposing the fixed boundary condition $\phi_i(x = 0) = \pi/2$ on the boson field [77,78].

One can now proceed in complete analogy to the weak- V case to obtain the renormalization group equation

$$\frac{dt}{d\ell} = \left(1 - \frac{1}{K} \right) t. \quad (2.157)$$

Again, there are three different regimes: (i) for $K > 1$ now $t(\ell \rightarrow \infty) \rightarrow \infty$, i.e. the tunneling amplitude becomes very big. This can be interpreted as indicating perfect transmission, e.g. $G = e^2 K / h$; (ii) the case $K = 1$ remains marginal, leading to a t -dependent conductance; (iii) for $K < 1$ t scales to zero, there thus is no transmission, and $G = 0$. The results obtained in the two limiting cases of small V (weak scattering) and of small t (weak tunneling) are clearly compatible: e.g. for $K < 1$, V becomes large, i.e. at sufficiently low energies one expects essentially a tunneling type behavior, and then from (2.157) the tunneling amplitude actually does scale to zero, giving

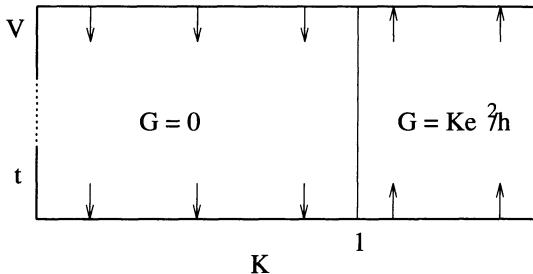


Fig. 2.10. “Phase diagram” of a localized inhomogeneity in a spinless Luttinger liquid, characterized by an exponent parameter K , according to [74]. The scaling trajectories calculated for weak V or t are indicated by arrows. It is clearly plausible to assume direct scaling from weak to strong coupling in the whole range of K

zero conductance in the low-energy (or low-temperature) limit. For $K \geq 1$ a similar compatibility of the two limiting cases is found. The global behavior can be represented by the “phase diagram” in Fig. 2.10. For electrons with spin but spin-independent interactions, results are very similar: the separation between zero and perfect transmission is at $K_\rho = 1$, with $K_\rho = 1$ again the marginal case. In the transmitting region the conductance is $G = 2K_\rho e^2/h$.

These considerations can be generalized to the case of two barriers [74,79]. In particular, assuming that there are two identical, weakly scattering barrier at $\pm d$, the effective scattering potential becomes $V_{\text{eff}}(q) = 2v(q) \cos(qd/2)$. Though in general this is non-zero when $V(q)$ is non-zero, for particular values of k_F , so that $\cos(k_F d) = 0$, this potential vanishes, giving rise to perfect transmission even for $K < 1$. This *resonant scattering* condition corresponds to an average particle number between the two barriers of the form $\nu + 1/2$, with integer ν , i.e. the “island” between the two barriers is in a degenerate state. If interactions between the electrons in the island are included, one can recover the physics of the Coulomb blockade [74,79].

For the chiral Luttinger liquid, discussed in Sect. 2.5.1, backscattering events a priori seem to be excluded because all the particles are moving in the same direction. In that sense the chiral Luttinger liquid can be considered as “perfect”. However, if the quantum hall device has a constriction that brings the two edges close to each other, scattering from one edge to the other becomes possible and is the equivalent of backscattering. Then similar considerations as made for the single-impurity case are possible [80], and in particular the crossover function describing the conductance through a resonance as a function of temperature [81,82] and the $I(V)$ characteristic [83] have been obtained.

Anderson localization of one-dimensional interacting fermions. The discussion of the previous section was concerned with the effect of at most two impurities, weak or strong. Clearly, in that case the effects of coherent

scattering from many impurities, which typically give rise to Anderson localization, are absent. We now turn to this more complicated case which had been studied in fact well before the single impurity work [84–87].

In the absence of electron-electron interactions, localization effects can be discussed in the framework of a scaling theory [73]. Under the assumption that at some short length scale one has elastic scattering of electrons off impurities, this theory leads to the following β -function for the variation of the conductance with linear dimension L :

$$\beta(G) = \frac{d \ln(G)}{d \ln(L)} = d - 2 - \frac{a}{G} + \dots , \quad (2.158)$$

where a is a constant and d the spatial dimensionality. In particular in one dimension this leads to a conductance decaying exponentially with the length of the system, exhibiting clearly the localized character of the all single-electron states (a fact first shown by Mott [72] and studied in great detail since [88,89]).

In an interacting one-dimensional system (as described by the Luttinger liquid picture of the previous section) now a number of questions arise: what is the influence of disorder on the phase diagram obtained previously? What are the transport properties? Can one have true superconductivity in one dimension, i.e. infinite conductivity in a disordered system? To answer these questions we discuss below the generalizations necessary to include disorder in our previous picture.

We start by the standard term in the Hamiltonian describing the coupling of a random potential to the electron density

$$H_{\text{imp}} = \sum_i \int dx V(x - R_i) \hat{\rho}(x) , \quad (2.159)$$

where the R_i are the random positions of impurity atoms, each acting with a potential V on the electrons. In one dimension one can distinguish two types of processes: (i) forward scattering, where the scattered particle remains in the vicinity of its Fermi point. As in the single-impurity case, this leads to a term proportional to $\partial\phi_\rho$ and can be absorbed by a simple redefinition of the ϕ_ρ field. The physical effects are minor, and in particular the DC conductivity remains infinite. (ii) backward scattering where an electron is scattered from k_F to $-k_F$ or *vice versa*. For small impurity density this can be represented by a complex field ξ with Gaussian distribution of width $D_\xi = n_i V (q = 2k_F)^2$:

$$H_b = \sum_\sigma \int dx \left(\xi(x) \psi_{R\sigma}^\dagger(x) \psi_{L\sigma}(x) + \text{h.c.} \right) . \quad (2.160)$$

This term has dramatic effects and in particular leads to Anderson localization in the noninteracting case [90].

From a perturbative expansion in the disorder one now obtains a set of coupled renormalization group equations [87]:

$$\begin{aligned}\frac{dK_\rho}{d\ell} &= -\frac{u_\rho}{2u_\sigma} K_\rho^2 \mathcal{D} \\ \frac{dK_\sigma}{d\ell} &= -\frac{1}{2}(\mathcal{D} + y^2) K_\sigma^2 \\ \frac{dy}{d\ell} &= 2(1 - K_\sigma)y - \mathcal{D} \\ \frac{d\mathcal{D}}{d\ell} &= (\beta - K_\rho - K_\sigma - y)\mathcal{D}\end{aligned}\tag{2.161}$$

where $\mathcal{D} = 2D_\xi\alpha/(\pi u_\sigma^2)(u_\sigma/u_\rho)^{K_\rho}$ is the dimensionless disorder, $y = g_{1\perp}/(\pi u_\sigma)$ is the dimensionless backscattering amplitude, and the K_ν are defined in (2.109). These equations are valid for *arbitrary* K_ν (the usual strength of bosonization), but to lowest order in \mathcal{D} and y .

As a first application of (2.161) one can determine the effect of the random potential on the “phase diagram”, as represented in Fig. 2.8. In fact, there are three different regimes:

1. For $K_\rho > 2$ and $g_{1\perp}$ sufficiently positive the fixed point is $\mathcal{D}^*, y^* = 0, K_\rho^* \geq 2$. Because the effective random potential vanishes this is a delocalized region, characterized as in the pure case by the absence of a gap in the spin excitations and dominant TS fluctuations.
2. For $K_\rho > 3$ and $g_{1\perp}$ small or negative one has $\mathcal{D}^* = 0, y \rightarrow -\infty, K_\rho^* \geq 3$. Again, this is a delocalized region, but now because $y \rightarrow -\infty$ there is a spin gap and one has predominant SS fluctuations.
3. In all other cases one has $\mathcal{D} \rightarrow \infty, y \rightarrow -\infty$. This corresponds to a localized regime. For small K_ρ the bosonized Hamiltonian in this regime is that of a charge density wave in a weak random potential with small quantum fluctuations parameterized by K_ρ . This region can therefore be identified as a weakly pinned CDW, also called a “charge density glass” (CDG). The transition from the CDG to the SS region then can be seen as depinning of the CDW by quantum fluctuations.

One should notice that the CDG is a nonmagnetic spin singlet, representing approximately a situation where localized single-particle states are doubly occupied. Though this is acceptable for attractive or possibly weakly repulsive interactions, for strong short-range repulsion single occupancy of localized states seems to be more likely. One then has a spin in each localized state, giving probably rise to a localized antiferromagnet with random exchange (RAF).

A detailed theory of the relative stability of the two states is currently missing and would certainly at least require higher-order perturbative treatment. The boundaries of the different regimes can be determined in many cases from (2.161), and the resulting phase diagram is shown in Fig. 2.11.

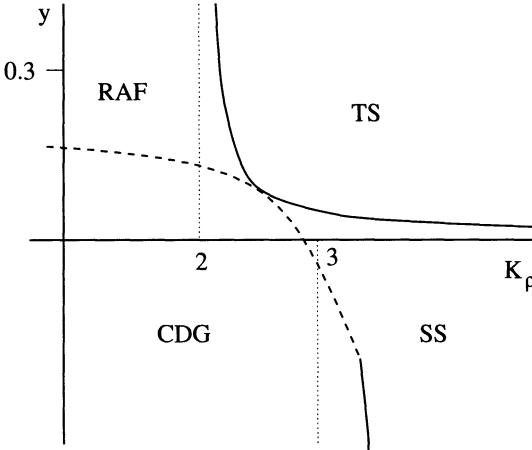


Fig. 2.11. Phase diagram of a Luttinger liquid in the presence of a weak random potential ($\mathcal{D} = 0.05$). The full lines represent results obtained directly from the scaling equations (2.162), the dashed lines are qualitative interpolations. The dotted lines are the phase boundaries in the limit $\mathcal{D} \rightarrow 0$

The localization length for small disorder can be obtained from standard scaling arguments: suppose that a system with some fixed disorder \mathcal{D}_0 has a localization length ξ_0 . Then in the general case one has $\xi(\mathcal{D}) = \xi_0 e^{\ell(\mathcal{D}_0, \mathcal{D})}$, where $\ell(\mathcal{D}_0, \mathcal{D})$ is the time it takes for the “bare” disorder \mathcal{D} to scale up to \mathcal{D}_0 . From this reasoning one finds, for the case without a spin gap ($g_1 > 0$) and weak disorder

$$\xi(\mathcal{D}) \propto (1/\mathcal{D})^{1/(2-K_\rho)} . \quad (2.162)$$

Note that for $K_\rho > 1$, i.e. superconducting fluctuations predominating in the pure case ξ is greater than the mean free path $\lambda \propto 1/\mathcal{D}$, there is a kind of diffusive regime, contrary to the noninteracting case. On the other hand, for $K_\rho < 1$ one has $\xi < \lambda$. In the vicinity of the TS–RAF boundary one has

$$\xi(\mathcal{D}) \propto \exp\left(\frac{K_\rho - 2}{\mathcal{D} - y(K_\rho - 2)}\right) . \quad (2.163)$$

The analogous results for the case with a spin gap ($g_1 < 0$) are

$$\xi(\mathcal{D}) \propto (1/\mathcal{D})^{1/(3-K_\rho)} \quad (2.164)$$

$$\xi(\mathcal{D}) \propto \exp\left(\frac{2\pi}{\sqrt{9\mathcal{D} - (K_\rho - 3)^2}}\right) . \quad (2.165)$$

There are two points to be noted about this result: (i) for $K_\rho = 0$ one has $\xi \propto \mathcal{D}^{-1/3}$, which is the same result as that found for the pinning length

of a classical CDW [91]. (ii) the results (2.162, 2.163) and (2.164, 2.165) are qualitatively different, both in the vicinity of the phase boundaries and in the localized states. The transitions are thus in different universality classes, and this strongly supports the idea that the localized phases reached through the transition are themselves different (RAF or CDG).

The temperature dependence of the DC conductivity can be obtained noting that at finite temperature there are no coherent effects on length scales larger than v_F/T . One therefore stops renormalization at $e^{\ell^*} = v_F/(\alpha T)$. As long as \mathcal{D} remains weak one can still use the Born approximation to obtain

$$\sigma(\mathcal{D}) = \sigma_0 \frac{e^\ell \mathcal{D}}{\mathcal{D}(\ell)}, \quad (2.166)$$

where $\sigma_0 = e^2 v_F^2 / 2\pi\hbar D_\xi$ is the lowest order conductivity. In the delocalized phases one then finds a conductivity diverging as $\sigma(T) \sim T^{-1-\gamma}$ where $\gamma = K_\rho^* - 2$ in the TS case and $\gamma = K_\rho^* - 3$ in the SS case. On the phase boundaries one has universally $\sigma \sim 1/T$. In the localized region \mathcal{D} diverges at low temperatures, and a perturbative calculation thus becomes meaningless. However, the conducting-localized crossover can still be studied at not too low temperatures [87]. In particular, the high-temperature conductivity is found to vary as $\sigma \sim T^{1-K_\rho}$. This is the perturbative result first found by Mattis [75] and by Luther and Peschel [76] and also reproduced by the single-impurity calculations [74]. The high-temperature behavior thus can be understood in terms of scattering off the individual impurities. On the other hand, at lower temperatures one necessarily comes into the region where \mathcal{D} increases sharply. This has its origin in coherent scattering from many impurities and ultimately gives rise to localization.

One can finally notice the effects of different types of interactions on localization. Roughly speaking, for forward scattering repulsion ($g_2 > 0$) enhances localization whereas attraction weakens it. In particular, strong attraction leads to vanishing effective random potential. The delocalized state then can be considered to be a true superconductor in the sense that there is infinite conductivity even in an impure system. The effect of backward scattering interactions is opposite to that of forward interactions.

2.5.3 The Spin-1/2 Chain as a Luttinger Liquid

One of the fundamental models of solid state physics is the Heisenberg model of insulating magnets. In the one-dimensional case (“spin chains”) its Hamiltonian takes the form

$$\begin{aligned} H &= \sum_{l=1}^L (S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z) \\ &= \sum_{l=1}^L \left(\frac{1}{2} (S_l^+ S_{l+1}^- + S_l^- S_{l+1}^+) + \Delta S_l^z S_{l+1}^z \right). \end{aligned} \quad (2.167)$$

Here $\mathbf{S}_l = (S_l^x, S_l^y, S_l^z)$ is a spin operator (we will first concentrate on the case of spin 1/2 so that $\mathbf{S}_l \cdot \mathbf{S}_l = 3/4$) acting on site l , Δ is an anisotropy parameter that allows one to treat the antiferromagnetic ($\Delta = 1$), the ferromagnetic ($\Delta = -1$), and general anisotropic cases, and periodic boundary conditions imply $\mathbf{S}_{L+1} = \mathbf{S}_1$. We notice that the Hamiltonian conserves the z -component of the total spin (for $|\Delta| = 1$ total spin is also conserved).

The spin model can be transformed into a model of spinless fermions, noting that S_l^+ and S_l^- anticommute. The *Jordan–Wigner transformation* [92] then relates spin to fermion operators (a_l, a_l^\dagger) via

$$S_l^+ = a_l^\dagger \exp \left(i\pi \sum_{j=1}^{l-1} a_j^\dagger a_j \right) , \quad S_l^z = a_l^\dagger a_l - \frac{1}{2} . \quad (2.168)$$

Presence or absence of a fermion now represent an up or down spin, and the exponential factor insures that spin operators on different sites commute, whereas fermionic operators of course anticommute. The transformation can now be used to rewrite the spin Hamiltonian (2.167) in terms of fermions as

$$H = \sum_{l=1}^L \left(\frac{1}{2} \left(a_l^\dagger a_{l+1} + a_{l+1}^\dagger a_l \right) + \Delta \left(a_l^\dagger a_l - \frac{1}{2} \right) \left(a_{l+1}^\dagger a_{l+1} - \frac{1}{2} \right) \right) . \quad (2.169)$$

The “spin-flip” terms thus give rise to motion of the fermions, whereas the S^z - S^z interaction leads to a fermion-fermion interaction between adjacent sites.

It is instructive to see in detail how one can pass to the continuum limit and then to a bosonic model starting from (2.169). We start by passing to momentum space in the standard way:

$$a_l = \frac{1}{\sqrt{L}} \sum_k a_k e^{ikl} , \quad (2.170)$$

with the momentum sum restricted to the first Brillouin zone: $-\pi < k \leq \pi$. Insertion into (2.169) then straightforwardly leads to

$$\begin{aligned} H &= \sum_k (-\cos k - \Delta) a_k^\dagger a_k \\ &\quad + \frac{\Delta}{L} \sum_{k_1} \delta(k_1 + k_2 - k_3 - k_4) e^{i(k_1 - k_4)} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \end{aligned} \quad (2.171)$$

$$\begin{aligned} &= \sum_k (-\cos k - \Delta) a_k^\dagger a_k \\ &\quad + \frac{\Delta}{2L} \sum_{k_1 k_2 q} (\cos q - \cos(k_1 - k_2 + q)) a_{k_1+q}^\dagger a_{k_2-q}^\dagger a_{k_2} a_{k_1} , \end{aligned} \quad (2.172)$$

where the δ -symbol insures momentum conservation modulo 2π : $\delta(x) = 1$ if $x = 0 \bmod 2\pi$, and $\delta(x) = 0$ otherwise. In order to be close to the standard case of a band minimum at $k = 0$ we have shifted the origin of k -space by π . This amounts to a rotation of every other spin in (2.167) by π around the z axis. In (2.172) the form factor appearing in the interaction has been properly symmetrized. This will be useful subsequently.

At least for weak interaction (small Δ) only interactions involving states close to the Fermi energy are important, and one therefore can then map (2.171) onto the spinless Luttinger model discussed in Sect. 2.4. For the g_2 interaction, processes with either $k_1 \approx k_4$ or $k_1 \approx k_3$ contribute (the \approx sign is meant to indicate that both momenta are close to the same Fermi point). To cast the second type of processes into the form of (2.88) one has to commute two fermion operators, giving rise to an extra minus sign. The coupling constant at small momentum transfer then is

$$g_2 = 2\Delta(1 - \cos 2k_F) . \quad (2.173)$$

Note that this vanishes when the spin chain is nearly fully spin-polarized ($k_F \rightarrow 0, \pi$).

The determination of the appropriate parameter g_4 is slightly less straightforward. Naively, one would expect that processes where all four states are in the vicinity of the same Fermi point contribute, giving $g_4 = 2\Delta$. However, as pointed out by Fowler [93], this is not correct: in fact, in the lattice model there are corrections to the bare fermion energy coming from the exchange part of the first-order Hartree–Fock selfenergy, given by

$$\Sigma_{HF}(k) = -\frac{2\Delta}{L} \sum_p \langle a_p^\dagger a_p \rangle = -\frac{2\Delta}{\pi} \sin k_F \cos k . \quad (2.174)$$

This leads to a renormalization of the Fermi velocity

$$v_F = \sin k_F \rightarrow \sin k_F + \frac{2\Delta}{\pi} \sin^2 k_F . \quad (2.175)$$

On the other hand, in the continuum model of Sect. 2.4, such Hartree–Fock selfenergy corrections must not be considered (formally by appropriately normal ordering the interaction terms), otherwise they would be infinite due to sums over infinitely many occupied states. The proper way to account for the finite velocity renormalization, (2.175), then is to include this via a properly chosen g_4 . In the present case, in order to reproduce (2.175) one then has to set

$$g_4 = g_2 = 2\Delta(1 - \cos 2k_F) . \quad (2.176)$$

Note that at half-filling this is twice the naive expectation.

Following the steps of Sect. 2.4, the low-energy excitations of the spin chain model then are described by the Hamiltonian [94]

$$H = \int dx \left(\frac{\pi u K}{2} \Pi(x)^2 + \frac{u}{2\pi K} (\partial_x \phi)^2 \right). \quad (2.177)$$

This is of course our standard Hamiltonian, i.e. *spin chains are Luttinger liquids* [23]. Specializing to the case of zero applied field, so that the total magnetization vanishes and one has $k_F = \pi/2$, the parameters are given perturbatively by

$$u = \sqrt{1 + \frac{4\Delta}{\pi}} \quad , \quad K = \frac{1}{\sqrt{1 + \frac{4\Delta}{\pi}}}. \quad (2.178)$$

These results agree to first order in Δ with the exact ones, to be discussed in Sect. 2.5.3 below, and are close to them over much of the parameter range.

For an exactly half-filled band extra umklapp scattering processes are possible, with $k_{1,2} \approx \pi/2$, $k_{3,4} \approx -\pi/2$, and *vice versa* [95–97]. In a continuum representation, these operators become

$$H_u = \Delta \int dx \left((\psi_R^\dagger \partial_x \psi_R^\dagger) (\psi_L \partial_x \psi_L) + \text{h.c.} \right) = \frac{\Delta}{2(\pi\alpha)^2} \int \cos 4\phi(x). \quad (2.179)$$

This operator has scaling dimension $4K$ and therefore is strongly irrelevant for small Δ . However, it becomes relevant for $K < 1/2$ and then in particular is responsible for the creation of a gap in the excitation spectrum and long-range antiferromagnetic order in the case of an Ising type anisotropy ($\Delta > 1$).

Physical properties of the spin 1/2 chain (small Δ). From the bosonized form of the Hamiltonian one immediately obtains the specific heat of the spin chain as

$$C(T) = -\frac{T}{L} \frac{\partial^2 F}{\partial T^2} = \frac{\pi k_B^2 T}{6u}. \quad (2.180)$$

The susceptibility for a field applied along the z -direction can also be obtained, noting that $\partial_x \phi$ is proportional to the fermion density, i.e. the z -component of the magnetization:

$$\chi = \frac{4\mu_B^2}{\pi} \frac{K}{u}. \quad (2.181)$$

Beyond these thermodynamic properties the fermionic analogy allows one to study correlation functions of the spin chain using the bosonization formalism developed in Sect. 2.4.1. We start by rewriting the fermion operators

in a continuum form:

$$a_l = \psi_R(x) + \psi_L(x), \quad (2.182)$$

where $x = la$ and in most of what follows the lattice constant a is set to unity. In order to calculate spin-spin correlation functions we further need a representation of the exponential factor in (2.168). This is given by

$$\exp\left(i\pi\sum_{j=1}^{l-1} a_j^\dagger a_j\right) = e^{ik_F x} e^{-i\phi(x)}, \quad (2.183)$$

where the first exponential factor comes from the mean value of the density and the second one represents the effect of fluctuations about that mean, see (2.95).

We can now use the continuum representations (2.182) and (2.183) together with the fermionic representation of the spin-1/2 operators, (2.168), to give a bosonic representation of the spins:

$$\begin{aligned} S_l^z &= \psi_R^+ \psi_R(x) + \psi_L^+ \psi_L(x) + \psi_L^+ \psi_R(x) + \psi_R^+ \psi_L(x) \\ &= -\frac{1}{\pi} \partial_x \phi + \frac{1}{\pi\alpha} \cos(2\phi(x) - 2k_F x) \\ S_l^+ &= \frac{1}{\sqrt{2\pi\alpha}} \left((-1)^n e^{-i\theta} + e^{i(2k_F - \pi)x} e^{-2i\phi - i\theta} \right). \end{aligned} \quad (2.184)$$

Here we have omitted the Klein factors which will be unimportant in the following, and have kept a finite α as a short-distance cutoff, to be taken of the order of a lattice constant. Further, in S_l^+ we have restored the factors $(-1)^l$ that were lost in going from (2.169) to (2.172). Using these expressions we can now follow the calculations of Sect. 2.4.1 to obtain the spin-spin correlation functions. In particular, at zero temperature [94]

$$\begin{aligned} \langle T_\tau S_l^z(\tau) S_0^z(0) \rangle &= \frac{K}{\pi^2} \frac{x^2 - u^2 \tau^2}{(x^2 + u^2 \tau^2)^2} + \frac{A \cos(2k_F x)}{(x^2 + u^2 \tau^2)^K} \\ \langle T_\tau S_l^+(\tau) S_0^-(0) \rangle &= B_1 \cos((2k_F - \pi)x) \frac{x^2 - u^2 \tau^2}{(x^2 + u^2 \tau^2)^{K+1/4K+1}} \\ &\quad + \frac{B_2 \cos \pi x}{(x^2 + u^2 \tau^2)^{1/4K}}. \end{aligned} \quad (2.185)$$

Here the constants A, B_1 cannot be reliably determined by the present methods, however in the nonoscillating part of the z - z correlation function there is no undetermined parameter, and very recently B_2 has been determined [98]. In most cases the dominant contributions in (2.185) come from the second, quickly oscillating terms. The alternation indicates the expected tendency towards antiferromagnetic order, but correlations do decay (with a rather slow power law), and there is thus no long-range order, as to be expected in one dimension.

Fourier transforming (2.185) gives, after analytic continuation to real frequencies and generalizing to nonzero temperature [99]

$$\begin{aligned}\chi_{\parallel}(q, \omega) &= -\frac{A \sin(\pi K)}{u} \left(\frac{2\pi T}{u} \right)^{2K-2} B \left(\frac{K}{2} - i \frac{\omega + uq'}{4\pi T}, 1-K \right) \\ &\quad B \left(\frac{K}{2} - i \frac{\omega - uq'}{4\pi T}, 1-K \right) + \frac{\pi A}{u(1-K)} \\ &= -\sin(\pi K) \Gamma^2(1-K) \frac{A}{u^{2K+1}} [u^2 q'^2 - (\omega + i\delta)^2]^{K-1} \quad \text{at } T = 0.\end{aligned}\tag{2.186}$$

Here $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is Euler's beta function, $q' = q - 2k_F$, and the last term has been introduced "by hand" in order to reproduce the known logarithmic result for $K = 1$. For the transverse $(+-)$ correlations, an analogous result holds, with $K \rightarrow 1/(4K)$ and $q' = q - \pi$. One should notice that (2.186) is valid uniformly for arbitrary ratios between the energies T , ω , and $u|q'|$, as long as all of them are small compared to the exchange energy (which here play the role of the bandwidth). At zero temperature, a finite imaginary part of χ_{\parallel} and χ_{\perp} exists in the V-shaped region $|\omega| \geq u|q'|$. This represents precisely the regions where the Bethe ansatz solution produces the two-spinon continuum [100]. The magnetic scattering cross section obtained from (2.186) is in excellent agreement with experiment [101–103].

At finite temperature, the susceptibilities take the general quantum critical scaling form [104]

$$\chi_{\parallel, \perp}(q, \omega) = \frac{A_{\parallel, \perp}}{T^{2-2K}} \phi_{\parallel, \perp} \left(\frac{uq'}{4\pi T}, \frac{\omega}{4\pi T} \right),\tag{2.187}$$

with the scaling functions $\phi_{\parallel, \perp}$ given by the products of beta functions in (2.186). Figure 2.12 shows plots of this function for the particularly relevant case of the isotropic antiferromagnet, $K = 1/2$. Note that these function has maxima for $x \approx y$. For $x \approx y \rightarrow \infty$ these maxima develop into the (zero-temperature) square root singularity of (2.186).

Generalizing the arguments leading to (2.184) it has been shown [105] that the nearest-neighbour interaction contains an oscillating part of the form

$$\mathbf{S}_l \cdot \mathbf{S}_{l+1} \approx \frac{1}{\pi\alpha} \sin(2\phi(x) - 2k_F x).\tag{2.188}$$

Correlation functions of this operator, which describes the tendency toward dimerization (or spin-Peierls order), then decay with the same power law as those of the z component of the spin.

An important consequence of the above discussion is that the low temperature correlations are only determined by two parameters: one, K , determines the power laws of the decay of correlations, the other one, u , is the velocity of the excitations. In the previous section, we have given explicit formulae

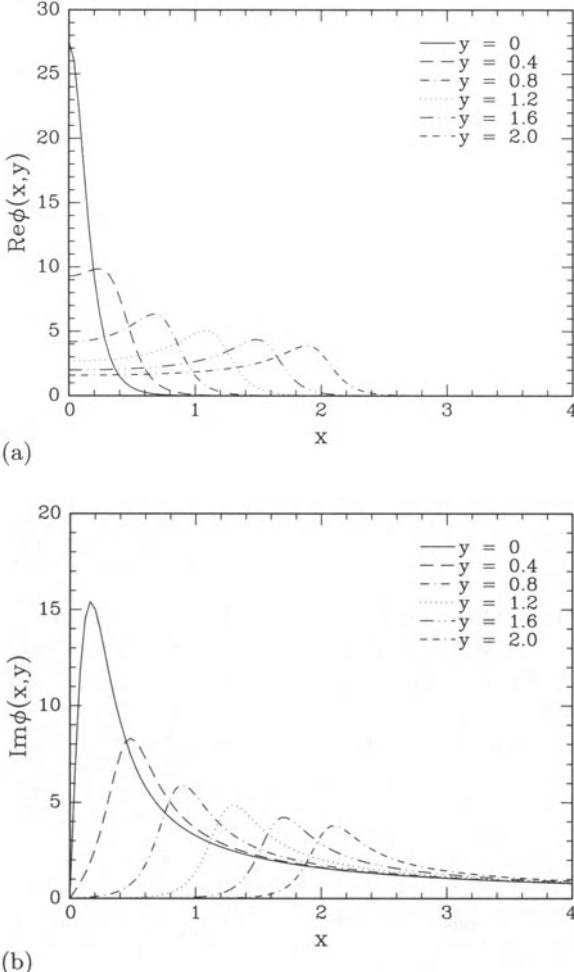


Fig. 2.12. Real (a) and imaginary (b) part of the quantum critical scaling function $\phi(x, y)$ for the isotropic Heisenberg antiferromagnet ($K = 1/2$)

for these constants, (2.178), however these can only be expected to be valid for small Δ where the linearization of the fermion spectrum is expected to be *quantitatively* reliable. It is clearly interesting to determine these parameters outside the perturbative regime. The exact Bethe wavefunction is, at least up to now, too complicated as to allow calculation of correlation functions (however, for some recent progress see [106,107]). One point where a non-perturbative result can be obtained is the isotropic Heisenberg antiferromagnet $\Delta = 1$ in zero field, so that $k_F = \pi/2$. Then spin rotation invariance requires the longitudinal and transverse correlation functions to be equal, and

thus $K = 1/2$ is needed in (2.185). Note that this implies that the dimerization correlations described by (2.188) also decay with a $1/r$ power law.

For general Δ a more indirect approach to the determination can be used [95]. One notices that the bosonized Hamiltonian leads to a variation of the ground state energy with the number of right- and left-going particles as

$$E(N, J) = \frac{\pi}{2L} \left(uKJ^2 + \frac{u}{K} N^2 \right), \quad (2.189)$$

where N and J are the sum and difference of the number of added right- and left-going particles, and equations (2.90) to (2.92) have been used. Haldane introduces the charge and current velocities $v_N = u/K$ and $v_J = uK$. On the other hand, the variation of the ground state energy with N and J can also be obtained from the Bethe solution (or even from a numerical solution on a finite lattice, if an exact solution is not available). In particular, for the spin-1/2 chain in zero field [108]

$$E(N, 0) = \frac{(\pi - \arccos \Delta) \sqrt{1 - \Delta^2}}{2 \arccos \Delta} \frac{N^2}{L}. \quad (2.190)$$

This fixes u/K . Note that the coefficient of N^2 in (2.189) is proportional to the inverse susceptibility. The variation with J can also be obtained, noting that $J \propto \partial_x \theta$ can be related to the stiffness constant of the x and y components of the spin (compare (2.184)). However, it is simpler to use directly the known result for the velocity of the elementary excitations [109]:

$$u = \frac{\pi \sqrt{1 - \Delta^2}}{2 \arccos \Delta}. \quad (2.191)$$

Alternatively, the velocity can also be obtained from the low-temperature specific heat [110], leading to the same result. This then fixes the correlation exponent as

$$K = \frac{\pi}{2(\pi - \arccos \Delta)}, \quad (2.192)$$

first obtained in [94] by a scaling argument using the exact solution of the fully anisotropic XYZ model [111]. The exact results (2.191) and (2.192) agree to linear order in Δ with (2.178), as expected, and even beyond first order the approximate results are quite close to the exact ones. One should also notice that to first order in the interaction Δ the coefficients in (2.189) can be calculated using the zeroth-order wavefunction, i.e. the noninteracting Slater determinant. This of course reproduces (2.176) and probably represents the “cheapest” way to calculate the anomalous exponent K . The exact expressions (2.191) and (2.192) are valid for vanishing magnetization, i.e. no applied field. Numerical results for a finite magnetization along z (i.e. a non-half-filled fermion band) have been given by Haldane [95].

The isotropic antiferromagnet ($\Delta = 1$). Up to now we have ignored the effect of the umklapp operator, (2.179). Indeed, as long as it is irrelevant, i.e. in the region of planar anisotropy $\Delta < 1$, umklapp interactions only lead to subleading corrections to correlation functions [38]. However, at the isotropic antiferromagnetic point $\Delta = 1$, the umklapp operator is only marginally irrelevant. Then the corresponding coupling constant is renormalized as [112,39,38,40]

$$\frac{dg}{d\ell} = -\pi b g^2 , \quad (2.193)$$

with solution

$$\pi b g(\ell) = \frac{\pi b g}{1 + \pi b g \ln L} = \frac{1}{\ln L/L_0} . \quad (2.194)$$

Here $b = 4/\sqrt{3}$ is a normalization constant fixed by the requirement that the correlations of the marginal operator decay as r^{-4} with unit coefficient, $L = e^\ell$ is the rescaled short distance cutoff (lattice constant), and $L_0 = \exp(-1/(\pi b g))$. The determination of the bare coupling constant g is not entirely trivial. Naively from (2.179) one would set it of order Δ . However, for the isotropic antiferromagnet $\Delta = 1$, giving a coupling of order unity which would be expected to be outside the perturbative domain of validity of (2.193). A precise determination can be achieved noting that the marginal operator affects the low-lying excited states in a finite system in a well-understood way involving only the combination $\pi b g(\ell)$ [112,39]. One finds in particular that the predicted linear variation of $\pi b g(\ell)$ with $\ln(L)$ is satisfied to within a few percent, and an order of magnitude better agreement is found if two-loop corrections are included [113]. From the two-loop calculation the bare coupling constant (at $L = 1$) can be estimated as $g \approx 0.24$. A priori, this would be expected to be an effective coupling, reproducing correctly the long-distance behavior, however in fact the spectra of all but the shortest chains are rather well fitted by this value. On the other hand, there are considerable uncertainties associated with this estimate: fitting the same data to the one-loop function one finds $g \approx 0.11$.

The umklapp operator leads to logarithmic corrections in the temperature dependence of the spin susceptibility, of the form [114]

$$\chi(T) = \frac{4\mu_B^2}{\pi^2} \left(1 + \frac{1}{2 \ln(T_0/T)} \right) , \quad (2.195)$$

with $T_0 \approx 7.7$, obtained from fitting (2.195) to exact finite-temperature Bethe ansatz results. Experimental results on the nearly perfectly one-dimensional compound Sr_2CuO_3 [115] are in excellent agreement with the logarithmic law (2.195) and with Bethe ansatz results [114] over a wide temperature regime. Similar logarithmic corrections also exist in the zero temperature magnetization curve $M(H)$ [108]. On the other hand, there are no such corrections in the low-temperature specific heat.

The marginal operator further produces multiplicative logarithmic corrections in various correlation functions, e.g. [39,38,40]

$$\langle T_\tau \mathbf{S}_x(\tau) \cdot \mathbf{S}_0(0) \rangle = \frac{3}{(2\pi)^{3/2}} \frac{(-1)^x}{x^2 + u^2 \tau^2} \ln^{1/2} \left(\frac{\sqrt{x^2 + u^2 \tau^2}}{L_0} \right), \quad (2.196)$$

where the prefactor has only been determined very recently [116]. An identical logarithmic correction is found in correlations of the dimerization operator, (2.188), however with exponent $-3/2$ instead of $1/2$, i.e. the dimerization fluctuations are logarithmically suppressed compared to the antiferromagnetic ones. Also the numerical prefactor in those correlations is not currently known. Similarly, the staggered susceptibility has a logarithmic correction factor:

$$\chi(\pi, 0; T) = \frac{A}{T} \ln^{1/2}(T_0/T). \quad (2.197)$$

Numerical investigations of correlation functions initially shed doubt on the existence of these logarithmic corrections [117], however more recent numerical [118–122] and analytic [106,107] work provides ample evidence for their existence. In particular, the constants in (2.197) have been determined as $A = 0.32 \pm 0.01$, $T_0 = 5.9 \pm 0.2$ [122]. The difference of T_0 with the value obtained from the susceptibility is possibly due to the fact that the susceptibility data were fitted in the asymptotic low-temperature region ($T \leq 10^{-2}$), whereas the staggered susceptibility was calculated at somewhat higher temperatures.

Apart from being directly accessible in neutron scattering experiments [101–103], the above staggered spin correlation function leads to characteristic temperature dependences in the longitudinal and transverse NMR relaxation times [121] (under the assumption of a hyperfine-dominated relaxation)

$$\frac{1}{T_1} \propto \left(\ln \frac{T_0}{T} \right)^{1/2}, \quad \frac{1}{T_{2G}} \propto \left(\frac{1}{T} \ln \frac{T_0}{T} \right)^{1/2}, \quad (2.198)$$

in good agreement with experiments on Sr_2CuO_3 [123]. In the absence of the marginal operator the logarithmic factors would be absent [124], leading in particular to a temperature-independent T_1 .

We finally mention that for $\Delta > 1$ the spins are preferentially aligned along the z -direction, and one then has a long-range ordered ground state of the Ising type. There is thus a phase transition exactly at the isotropic point $\Delta = 1$. In the fermionic language, this corresponds to a metal-insulator transition [125].

We can summarize this section by noting that quantum spin chains provide one of the experimentally best established cases of Luttinger liquid behavior. This largely due to two facts: (i) the relative ease with which one can define a microscopic Hamiltonian for a given experimental system: one

rarely has to go beyond a slightly modified Heisenberg model, with very few interaction constants to be determined; this is to be compared with the difficulties one encounters in conducting systems: long-range interactions, electron-phonon interactions, etc.; (ii) the availability of a large number of experimental techniques which give results directly comparable with experiment and, concomitantly, the possibility of using either very well-controlled (often exact) theoretical methods or numerical approaches which for spin systems are much more reliable than for itinerant fermions.

2.6 Spin Ladders and Coupled Luttinger Liquids

2.6.1 Coupled Spin Chains

Investigating models of coupled parallel chains is of interest for a number of reasons: (i) quasi-one-dimensional antiferromagnets always have some form of interchain coupling, usually leading to three-dimensional ordering at sufficiently low temperatures [126–128]; (ii) there is a number of “spin-ladder” compounds containing a small number of coupled chains [129,130]; (iii) coupled spin-1/2 chain models can be used to describe higher spin quantum numbers [131].

Consider N coupled spin-1/2 chains with spin degrees of freedom \mathbf{S}_j , $j = 1 \dots N$, described by the Hamiltonian

$$H = \sum_{j=1}^N H(\mathbf{S}_j) + \sum_{j < k} \lambda_{jk} H_c(\mathbf{S}_j, \mathbf{S}_k), \quad H_c(\mathbf{S}_j, \mathbf{S}_k) = \sum_i \mathbf{S}_{j,i} \cdot \mathbf{S}_{k,i}, \quad (2.199)$$

where i labels sites along the chains, j, k label the chains, and $H(\mathbf{S}_j)$ is of the form (2.167). For $\lambda_{jk} \equiv -1$ the ground state of this model is exactly that of the spin- $N/2$ chain, each site being in a state of total spin $N/2$ [131], but the model can be considered for general λ , both ferromagnetic ($\lambda < 0$) and antiferromagnetic ($\lambda > 0$). Performing now the Jordan–Wigner transformation for the \mathbf{S}_j separately and going to the boson representation the Hamiltonian becomes [132]

$$\begin{aligned} H = & \int dx \left[\frac{\pi u K}{2} \boldsymbol{\Pi}(x)^2 + \frac{u}{2\pi K} (\partial_x \boldsymbol{\phi})^2 + \frac{u}{2\pi \bar{K}} (\partial_x \bar{\boldsymbol{\phi}})^2 \right] \\ & + \frac{1}{(\pi\alpha)^2} \sum_{j < k} \int dx \{ \lambda_{1,jk} \cos(2(\phi_j + \phi_k)) + \lambda_{2,jk} \cos(2(\phi_j - \phi_k)) \\ & + \lambda_{3,jk} \cos(\theta_j - \theta_k) \}. \end{aligned} \quad (2.200)$$

Here $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_N)$, $\bar{\boldsymbol{\phi}} = \sum_j \phi_j / \sqrt{N}$, the constants u, K, \bar{K} all depend on the different constants in the original Hamiltonian, and the $\lambda_{1,2,3;jk}$ are all proportional to λ_{jk} .

Elementary power counting, using the result (2.196) for the spin correlations, shows that the coupling term H_c always is a strongly relevant perturbation. Consequently, an explicit renormalization group calculation [132] shows that either λ_2 or λ_3 always scale to strong coupling, i.e. the “relative” degrees of freedom $\phi_j - \phi_k$ all acquire a gap. In particular, for not too strong anisotropy, the λ_3 operator dominates, giving rise to long-range order in the $\theta_j - \theta_k$ and correspondingly exponential decay of the $\phi_j - \phi_k$ correlations. Integrating out these massive degrees of freedom an effective Hamiltonian for the “global” $\bar{\phi}$ mode is found:

$$H = \frac{u}{2} \int \left\{ \pi K \bar{\Pi}(x)^2 + \frac{1}{\pi K} (\partial_x \bar{\phi})^2 + g \cos(\mu \sqrt{N} \bar{\phi}) - \frac{\sqrt{N}}{\pi} h \partial_x \bar{\phi} \right\}, \quad (2.201)$$

where $\mu = 2$ for even N and $\mu = 4$ for odd N , the coefficients u, K, g are renormalized, and h is an external magnetic field applied along the z direction. Similarly, the leading contribution to spin correlations comes from the operators

$$S^+(x) \propto e^{i\pi x} e^{-i\bar{\theta}/\sqrt{N}}, \quad S^z(x) \propto e^{i\pi x} \cos(2\sqrt{N}\bar{\phi}), \quad (2.202)$$

where the second equation applies to odd N only.

From this a number of important conclusions can be drawn [132]. We first notice that massless excitations and the corresponding slow algebraic decay of correlation functions are only possible if the cos term in (2.201) is irrelevant. Moreover from (2.202) it follows that spin correlation functions are isotropic only if $K = 1/(2N)$, implying a decay as $1/x$, as in the $S = 1/2$ case. For the case of odd N (equivalently, for ferromagnetic λ , for half-odd-integer S) this is indeed the correct behavior: for $\mu = 4$ the cos term is marginally irrelevant. Thus both antiferromagnetic spin chains with half-odd-integer S and for odd numbers of coupled $S = 1/2$ chains massless behavior is predicted, with correlations asymptotically decaying like those of a spin-1/2 chain [132]. There is both numerical [133–136] and experimental [129] evidence that this is correct.

On the other hand, for even N (equivalently, integer S) the cos term in (2.201) is strongly relevant and therefore generates a gap Δ_s in the spin excitations. For the integer- S spin chains this is the well-known and verified Haldane prediction [137], but there also is a gap for any even number of coupled chains [132,138]. Analogous conclusions concerning qualitative differences between even and odd numbers of coupled chains have also been reached based on the nonlinear σ -model [139]. The gap implies exponential decay of spin correlations. Numerical [135,136] and, at least for $N = 2$, experimental work [129] confirms this picture. Another prediction, again valid both for integer- S antiferromagnets and even numbers of coupled chains, concerns the effect of an applied magnetic field: [132,140,141] as long as the field

is smaller than a critical field $h_c \propto \Delta_s$, the ground state is unchanged and has zero magnetization. However, beyond h_c the magnetization is expected to increase as $M \propto \sqrt{h - h_c}$. Experiments on $S = 1$ antiferromagnetic chains confirm this prediction [142].

A natural question left open by the above considerations concerns quasi-one-dimensional antiferromagnets like KCuF₃ [126,127] or Sr₂CuO₃ [128,115], which can be considered as the $N \rightarrow \infty$ limit of the above model. One clearly expects (and observes) true antiferromagnetic order at sufficiently low temperatures, at first sight in contradiction both with the exponential decay of spin correlation predicted for even N and the universal $1/x$ law for odd N . However, one should note that on the one hand the correlation length of the even- N systems is expected to increase quickly with increasing N , and that on the other hand the $1/x$ correlation law of the odd- N systems also is expected to be only valid beyond a correlation length $\xi(N)$ which increases with N . In the thermodynamic limit $N \rightarrow \infty$ this then is perfectly consistent with the existence of long-range order. Theoretical treatments of magnetic order in quasi-one-dimensional antiferromagnets can be based on a mean-field treatment of the interchain interaction [143,144] which gives quantitative predictions for systems like KCuF₃ or Sr₂CuO₃ [145].

2.6.2 Two Coupled Luttinger Liquids

It is clearly of interest to see what happens to the peculiar one-dimensional behavior when one puts chains in parallel. This question is of relevance for the understanding of quasi-one-dimensional conductors [44], doped spin ladders [146], few-channel quantum wires [64], and generally for the understanding of possible non-Fermi-liquid behavior and correlation-induced superconductivity in higher-dimensional solids. Of particular interest is the effect of an interchain single-particle tunneling term of the form

$$H_{ij} = -t_\perp \int dx (\psi_{rsi}^\dagger \psi_{rsj} + h.c.) , \quad (2.203)$$

where ψ_{rsj} is the fermion field operator for right ($r = +$) or left ($r = -$) going particles of spin s on chain i . Simple scaling arguments [147] lead to the “phase diagram” shown in Fig. 2.13. The dashed line represents the crossover below which single-particle tunneling becomes strongly relevant and below which one thus expects Fermi liquid like behavior (an alternative interpretation is due to Anderson [148,149]). The full lines indicate where two-particle or particle-hole hopping becomes relevant. The most plausible interpretation is that this is the temperature where three-dimensional long-range order of some type sets in. A more detailed discussion of this in quasi-one-dimensional systems (a thermodynamically large number of chains) has been given elsewhere [13,150]. It is also worth noting that a different approach to the crossover to higher dimensions, working with continuously varying spatial dimension, comes to similar conclusions [151].

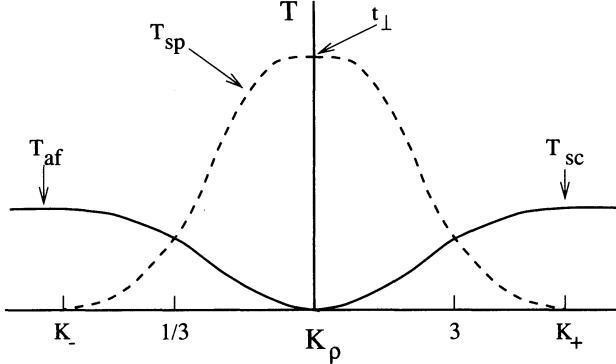


Fig. 2.13. Qualitative phase diagram in the temperature- K_ρ plane for Luttinger liquids coupled by interchain hopping. $K_\pm = 3 \pm \sqrt{8}$

As a model for doped spin-ladder systems, as well as a first step towards a many-chain system, one can study the two-chain case [152–154]. The tunneling term, (2.203), then leads to a splitting of the single-particle bands into symmetric and antisymmetric combinations which we label by transverse wavenumbers $k_\perp = 0, \pi$. Now each k_\perp mode can be bosonized separately. Introducing the linear combinations $\phi_{\nu\pm} = (\phi_{\nu 0} \pm \phi_{\nu\pi})/\sqrt{2}$ ($\nu = \rho, \sigma$) the Hamiltonian (including t_\perp) then takes the form

$$\begin{aligned}
 H &= H_0 + H_{\text{int},1} + H_{\text{int},2} \\
 H_0 &= \frac{\pi v_F}{2} \sum_{\substack{\nu=\rho,\sigma \\ \alpha=\pm}} \int dx \left[\Pi_{\nu\alpha}^2 + \frac{1}{\pi^2} (\partial_x \phi_{\nu\alpha})^2 \right] \\
 H_{\text{int},1} &= -\frac{g_1}{4} \int dx \left[\frac{1}{\pi^2} \{(\partial_x \phi_{\rho+})^2 + (\partial_x \phi_{\sigma+})^2\} - \Pi_{\rho+}^2 - \Pi_{\sigma+}^2 \right] \\
 &\quad + \frac{g_1}{2(\pi\alpha)^2} \int dx \{ \cos 2\phi_{\sigma+} (\cos 2\theta_{\rho-} + \cos 2\phi_{\sigma-} - \cos 2\theta_{\sigma-}) \\
 &\quad - \cos 2\theta_{\rho-} \cos 2\theta_{\sigma-} \} \\
 H_{\text{int},2} &= \frac{1}{4} \int dx \sum_{\gamma=\pm} g_\gamma^{(2)} \left[\frac{1}{\pi^2} (\partial_x \phi_{\rho\gamma})^2 - \Pi_{\rho\gamma}^2 \right] \\
 &\quad + \frac{g_{00\pi\pi}^{(2)}}{2(\pi\alpha)^2} \int dx \cos 2\theta_{\rho-} (\cos 2\phi_{\sigma-} + \cos 2\theta_{\sigma-}) . \tag{2.204}
 \end{aligned}$$

Here $g_\gamma^{(2)} = g_{0000}^{(2)} + \gamma g_{0\pi\pi 0}^{(2)}$, and $g_{abcd}^{(i)}$ is the coupling constant for an interaction scattering two particles from k_\perp -states (a, b) into (d, c) . The signs of the different interaction terms in (2.204) have been determined following the reasoning explained in the appendix. Here we consider a case where there is only intrachain interaction, implying that all *bare* coupling constants are independent of k_\perp .

For the pure forward scattering model ($g_1 \equiv 0$) the only nonlinear interaction ($g_{00\pi\pi}^{(2)}$) scales to infinity, leading to a gap in the ($\rho-$) modes and in *half* of the ($\sigma-$) modes. The remaining ($\sigma-$) modes are protected by the duality symmetry under $\phi_{\sigma-} \leftrightarrow \theta_{\sigma-}$, the ($\sigma-$) sector is in fact a critical point of the Ising type for $g_1 \equiv 0$ [154]. For repulsive interactions here the dominant fluctuations are of CDW type, with decay proportional to $r^{-(3+2K_\rho)/4}$, and $K_\rho^2 = (\pi v_F - g_2 + g_1/2) / (\pi v_F + g_2 - g_1/2)$. This state can be labeled by the number of massless modes as $C1S1\frac{1}{2}$, where quite generally $CnSm$ denotes a state with n massless charge and m massless spin modes [155].

For nonzero g_1 all interactions scale to strong coupling, only the total charge mode remains massless ($C1S0$), reflecting the translational invariance of the system, and all spin excitations have a gap. The physics in this regime can be determined looking for the semiclassical minima of the different cos terms in (2.204). One then finds that the CDW correlations now decay exponentially, and for the interesting case $g_1 > 0, g_2 > g_1/2$, corresponding to purely repulsive interaction, the strongest fluctuations are now of “d-type” superconducting pairing [152,154], with decay as $r^{-1/(2K_\rho)}$. Labeling this state as “d-type” seems appropriate because the pairing amplitudes at $k_\perp = 0$ and π intervene with opposite sign. In real space, this corresponds to pairs formed by two fermions on the two different chains. Note that even for weak interactions where $K_\rho \rightarrow 1$ this decay is very slow. The $4k_F$ component of the density correlations also has a power law decay, however with an exponent $2K_\rho$, much bigger than the SCd exponent [154,155]. The full phase diagram in the g_1 - g_2 plane is shown in Fig. 2.14. For $g_1 < 0$ the diagram is identical to the single-chain case, however, for $g_1 > 0$ the behavior is changed dramatically, and in particular superconductivity is predicted for repulsive interactions, for example for the Hubbard model which would be represented as $g_1 = g_2$ in the present language.

Remarkably, results basically identical to this weak coupling analysis can be obtained assuming strong repulsive interactions in the individual chains, so that one is for example in the regime where interchain electron-hole pair hopping is more relevant than single particle tunneling, $K_\rho < 1/3$ in Fig. 2.13. It is then more appropriate to bosonize the degrees of freedom of individual chains, rather than working in k_\perp -space. Under renormalization one then generates an interchain interaction of the form [147,156]

$$\begin{aligned} H_{jk} = & \frac{1}{(2\pi\alpha)^2} \cos(\sqrt{2}(\phi_{\rho j} - \phi_{\rho k})) \\ & \times \left\{ J_\perp [\cos(\sqrt{2}(\theta_{\sigma j} - \theta_{\sigma k})) + \cos(\sqrt{2}\phi_{\sigma j}) \cos(\sqrt{2}\phi_{\sigma k})] \right. \\ & \left. + V \sin(\sqrt{2}\phi_{\sigma j}) \sin(\sqrt{2}\phi_{\sigma k}) \right\}, \end{aligned} \quad (2.205)$$

where j, k now are chain indices. Remarkably, this term leads to properties identical to those found in weak coupling [154]. First, for $g_1 = 0$ one has $V = J_\perp$. Then H_{jk} is invariant under the duality $\phi_{\sigma j} \leftrightarrow \theta_{\sigma j}$, and one has an

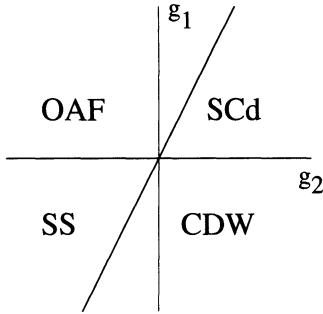


Fig. 2.14. Phase diagram of the two-chain model. The different dominating fluctuations are: SCd: “d-type” pairing; OAF: orbital antiferromagnetism; SS: “s-type” pairing; CDW: charge density wave. The critical lines $g_1 = 0$ and $g_1 = 2g_2$ are of Ising type

Ising critical theory. Secondly, for $g_1 > 0$ one finds $V \neq J_\perp$. Then the fields $\phi_{\rho 1} - \phi_{\rho 2}$, $\phi_{\sigma 1} + \phi_{\sigma 2}$, and $\theta_{\sigma 1} - \theta_{\sigma 2}$ become long-range ordered and one is in a C1S0 state. Power law correlations again exist for SCd and $4k_F$ density fluctuations, with the same scaling relation between the two exponents as in weak coupling. However, because now $K_\rho < 1/3$ the $4k_F$ density fluctuations actually dominate. The equivalent results in the weak and strong coupling regime very strongly suggest that *the two-chain model is in the same phase for weak and strong repulsion*. This point is further supported by considering the “t-J ladder model” for strong interchain exchange [157].

Numerical work on the two-chain model is in agreement with the existence of d-type pairing [158–162], the evidence for the $4k_F$ density fluctuations is however inconclusive [163]. Concerning experimental observation, one should notice that the SCd state becomes localized by weak disorder [164].

2.6.3 Summary

In this section we have discussed a number of results, mainly analytical, on the effect of different forms of interchain coupling on the Luttinger liquid behavior of strictly one-dimensional systems. As far as spin chains are concerned, the most spectacular result is the “oscillation” between even and odd numbers of chains, reminiscent of (and formally related to) the Haldane phenomenon in spin-S antiferromagnetic chains. There is both experimental and numerical evidence for this behavior, as discussed above.

For conducting chains, we have only discussed the two-chain case. The most interesting conclusion here was that for the Hubbard model (and a rather wide class of its generalizations) with purely repulsive interactions a d-wave superconducting state is predicted from weak up to rather strong repulsion. This seems to be one of the first cases where there is a reliable theoretical argument in favour of superconductivity in the (repulsive) Hubbard model. Only for very strong repulsion does a $4k_F$ CDW predominate. These results in principle apply directly to doped spin ladders [146], and to few-channel quantum wires. Concerning doped spin ladders, one of the most

interesting experimental questions certainly is whether the superconducting state is indeed of d type as predicted.

A number of results exist for larger numbers of chains [165–168]. In the perturbative region for weak repulsion generally again a d-wave superconducting state is found, however for stronger coupling different phases are found [166].

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Appendix: When to Bosonize in Peace

The fundamental ingredient for the following is the expression (2.96) for the single-fermion field operators

$$\begin{aligned} \psi_{\pm,\sigma}(x) &= \frac{1}{\sqrt{L}} \sum_k a_{\pm,\sigma,k} e^{ikx} \\ &= \lim_{\alpha \rightarrow 0} \frac{\eta_{\pm,\sigma}}{\sqrt{2\pi\alpha}} \exp [\pm ik_F x \mp i\phi_\sigma(x) + i\theta_\sigma(x)] , \end{aligned} \quad (2.206)$$

where σ can be the spin or some other internal degree of freedom of the fermions and we have replaced the U -operators by the Majorana (“real”) fermion operators $\eta_{\pm,\sigma}$ introduced to guarantee proper anticommutation between the ψ ’s [26]. They satisfy the anticommutation relation

$$[\eta_r, \eta_s]_+ = 2\delta_{r,s} , \quad (2.207)$$

where r and s are compound indices containing both the chirality \pm and the internal degree of freedom σ . Eq. (2.207) implies in particular $(\eta_r)^2 = 1$. The Majorana fermions can be represented by standard (Dirac) fermion operators c_r as $\eta_r = c_r^\dagger + c_r$. Note that there is just one isolated fermionic degree of freedom per branch, and that these degrees of freedom will not appear in the bosonized Hamiltonian if properly handled.

Using (2.206) and its generalization to cases with spin and other “internal” degrees of freedom like perpendicular momentum indices in coupled chain problems, a typical fermion interaction term becomes

$$\begin{aligned} \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\gamma \psi_\delta &= \eta_\alpha \eta_\beta \eta_\gamma \eta_\delta \times (\text{boson operators}) \\ &= h_{\alpha\beta\gamma\delta} \times (\text{boson operators}) , \end{aligned} \quad (2.208)$$

where the second equality defines $h_{\alpha\beta\gamma\delta}$. This operator, responsible for taking into account fermion anticommutation properly, is nevertheless carefully passed under the rug in the vast majority of the literature, thus leaving a purely bosonic Hamiltonian to be considered, as implied in the term “bosonization”. I will here investigate under which conditions this is allowed.

The relevant situation is that all the indices $\alpha, \beta, \gamma, \delta$ in $h_{\alpha\beta\gamma\delta}$ are different from each other. Otherwise the anticommutation rule

$$[\eta_r, \eta_s]_+ = 2\delta_{r,s} \Rightarrow \eta_r^2 = 1, \quad (2.209)$$

allows to simplify $h_{\alpha\beta\gamma\delta}$. I will therefore only consider the general case. First note that

$$h_{\alpha\beta\gamma\delta}^2 = 1, \quad (2.210)$$

$h_{\alpha\beta\gamma\delta}$ thus has eigenvalues ± 1 . Secondly,

$$[h_{\alpha\beta\gamma\delta}, h_{\kappa\lambda\mu\nu}]_\pm = 0, \quad (2.211)$$

where according to whether an even or odd number of pairs of indices taken from the two sets $(\alpha, \beta, \gamma, \delta)$ and $(\kappa, \lambda, \mu, \nu)$ are equal the commutator (even case) or anticommutator (odd case) is to be used. Finally, permutation of indices leads to sign changes:

$$h_{\alpha\beta\gamma\delta} = -h_{\beta\alpha\gamma\delta} = -h_{\alpha\gamma\beta\delta} = -h_{\alpha\beta\delta\gamma}. \quad (2.212)$$

It is now clear that if all the h 's occurring in a given Hamiltonian commute, they can be simultaneously diagonalized, which means that it will be possible to replace each of the h 's by ± 1 , leading to a purely bosonic Hamiltonian. This clearly is the case if all the h 's occurring have an even number of indices in common. In the opposite case some of the h 's do not commute, therefore can not be simultaneously diagonalized and not be eliminated from the Hamiltonian. Bosonization then is not possible.

As a simple example consider the single-chain Luttinger model with spin, Sect. 2.4.2: the four allowed values of the discrete indices are $1 \equiv (+, \uparrow)$, $2 \equiv (+, \downarrow)$, $3 \equiv (-, \uparrow)$, $4 \equiv (-, \downarrow)$. Consequently, only h_{1234} can occur, and according to the eigenvalue chosen the backward scattering interaction takes the form $\pm g_1 \cos(\sqrt{8}\phi_\sigma)$. The choice of eigenvalue of h_{1234} affects however the expressions for correlation functions: for example $h_{1234} = \pm 1$ implies $\eta_1\eta_3 = \pm\eta_2\eta_4$, and consequently the $2k_F$ charge density operator contains either a factor $\cos(\sqrt{2}\phi_\sigma)$ (plus sign) or $\sin(\sqrt{2}\phi_\sigma)$. A similar discrete “gauge covariance” exists of course for all correlation functions.

In more complicated cases like the two-chain problem, more than one h -operator occurs. Even if they all commute, as is the case for the two-chain problem, additional constraints on the permissible eigenvalues of the h 's exist due to the existence of relations of the type

$$h_{\alpha\beta\gamma\delta} h_{\kappa\lambda\mu\nu} h_{\pi\rho\sigma\tau} = \pm 1, \quad (2.213)$$

and similar relations involving more than three h 's. However, a discrete gauge freedom of the type mentioned above often remains. For the particular case of fermions with an internal SU(N) symmetry [26], bosonization can be performed without problem and all the h -operators can be given eigenvalue +1, a fact not noticed in the original work.

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3 Quantum Number Fractionalization in Antiferromagnets

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and Olivia L. White

3.1 Introduction

In these lectures we shall derive and discuss a set of exact eigenstates of the Haldane-Shastry [1,2] model, a realization of the spin-1/2 Heisenberg chain in which the quantum-disordered spin liquid ground state and the neutral, spin-1/2 excitations of such systems are particularly easy to understand. This behavior is not unique to the model, and in particular occurs in the Bethe solution of the near-neighbour Heisenberg chain, where it was first discovered [3,4], but it is more accessible in this form. The model also makes the relationship of the spin chain to the fractional quantum Hall effect transparent [5]. The key results are shown in Fig. 3.1.

1. The ground state has the same functional form as the fractional quantum Hall ground state.
2. This ground state has quantum disorder and has the same relation to the antiferromagnetically ordered state that a quantum liquid has to a conventional crystal.
3. The elementary excitations of this state are spin-1/2 particles, spinons, and not spin waves, which are the elementary excitations of an ordered antiferromagnet. Spinons have
 - (a) A relativistic band structure with a Dirac point at momentum $\pm\pi/2$.
 - (b) 1/2 fractional statistics.

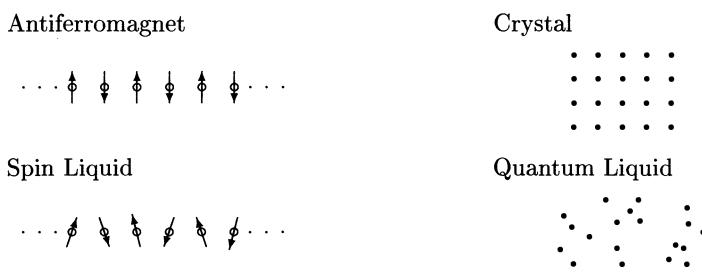


Fig. 3.1. Illustration of the quantum spin liquid ground state

- (c) $N/2$ allowed momenta rather than N .
- 4. The ground state is effectively 2-fold degenerate. The missing $N/2$ states of the spinon are excitations of the other ground state.
- 5. The Hamiltonian has a factorization which may be constructed as supersymmetric.

3.2 Haldane–Shastry Hamiltonian

Let a lattice of N sites be wrapped onto a unit circle, as shown in Fig. 3.2, so that each site may be expressed as a complex number z_α satisfying

$$z_\alpha^N - 1 = 0, \quad (3.1)$$

and let each site possess a single unpaired half-integral spin with its corresponding spin operator \mathbf{S}_α . The Haldane–Shastry Hamiltonian is then given by

$$\mathcal{H}_{\text{HS}} = J \left(\frac{2\pi}{N} \right)^2 \sum_{\alpha < \beta}^N \frac{\mathbf{S}_\alpha \cdot \mathbf{S}_\beta}{|z_\alpha - z_\beta|^2}. \quad (3.2)$$

This Hamiltonian is translationally invariant, satisfies

$$[\mathcal{H}_{\text{HS}}, \mathbf{S}] = 0 \quad \mathbf{S} = \sum_{\alpha}^N \mathbf{S}_\alpha, \quad (3.3)$$

and possesses the special internal symmetry

$$[\mathcal{H}_{\text{HS}}, \mathbf{A}] = 0 \quad \mathbf{A} = \frac{i}{2} \sum_{\alpha \neq \beta} \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) (\mathbf{S}_\alpha \times \mathbf{S}_\beta). \quad (3.4)$$

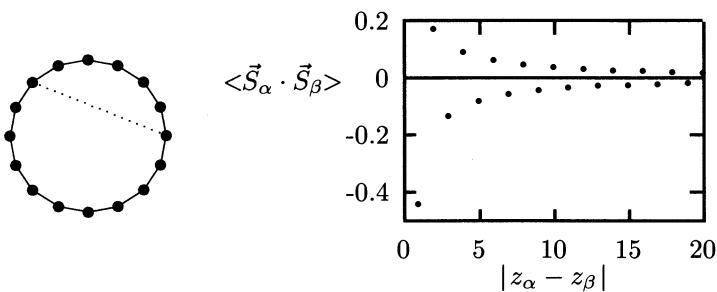


Fig. 3.2. *Left:* Illustration of Haldane–Shastry model. *Right:* Magnetic correlation function defined by 3.28

Proof

We proceed by applying the commutation relations

$$[\mathbf{S}_1, (\mathbf{S}_1 \cdot \mathbf{S}_2)] = i(\mathbf{S}_1 \times \mathbf{S}_2) \quad (3.5)$$

$$[(\mathbf{S}_1 \times \mathbf{S}_2), (\mathbf{S}_1 \cdot \mathbf{S}_2)] = \frac{i}{2}(\mathbf{S}_1 - \mathbf{S}_2) , \quad (3.6)$$

the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} , \quad (3.7)$$

the fact that

$$\left[\frac{z_j + z_k}{z_j - z_k} - \frac{z_\ell + z_k}{z_\ell - z_k} \right] \frac{1}{|z_j - z_\ell|^2} = \frac{z_j z_k z_\ell}{(z_j - z_k)(z_\ell - z_k)(z_j - z_\ell)} , \quad (3.8)$$

which is totally antisymmetric under permutation of j , k , and ℓ , and

$$\sum_{j \neq k} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_j - z_k|^2} = - \sum_{\alpha=1}^{N-1} \frac{z_\alpha(z_\alpha + 1)}{(z_\alpha - 1)^3} = 0 . \quad (3.9)$$

We have

$$\begin{aligned} & \sum_{j \neq k} \sum_{\alpha \neq \beta} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_\alpha - z_\beta|^2} [(\mathbf{S}_j \times \mathbf{S}_k), (\mathbf{S}_\alpha \cdot \mathbf{S}_\beta)] \\ &= 4i \sum_{j \neq k \neq \ell} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_j - z_\ell|^2} \left[(\mathbf{S}_j \cdot \mathbf{S}_k) \mathbf{S}_\ell - (\mathbf{S}_\ell \cdot \mathbf{S}_k) \mathbf{S}_j \right] \\ &+ i \sum_{j \neq k} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_j - z_k|^2} (\mathbf{S}_j - \mathbf{S}_k) = 0 . \quad \square \end{aligned} \quad (3.10)$$

It follows from the commutation relations

$$[S^a, S^b] = i \epsilon^{abc} S^c \quad [S^a, A^b] = i \epsilon^{abc} A^c , \quad (3.11)$$

that \mathcal{H}_{HS} , S^2 , and $(\mathbf{A} \cdot \mathbf{S})$ all commute with each other.

3.3 Ground State

Let us imagine the spin system to be a 1-dimensional string of boxes populated by hard-core bosons, the \downarrow spin state corresponding to an empty box and the \uparrow spin state corresponding to an occupied one. The ground state wavefunction is a rule by which a complex number is assigned to each boson configuration. The total number of bosons is conserved, as it is physically the

same thing as the eigenvalue of S^z . Let N be even and let $z_1, \dots, z_{N/2}$ denote the locations of $N/2$ bosons, the number appropriate for a spin singlet. Then the Haldane–Shastry ground state is

$$\Psi(z_1, \dots, z_{N/2}) = \prod_{j < k}^{N/2} (z_j - z_k)^2 \prod_{j=1}^{N/2} z_j . \quad (3.12)$$

Its energy is

$$\mathcal{H}_{\text{HS}}|\Psi\rangle = -J \left(\frac{\pi^2}{24} \right) \left(N + \frac{5}{N} \right) |\Psi\rangle . \quad (3.13)$$

Proof

We begin by observing that $[S_\alpha^+ S_\beta^- \Psi](z_1, \dots, z_{N/2})$ is identically zero unless one of the arguments $z_1, \dots, z_{N/2}$ equals z_α . We have

$$\begin{aligned} & \left[\left\{ \sum_{\beta \neq \alpha}^N \frac{S_\alpha^+ S_\beta^-}{|z_\alpha - z_\beta|^2} \right\} \Psi \right] (z_1, \dots, z_{N/2}) \\ &= \sum_{j=1}^{N/2} \sum_{\beta \neq j}^N \frac{1}{|z_j - z_\beta|^2} \Psi(z_1, \dots, z_{j-1}, z_\beta, z_{j+1}, \dots, z_{N/2}) \\ &= \sum_{j=1}^{N/2} \sum_{\ell=0}^{N-2} \left\{ \sum_{\beta \neq j}^N \frac{z_\beta (z_\beta - z_j)^\ell}{\ell! |z_j - z_\beta|^2} \right\} \left(\frac{\partial}{\partial z_j} \right)^\ell \left\{ \Psi(z_1, \dots, z_{N/2}) / z_j \right\} \\ &= \sum_{j=1}^{N/2} \left\{ \frac{(N-1)(N-5)}{12} z_j - \frac{N-3}{2} z_j^2 \frac{\partial}{\partial z_j} + \frac{1}{2} z_j^3 \frac{\partial^2}{\partial z_j^2} \right\} \\ &\quad \times \left\{ \Psi(z_1, \dots, z_{N/2}) / z_j \right\} \\ &= \left\{ \frac{N(N-1)(N-5)}{24} - \frac{N-3}{2} \sum_{j \neq k}^{N/2} \frac{2z_j}{z_j - z_k} \right. \\ &\quad \left. + \frac{1}{2} \left[\sum_{j \neq k \neq m}^{N/2} \frac{4z_j^2}{(z_j - z_k)(z_j - z_m)} + \sum_{j \neq k}^{N/2} \frac{2z_j^2}{(z_j - z_k)^2} \right] \right\} \Psi(z_1, \dots, z_{N/2}) \\ &= \left\{ \frac{N(N-1)(N-5)}{24} - \frac{N(N-3)}{4} \left(\frac{N}{2} - 1 \right) \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{2N}{3} \left(\frac{N}{2} - 1 \right) \left(\frac{N}{2} - 2 \right) + \frac{N}{2} \left(\frac{N}{2} - 1 \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{j \neq k}^{N/2} \frac{2}{|z_j - z_k|^2} \Big] \Big\} \Psi(z_1, \dots, z_{N/2}) \\
& = \left\{ -\frac{N}{8} - \sum_{j \neq k}^{N/2} \frac{1}{|z_j - z_k|^2} \right\} \Psi(z_1, \dots, z_{N/2}) . \tag{3.14}
\end{aligned}$$

Here we have used the fact that

$$\frac{1}{|z_\alpha - z_\beta|^2} = -\frac{z_\alpha z_\beta}{(z_\alpha - z_\beta)^2} , \tag{3.15}$$

i.e. that the Hamiltonian is effectively analytic in the spin coordinates, and the sums (with sites labeled so that $z_N = 1$)

$$\sum_{\alpha=1}^{N-1} \frac{1}{|z_\alpha - 1|^2} = \frac{N^2 - 1}{12} , \tag{3.16}$$

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{(z_\alpha - 1)^2} = -\frac{(N-1)(N-5)}{12} , \tag{3.17}$$

$$\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{(z_\alpha - 1)} = \frac{N-3}{2} , \tag{3.18}$$

$$\sum_{\alpha=1}^{N-1} z_\alpha^2 = -1 , \tag{3.19}$$

$$\sum_{\alpha=1}^{N-1} z_\alpha^2 (z_\alpha - 1) = \dots = \sum_{\alpha=1}^{N-1} z_\alpha^2 (z_\alpha - 1)^{N-3} = 0 , \tag{3.20}$$

$$\frac{z_\alpha^2}{(z_\alpha - z_\beta)(z_\alpha - z_\gamma)} + \frac{z_\beta^2}{(z_\beta - z_\alpha)(z_\beta - z_\gamma)} + \frac{z_\gamma^2}{(z_\gamma - z_\alpha)(z_\gamma - z_\beta)} = 1 , \tag{3.21}$$

which are worked out in Appendix A. We also have

$$\begin{aligned}
& \left[\left\{ \sum_{\beta \neq \alpha}^N \frac{S_\alpha^z S_\beta^z}{|z_\alpha - z_\beta|^2} \right\} \Psi \right] (z_1, \dots, z_{N/2}) \\
& = \left\{ -\frac{N(N^2 - 1)}{48} + \sum_{j \neq k}^{N/2} \frac{1}{|z_j - z_k|^2} \right\} \Psi(z_1, \dots, z_{N/2}) . \tag{3.22}
\end{aligned}$$

This completes the proof, since

$$\mathcal{H}_{\text{HS}} = \frac{1}{2} J \left(\frac{2\pi}{N} \right)^2 \left\{ \sum_{\alpha \neq \beta} \frac{S_\alpha^+ S_\beta^-}{|z_\alpha - z_\beta|^2} + \sum_{\alpha \neq \beta} \frac{S_\alpha^z S_\beta^z}{|z_\alpha - z_\beta|^2} \right\}. \quad \square \quad (3.23)$$

Important Properties

- Reality:** Since z_α lies on the unit circle we have

$$\begin{aligned} \Psi^*(z_1, \dots, z_{N/2}) &= \prod_{j < k}^{N/2} (z_j^* - z_k^*)^2 \prod_j^{N/2} z_j^* \\ &= \prod_{j < k}^{N/2} (z_k - z_j)^2 \prod_j^{N/2} z_j^{1-N} = \Psi(z_1, \dots, z_{N/2}). \end{aligned} \quad (3.24)$$

Thus Ψ is real despite being a polynomial in the complex variables z_j .

- Translational Invariance:** Ψ is translated one lattice spacing by multiplying each of its arguments by $z = \exp(i2\pi/N)$. Since it is a homogeneous polynomial of degree $(N/2)^2$ we have

$$\Psi(z_1 z, \dots, z_{N/2} z) = \exp(iN\pi/2) \Psi(z_1, \dots, z_{N/2}). \quad (3.25)$$

The crystal momentum of the state, i.e. the phase it acquires under translation, is thus 0 or π , depending on the value of the even integer N .

- Spin Rotational Invariance:** To prove Ψ is a spin singlet it suffices to show that it is an eigenstate of S^z with eigenvalue zero and that it is destroyed by the spin lowering operator S^- . The former is true for any wavefunction in which the number of z_j is constrained to $N/2$. For the latter we have

$$\begin{aligned} [S^- \Psi](z_2, \dots, z_{N/2}) &= \sum_{\alpha=1}^N \Psi(z_\alpha, z_2, \dots, z_{N/2}) \\ &= \lim_{z_1 \rightarrow 0} \sum_{\ell=1}^{N-1} \frac{1}{\ell!} \left\{ \sum_{\alpha=1}^N z_\alpha^\ell \right\} \frac{\partial^\ell}{\partial z_1^\ell} \Psi(z_1, z_2, \dots, z_{N/2}) = 0, \end{aligned} \quad (3.26)$$

since

$$\sum_{\alpha=1}^N z_\alpha^\ell = N \delta_{\ell 0} \pmod{N}. \quad (3.27)$$

This implies that the wavefunction is the same with the roles of \uparrow and \downarrow reversed or, more generally, with the quantization axis taken to be an arbitrary direction in spin space.

- 4. Quantum Disorder:** In Fig. 3.2 we plot the spin-spin correlation function

$$\langle \mathbf{S}_\alpha \cdot \mathbf{S}_\beta \rangle = \frac{3}{2} \left(\frac{N}{2} - 1 \right) \frac{\sum_{z_3, \dots, z_{N/2}} |\Psi(z_\alpha, z_\beta, z_3, \dots, z_{N/2})|^2}{\sum_{z_2, \dots, z_{N/2}} |\Psi(z_\alpha, z_2, \dots, z_{N/2})|^2} - \frac{3}{4}. \quad (3.28)$$

The convergence of this function to zero as $|z_\alpha - z_\beta| \rightarrow \infty$ shows that $|\Psi\rangle$ has no long-range order and is a spin liquid. The fall-off is slow, however, and this is important. Strongly-disordered spin liquids, i.e. with exponential decay of correlations on a length scale ξ , are easy to construct when the spin in the unit cell is integral. They have an energy gap $\Delta = \hbar v / \xi$, where v is the spin-wave velocity of a nearby ordered state. But strongly-disordered spin liquids *cannot* be stabilized with short-range interactions when the spin in the unit cell is half-integral. The excitation spectrum in this case is always gapless [6].

- 5. Factorizability:** The fact that $|\Psi\rangle$ is a product of pair factors makes a number of its properties easy to calculate by semi-classical Monte-Carlo techniques. Let us, for example, consider (3.28). Writing

$$|\Psi(z_1, \dots, z_{N/2})|^2 = \exp \left[-\phi(z_1, \dots, z_{N/2}) \right]$$

$$\phi(z_1, \dots, z_{N/2}) = -4 \sum_{j < k} \ln |z_j - z_k|, \quad (3.29)$$

we see that the summand is the Boltzmann factor of an equivalent finite-temperature classical lattice gas, and that we are computing the joint probability for two of these particles to reside at sites z_α and z_β . This may be done by generating a time sequence of configurations using a rule that obeys detailed balance, and then simply counting how many times the sites z_α and z_β are simultaneously occupied. The simplest such algorithm is the following. Let the current configuration be $z_1, \dots, z_{N/2}$.

- (a) Loop on particles j .
- (b) For this particle, roll the dice to choose a direction. Compute

$$z'_j = z_j \exp(\pm i 2\pi/N) \quad (3.30)$$

depending on the outcome.

- (c) Compute

$$f = \prod_{k \neq j} \left| \frac{z'_j - z_k}{z_j - z_k} \right|^4. \quad (3.31)$$

- (d) If $f > 1$ update z_j to z'_j .
- (e) If $f < 1$, roll the dice to generate a real number x between 0 and 1. Update z_j to z'_j if $x < f$ but do nothing otherwise.

Fig. 3.2 was generated using this algorithm.

6. **Degeneracy:** The Haldane–Shastry ground state is not degenerate, but it is nearly so. The alternate ground state is

$$\Psi'(z_1, \dots, z_{N/2}) = \prod_{j < k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2 \right]. \quad (3.32)$$

It has crystal momentum π greater than that of Ψ and has energy

$$\mathcal{H}_{\text{HS}} |\Psi'\rangle = -J \left(\frac{\pi^2}{24} \right) \left(N - \frac{7}{N} \right) |\Psi'\rangle. \quad (3.33)$$

It is equivalent to the original vacuum plus a pair of spinons excited out of the vacuum into a singlet with total momentum π .

3.4 Spinons

Let the number of sites N be odd and let

$$\Psi_\alpha(z_1, \dots, z_M) = \prod_j^M (z_\alpha - z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_j^M z_j, \quad (3.34)$$

where $M = (N - 1)/2$. This is a \downarrow spin on site α surrounded by an otherwise featureless singlet sea (Fig. 3.3). We have

$$\sum_{\beta \neq \alpha}^N S_\beta^- \Psi_\alpha = 0, \quad (3.35)$$

per (3.26). The combination of these states given by

$$\Psi_m(z_1, \dots, z_M) = \sum_\alpha^N (z_\alpha^*)^m \Psi_\alpha(z_1, \dots, z_M), \quad (3.36)$$

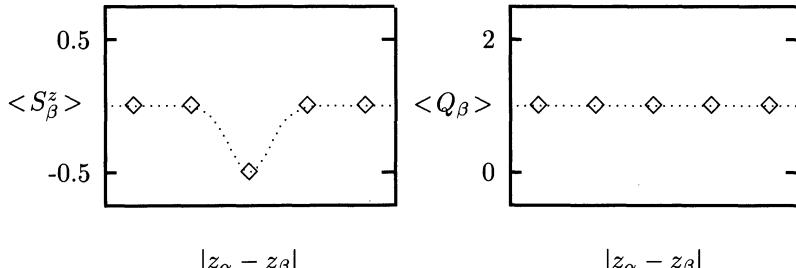


Fig. 3.3. Spin and charge profiles of the localized spinon $|\Psi_\alpha\rangle$ defined by (3.34). The *dotted lines* are a guide to the eye

with $0 \leq m \leq (N - 1)/2$, is an exact eigenstate of the Hamiltonian with eigenvalue

$$\begin{aligned} \mathcal{H}_{\text{HS}}|\Psi_m\rangle \\ = \left\{ -J \left(\frac{\pi^2}{24} \right) \left(N - \frac{1}{N} \right) + \frac{J}{2} \left(\frac{2\pi}{N} \right)^2 m \left(\frac{N-1}{2} - m \right) \right\} |\Psi_m\rangle . \end{aligned} \quad (3.37)$$

Proof

Following Haldane [1] we consider a wavefunction of the general form

$$\Psi(z_1, \dots, z_M) = \Phi(z_1, \dots, z_M) \prod_{j < k}^M (z_j - z_k)^2 \prod_j^M z_j , \quad (3.38)$$

where $M = (N - 1)/2$ and Φ is a homogeneous symmetric polynomial of degree less than $N - 2M + 2$. This latter condition causes Ψ to be a polynomial of degree less than $N + 1$ in each of its variables z_j , and thus allows the Taylor expansion technique used for the ground state to be applied. Doing so, we find that

$$\mathcal{H}_{\text{HS}}\Psi = \frac{J}{2} \left(\frac{2\pi}{N} \right)^2 \left\{ \lambda + \frac{N}{48}(N^2 - 1) + \frac{M}{6}(4M^2 - 1) - \frac{N}{2}M^2 \right\} \Psi , \quad (3.39)$$

provided that Φ satisfies

$$\frac{1}{2} \left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi}{\partial z_j^2} + \sum_{j \neq k}^M \frac{4z_j^2}{z_j - z_k} \frac{\partial \Phi}{\partial z_j} \right\} - \frac{N-3}{2} \sum_j^M z_j \frac{\partial \Phi}{\partial z_j} = \lambda \Phi . \quad (3.40)$$

Let us now consider the polynomial

$$\Phi_A(z_1, \dots, z_M) = \prod_j^M (z_A - z_j) = \sum_{m=0}^M z_A^m P_m(z_1, \dots, z_M) , \quad (3.41)$$

where z_A is not necessarily a lattice site. We have

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi_A}{\partial z_j^2} + \sum_{j \neq k}^M \frac{4z_j^2}{z_j - z_k} \frac{\partial \Phi_A}{\partial z_j} \right\} - \frac{N-3}{2} \sum_j^M z_j \frac{\partial \Phi_A}{\partial z_j} \\ &= 2 \sum_{j < k} \left[\frac{z_j^2}{(z_j - z_A)(z_j - z_k)} + \frac{z_k^2}{(z_k - z_A)(z_k - z_j)} \right] \Phi_A \end{aligned}$$

$$\begin{aligned}
& -\frac{N-3}{2} \sum_j^M \frac{z_j}{z_j - z_A} \Phi_A \\
&= \left\{ M(M-1) - z_A^2 \frac{\partial^2}{\partial z_A^2} - \frac{N-3}{2} \left[M - z_A \frac{\partial}{\partial z_A} \right] \right\} \Phi_A \\
&= \sum_{m=0}^M m \left(\frac{N-1}{2} - m \right) z_A^m P_m . \tag{3.42}
\end{aligned}$$

The proof is completed by multiplying both sides of this equation by $(z_A^*)^n$ and then summing on lattice sites z_A . \square

The state $|\Psi_m\rangle$ is a propagating \downarrow spinon with crystal momentum

$$q = \frac{\pi}{2}N - \frac{2\pi}{N} \left(m + \frac{1}{4} \right) \pmod{2\pi}, \tag{3.43}$$

per the definition

$$\Psi_m(z_1 z, \dots, z_M z) = \exp(iq) \Psi_m(z_1, \dots, z_M). \tag{3.44}$$

Rewriting the eigenvalue as

$$\mathcal{H}|\Psi_m\rangle = \left\{ -J \left(\frac{\pi^2}{24} \right) \left(N + \frac{5}{N} - \frac{3}{N^2} \right) + E_q \right\} |\Psi_m\rangle, \tag{3.45}$$

we obtain the dispersion relation

$$E_q = \frac{J}{2} \left[\left(\frac{\pi}{2} \right)^2 - q^2 \right] \pmod{\pi} \tag{3.46}$$

plotted in Fig. 3.4. Note that the momenta available to the spinon span only the inner or outer half of the Brillouin zone, depending on whether $N-1$ is divisible by 4. The spinon dispersion at low energies is linear in q with a velocity

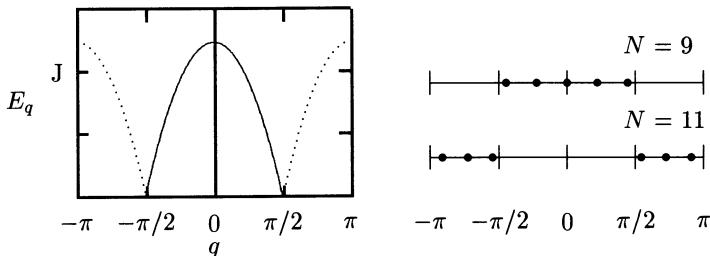


Fig. 3.4. *Left:* Spinon dispersion given by (3.46). *Right:* Allowed values of q for adjacent odd N

$$v_{\text{spinon}} = \frac{\pi}{2} \frac{Jb}{\hbar}, \quad (3.47)$$

where $b = 2\pi/N$ is the bond length.

The existence of spinons is an automatic consequence of quantum disorder whenever the total spin in the unit cell is half-integral. The spin liquid ground state must be a singlet because otherwise it is ferromagnetic. A singlet is possible only if the total number of lattice sites N is even. If N is odd then the total spin can be no less than $1/2$. But since there can be no physical difference between even and odd in the limit of large N , the system must have had a neutral spin- $1/2$ excitation to begin with.

The ground state of the odd- N spin chain is 4-fold degenerate and is given by $|\Psi_m\rangle$ for $m = 0$ and $(N - 1)/2$ and their \uparrow counterparts. This corresponds physically to a “left-over” spinon with momentum $\pm\pi$.

Spinons maintain their identity when more than one of them is present. When N is even, for example, the states

$$|\Psi_{mn}\rangle = \sum_{\alpha=1}^N \sum_{\beta=1}^N f_{mn}(z_\alpha^* z_\beta) (z_\alpha^*)^m (z_\beta^*)^n |\Psi_{\alpha\beta}\rangle, \quad (3.48)$$

where

$$\Psi_{\alpha\beta}(z_1, \dots, z_M) = \prod_j^M (z_\alpha - z_j)(z_\beta - z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_j^M z_j, \quad (3.49)$$

with $M = N/2 - 1$ and

$$f_{mn}(z) = \sum_{\ell=0}^{N/2} a_\ell z^\ell - \frac{1}{2} a_{N/2} z^{N/2}$$

$$a_\ell = \frac{m - n + 2\ell}{2\ell[\ell + m - n - 1/2]} \sum_{k=0}^{\ell-1} a_k \quad (a_0 = 1) \quad (3.50)$$

with $m \geq n$, are eigenstates of the Hamiltonian with eigenvalue

$$\mathcal{H}_{\text{HS}} |\Psi_{mn}\rangle = \left\{ -J \left(\frac{\pi^2}{24} \right) \left(N - \frac{19}{N} + \frac{24}{N^2} \right) + \frac{J}{2} \left(\frac{2\pi}{N} \right)^2 \right.$$

$$\times \left. \left[m \left(\frac{N}{2} - 1 - m \right) + n \left(\frac{N}{2} - 1 - n \right) - \left| \frac{m-n}{2} \right| \right] \right\} |\Psi_{mn}\rangle. \quad (3.51)$$

Proof

Following the procedure we use for one spinon we take Φ to be a superposition of states of the form

$$\begin{aligned}\Phi_{AB} &= \prod_j^M (z_A - z_j)(z_B - z_j) \\ &= \sum_{m=0}^M \sum_{n=0}^M z_A^m z_B^n P_m(z_1, \dots, z_M) P_n(z_1, \dots, z_M),\end{aligned}\quad (3.52)$$

where z_A and z_B are not necessarily lattice sites. We find that

$$\begin{aligned}&\frac{1}{2} \left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi_{AB}}{\partial z_j^2} + \sum_{j \neq k}^M \frac{4z_j^2}{z_j - z_k} \frac{\partial \Phi_{AB}}{\partial z_j} \right\} - \frac{N-3}{2} \sum_j^M z_j \frac{\partial \Phi_{AB}}{\partial z_j} \\ &= \left\{ -\frac{z_A^2}{z_A - z_B} \frac{\partial}{\partial z_A} - \frac{z_B^2}{z_B - z_A} \frac{\partial}{\partial z_B} - z_A^2 \frac{\partial^2}{\partial z_A^2} - z_B^2 \frac{\partial^2}{\partial z_B^2} \right. \\ &\quad \left. + \left(\frac{N-3}{2} \right) \left[z_A \frac{\partial}{\partial z_A} + z_B \frac{\partial}{\partial z_B} \right] + \left[2M^2 - M(N-2) \right] \right\} \Phi_{AB} \\ &= \sum_{m=0}^M \sum_{n=0}^M \left\{ m \left(\frac{N}{2} - 1 - m \right) + n \left(\frac{N}{2} - 1 - n \right) \right. \\ &\quad \left. - \left(\frac{m-n}{2} \right) \frac{z_A + z_B}{z_A - z_B} \right\} z_A^m z_B^n P_m P_n,\end{aligned}\quad (3.53)$$

and thus that

$$\begin{aligned}&\frac{1}{2} \left\{ \sum_j^M z_j^2 \frac{\partial^2 \Phi_{mn}}{\partial z_j^2} + \sum_{j \neq k}^M \frac{4z_j^2}{z_j - z_k} \frac{\partial \Phi_{mn}}{\partial z_j} \right\} - \frac{N-3}{2} \sum_j^M z_j \frac{\partial \Phi_{mn}}{\partial z_j} \\ &= \left\{ m \left(\frac{N}{2} - 1 - m \right) + n \left(\frac{N}{2} - 1 - n \right) + \frac{m-n}{2} \right\} \Phi_{mn} \\ &\quad - \sum_{\ell=0}^n (m-n+2\ell) \Phi_{m+\ell, n-\ell},\end{aligned}\quad (3.54)$$

for $m \geq n$, where

$$\Phi_{mn} = \sum_{\alpha=1}^N \sum_{\beta=1}^N (z_\alpha^*)^m (z_\beta^*)^n \Phi_{\alpha\beta} = N^2 P_m P_n.\quad (3.55)$$

In obtaining this last expression we have used the identity

$$\frac{x+y}{x-y} (x^m y^n - x^n y^m) = 2 \sum_{\ell=0}^{m-n} x^{m-\ell} y^{n+\ell} - (x^m y^n + x^n y^m).\quad (3.56)$$

The solution of (3.40) is then

$$\Phi = \sum_{\ell=0}^n a_\ell \Phi_{m+\ell, n-\ell}$$

$$\lambda = m \left(\frac{N}{2} - 1 - m \right) + n \left(\frac{N}{2} - 1 - n \right) - \frac{m-n}{2}, \quad (3.57)$$

where the coefficients a_ℓ are given by (3.50). Such a simple solution is possible because the matrix to which (3.54) corresponds is lower triangular, i.e. takes the form

$$\text{Matrix} = \begin{bmatrix} E_0 & 0 & 0 & 0 & \dots \\ v_{10} & E_1 & 0 & 0 & \\ v_{20} & v_{21} & E_2 & 0 & \\ v_{30} & v_{31} & v_{32} & E_3 & \dots \\ \vdots & & & \vdots & \end{bmatrix}. \quad (3.58)$$

The eigenvalues of such a matrix are its diagonal elements, and the corresponding eigenvectors are generated by recursion. It should be noted that the upper bound on the sum in (3.50) is flexible, as Φ_{mn} is identically zero unless $0 \leq m, n \leq M \pmod{N}$. We have chosen the largest possible value so as to optimize the smoothness and short-rangeness of $f_{mn}(z)$. \square

It may be seen in Fig. 3.5 that $f_{mn}(z_\alpha/z_\beta)$ exhibits a scattering resonance, an enhancement when $z_\alpha^* z_\beta \simeq 1$, indicating that the spinons attract each other. This attractive force may also be inferred from the energy eigenvalue if we rewrite it as

$$\mathcal{H}_{\text{HS}} |\Psi_{mn}\rangle = \left\{ -J \left(\frac{\pi^2}{24} \right) \left(N + \frac{5}{N} \right) + E_{q_1} + E_{q_2} + V_{q_1-q_2} \right\} |\Psi_{mn}\rangle, \quad (3.59)$$

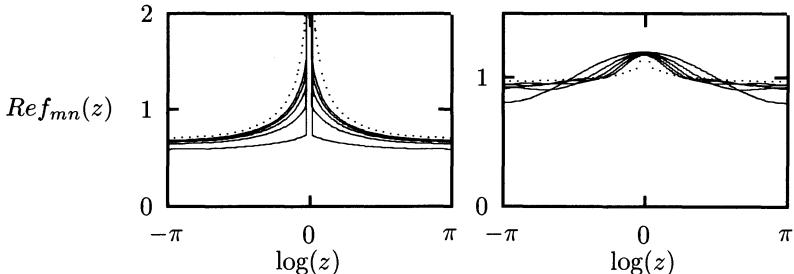


Fig. 3.5. *Left:* Real part of spinon pair internal wavefunction $f_{mn}(z)$ defined by (3.50) for the case of $N = 100$ and $m - n = 2, 4, \dots, 10$ (solid) and 50 (dots). *Right:* The “adjoint” function $\bar{f}_{mn}(z)$ defined in Problem 7

where

$$q_1 = \frac{\pi}{2} - \frac{2\pi}{N} \left(m + \frac{1}{2} \right) \quad q_2 = \frac{\pi}{2} - \frac{2\pi}{N} \left(n + \frac{1}{2} \right) \quad (3.60)$$

are the spinon momenta, E_q is defined as in (3.46), and

$$V_q = -J \frac{\pi}{N} |q| . \quad (3.61)$$

Note that this potential vanishes as $N \rightarrow \infty$, as expected for particles that interact only when they are close together, and that the total crystal momentum, as defined by (3.44), is

$$q = \frac{\pi}{2}(N-2) + q_1 + q_2 \pmod{2\pi} , \quad (3.62)$$

a value greater by π than that of the ground state when $q_1 = q_2 = 0$.

For every triplet state $|\Psi_{mn}\rangle$ defined as in (3.48) there is a corresponding singlet $\Lambda^z S^+ |\Psi_{mn}\rangle$, where Λ is defined as in (3.4), with exactly the same energy eigenvalue. For example, for the case of $m = N/2 - 1$ and $n = 0$ we have

$$[\Lambda^z S^+ \Psi_{mn}](z_1, \dots, z_{N/2}) = -N^2 \prod_{j < k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2 \right] , \quad (3.63)$$

which may be seen to be the alternate ground state defined in (3.32).

Proof

We begin by using the sum rules

$$\prod_j^M z_j \prod_j^{N-M} \eta_j = 1 \quad z_j \prod_{k \neq j}^M (z_j - z_k) \prod_k^{N-M} (z_j - \eta_k) = N , \quad (3.64)$$

to rewrite the wavefunction in terms of the \downarrow spin locations η_j , per

$$\Psi_{mn}(z_1, \dots, z_{N/2-1}) = \prod_{j < k}^{N/2-1} (z_j - z_k)^2 \prod_j^{N/2-1} z_j^2 = \prod_{j < k}^{N/2+1} (\eta_j - \eta_k)^2 . \quad (3.65)$$

We then have

$$\begin{aligned} [S^+ \Psi_{mn}](z_1, \dots, z_{N/2}) &= \sum_{\alpha} \Psi_{mn}(\eta_1, \dots, \eta_{N/2}, z_{\alpha}) \\ &= N \prod_{j < k}^{N/2} (\eta_j - \eta_k)^2 \left[1 + \prod_{j=1}^{N/2} \eta_j^2 \right] = N \prod_{j < k}^{N/2} (z_j - z_k)^2 \left[1 + \prod_{j=1}^{N/2} z_j^2 \right] , \end{aligned} \quad (3.66)$$

and thus

$$\begin{aligned}
& [\Lambda^z S^+ \Psi_{mn}](z_1, \dots, z_{N/2}) \\
&= \frac{1}{2} \sum_{\alpha \neq \beta} \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) [S_\alpha^+ S_\beta^- S^+ \Psi_{mn}](z_1, \dots, z_{N/2}) \\
&= \frac{1}{2} \sum_j \sum_{\beta \neq j} \left(\frac{z_j + z_\beta}{z_j - z_\beta} \right) [S^+ \Psi_{mn}](z_1, \dots, z_{j-1}, z_\beta, z_{j+1}, \dots, z_{N/2}) \\
&= N \sum_j^{N/2} z_j \frac{\partial}{\partial z_j} \left\{ \prod_{j < k}^{N/2} (z_j - z_k)^2 \left[1 + \prod_{j=1}^{N/2} z_j^2 \right] \right\} \\
&= -N^2 \prod_{j < k}^{N/2} (z_j - z_k)^2 \left[1 - \prod_{j=1}^{N/2} z_j^2 \right]. \quad \square
\end{aligned} \tag{3.67}$$

The singlet has no simple exact representation in terms of $|\Psi_{\alpha\beta}\rangle$ but is reasonably approximated by

$$\begin{aligned}
& \Lambda^z S^+ |\Psi_{mn}\rangle \\
& \simeq \frac{1}{2} \sum_{\alpha \neq \beta}^N f_{mn}(z_\alpha/z_\beta) (z_\alpha^*)^m (z_\beta^*)^n \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) (S_\alpha^+ - S_\beta^+) |\Psi_{\alpha\beta}\rangle.
\end{aligned} \tag{3.68}$$

That it is an energy eigenstate follows from the conservation of Λ and the non-commutativity of Λ and S implicit in (3.11). That it is a singlet follows from

$$\Lambda^z |\Psi_{mn}\rangle = \left\{ \frac{N-2}{2} - m - n \right\} |\Psi_{mn}\rangle, \tag{3.69}$$

i.e. that $|\Psi_{mn}\rangle$ is an eigenstate of Λ^z , an important result we shall revisit. This implies that the spin-2 representation contained in the 9 states $\Lambda^\mu S^\nu |\Psi_{mn}\rangle$ ($\mu, \nu = 1, 2, 3$) must be identically zero and the spin-1 representation must be just $|\Psi_{mn}\rangle$ itself.

Spinons are semions, i.e. particles obeying 1/2 fractional statistics. Since the 2-spinon wavefunction

$$\Psi_{AB} = \prod_j (z_j - z_A)(z_j - z_B) \prod_{j < k} (z_j - z_k)^2, \tag{3.70}$$

where z_A and z_B are not necessarily lattice sites, has the property

$$\Psi_{AB}^*(z_1, \dots, z_{N/2-1}) = (z_A z_B)^{1-N/2} \Psi_{AB}(z_1, \dots, z_{N/2-1}), \tag{3.71}$$

the Berry phase vector potential for adiabatic motion of spinon A in the presence of B is

$$\frac{1}{2} \left[\langle \psi_{AB} | z_A \frac{\partial}{\partial z_A} \psi_{AB} \rangle + \langle z_A \frac{\partial}{\partial z_A} \psi_{AB} | \psi_{AB} \rangle \right] / \langle \psi_{AB} | \psi_{AB} \rangle = \frac{1}{2} \left(1 - \frac{N}{2} \right). \quad (3.72)$$

The phase to “exchange” the spinons by moving A all the way around the loop is thus

$$\Delta\phi = \oint \frac{1}{2} \left(1 - \frac{N}{2} \right) \frac{dz_A}{z_A} = \pm \frac{\pi}{2} i \pmod{2\pi}. \quad (3.73)$$

This number is 0 or π for bosons or fermions. Fractional statistics is actually the long-range force between the spinons manifested in the resonant enhancements of Fig. 3.5 and the potential $V_{q_1-q_2}$ in (3.59), and has nothing to do with the symmetry or antisymmetry of $|\Psi_{\alpha\beta}\rangle$ under interchange of α and β . It does have to do with state-counting. The number of states available to $\ell \downarrow$ spinons, determined by counting the number of distinct symmetric polynomials of the form

$$\Phi_{z_{A_1}, \dots, z_{A_\ell}}(z_1, \dots, z_{(N-\ell)/2}) = \prod_j^{(N-\ell)/2} (z_j - z_{A_1}) \times \dots \times (z_j - z_{A_\ell}), \quad (3.74)$$

is

$$\mathcal{N}_\ell^{\text{semi}} = \binom{N/2 + \ell/2}{\ell}. \quad (3.75)$$

This is just halfway between the numbers

$$\mathcal{N}_\ell^{\text{fermi}} = \binom{N/2}{\ell} \quad \mathcal{N}_\ell^{\text{bose}} = \binom{N/2 + \ell}{\ell}, \quad (3.76)$$

likewise calculated assuming that the number of states available for one particle is $N/2$.

3.5 Annihilation Operators

The operators

$$\boldsymbol{\Omega}_\alpha = \frac{1}{2} \sum_{\beta \neq \alpha} \left(\frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) [i(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta) + \boldsymbol{S}_\beta] \quad (3.77)$$

annihilate the Haldane-Shastry ground state, i.e. satisfy

$$\boldsymbol{\Omega}_\alpha |\Psi\rangle = 0, \quad (3.78)$$

for all α .

Proof

We have as before that $[S_\alpha^+ S_\beta^- \Psi](z_1, \dots, z_{N/2})$ is zero unless one of the arguments $z_1, \dots, z_{N/2}$ equals z_α , but in this case we do not sum over z_α . Instead we have

$$\begin{aligned} \sum_{\beta \neq \alpha}^N \frac{z_\alpha}{z_\alpha - z_\beta} [S_\alpha^+ S_\beta^- \Psi](z_\alpha, z_2, \dots, z_{N/2}) &= \sum_{\beta \neq \alpha}^N \frac{z_\alpha}{z_\alpha - z_\beta} \Psi(z_\beta, z_2, \dots, z_{N/2}) \\ &= \sum_{\ell=0}^{N-2} \left\{ \frac{1}{\ell!} \sum_{\beta \neq \alpha}^N \frac{z_\alpha z_\beta (z_\beta - z_\alpha)^\ell}{z_\alpha - z_\beta} \right\} \frac{\partial^\ell}{\partial z_\alpha^\ell} \left\{ \frac{\Psi(z_\alpha, z_2, \dots, z_{N/2})}{z_\alpha} \right\} \\ &= \left\{ -\frac{N-1}{2} + 2 \sum_{j \neq \alpha}^{N/2} \frac{z_\alpha}{z_\alpha - z_j} \right\} \Psi(z_\alpha, z_2, \dots, z_{N/2}). \end{aligned} \quad (3.79)$$

However $(1/2 + S_\alpha^z)|\Psi\rangle$ is also identically zero unless one of the arguments $z_1, \dots, z_{N/2}$ equals z_α . We thus have

$$\begin{aligned} \sum_{\beta \neq \alpha}^N \left[\frac{z_\alpha}{z_\alpha - z_\beta} \left(\frac{1}{2} + S_\alpha^z \right) \left(\frac{1}{2} + S_\beta^z \right) \Psi \right] (z_\alpha, z_2, \dots, z_{N/2}) \\ = \sum_{j \neq \alpha} \frac{z_\alpha}{z_\alpha - z_j} \Psi(z_\alpha, z_2, \dots, z_{N/2}). \end{aligned} \quad (3.80)$$

Subtracting these from each other we find that

$$\begin{aligned} \left\{ \sum_{\beta \neq \alpha}^N \frac{z_\alpha}{z_\alpha - z_\beta} \left[S_\alpha^+ S_\beta^- - 2 \left(\frac{1}{2} + S_\alpha^z \right) \left(\frac{1}{2} + S_\beta^z \right) \right] \right. \\ \left. + \frac{N-1}{2} \left(\frac{1}{2} + S_\alpha^z \right) \right\} |\Psi\rangle = 0 \end{aligned} \quad (3.81)$$

for all α . However since $|\Psi\rangle$ is a spin singlet the irreducible representations of the rotation group present in this operator must destroy $|\Psi\rangle$ separately. The scalar component is identically zero. The vector component is

$$\sum_{\beta \neq \alpha}^N \frac{z_\alpha}{z_\alpha - z_\beta} [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta] |\Psi\rangle = 0. \quad (3.82)$$

The rank-2 tensor component is the product of the two and therefore also zero. Since $|\Psi\rangle$ is also its own time-reverse it must be destroyed by the time-reverse of the vector operator, i.e.

$$\sum_{\beta \neq \alpha} \frac{z_\alpha^*}{z_\alpha^* - z_\beta^*} [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta] = - \sum_{\beta \neq \alpha} \frac{z_\beta}{z_\alpha - z_\beta} [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta]. \quad (3.83)$$

The difference of these is the trivial operator $\mathbf{S}_\alpha \times \mathbf{S}$, and their sum is $2\boldsymbol{\Omega}_\alpha$. \square

These operators satisfy

$$\mathbf{S}_\alpha \times \boldsymbol{\Omega}_\alpha = -\frac{i}{2}\boldsymbol{\Omega}_\alpha \quad \mathbf{S}_\alpha \cdot \boldsymbol{\Omega}_\alpha = 0 \quad (3.84)$$

and are related to $\boldsymbol{\Lambda}$ by

$$\sum_\alpha \boldsymbol{\Omega}_\alpha = \boldsymbol{\Lambda} . \quad (3.85)$$

They are not symmetries of the Hamiltonian but supercharges, for we have

$$\sum_\alpha \boldsymbol{\Omega}_\alpha^\dagger \cdot \boldsymbol{\Omega}_\alpha = \frac{3}{2} \left[3 \sum_{\alpha \neq \beta} \frac{\mathbf{S}_\alpha \cdot \mathbf{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{N(N^2 + 5)}{16} - \frac{(N+1)}{4} S^2 \right] , \quad (3.86)$$

exactly.

Proof

Since

$$\boldsymbol{\Omega}_\alpha = \sum_{\beta \neq \alpha} \frac{z_\alpha}{z_\alpha - z_\beta} [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta] - \frac{1}{2} [i(\mathbf{S}_\alpha \times \mathbf{S}) + \mathbf{S}] , \quad (3.87)$$

per the previous discussion, we have

$$\begin{aligned} & \sum_\alpha \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \frac{[i(\mathbf{S}_\alpha \times \mathbf{S}_\gamma) + \mathbf{S}_\gamma]^\dagger \cdot [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta]}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} \\ &= \sum_\alpha \boldsymbol{\Omega}_\alpha^\dagger \cdot \boldsymbol{\Omega}_\alpha + \frac{3}{2} \mathbf{S} \cdot \boldsymbol{\Lambda} + \frac{3}{8} (N-1) S^2 . \end{aligned} \quad (3.88)$$

In evaluating this expression we have used

$$[i(\mathbf{S}_\alpha \times \mathbf{S}_\gamma) + \mathbf{S}_\gamma]^\dagger = [i(\mathbf{S}_\gamma \times \mathbf{S}_\alpha) + \mathbf{S}_\gamma] , \quad (3.89)$$

$$\sum_\alpha [i(\mathbf{S} \times \mathbf{S}_\alpha) + \mathbf{S}] \cdot \boldsymbol{\Omega}_\alpha = \sum_\alpha [i\mathbf{S} \cdot (\mathbf{S}_\alpha \times \boldsymbol{\Omega}_\alpha) + \mathbf{S} \cdot \boldsymbol{\Omega}_\alpha] = \frac{3}{2} \mathbf{S} \cdot \boldsymbol{\Lambda} , \quad (3.90)$$

and

$$\begin{aligned} & \sum_\alpha [i(\mathbf{S} \times \mathbf{S}_\alpha) + \mathbf{S}] \cdot [i(\mathbf{S}_\alpha \times \mathbf{S}) + \mathbf{S}] = \frac{3}{2} \sum_\alpha [S^2 - i\mathbf{S} \cdot (\mathbf{S}_\alpha \times \mathbf{S})] \\ &= \frac{3}{2} [N-1] S^2 , \end{aligned} \quad (3.91)$$

which follows from

$$\begin{aligned}
& [i(\mathbf{S}_\gamma \times \mathbf{S}_\alpha) + \mathbf{S}_\gamma] \cdot [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta] \\
&= -(\mathbf{S}_\gamma \times \mathbf{S}_\alpha) \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta) + i\mathbf{S}_\gamma \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta) + i(\mathbf{S}_\gamma \times \mathbf{S}_\alpha) \\
&\quad \cdot \mathbf{S}_\beta + \mathbf{S}_\gamma \cdot \mathbf{S}_\beta \\
&= \left(1 + \frac{3}{4}\right)(\mathbf{S}_\gamma \cdot \mathbf{S}_\beta) - (\mathbf{S}_\gamma \cdot \mathbf{S}_\alpha)(\mathbf{S}_\alpha \cdot \mathbf{S}_\beta) + 2i\mathbf{S}_\gamma \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta) \\
&= \frac{3}{2}[\mathbf{S}_\gamma \cdot \mathbf{S}_\beta + i\mathbf{S}_\gamma \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta)] . \tag{3.92}
\end{aligned}$$

The 2-spin sum is

$$\begin{aligned}
\sum_{\beta \neq \gamma \neq \alpha} \frac{\mathbf{S}_\beta \cdot \mathbf{S}_\gamma}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} &= - \sum_{\alpha \neq \beta \neq \gamma} \frac{z_\alpha z_\gamma}{(z_\alpha - z_\gamma)(z_\alpha - z_\beta)} \mathbf{S}_\gamma \cdot \mathbf{S}_\beta \\
&= - \sum_{\beta \neq \gamma} \frac{z_\gamma}{z_\beta - z_\gamma} \mathbf{S}_\gamma \cdot \mathbf{S}_\beta \sum_{\alpha \neq \beta, \gamma} \left[\frac{z_\alpha}{z_\alpha - z_\beta} - \frac{z_\alpha}{z_\alpha - z_\gamma} \right] \\
&= - \sum_{\beta \neq \gamma} \frac{z_\gamma}{z_\beta - z_\gamma} \mathbf{S}_\gamma \cdot \mathbf{S}_\beta \left[\frac{z_\beta}{z_\beta - z_\gamma} - \frac{z_\gamma}{z_\gamma - z_\beta} \right] \\
&= - \sum_{\beta \neq \gamma} \frac{z_\gamma(z_\gamma + z_\beta)}{(z_\beta - z_\gamma)^2} \mathbf{S}_\gamma \cdot \mathbf{S}_\beta = -\frac{1}{2} \sum_{\beta \neq \gamma} \left(\frac{z_\beta + z_\gamma}{z_\beta - z_\gamma} \right)^2 \mathbf{S}_\beta \cdot \mathbf{S}_\gamma \\
&= -\frac{1}{2} S^2 + \frac{3}{8} N + 2 \sum_{\alpha \neq \beta} \frac{\mathbf{S}_\alpha \cdot \mathbf{S}_\beta}{|z_\alpha - z_\beta|^2} . \tag{3.93}
\end{aligned}$$

The 3-spin sum is

$$\begin{aligned}
i \sum_{\alpha \neq \beta \neq \gamma} \frac{\mathbf{S}_\gamma \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta)}{(z_\alpha^* - z_\gamma^*)(z_\alpha - z_\beta)} &= i \sum_{\alpha \neq \beta \neq \gamma} \frac{z_\alpha z_\gamma}{(z_\alpha - z_\gamma)(z_\alpha - z_\beta)} \mathbf{S}_\alpha \cdot (\mathbf{S}_\gamma \times \mathbf{S}_\beta) \\
&= \frac{i}{2} \sum_{\alpha \neq \beta \neq \gamma} \frac{z_\alpha(z_\gamma - z_\beta)}{(z_\alpha - z_\beta)(z_\alpha - z_\gamma)} \mathbf{S}_\alpha \cdot (\mathbf{S}_\gamma \times \mathbf{S}_\beta) \\
&= \frac{i}{2} \sum_{\alpha \neq \beta \neq \gamma} \left[\frac{z_\alpha}{z_\alpha - z_\gamma} - \frac{z_\alpha}{z_\alpha - z_\beta} \right] \mathbf{S}_\alpha \cdot (\mathbf{S}_\gamma \times \mathbf{S}_\beta) \\
&= \frac{i}{2} \sum_{\alpha \neq \beta \neq \gamma} \left(\frac{z_\alpha + z_\gamma}{z_\alpha - z_\gamma} \right) (\mathbf{S}_\alpha \times \mathbf{S}_\gamma) \cdot \mathbf{S}_\beta = \mathbf{A} \cdot \mathbf{S} . \tag{3.94}
\end{aligned}$$

Note that in the last step we have used the identity

$$(\mathbf{S}_\alpha \times \mathbf{S}_\gamma) \cdot (\mathbf{S}_\alpha + \mathbf{S}_\gamma) = 0 . \tag{3.95}$$

Putting these results together, we find that

$$\begin{aligned}
& \sum_{\alpha} \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \frac{1}{(z_{\alpha}^* - z_{\gamma}^*)(z_{\alpha} - z_{\beta})} [-i(\mathbf{S}_{\alpha} \times \mathbf{S}_{\gamma}) + \mathbf{S}_{\gamma}] \cdot [i(\mathbf{S}_{\alpha} \times \mathbf{S}_{\beta}) + \mathbf{S}_{\beta}] \\
&= \frac{3}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \sum_{\gamma \neq \alpha} \frac{1}{(z_{\alpha}^* - z_{\gamma}^*)(z_{\alpha} - z_{\beta})} [\mathbf{S}_{\gamma} \cdot \mathbf{S}_{\beta} + i\mathbf{S}_{\gamma} \cdot (\mathbf{S}_{\alpha} \times \mathbf{S}_{\beta})] \\
&= \frac{3}{2} \left\{ \sum_{\beta \neq \alpha} \frac{1}{|z_{\alpha} - z_{\beta}|^2} \left[\frac{3}{4} + 3 \mathbf{S}_{\alpha} \cdot \mathbf{S}_{\beta} \right] - \frac{S^2}{2} + \frac{3}{8} N + \mathbf{A} \cdot \mathbf{S} \right\} \\
&= \frac{3}{2} \left[3 \sum_{\alpha \neq \beta} \frac{\mathbf{S}_{\alpha} \cdot \mathbf{S}_{\beta}}{|z_{\alpha} - z_{\beta}|^2} + \frac{N(N^2 + 5)}{16} - \frac{S^2}{2} + \mathbf{S} \cdot \mathbf{A} \right]. \quad \square \tag{3.96}
\end{aligned}$$

Since $\langle \Phi | \boldsymbol{\Omega}_{\alpha}^{\dagger} \cdot \boldsymbol{\Omega}_{\alpha} | \Phi \rangle$ is non-negative for any wavefunction $|\Phi\rangle$, this provides an explicit demonstration that $|\Psi\rangle$ is the true ground state. The annihilation operators and their equivalence to \mathcal{H}_{HS} when squared and summed were originally discovered by Shastry [7]. They are modeled after a similar set of operators discovered for the 2-dimensional chiral spin liquid, although there was a minus-sign error in the original paper which caused the operators to be mistakenly reported as scalars [8]. They are lattice versions of the Knizhnik–Zamolodchikov operators known from studies of the Calogero–Sutherland model, the 1-dimensional Bose gas with inverse-square repulsions [9,10].

3.6 Spin Current

The operator \mathbf{A} is a scaled spin current. Its action on the propagating spinon of (3.36), for example, is

$$\Lambda^z |\Psi_m\rangle = \left\{ \frac{N-1}{4} - m \right\} |\Psi_m\rangle, \tag{3.97}$$

which is proportional to the spinon velocity

$$\frac{dE_q}{dq} = \frac{2\pi J}{N} \left\{ \frac{N-1}{4} - m \right\}. \tag{3.98}$$

Its action on the 2-spinon state given by (3.69) is similarly the sum of the two spinon velocities.

Proof

We have, with $M = (N - 1)/2$,

$$\begin{aligned} [\Lambda^z \Psi_A](z_1, \dots, z_M) &= \frac{1}{2} \sum_j^M \sum_{\beta \neq j} \left(\frac{z_j + z_\beta}{z_j - z_\beta} \right) \Psi_A(z_1, \dots, z_{j-1}, z_\beta, z_{j+1}, \dots, z_M) \\ &= \frac{1}{2} \sum_j^M \sum_\ell \frac{1}{\ell!} \left[\sum_{\beta \neq j} \left(\frac{z_j + z_\beta}{z_j - z_\beta} \right) (z_\beta - z_j)^\ell z_\beta \right] \frac{\partial^\ell}{\partial z_j^\ell} \left\{ \Psi_A(z_1, \dots, z_M) / z_j \right\} \\ &= \left\{ \frac{N-1}{4} - z_A \frac{\partial}{\partial z_A} \right\} \Psi_A(z_1, \dots, z_M), \end{aligned} \quad (3.99)$$

and thus

$$\Lambda^z |\Psi_m\rangle = \sum_{z_A} (z_A^*)^m \Lambda^z |\Psi_A\rangle = \left\{ \frac{N-1}{4} - m \right\} |\Psi_m\rangle. \quad \square \quad (3.100)$$

A more traditional description of this current may be constructed by interpolating the spin operators into the interstices by means of the formula

$$\boldsymbol{\sigma}(z) = \left[\frac{z^{N/2} - z^{-N/2}}{2N} \right] \sum_\beta \left(\frac{z + z_\beta}{z - z_\beta} \right) \boldsymbol{S}_\beta. \quad (3.101)$$

The Hamiltonian is then

$$\frac{1}{2\pi i} \oint \left[z \frac{d\boldsymbol{\sigma}}{dz} \right] \cdot \left[z \frac{d\boldsymbol{\sigma}}{dz} \right] \frac{dz}{z} = -\frac{2}{N} \sum_{\alpha \neq \beta}^N \frac{\boldsymbol{S}_\alpha \cdot \boldsymbol{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{3}{8}(N-1) + \frac{S^2}{8}, \quad (3.102)$$

where the integral is performed over the unit circle. We also have spin density and spin current density operators

$$\boldsymbol{\rho}(z) = -i\boldsymbol{\sigma}(z) \times \boldsymbol{\sigma}(z)$$

$$\mathbf{j}(z) = \frac{1}{2i} \left\{ \boldsymbol{\sigma} \times \left[z \frac{d\boldsymbol{\sigma}}{dz} \right] - \left[z \frac{d\boldsymbol{\sigma}}{dz} \right] \times \boldsymbol{\sigma} \right\}, \quad (3.103)$$

which satisfy the continuity equation

$$\lim_{z \rightarrow z_\alpha} \left\{ z \frac{d\mathbf{j}}{dz} + \left[\sum_{\alpha \neq \beta} \frac{\boldsymbol{S}_\alpha \cdot \boldsymbol{S}_\beta}{|z_\alpha - z_\beta|^2}, \boldsymbol{\rho} \right] \right\} = 0. \quad (3.104)$$

The zero-momentum component of this conserved current density is

$$\frac{1}{2\pi i} \oint \mathbf{j} \frac{dz}{z} = \mathbf{A}. \quad (3.105)$$

3.7 Supersymmetry

We shall now consider the generalization of the Haldane–Shastry Hamiltonian

$$\begin{aligned} \mathcal{H}_{KY} = J \left(\frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta}^N \frac{1}{|z_\alpha - z_\beta|^2} P \left\{ \mathbf{S}_\alpha \cdot \mathbf{S}_\beta \right. \\ \left. - \frac{1}{4} \sum_s (c_{\alpha s}^\dagger c_{\beta s} + c_{\beta s}^\dagger c_{\alpha s}) + \frac{1}{2}(n_\alpha + n_\beta) - \frac{1}{4} n_\alpha n_\beta - \frac{3}{4} \right\} P \end{aligned} \quad (3.106)$$

first studied by Kuramoto and Yokoyama [11], where

$$P = \prod_\alpha (1 - c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow} c_{\alpha\uparrow}) , \quad (3.107)$$

is the Gutzwiller projector, and site occupation and spin operators are

$$\begin{aligned} n_\alpha &= c_{\alpha\uparrow}^\dagger c_{\alpha\uparrow} + c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow} & S_\alpha^x &= \frac{1}{2}(c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow} + c_{\alpha\downarrow}^\dagger c_{\alpha\uparrow}) \\ S_\alpha^y &= \frac{1}{2i}(c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow} - c_{\alpha\downarrow}^\dagger c_{\alpha\uparrow}) & S_\alpha^z &= \frac{1}{2}(c_{\alpha\uparrow}^\dagger c_{\alpha\uparrow} - c_{\alpha\downarrow}^\dagger c_{\alpha\downarrow}) . \end{aligned} \quad (3.108)$$

Thus each site can be \uparrow , \downarrow , or unoccupied - but not doubly occupied - and the hole can tunnel to nearby sites by means of the same inverse-square matrix element characterizing the spin exchange. This equivalence of the energy scales for magnetism and charge transport causes the Hamiltonian to be supersymmetric in the sense of (3.86) and also in the more traditional one of commuting with electron or hole injection per

$$\sum_\alpha [\mathcal{H}_{KY}, P c_{\alpha s} P] = 0 . \quad (3.109)$$

Proof

It suffices to show that

$$[\mathcal{H}_{\alpha\beta}, (c_{\alpha\uparrow} + c_{\beta\uparrow})] |\psi\rangle = 0 , \quad (3.110)$$

where

$$\begin{aligned} \mathcal{H}_{\alpha\beta} = P \left\{ \mathbf{S}_\alpha \cdot \mathbf{S}_\beta - \frac{1}{4} \sum_s (c_{\alpha s}^\dagger c_{\beta s} + c_{\beta s}^\dagger c_{\alpha s}) - \frac{1}{4} n_\alpha n_\beta \right. \\ \left. + \frac{1}{2}(n_\alpha + n_\beta - 1) \right\} P \end{aligned} \quad (3.111)$$

for the 9 configurations $|\psi\rangle$ on sites α and β allowed by the projector P . Denoting the state with no electron on either site by $|0\rangle$ we have

- **Case 1:**

$$|\psi_1\rangle = c_{\alpha\uparrow}^\dagger c_{\beta\uparrow}^\dagger |0\rangle \quad (3.112)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_1\rangle = \frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_1\rangle = \frac{1}{2}(c_{\beta\uparrow}^\dagger - c_{\alpha\uparrow}^\dagger)|0\rangle$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_1\rangle = \mathcal{H}_{\alpha\beta}(c_{\beta\uparrow}^\dagger - c_{\alpha\uparrow}^\dagger)|0\rangle = \frac{1}{2}(c_{\beta\uparrow}^\dagger - c_{\alpha\uparrow}^\dagger)|0\rangle$$

- **Case 2:**

$$|\psi_2\rangle = (c_{\alpha\uparrow}^\dagger c_{\beta\downarrow}^\dagger - c_{\alpha\downarrow}^\dagger c_{\beta\uparrow}^\dagger)|0\rangle \quad (3.113)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_2\rangle = -\frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_2\rangle = -\frac{1}{2}(c_{\beta\downarrow}^\dagger + c_{\alpha\downarrow}^\dagger)|0\rangle$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_2\rangle = \mathcal{H}_{\alpha\beta}(c_{\beta\downarrow}^\dagger + c_{\alpha\downarrow}^\dagger)|0\rangle = -\frac{1}{2}(c_{\beta\downarrow}^\dagger + c_{\alpha\downarrow}^\dagger)|0\rangle$$

- **Case 3:**

$$|\psi_3\rangle = (c_{\alpha\uparrow}^\dagger c_{\beta\downarrow}^\dagger + c_{\alpha\downarrow}^\dagger c_{\beta\uparrow}^\dagger)|0\rangle \quad (3.114)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_3\rangle = \frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_3\rangle = \frac{1}{2}(c_{\beta\downarrow}^\dagger - c_{\alpha\downarrow}^\dagger)|0\rangle$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_3\rangle = \mathcal{H}_{\alpha\beta}(c_{\beta\downarrow}^\dagger - c_{\alpha\downarrow}^\dagger)|0\rangle = \frac{1}{2}(c_{\beta\downarrow}^\dagger - c_{\alpha\downarrow}^\dagger)|0\rangle$$

- **Case 4:**

$$|\psi_4\rangle = c_{\alpha\uparrow}^\dagger c_{\beta\uparrow}^\dagger |0\rangle \quad (3.115)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_4\rangle = \frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_4\rangle = 0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_4\rangle = 0$$

- **Cases 5 and 6:**

$$|\psi_5\rangle = c_{\alpha\uparrow}|0\rangle \quad (3.116)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_5\rangle = -\frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})c_{\beta\uparrow}^\dagger|0\rangle = -\frac{1}{2}|0\rangle$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_5\rangle = \mathcal{H}_{\alpha\beta}|0\rangle = -\frac{1}{2}|0\rangle$$

- **Cases 7 and 8:**

$$|\psi_7\rangle = c_{\alpha\downarrow}|0\rangle \quad (3.117)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_7\rangle = -\frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})c_{\beta\downarrow}^\dagger|0\rangle = 0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_7\rangle = 0$$

- **Case 9:**

$$|\psi_9\rangle = |0\rangle \quad (3.118)$$

$$(c_{\alpha\uparrow} + c_{\beta\uparrow})\mathcal{H}_{\alpha\beta}|\psi_9\rangle = -\frac{1}{2}(c_{\alpha\uparrow} + c_{\beta\uparrow})|0\rangle = 0$$

$$\mathcal{H}_{\alpha\beta}(c_{\alpha\uparrow} + c_{\beta\uparrow})|\psi_9\rangle = 0 . \quad \square$$

3.8 Holons

Holons are charged, spin-0 elementary excitations of the Kuramoto–Yokoyama Hamiltonian made by removing an electron from the center of a spinon. The localized holon $c_{\alpha\downarrow}|\Psi_\alpha\rangle$, where $|\Psi_\alpha\rangle$ is defined as in (3.34), is shown in Fig. 3.6. It is a natural complement to the spinon from which it was made, for the two states differ only in the disposition of the central site, which does not fluctuate and is not involved in any way in the quantum number fractionalization. This state is an exact spin singlet by virtue of (3.35). Let N be odd and let $M = (N - 1)/2$ as in (3.36). Then the propagating holon wavefunction

$$\Psi_m^{\text{holon}}(z_1, \dots, z_M|h) = (h^*)^m \prod_j^M (h - z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_j^M z_j , \quad (3.119)$$

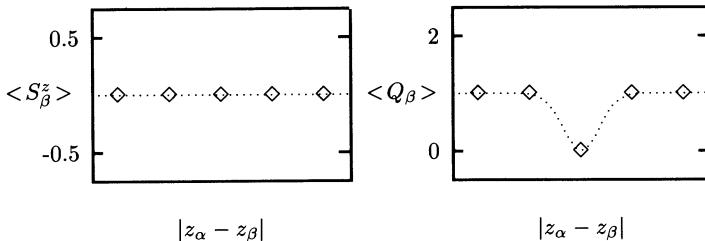


Fig. 3.6. Spin and charge profiles of the localized holon $c_{\alpha\downarrow}|\Psi_\alpha\rangle$. The dotted lines are a guide to the eye

where z_1, \dots, z_M denote the positions of the \uparrow sites and h denotes the position of the empty site, all others being \downarrow , satisfies

$$\begin{aligned} \mathcal{H}_{\text{KY}} |\Psi_m^{\text{holon}}\rangle \\ = \left\{ -J \left(\frac{\pi^2}{24} \right) \left(N - \frac{1}{N} \right) + \frac{J}{2} \left(\frac{2\pi}{N} \right)^2 m \left(\frac{N+1}{2} + m \right) \right\} |\Psi_m^{\text{holon}}\rangle, \end{aligned} \quad (3.120)$$

provided that $-(N+1)/2 \leq m \leq 0$.

Proof

To simplify the notation let us negate the power of h in (3.119) so that the holon wavefunction becomes

$$\Psi_n(z_1, \dots, z_M|h) = h^n \prod_j^M (h - z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_j^M z_j, \quad (3.121)$$

with $0 \leq n \leq (N+1)/2$ and $M = (N-1)/2$. The effect of the spin-exchange part of the Hamiltonian is the same as for the spinon. Taylor expanding as usual we obtain

$$\begin{aligned} \sum_{\alpha \neq \beta}^N \left[\frac{S_\alpha^+ S_\beta^-}{|z_\alpha - z_\beta|^2} \Psi_n \right] (z_1, \dots, z_M|h) &= \left\{ \left[\frac{1-N^2}{24} - \sum_{j \neq k}^M \frac{1}{|z_j - z_k|^2} \right] h^n \right. \\ &\quad \left. + \frac{N-3}{2} h^{n+1} \frac{\partial}{\partial h} - h^{n+2} \frac{\partial^2}{\partial h^2} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M|h)}{h^n} \right\}. \end{aligned} \quad (3.122)$$

The charge-exchange terms also behave similarly. The operator $c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger$ gives zero unless the holon resides at z_α . For this case we have

$$\begin{aligned} \sum_{\alpha \neq \beta}^N \left[\frac{P c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger P}{|z_\alpha - z_\beta|^2} \Psi_n \right] (z_1, \dots, z_M|z_\alpha) &= \sum_{\beta \neq \alpha}^N \frac{1}{|z_\alpha - z_\beta|^2} \Psi_n(z_1, \dots, z_M|z_\beta) \\ &= \sum_{\beta \neq \alpha}^N \sum_{\ell=0}^{M+1} \frac{z_\beta^n (z_\beta - z_\alpha)^\ell}{\ell! |z_\beta - z_\alpha|^2} \left(\frac{\partial}{\partial z_\alpha} \right)^\ell \left\{ \frac{\Psi_n(z_1, \dots, z_M|z_\alpha)}{z_\alpha^n} \right\} \\ &= \left\{ \left[\frac{N^2 - 1}{12} + \frac{n(n-N)}{2} \right] z_\alpha^n \right. \\ &\quad \left. - \left[\frac{N-1}{2} - n \right] z_\alpha^{n+1} \frac{\partial}{\partial z_\alpha} + \frac{1}{2} z_\alpha^{n+2} \frac{\partial^2}{\partial z_\alpha^2} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M|z_\alpha)}{z_\alpha^n} \right\}. \end{aligned} \quad (3.123)$$

Summing on α we then obtain

$$\begin{aligned} \sum_{\alpha \neq \beta}^N \left[\frac{P c_{\alpha\downarrow} c_{\beta\downarrow}^\dagger P}{|z_\alpha - z_\beta|^2} \Psi_n \right] (z_1, \dots, z_M|h) &= \left\{ \left[\frac{N^2 - 1}{12} + \frac{n(n-N)}{2} \right] h^n \right. \\ &\quad \left. - \left[\frac{N-1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h} + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M|h)}{h^n} \right\}. \end{aligned} \quad (3.124)$$

For the other charge-exchange channel we use the fact that $|\Psi_n\rangle$ is the same written in the \downarrow coordinates η_1, \dots, η_M by virtue of being a singlet. This gives

$$\sum_{\alpha \neq \beta}^N \left[\frac{P c_{\alpha\uparrow} c_{\beta\uparrow}^\dagger P}{|z_\alpha - z_\beta|^2} \Psi_n \right] (\eta_1, \dots, \eta_M | h) = \left\{ \left[\frac{N^2 - 1}{12} + \frac{n(n - N)}{2} \right] h^n - \frac{1}{2} \left[\frac{N - 1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h} + \frac{1}{4} h^{n+2} \frac{\partial^2}{\partial h^2} \right\} \left\{ \frac{\Psi_n(\eta_1, \dots, \eta_M | h)}{h^n} \right\}. \quad (3.125)$$

It remains only to rewrite this expression in terms of the \uparrow coordinates z_1, \dots, z_M . We have

$$\begin{aligned} h^{n+1} \frac{\partial}{\partial h} \left\{ \frac{\Psi_n(\eta_1, \dots, \eta_M | h)}{h^n} \right\} &= \sum_j^M \frac{h}{h - \eta_j} \Psi_n(\eta_1, \dots, \eta_M | h) \\ &= \left\{ \frac{N - 1}{2} - \sum_j^M \frac{h}{h - z_j} \right\} \Psi_n(z_1, \dots, z_M | h) \\ &= \left\{ \left(\frac{N - 1}{2} \right) h^n - h^{n+1} \frac{\partial}{\partial h} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M | h)}{h^n} \right\}, \end{aligned} \quad (3.126)$$

and

$$\begin{aligned} h^{n+2} \frac{\partial^2}{\partial h^2} \left\{ \frac{\Psi_n(\eta_1, \dots, \eta_M | h)}{h^n} \right\} &= \sum_{j \neq k}^M \frac{h^2}{(h - \eta_j)(h - \eta_k)} \Psi_n(\eta_1, \dots, \eta_M | h) \\ &= \sum_j^M \frac{h}{h - \eta_j} \left\{ \frac{N - 1}{2} - \sum_k^M \frac{h}{h - z_k} - \frac{h}{h - \eta_j} \right\} \Psi_n(\eta_1, \dots, \eta_M | h) \\ &= \left\{ \left[\frac{N - 1}{2} - \sum_k^M \frac{h}{h - z_k} \right]^2 + \frac{(N - 1)(N - 5)}{12} \right. \\ &\quad \left. + \sum_j^M \left(\frac{h}{h - z_j} \right)^2 \right\} \Psi_n(z_1, \dots, z_M | h) \\ &= \left\{ \left[\frac{(N - 1)(N - 2)}{3} + 2 \sum_j^M \left(\frac{h}{h - z_j} \right)^2 \right] h^n \right. \\ &\quad \left. - (N - 1) h^{n+1} \frac{\partial}{\partial h} + h^{n+2} \frac{\partial^2}{\partial h^2} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M | h)}{h^n} \right\}, \end{aligned} \quad (3.127)$$

which gives

$$\sum_{\alpha \neq \beta}^N \left[\frac{P c_{\alpha \uparrow} c_{\beta \uparrow}^{\dagger} P}{|z_{\alpha} - z_{\beta}|^2} \Psi_n \right] (z_1, \dots, z_M | h) = \left\{ \frac{n(n-1)}{2} h^n + \sum_j^M \frac{h^{n+2}}{(h - z_j)^2} \right. \\ \left. - n h^{n+1} \frac{\partial}{\partial h} + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \right\} \left\{ \frac{\Psi_n(z_1, \dots, z_M | h)}{h^n} \right\}. \quad (3.128)$$

For the Ising and electrostatic energies we have

$$\sum_{\alpha \neq \beta}^N \left[\frac{S_{\alpha}^z S_{\beta}^z}{|z_{\alpha} - z_{\beta}|^2} \Psi_n \right] (z_1, \dots, z_M | h) \\ = \left\{ \sum_{j \neq k}^M \frac{1}{|z_j - z_k|^2} + \sum_j^M \frac{1}{|h - z_j|^2} - \frac{N(N^2 - 1)}{48} \right\} \Psi_n(z_1, \dots, z_M | h) \quad (3.129)$$

and

$$\sum_{\alpha \neq \beta}^N \frac{1}{|z_{\alpha} - z_{\beta}|^2} \left[\frac{1}{2}(n_{\alpha} + n_{\beta}) - \frac{1}{4} n_{\alpha} n_{\beta} - \frac{3}{4} \right] \Psi_n(z_1, \dots, z_M | h) \\ = \frac{1 - N^2}{24} \Psi_n(z_1, \dots, z_M | h). \quad (3.130)$$

Adding these contributions together we obtain

$$\sum_{\alpha \neq \beta}^N \frac{1}{|z_{\alpha} - z_{\beta}|^2} P \left[S_{\alpha}^+ S_{\beta}^- + S_{\alpha}^z S_{\beta}^z + \sum_s c_{\alpha s} c_{\beta s}^{\dagger} + \frac{1}{2}(n_{\alpha} + n_{\beta}) \right. \\ \left. - \frac{1}{4} n_{\alpha} n_{\beta} - \frac{3}{4} \right] P \Psi_n(z_1, \dots, z_M | h) \\ = \left[-\frac{N(N^2 - 1)}{48} + n \left(n - \frac{N + 1}{2} \right) \right] \Psi_n(z_1, \dots, z_M | h). \quad \square \quad (3.131)$$

The propagating holon wavefunction differs from that of the spinon in being defined for *all* values of m , including those for which it is not an eigenstate of the \mathcal{H}_{KY} . For these other values of m we observe that

$$0 < \left| \langle \Psi_m | \sum_{\alpha}^N P c_{\alpha \downarrow}^{\dagger} P | \Psi_m^{\text{holon}} \rangle \right|^2 < 1, \quad (3.132)$$

i.e. that a supersymmetric rotation of the spinon, which is an eigenstate of \mathcal{H}_{KY} , has a nonzero projection onto the corresponding holon. The physical meaning of this is that the holon can lose energy by spontaneous emission of

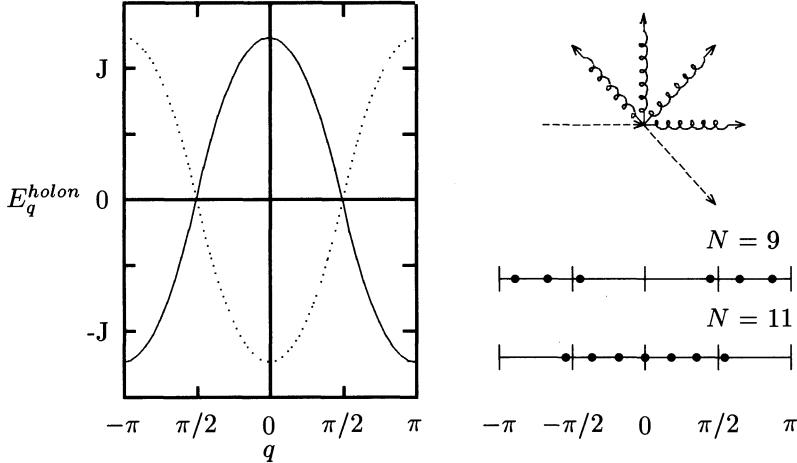


Fig. 3.7. *Left:* Holon dispersion relation defined by (3.134). *Right:* Allowed values of q for adjacent odd N . Only the negative-energy holons are eigenstates of the Hamiltonian as positive-energy holons can lose energy by spontaneous emission of spinons

spinons. Let us assign to this holon the energy of the exact eigenstate onto which it projects and write

$$E_m^{\text{holon}} = \frac{J}{2} \left(\frac{2\pi}{N} \right)^2 \begin{cases} m \left(\frac{N+1}{2} + m \right) & -\frac{N+1}{2} \leq m \leq 0 \\ m \left(\frac{N-1}{2} - m \right) & 0 \leq m \leq \frac{N-1}{2} \end{cases}, \quad (3.133)$$

or

$$E_q^{\text{holon}} = \begin{cases} E_q & |q| < \pi/2 \pmod{2\pi} \\ -E_q & |q| \geq \pi/2 \pmod{2\pi} \end{cases}, \quad (3.134)$$

the crystal momentum q and spinon energy E_q are defined as in (3.43) and (3.46). This is shown in Fig. 3.7. The positive-energy part of the holon band is then unstable because it has negative curvature and thus does not satisfy the Landau criterion for spontaneous decay. A negative-energy holon, on the other hand, is forbidden from decaying by momentum conservation.

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Appendix A: Fourier Sums

Since the lattice sites z_α are roots of unity we have

$$\prod_{\alpha}^N (z - z_\alpha) = z^N - 1 . \quad (3.135)$$

Then for $0 < m \leq N$ we have

$$\begin{aligned} \sum_{\alpha=1}^{N-1} \frac{z_\alpha^m}{z_\alpha - 1} &= \frac{N}{2\pi i} \oint_C \frac{z^{m-1} dz}{(z-1)(z^N-1)} = -\frac{N}{2\pi i} \oint_{C'} \frac{z^{m-1} dz}{(z-1)(z^N-1)} \\ &= -\frac{N}{2\pi i} \oint \left\{ \frac{1 + \binom{m-1}{1} x + \binom{m-1}{2} x^2 + \dots}{\binom{N}{1} + \binom{N}{2} x + \binom{N}{3} x^2 + \dots} \right\} \frac{dx}{x^2} \\ &= \frac{N+1}{2} - m , \end{aligned} \quad (3.136)$$

and

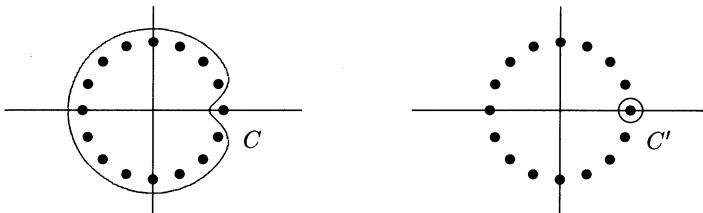


Fig. 3.8. Contours used in (3.136) and (3.137)

$$\begin{aligned}
\sum_{\alpha=1}^{N-1} \frac{z_\alpha^m}{|z_\alpha - 1|^2} &= - \sum_{\alpha=1}^{N-1} \frac{z_\alpha^{m+1}}{(z_\alpha - 1)^2} \\
&= - \frac{N}{2\pi i} \oint_C \frac{z^m dz}{(z-1)^2(z^N-1)} = \frac{N}{2\pi i} \oint_{C'} \frac{z^m dz}{(z-1)^2(z^N-1)} \\
&= \frac{1}{2\pi i} \oint \left\{ \frac{1 + \binom{m-1}{1} x + \binom{m-1}{2} x^2 + \dots}{\binom{N}{1} + \binom{N}{2} x + \binom{N}{3} x^2 + \dots} \right\} \frac{dx}{x^3} \\
&= \frac{N^2 - 1}{12} - \frac{m(N-1)}{2} + \frac{m(m-1)}{2}. \tag{3.137}
\end{aligned}$$

Appendix B: Problems

1. Let $\eta_1, \dots, \eta_{N/2}$ be the sites complementary to $z_1, \dots, z_{N/2}$. Show that

$$\prod_{j < k}^{N/2} (z_j - z_k)^2 \prod_j^{N/2} z_j = \prod_{j < k}^{N/2} (\eta_j - \eta_k)^2 \prod_j^{N/2} \eta_j. \tag{3.138}$$

2. Show that for any polynomial $p(z)$ of degree less than N

$$\sum_{\beta \neq \alpha} \frac{p(z_\beta)}{z_\alpha - z_\beta} = \frac{1}{z_\alpha} \left[Np(0) - \left(\frac{N+1}{2} \right) p(z_\alpha) \right] + \frac{\partial p}{\partial z}(z_\alpha). \tag{3.139}$$

3. Show that for $\alpha \neq \beta \neq \gamma$

$$(\mathbf{S}_\alpha \times \mathbf{S}_\beta)^\dagger = (\mathbf{S}_\alpha \times \mathbf{S}_\beta) \tag{3.140}$$

$$\mathbf{S}_\alpha \cdot (\mathbf{S}_\beta \times \mathbf{S}_\gamma) = \mathbf{S}_\gamma \cdot (\mathbf{S}_\alpha \times \mathbf{S}_\beta) = -\mathbf{S}_\alpha \cdot (\mathbf{S}_\gamma \times \mathbf{S}_\beta) \tag{3.141}$$

$$[(\mathbf{S}_\alpha \cdot \mathbf{S}_\beta), (\mathbf{S}_\alpha \times \mathbf{S}_\beta)] = 0 \tag{3.142}$$

$$[(\mathbf{S}_\alpha \cdot \mathbf{S}_\gamma), (\mathbf{S}_\alpha \times \mathbf{S}_\beta)] = i \left[(\mathbf{S}_\alpha \cdot \mathbf{S}_\beta) \mathbf{S}_\gamma - (\mathbf{S}_\gamma \cdot \mathbf{S}_\beta) \mathbf{S}_\alpha \right]. \tag{3.143}$$

4. Show that the operator

$$\begin{aligned}
&\mathbf{R}_\alpha(z) \\
&= \sum_{\beta \neq \alpha} \left[\left(\frac{1-z}{2} \right) \frac{z_\alpha}{z_\alpha - z_\beta} + \left(\frac{1+z}{2} \right) \frac{z_\beta}{z_\alpha - z_\beta} \right] [i(\mathbf{S}_\alpha \times \mathbf{S}_\beta) + \mathbf{S}_\beta], \tag{3.144}
\end{aligned}$$

where z is an arbitrary complex number, satisfies

$$\mathbf{R}_\alpha(z)|\Psi\rangle = 0, \quad (3.145)$$

where $|\Psi\rangle$ is the Haldane-Shastry ground state, and

$$\begin{aligned} \sum_\alpha \mathbf{R}_\alpha^\dagger(z) \cdot \mathbf{R}_\alpha(z) &= \frac{3}{2} \left\{ 3 \sum_{\alpha \neq \beta} \frac{\mathbf{S}_\alpha \cdot \mathbf{S}_\beta}{|z_\alpha - z_\beta|^2} + \frac{N(N^2 + 5)}{16} \right. \\ &\quad \left. + \left[\frac{N-1}{4} \left(|z|^2 - 1 \right) - \frac{1}{2} \right] S^2 - \left(\frac{z + z^*}{2} \right) \mathbf{S} \cdot \mathbf{A} \right\}. \end{aligned} \quad (3.146)$$

5. Show that the ground state wavefunction for a noninteracting fermi sea on this lattice is

$$\Phi(z_1, \dots, z_{N/2}, \eta_1, \dots, \eta_{N/2}) = \prod_j^{N/2} (z_j \eta_j)^{-N/2} \prod_{j \leq k}^{N/2} (z_j - z_k)(\eta_j - \eta_k). \quad (3.147)$$

Then show that the Haldane-Shastry ground state is the Gutzwiller projection of $|\Phi\rangle$, i.e. that $|\Psi_{\text{HS}}\rangle = P |\Phi\rangle$, where P is defined as in (3.107). Hint: If p denotes a permutation of $N/2$ things and $\text{sgn}(p)$ is its sign, then

$$\sum_p^{(N/2)!} \text{sgn}(p) z_{p(1)}^0 \times \dots \times z_{p(N/2)}^{N/2-1} = \prod_{j < k}^{N/2} (z_j - z_k). \quad (3.148)$$

6. For lattice site z_α different from 1 show that

$$\begin{aligned} \frac{2}{J} \left(\frac{N}{2\pi} \right)^2 \sum_{m=-(N+1)/2}^{(N-1)/2} E_m^{\text{holon}} z_\alpha^m &= \frac{z_\alpha}{(1-z_\alpha)^2} \left\{ \frac{z_\alpha^{(N+1)/2} - z_\alpha^{(N-1)/2}}{2} \right. \\ &\quad \left. + \left[\frac{1+z_\alpha}{1-z_\alpha} + \frac{1}{2} \right] \left[z_\alpha^{(N+1)/2} + z_\alpha^{(N-1)/2} - 2 \right] \right\}, \end{aligned} \quad (3.149)$$

where E_m^{holon} is given by (3.133). Then use this result to show that

$$\begin{aligned} \langle \psi_\alpha | \psi_\beta \rangle &= \sum_{z_1} \dots \sum_{z_M} \psi_\alpha^*(z_1, \dots, z_M) \psi_\beta(z_1, \dots, z_M) \\ &= \frac{2 \langle \psi_\alpha | \psi_\alpha \rangle}{N} \left\{ \frac{1 - (z_\alpha^* z_\beta)^{(N+1)/2}}{1 - z_\alpha^* z_\beta} - \frac{1 + (z_\alpha^* z_\beta)^{(N-1)/2}}{4} \right\}, \end{aligned} \quad (3.150)$$

where $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ are localized spinon wavefunctions defined per (3.34).

Hint: The Fourier sum measures $\langle \psi_\alpha | c_{\alpha\downarrow}^\dagger \mathcal{H}_{\text{KY}} c_{\beta\downarrow} | \psi_\beta \rangle$.

7. The amplitudes a_ℓ defined by (3.50) are the right eigenvectors of the non-hermitian matrix of (3.58). Show that the left eigenvectors of this matrix are

$$b_\ell = \frac{1}{2\ell[\ell + m - n - 1/2]} \sum_{k=\ell+1}^0 (m - n + 2k)b_k \quad (b_0 = 0), \quad (3.151)$$

for $\ell \leq 0$. The size of these “adjoint” coefficients may be judged from Fig. 3.5, where we have plotted the function $\bar{f}_{mn}(z) = \sum_\ell b_\ell z^\ell$. Then show that

$$\sum_{\alpha}^N z_{\alpha}^k S_{\alpha}^- |\psi\rangle = \sum_{j=0}^{k/2} c_j |\psi_{k-j,j}\rangle, \quad (3.152)$$

where $|\psi\rangle$ is the Haldane-Shastry ground state $|\psi_{mn}\rangle$ is the 2-spinon eigenstate defined by (3.48), and c_j are a set of coefficients. Obtain an expression these in terms of the b_ℓ .

8. Let A_β and B_β be any quantum operators and let

$$\begin{aligned} a(z) &= \left[\frac{z^{N/2} - z^{-N/2}}{2N} \right] \sum_{\beta}^N \left(\frac{z + z_{\beta}}{z - z_{\beta}} \right) A_{\beta} \\ b(z) &= \left[\frac{z^{N/2} - z^{-N/2}}{2N} \right] \sum_{\beta}^N \left(\frac{z + z_{\beta}}{z - z_{\beta}} \right) B_{\beta}. \end{aligned} \quad (3.153)$$

Show that

$$\frac{N}{2\pi i} \oint a^\dagger(z) b(z) \frac{dz}{z} = \sum_{\beta}^N A_{\beta}^\dagger B_{\beta}. \quad (3.154)$$

9. Show that the operator $\sigma(z)$ defined by (3.101) is hermitian whenever $|z| = 1$.
 10. Let N be odd and let z_1, \dots, z_M and η_1, \dots, η_M with $M = (N - 1)/2$ be distinct lattice sites. Show that

$$\sum_{j \neq k}^M \frac{1}{|\eta_j - \eta_k|^2} = \sum_{j \neq k}^M \frac{1}{|z_j - z_k|^2} + \sum_{j=1}^M \frac{1}{|h - z_j|^2} - \frac{N^2 - 1}{12}, \quad (3.155)$$

where h denotes the one site not occupied by z_j or η_j .

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4 Conformal Field Theory Approach to Quantum Impurity Problems

Ian Affleck

There are several problems in condensed matter (and particle) physics in which a local degree of freedom interacts with a gapless (critical) continuum. Some of these are intrinsically 1 dimensional (1D): quantum wires, quantum Hall effect edge states, quantum spin chains. Others can be reduced to 1D because the bulk excitations are harmonic and can be expanded in spherical (or other) harmonics giving a radial (1D) problem: the Kondo problem (1,2,3, ... impurities), X-ray edge singularities, the monopole-baryon system (Callan–Rubakov effect). These problems typically exhibit infrared singularities causing the breakdown of perturbation theory in the impurity-bulk interactions. A powerful method to study the long distance, long time behaviour is the renormalization group (RG) together with boundary conformal field theory (BCFT), which I developed with various collaborators, especially Andreas Ludwig [1,2]. Other useful methods are the Bethe ansatz [3,4], exact integrability [5], the study of special models which are fully harmonic (called the Toulouse limit in the Kondo problem) [6] and numerical techniques [7]. In these lectures I will discuss the Kondo problem (including its multi-channel generalizations). This is the model for which BCFT has been most useful so far. New applications of these techniques may arise as experiments on nanostructures progress.

4.1 Old Results on the Kondo Problem

The Kondo (or s-d) model is a model for dilute magnetic impurities in a metal, e.g. $\text{Cu}_{1-x}\text{Fe}_x$ with $x \approx 10^{-5} - 10^{-3}$. The resistivity, susceptibility, specific heat and other quantities contain a term linear in x , given by the single impurity model:

$$H = \sum_{\mathbf{k}\alpha} \psi_{\mathbf{k}}^{\dagger\alpha} \psi_{\mathbf{k}\alpha} \epsilon(\mathbf{k}) + \lambda \mathbf{S} \cdot \sum_{\mathbf{k}\mathbf{k}'} \psi_{\mathbf{k}}^{\dagger} \frac{\boldsymbol{\sigma}}{2} \psi_{\mathbf{k}'} , \quad (4.1)$$

where $\psi_{\mathbf{k}\alpha}$'s are conduction electron annihilation operators (of momentum \mathbf{k} , spin α), ϵ_k is the dispersion relation, assumed spherically symmetry, λ is the Kondo coupling constant and \mathbf{S} represents the spin of the magnetic impurity with

$$[S^a, S^b] = i\epsilon^{abc} S^c .$$

We consider the case $S = 1/2$. The interaction term represents an impurity spin interacting with the electron spin at $\mathbf{x} = 0$.

4.1.1 Reduction to 1D

For small λ only momentum $k \approx k_F$, the Fermi wave-vector, is important, e.g.

$$\epsilon(k) = \frac{k^2}{2m} - \epsilon_F \approx v_F(k - k_F). \quad (4.2)$$

Assuming a δ -function Kondo interaction, only the s-wave interacts with the impurity so we can reduce the problem to 1D, i.e.

$$\psi(\mathbf{k}) = \frac{1}{\sqrt{4\pi k}} \psi_0(k) + \text{higher harmonics}. \quad (4.3)$$

Upon linearizing the dispersion relation we get a Lorentz invariant Dirac fermion:

$$\Psi_{L,R}(r) \equiv \int_{-\Lambda}^{\Lambda} dk e^{\pm ikr} \psi_0(k + k_F), \quad \Rightarrow \quad \psi_L(0) = \psi_R(0). \quad (4.4)$$

($\Lambda \ll k_F$ is a reduced cut-off.) The low energy Hamiltonian is:

$$H = \frac{v_F}{2\pi} \int_0^\infty dr \left(\psi_L^\dagger i \frac{d}{dr} \psi_L - \psi_R^\dagger i \frac{d}{dr} \psi_R \right) + v_F \lambda \psi_L^\dagger(0) \frac{\sigma}{2} \psi_L(0) \cdot \mathbf{S}. \quad (4.5)$$

(λ was rescaled by the density of states; I will set $v_F = 1$.) In the imaginary time formulation, $\psi_{L,R}$ are analytic functions of the complex variable z (z^*) only:

$$\psi_L = \psi_L(x, \tau) = \psi_L(z = \tau + ix), \quad \psi_R(x, \tau) = \psi_R(z^* = \tau - ix), \quad (4.6)$$

where τ is imaginary time and $x = r$. We have normalized the fields so that:

$$\langle \psi_L(z) \psi_L^\dagger(0) \rangle = \frac{1}{z}, \quad \langle \psi_R(z^*) \psi_R^\dagger(0) \rangle = \frac{1}{z^*}. \quad (4.7)$$

An alternative formulation involves left-movers only, defined on the entire real axis:

$$\psi_L(\tau, -r) = \psi_R(\tau, r) \quad (r > 0), \quad (4.8)$$

with the free part of the Hamiltonian becoming:

$$H_0 = \frac{v_F}{2\pi} \int_{-\infty}^\infty dx \psi_L^\dagger i \frac{d}{dx} \psi_L \quad (4.9)$$

see Fig. 4.1.

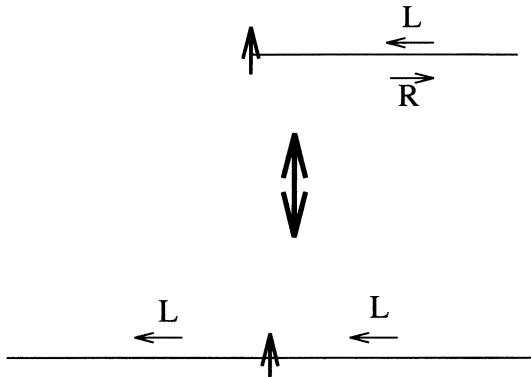


Fig. 4.1. Reflecting the left-movers to the negative axis

4.1.2 Perturbative RG

Since S doesn't appear in the unperturbed Hamiltonian, $S(\tau)$ is independent of τ , in the calculation of Dyson–Feynman graphs. The time-ordered product gives:

$$\begin{aligned} T[S^a(\tau)S^b(\tau')] &= \frac{1}{2}\{S^a, S^b\} + \text{sn}(\tau - \tau')\frac{1}{2}[S^a(\tau), S^b(\tau')] \\ &= \frac{1}{4}\delta^{ab} + \text{sn}(\tau - \tau')\frac{i}{2}\epsilon^{abc}S^c, \end{aligned} \quad (4.10)$$

where sn denotes the sign function. The δ -function term doesn't contribute to the lowest order coupling constant renormalization, the second diagram in Fig. 4.2. This diagram gives the integral:

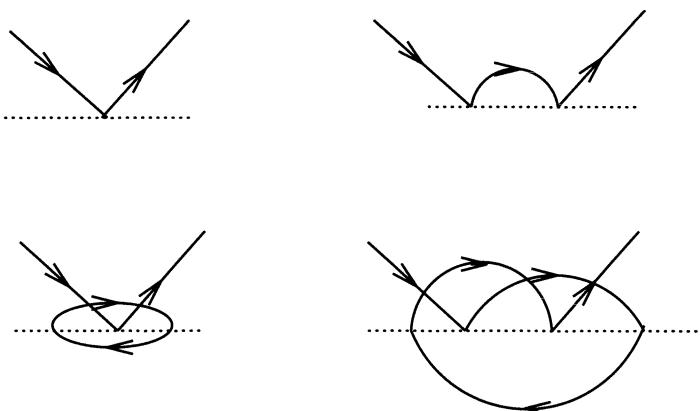


Fig. 4.2. Feynman diagrams contributing to renormalization of the Kondo coupling constant to third order

$$\int d\tau_1 d\tau_2 \frac{\text{sn}(\tau_1 - \tau_2)}{\tau_1 - \tau_2} \propto \ln D , \quad (4.11)$$

where D is an energy cut-off. As we reduce the bandwidth, D , the change in the effective coupling constant is given by:

$$\frac{d\lambda}{d \ln D} = -\lambda^2 + \dots , \quad (4.12)$$

corresponding to the RG flow of Fig. 4.3. Integrating the equation gives:

$$\lambda_{\text{eff}}(D) = \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{D}} . \quad (4.13)$$

This growth of the effective coupling at low energies, in the antiferromagnetic case, explains the growth of the resistivity at low temperatures. Essentially this explanation of the unusual growth of the resistivity with reducing temperature was given by Kondo [8] in 1964; it became known as the Kondo effect. Note that $\lambda_{\text{eff}}(T) \approx 1$ at

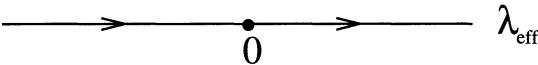


Fig. 4.3. RG flow of the Kondo coupling

$$T = T_K \equiv D_0 e^{-1/\lambda_0} \ll D_0 . \quad (4.14)$$

Here we are using the RG from the condensed matter viewpoint (rather than the slightly different quantum field theory viewpoint). The ultraviolet cut-off is held fixed at the physical bandwidth (essentially an inverse lattice spacing). To study physical properties at a lower energy scale, T , we integrate out the high energy degrees of freedom to derive an effective theory at energy scale T . Formally, this can be regarded as the lowering the effective cut-off down to this scale.

4.1.3 Local Fermi Liquid Theory

A picture of the asymptotic low T behaviour of the Kondo model emerged from work by Anderson, Wilson, Nozières and others. Here I follow closely the discussion of Nozières [9]. It is believed that $\lambda_{\text{eff}} \rightarrow \infty$ at $T \rightarrow 0$. But what does such a statement really mean? To make this more precise it is very convenient to consider a lattice model. The motivation for this very closely parallels the motivation for considering gauge theories on a lattice to understand confinement. (This is presumably no coincidence since Wilson

pioneered this approach to both problems.) For convenience we consider the 1D case although this is unimportant. Thus we take the Hamiltonian:

$$H = -t \sum_i (\psi_i^\dagger \psi_{i+1} + \psi_{i+1}^\dagger \psi_i) + \lambda \mathbf{S} \cdot \psi_0^\dagger \frac{\boldsymbol{\sigma}}{2} \psi_0 . \quad (4.15)$$

Consider the limit $\lambda >> |t|$. The ground state of the interaction term will be the following configuration: one electron at the site 0 forms a singlet with the impurity: $| \uparrow\downarrow \rangle - | \downarrow\uparrow \rangle$. The other electrons can do anything (except enter site 0). Now include a small finite hopping term, t . The other electrons (apart from the one making the singlet) go into free electron (Bloch) wave-functions that vanish at the origin. That is, the parity even wave-functions change from $\cos ki$ (at $\lambda = 0$) to $|\sin ki|$ (at $\lambda = \infty$). This corresponds to a $\pi/2$ phase shift or a change in *boundary conditions* (b.c.):

$$\begin{aligned} \psi_L(0) &= \psi_R(0), & \lambda = 0 \\ \psi_L(0) &= -\psi_R(0), & \lambda = \infty . \end{aligned} \quad (4.16)$$

The strong coupling fixed point is the same as the weak coupling fixed point except for a change in boundary conditions (and the removal of the impurity).

This change in b.c.'s corresponds to a change in the finite-size spectrum (FSS). This observation was used extensively by Wilson and Nozières in the mid 70's. By studying the finite-size spectrum numerically we can test hypotheses regarding the RG flow. This technique has since become standard in the CFT study of two-dimensional critical phenomena. To study the FSS we impose another convenient, fixed, b.c. at the distance l away from the impurity,

$$\psi_L(l) = -\psi_R(l) . \quad (4.17)$$

It is convenient to use the equivalent description in terms of left-movers only:

$$\begin{aligned} \psi_L(l) &= -\psi_L(-l) \\ \psi_L(0^+) &= \psi_L(0^-), & \lambda = 0 \\ \psi_L(0^+) &= -\psi_L(0^-), & \lambda = \infty . \end{aligned} \quad (4.18)$$

For $\lambda = 0$ the boundary conditions are simply anti-periodic. For $\lambda = \infty$, taking into account the sign flip at the origin, the b.c.'s are effectively periodic. Thus the allowed wave-vectors are:

$$\begin{aligned} k &= \frac{\pi}{l}(n + 1/2) & \lambda = 0 \\ k &= \frac{\pi n}{l} & \lambda = \infty . \end{aligned} \quad (4.19)$$

N.B. k here is shifted by k_F . The two sets of finite-size energy levels are drawn in Figs. 4.4 and 4.5.

Wilson's numerical RG scheme [7] involves calculating the low-lying spectrum numerically and looking for this shift. This indicates that λ renormalizes

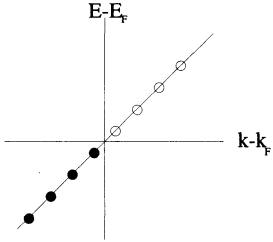


Fig. 4.4. Free fermion energy levels with antiperiodic boundary conditions

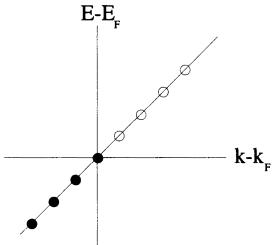


Fig. 4.5. Free fermion energy levels with periodic boundary conditions

to ∞ even if it is initially small. However, if the bare coupling is small but flows to large values for $T \leq T_K$, we expect this *effective b.c.* description to work only at low energies ($< T_K$) and long distances $> \xi_K \approx v_F/T_K >> 1/k_F$. We imagine a screening electron in a very spread out wave-function, as shown in Fig. 4.6. The effective b.c. description works for long wave-length probes. Only the very low energy excitations are given by this *local Fermi liquid* description.

This observation determines the $T = 0$ resistivity for an array of Kondo impurities at random locations of low density n_i . It is the same as for non-magnetic s-wave scatterers with a $\pi/2$ phase shift at the Fermi energy. $\delta = \pi/2$ gives the so-called unitary limit resistivity:

$$\rho_u = \frac{3n_i}{\pi\nu^2v_F^2e^2}, \quad (4.20)$$

where ν is the density of states at the Fermi surface.

The low- T behaviour, so far, seems trivial. Much of the interesting behaviour comes from the leading irrelevant operator. The impurity spin has disappeared (screened) from the description of the low- T physics. However certain interactions between electrons are generated (at the impurity site only) in the process of eliminating the impurity spin. We write the effective Hamiltonian as:

$$H_{\text{eff}} = H_{\text{FP}} + \sum \lambda_i \mathcal{O}_i(0), \quad (4.21)$$

where H_{FP} is the fixed point Hamiltonian including the boundary condition. In this case, it is simply the free Fermion Hamiltonian. The \mathcal{O}_i 's are various local operators, defined at the origin. At the screened Fermi liquid Kondo

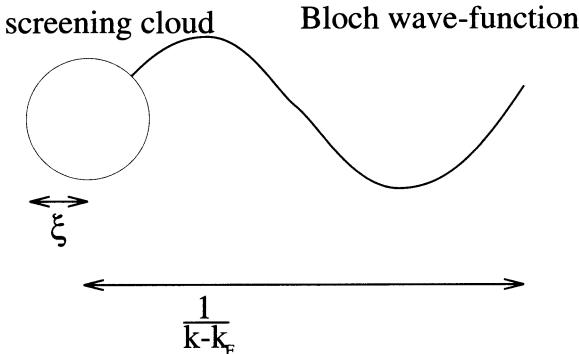


Fig. 4.6. Non-interacting Bloch states with a vanishing boundary condition occur for $|k - k_F| \ll v_F/T_K$

fixed point these operators must be constructed out of Fermion fields only, not S , since the impurity spin has been screened and hence eliminated from the low energy theory. The allowed operators must obey all the symmetries of the theory. As usual, we define the scaling dimensions, x_i by:

$$\langle \mathcal{O}_i(\tau) \mathcal{O}_i^\dagger(0) \rangle \propto \frac{1}{|\tau|^{2x_i}}. \quad (4.22)$$

The RG eigenvalue of λ_i is $1 - x_i$; the term 1 rather than the usual 2 in (1+1) dimensions occurs because the interaction is at the origin only, being accompanied by a Dirac delta-function. We now can easily enumerate all possible operators, using the left-moving formalism. The only operator of dimension $x = 1$ allowed by spin rotation symmetry is:

$$x = 1 : \psi^{\dagger\alpha}(0)\psi_\alpha(0). \quad (4.23)$$

This is actually forbidden by particle-hole symmetry. Even if particle-hole symmetry is broken, as it would be for a realistic model, it turns out that, in any event, this operator is strictly marginal (doesn't renormalize). This will be discussed in a later section, using bosonization. There are apparently two operators of next lowest dimension, $x = 2$:

$$i\psi^{\dagger\alpha} \frac{d}{dx} \psi_\alpha(0) - i \frac{d}{dx} \psi^{\dagger\alpha} \psi_\alpha(0), \quad (4.24)$$

corresponding to a k -dependent phase shift and

$$\psi^{\dagger\uparrow}\psi_\uparrow\psi^{\dagger\downarrow}\psi_\downarrow, \quad (4.25)$$

corresponding to a local electron-electron interaction induced by an impurity spin-flip. Since both these operators have $x_i = 2$, it follows that the Fermi liquid fixed point is stable (given our assumption about the $x = 1$ operator).

Note that, by contrast, the high energy, zero Kondo coupling, fixed point is unstable. A marginally relevant dimension 1 operator (the Kondo interaction) can occur there because of the presence of the impurity spin. The screening of the impurity spin eliminates this marginally relevant interaction. The coupling constants have dimensions of inverse energy. In the limit of small bare Kondo coupling, we expect that these infrared irrelevant coupling constants will be of order the largest inverse energy scale in the problem, $1/T_K$. In this limit, $1/T_K \gg 1/D$, the ratio of these two leading irrelevant operators can be fixed, for example using spin-charge separation, as we show later. Thus there is only one unknown dimensionless parameter of order 1:

$$\lambda_1 = n_W/T_K . \quad (4.26)$$

The dimensionless number, n_W is known as the Wilson number. Its actual value depends, of course, on the precise definition of T_K . If we define T_K by the high temperature logarithmic behaviour then n_W can be fixed from the low temperature behaviour. Calculation of the actual value of n_W requires more powerful methods than these general RG arguments since it is a function of the crossover between the two fixed points. It was first calculated numerically by Wilson and then exactly from the Bethe ansatz.

The basic strategy of the RG is to calculate various quantities in lowest order perturbation theory in λ_1 : specific heat, susceptibility, resistivity at $T > 0, \dots$. The low T behaviour of all quantities is determined in terms of one unknown parameter n_W . Various ratios of quantities, in which λ_1 drops out, are universal. Note that, since the operator is irrelevant, perturbation theory in λ_1 is infrared finite and valid at $T \rightarrow 0$, unlike perturbation theory in λ . Once the impurity is screened, only irrelevant interactions can occur.

Thus the high temperature behaviour is determined by ordinary (RG improved) perturbation theory in λ while the low temperature behaviour is determined by perturbation theory in λ_1 at the Fermi liquid fixed point. The crossover of various quantities at intermediate temperatures of $O(T_K)$ is given by universal scaling functions which require other methods for their computation: either numerical or using the exact integrability. In the following three figures, Figs. 4.7, 4.8 and 4.9, we sketch the resulting behaviour. The entropy (whose derivative gives the specific heat) and susceptibility have been calculated accurately from the Bethe ansatz and agree with the RG predictions. The resistivity is more difficult to calculate since it is not an equilibrium property.

4.2 Some Basic Conformal Field Theory/ Luttinger Liquid Techniques

It is very useful to bosonize free fermions to understand the Kondo effect. This allows separation of spin and charge degrees of freedom which greatly simplifies the problem.

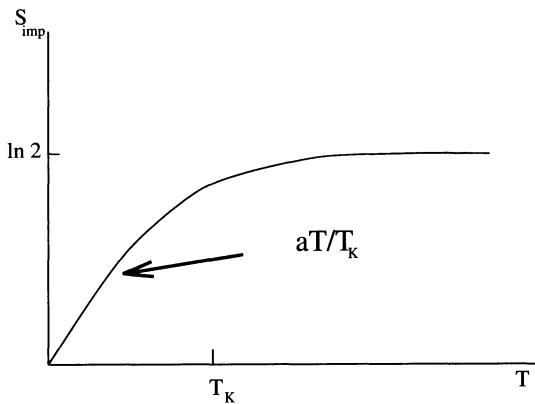


Fig. 4.7. Qualitative behaviour of the impurity entropy

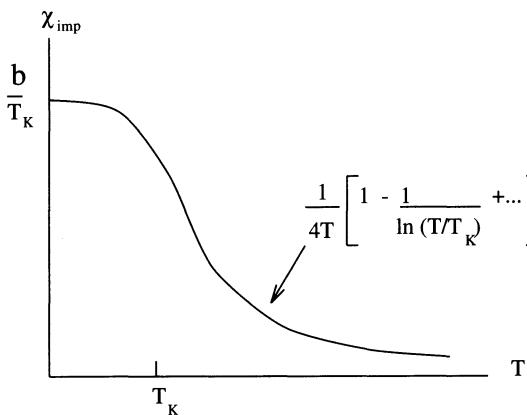


Fig. 4.8. Qualitative behaviour of the impurity susceptibility

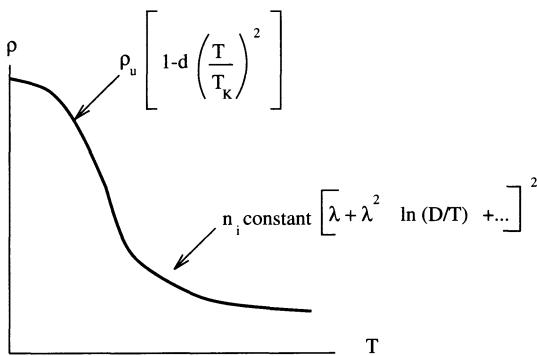


Fig. 4.9. Qualitative behaviour of the resistivity

We start by considering a left-moving spinless fermion field with Hamiltonian density:

$$\mathcal{H} = \frac{1}{2\pi} \psi_L^\dagger i \frac{d}{dx} \psi_L . \quad (4.27)$$

Define the current (=density) operator,

$$\begin{aligned} J_L(x-t) &= : \psi_L^\dagger \psi_L : (x, t) \\ &= \lim_{\epsilon \rightarrow 0} [\psi_L(x) \psi_L(x+\epsilon) - \langle 0 | \psi_L(x) \psi_L(x+\epsilon) | 0 \rangle] . \end{aligned} \quad (4.28)$$

(Henceforth we generally drop the subscripts “L”.) We will reformulate the theory in terms of currents (key to bosonization). Consider:

$$\begin{aligned} J(x) & J(x+\epsilon) \quad \text{as } \epsilon \rightarrow 0 \\ &=: \psi^\dagger(x) \psi(x) \psi^\dagger(x+\epsilon) \psi(x+\epsilon) : \\ &+ [: \psi^\dagger(x) \psi(x+\epsilon) : + : \psi(x) \psi^\dagger(x+\epsilon) :] G(\epsilon) + G(\epsilon)^2 \\ G(\epsilon) &= \langle 0 | \psi(x) \psi^\dagger(x+\epsilon) | 0 \rangle = \frac{1}{-i\epsilon} . \end{aligned} \quad (4.29)$$

By Fermi statistics the 4-Fermi term vanishes as $\epsilon \rightarrow 0$

$$: \psi^\dagger(x) \psi(x) \psi^\dagger(x) \psi(x) : = - : \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) : = 0 . \quad (4.30)$$

The second term becomes a derivative,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left[J(x) J(x+\epsilon) + \frac{1}{\epsilon^2} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{-i\epsilon} [: \psi^\dagger(x) \psi(x+\epsilon) : - : \psi^\dagger(x+\epsilon) \psi(x) :] \\ &= 2i : \psi^\dagger \frac{d}{dx} \psi : \\ \mathcal{H} &= \frac{1}{4\pi} J(x)^2 + \text{constant} . \end{aligned} \quad (4.31)$$

Now consider the commutator, $[J(x), J(y)]$. The quartic and quadratic terms cancel. We must be careful about the divergent c-number part,

$$\begin{aligned} [J(x), J(y)] &= -\frac{1}{(x-y-i\delta)^2} + \frac{1}{(x-y+i\delta)^2} \quad (\delta \rightarrow 0^+) \\ &= \frac{d}{dx} \left[\frac{1}{x-y-i\delta} - \frac{1}{x-y+i\delta} \right] \\ &= 2\pi i \frac{d}{dx} \delta(x-y) . \end{aligned} \quad (4.32)$$

Now consider the free massless boson theory with Hamiltonian density (setting $v_F = 1$):

$$\mathcal{H} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 , \quad \left[\phi(x), \frac{\partial}{\partial t} \phi(y) \right] = i\delta(x-y) . \quad (4.33)$$

We can again decompose it into the left and right-moving parts,

$$\begin{aligned} (\partial_t^2 - \partial_x^2)\phi &= (\partial_t + \partial_x)(\partial_t - \partial_x)\phi \\ \phi(x, t) &= \phi_L(x + t) + \phi_R(x - t) \\ (\partial_t - \partial_x)\phi_L &\equiv \partial_- \phi_L = 0, \quad \partial_+ \phi_R = 0 \\ \mathcal{H} &= \frac{1}{4}(\partial_- \phi)^2 + \frac{1}{4}(\partial_+ \phi)^2 = \frac{1}{4}(\partial_- \phi_R)^2 + \frac{1}{4}(\partial_+ \phi_L)^2. \end{aligned} \quad (4.34)$$

Consider the Hamiltonian density for a left-moving boson field:

$$\begin{aligned} \mathcal{H} &= \frac{1}{4}(\partial_+ \phi_L)^2 \\ [\partial_+ \phi_L(x), \partial_+ \phi_L(y)] &= [\dot{\phi} + \phi', \dot{\phi} + \phi'] = 2i \frac{d}{dx} \delta(x - y). \end{aligned} \quad (4.35)$$

Comparing to the Fermionic case, we see that:

$$J_L = \sqrt{\pi} \partial_+ \phi_L = \sqrt{\pi} \partial_+ \phi, \quad (4.36)$$

since the commutation relations and Hamiltonian are the same. That means the operators are the same with appropriate boundary conditions. It is also possible to represent fermion operators in terms of the boson,

$$\psi_L \sim e^{i\sqrt{4\pi}\phi_L}, \quad (4.37)$$

which gives the correct Green's function.

Let's compare the spectra. We begin with anti-periodic b.c.'s on the fermions:

$$\begin{aligned} \psi(l) &= -\psi(-l) \quad (\text{i.e. } \psi_L(l) + \psi_R(l) = 0), \\ k &= \frac{\pi}{l} \left(n + \frac{1}{2} \right), \quad n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (4.38)$$

[(See Fig. 4.4). Note that we have shifted k by k_F .] Consider the minimum energy state of charge Q (relative to the ground state). See Fig. 4.10. We have the single Fermion energy:

$$E = v_F k, \quad (4.39)$$

so:

$$E(Q) = v_F \frac{\pi}{l} \sum_{n=0}^{Q-1} \left(n + \frac{1}{2} \right) = \frac{v_F \pi}{2l} Q^2. \quad (4.40)$$

Now consider particle-hole excitations relative to the Q -ground state: the most general particle-hole excitation is obtained by raising n_m electrons by m levels, then n_{m-1} electrons by $m-1$ levels, etc. (See Fig. 4.11.)

$$E = \frac{\pi v_F}{l} \left(\frac{1}{2} Q^2 + \sum_{m=1}^{\infty} n_m \cdot m \right). \quad (4.41)$$

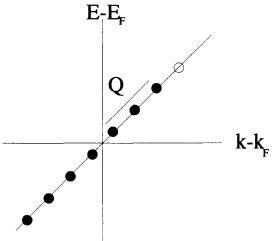


Fig. 4.10. A state with Q extra electrons added to the ground state

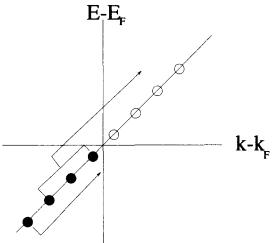


Fig. 4.11. A particle-hole excitation in which three electrons are raised four levels and then one electron is raised three levels

Now consider the bosonic spectrum. What are the boundary conditions? Try the periodic one,

$$\phi(l) = \phi(-l) \Rightarrow k = \frac{\pi m}{l} . \quad (4.42)$$

The m^{th} single particle level has $E_m = v_F k_m$. The total energy is

$$E = \frac{\pi v_F}{l} \left(\sum_1^{\infty} n_m \cdot m \right), \quad n_m = \text{occupation number : } 0, 1, 2, \dots \quad (4.43)$$

Where does the Q^2 term in (4.41) come from? We need more general boundary condition on the boson field. Let ϕ be an angular variable:

$$\begin{aligned} \phi_L(-l) &= \phi_L(l) + \sqrt{\pi} Q, \quad Q = 0, \pm 1, \pm 2, \dots \\ \Rightarrow \phi_L(x+t) &= \frac{\sqrt{\pi}}{2} \frac{Q}{l} \cdot (x+t) + \sum_{m=1}^{\infty} \frac{1}{\sqrt{4\pi m}} (e^{-i\frac{\pi m}{l}(x+t)} a_m + \text{h.c.}) , \end{aligned} \quad (4.44)$$

where a_n 's are the annihilation operators and Q is the winding number,

$$E = \int_{-l}^l dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right] = \frac{\pi}{l} \left[\frac{1}{2} Q^2 + \dots \right] . \quad (4.45)$$

Here we have set $v_F = 1$. We have the following correspondence:
soliton \leftrightarrow electron, oscillator \leftrightarrow particle-hole pair.

Note that the angular nature of ϕ , as defined in (4.44), is perfectly consistent with the bosonization formula of (4.37) and the single-valuedness of ψ .

For the Kondo effect we are also interested in periodic b.c.'s on the fermions: (see Fig. 4.5.)

$$\begin{aligned}\psi_L(l) &= +\psi_L(-l), \quad k = \frac{\pi}{l}n, \quad (\text{for fermions}) \\ E &= \frac{\pi v_F}{l} \left[\frac{Q(Q-1)}{2} + \sum_1^\infty n_m m \right].\end{aligned}\quad (4.46)$$

We have the degenerate ground state, $Q = 0$ or 1 , which corresponds to an anti-periodic boundary condition on ϕ ,

$$\begin{aligned}\phi(l) &= \phi(-l) + \sqrt{\pi} \left(Q - \frac{1}{2} \right) \\ E &= \frac{\pi}{l} \frac{1}{2} \left(Q - \frac{1}{2} \right)^2 + \dots = \frac{\pi}{l} \left(\frac{1}{2} Q(Q-1) + \text{const.} + \dots \right).\end{aligned}\quad (4.47)$$

Now we include spin, i.e. we have 2-component electrons,

$$\mathcal{H} = iv_F \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha, \quad (\alpha = 1, 2, \quad \text{summed}). \quad (4.48)$$

Now we have charge and spin currents (or densities). We can write \mathcal{H} in a manifestly $SU(2)$ invariant way, quadratic in charge and spin currents:

$$J := \psi^{\alpha\dagger} \psi_\alpha : , \quad \mathbf{J} = \psi^{\dagger\alpha} \frac{\boldsymbol{\sigma}_\alpha^\beta}{2} \psi_\beta . \quad (4.49)$$

Using:

$$\begin{aligned}\boldsymbol{\sigma}_\alpha^\beta \cdot \boldsymbol{\sigma}_\gamma^\delta &= 2\delta_\gamma^\beta \delta_\alpha^\delta - \delta_\beta^\alpha \delta_\gamma^\delta \\ \mathbf{J}^2 &= -\frac{3}{4} : \psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta : + \frac{3i}{2} \psi^{\alpha+} \frac{d}{dx} \psi_\alpha + c\text{-number}, \\ J^2 &= : \psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta : + 2i \psi^{\alpha+} \frac{d}{dx} \psi_\alpha + c\text{-number}, \\ \mathcal{H} &= \frac{1}{8\pi} J^2 + \frac{1}{6\pi} \mathbf{J}^2 ,\end{aligned}\quad (4.50)$$

we have the following commutation relations,

$$\begin{aligned}[J(x), J(y)] &= 4\pi i \delta'(x-y), \quad (\text{twice the result for the spinless case}) \\ [J(x), J^z(y)] &= \frac{1}{2} [J_\uparrow + J_\downarrow, J_\uparrow - J_\downarrow] = 0 .\end{aligned}\quad (4.51)$$

From $[J, \mathbf{J}] = 0$, we see that \mathcal{H} is sum of commuting charge and spin parts.

$$\begin{aligned}[J^a(x), J^b(y)] &= 2\pi \psi^\dagger \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] \psi \cdot \delta(x-y) + \text{tr} \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] 2\pi i \frac{d}{dx} \delta(x-y) \\ &= 2\pi i \epsilon^{abc} J^c(x) \cdot \delta(x-y) + \pi i \delta^{ab} \frac{d}{dx} \delta(x-y) .\end{aligned}\quad (4.52)$$

We obtain the Kac-Moody algebra of central charge $k = 1$. More generally the coefficient of the second term is multiplied by an integer k . Fourier transforming,

$$\mathbf{J}_n \equiv \frac{1}{2\pi} \int_{-l}^l dx e^{in\frac{\pi}{l}x} \mathbf{J}(x), \quad [J_n^a, J_m^b] = i\epsilon^{abc} J_{n+m}^c + \frac{1}{2} n \delta^{ab} \delta_{n,-m}. \quad (4.53)$$

we have an ∞ -dimensional generalization of the ordinary $SU(2)$ Lie algebra. The spin part of the Hamiltonian is

$$H_s = \frac{\pi}{l} \frac{1}{3} \sum_{n=-\infty}^{\infty} : \mathbf{J}_{-n} \cdot \mathbf{J}_n : . \quad (4.54)$$

The spectrum of H_s is again determined by the algebra obeyed by the \mathbf{J}_n 's together with boundary conditions. The construction is similar to building representations of $SU(2)$ from commutation relations, i.e. constructing raising operator, etc.

In the $k = 1$ case we are considering here it is simplest to use:

$$\begin{aligned} \mathbf{J}(x)^2 &= 3(J^z(x))^2 \\ \mathcal{H} &= \frac{1}{8\pi} J^2 + \frac{1}{2\pi} (J^z)^2 \\ &= \frac{1}{4\pi} (J_\uparrow^2 + J_\downarrow^2) \\ &= \frac{1}{4} ((\partial_+ \phi_\uparrow)^2 + (\partial_+ \phi_\downarrow)^2) \\ &= \frac{1}{4} \left[\left(\partial_+ \left(\frac{\phi_\uparrow + \phi_\downarrow}{\sqrt{2}} \right) \right)^2 + \left(\partial_+ \left(\frac{\phi_\uparrow - \phi_\downarrow}{\sqrt{2}} \right) \right)^2 \right] \\ &= \frac{1}{4} ((\partial_+ \phi_c)^2 + (\partial_+ \phi_s)^2). \end{aligned} \quad (4.55)$$

Now we have introduced two commuting charge and spin free massless bosons. $SU(2)$ symmetry is now concealed but boundary condition on ϕ_s must respect it. Consider the spectrum of fermion theory with boundary condition $\psi(l) = -\psi(-l)$,

$$E = \frac{\pi V}{l} \left[\frac{Q_\uparrow^2}{2} + \frac{Q_\downarrow^2}{2} + \sum_{m=-\infty}^{\infty} m(n_m^\uparrow + n_m^\downarrow) \right]. \quad (4.56)$$

Change over to ϕ_c and ϕ_s . We can relabel occupation numbers,

$$\begin{aligned} n_m^\uparrow, n_m^\downarrow &\longrightarrow n_m^c, n_m^s \\ Q &= Q_\uparrow + Q_\downarrow \\ S^z &= \frac{1}{2}(Q_\uparrow - Q_\downarrow) \\ E &= \frac{\pi v_F}{l} \left[\frac{1}{4}Q^2 + (S^z)^2 + \sum_1^\infty mn_m^c + \sum_1^\infty mn_m^s \right] \end{aligned} \quad (4.57)$$

$$\begin{aligned} E &= E_c + E_s \\ \phi_c &= \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{Q}{l} (x + t) + \dots \\ \phi_s &= \frac{\pi}{\sqrt{2}} \frac{S^z}{l} (x + t) + \dots \end{aligned} \quad (4.58)$$

Actually charge and spin bosons are not completely decoupled; we must require $Q = 2S^z \pmod{2}$, to correctly reproduce the free fermion spectrum. We see that the boundary conditions on ϕ_c and ϕ_s are coupled. Now consider the phase-shifted case,

$$E = \frac{\pi v_F}{l} \left[\frac{1}{4}(Q - 1)^2 + (S^z)^2 + \dots \right]. \quad (4.59)$$

Redefine $Q - 1 \rightarrow Q$ so

$$E = \frac{\pi v_F}{l} \left[\frac{1}{4}Q^2 + (S^z)^2 + \dots \right], \quad (4.60)$$

the same as before the phase shift, (4.57). One of the 0-energy single-particle states is filled, for $Q = 0$ and there are 4 ground states,

$$(Q, S^z) = \left(0, \pm \frac{1}{2} \right), \quad (\pm 1, 0). \quad (4.61)$$

Now $Q = 2S^z + 1 \pmod{2}$; i.e. we “glue” together charge and spin excitations in two different ways, either

$$\begin{aligned} &(\text{even, integer}) \oplus (\text{odd, half-integer}) \\ \text{or } &(\text{even, half-integer}) \oplus (\text{odd, integer}), \end{aligned} \quad (4.62)$$

depending on the boundary conditions. The $\pi/2$ phase shift simply reverses these “gluing conditions”.

The set of all integer spin states form a “conformal tower”. They can be constructed from the Kac-Moody algebra by applying the raising operators J_{-n} to the lowest (singlet) state, with all spacings $\frac{\pi v_F}{l} \cdot (\text{integer})$. Likewise for all half-integer spin states, $(S^z)^2 = 1/4 + \text{integer}$. Likewise for even and odd charge states. The K-M algebra determines uniquely conformal towers but boundary conditions determine which conformal towers occur in the spectrum and in which spin-charge combinations.

4.3 CFT Approach to the Kondo Problem

In this section, I sketch how some of the field theory techniques reviewed in the previous section can be applied to gain new insight into the Kondo problem.

4.3.1 Current Algebra Formulation and the Infrared Fixed Point

The chiral one-dimensional Hamiltonian density of (4.9) and (4.5) is:

$$\mathcal{H} = \psi_L^{\dagger\alpha} i \frac{d}{dx} \psi_{L\alpha} + \lambda \psi_L^{\dagger\alpha} \frac{\sigma_\alpha^\beta}{2} \psi_{L\beta} \cdot \mathbf{S} \delta(x) \quad (\text{left-movers only}) . \quad (4.63)$$

We rewrite it in terms of spin and charge currents only,

$$\mathcal{H} = \frac{1}{8\pi} J^2 + \frac{1}{6\pi} (\mathbf{J})^2 + \lambda \mathbf{J} \cdot \mathbf{S} \delta(x) . \quad (4.64)$$

The Kondo interaction involves spin fields only, not charge fields: $H = H_s + H_c$. Henceforth we only consider the spin part. In Fourier transformed form,

$$\begin{aligned} H_s &= \frac{\pi}{l} \left(\frac{1}{3} \sum_{n=-\infty}^{\infty} \mathbf{J}_{-n} \cdot \mathbf{J}_n + \lambda \sum_{n=-\infty}^{\infty} \mathbf{J}_n \cdot \mathbf{S} \right) \\ [J_n^a, J_m^b] &= i\epsilon^{abc} J_{n+m}^c + \frac{n}{2} \delta^{ab} \delta_{n,-m} \\ [S^a, S^b] &= i\epsilon^{abc} S^c \\ [S^a, J_n^b] &= 0 . \end{aligned} \quad (4.65)$$

From calculating Green's functions for $\mathbf{J}(x)$ we could again reproduce perturbation theory $d\lambda/d\ln D = -\lambda^2 + \dots$. That is a small $\lambda > 0$ grows. What is the infrared stable fixed point? Consider $\lambda = 2/3$, where we may “complete the square”.

$$\begin{aligned} H &= \frac{\pi V}{3l} \sum_{n=-\infty}^{\infty} \left[(\mathbf{J}_{-n} + \mathbf{S}) \cdot (\mathbf{J}_n + \mathbf{S}) - \frac{3}{4} \right] \\ [J_n^a + S^a, J_m^b + S^b] &= i\epsilon^{abc} (J_{n+m}^c + S^c) + \frac{n}{2} \delta^{ab} \delta_{n,-m} . \end{aligned} \quad (4.66)$$

H is quadratic in the new currents, $\mathcal{J}_n \equiv \mathbf{J}_n + \mathbf{S}$, which obey the same Kac-Moody algebra! What is the spectrum of $H(\lambda = 2/3)$? We must get back to Kac-Moody conformal towers for integer and half-integer spin. This follows from the KM algebra and the form of H (i.e. starting from the lowest state we produce the entire tower by applying the raising operators, \mathbf{J}_{-n}).

Thus we find that *the strong-coupling fixed point is the same as the weak-coupling fixed point*. However, the total spin operator is now $\mathcal{J}_0 = \mathbf{J}_0 + \mathbf{S}$. We

consider impurity spin magnitude, $s = 1/2$. Any integer-spin state becomes a 1/2-integer spin state and vice versa.

$$\text{Integer} \leftrightarrow 1/2\text{-Integer}. \quad (4.67)$$

Presumably $\lambda = 2/3$ is the strong coupling fixed point in this formulation of the problem. ∞ -coupling can become finite coupling under a redefinition, e.g.

$$\lambda_{\text{Lattice}} = \frac{\lambda_{KM}}{1 - 3/2\lambda_{KM}}. \quad (4.68)$$

We expect the low-energy, large l spectrum to be KM conformal towers for any λ . The effect of the Kondo interaction is to interchange the two conformal towers, Integer \leftrightarrow 1/2-Integer. See Fig. 4.12. This is equivalent to a $\pi/2$ phase-shift,

$$\begin{aligned} &(\text{even, integer}) \oplus (\text{odd, } 1/2\text{-integer}) \\ &\qquad\Downarrow \\ &(\text{even, } 1/2\text{-integer}) \oplus (\text{odd, integer}). \end{aligned} \quad (4.69)$$

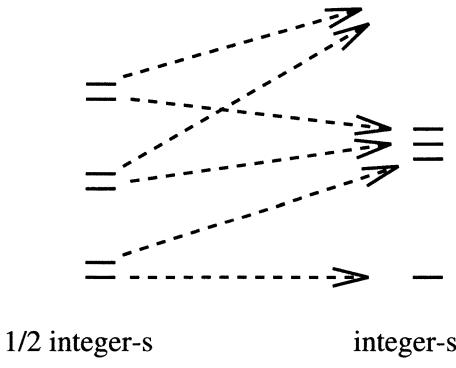


Fig. 4.12. At $\lambda = 2/3$ the 1/2-integer-spin conformal tower is mapped into the integer-spin conformal tower

An immediate consequence of this approach is a simple understanding of why there is only one leading irrelevant operator with a coefficient of $O(1/T_K)$.

First consider the dimension one operator discussed in Sect. I(C), which can be written $J(0)$, the charge current, and corresponds to potential scattering, in addition to Kondo scattering. Because this is purely a charge operator

it doesn't couple to the renormalization of the Kondo interaction which takes place entirely in the spin sector. Furthermore, it can be bosonized as $\partial_- \phi_c$. Such a term leaves the charge sector as a free theory and so doesn't lead to any renormalization. It does produce an additional phase shift, but one that doesn't renormalize.

The two independent dimension two operators, allowed by symmetry at the screened fixed point are quadratic in spin and charge operators: \mathbf{J}^2 and J^2 . Since our bosonization/current algebra approach makes it clear that the Kondo interaction only involves the spin sector, we would expect, at least naively, that the charge operator could not be generated, so that the unique leading irrelevant operator is \mathbf{J}^2 .

Actually, these statements are slightly too naive since exact spin-charge separation only occurs for specially chosen ultraviolet cut-offs and ignoring irrelevant operators in perturbation theory. However, the leading infrared divergences of perturbation theory don't depend on the details of the ultraviolet cut-off, and therefore occur purely in the spin sector. The generation of the charge operator, J^2 , should be suppressed by a power of the ultraviolet cut-off, so we expect its coefficient to be $\propto 1/D \ll 1/T_K$. Similarly, we expect only a finite (non infrared singular) renormalization of the potential scattering part of the phase shift, due to the Kondo interaction.

Now we calculate the specific heat and susceptibility to 1st order in λ_1 , the coupling constant multiplying \mathbf{J}^2 in the effective Hamiltonian. Consider the susceptibility of left-moving free fermions:

$$\begin{aligned} \text{0-th order} \quad M &= \frac{1}{2}(n_\uparrow - n_\downarrow) = l \int d\epsilon \nu(\epsilon) \left[n\left(\epsilon + \frac{\hbar}{2}\right) - n\left(\epsilon - \frac{\hbar}{2}\right) \right] \\ &\quad \chi = \frac{l}{2\pi} \quad (\text{for } T \ll D) \\ \text{1st order} \quad \chi &= \frac{1}{3T} \langle \left[\int dx \mathcal{J}(x) \right]^2 \rangle_{\lambda_1} \\ &= \chi_0 - \frac{\lambda_1}{3T^2} \langle \left[\int dx \mathcal{J}(x) \right]^2 \mathcal{J}(0)^2 \rangle + \dots \end{aligned} \quad (4.70)$$

A simplifying trick is to replace:

$$\delta\mathcal{H} = \lambda_1 \mathcal{J}^2(0) \delta(x) \longrightarrow \frac{\lambda_1}{2l} \mathcal{J}^2(x), \quad (4.71)$$

which gives the same result to first order in λ_1 (only) by translational invariance of H at $\lambda_1 = 0$. Now the Hamiltonian density changes into

$$\mathcal{H} \rightarrow \left(\frac{1}{6\pi} + \frac{\lambda_1}{2l} \right) \mathcal{J}^2(x). \quad (4.72)$$

We simply rescale H by a factor

$$H \rightarrow \left(1 + \frac{3\pi\lambda_1}{l} \right) H. \quad (4.73)$$

Equivalently in a thermal average,

$$T \rightarrow \frac{T}{1 + 3\pi\lambda_1/l} \equiv T(\lambda_1) \quad (4.74)$$

$$\begin{aligned} \chi(\lambda_1, T) &= \frac{1}{3T} \left\langle \left(\int \mathcal{J} \right)^2 \right\rangle_{T(\lambda_1)} \\ &= \frac{1}{1 + 3\pi\lambda_1/l} \chi(0, T(\lambda_1)) \\ &\approx \left[1 - \frac{3\pi\lambda_1}{l} \right] \chi_0 \\ &= \frac{l}{2\pi} - \frac{3\lambda_1}{2}, \end{aligned} \quad (4.75)$$

where in the last equality the first term represents the bulk part and the second one, of order $\sim 1/T_K$, comes from the impurity part.

Now consider the specific heat. The zeroth order, free fermion result can be written as a sum of equal contributions from the charge and spin bosons:

$$0\text{-th order} \quad C = C_c + C_s, \quad C_c = C_s = \frac{\pi l T}{3}. \quad (4.76)$$

Each free left-moving boson makes an identical contribution.

$$\begin{aligned} \text{1st order in } \lambda_1 \quad C_s(\lambda_1, T) &= \frac{\partial}{\partial T} \left\langle H(\lambda_1) \right\rangle_{\lambda_1} \\ &= C_s(0, T(\lambda_1)) \\ &= \frac{\pi l}{3} \frac{T}{1 + 3\pi\lambda_1/l} \\ &\approx \frac{\pi l T}{3} - \pi^2 \lambda_1 T \end{aligned} \quad (4.77)$$

$$\frac{\delta C_s}{C_s} = -\frac{3\pi\lambda_1}{l} = 2 \frac{\delta C_s}{C}. \quad (4.78)$$

While both the impurity susceptibility and specific heat contain λ_1 , this drops out in the “Wilson Ratio”:

$$R_W \equiv \frac{\delta\chi/\chi}{\delta C/C} = 2 = \frac{C}{C_s}. \quad (4.79)$$

Note that R_W effectively measures the fraction of C coming from the spin degrees of freedom. A similar, but more complicated calculation also gives the T^2 term in the resistivity, which turns out to be proportional to λ_1^2 .

4.3.2 BCFT and the Multi-Channel Kondo Problem

This simple interchange of half-integer spin and integer spin conformal towers takes us from the zero coupling to strong coupling fixed point in the Kondo problem. This is a simple example of what is known in conformal field theory as fusion. Fusion is based on the operator product expansion (OPE). The free spin boson is equivalent to the $k=1$ Wess–Zumino–Witten model. This contains only one primary operator, with $SU(2)$ quantum number $j = 1/2$. The operator product expansion of this operator with itself gives only the identity operator (plus descendants with respect to the Kac-Moody algebra). The OPE of the identity operator with the primary field gives, as always, the primary field. Thus these fusion rules may be summarized as:

$$\begin{aligned} 0 \times 1/2 &\rightarrow 1/2 \\ 1/2 \times 1/2 &\rightarrow 0 , \end{aligned} \tag{4.80}$$

where 0 and $1/2$ represent the identity operator (spin 0) and the primary field (spin $1/2$). In conformal field theory we identify conformal towers with primary fields. The two conformal towers in the $k = 1$ WZW model, containing all states of integer and half-integer spin, are identified with the identity operator and the $j = 1/2$ primary operator respectively. Thus the fusion rules describe how the spectrum changes in passing from ultraviolet to infrared fixed points. This conformal field theory process of fusion corresponds to the physical process of screening of the impurity spin. Since this screening only occurs at low temperatures, we may say that the Kondo effect corresponds to “cold fusion”.

A considerably less trivial application of conformal field theory methods is to the multi-channel Kondo problem [12]. In real metals there are often several “channels” of electrons –e.g. different d-shell orbitals. A very simple and symmetric model is:

$$H = \sum_{\mathbf{k}, \alpha, i} \epsilon_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger \alpha i} \psi_{\mathbf{k} \alpha i} + \lambda \mathbf{S} \cdot \sum_{\mathbf{k}, \mathbf{k}', \alpha, \beta, i} \psi_{\mathbf{k}}^{\dagger \alpha i} \sigma_{\alpha}^{\beta} \psi_{\mathbf{k}' \beta i} , \tag{4.81}$$

where $i = 1, 2, 3, \dots, k$. This model has $SU(2) \times SU(k) \times U(1)$ symmetry. Realistic systems do not have this full symmetry. To understand the potential applicability of this model we need to analyse the relevance of various types of symmetry breaking [10]. In the $k = 2$ case, an interesting possible experimental application of the model was proposed by Ralph, Ludwig, von Delft and Buhrman [11]. In general, we let the impurity have an arbitrary spin, s , as well.

Perturbation theory in λ is similar to the result mentioned before:

$$\begin{aligned} \frac{d\lambda}{d \ln D} &= -\lambda^2 + \frac{k}{2} \lambda^3 + O[ks(s+1)\lambda^4] \\ S^2 &= s(s+1) . \end{aligned} \tag{4.82}$$

Does $\lambda \rightarrow \infty$ as $T \rightarrow 0$? Let's suppose it does and check consistency. What is the ground state for the lattice model of (4.15), generalized to arbitrary k and s , at $\lambda/t \rightarrow \infty$? In the limit we just consider the single-site model:

$$H = \lambda \mathbf{S} \cdot \psi_0^\dagger \frac{\boldsymbol{\sigma}}{2} \psi_0 , \quad (4.83)$$

for $\lambda > 0$ (antiferromagnetic case) the minimum energy state has maximum spin for electrons at $\mathbf{0}$, i.e. spin = $k/2$. Coupling this spin- $k/2$ to a spin- s , we don't get a singlet if $s \neq k/2$, but rather an effective spin of size $|s - k/2|$. (See Fig. 4.13.) The impurity is underscreened ($k/2 < s$) or overscreened ($k/2 > s$).

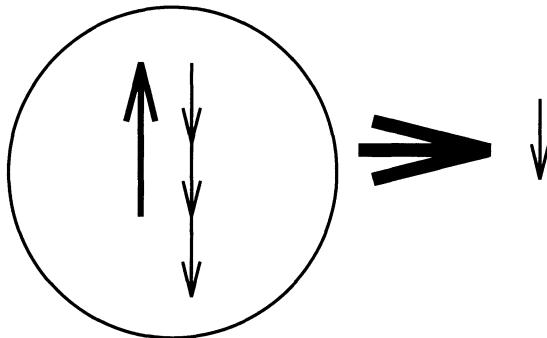


Fig. 4.13. Formation of an effective spin at strong Kondo coupling. $k = 3$, $s = 1$ and $s_{\text{eff}} = 1/2$

Now let $t/\lambda \ll 1$ be finite. Sites ± 1 can exchange an electron with 0. This gives an effective Kondo interaction:

$$\lambda_{\text{eff}} \sim \frac{t^2}{\lambda} \ll 1 .$$

See Fig. 4.14. What is the sign of λ_{eff} ? The coupling of the electron spins is antiferromagnetic: $\lambda_{\text{eff}} \mathbf{S}_{\text{el},0} \cdot \mathbf{S}_{\text{el},1}$, with $\lambda_{\text{eff}} > 0$ (as in the Hubbard model). But we must combine spins

$$\mathbf{S}_{\text{eff}} = \mathbf{S} + \mathbf{S}_{\text{el},0} . \quad (4.84)$$

For $k/2 < s$, $\mathbf{S}_{\text{eff}} \parallel -\mathbf{S}_{\text{el},0}$ but, for $k/2 > s$, $\mathbf{S}_{\text{eff}} \parallel +\mathbf{S}_{\text{el},0}$. So, ultimately, $\lambda_{\text{eff}} < 0$ in the underscreened case and $\lambda_{\text{eff}} > 0$ in the overscreened case. In the first (underscreened) case, the assumption $\lambda \rightarrow \infty$ was consistent since a ferromagnetic $\lambda_{\text{eff}} \rightarrow 0$ under renormalization and this implies $\lambda \rightarrow \infty$, since $\lambda_{\text{eff}} \sim -t/\lambda^2$. In this case we expect the infrared fixed point to correspond to a decoupled spin of size $s_{\text{eff}} = s - k/2$ and free electrons with a $\pi/2$ phase shift. In the second (overscreened) case the ∞ -coupling fixed point is

not consistent. Hence the fixed point occurs at intermediate coupling: this fixed point does not correspond to a simple boundary condition on electrons, instead it is a non-Fermi-liquid fixed point. See Fig. 4.15.

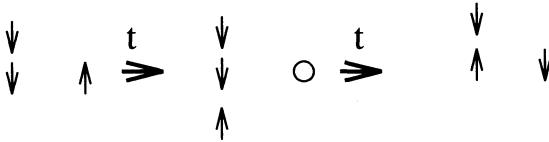


Fig. 4.14. Effective Kondo interaction of $O(t^2)$

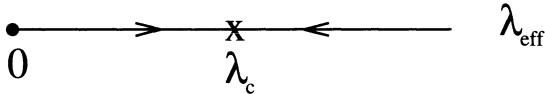


Fig. 4.15. RG flow of the Kondo coupling in the overscreened case

For the $k = 2$, $s = 1/2$ case we may think of the electrons (one from each channel) in the first layer around the impurity as aligning antiferromagnetically with the impurity. This overscreens it, leaving an effective $s = 1/2$ impurity. The electrons in the next layer then overscreen this effective impurity, etc. At each stage we have an effective $s = 1/2$ impurity. (See Fig. 4.16.) Note in this special case that there is a duality between the weak and strong coupling unstable fixed points: they both contain an $s = 1/2$ impurity.

The application of conformal field theory methods to the multi-channel Kondo problem begins by a bosonization technique which isolates the relevant (spin) degrees of freedom. It turns out that the system of 2 spins and k flavours of free electrons can be bosonized in terms of decoupled spin, charge and flavour degrees of freedom. Only the spin degrees of freedom couple to the impurity spin, as in the single channel case. For k channels, it turns out that the spin degrees of freedom correspond to a somewhat different conformal field theory, not equivalent to a free boson, called the $SU(2)_k$ WZW model. This basically follows from the fact that the spin currents:

$$\mathbf{J} \equiv \sum_{i=1}^k \mathbf{J}_i , \quad (4.85)$$

obey the Kac-Moody algebra of (4.52) with central charge k . This theory has $k + 1$ conformal towers labelled by the spin of the corresponding primary field (and of the lowest energy state in the tower), $j = 0, 1/2, 1, \dots, k/2$. The free field spectrum (zero Kondo coupling) can be written in terms of certain combinations of spin charge and flavour conformal towers, worked out by Altshuler, Bauer and Itzykson [13]. It can be shown that the underscreened

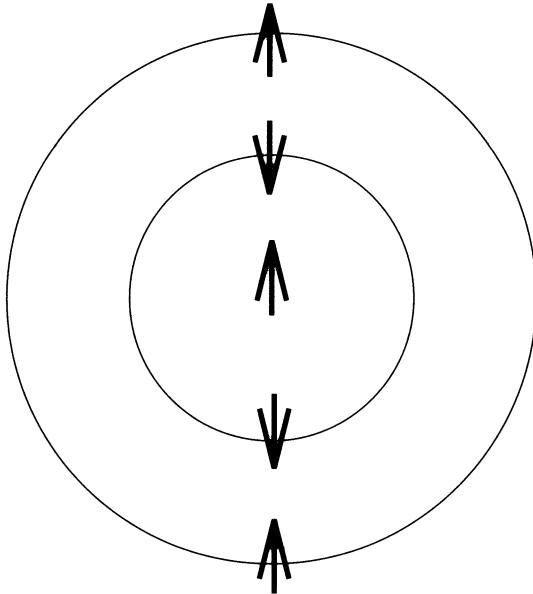


Fig. 4.16. The overscreened case with $s = 1/2$, $k = 2$

(or exactly screened) infrared fixed point, for an impurity of spin $s \leq k/2$, is obtained by fusion with the spin $j = s$ operator. The fusion rules in this case take the form:

$$s \times j \rightarrow |s - j|, |s - j| + 1, |s - j| + 2, \dots \min\{s + j, k - s - j\}. \quad (4.86)$$

This is a restriction of the ordinary spin addition rules which, in particular, is consistent with the fact that no conformal towers exist with spin $j > k/2$. The fact that the conformal towers have spin $j \leq k/2$ corresponds nicely to the underscreening/overscreening phenomena. In the underscreened case, $s > k/2$, fusion with spin $k/2$ occurs, with an effective impurity spin, of size $s - k/2$, left over. The overscreened fixed points are of “non-Fermi liquid” type. This means that there are operators at these fixed points which cannot be expressed as products of free fermion fields. This can be seen from the large- k limit of the β -function in (4.82). It can be shown that the terms of higher order in λ than cubic are small at the fixed point which occurs at $\lambda_c \approx 2/k$. The slope of the β -function at the fixed point $\approx 2/k$. This gives the dimension of the leading irrelevant operator. Since this is not an integer, or half-integer, this cannot correspond to any product of free fermion fields.

So far, we have only described how the finite size spectrum can be obtained from fusion but it turns out that this method gives a complete solution of all critical properties of the infrared fixed point, when combined with the full range of techniques of boundary conformal field theory. Here, we just mention one crucial construction: the boundary state, introduced by Cardy [14]. It

is very convenient to consider a conformally invariant system defined on a cylinder of circumference β in the τ -direction and length l in the x direction, with conformally invariant boundary conditions A and B at the two ends. (See Fig. 4.17.) From the quantum mechanical point of view, this corresponds to a finite temperature, $T = 1/\beta$. The partition function for this system is:

$$Z_{AB} = \text{tr } e^{-\beta H_{AB}^l}, \quad (4.87)$$

where we are careful to label the Hamiltonian by the boundary conditions as well as the length of the spatial interval, both of which help to determine the spectrum. Alternatively, we may make a modular transformation, $\tau \leftrightarrow x$. Now the spatial interval, of length β , is periodic. We write the corresponding Hamiltonian as H_P^β . The system propagates for a time interval l between initial and final states A and B . Thus we may equally well write:

$$Z_{AB} = \langle A | e^{-l H_P^\beta} | B \rangle. \quad (4.88)$$

Equating these two expressions, (4.87) and (4.88) gives powerful constraints which allow us to determine the conformally invariant spectrum and the boundary states, in many cases. Once the boundary state has been determined, all universal critical information about the boundary condition can be calculated. This essentially reduces to the calculation of various critical amplitudes which are expressed as matrix elements in the boundary state [14,15]. The interested reader is referred to a longer review article [16] for a more detailed exposition of the application of these methods to the multi-channel Kondo problem.

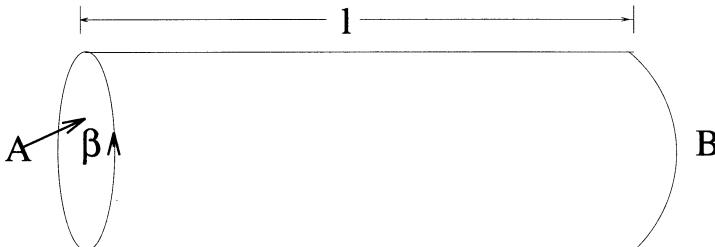


Fig. 4.17. Cylinder of length l , circumference β with boundary conditions A and B at the two ends

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5 Quantum Magnetism Approaches to Strongly Correlated Electrons

Assa Auerbach, Federico Berruto, and Luca Capriotti

Problems of strongly interacting electrons can be greatly simplified by reducing them to effective quantum spin models. The initial step is renormalization of the Hamiltonian into a lower energy subspace. The positive and negative U Hubbard models are explicitly transformed into the Heisenberg and $-x - xz$ models respectively. Basic tools of quantum magnetism are introduced and used: spin coherent states path integral, spin wave theory, and continuum theory of rotators. The last lecture concerns pseudospin approaches to superconductivity and superfluidity. The SO(3) rotator theory for the $-x - xz$ model describes the charge density wave to superconductor transition for e.g. doped bismuthates. Analogously, Zhang's theory for collective modes of high T_c cuprates describes the antiferromagnet to d -wave superconductor transition using SO(5) rotators. Finally, the Magnus force on two dimensional vortices and their momentum, are derived from the Berry phase of the spin path integral.

5.1 Deriving the Effective Hamiltonian

Let us consider the Hubbard model for conduction electrons hopping on a lattice with short range interactions

$$\begin{aligned}\mathcal{H} &= \mathcal{T} + \mathcal{U} \\ \mathcal{T} &= -t \sum_{\langle ij \rangle s=\uparrow,\downarrow} c_{i,s}^\dagger c_{j,s} \\ \mathcal{U} &= U \sum_i n_{i,\uparrow} n_{i,\downarrow}.\end{aligned}\tag{5.1}$$

It is always tempting to reduce the interaction term \mathcal{U} to fermion bilinears (single electron terms) using the Hartree Fock (HF) variational approximation. However this approach is known to be seriously flawed in several important cases. For example, while the HF spin density wave is energetically favourable for $U > 0$ at half filling, it breaks spin symmetry too readily in one and two dimensions, in violation of the Mermin Wagner theorem. This implies that Fock states might be too restrictive as a variational basis. We can illustrate this point using a simple toy model: the Hubbard model on

two sites. It will also teach us something about onsite interactions and their effect on spin correlations.

5.1.1 Two-Site Hubbard Model

The two-site Hubbard model with two electrons, is

$$H = -t \sum_{s=\uparrow,\downarrow} (c_{1,s}^\dagger c_{2,s} + c_{2,s}^\dagger c_{1,s}) + U \sum_{i=1,2} n_{i,\uparrow} n_{i,\downarrow}, \quad (5.2)$$

where we take $U > 0$. The total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, commutes with the Hamiltonian, we restrict ourselves to the $S = 0$ and $S = 1$ subspaces. The singlet subspace is spanned by three states

$$\frac{c_{1,\uparrow}^\dagger c_{2,\downarrow}^\dagger - c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger}{\sqrt{2}} |0\rangle, \quad c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger |0\rangle, \quad c_{2,\uparrow}^\dagger c_{2,\downarrow}^\dagger |0\rangle. \quad (5.3)$$

In this subspace, the Hamiltonian is

$$H_{S=0} = \begin{pmatrix} 0 & \sqrt{2}t & \sqrt{2}t \\ \sqrt{2}t & U & 0 \\ \sqrt{2}t & 0 & U \end{pmatrix}.$$

Diagonalizing the Hamiltonian in the singlet sector, one gets

$$H_{S=0} \longrightarrow \begin{pmatrix} (U - \sqrt{U^2 + 16t^2})/2 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & (U + \sqrt{U^2 + 16t^2})/2 \end{pmatrix},$$

which, in the *strong coupling limit* ($U \gg t$) is

$$H_{S=0} \longrightarrow [U \gg t] \begin{pmatrix} -4t^2/U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U + 4t^2/U \end{pmatrix}.$$

The triplet subspace is spanned by the states:

$$\frac{c_{1,\uparrow}^\dagger c_{2,\downarrow}^\dagger + c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger}{\sqrt{2}} |0\rangle, \quad c_{1,\uparrow}^\dagger c_{2,\uparrow}^\dagger |0\rangle, \quad c_{1,\downarrow}^\dagger c_{2,\downarrow}^\dagger |0\rangle, \quad (5.4)$$

which all have zero energy.

The comparison of the spectrum and wave functions of the non-interacting case and the strong coupling limit is displayed in Figs. 5.1 and 5.2.

The non-interacting groundstate is the two-site version of the Hartree–Fock two electron “Fermi sea”. This state, and the lowest excited multiplet, contain sizeable contributions from the doubly occupied singlets

$$\frac{1}{\sqrt{2}} (|\uparrow\downarrow_1, 0_2\rangle, \quad |0_1, \uparrow\downarrow_2\rangle). \quad (5.5)$$

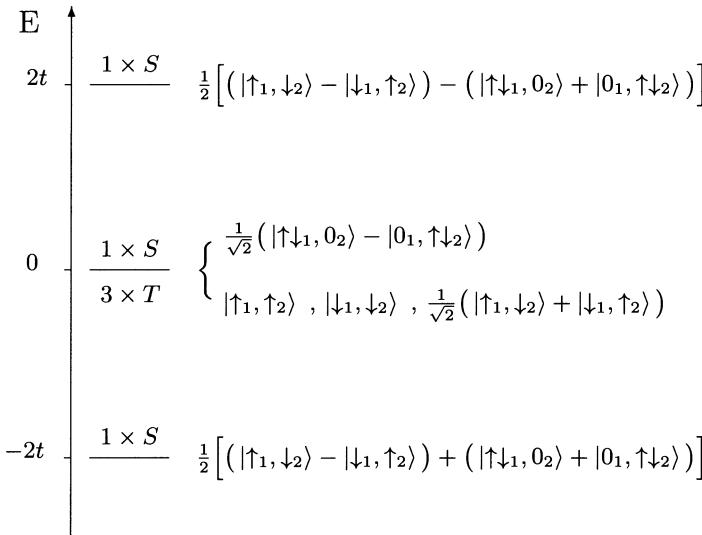


Fig. 5.1. Noninteracting ($U=0$) eigenstates of the two-site Hubbard model. The ground state is the “two electron Fermi sea”

The Hubbard interaction pushes these states to energies of order U . At strong coupling therefore, the ground state becomes a *valence bond singlet*, of singly occupied sites with no charge fluctuations. It also *cannot be expressed, even approximately, as a Fock state*.

The lessons to be learned from this toy model is that repulsive interactions can

- enhance magnetic correlations,
- reduce double occupancies in the ground state,
- separate spin and charge excitations.

5.1.2 Renormalization to Low Energy Subspace

Low temperature phases and their interesting DC transport properties, are determined by the ground state and low energy excitations. This chapter is slightly formal, as it shows what is precisely meant by the Renormalization Group (RG) transformation which replaces an original non diagonal Hamiltonian with an effective one for a lower energy subspace. Let us consider any Hamiltonian written as

$$H = H_0 + V, \quad (5.6)$$

where H_0 is diagonal and V is a non diagonal perturbation. We define the Hilbert space using the eigenstates of H_0 , and $P_0(\Lambda)$ is projector onto the subspace with energies less than Λ .

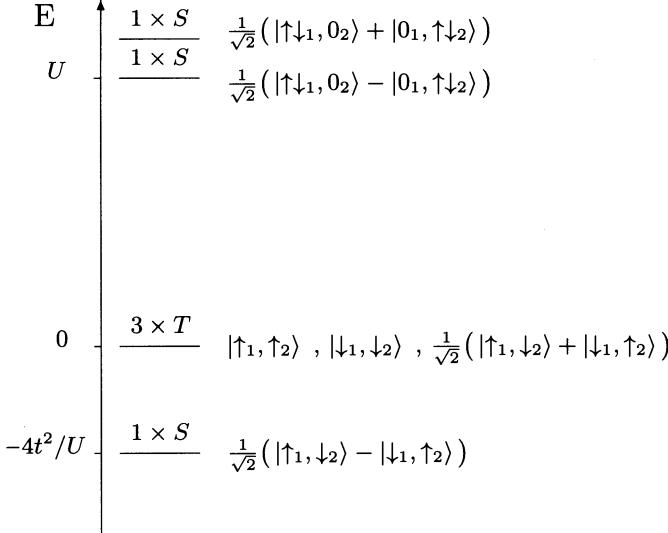


Fig. 5.2. Eigenstates of the two-sites Hubbard model in the strong coupling limit

The resolvent operator, $G = (E - H)^{-1}$, projected onto the latter subspace is given by a well known matrix inversion identity [1]

$$\begin{aligned}
 G_{00}(E) &= P_0 G(E) P_0 \\
 &= P_0 \begin{pmatrix} E - P_0(H_0 + V)P_0 & P_0 V(1 - P_0) \\ (1 - P_0)V P_0 & (1 - P_0)(E - (H_0 + V))(1 - P_0) \end{pmatrix}^{-1} P_0 \\
 &= \{E - P_0(H_0 + V)P_0 \\
 &\quad - P_0 V(1 - P_0)[(1 - P_0)(E - (H_0 + V))(1 - P_0)]^{-1}(1 - P_0)V P_0\}^{-1} \\
 &\equiv (E - \mathcal{H}_{\text{eff}}(E))^{-1}, \tag{5.7}
 \end{aligned}$$

where the last equality defines the *effective Hamiltonian* $\mathcal{H}_{\text{eff}}(E)$. Then the spectrum of H , which corresponds to states with non-zero weights in the subspace considered, is given by the zeros of the characteristic polynomial of $\mathcal{H}_{\text{eff}}(E)$,

$$\det(E_n - \mathcal{H}_{\text{eff}}(E_n)) \tag{5.8}$$

that is by the poles of the function $\text{Tr } G(E)_{00}$. The effective Hamiltonian can be also written as

$$\begin{aligned}\mathcal{H}_{\text{eff}}(E) &= P_0 \left\{ (H_0 + V) + V(1 - P_0) \right. \\ &\quad \times \left[(1 - P_0) \left(1 - (E - H_0)^{-1} V(1 - P_0) \right) \right]^{-1} (1 - P_0)(E - H_0)^{-1} V \left. \right\} P_0 \\ &= P_0 \left\{ (H_0 + V) + V \sum_{n=1}^{\infty} \left[\frac{1 - P_0}{E - H_0} V \right]^n \right\} P_0.\end{aligned}\quad (5.9)$$

If P_0 projects onto the ground state manifold of H_0 , (5.9) defines the *Brillouin-Wigner* perturbation theory¹. For two cut-off energies $\Lambda' < \Lambda$, (5.9) is a Renormalization Group transformation

$$\mathcal{H}(\Lambda) \rightarrow \mathcal{H}(\Lambda') = H_0(\Lambda') + V(\Lambda') \quad (5.10)$$

where

$$\mathcal{H}(\Lambda') = P(\Lambda') \left\{ \mathcal{H}(\Lambda) - V(\Lambda) \sum_{n=1}^{\infty} \left[\frac{1 - P(\Lambda')}{E - H_0(\Lambda)} V(\Lambda) \right]^n \right\} P(\Lambda'). \quad (5.11)$$

After doing the best job we can to evaluate $\mathcal{H}(\Lambda')$ (it is clear one needs to truncate the infinite sum and do something about the energy dependence of the denominators), the terms separate naturally into a diagonal operator H_0 and residual interactions $V(\Lambda')$. Sometimes we are lucky, and complicated terms of $V(\Lambda')$ become relatively smaller as Λ' is reduced. These are called *irrelevant interactions* which scale to zero. We end this Section by remarking that the RG transformation should preserve all the symmetries of the Hamiltonian. If \mathcal{H} has explicit symmetry-breaking terms, those terms may grow or shrink under the RG transformation rendering the low energy correlations less or more symmetrical, as the case may be.

5.1.3 From Hubbard to $t - J$ and Heisenberg Models

As an explicit derivation of an effective Hamiltonian outlined in Sect. 5.1.2, we consider the Hubbard model $\mathcal{H} = \mathcal{T} + \mathcal{U}$ of (5.1) in the strong coupling regime ($U/t \gg 1$). The diagonal part H_0 we choose as \mathcal{U} , the onsite interactions. This term divides the Fock space into two subspaces, the singly occupied and empty sites configurations

$$S = \left\{ |n_{1,\uparrow}, n_{1,\downarrow}, n_{2,\uparrow}, n_{2,\downarrow}, \dots\rangle : \forall i, n_{i,\uparrow} + n_{i,\downarrow} \leq 1 \right\}, \quad (5.12)$$

and configurations with one or more doubly occupied sites

$$D = \left\{ |n_{1,\uparrow}, n_{1,\downarrow}, n_{2,\uparrow}, n_{2,\downarrow}, \dots\rangle : \exists i, n_{i,\uparrow} + n_{i,\downarrow} = 2 \right\}. \quad (5.13)$$

¹ Note that the sum $\sum_{n=1}^{\infty}$ in (5.9) does not correspond to a perturbation series for the ground state energy, since the terms depend on E .

The hopping term \mathcal{T} couples the S and D subspaces by moving an electron into, or out of, a doubly occupied state. We define P_0 to project onto the ground state manifold of subspace S , and thus

$$G_{00}(E) = P_0 G(E) P_0 = (E - \mathcal{H}_{\text{eff}}(E))^{-1}, \quad (5.14)$$

where the effective Hamiltonian \mathcal{H}_{eff} , is given by (5.7)

$$\mathcal{H}_{\text{eff}}(E) = P_0 \mathcal{T} P_0 + P_0 \mathcal{T} \left[(1 - P_0)(E - (\mathcal{U} + T))(1 - P_0) \right]^{-1} \mathcal{T} P_0. \quad (5.15)$$

In the strong coupling limit, expanding the effective Hamiltonian to zeroth order in E/U and to second order in t/U one gets

$$\mathcal{H}_{\text{eff}}(E) \xrightarrow{t/U \ll 1} \mathcal{H}^{t-J}, \quad (5.16)$$

$$\mathcal{H}^{t-J} = P_0 \left(\mathcal{T} - \frac{t^2}{U} \sum_{i,j,k,s,s'} c_{i,s}^\dagger c_{j,s} n_{j,\uparrow} n_{j,\downarrow} c_{j,s'}^\dagger c_{k,s'} \right) P_0, \quad (5.17)$$

i.e., the low energy excitations of the Hubbard model are described by the Hamiltonian of the so called $t - J$ model.

The fermion operators appearing in (5.17), can be rearranged in the following way:

$$\mathcal{H}^{t-J} = P_0 (\mathcal{T} + \mathcal{T}' + \mathcal{H}^H) P_0, \quad (5.18)$$

$$\mathcal{T}' = -\frac{t^2}{2U} \sum_{i,j,k}^{\neq k} \left[\sum_s c_{i,s}^\dagger c_{k,s} n_j - c_i^\dagger \boldsymbol{\sigma} c_k \cdot c_j^\dagger \boldsymbol{\sigma} c_j \right], \quad (5.19)$$

$$\mathcal{H}^H = \frac{J}{2} \sum_{\langle i,j \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{n_i n_j}{4} \right), \quad (5.20)$$

where $J = 4t^2/U$ and the $S = 1/2$ spin operators \mathbf{S}_i are

$$S_i^\alpha = \frac{1}{2} \sum_{s,s'} c_{i,s}^\dagger \sigma_{s,s'}^\alpha c_{i,s'}, \quad (5.21)$$

σ^α being the Pauli matrices.

At half filling, i.e., when $n_i = 1$, P_0 annihilates \mathcal{T} and \mathcal{T}' since there can be no hopping processes within subspace S when there are no empty sites. The transport of charge is prevented by an energy gap of order U . This is the *Mott insulator*, which describes the undoped (parent compounds) of the high T_c superconductors of the cuprate family.

In this limit, the t-J model simply reduces to the spin $S = 1/2$ Heisenberg model

$$\mathcal{H}^{t-J} \rightarrow \frac{J}{2} \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \text{const.} \quad (5.22)$$

As in the two site Hubbard model of two electrons (see Sect. 5.1.1), the low energy excitations are purely magnetic.

5.1.4 The Negative- U Hubbard Model

The negative- U Hubbard model describes local attractive interactions between electrons which could be produced by several microscopic mechanisms e.g., phonons, plasmons or spin fluctuations. We choose, for convenience to write the model as follows

$$\begin{aligned} \mathcal{H}^{-U} = & -t \sum_{\langle i,j \rangle, s} c_{i,s}^\dagger c_{j,s} - \frac{U}{2} \sum_i (n_i - 1)^2 \\ & + \frac{1}{2} \sum_{\langle i,j \rangle} V_{ij} (n_i - 1)(n_j - 1) - \mu \sum_i n_i , \end{aligned} \quad (5.23)$$

where the negative- U term favours pairs of electrons on the same site in competition with the hopping term which delocalizes the electrons; V_{ij} is intersite Coulomb interactions and μ the chemical potential. The following canonical transformation

$$c_{i,\uparrow} \longrightarrow \tilde{c}_{i,\uparrow} , \quad (5.24)$$

$$c_{i,\downarrow} \longrightarrow \tilde{c}_{i,\downarrow}^\dagger , \quad (5.25)$$

maps the negative- U to a positive- U Hamiltonian:

$$\begin{aligned} \mathcal{H}^{-U} \rightarrow \mathcal{H}^{+U} = & -t \sum_{\langle i,j \rangle} (\tilde{c}_{i,\uparrow}^\dagger \tilde{c}_{j,\uparrow} - \tilde{c}_{i,\downarrow}^\dagger \tilde{c}_{j,\downarrow}) + \frac{U}{2} \sum_i (\tilde{n}_i - 1)^2 \\ & + \frac{1}{2} \sum_{i,j} J_{ij}^a \tilde{S}_i^z \tilde{S}_j^z - h \sum_i \tilde{S}_i^z - N \left(\mu + \frac{U}{2} \right) , \end{aligned} \quad (5.26)$$

where

$$J_{ij}^a = 4V_{ij} , \quad (5.27)$$

$$h = 2\mu , \quad (5.28)$$

$$\tilde{S}_i^z = \frac{1}{2}(\tilde{n}_i - 1) , \quad (5.29)$$

and N is the total number of sites.

Following the derivations of Sect. 5.1.3, and using the fact that \mathcal{H}^{+U} is at *half filling* (for a proof see Sect. 3.3.1 in [2]), this model at large $|U|/t$ can be directly mapped onto an effective pseudospin model

$$\begin{aligned} \mathcal{H}^{+U} \rightarrow & \mathcal{H}^{-x-xz} + \mathcal{O}(t^2/U) \\ \mathcal{H}^{-x-xz} = & \frac{1}{2} \sum_{ij}^{nn} \left[J_{ij}^z \tilde{S}_i^z \tilde{S}_j^z - J_{ij}^x (\tilde{S}_i^x \tilde{S}_j^x + \tilde{S}_i^y \tilde{S}_j^y) \right] - \sum_i h_i \tilde{S}_i^z \end{aligned} \quad (5.30)$$

where the *pseudospin* operators are

$$\begin{aligned}\tilde{S}_i^z &= \frac{1}{2}(\tilde{n}_i - 1) \\ \tilde{S}_i^x &= \frac{1}{2}(\tilde{c}_{i\uparrow}^\dagger \tilde{c}_{i\downarrow} + \tilde{c}_{i\downarrow} \tilde{c}_{i\uparrow}) \\ \tilde{S}_i^y &= \frac{1}{2i}(\tilde{c}_{i\uparrow}^\dagger \tilde{c}_{i\downarrow}^\dagger - \tilde{c}_{i\downarrow} \tilde{c}_{i\uparrow}).\end{aligned}\quad (5.31)$$

We see that the local charge operator and the pair operator have the same commutation relations as angular momenta along the z -axis and xy plane respectively. The quantum properties of the pseudospins explain Josephson commutation relation between charge and superconducting phase, $[N, \phi] = 1$.

At weak coupling $|U|/t < 1$, it can be also argued that the negative- U model renormalizes onto a similar effective model as (5.30) albeit with different lattice constant and interaction parameters. The Fermi sea is unstable with respect to attractive interactions as seen diagrammatically by the divergence of the vertex function in the BCS or charge density wave channels. Since the attractive interaction scales to strong coupling, at the scale where the cut-off energy equals the BCS gap, the effective Hamiltonian can be transformed to the $-x - xz$ model to obtain the strong coupling fixed point Hamiltonian. This procedure however, has not yet been carried out, to the best of our knowledge.

In Sect. 5.3, we shall use the classical \mathcal{H}^{-x-xz} model to describe superconductivity, and charge density wave phases.

5.2 Quantum Magnetism

This lecture is technical in nature. It contains a brief review of the spin path integral and how to obtain the classical and semiclassical approximations to it. A fuller background for this subject, with compatible notations, can be found in [2]. Here, a new emphasis is placed on anisotropic models and their rotator representation.

5.2.1 Spin Coherent States

Path integrals provide formal expressions which can lead to useful approximation schemes. A path integral representation of spin models can be constructed using *spin coherent states*. Let us consider the eigenstates $|S, m\rangle$ of \mathbf{S}^2 and S^z with eigenvalues $S(S+1)$ and m , respectively. Spin coherent states are a family of spin states labelled by a unit vector $\hat{\Omega} = (\theta, \phi)$, where θ and ϕ are the latitude and longitude angles respectively. They are defined by applying the $SU(2)$ rotation operator to the highest weight state² in representation

² The phase $e^{-iS\phi}$ represents a gauge choice with one singularity on the sphere: at the south pole.

S :

$$\begin{aligned} |\hat{\Omega}\rangle_S &\equiv e^{i\phi S^z} e^{i\theta S^y} e^{-i\phi S^z} |S, S\rangle \\ &= e^{-iS\phi} \sqrt{(2S)!} \sum_{m=-S}^{+S} \frac{u(\theta, \phi)^{S+m} v(\theta, \phi)^{S-m}}{\sqrt{(S+m)!(S-m)!}} |S, m\rangle , \end{aligned} \quad (5.32)$$

with

$$u(\theta, \phi) = \cos(\theta/2)e^{i\phi/2} , \quad (5.33)$$

$$v(\theta, \phi) = \sin(\theta/2)e^{-i\phi/2} . \quad (5.34)$$

Using (5.32), two useful identities can be readily proven:

- The resolution of identity

$$\frac{2S+1}{4\pi} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi |\hat{\Omega}\rangle\langle\hat{\Omega}| = \sum_{m=-S}^{+S} |S, m\rangle\langle S, m| = I_S . \quad (5.35)$$

- The overlap of two states with closeby unit vectors

$$\langle\hat{\Omega}|\hat{\Omega}'\rangle \simeq \left(\frac{1 + \hat{\Omega} \cdot \hat{\Omega}'}{2} \right)^S e^{(iS(1-\cos\bar{\theta})(\phi-\phi'))} \quad (5.36)$$

where $\bar{\theta}$ is the average latitude of the two vectors.

5.2.2 Spin Path Integral

The partition function of a single spin with Hamiltonian H is

$$Z = Tr [e^{-\beta H}] = Tr \left[\underbrace{e^{-\epsilon H} e^{-\epsilon H} \dots e^{-\epsilon H}}_{N_\epsilon = \beta/\epsilon \text{ times}} \right] , \quad (5.37)$$

with $\beta = 1/T$, T being the temperature. By inserting $N_\epsilon - 1$ resolutions of the identity (5.35), and in the limit $\epsilon \rightarrow 0$, by expanding each exponential to first order, one gets

$$Z \simeq \int d\hat{\Omega}_1 \dots d\hat{\Omega}_{N_\epsilon} \prod_{n=0}^{N_\epsilon-1} \langle\hat{\Omega}(\tau_n)|1 - \epsilon H|\hat{\Omega}(\tau_{n+1})\rangle , \quad (5.38)$$

with the boundary condition $\hat{\Omega}_0 = \hat{\Omega}_{N_\epsilon}$. In the limit $N_\epsilon \rightarrow \infty$ (and $\epsilon \rightarrow 0, \beta = N_\epsilon \epsilon = \text{const.}$) (5.38) defines a path integral

$$Z = \oint \mathcal{D}\hat{\Omega}(\tau) \exp \left[- \int_0^\beta d\tau \left(iS(1 - \cos\theta(\tau))\dot{\phi}(\tau) + H[\hat{\Omega}(\tau)] \right) \right] . \quad (5.39)$$

The time dependent term in (5.39)

$$iS\omega[\hat{\Omega}] = iS \int_0^\beta d\tau (1 - \cos \theta) \dot{\phi} \quad (5.40)$$

derives from the overlap between coherent states (5.36). It is known as the *Berry phase* of the spin history and it is geometric, i.e., depends on the trajectory of $\hat{\Omega}(\tau)$ on the unit sphere. In fact it measures the area enclosed by the path $\hat{\Omega}(\tau)$ on the unit sphere (Fig. 5.3).

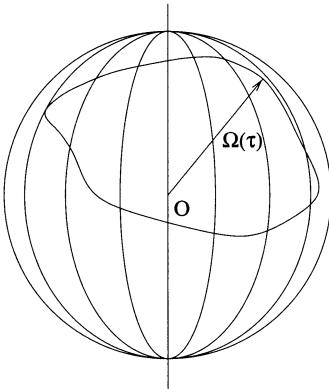


Fig. 5.3. The Berry phase $\omega[\hat{\Omega}]$ measures the area enclosed by the trajectory of $\hat{\Omega}(\tau)$ on the unit sphere which does not include the south pole

The classical Hamiltonian is defined as

$$H[\hat{\Omega}(\tau)] = \langle \hat{\Omega}(\tau) | H | \hat{\Omega}(\tau) \rangle. \quad (5.41)$$

An implicit assumption in (5.39) is that the path integral is dominated by smooth (differentiable) paths. This turns out to be unjustified, since discontinuous paths matter for the correct ordering of quantum operators. For that reason, path integral results should be checked whenever possible against operator methods. Ordering ambiguities give rise to erroneous quantum corrections to energies and local spin correlations. They do not effect, however, long distance and long timescale correlation functions.

The partition function for a system of N spins is

$$Z = \oint \prod_{i=1}^N \mathcal{D}\hat{\Omega}_i(\tau) \exp \left[iS \sum_{i=1}^N \omega[\hat{\Omega}_i] - \int_0^\beta d\tau H[\{\hat{\Omega}_i(\tau)\}] \right]. \quad (5.42)$$

For example, the nearest neighbour Heisenberg model partition function is

$$Z_H = \oint \mathcal{D}\hat{\Omega} \exp \left[iS \sum_{i=1}^N \omega[\hat{\Omega}_i] - \frac{J}{2} \int_0^\beta \sum_{\langle i,j \rangle} \hat{\Omega}_i \cdot \hat{\Omega}_j \right]. \quad (5.43)$$

The spin coherent states path integral (5.42) are convenient starting points for deriving semiclassical, i.e. large S , approximations. The integration variables are unit vectors, i.e. classical spins. The quantum effects enter through their time dependent fluctuations. Keeping $JS^2 \rightarrow J$ fixed and sending $S \rightarrow \infty$, suppresses the contributions of fluctuating paths with $\dot{\hat{\Omega}} \neq 0$. This leaves integration over frozen spin configurations precisely as in the classical partition function

$$Z \xrightarrow{S \rightarrow \infty} \int \prod_{i=1}^N d\hat{\Omega}_i e^{-\beta H[\{\hat{\Omega}_i\}]}. \quad (5.44)$$

Now it is possible to use S as the control parameter for a systematic expansion of the partition function. In particular, applying the saddle point approximation (analytically continued to *real* time $t = i\tau$) yields

$$S \frac{\delta \omega[\hat{\Omega}_i]}{\delta \dot{\hat{\Omega}}_i} - \int_0^t dt \frac{\delta H}{\delta \dot{\hat{\Omega}}_i} = 0, \quad (5.45)$$

which are the *Euler–Lagrange* equations of motion for classical spins

$$S \hat{\Omega}_i^{\text{cl}}(t) \times \dot{\hat{\Omega}}_i^{\text{cl}}(t) = \frac{\partial}{\partial \hat{\Omega}_i} H \left[\{\hat{\Omega}_j^{\text{cl}}\} \right]. \quad (5.46)$$

5.2.3 Spin Wave Theory

When spin symmetry is broken, either spontaneously or by explicit symmetry breaking terms, it is quite natural to use the semiclassical expansion of the path integral. We consider small fluctuations around a classical spin configuration, $\hat{\Omega}_i^{\text{cl}}$ which minimizes $H[\Omega]$.

$$\hat{\Omega}_i(\tau) = \hat{\Omega}_i^{\text{cl}} + \delta\hat{\Omega}_i(\tau). \quad (5.47)$$

To leading Gaussian order, the partition function is approximated by

$$Z \simeq e^{-\beta H[\Omega^{\text{cl}}]} \int \mathcal{D}\delta\hat{\Omega}(\tau) \exp \left[iS \sum_{i=1}^N \delta^2 \omega[\hat{\Omega}_i] - \int_0^\beta d\tau \delta^2 H[\{\hat{\Omega}_i(\tau)\}] \right], \quad (5.48)$$

where

$$\begin{aligned}\delta^2\omega[\hat{\Omega}_i] &= \frac{1}{2} \sum_{i=1}^N \int_0^\beta d\tau \hat{\Omega}_i \cdot \delta\dot{\hat{\Omega}}_i \times \delta\hat{\Omega}_i , \\ \delta^2 H[\hat{\Omega}_i] &= \frac{1}{2} \sum_{\langle i,j \rangle} \delta\hat{\Omega}_i \frac{\delta^2 H[\{\hat{\Omega}_i\}]}{\delta\hat{\Omega}_i \delta\hat{\Omega}_j} \delta\hat{\Omega}_j .\end{aligned}\quad (5.49)$$

$\delta\hat{\Omega}_i$ which are perpendicular to $\hat{\Omega}_i$ can be projected onto the two tangential unit vectors which define the harmonic oscillator degrees of freedom

$$q_i = \delta\hat{\Omega}_i \cdot \hat{\phi}_i \quad (5.50)$$

$$p_i = S\delta\hat{\Omega}_i \cdot \hat{\theta}_i , \quad (5.51)$$

These variables can be used to represent the Gaussian fluctuations as

$$Z \simeq e^{-\beta H[\hat{\Omega}^{cl}]} \times \int \mathcal{D}p_i \mathcal{D}q_i \exp \left[\int_0^\beta d\tau \left(i \frac{\mathbf{p} \cdot \dot{\mathbf{q}} - \dot{\mathbf{p}} \cdot \mathbf{q}}{2} - \frac{1}{2} (\mathbf{q}, \mathbf{p}) H^{(2)} \left(\frac{\mathbf{q}}{\mathbf{p}} \right) \right) \right] \quad (5.52)$$

where $H^{(2)}$ is a dynamical matrix of coupled harmonic oscillators

$$H^{(2)} = \begin{pmatrix} K & P \\ P^t & M^{-1} \end{pmatrix} , \quad (5.53)$$

where

$$K_{ij} = \left. \frac{\partial^2 H}{\partial q_i \partial q_j} \right|_{\mathbf{q}=\mathbf{p}=0} , \quad (5.54)$$

$$M_{ij}^{-1} = \left. \frac{\partial^2 H}{\partial p_i \partial p_j} \right|_{\mathbf{q}=\mathbf{p}=0} , \quad (5.55)$$

are the force constant and reciprocal mass matrices respectively and

$$P_{ij} = \left. \frac{\partial^2 H}{\partial p_i \partial q_j} \right|_{\mathbf{q}=\mathbf{p}=0} , \quad (5.56)$$

couples coordinates and momenta. Eq. (5.52) is the harmonic spin wave partition function of any quantum spin Hamiltonian, whose classical ground state is known. By diagonalizing its action one readily obtains the spin wave excitation energies and wavefunctions, and spin correlations can be evaluated to the subleading order in S^{-1} . The complexity of the calculation depends on the lattice symmetry of the classical ground state, i.e. such as the size of its magnetic unit cell. For example, for the Néel state, it is two lattice unit cells.

For completeness, we work out the spin wave dispersion of the antiferromagnetic Heisenberg model with a Néel state given by

$$(\theta_i^{\text{cl}}, \phi_i^{\text{cl}}) = \begin{cases} (\pi/2, 0) & i \in A \\ (\pi/2, \pi) & i \in B \end{cases}, \quad (5.57)$$

where A and B are the two sublattices in which the lattice can be divided. The harmonic degrees of freedom are

$$q_i(t) = \begin{cases} \phi_i(t) & i \in A \\ \pi + \phi_i(t) & i \in B \end{cases} \quad (5.58)$$

and

$$p_i(t) = S \cos \theta_i(t). \quad (5.59)$$

The dynamical matrix of the model (5.43) is

$$\begin{aligned} H^{(2)} &= \frac{J}{2} \sum_{\langle i,j \rangle} \left[\frac{(p_i - p_j)^2}{S^2} - (q_i - q_j)^2 \right] \\ &= \frac{1}{2} \sum_{\mathbf{k}} (p_{\mathbf{k}}, q_{\mathbf{k}}) \begin{pmatrix} zJ(1 + \gamma_{\mathbf{k}}) & 0 \\ 0 & zJS^2(1 - \gamma_{\mathbf{k}}) \end{pmatrix} \begin{pmatrix} p_{\mathbf{k}} \\ q_{\mathbf{k}} \end{pmatrix}, \end{aligned} \quad (5.60)$$

where

$$\gamma_{\mathbf{k}} = \frac{1}{z} \sum_{\mathbf{d}} e^{i\mathbf{k}\cdot\mathbf{d}}, \quad (5.61)$$

z and \mathbf{d} being respectively the coordination number and the vector connecting one site to its nearest-neighbours. The dispersion relation of the small fluctuations around the ground state configuration, i.e., of the *spin waves*, can be found solving the characteristic equation

$$\det \begin{pmatrix} S^{-2}zJ(1 + \gamma_{\mathbf{k}}) & i\omega_{\mathbf{k}} \\ -i\omega_{\mathbf{k}} & zJ(1 - \gamma_{\mathbf{k}}) \end{pmatrix} = 0, \quad (5.62)$$

and is $\omega_{\mathbf{k}} = \frac{1}{S}zJ\sqrt{1 - \gamma_{\mathbf{k}}^2}$, for two distinct spin wave modes.

5.2.4 Continuum Theory for Anisotropic Models

Spin wave theory is restricted to the ordered phases of the Heisenberg model. However, one can still use a semiclassical approach even in the absence of spontaneously broken symmetry. A short range classical Hamiltonian is mostly sensitive to short-range correlations. Thus the Heisenberg Hamiltonian, in the large S limit, has at least short range antiferromagnetic order. In

the path integral approach it is possible to utilize the short lengthscale correlations to define a continuum theory without assuming broken symmetry.

In this Section we spend some time preparing the ground for Lecture 5.3. To that end, we derive the continuum theory for the anisotropic xxz model in a magnetic field. The resulting path integral will be later used in the context of quantum properties of superconductors. The continuum theory is shown to be equivalent to $\text{SO}(3)$ quantum rotators. The rotator formulation is readily generalizable to $\text{SO}(5)$ symmetry, which is the topic of Sect. 5.3.3. Subsequently, for the isotropic case we review Haldane's mapping of the quantum Heisenberg antiferromagnet in d dimensions into the nonlinear sigma model (NLSM) in $d+1$ dimensions, and the main results which can be obtained by that mapping.

The first step is to parametrize the spins using two continuous vector fields \hat{n} and \mathbf{L} ,

$$\hat{\Omega}_i^\alpha = \eta_i \hat{n}^\alpha(\mathbf{x}_i) \sqrt{1 - \left| \frac{\mathbf{L}(\mathbf{x}_i)}{S} \right|^2 + \frac{L^\alpha(\mathbf{x}_i)}{S}} , \quad (5.63)$$

where $\eta_i = e^{i\pi \cdot \mathbf{x}_i}$ has opposite signs on the two sublattices. Each pair of neighbouring spins (4 degrees of freedom) is replaced by \hat{n}, \mathbf{L} . We can choose \hat{n} to be a unimodular ($|\hat{n}|^2 = 1$) Néel field (2 degrees of freedom), and \mathbf{L} is the perpendicular canting field, with the constraint $\mathbf{L} \cdot \hat{n} = 0$ (2 degrees of freedom). The spin measure of (5.42) becomes

$$\mathcal{D}\hat{\Omega} \rightarrow \mathcal{D}\hat{n} \mathcal{D}\mathbf{L} \delta(\mathbf{L} \cdot \hat{n}) , \quad (5.64)$$

where the δ functionals are local space-time constraints.

Let us consider a general anisotropic spin model in a magnetic field \mathbf{h} ,

$$\mathcal{H} = \frac{1}{2} \sum_{ij,\alpha} J_{ij}^\alpha \hat{\Omega}_i^\alpha \hat{\Omega}_j^\alpha - \mathbf{h} \cdot S \sum_i \hat{\Omega}_i . \quad (5.65)$$

Using (5.63) and expanding to lowest order in L^α , $\partial_i n^\alpha$ and $\partial_i L^\alpha$ we obtain the energy density

$$\begin{aligned} \mathcal{H} &\rightarrow \int d^d x \, H[\mathbf{L}, \hat{n}] \\ H &= E^{\text{cl}}[n] + \frac{1}{2} \sum_\alpha \left(\chi_\alpha^{-1} (L^\alpha)^2 + \rho_s^\alpha (\partial_i n^\alpha)^2 \right) - a^{-d} \mathbf{h} \cdot \mathbf{L} , \end{aligned} \quad (5.66)$$

where the energy, spin stiffness, and susceptibility parameters are respectively:

$$\begin{aligned} E^{\text{cl}}[n] &\equiv -\frac{1}{2\mathcal{N}a^d} \sum_{\alpha} (n^{\alpha})^2 \left(\sum_{ij} J_{ij}^{\alpha} \eta_i \eta_j \right), \\ \rho^{\alpha} &\equiv \frac{1}{2d\mathcal{N}a^d} \sum_{ij} J_{ij}^{\alpha} \eta_i \eta_j |\mathbf{x}_i - \mathbf{x}_j|^2, \\ \chi_{\alpha} &= \frac{1}{\mathcal{N}a^d} \left(\sum_{ij} J_{ij}^{\alpha} \right) - 2S^{-2} E^{\text{cl}} \end{aligned} \quad (5.67)$$

a and \mathcal{N} are the lattice constant and size respectively. Expansion of the Berry phase term to the same order yields two terms

$$iS \sum_i \omega \left[\hat{\Omega}_i \right] = -i\Upsilon + i \int_0^{\beta} d\tau \int d^d x \hat{n} \times \dot{\hat{n}} \cdot \mathbf{L}, \quad (5.68)$$

where

$$\Upsilon(\hat{n}) = S \int d^d x e^{i\pi \cdot \mathbf{x}} \omega(\hat{n}(\mathbf{x})). \quad (5.69)$$

Collecting the terms together, we have the path integral

$$Z^{xxz} = \int \mathcal{D}\hat{n} \mathcal{D}\mathbf{L} \delta(\mathbf{L} \cdot \hat{n}) e^{-i\Upsilon} \exp \left(\int d\tau d^d x i\hat{n} \times \dot{\hat{n}} \cdot \mathbf{L} - H[\mathbf{L}, \hat{n}] \right). \quad (5.70)$$

5.2.5 Anisotropic Quantum Rotators

Equation (5.70) can be physically understood as a path integral of *rotators*. Consider the phase space path integral over an N dimensional field \mathbf{n} , and canonical momenta \mathbf{p} with a “Mexican hat” potential

$$\begin{aligned} Z &= \int \mathcal{D}\mathbf{p} \mathcal{D}\mathbf{n} \exp \left(\int d\tau d^d x i\dot{\mathbf{n}} \cdot \mathbf{p} - H[\mathbf{p}, \mathbf{n}] \right) \\ H^{MH} &= H^{\text{rot}}[\mathbf{p}, \mathbf{n}] + K(|\mathbf{n}| - 1)^2. \end{aligned} \quad (5.71)$$

If K is taken to be very large, fluctuations of $\delta n_{||} = |\mathbf{n}| - 1$, and its conjugate momentum $\mathbf{p}_{||}$ become high frequency harmonic oscillators, which can be integrated out in the adiabatic approximation. This leaves us with the slow degrees of freedom $\hat{n} = \mathbf{n}/|\mathbf{n}|$, and \mathbf{p}_{\perp} , and a renormalized Hamiltonian

$$Z_{K \rightarrow \infty} \propto \int \mathcal{D}\hat{n} \mathcal{D}\mathbf{p} \delta(\mathbf{p} \cdot \hat{n}) \exp \left(\int d\tau d^d x i\dot{\hat{n}} \cdot \mathbf{p}_{\perp} - \tilde{H}^{\text{rot}}[\mathbf{p}_{\perp}, \hat{n}] \right). \quad (5.72)$$

A Faddeev–Jackiw quantization of a particle on an N -sphere leads to the same constraints, as shown in [3].

For the $N = 3$ model, the angular momenta and the transverse momenta are both vectors, related by

$$\mathbf{L} = \hat{\mathbf{n}} \times \mathbf{p}_\perp, \quad \mathbf{p}_\perp = \hat{\mathbf{L}} \times \mathbf{n}. \quad (5.73)$$

A change of variables $\mathbf{p}_\perp \rightarrow \mathbf{L}$ has a unit Jacobian

$$\mathcal{D}\mathbf{p}\delta(\mathbf{p} \cdot \hat{\mathbf{n}}) = \mathcal{D}\mathbf{L}\delta(\mathbf{L} \cdot \hat{\mathbf{n}}). \quad (5.74)$$

Substituting (5.73) and (5.74) into (5.72) yields a path integral of the form (5.70), without the Berry phase e^{-iY} .

For general N , the $N(N - 1)/2$ angular momenta are defined as

$$L_{ab} \equiv n_a p_b - n_b p_a, \quad a, b = 1, \dots, N. \quad (5.75)$$

Note, that it is not useful for $N > 3$ to write the path integral measure in terms of L_{ab} , because they are not independent degrees of freedom, and more constraints are required. On the other hand, the *transverse momenta*, which obey $\sum_a n_a(p_\perp)_a = 0$, can be expressed as

$$(p_\perp)_a = \sum_b L_{ab} n_b. \quad (5.76)$$

Therefore it is always possible to express the rotator Hamiltonian \tilde{H}^{rot} of (5.72) in terms of L_{ab} and $\hat{\mathbf{n}}$.

The opposite direction, might be even more useful. For example, as in (5.66), the starting point could be a Hamiltonian whose kinetic energy is expressed using symmetry generators

$$H^{\text{rot}}[\mathbf{L}, \hat{\mathbf{n}}] = E^{\text{cl}}[\hat{\mathbf{n}}] + \frac{1}{2} \sum_{a < b} \chi_{ab}^{-1} L_{ab}^2 + \frac{1}{2} \sum_a \rho_s^a (\partial_i n_a)^2 - a^{-d} \sum_{a < b} h_{ab} L_{ab}, \quad (5.77)$$

where h_{ab} and χ_{ab} are SO(N) fields and susceptibilities respectively. (For SO(3) their vector notation is given by $X^a \equiv \sum_{bc} \epsilon^{abc} X_{bc}$). Substituting (5.75) into H^{rot} yields

$$\begin{aligned} H^{\text{rot}}[\mathbf{p}, \hat{\mathbf{n}}] &= E^{\text{cl}}[n] + \frac{1}{2} \sum_{ab} M_{ab}^{-1} p_a p_b + \sum_a \rho_s^a (\partial_i n^a)^2 \\ &\quad - a^{-d} \sum_{a < b} h_{ab} (n_a p_b - n_b p_a), \\ M_{ab}^{-1} &\equiv \delta_{ab} \left(\sum_c \chi_{ac}^{-1} n_c^2 \right) - \chi_{ab}^{-1} n_a n_b \end{aligned} \quad (5.78)$$

where $M[\hat{n}]$ is an anisotropic “mass” matrix in the Cartesian basis.

We note that by (5.78), the path integral (5.72) is Gaussian in momenta p_a . One must be careful and integrate only over the transverse components to \hat{n} . For a given direction \hat{n} , we choose the transverse basis \hat{e}_i , $i = 1, N - 1$ which obeys the following conditions

$$\begin{aligned} \hat{e}_i \cdot \hat{n} &= 0, \quad i = 1, N - 1 \\ \sum_{a,b}^N \hat{e}_i^a M_{ab}^{-1} \hat{e}_j^b &= \delta_{ij} \tilde{M}_i^{-1}[\hat{n}] . \end{aligned} \quad (5.79)$$

This is always possible since the first condition leaves the freedom to perform an $SO(N-1)$ rotation on the transverse basis. For an arbitrary transverse basis $\{\hat{f}_i\}$, we find the rotation which diagonalizes $\hat{f}_i M^{-1} \hat{f}_j$, and the resulting eigenbasis is chosen as $\{\hat{e}_i\}$.

Thus, we parametrize

$$\mathbf{p} = \sum_i p_i \hat{e}_i \quad (5.80)$$

and integrate unrestrictedly over $\int \mathcal{D}p_i$ to obtain

$$\begin{aligned} Z &= \int \mathcal{D}\hat{n} \exp \left[- \int_0^\beta d\tau \int d^d x \left(E^{cl}[n] + \sum_{l,a} \frac{1}{2} \rho^\alpha (\partial_l n^a)^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \sum_j M_j \left(\hat{n} \cdot \hat{e}_j - i h_j \right)^2 \right) \right] \\ h_j &\equiv a^{-d} \sum_{a < b} h_{ab} (n_b \hat{e}_j^a - n_a \hat{e}_j^b) . \end{aligned} \quad (5.81)$$

This expression is ready for the evaluation of the classical (time independent) ground state \hat{n}^{cl} as a function of applied field h_{ab} , and a spinwave expansion about it. In the next Section however, we shall only deal with the isotropic case.

5.2.6 Haldane’s Mapping

We now return to (5.81) but specialize to the *isotropic* Heisenberg model without a field, where life simplifies considerably.

The inverse mass matrix is simply

$$M^{-1} = \chi^{-1} \delta_{ab} - n_a n_b . \quad (5.82)$$

Any choice of transverse basis \hat{e}_i yields a diagonal $M_i = \chi$ in (5.79). Omitting the constant E^{cl} , and inserting the Berry phase $e^{-i\Upsilon[\hat{n}]}$ the partition function

(5.81) reduces to Haldane's result

$$\begin{aligned} Z &= \int \mathcal{D}\hat{n} e^{-i\Upsilon(\hat{n})} \exp \left(-\frac{1}{2} \int_0^\beta d\tau \int d^d x (\chi |\dot{\hat{n}}|^2 + \rho_S \sum_{l=1}^d |\partial_l \hat{n}|^2) \right) \\ &= \int \mathcal{D}\hat{n} e^{-i\Upsilon(\hat{n})} \exp \left(-\frac{a^{1-d}}{2f} \int dx_0 d^d x (\partial_\mu \hat{n})^2 \right), \end{aligned} \quad (5.83)$$

where $x_0 = c\tau$, and $c = \sqrt{\rho_S/\chi}$ is the *spin wave velocity*. For the nearest neighbour model with interaction J , one obtains from (5.67):

$$\chi = S^2/(4dJa^d), \quad (5.84)$$

$$\rho_S = Ja^{2-d}, \quad (5.85)$$

$$f = \frac{ca^{1-d}}{\rho_S} = 2\sqrt{d}S^{-1}. \quad (5.86)$$

Equation (5.83) is the partition function for a NLSM with an additional Berry phase term.

In $1+1$ dimensions the NLSM is disordered for all f , as required by the classical Mermin Wagner theorem. Its correlations are known to fall off exponentially at large distances with a correlation length which goes as $\xi \propto e^{2\pi/f}$. By the (Lorentz) symmetry of the action between spatial and temporal dimensions, this implies a gap (*Haldane's gap*) for all excitations above the ground state. However one should also consider the effects of the phases brought about by the term $\Upsilon(\hat{n})$.

For $d = 1$, $\Upsilon(\hat{n})$ is a topological winding number of the two dimensional NLSM. For all continuous fields, it yields $e^{-i\Upsilon(\hat{n})} = e^{-i2\pi Sk}$ with k an integer number. Thus, the Berry phase factor is 1 for all integer S , while it can be ± 1 for half integer spins. As a result, it produces interference effects for half odd integer spins, and drastically changes the ground state properties and the elementary excitations spectrum of the Heisenberg chain.

In $d = 2$ the topological phase is zero for all continuous fields. For the nearest neighbours Heisenberg model, Neves and Perez [4] proved that the ground state is ordered for all $S \geq 1$. Also, series expansions and numerical simulations provide evidence of the presence of an ordered ground state even for $S = 1/2$.

5.2.7 Spin Liquid States

For most antiferromagnetic Heisenberg models the ground state is not explicitly known. While for finite bipartite lattices, ground state theorems require the ground state to be a total singlet, and have positivity conditions (Marshall's signs). The discussion of the previous Section expects them to exhibit long range order in two dimensions. Considering these conditions, particularly useful variational states for the $S = 1/2$ antiferromagnetic Heisenberg

model are Fazekas and Anderson's *Resonating Valence Bond* (RVB) states [5]

$$|\{d_\alpha\}\rangle = \sum_{\alpha} d_\alpha |\alpha\rangle , \quad (5.87)$$

where

$$|\alpha\rangle = \prod_{(i,j) \in \Lambda_\alpha}^{\text{i} \epsilon A, j \epsilon B} \frac{1}{\sqrt{2}} \left(|\uparrow_i\rangle |\downarrow_j\rangle - |\downarrow_i\rangle |\uparrow_j\rangle \right) , \quad (5.88)$$

and d_α can be chosen to have the form

$$d_\alpha = \left(\prod_{(i,j) \in \Lambda_\alpha} u_{ij} \right) . \quad (5.89)$$

Since the states $|\alpha\rangle$ are not orthogonal to each other, it is not possible to evaluate correlations of RVB states analytically. Monte Carlo simulations of Liang, Doucot and Anderson [6], and Havilio [7], have found that the RVB states have long range Néel order for u_{ij} that decay slower than

$$u_{ij} \simeq |x_i - x_j|^{-p} , p \leq 3 . \quad (5.90)$$

The RVB states (5.87) can thus be used as variational ground states for both ordered and disordered phases. This makes them appealing candidates for studying the transitions from the Néel phase to possible quantum disordered phases, particularly in the presence of hole doping [7].

5.3 Pseudospins and Superconductivity

The $-x - xz$ model, encountered in Lecture 5.1, will be considered as an effective Hamiltonian for the low temperature, long wavelength properties of *s*-wave superconductors and charge density waves. Such transitions are observed in e.g. doped bismuthates $\text{Ba}_{1-x}\text{K}_x\text{BiBO}_3$. Recall (5.30):

$$\mathcal{H}^{-x-xz} = \frac{1}{2} \sum_{ij}^{nn} \left[J_{ij}^z \tilde{S}_i^z \tilde{S}_j^z - J_{ij}^x (\tilde{S}_i^x \tilde{S}_j^x + \tilde{S}_i^y \tilde{S}_j^y) \right] - \sum_i h_i \tilde{S}_i^z . \quad (5.91)$$

\tilde{S} are spin 1/2 operators, and $h = 2\mu$, where μ is the electron chemical potential. As shown later, it can also be used to describe superfluids. In this lecture we use the continuum rotator model, and its classical limit to obtain the phase diagram. The generalization to Zhang's SO(5) rotators, which describe the collective modes of high T_c cuprates, and the transition from antiferromagnetism to *d*-wave superconductivity, is straightforward. The last Section uses the $-x - xz$ model to describe dynamics of vortices in superfluids.

5.3.1 Charge Density Wave to Superconductivity

In *bipartite lattice* with sublattices *A* and *B* (e.g. square, simple cubic, etc.), the negative J^x terms of the $-x - xz$ model can be rotated to positive terms by a global $e^{iS^z\pi}$ rotation on sublattice *B*. Obviously, for frustrated (non bipartite) lattices the xxz and the $-x - xz$ models are *not* equivalent. For example, the triangular, Kagomé, and face centered cubic lattices [8], the $-x - xz$ model prefers to order in the xy plane even when $J^z > |J^x|$ at zero doping³.

Here we specialize to the classical model on a bipartite (square or cubic) lattice

$$\begin{aligned} H &= \frac{1}{z} \sum_{\langle i,j \rangle}^{nn} \left[J^z \hat{\Omega}_i^z \hat{\Omega}_j^z - J^x (\hat{\Omega}_i^x \hat{\Omega}_j^x + \hat{\Omega}_i^y \hat{\Omega}_j^y) \right] \\ &\quad - hS \sum_i \hat{\Omega}_i^z + H^{nnn} \\ H_{nnn} &= \frac{1}{z'} \sum_{i,j}^{nnn} K \hat{\Omega}_i^z \hat{\Omega}_j^z, \end{aligned} \tag{5.92}$$

where z and z' are nearest neighbour (nn) and next nearest neighbour (nnn) coordination numbers respectively.

By (5.31) we note that the electron charge expectation value and the superconducting order parameters are

$$\begin{aligned} \langle n_i \rangle - 1 &= \hat{\Omega}_i^z \\ \langle c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \rangle &= \hat{\Omega}_i^x + i \hat{\Omega}_i^y \\ x &= -\mathcal{N}^{-1} \sum_i \hat{\Omega}_i^z, \end{aligned} \tag{5.93}$$

x is the hole *doping concentration* away from half filling.

At finite temperatures, molecular mean field theory for $S = 1/2$ [9] gives the critical temperatures as a function of doping x to be

$$\begin{aligned} T_c^{\text{CDW}}(x) &= \frac{1}{4} (J^z - K)(1 - x^2) \\ T_c^{\text{SC}}(x) &= \frac{J^x x}{4 \operatorname{arctanh}(x)}. \end{aligned} \tag{5.94}$$

The two curves meet at T^* (See Fig. 5.4). This point is tetracritical or bicritical, depending on whether the transition (as a function of field h) is second or first order respectively. In Fig. 5.4 this is reflected by the nature of the intermediate region which would be either a mixed (“supersolid”) phase, or phase separation between pure CDW and SC domains respectively. The next Section is devoted to determining the criteria for the order of the transition.

³ This helps to explain superconductivity in K_3C_{60} which is an FCC compound with a half filled conduction band.

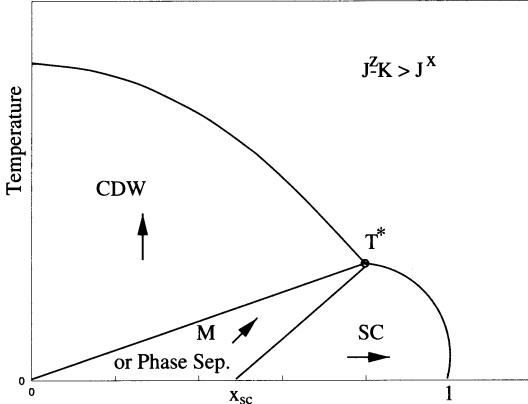


Fig. 5.4. Typical mean field phase diagram (doping concentration versus temperature) of the $-x - xz$ model. The arrows denote directions of rotator field \hat{n} in the xz plane

Near T^* , the lowest order expansion of the Ginzburg–Landau free energy functional has $SO(3)$ symmetry. Thus even though the model might have high anisotropy $J^z \gg J^x$, near the bicritical point the Heisenberg symmetry is approximately restored. This “symmetry restoring” is also argued to happen near the multicritical point of the anisotropic $SO(5)$ model [10].

5.3.2 Order of Transition from Rotator Theory

Finding out the order of the CDW-SC transition at zero T , requires the energy of the putative mixed state (M), which interpolates between the pure CDW and pure SC. The rotator partition function (5.81) comes in handy for that purpose. We write (5.92) as (5.65) by letting $-J^x \rightarrow +J^x$, and take the continuum limit with unit lattice constant $a = 1$ following the prescriptions of Sect. 5.2.4. The M ground state is parametrized by one Néel angle $\hat{n} = (\sin \theta, \cos \theta)$, where $0 < \theta < \pi/2$. $\theta = 0$, $\theta = \pi/2$ are the pure CDW, SC respectively.

First we evaluate the inverse mass matrix of (5.78)

$$M_{ab}^{-1} = \begin{pmatrix} \chi_y^{-1} \cos^2 \theta & 0 & -\chi_y^{-1} \cos \theta \sin \theta \\ 0 & \chi_y^{-1} \cos^2 \theta + \chi_z^{-1} \sin^2 \theta & 0 \\ -\chi_y^{-1} \cos \theta \sin \theta & 0 & \chi_y^{-1} \sin^2 \theta \end{pmatrix}. \quad (5.95)$$

The transverse vectors which diagonalize the projected inverse mass matrix are $\hat{e}_1 = \hat{y} \times \hat{n}$ and $\hat{e}_2 = \hat{y}$. The mass eigenvalues are

$$\tilde{M} = \begin{pmatrix} \chi_y & 0 \\ 0 & (\chi_y^{-1} \cos^2 \theta + \chi_z^{-1} \sin^2 \theta)^{-1} \end{pmatrix}. \quad (5.96)$$

Set the lattice constant to unity $a = 1$, using (5.67) and (5.92) we obtain

$$\begin{aligned} h_j &= -h \sin \theta \delta_{j,2} \\ \chi_z^{-1}(\theta) &= \frac{2}{S^2} ((J^z + K) + (J^z - K) \cos^2 \theta + |J^x| \sin^2 \theta) \\ \chi_y^{-1}(\theta) &= \frac{2}{S^2} (|J^x| + (J^z - K) \cos^2 \theta + |J^x| \sin^2 \theta) . \end{aligned} \quad (5.97)$$

Here we have set the lattice constant to unity $a = 1$.

Thus, the classical ground state θ^{cl} minimizes the energy

$$\begin{aligned} E^{\text{rot}}[\theta, h] &= -(J^z - K) \cos^2 \theta - J^x \sin^2 \theta - \frac{1}{2} \sum_{j=1,2} M_j h_j^2 \\ &= -(J^z - K) + (J^z - K - J^x) \sin^2 \theta \\ &\quad - \frac{\frac{1}{4} S^2 h^2 \sin^2 \theta}{J^z - K + J^x + 2K \sin^2 \theta} . \end{aligned} \quad (5.98)$$

A simple analysis of (5.98) reveals that the order of the transition depends on the sign of K :

For $K < 0$,

$$\left. \frac{\partial^2 E}{\partial (\sin^2 \theta)^2} \right|_{K < 0} < 0, \quad 0 < \theta < \pi/2 \quad (5.99)$$

i.e., there is a *first* order transition between $\theta^{\text{cl}} = 0$ to $\theta^{\text{cl}} = \pi/2$ called a “*spin flop*”. This happens at a magnetic field h_{sc} given by

$$\begin{aligned} E^{\text{rot}}[0, h_{\text{sc}}] &= E^{\text{rot}}[\pi/2, h_{\text{sc}}] \\ \rightarrow h_{\text{sc}} &= 2S^{-1} \sqrt{(J^z - K - J^x)(J^z + K + J^x)} . \end{aligned} \quad (5.100)$$

For any h , the doping concentration is given by

$$x(h) = 2S^{-1} \frac{\partial \min_\theta E}{\partial h} \quad (5.101)$$

In the case of a first order transition, there is phase separation between the doping concentrations of pure CDW and pure SC at densities

$$x_{\text{cdw}} = 0, \quad x_{\text{sc}} = \sqrt{\frac{J^z - K - J^x}{J^z + K + J^x}} . \quad (5.102)$$

On the other hand, for $K > 0$, $E(\theta)$ is minimized at

$$\sin^2 \theta^{\text{cl}} = \left(\frac{1}{2} h S \sqrt{\frac{J^z - K + J^x}{J^z - K - J^x}} - (J^z - K + J^x) \right) / 2K , \quad (5.103)$$

which indicates the existence of a mixed phase for fields in the range $h_{\text{cdw}} < h < h_{\text{sc}}$, where

$$\begin{aligned} h_{\text{cdw}} &= \frac{2}{S} \sqrt{(J^z - K + J^x)(J^z - K - J^x)}, \quad x_{\text{cdw}} = 0. \\ h_{\text{sc}} &= \frac{2}{S} \sqrt{\frac{J^z - K - J^x}{J^z - K + J^x}} (J^z + K + J^x), \quad x_{\text{sc}} = \sqrt{\frac{J^z - K - J^x}{J^z - K + J^x}}. \end{aligned} \quad (5.104)$$

For the pure nearest neighbour model ($K = 0$), it is easy to see by (5.98), that at $h_{\text{sc}} = h_{\text{cdw}}$ the mixed state is degenerate with a Maxwell construction of phase separation into SC and CDW. This degeneracy is lifted in the quantum version of the same model, which favours phase separation [11].

5.3.3 SO(5) Rotator Theory and High-T_c Superconductors

Recently an SO(5) theory of high T_c cuprate superconductors has been proposed by Shou-Cheng Zhang [10]. The order parameters of antiferromagnetism (AFM) and *d*-wave superconductivity (dSC) are written as the cartesian components of a 5 dimensional vector

$$\begin{aligned} \hat{n} &= (n_1, n_2, n_3, n_4, n_5), \quad |\hat{n}| = 1 \\ (n_2, n_3, n_4) &= \langle \sum_i e^{i\pi x_i} S_i \rangle \\ n_1 + i n_5 &= \langle \sum_{\mathbf{p}} g(\mathbf{p}) c_{\mathbf{p}\uparrow}^\dagger c_{\mathbf{p}\downarrow}^\dagger \rangle. \end{aligned} \quad (5.105)$$

$g(\mathbf{p})$ is the Fourier transform of the pair wavefunction which for *d*-wave pairing transforms as $\cos(p_x) - \cos(p_y)$ under lattice rotations. Zhang has found explicit, second quantized constructions for the 10 SO(5) generators $\{L_{a,b}\}_{ab}$, $a, b = 1, \dots, 5$ which rotate \hat{n} in 5 dimensions. Particularly interesting are the Π operators, i.e. generators $L_{1,a}$, $a = 2, 3, 4$, which rotate between the AFM and dSC hyperplanes. These are expected to create new low lying pseudo-Goldstone modes near the transition.

Zhang has proposed that the long wave fluctuations are described by an effective rotator Hamiltonian of the form (5.78),

$$H[L, n] = \frac{1}{2} \sum_{a < b} \chi_{ab}^{-1} L_{ab}^2 + \frac{1}{2} \rho \sum_a |\nabla n_a|^2 + g(n_1^2 + n_5^2) - 2\mu L_{15}, \quad (5.106)$$

where L_{ab} are the generators of SO(5) algebra, and L_{15} is the charge operator whose expectation value yields half the doping concentration $\langle L_{15} \rangle = x/2$. Using the substitution (5.75), the momenta can be integrated out of the

partition function, leaving us with

$$Z = \int \mathcal{D}\hat{n} \exp \left[- \int_0^\beta d\tau \int d^d x \left(\frac{1}{2} \sum_{j=1}^4 M_j \left(\hat{n} \cdot \hat{e}_j - i h_j \right)^2 + \frac{1}{2} \rho \sum_{a=1}^5 |\nabla n_a|^2 + g(n_1^2 + n_5^2) \right) \right]$$

$$h_j \equiv 2\mu(n_5 \hat{e}_j^1 - n_1 \hat{e}_j^5). \quad (5.107)$$

We set $\hat{n} = 0$, and search for the classical ground state.

Let us first consider the SO(5) symmetric model, where all $\chi_{ab} = \chi$, and $g = 0$. For any finite $\mu \neq 0$, the ground state flops into (n_1, n_5) plane, i.e. is superconducting.

Experimentally, a transition from AFM to dSC is observed at low hole concentrations x in many of the high T_c systems. Appealing to the analogy with the $-x - xxz$ model, this suggests that the symmetry breaking terms of $SO(5) \rightarrow SO(3) \times SO(2)$ should not be forbiddingly large. Symmetry breaking terms can be included by $g > 0$, and letting the charge susceptibility $\chi_c = \chi_{1,5}$ be different than all other susceptibilities $\chi_{ab} = \chi, a, b \neq 1, 5$. Without loss of generality, we can choose the order parameter to tilt between the AFM and dSC hyperplanes $\hat{n} = (\sin \theta, \cos \theta, 0, 0, 0)$. Following the same derivation as for xxz rotators (5.98), the classical energy is

$$E[\theta, \mu] = g \sin \theta^2 - \frac{2 \sin^2 \theta \mu^2}{\chi^{-1} \cos^2 \theta + \chi_c^{-1} \sin^2 \theta}. \quad (5.108)$$

It is now straightforward to verify that the ground state is in the (n_2, n_3, n_4) sphere at $\mu = 0$, and will “flop” into the SC state $\theta = \pi/2$ at large enough μ .

It also follows, using the same path as in Sect. 5.3.2, that the order of the transition depends on the relative magnitudes of susceptibilities⁴:

$$\begin{aligned} \chi_c > \chi &\Rightarrow \text{First order transition at } \mu_{\text{dsc}} = \sqrt{g/(2\chi_c)} \\ &\text{phase separation at } 0 < x < 4\sqrt{g\chi_c/2} \\ \chi_c < \chi &\Rightarrow \text{Mixed phase for } \sqrt{g/(2\chi)} < \mu < \sqrt{g\chi/2} (\chi_c^{-1} + \chi^{-1}) \\ &\text{at densities } 0 < x < 4\sqrt{g\chi/2} \end{aligned} \quad (5.109)$$

In the mixed phase, the relation between the SC order parameter and the doping concentration is linear

$$\langle n_1^2 + n_5^2 \rangle = \sin^2 \theta^{\text{cl}} = \frac{x}{4\sqrt{g\chi/2}}. \quad (5.110)$$

Before attempting to compare these results to experiments, we must remember that this is merely the classical approximation.

⁴ These results differ somewhat from the phase boundaries in [10].

5.3.4 Vortex Dynamics in Superfluids

We henceforth restrict ourselves to two dimensions. The quantum $-x - xz$ Hamiltonian (5.30) for spin 1/2 can be written in terms of Holstein–Primakoff bosons defined as

$$S^z = a^\dagger a - \frac{1}{2}, \quad (5.111)$$

$$S^- = \sqrt{1 - a^\dagger a} a, \quad (5.112)$$

$$S^+ = a^\dagger \sqrt{1 - a^\dagger a}, \quad (5.113)$$

yielding

$$\begin{aligned} \mathcal{H}^{-x-xz} = & \frac{1}{2} \sum_{ij} J_{ij}^z \left(\frac{1}{2} - a_i^\dagger a_i \right) \left(\frac{1}{2} - a_j^\dagger a_j \right) \\ & - J_{ij}^x [a_i^\dagger a_j \sqrt{(1 - a_i^\dagger a_i)(1 - a_j^\dagger a_j)} + \text{h.c.}] - h \sum_i \left(\frac{1}{2} - a_i^\dagger a_i \right). \end{aligned} \quad (5.114)$$

The partition function can be written as a Bose coherent state path integral

$$Z = \int \mathcal{D}^2 z \exp \left[-i \int d\tau \left(\sum_i z_i^* \partial_\tau z_i - H^{-x-xz}[z_i^*(\tau), z_i(\tau)] \right) \right], \quad (5.115)$$

where $\mathcal{D}z_i$ is an integration over the complex plane. Taking the continuum limit, $z_i \rightarrow \phi(x_i)$, we can write

$$Z = \int \mathcal{D}^2 \phi e^{-S[\phi]}, \quad (5.116)$$

where

$$S = \int d\tau d^d x \left[\phi^* i \partial_\tau \phi + \frac{1}{2} a |\nabla \phi|^2 + V(|\phi|) \right], \quad (5.117)$$

is the time dependent Ginzburg–Landau action with

$$V(\phi, \phi^*) = z J^z a^2 \left(\frac{1}{2} - |\phi|^2 \right)^2 - z J^x a^2 |\phi|^2 (1 - |\phi|^2) + \mu |\phi|^2. \quad (5.118)$$

The classical equation of motion is given by analytically continuing $\tau \rightarrow it$ and finding the saddle point

$$\frac{\delta S}{\delta \phi^*}(\phi) = 0 \rightarrow i \partial_t \phi = a \nabla^2 \phi + \frac{\delta V}{\delta \phi^*}(\phi, \phi^*), \quad (5.119)$$

which is known as the Gross–Pitaevski or Non Linear Schrödinger equation, whose solutions $\phi(x, t)$ describe collective modes (phase fluctuations), and dynamics of vortex configurations.

A different approach to dynamics of superfluids, is to use the quantum spin model (5.114) on a lattice with a lattice spacing a which is smaller than the interparticle spacing, as represented by the spin coherent states path integral (5.42).

$$Z = \int \mathcal{D} \cos \theta_i \mathcal{D} \phi_i \exp \left[\int_0^\beta d\tau \left(i \sum_i (1 - \cos \theta_i(\tau)) \dot{\phi}_i(\tau) - H[\hat{\Omega}_i] \right) \right]. \quad (5.120)$$

A superfluid state is characterized by an ordered state in the xy plane, with a constant two dimensional boson number density $a^2 \rho_s = |\phi|^2$. Using (5.113), the average spin direction is related to ρ_s by

$$\frac{1}{2}(1 - \langle \cos \theta \rangle) = \rho_s a^2. \quad (5.121)$$

A vortex configuration can be parametrized by the azimuthal angles at lattice points i by

$$\phi_i(t) = \arg(x_i - x_V(t)), \quad (5.122)$$

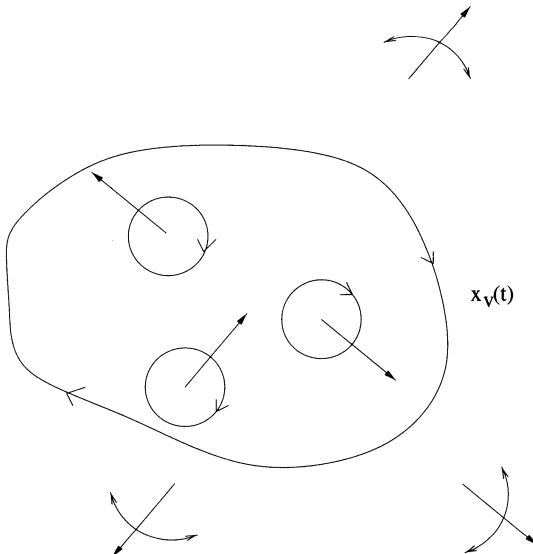


Fig. 5.5. Berry phases due to a moving vortex

where $x_V(t)$ is the vortex core trajectory. As one can see in Fig.5.5, the Berry phase of a vortex path $x_V(t)$ can be written

$$\omega = \rho_s a^2 \sum_{i \in S_c} \oint d\phi_i = 2\pi\rho_s a^2 N_c , \quad (5.123)$$

where the sum is extended to the N_c lattice sites included by the vortex path since the contribution of the others is zero. This Berry phase generates a Magnus force on the moving vortex. This is evident when we write it as an integral over a gauge potential

$$\frac{\omega}{2\pi} = \int_c dx \mathbf{A} \cdot \dot{\mathbf{x}} = \int_c \mathbf{A} \cdot d\mathbf{l} = BS_c , \quad (5.124)$$

where S_c is the area included by the path c , and B is an effective magnetic field (in dimensions of unit flux quantum). Thus, comparing (5.123) and (5.124), one can define B to be simply

$$B \equiv \rho_s \quad (5.125)$$

and the Magnus force acting on the vortex can be written

$$\mathbf{F}_{\text{Magnus}} = B\hat{z} \times \dot{\mathbf{x}}_V = 2\pi\rho_s \hat{z} \times \dot{\mathbf{x}}_V . \quad (5.126)$$

In the absence of any other time derivative terms, the vortex moves like a massless particle in a strong magnetic field restricted to the lowest Landau level. The semiclassical momentum \mathbf{p} of a vortex configuration can be evaluated by computing the expectation value of the translation operator $T_{\mathbf{a}}$

$$\begin{aligned} e^{i\mathbf{a} \cdot \mathbf{P}} &\equiv \langle \hat{\Omega} | T_{\mathbf{a}} | \hat{\Omega} \rangle = \langle \hat{\Omega}(\mathbf{x}) | \hat{\Omega}(\mathbf{x} + \mathbf{a}) \rangle \\ &\simeq \exp \left[iS \sum_i (1 - \cos \theta_i) (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_i + \mathbf{a})) \right] \\ &\simeq \exp \left(i \int d^2x \rho_s \nabla \phi \cdot \mathbf{a} \right) = e^{i2\pi\rho_s \mathbf{a} \cdot \hat{z} \times \mathbf{x}_V} . \end{aligned} \quad (5.127)$$

Similarly, the momentum of a vortex-antivortex pair at positions $\mathbf{x}_V, \mathbf{x}_{\bar{V}}$ can be computed

$$\mathbf{P} = 2\pi\rho_s \hat{z} \times (\mathbf{x}_V - \mathbf{x}_{\bar{V}}) . \quad (5.128)$$

Since on a lattice the only distinguishable momenta are within the first Brillouin zone, (5.128) implies that vortex-antivortex pair configurations can tunnel between different separations which belong to the discrete family

$$\mathbf{x}'_V - \mathbf{x}'_{\bar{V}} = -(2\pi\rho_s)^{-1} (\mathbf{P} + \mathbf{G}) \times \hat{z} , \quad (5.129)$$

where \mathbf{G} is any reciprocal lattice vector. This is precisely an umklapp scattering of the superfluid current by the lattice. This amounts to quantum dissipation of the supercurrent due to continuous translation symmetry breaking of a lattice potential [11].

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6 Introduction to Some Common Topics in Gauge Theory and Spin Systems

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6.1 Overview

The theme of this chapter¹ is an introduction to some ideas and techniques of gauge field theory which have some overlap with similar structures in the statistical physics of spin systems. The emphasis is on strong coupling dynamics and the development of non-perturbative techniques.

The text is roughly divided into two parts. First is a pedagogical introduction to the most basic aspects of gauge theories. The second part explores the analogies between the strong coupling expansion of lattice gauge theory and quantum spin systems.

This is not intended to be an exhaustive review of all results, techniques and applications of gauge theory, even in the context of condensed matter physics applications. Our main aim is pedagogy and a summary of results on the strong coupling limit. Also, the citations are a reflection of our own taste and experience and are not intended to be complete. We take this opportunity to apologize to all of those authors whose relevant works we have not managed to cite.

Gauge field theories based on non-Abelian gauge groups were primarily developed in elementary particle physics to explain the interactions of elementary particles. They do, however, play an increasingly important role in condensed matter physics. The present lectures review a subset of all applications of gauge theory to condensed matter - their appearance as constraint fields in the description of spin systems.

Gauge theories have played a major role in elementary particle physics where they are the main component of the standard model. That model has, through many rigorous tests over the past twenty five years, become the commonly accepted model for the known elementary particles and their interactions. Of course, a part of the standard model is quantum electrodynamics, the interaction which is ultimately responsible for all phenomena in condensed matter physics. It is also responsible for atomic structure where its effects are tested experimentally to such high accuracy that it is presently the most accurate theory in science.

¹ Lecture notes compiled by Riccardo Maciocco and Francesco Paolo Mancini.

The modern theory of fundamental interactions is gauge theory. Examples of gauge theories include

- Einstein's theory of gravitation (1916)
- Weyl's formulation of electromagnetism (1919)
- Yang–Mills fields (1954)
 - electro-weak gauge theory (1967)
 - quantum chromodynamics (1972).

These theories are based on the idea of local gauge invariance, which is implemented in a field theory by introducing a gauge field. The gauge field is then the mediator of an interaction. The elementary particles which mediate interactions are the gravitons, photons, W and Z Bosons and the gluons. These gauge fields have properties which are distinct from those of the matter fields whose interactions they mediate - the quarks and leptons.

Gauge theories feature four dynamical scenarios which typify their behaviour.

- Most familiar is the behaviour of quantum electrodynamics which features a Coulomb interaction between charged particles, the Faraday, Gauss and Ampere laws of electromagnetism. This is known as the Coulomb phase of the gauge theory. It is characterised by the infinite range of the Coulomb interaction between charged particles². The potential energy which is stored in the Coulomb interaction between particles with charges e_1 and e_2 is

$$V(\mathbf{r}) = \frac{e_1 e_2}{4\pi |\mathbf{r}|}. \quad (6.1)$$

- Second is spontaneous symmetry breaking and the Higgs mechanism. This mechanism actually first became known in condensed matter physics in the context of super conductivity theory. In particle physics, the Higgs mechanism is an integral component of the standard model as it explains how the gauge bosons which mediate the weak interactions get their masses, as well as why the $SU(2)$ symmetry of the electro-weak model is not seen in the elementary particle spectrum. On the theoretical level it is a well understood non-perturbative but weak coupling phenomenon. It is characterised by short-ranged gauge interactions, with L called the coherence length. It describes electro magnetic interactions in the super conducting phase.
- The third feature is confinement. The strong interactions are described by a non-Abelian gauge theory which is confining. Confinement is an observed property of the strong interactions and is an unproven, but widely believed feature of most non-Abelian gauge theories in four and lower spacetime dimensions. The idea is that the gauge interaction is so strong that charged particles are permanently confined into neutral bound

² When a force law behaves as $\mathbf{F}(\mathbf{r}) \sim -\nabla e^{-|\mathbf{r}|/L}/|\mathbf{r}|$ it is said to have range L .

states. In the gauge theory of the strong interactions, QCD, quarks and other particles which carry colour charges are not seen in the asymptotic states as real particles but are permanently confined into colour-neutral bound states. The theoretical mechanism behind confinement is notoriously difficult to analyse. It is intrinsically a strong coupling phenomenon. In fact it requires interactions strong enough to permanently confine particles in bound states - so that the binding energy is actually infinite. The potential energy of a pair of charged particles rises linearly with their separation,

$$V(\mathbf{r}) = \sigma |\mathbf{r}| \quad (6.2)$$

where σ is a parameter with the dimension of momentum squared, usually called the string tension.

- Closely related to confinement is dynamical chiral symmetry breaking. This is the mechanism which creates the π -mesons and which keeps their masses light. Since the up and down quarks are light but not quite massless, chiral symmetry is actually only an approximate symmetry of particle physics. Otherwise the pion which is the Goldstone boson arising from the symmetry breaking would be strictly massless, rather than just being light. The theory behind chiral symmetry breaking is only partially understood. This is a result of the fact that it is a phenomenon which occurs at strong coupling where the standard tools of perturbation theory are not reliable. In the following Sections we shall find analogs in spin systems of strong interaction behaviour, particularly chiral symmetry breaking. There are conjectures that if one could adjust the charge of the electron in ordinary electrodynamics, there would be some critical value of the fine structure constant $\alpha = e^2/4\pi$ where there would be a chiral phase transition and the electron would become very massive. This transition would break the approximate chiral symmetry of quantum electrodynamics, which is there because the electron is very light. It is this behaviour that we shall find analogs of in quantum spin systems where the appearance of anti ferromagnetic order will be the analog of chiral symmetry breaking.

6.2 What Is a Gauge Theory?

In this Section we shall present a very elementary discussion of gauge theory, including Yang–Mills theory. This is intended as a brief introduction to these concepts for the uninitiated. The more sophisticated reader should skip to the following Sections. This Section is meant to introduce a minimal framework of ideas which are needed to understand the sections that follow. For more details on electrodynamics, classical field theories and gauge field theories, the reader is referred to the textbooks by J.D. Jackson [22] and I. Aitchison and D. Hey [1].

6.2.1 Electrodynamics

Historically, the first gauge theory is Maxwell's electrodynamics. It was formulated as a physical theory by Maxwell [36] and its gauge invariance was elevated to the role of a physical principle by Weyl [56]. It is actually a special example of gauge theory with the gauge group $U(1)$. Let us begin by reminding the reader of the relativistic form of Maxwell's equations. There are two kinds of fields, the electric field, $\mathbf{E}(\mathbf{x}, t)$ which is a vector field and the magnetic field $\mathbf{B}(\mathbf{x}, t)$ which is a pseudo-vector. For a given distribution of electric charges, $\rho(\mathbf{x}, t)$, and currents, $\mathbf{J}(\mathbf{x}, t)$, Maxwell's equations are³

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} - \frac{\partial}{\partial t} \mathbf{E} &= \mathbf{J}.\end{aligned}\tag{6.3}$$

These are differential equations which, given boundary conditions which are usually dictated by the physical situation, can be used to solve for the electric and magnetic fields once the charge and current distributions are known. They are an internally consistent set of equations only when the charge and current densities obey a continuity equation,

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0.\tag{6.4}$$

In order to write these equations in relativistic form, it is first necessary to notice that the electric and magnetic fields can be written as components of an antisymmetric four-by-four tensor⁴ $F_{\mu\nu}(x, t)$ where $\mu, \nu = 0, 1, 2, 3$ with

³ We use a system of units where the speed of light is set equal to one. In these units, both space and time are measured in distance units, such as centimeters. Later in these lectures, we shall also sometimes use the units where Planck's constant, \hbar is set equal to one. In those units, energies and momenta are measured in inverse distance units, such as inverse centimeters. We also assume that electric charge is measured in a system of units where the electric permittivity and magnetic permeability of free space are equal to one, $\epsilon_0 = 1$ and $\mu_0 = 1$. In this Section we shall follow these conventions. These constants can always be restored in any formulae using dimensional considerations.

⁴ We shall use standard relativistic notation through this paper. The spacetime coordinates are labelled by x^μ where, in D spacetime dimensions, $\mu = 0, 1, \dots, D-1$ with 0 denoting the time coordinate. In the following, we shall have occasion to use spacetime dimensions which are less than four, $D = 2, 3, 4$. (Please note that Maxwell's equations could have a very different form in different dimensions.) We will typically use Greek letters $\mu, \nu, \rho, \sigma, \lambda, \dots$ to denote spacetime indices. In order to refer to the spatial components of a vector, we shall often use Roman indices, for example x^i with $i, j, k, \dots = 1, \dots, D-1$. Also, a spatial vector will

space $i = 1, 2, 3$ and time 0 indices:

$$E_i = F^{0i}, \quad B_1 = F_{23}, \quad B_2 = F_{31}, \quad B_3 = F_{12} \quad (6.5)$$

or

$$F^{\mu\nu}(x) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \quad (6.6)$$

The charge density and current density can also be combined to form a four-vector with space-time components

$$j^\mu(\mathbf{x}, t) = (\rho(\mathbf{x}, t), \mathbf{j}(\mathbf{x}, t)). \quad (6.7)$$

Using the four-current and the field strength tensor, Maxwell's equations can be presented in relativistically covariant form

$$\begin{aligned} \partial^\nu F_{\mu\nu}(x) &= j_\mu(x) \\ \partial_\mu F_{\nu\lambda}(x) + \partial_\nu F_{\lambda\mu}(x) + \partial_\lambda F_{\mu\nu}(x) &= 0. \end{aligned} \quad (6.8)$$

Since $F_{\mu\nu}$ is a symmetric tensor, the first of these equations is consistent if the current is conserved,

$$\partial_\mu j^\mu = 0. \quad (6.9)$$

The second of the relativistic Maxwell equations (called the Bianchi identity) is integrated by introducing the four-vector potential A_μ ,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \quad (6.10)$$

sometimes be denoted by \mathbf{x} . The spacetime index on a vector is raised and lowered using the Minkowski metric for the appropriate spacetime dimension. For example, in four spacetime dimensions, the metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}$$

so that the scalar product of two four-vectors is a Lorentz invariant quantity,

$$u^\mu v_\mu = g^{\mu\nu} u_\mu v_\nu = u_0 v_0 - \mathbf{u} \cdot \mathbf{v}.$$

Here, and in the bulk of this paper, we use the Einstein summation convention, where repeated indices are assumed to be summed over. In other dimensions, we adjust the number of diagonal entries “-1” accordingly. For more details on this relativistic notation, see the textbook by J.D. Jackson [22].

The Maxwell equation with sources is then given by

$$\partial^2 A_\mu(x) - \partial_\mu \partial^\nu A_\nu(x) = j_\mu(x). \quad (6.11)$$

The introduction of the four-vector potential is accompanied by gauge invariance. If we make the substitution

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \Lambda(x), \quad (6.12)$$

called a gauge transformation, the field strength tensor is unchanged. This means that if we find a four-vector field $A_\mu(x)$ which satisfies Maxwell's equations, there is a family of solutions of the equations whose members are related by gauge transformations.

A representative of the set of all gauge equivalent field configurations can be chosen by imposing a gauge condition. For example, for any field configuration A_μ , we can always find another configuration which obeys the covariant gauge condition

$$\partial_\mu A^\mu(x) = 0. \quad (6.13)$$

In this covariant gauge, the field equation has a particularly simple form,

$$\partial^2 A_\mu = j_\mu, \quad (6.14)$$

which can be integrated by introducing the appropriate Green function for the relativistic wave operator.

Maxwell's equations can be derived from an action using a variational principle if, first of all, we define the field strength tensor as the one derived from the vector potential as in (6.10) and we take the action as

$$S = - \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right), \quad (6.15)$$

where we treat A_μ as the fundamental field. This action exhibits the symmetries of electrodynamics - explicitly. Lorentz invariance and gauge invariance (only when the current is conserved).

Exercises:

- A Lorentz transformation is a linear transformation, of the coordinates

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu, \quad (6.16)$$

which preserves the proper inner product,

$$x^\mu g_{\mu\nu} y^\nu = x^0 y^0 - \mathbf{x} \cdot \mathbf{y}, \quad (6.17)$$

that is,

$$\Lambda^\mu_\rho \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}. \quad (6.18)$$

Show that the set of all matrices with this property, together with ordinary matrix multiplication forms a group. (That is, the set is closed under multiplication, contains an identity, each element has an inverse which is also in the set and the product is associative.) This group is called the Lorentz group.

- Show that, if we consider a Lorentz transformation matrix Λ^μ_ν , as defined in the previous problem, which differs infinitesimally from the identity,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \quad (6.19)$$

then $\omega_{\mu\nu} \equiv g_{\mu\rho}\omega^\rho_\nu$ must be an anti-symmetric matrix. Show that given two Lorentz transformations implemented by Λ and $\tilde{\Lambda}$ the difference $\hat{\Lambda} = \Lambda\tilde{\Lambda}\Lambda^{-1}\tilde{\Lambda}^{-1}$ is a Lorentz transformation and if

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \quad \tilde{\Lambda}^\mu_\nu = \delta^\mu_\nu + \tilde{\omega}^\mu_\nu \quad (6.20)$$

then

$$\hat{\Lambda}^\mu_\nu = \delta^\mu_\nu + [\omega, \tilde{\omega}]^\mu_\nu. \quad (6.21)$$

This means that $[\omega, \tilde{\omega}]^\mu_\nu$ must be another Lorentz transformation. A convenient basis for real traceless matrices is

$$(T^{\alpha\beta})_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu,$$

where the pair of indices $(\alpha\beta)$ also labels the generator. The structure constants $f_{\epsilon\eta}^{\alpha\beta,\gamma\delta}$ are defined by

$$[T^{\alpha\beta}, T^{\gamma\delta}] = f_{\epsilon\eta}^{\alpha\beta,\gamma\delta} T^{\epsilon\eta}.$$

Find an explicit expression for the structure constants.

- As well as the electro magnetic field tensor which we have discussed in the above Section, one can also form the dual tensor

$$G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho} \quad (6.22)$$

or, explicitly,

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & -E_2 & -E_1 & 0 \end{pmatrix}. \quad (6.23)$$

Note that this tensor is formed by putting $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$, the so-called electric-magnetic duality operation. Using the electro magnetic and dual tensors, we can form the following invariants, $F^{\mu\nu}F_{\mu\nu}$, $G^{\mu\nu}G_{\mu\nu}$ and $F^{\mu\nu}G_{\mu\nu}$. Compute these invariants in terms of the spatial vectors \mathbf{E} and \mathbf{B} .

- Maxwell's equations can be derived using a variational principle beginning with the action

$$S = - \int d^D x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right) \quad (6.24)$$

if one takes $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and varies with respect to A_μ . Use the generalization of this procedure to derive Maxwell's equations in two, three and five spacetime dimensions. Using the definitions $E^i \equiv F^{0i}$ and $B^{ij} \equiv F^{ij}$ ⁵ derive Maxwell's equations for \mathbf{E} and B^{ij} in D=2,3,4 and 5 spacetime dimensions.

- Consider an infinitesimal transformation of the coordinates, $x^\mu \rightarrow \tilde{x}^\mu(x)$ a tensor field transforms as

$$t'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^\mu} \frac{\partial x^{\sigma'}}{\partial x^\nu} t_{\rho\sigma}(x) .$$

If the transformation is infinitesimal, $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + f^\mu(x)$, show that the tensor resulting infinitesimal transformation of the tensor field is by its Lie derivative with respect to the vector field $f^\mu(x)$,

$$\delta t_{\mu\nu}(x) = -f^\lambda(x) \partial_\lambda t_{\mu\nu}(x) - \partial_\mu f^\lambda(x) t_{\lambda\nu}(x) - \partial_\nu f^\lambda(x) t_{\mu\lambda}(x) . \quad (6.25)$$

In particular, the spacetime metric tensor $g_{\mu\nu}(x)$ should transform in this way. If a spacetime geometry has symmetries, there are some coordinate transformations for which $\delta g_{\mu\nu}(x) = 0$. An infinitesimal Lorentz transformation is a coordinate transformation under which the metric of Minkowski space is invariant. Use the transformation law for tensors given above to show that an infinitesimal Lorentz transformation is a coordinate transform when the vector function $f_\mu(x) \equiv g_{\mu\nu} f^\nu(x)$ obeys the Killing equation,

$$\partial_\mu f_\nu(x) + \partial_\nu f_\mu(x) = 0 . \quad (6.26)$$

Show that the only solutions of this equations correspond to infinitesimal translations and Lorentz transformations, $x^\mu \rightarrow x^\mu + a^\mu + \omega^\mu_\nu x^\nu$ where $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

- Under an infinitesimal general coordinate transformation, a vector field transforms by its Lie derivative,

$$\delta A_\mu(x) = -f^\nu \partial_\nu A_\mu - \partial_\mu f^\nu A_\nu . \quad (6.27)$$

If A_μ is the vector potential, this transformation is not gauge invariant. Show that we can augment this coordinate transformation with a gauge

⁵ The magnetic field is a spatial vector only in four spacetime dimensions. In other dimensions it must be treated as an antisymmetric tensor.

transformation, $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, where Λ depends on A_μ in such a way that the total combined transformation

$$\delta A_\mu = -f^\nu F_{\nu\mu} \quad (6.28)$$

is gauge invariant.

- Show that, if $\delta A_\mu = -f^\nu F_{\nu\mu}$, the change of the Lagrangian density under this transformation can be written (without aid of the equations of motion) in the form of the spacetime divergence of a vector field,

$$\delta \mathcal{L} = \partial_\mu K^\mu , \quad (6.29)$$

where

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (6.30)$$

and f^μ obeys the Killing equation

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 0 . \quad (6.31)$$

Show that, in four dimensions only, it is actually sufficient for f^μ to satisfy the conformal Killing equation

$$\partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{D} g_{\mu\nu} \partial_\lambda f^\lambda = 0 . \quad (6.32)$$

Find solutions of this equation. Show that they correspond to infinitesimal translations and Lorentz transformations which were identified in a previous problem as well as conformal transformations. There are two special conformal transformations, the dilatation

$$f^\mu = \sigma x^\mu , \quad (6.33)$$

and the special conformal transformation,

$$f^\mu = 2\lambda_\nu x^\nu x^\mu - \lambda^\mu x^\nu x_\nu . \quad (6.34)$$

The finite transformations corresponding to these are

$$x^\mu \rightarrow e^\sigma x^\mu , \quad x^\mu \rightarrow \frac{x^\mu + \lambda^\mu x^\nu x_\nu}{1 + 2\lambda_\rho x^\rho + \lambda^\kappa \lambda_\kappa x^\sigma x_\sigma} , \quad (6.35)$$

respectively. The conformal symmetry of the Maxwell theory in four dimensions arises from the fact that it contains no dimensional parameters, so there is no parameter to fix the scaling dimension of the fields. Then, any solution of Maxwell's equations can be transformed to another solution by a conformal transformation. This symmetry is usually broken by coupling to currents which are generally not conformally invariant. It is also broken when the theory is quantized because of the necessity of introducing an ultraviolet cutoff in order to properly define the theory.

- In 2+1-dimensions, it is possible to add the Chern–Simons term

$$\frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \quad (6.36)$$

to the Lagrangian density. This term is not invariant under gauge transformations, but transforms by a total derivative. The resulting equations of motion, on the other hand, are gauge invariant. Find Maxwell’s equation when the usual Maxwell action, generalized to 3 spacetime dimensions, is modified by adding the Chern–Simons term. Solve the resulting equations with vanishing sources and show that the physical excitations propagate like massive fields, where the mass is related to the parameter m [13].

- Show that, under an arbitrary coordinate transformation, the Chern–Simons term defined in the above exercise varies like the divergence of a three-vector. This means that, if we formulated a three dimensional gauge theory where the Chern–Simons term were the only term in the action, the theory would be invariant under arbitrary coordinate transformations. This is a feature of a topological field theory.

6.2.2 Geometry

Gauge fields have a natural interpretation in terms of the geometry of fiber bundles. Before we go on to consider more complicated gauge groups, let us use the example of electro magnetic $U(1)$ gauge theory to give an intuitive illustration of this geometrical structure. Gauge fields take their natural form in the geometry of a vector bundle.

The goal of the construction is to formulate quantum mechanics in such a way that we can adjust the phase of the wave function independently at any point of space and at any time, as

$$\psi(\mathbf{x}, t) \rightarrow e^{i\chi(\mathbf{x}, t)} \psi(\mathbf{x}, t). \quad (6.37)$$

This, together with some other consistency requirements, like locality of equations of motion will lead naturally to the electro magnetic gauge theory.

That the symmetry (6.37) is a reasonable thing to require in the first place is a result of the fact that the physical information that is carried by wave functions is unchanged. For example, the probability density

$$\rho(\mathbf{x}, t) d^3x = \psi^\dagger(\mathbf{x}, t) \psi(\mathbf{x}, t) d^3x, \quad (6.38)$$

which measures the probability that, at time t , a particle is in the volume d^3x , centered at point \mathbf{x} , is invariant under the phase redefinition. If we have two wave-functions, the overlap density

$$\psi_1^\dagger(\mathbf{x}, t) \psi_2(\mathbf{x}, t) d^3x \quad (6.39)$$

is invariant if both wave-functions are transformed by the same phase function.

However, the Schrödinger equation is not invariant under (6.37). In order to make it invariant, it is necessary to find a meaningful definition of derivatives of a wave-function which incorporates the symmetry (6.37). To do this, we need to have a sense in which wave-functions at neighbouring points of space would be considered “close” to each other, i.e. we must define the sense in which the wave-function at a neighbouring point, $(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$ is close to the wave-function at point (\mathbf{x}, t) . One way to frame the question is to ask what one would mean by a constant function from this point of view.

Since the modulus of the wave-function is unaffected by the symmetry that we want to impose, it is clear that a constant function should have constant modulus,

$$|\psi(\mathbf{x} + \delta\mathbf{x}, t + \delta t)| = |\psi(\mathbf{x}, t)| . \quad (6.40)$$

However, for the phase of the wave-function, we would like to preserve the notion that the phase is re-definable at each point of the space and time. This means that we have to choose the definition of constant function in a way that is also re-definable.

The only way to do this is to introduce an additional degree of freedom, a vector field $A_\mu(\mathbf{x}, t)$, so that ψ is a constant function if its modulus is constant and if

$$\arg \psi(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = \arg \psi(\mathbf{x}, t) + \mathbf{A}(\mathbf{x}, t) \cdot \delta\mathbf{x} + A_0(\mathbf{x}, t)\delta t . \quad (6.41)$$

This introduces the vector field $A_\mu(\mathbf{x}, t)$ which, given a wave-function $\psi(\mathbf{x}, t)$, tells us which wave-function at a neighbouring point $(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$ is the closest one. The 4-vector A_μ should have some specified value for all (\mathbf{x}, t) , that is, it should be a vector field.

When the phase of the wave-function is re-defined as in (6.37), the relationship in (6.41) should remain unmodified. However,

$$\begin{aligned} \arg \psi(\mathbf{x} + \delta\mathbf{x}, t + \delta t) &= \arg \psi(\mathbf{x}, t) + \mathbf{A}(\mathbf{x}, t) \cdot \delta\mathbf{x} + A_0(\mathbf{x}, t)dt + \chi(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - \chi(\mathbf{x}, t) \\ &= \arg \psi(\mathbf{x}, t) + \mathbf{A}(\mathbf{x}, t) \cdot \delta\mathbf{x} + A_0(\mathbf{x}, t)dt + \delta\mathbf{x} \cdot \nabla \chi(\mathbf{x}, t) + \delta t \frac{\partial}{\partial t} \chi(\mathbf{x}, t) . \end{aligned} \quad (6.42)$$

This equation would be un-changed if the vector field A_μ is also replaced by the transformed vector field

$$\mathbf{A}(\mathbf{x}, t) \rightarrow \mathbf{A}(\mathbf{x}, t) - \nabla \chi(\mathbf{x}, t) \quad A_0(\mathbf{x}, t) \rightarrow A_0(\mathbf{x}, t) - \frac{\partial}{\partial t} \chi(\mathbf{x}, t) . \quad (6.43)$$

This combination of the phase transformation of the wave-function (6.37) and the transformation of the vector field (6.43) is called the *gauge transformation*. $A_\mu(\mathbf{x}, t)$ is called a *gauge field*.

Now that we have the gauge field, $A_\mu(\mathbf{x}, t)$, and an invariant way to compare functions as neighbouring spacetime points, we can define derivative. That is done by taking the difference of the function at two neighbouring points, and including the difference of phases, so that if the neighbouring functions happened to be the “closest” possible, then the derivative would vanish:

$$\begin{aligned} dx^\mu D_\mu \psi(\mathbf{x}, t) &= \lim_{|\delta\mathbf{x}|, \delta t \rightarrow 0} \psi(\mathbf{x}^\mu + dx^\mu) - e^{i dx^\mu A_\mu(\mathbf{x}, t)} \psi(\mathbf{x}, t) \\ &= dx^\mu \left(\frac{\partial}{\partial x^\mu} + i A_\mu(\mathbf{x}) \right) \psi(\mathbf{x}) . \end{aligned} \quad (6.44)$$

This derivative transforms co-variantly under the gauge transform,

$$D_\mu \psi(\mathbf{x}) \rightarrow e^{i \chi(\mathbf{x})} D_\mu \chi(\mathbf{x}) \quad (6.45)$$

It can be used to formulate a gauge invariant Schrödinger equation, and other gauge invariant objects like the conserved probability current.

In a dynamical theory, say the quantum mechanics which describes the motion of a particle, the wave-function is the solution of a differential equation, the Schrödinger equation. If this theory is formulated in a gauge invariant way, it is necessary to specify the vector field $A_\mu(x)$. It can be treated simply as an external field in which the particle moves. On the other hand, to formulate a complete theory, it may be desirable to find a dynamical law which also determines the vector field $A_\mu(x)$. This dynamical law could be Maxwell’s electrodynamics where, for example, the current is given by the probability current for the particle.

Now that we have formulated the concept of derivative, there is a final geometrical question we can ask, whether there exists such a thing as a constant function. A wave-function which is a constant would have the property

$$D_\mu \psi(\mathbf{x}) = 0 . \quad (6.46)$$

A condition for the existence of such a function is that the commutator of covariant derivatives should vanish:

$$[D_\mu, D_\nu] \psi(\mathbf{x}) = -i F_{\mu\nu}(\mathbf{x}) \psi(\mathbf{x}) = 0 , \quad (6.47)$$

where the curvature tensor is

$$F_{\mu\nu}(\mathbf{x}) = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (6.48)$$

which is the same quantity as the electro magnetic field tensor. Geometrically it is interpreted as the obstruction to finding constant wave-functions.

Exercise:

- Show that the set of all changes of phase of the wave-function, together with the natural composition law, $e^{i\chi} \odot e^{iA} = e^{i(\chi+A)}$ forms a group. This group is called $U(1)$. When gauge theory is based on this local invariance group, it is called the gauge group. Since $U(1)$ is a commutative (Abelian) group, gauge theory based on a $U(1)$ gauge group is called Abelian gauge theory.

6.2.3 Schrödinger Equation for a Charged Particle

Consider the quantum mechanics of a non-relativistic charged particle. The Schrödinger equation for a free particle is given by

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t). \quad (6.49)$$

The probability density obeys a continuity equation

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = 0, \quad (6.50)$$

where

$$\rho(\mathbf{x}, t) = \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) \quad (6.51)$$

$$\mathbf{j}(\mathbf{x}, t) = i\frac{\hbar}{2m} [\psi^*(\mathbf{x}, t)\nabla\psi(\mathbf{x}, t) - \nabla\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)], \quad (6.52)$$

and we define the probability current as

$$J^\mu(\mathbf{x}, t) = (\rho(\mathbf{x}, t), \mathbf{j}(\mathbf{x}, t)). \quad (6.53)$$

The conservation law for the probability current follows from the Schrödinger equation. Using the Noether procedure, it can also be seen to be a consequence of the invariance of the action,

$$S = \int dt d^3x \psi^* \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) \psi. \quad (6.54)$$

The Schrödinger equation can be derived from this action using the variational principle. Consider

$$\psi(\mathbf{x}, t) \rightarrow \psi(\mathbf{x}, t) + \delta\psi(\mathbf{x}, t). \quad (6.55)$$

Then, the equation of motion is satisfied if the linear variation of the action vanishes,

$$\begin{aligned} \delta S &= \int dt d^3x \left(\delta\psi^* \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) \psi \right. \\ &\quad \left. + \delta\psi \left(-i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) \psi^* \right) = 0. \end{aligned} \quad (6.56)$$

The coefficient of $\delta\psi^*$ must vanish, resulting in the Schrödinger equation (6.49).

Now, consider an infinitesimal transformation of the phase of the wave-function with a parameter which depends on the space-time coordinate,

$$\psi(\mathbf{x}, t) \rightarrow \psi(\mathbf{x}, t) + i\Lambda(\mathbf{x}, t)\psi(\mathbf{x}, t) \quad (6.57)$$

or

$$\delta\psi = i\Lambda\psi, \quad \delta\psi^* = -i\Lambda\psi^*. \quad (6.58)$$

Then, to leading order in $\Lambda(\mathbf{x}, t)$, the action transforms as

$$\delta S = \int dt d^3x \partial_\mu \Lambda j^\mu(\mathbf{x}, t). \quad (6.59)$$

To obtain this equation we are not allowed integrations by parts, or the use of the equation of motion. It encodes the information that the action has a symmetry when Λ is a constant in the fact that the linear variation depends only on the derivatives of Λ . Now, if we also assume that $\Lambda(\mathbf{x}, t)$ is a function which vanishes on the boundaries of the integration region, so that we can integrate by parts in the above expression, and we now assume that the equation of motion is satisfied by the wave-functions, then the linear variation of the action must vanish. This is only possible with Λ of arbitrary profile if $\partial_\mu j^\mu = 0$. This is the content of *Noether's theorem* which relates symmetry to conservation laws. The coupling of the electro magnetic field to the quantum mechanical particle is accomplished by promoting the phase symmetry which we have been discussing to a local symmetry. A change in the phase of the wave function by a spacetime dependent phase would be an exact symmetry of the action, and the Schrödinger equation, if it were not for the terms with derivatives operating on the wave-function. The electro magnetic field, $A_\mu(\mathbf{x}, t)$ is coupled by forming covariant derivatives of the gauge fields,

$$\frac{\partial}{\partial t} - ieA_0(\mathbf{x}, t) \quad (6.60)$$

and

$$\nabla - ie\mathbf{A}(\mathbf{x}, t). \quad (6.61)$$

Then, if we perform a gauge transformation of the gauge field as well as a phase transformation of the wave-function field, the action

$$S = \int dt d^3x \left\{ \psi^* \left(i\hbar \left(\frac{\partial}{\partial t} - ieA_0(\mathbf{x}, t) \right) + \frac{\hbar^2}{2m} (\nabla - ie\mathbf{A}(\mathbf{x}, t))^2 \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}. \quad (6.62)$$

is invariant under gauge transformations. Here, we have added the action of the electro magnetic field. This theory describes the quantum mechanics of a single charged non-relativistic particle coupled to a classical electro magnetic field, which is in turn dynamical in the sense that the Maxwell equations should be solved using the current density of the particle. A fully quantum version of the electro magnetic field follows by making the field a quantum variable. Its coupling to the quantum mechanical particle then is only consistent in the context of second quantization.

Exercises:

- The action in (6.62) has the phase symmetry under $\psi(\mathbf{x}, t) \rightarrow e^{iA} \psi(\mathbf{x}, t)$. Use Noether's method to find the conserved current in this case, which should be gauge invariant versions of $\rho(\mathbf{x}, t)$ and $\mathbf{j}(\mathbf{x}, t)$.
- Use the variational principle to derive the equation of motion for both the gauge field and the wave-function that results from (6.62). Show that conservation of the current derived in the above problem is a direct consequence of these equations of motion.

6.2.4 Bohm–Aharonov Phase

Upon transport on a path, Γ , through spacetime, which we represent by a four-vector function of a curve parameter, τ , $x^\mu(\tau)$, the wave-function of a charged particle accumulates a Bohm–Aharonov phase

$$\psi(\mathbf{x}, t) \rightarrow e^{ie\phi_\Gamma} \psi(\mathbf{x}, t), \quad (6.63)$$

where the phase depends on Γ

$$\phi = \int_0^1 d\tau \frac{dx^\mu(\tau)}{d\tau} A_\mu(x(\tau)). \quad (6.64)$$

This is actually a semi-classical statement, generally true only for the slow motion of a heavy particle. To see how it works, consider the ansatz for the wave-function,

$$\psi(\mathbf{x}, t) = \exp\left(-\frac{i}{\hbar} s(\mathbf{x}, t)\right), \quad (6.65)$$

where $s(\mathbf{x}, t)$ is generally a complex function. The Schrödinger equation has the form

$$\begin{aligned} \left(\frac{\partial s(\mathbf{x}, t)}{\partial t} + eA_0(\mathbf{x}, t) \right) &= \frac{1}{2m} (\nabla s(\mathbf{x}, t) + e\mathbf{A}(\mathbf{x}, t))^2 \\ &+ \frac{i\hbar}{2m} \nabla \cdot (\nabla s(\mathbf{x}, t) + e\mathbf{A}(\mathbf{x}, t)) . \end{aligned} \quad (6.66)$$

If we put \hbar to zero, there is generally no solution of this equation. However, if there is a region of space where $F_{\mu\nu} = 0$ (note that this does not imply that $A_\mu = 0$ there) then, by choosing a path Γ through that region, this equation is solved by letting

$$s(x) = -e \int_{\Gamma}^x dy^\mu A_\mu(y) . \quad (6.67)$$

This is the Bohm–Aharonov phase.

Exercises:

- The Bohm–Aharonov phase for a spacetime path $x^\mu(\tau)$ when the electro magnetic field is a constant, $A_\mu = -F_{\mu\nu}/2$ is given by

$$-\frac{e}{\hbar} \int d\tau F_{\mu\nu} \dot{x}^\mu(\tau) x^\nu(\tau) . \quad (6.68)$$

Find an equation for the paths which minimize this phase. Solve this equation for the case that the constant field strength is either purely electric or purely magnetic.

- Consider the action for a relativistic particle coupled to an electro magnetic field,

$$S = -m \int d\tau \sqrt{\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}} D + e \int d\tau \frac{dx^\mu}{d\tau} A_\mu(x(\tau)) . \quad (6.69)$$

Use the variational principle to derive the relativistic form of the Lorentz force law

$$\frac{d}{d\tau} (P^\mu) = e F^{\mu\nu}(x(\tau)) \dot{x}_\nu \quad (6.70)$$

where

$$P^\mu = m \frac{dx^\mu/d\tau}{\sqrt{\frac{dx^\nu}{d\tau} \frac{dx_\nu}{d\tau}}} \quad (6.71)$$

is the 4-momentum.

6.2.5 Classical Yang–Mills Theory

So far, we have discussed the subject of Abelian gauge theories. Non-Abelian gauge theory was originally formulated by Yang and Mills [60] as a generalization of Abelian gauge theories to the situation where the quantum mechanical

wave-function has many components which, for example, can be assembled into a vector

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix} \quad (6.72)$$

the gauge invariance which is required is generalized from multiplying the wave-function by a space-time dependent phase to multiplying it by a unitary matrix,

$$\psi_\alpha(x) \rightarrow \sum_{\beta=1}^N U_{\alpha\beta}(x) \psi_\beta(x), \quad (6.73)$$

where $\psi_\alpha(x)$ denote the components of $\psi(x)$. Here, $U(x)$ is an $N \times N$ unitary matrix, i.e. one with the property that

$$U(x)U^\dagger(x) = 1 = U^\dagger(x)U(x). \quad (6.74)$$

This gauge transformation has the property that it leaves the probability measure

$$\sum_{\alpha=1}^N \psi_\alpha^\dagger(x) \psi_\alpha(x) d^3x \quad (6.75)$$

invariant. Note that the summation over indices is an important part of this expression. If we are to have a gauge invariant quantum mechanics, the different individual wave functions are really indistinguishable and the probability that the quantum system exists in any of the states labelled by the indices of the wave-function is not a sensible quantity. These sub-states are called “colours” and gauge symmetry requires that we discuss colourless objects, that is objects where we have averaged over the colours in a way that the result is invariant under gauge transformations.

As in the case of Abelian gauge theory, the definition of derivative requires that we introduce a vector field. However, now this vector field acts on N -component wave function. This requires that we introduce an Hermitean matrix valued vector field $A_\mu^{\alpha\beta}(x)$ such that the covariant derivative is defined as

$$D_\mu^{\alpha\beta} \equiv \delta^{\alpha\beta} \partial_\mu - ie A_\mu^{\alpha\beta}(x). \quad (6.76)$$

The curvature of this gauge field is found from the commutator of two derivatives,

$$[D_\mu, D_\nu] = -i F_{\mu\nu}(x) \quad (6.77)$$

and is given by the expression

$$F_{\mu\nu}^{\alpha\beta}(x) = \partial_\mu A_\nu^{\alpha\beta}(x) - \partial_\nu A_\mu^{\alpha\beta}(x) + i [A_\mu(x), A_\nu(x)]^{\alpha\beta} . \quad (6.78)$$

Then, given the gauge transformation of the wave-function with the unitary matrix $U(x)$, the gauge field should transform as

$$A_\mu(x) \Rightarrow U(x) (A_\mu(x) - i\partial_\mu) U^{-1}(x) . \quad (6.79)$$

This makes the covariant derivative transform co-variantly,

$$\sum_{\beta=1}^N D_\mu^{\alpha\beta} \psi_\beta(x) \Rightarrow \sum_{\beta,\gamma=1}^N U^{\alpha\gamma}(x) D_\mu^{\gamma\beta} \psi_\beta(x) . \quad (6.80)$$

Furthermore, the curvature also transforms co-variantly

$$F_{\mu\nu}(x) \Rightarrow U(x) F_{\mu\nu}(x) U^{-1} . \quad (6.81)$$

With the use of covariant derivatives, we can form a gauge invariant Schrödinger equation,

$$i\hbar D_0 \psi(x) = -\frac{\hbar^2}{2m} \mathbf{D}^2 \psi(x) . \quad (6.82)$$

The gauge field can also be determined by a dynamical law. Gauge invariance of such a law is guaranteed if the equations which determine $A_\mu(x)$ are obtained by a variational principle which begins with a gauge invariant action functional. The simplest such action is the Yang–Mills action,

$$S_{\text{YM}} = - \int d^4x \frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) . \quad (6.83)$$

The variation of this action by the classical field $A_\mu(x)$ yields the equation of motion

$$\partial_\mu F_{\mu\nu}(x) + i [A_\mu(x), F_{\mu\nu}(x)] = 0 \quad (6.84)$$

and, because it is constructed from the gauge field, the curvature tensor also obeys the Bianchi identity

$$\begin{aligned} & \partial_\mu F_{\nu\lambda}(x) + i [A_\mu(x), F_{\nu\lambda}(x)] + \partial_\nu F_{\lambda\mu}(x) + i [A_\nu(x), F_{\lambda\mu}(x)] + \\ & + \partial_\lambda F_{\mu\nu}(x) + i [A_\lambda(x), F_{\mu\nu}(x)] = 0 . \end{aligned} \quad (6.85)$$

These are the generalization of the (source free) Maxwell equations of electrodynamics. Note that these equations are non-linear even in the absence of charged sources. This can be interpreted as the Yang–Mills field carrying the charge that it couples to, usually called colour charge, and thereby interacting with itself. Also, note a very essential difference between these equations and

those of Maxwell's electrodynamics. Maxwell's equations can be formulated entirely in terms of the field strengths, whereas, in the case of Yang–Mills theory, the vector potential field appears explicitly and plays an essential role in the field equations.

Here, we have discussed a Yang–Mills theory based on unitary symmetry. It is possible to consider a more general situation where the wave-function transforms by matrices which are in a unitary representation of some Lie group. Then, the gauge field $A_\mu(x)$ takes values in a unitary representation of a Lie algebra. It is a Hermitian matrix and can be expanded in terms of the Hermitian generators of the algebra,

$$A_\mu(x) = \sum_{a=1}^{\dim g} A_\mu^a(x) T^a, \quad (6.86)$$

where $\dim g$ is the dimension of the group and

$$T^a = T^{a\dagger} \quad (6.87)$$

are the generators of the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c \quad (6.88)$$

with f^{abc} the structure constants. For a compact, semi-simple Lie group, the generators can be normalized so that⁶

$$\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}. \quad (6.89)$$

The components of the vector field A_μ can always be projected out of it by taking a trace,

$$A_\mu^a(x) = 2\text{tr}(A_\mu(x) T^a). \quad (6.90)$$

Also, the field strength tensor can be expanded,

$$F_{\mu\nu}(x) = \sum_{a=1}^{\dim g} T^a F_{\mu\nu}^a \quad (6.91)$$

⁶ For the group $SU(2)$ of two by two unitary matrices with unit determinant, the generators can be taken to be proportional to the Pauli matrices, $T^a = \sigma^a/2$, where $a = 1, 2, 3$. The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and obey the identity

$$\sigma^a \sigma^b = \delta^{ab} + i \epsilon^{abc} \sigma^c$$

from which their algebraic properties can be deduced.

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + f^{abc} A_\mu^b(x) A_\nu^c(x) . \quad (6.92)$$

Exercises:

- Verify that

$$(D_\mu D_\nu - D_\nu D_\mu) \psi(x) = -i F_{\mu\nu}(x) \psi(x) . \quad (6.93)$$

Also, consider an $N \times N$ matrix-valued function $M(x)$. Show that

$$[D_\mu, [D_\nu, M(x)]] - [D_\nu, [D_\mu, M(x)]] = -i [F_{\mu\nu}(x), M(x)] . \quad (6.94)$$

- Suppose that A_μ is changed by an infinitesimal amount, $A_\mu(x) \rightarrow A_\mu(x) + \delta A_\mu(x)$. Show that, to linear order, the variation of $F_{\mu\nu}$ is given by

$$\delta F_{\mu\nu} = [D_\mu, \delta A_\nu] - [D_\nu, \delta A_\mu] . \quad (6.95)$$

- In the problem above, consider the infinitesimal variation of $A_\mu(x)$ where $\delta_\Omega A_\mu(x) = [D_\mu, \Omega(x)]$. Show that

$$\delta_\Omega F_{\mu\nu}(x) = [\Omega(x), F_{\mu\nu}(x)] . \quad (6.96)$$

Use this fact to show that, to linear order

$$\delta_\Omega \text{tr}(F_{\mu\nu} F^{\mu\nu}) = 0 . \quad (6.97)$$

- Show that the Bianchi identity (6.85) can be written as

$$[D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] + [D_\lambda, F_{\mu\nu}] = 0 \quad (6.98)$$

and follows as a similar identity for commutators of covariant derivative operators,

$$[D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\nu]] = 0 . \quad (6.99)$$

- Aside from symmetry under gauge transformations, the Yang–Mills action is symmetric under a global transformation, which could be regarded as a special case of a gauge transform where the transformation matrix is a constant,

$$\delta A_\mu(x) = i [\Omega, A_\mu(x)] . \quad (6.100)$$

Use the Noether method to find the conserved current which is a consequence of this symmetry. Verify explicitly using the equation of motion that this current is indeed conserved.

- Consider the change of A_μ under a space-time transformation,

$$\delta A_\mu(x) = -f^\nu(x)\partial_\nu A_\mu(x) - \partial_\mu f^\nu(x)A_\nu(x). \quad (6.101)$$

Show that this transformation can be augmented by a gauge transformation so that the total change in A_μ is given by

$$\delta A_\mu(x) = -f^\nu(x)F_{\nu\mu}(x). \quad (6.102)$$

Show that this is a symmetry transformation of the Yang–Mills action when f^μ satisfies the conformal Killing equation (6.32). Show that the Noether current corresponding to this symmetry is given by

$$J^\mu = f^\nu T_\nu^\mu, \quad (6.103)$$

where

$$T_{\mu\nu} = \text{tr} \left(F_{\mu\lambda} F_\nu^\lambda + \frac{g_{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma} \right). \quad (6.104)$$

Show that, conservation of the energy momentum tensor, $\partial_\mu T^{\mu\nu}$ follows from translation invariance alone, symmetry of the energy momentum tensor $T^{\mu\nu} = T^{\nu\mu}$ follows from Lorentz invariance and tracelessness of the tensor, $g_{\mu\nu} T^{\nu\mu} = 0$ follows from conformal invariance. Use the equation of motion to verify that the energy-momentum tensor is indeed conserved.

6.3 Relativistic Fermions

In this Section, we will review some of the salient features of quantization of relativistic fermions with emphasis on the information which is needed to study the following sections. This is meant only as a guide to the student. Those in need of a more thorough education on this subject are advised to consult one of the many excellent textbooks on relativistic quantum mechanics or quantum field theory, such as [9], [11], [42] or [35].

In relativistic quantum mechanics, the electron wave-function obeys the Dirac equation

$$(i\gamma^\mu \partial_\mu + m) \psi(\mathbf{x}, t) = 0. \quad (6.105)$$

We will consider this equation in various space-time dimensions, $1+1$, $2+1$ and $3+1$.

The γ -matrices obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (6.106)$$

where $g^{\mu\nu}$ is the inverse of the Minkowski metric in the appropriate spacetime dimension.

The wave-function which obeys the Dirac equation describes the propagation of a relativistic particle. To see that, consider the equation (6.105) and multiply from the left by the conjugate Dirac operator

$$i\gamma^\mu \partial_\mu - m \quad (6.107)$$

to obtain the relativistic wave equation

$$(-\partial^\mu \partial_\mu - m^2) \psi(\mathbf{x}, t) = 0. \quad (6.108)$$

This wave equation is solved by plane waves of the form

$$\psi(\mathbf{x}, t) = e^{iE(\mathbf{k})t - i\mathbf{k} \cdot \mathbf{x}}, \quad (6.109)$$

where the dispersion relation is

$$E(\mathbf{k}) = \pm \sqrt{\mathbf{k}^2 + m^2}. \quad (6.110)$$

Here, m is the mass of the particle. The full Dirac spinor has the form

$$\psi(\mathbf{x}, t) = \psi(\mathbf{k}) e^{iE(\mathbf{k})t - i\mathbf{k} \cdot \mathbf{x}}, \quad (6.111)$$

where

$$(\gamma^0 \sqrt{\mathbf{k}^2 + m^2} - \boldsymbol{\gamma} \cdot \mathbf{k} + m) \psi_>(\mathbf{k}) = 0 \quad (6.112)$$

and

$$(-\gamma^0 \sqrt{\mathbf{k}^2 + m^2} - \boldsymbol{\gamma} \cdot \mathbf{k} + m) \psi_<(\mathbf{k}) = 0. \quad (6.113)$$

There are typically several linearly independent wave-functions corresponding to the same energy. In four spacetime dimensions, for example, there are two, corresponding to the two spin states of the electron or the positron. The wave-functions should also carry this label (we will suppress it here). The Hamiltonian whose eigenvalues are the allowed energies of a relativistic fermion can be deduced by multiplying the Dirac equation from the left by γ^0 and comparing with the Schrödinger equation. The Dirac Hamiltonian is

$$H = i\boldsymbol{\gamma}^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \gamma^0 m. \quad (6.114)$$

When operating on wave-functions of the form $\psi(x) = \psi(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}$ the Dirac Hamiltonian has the form

$$H = \boldsymbol{\alpha} \cdot \mathbf{k} + \beta m, \quad (6.115)$$

where

$$\boldsymbol{\alpha} = -\gamma^0 \boldsymbol{\gamma} \quad (6.116)$$

and

$$\beta = \gamma^0 . \quad (6.117)$$

It is a Hermitian matrix and its eigenvalues are the energies $E(\mathbf{k})$.

To see that the Hamiltonian has a symmetric spectrum, we need to construct a unitary matrix which anti-commutes with it. That matrix is explicitly

$$\Sigma(\mathbf{k}) \equiv \frac{i\boldsymbol{\gamma} \cdot \mathbf{k}}{|\mathbf{k}|} . \quad (6.118)$$

If

$$H(\mathbf{k})\psi_E(\mathbf{k}) = E\psi_E(\mathbf{k}) \quad (6.119)$$

and

$$\psi_E^\dagger(\mathbf{k})\psi_E(\mathbf{k}) = 1 \quad (6.120)$$

then

$$H(\mathbf{k}) \Sigma(\mathbf{k})\psi_E(\mathbf{k}) = -E \Sigma(\mathbf{k})\psi_E(\mathbf{k}) \quad (6.121)$$

and

$$\psi_E^\dagger(\mathbf{k})\Sigma^\dagger(\mathbf{k})\Sigma(\mathbf{k})\psi_E(\mathbf{k}) = 1 . \quad (6.122)$$

This proves that the spectrum of the Dirac Hamiltonian is always symmetric.⁷

Thus, the wave functions describe particles with both positive and negative energy states. The negative energy states pose a problem, that any single particle state is unstable - under the influence of an external perturbation, it could decay to states with negative energy of arbitrarily large magnitude, thereby emitting energy which is arbitrarily large. In the classical theory, this problem is unsurmountable - there is no reasonable classical theory of relativistic spinor-s. In the quantum theory, this problem is solved by Dirac's hole theory which constructs the ground state as that many-body state where all negative energy states are occupied. That this state is stable requires that the fermions obey Fermi statistics and then makes use of the Pauli principle, that only one fermion is allowed to occupy each state. The state with all negative energy states occupied is taken as the ground state of the quantum system. This feature is generic to the Dirac equation in any spacetime dimensions. It is also similar to the ground state of a degenerate Fermi gas in non-relativistic many-body physics where the analog of the zero energy state is the Fermi surface. The necessity of defining the ground state in this

⁷ For continuum states of the Hamiltonian, this statement actually proves that the spectrum has symmetric support on the real line. More work is necessary to prove that the density of states is symmetric.

way leads to the existence of two kinds of excitations, particles, which occur when a positive energy state is occupied, and anti-particles, which exist when a negative energy state is unoccupied. The latter are the analog of particles and holes in the case of the degenerate Fermi gas. Both particle and anti-particle excitations have larger energy than the Dirac ground state. It was this consideration that led Dirac to conjecture the existence of the positron. The complicated ground state is most conveniently defined using second quantization. In that language, we define an operator

$$\psi(\mathbf{x}, t) = \int d\mathbf{k} \left(\psi_>(\mathbf{k}) e^{iE(\mathbf{k})t - i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{k}) + \psi_<(\mathbf{k}) e^{-iE(\mathbf{k})t - i\mathbf{k}\cdot\mathbf{x}} b^\dagger(\mathbf{k}) \right), \quad (6.123)$$

where $a(\mathbf{k}), a^\dagger(\mathbf{k})$ and $b(\mathbf{k}), b^\dagger(\mathbf{k})$ are pairs of creation and annihilation operators which obey the anti-commutation relations

$$\begin{aligned} \{a(\mathbf{k}), a^\dagger(\mathbf{q})\} &= \delta(\mathbf{k} - \mathbf{q}) \\ \{a(\mathbf{k}), a(\mathbf{q})\} &= 0 \\ \{b(\mathbf{k}), b^\dagger(\mathbf{q})\} &= \delta(\mathbf{k} - \mathbf{q}) \\ \{b(\mathbf{k}), b(\mathbf{q})\} &= 0. \end{aligned} \quad (6.124)$$

These anti-commutation relations take care of Fermi statistics. They also imply the anti-commutator,

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta_{ab} \delta(\mathbf{x} - \mathbf{y}). \quad (6.125)$$

The ground state, or vacuum is the state which is annihilated by all operators $a(\mathbf{k}), b(\mathbf{k})$,

$$a(\mathbf{k})|0\rangle = 0. \quad (6.126)$$

$$b(\mathbf{k})|0\rangle = 0. \quad (6.127)$$

Then many particle and anti-particle states are formed by operating creation operators

$$a^\dagger(\mathbf{k}_1) \dots a^\dagger(\mathbf{k}_m) b^\dagger(\mathbf{q}_1) \dots b^\dagger(\mathbf{q}_n) |0\rangle. \quad (6.128)$$

This state is an eigenstate of the second-quantized Dirac Hamiltonian, which can be written as

$$H = \int d\mathbf{x} \psi^\dagger(\mathbf{x}, t) (-i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \gamma^0 m) \psi(\mathbf{x}, t) \quad (6.129)$$

or

$$H = \int d\mathbf{k} \sqrt{\mathbf{k}^2 + m^2} (a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \quad (6.130)$$

with eigenvalue

$$\sqrt{\mathbf{k}_1^2 + m^2} + \dots + \sqrt{\mathbf{q}_n^2 + m^2}. \quad (6.131)$$

All energy eigenvalues are positive. The fermion and anti-fermion excitations thus have identical spectra. These excitations are distinguished by their charges, which are eigenvalues of the operator

$$Q = \int d\mathbf{x} \psi^\dagger(\mathbf{x}, t) \psi(\mathbf{x}, t) = \int d\mathbf{k} (a^\dagger(\mathbf{k}) a(\mathbf{k}) - b^\dagger(\mathbf{k}) b(\mathbf{k})). \quad (6.132)$$

6.3.1 1+1 Dimensions

The minimal representations of the Dirac matrix algebra in 1+1 dimensions is constructed from the Pauli matrices as, for example,

$$\gamma^0 = \sigma^2, \gamma^1 = i\sigma^1. \quad (6.133)$$

With this choice the two dimensional Dirac operator

$$i\gamma^\mu \partial_\mu + m = \begin{pmatrix} m & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & m \end{pmatrix} \quad (6.134)$$

is real. This means that the wave-function can be chosen to be real. In this case the fermion is called a *Majorana spinor*. A complex spinor which obeys the same equation, and can be thought of as a combination of two Majorana spinor-s - its real and imaginary part - is called a *Dirac spinor*. The electro magnetic field cannot be coupled to a Majorana spinor, as the coupling term in the equation, is complex and would lead to an inconsistency.⁸

If the mass of the fermion were zero, then the Dirac equation would have a simpler form,

$$i\gamma^\mu \partial_\mu \psi(x, t) = 0 \text{ or } \begin{pmatrix} 0 & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0, \quad (6.135)$$

which decomposes into two equations,

$$(\partial_0 - \partial_1) \psi_R = 0 \quad (6.136)$$

$$(\partial_0 + \partial_1) \psi_L = 0. \quad (6.137)$$

⁸ A coupling to a gauge field which is purely imaginary would be allowed. This will be possible in the context of Yang–Mills theory, where the gauge fields are matrices, when they are purely imaginary antisymmetric matrices. This occurs, for example, for gauge groups in the adjoint representation. It is not possible for quantum electrodynamics.

The two independent fermion fields described by ψ_L and ψ_R are called *Weyl spinors*. Since the equations are real, the spinors can also be chosen to be real. In this case, they are called *Majorana–Weyl spinors*. Only Weyl spinors can be coupled to electro magnetic fields and the complex field equations which their wave functions obey are

$$(\partial_0 - \partial_1 + ieA_0 - ieA_1) \psi_R = 0 \quad (6.138)$$

$$(\partial_0 + \partial_1 + ieA_0 + ieA_1) \psi_L = 0. \quad (6.139)$$

Again, these are consistent equations only when the wave-functions are complex.

The action from which the Dirac equation can be derived is

$$S_{\text{Dirac}} = \int d^2x \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi. \quad (6.140)$$

If the spinor is complex, this action has a symmetry under changing the phase of the spinor, $\psi(x) \rightarrow e^{i\chi}\psi$ and the current

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad (6.141)$$

is conserved.

In the case where the mass is zero, there is actually a higher symmetry. The action has the form

$$S_D = \int d^2x \left(\psi_L^\dagger (\partial_0 + \partial_1) \psi_L + \psi_R^\dagger (\partial_0 - \partial_1) \psi_R \right). \quad (6.142)$$

In this equation, the phases of the left and right moving spinors can be re-defined separately,

$$\psi_L \rightarrow e^{i\chi_L} \psi_L \quad \psi_R \rightarrow e^{i\chi_R} \psi_R. \quad (6.143)$$

This enhanced symmetry is called *chiral symmetry*. The resulting conserved currents are

$$J_L^\mu = (\psi_L^\dagger \psi_L, \psi_L^\dagger \psi_L) \quad J_R^\mu = (\psi_R^\dagger \psi_R, -\psi_R^\dagger \psi_R). \quad (6.144)$$

This is a $U(1) \times U(1)$ symmetry.

If there is more than one kind of fermion, the maximal symmetry group is found by decomposing each fermion into its real and imaginary parts, so that we have a number N_L of left-moving Weyl–Majorana fermions and N_R right-moving Weyl–Majorana fermions has the action

$$S_D = i \int d^2x \left(\sum_{a=1}^{N_L} \psi_{La} (\partial_0 + \partial_1) \psi_{aL} + \sum_{a=1}^{N_R} \psi_{Ra} (\partial_0 - \partial_1) \psi_{Ra} \right). \quad (6.145)$$

This action has an $O(N_L) \times O(N_R)$ symmetry. A symmetry which arises from existence of a number of copies of identical fields is usually called a *flavour symmetry* and the different species are called *flavours*.

Even at the classical level, where Fermi statistics are not important, it is very convenient to treat classical spinor fields as anti-commuting objects,

$$\psi_a(x, t)\psi_b(y, t') = -\psi_b(y, t')\psi_a(x, t) \quad (6.146)$$

together with the convention that complex conjugation inverts the order. This convention renders the action (6.145) and the Hamiltonian which would be derived from it both real at the classical level. It also determines the way that gauge fields can couple to bilinear fermion currents. For example, the current

$$J_{ab}^\mu = (\psi_{La}(x, t)\psi_{Lb}(x, t), \psi_{La}(x, t)\psi_{Lb}(x, t)) \quad (6.147)$$

is antisymmetric in the indices a and b and can be consistently coupled to a gauge field which is itself an antisymmetric matrix, so that the term that one would add to the Lagrangian density is

$$A_\mu^{ab}(x, t)J_{ba}^\mu(x, t). \quad (6.148)$$

The gauge field is a hermitian antisymmetric matrix when it is in the adjoint representation of a Lie group. Its components are related to the structure constants of the group

$$A_\mu^{ab} = \sum_{c=1}^{\dim g} -if^{abc}A_\mu^c. \quad (6.149)$$

Gauge fields which are in complex representations of a gauge group must be coupled to currents from complex fermions, generally the coupling term is

$$(A_{ab}^0 + A_{ab}^1)\psi_{Lb}^\dagger\psi_{La} \quad (6.150)$$

or

$$\left(\tilde{A}_{ab}^0 - \tilde{A}_{ab}^1\right)\psi_{Rb}^\dagger\psi_{Ra}. \quad (6.151)$$

When the fermions are coupled to gauge and gravitational field, quantization breaks some of the chiral symmetries. For example, gauge anomalies break the group which the gauge fields are coupled to one where left and right symmetries have identical groups. Furthermore, when coupled to gravitational fields, there must be an equal number of right and left-moving fermions in order that general coordinate invariance is not violated by gravitational anomalies.

Electrodynamics with massless spinors in 1+1 dimensions is called the Schwinger model. It is of great interest as a solvable quantum field theory that has gauge invariance. Its action has the form

$$S = \int d^2x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \sum_{a=1}^N (-i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + e A_\mu \bar{\psi}_a \gamma^\mu \psi_a) \right\}. \quad (6.152)$$

When $N > 1$ it is called the multi-flavour Schwinger model. It has $U(1)$ gauge symmetry and $[U(N) \times U(N)] / U(1)$ flavour symmetry. This symmetry is broken to $[U(N) \times U(N)] / [U(1) \times U(1)]$ flavour symmetry by the axial anomaly.

6.3.2 2+1 Dimensions

In 2+1 dimensions, we can still find a representation of the Dirac matrices in terms of Pauli matrices, say

$$\gamma^0 = \sigma^2 \quad \gamma^1 = i\sigma^3 \quad \gamma^2 = i\sigma^1. \quad (6.153)$$

In this case, the Dirac equation is a real,

$$\begin{pmatrix} -\partial_1 + m & \partial_t - \partial_2 \\ \partial_t + \partial_2 & \partial_1 + m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad (6.154)$$

and the spinor field that solves it can still be taken to be either real (Majorana) or complex (Dirac).

This equation has the further feature that it is not invariant under parity. A parity transformation can be implemented by reflecting one of the coordinates, say $x_1 \rightarrow -x_1$. If we take the parity transformation of the spinor as

$$\psi'(t, -x_1, x_2) = \sigma^3 \psi(t, x_1, x_2) \quad (6.155)$$

then the transformed field obeys the equation where the mass has opposite sign,

$$\begin{pmatrix} -\partial_1 - m & \partial_t - \partial_2 \\ \partial_t + \partial_2 & \partial_1 - m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \quad (6.156)$$

Thus, only the massless two-component spinor can have a parity invariant equation of motion.

On the other hand, unlike what happened in 1+1 dimensions (and all other even spacetime dimensions), the massless Dirac equation cannot be written as two independent equations and there is no analog of Weyl fermion in 2+1 dimensions.

Also, the only way to make a parity invariant massive spinor field is to take two two-component fields having Dirac equations

$$(i\gamma^\mu \partial_\mu + m) \psi_1 = 0 \quad (i\gamma^\mu \partial_\mu - m) \psi_2 = 0 \quad (6.157)$$

and the parity transformation

$$\psi'_1(t, -x_1, x_2) = \sigma^3 \psi_2(t, x_1, x_2) \quad \psi'_2(t, -x_1, x_2) = \sigma^3 \psi_1(t, x_1, x_2). \quad (6.158)$$

Both fields ψ_1 and ψ_2 could be taken to be Majorana spinors. This would be fewest possible degrees of freedom of the Fermi field which have parity symmetry. They cannot be consistently coupled to a gauge field unless $m = 0$. Then their Dirac equation has an $O(2) = U(1)$ symmetry and it can be modified as

$$\begin{aligned} i\gamma_\mu \partial_\mu \psi_1 + ie\gamma^\mu A_\mu \psi_2 &= 0 \\ i\gamma_\mu \partial_\mu \psi_2 - ie\gamma^\mu A_\mu \psi_1 &= 0 \end{aligned} \quad (6.159)$$

or, if we form the complex spinor $\psi = \psi_1 + i\psi_2$,

$$i\gamma^\mu \partial_\mu \psi + e\gamma^\mu \psi = 0. \quad (6.160)$$

A parity violating mass term can be added to this equation and it would be consistent with gauge invariance. Also, a parity conserving mass could be added, but it would break the gauge invariance. There is no way to add a mass term which is both parity invariant and gauge invariant. This fact is at the basis of the parity anomaly [41].

An often considered case is when there are two complex spinors [47, 48]. Then, if the theory is massless, there is a $U(2)$ symmetry which transforms the 4-component spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Actually, the maximal symmetry is seen by decomposing the spinors into their real and imaginary parts. There are four such fields and the maximal symmetry is $O(4)$. This symmetry is broken to $U(1) \times U(1)$ by the presence of a parity invariant mass term. Of course, there is also the possibility of introducing a parity variant mass, with the same sign for both spinors, and thereby preserving the $O(4)$ symmetry at the expense of parity.

Three dimensional electrodynamics couples N species of complex fermions to an electro magnetic gauge field,

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^N \bar{\psi}_a \gamma^\mu (i\partial_\mu + eA_\mu) \psi_a \right). \quad (6.161)$$

If N is even, one can introduce a parity invariant mass by taking a mass m for $N/2$ species of fermions and $-m$ for the other $N/2$ species. Then, parity is formulated as a combination of the spacetime parity transformation (6.155)

and an interchange of the two kinds of fermions, similar to (6.158). More generally, we could introduce a mass matrix, and mass term

$$\bar{\psi}_a M^{ab} \psi_b . \quad (6.162)$$

A parity transformation can still be formulated if there is $N \times N$ unitary matrix with the properties

$$UU^\dagger = 1 = U^\dagger U \quad U^\dagger MU = -M . \quad (6.163)$$

Such a matrix will exist only when the matrix M has even rank. Whether such fermion mass terms can be generated dynamically by spontaneous breaking of some of the flavour symmetry is an important question in three spacetime dimensional quantum electrodynamics.

6.3.3 3+1 Dimensions

In 3+1 dimensions, the minimal representation of the Dirac matrices is as four-by-four matrices. An example is

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} & \gamma^1 &= \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} & \gamma^3 &= \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} . \end{aligned}$$

With this choice, the Dirac equation is real, and the spinor can be chosen to be either Majorana or Dirac. When the mass is zero, with this representation of the Dirac matrices, the Dirac equation cannot be presented as two equations. However, in that case, it can be block-diagonalized by a similarity transformation. The resulting two blocks lead to the Weyl equations for two-component spinors,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) U_L &= 0 \\ \left(-\frac{\partial}{\partial t} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) U_R &= 0 , \end{aligned} \quad (6.164)$$

where U_L and U_R are left and right-handed Weyl spinors. Note that these equations are complex, so their solutions are necessarily complex. Thus, in four spacetime dimensions there is no Majorana–Weyl spinor. If there are N_L species of left handed spinors and N_R species of right handed spinors, the action is

$$S = \int d^4x \left\{ \sum_1^{N_L} \psi_{La}^\dagger (i\partial_t - i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \psi_{La} + \sum_1^{N_R} \psi_{Rb}^\dagger (-i\partial_t + i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \psi_{Rb} \right\} . \quad (6.165)$$

This action has $U(N_L) \times U(N_R)$ flavour symmetry, also sometimes called *chiral symmetry*. Some of this symmetry is violated by anomalies upon coupling to gauge fields or gravity.

Exercises:

- Consider the coordinates of 1+1 dimensional Minkowski space as the combinations

$$x^+ = \frac{1}{\sqrt{2}}(t+x) \quad x^- = \frac{1}{\sqrt{2}}(t-x) . \quad (6.166)$$

Show that the conformal Killing equation in 1+1 dimensions is solved by any functions with the property

$$\frac{\partial}{\partial x^+} f^- = 0 \quad \frac{\partial}{\partial x^-} f^+ = 0 , \quad (6.167)$$

where $f^\pm = \frac{1}{\sqrt{2}}(f^0 \pm f^1)$, so that

$$\tilde{x}^+ = x^+ + f^+(x^+) \quad \tilde{x}^- = x^- + f^-(x^-) \quad (6.168)$$

is an infinitesimal conformal transformation. Unlike in higher spacetime dimensions, the conformal group in 1+1 dimensions has an infinite number of independent transformations.

- Consider the action for a 1+1 dimensional Majorana–Weyl spinor

$$S = \int d^2x i\psi(x)(\partial_0 - \partial_1)\psi(x) \quad (6.169)$$

and an infinitesimal coordinate transformation to new coordinates $\tilde{x}^\mu = x^\mu + f^\mu(x)$. Find the transformation of the spinor which is a symmetry of the action when the coordinate transformation is a conformal transformation.

- For the problem above, use the appropriate generalization of the Noether procedure to identify the conserved current which is associated with conformal invariance.
- Consider two consecutive coordinate transformations,

$$x^{\mu'} = x^\mu + f^\mu(x) \quad x^{\mu''} = x^{\mu'} + g^\mu(x^{\mu'}) . \quad (6.170)$$

Show that the difference between the resulting vector $x^{\mu''}$ and the one which we would obtain if we do the composition in the opposite order is another coordinate transformation where the transformation function is the Lie bracket

$$\{f, g\}^\mu = f^\nu \partial_\nu g^\mu - g^\nu \partial_\nu f^\mu . \quad (6.171)$$

- The equal time canonical Poisson bracket of Majorana–Weyl spinors is given by

$$\{\psi(x, t), \psi(y, t)\}_{PB} = -\frac{i}{2}\delta(x - y) . \quad (6.172)$$

(For fermionic fields the Poisson bracket should be the one appropriate to anti-commuting functions. This is how it can be an even function of $x - y$.) Using this bracket, compute the algebra of the charges corresponding to conformal transformations. Find the relationship of this algebra to the Lie bracket of consecutive coordinate transformations discussed in the exercise above.

- Consider the equations of motion of the multi-flavour Schwinger model (6.152). Show that, at the classical level, they lead to conservation of the two $U(1)$ currents,

$$J_V^\mu = \sum_a \bar{\psi}_a \psi_a , \quad J_A^\mu = \sum_a \bar{\psi}_a \gamma^0 \gamma^1 \gamma^\mu \psi_a = \epsilon^{\mu\nu} J_{V\nu} \quad (6.173)$$

$$\partial_\mu J_V^\mu = 0 , \quad \partial_\mu J_A^\mu = 0 . \quad (6.174)$$

Using gauge symmetry, show that this implies that the classical model has only trivial solutions which, up to a gauge transform, are

$$A_\mu = 0 , \quad \psi = \begin{pmatrix} u(x^-) \\ v(x^+) \end{pmatrix} , \quad (6.175)$$

where u and v are arbitrary functions. In the quantum field theory, the axial anomaly modifies the second equation in (6.174) to

$$\partial_\mu J_A^\mu = \frac{e^2 N}{\pi} \epsilon_{\mu\nu} F^{\mu\nu} . \quad (6.176)$$

Show that in this case, the Maxwell equations imply that the field strength tensor obeys a massive wave equation.

- Find the action for $SU(N_C)$ Yang–Mills theory coupled to N_F species of massless Dirac spinors in 3+1 dimensions. Identify the conserved currents for the $U(N_F) \times U(N_F)$ chiral symmetry. In this case, the axial $U(1)$ subgroup is broken by anomalies when the theory is quantized.

6.4 Quantized Yang–Mills Theory

There are several possible starting points for the quantization of Yang–Mills theory. Conceptually, the most straightforward is the Hamiltonian formalism, which we shall now review. When we quantize a classical system, we promote the degrees of freedom, which in our case were the gauge fields, to random

variables. The dynamical laws, in the form of equations of motion, which governed the classical system become dynamical laws for computing the time evolution of the distribution which is used to take correlators of the random variables.

In a relativistic field theory, rather than being random functions, the field variables are random tempered distributions. As such, there is no easy way to define quantities like the product of two fields at the same point and correlations functions of separated fields typically suffer from divergences. The machinery of renormalization theory has been developed to deal with this problem. In following sections we shall review one approach, that of lattice regularizations. In this Section, however, we will ignore these difficulties and discuss how one would quantize Yang–Mills theory by Hamiltonian methods. Expressions such as operator products which are not properly defined are treated formally. In later sections, we will revisit this subject using a lattice regularizations where all objects are well defined.

The Yang–Mills equations of motion could be derived using the standard variational principle from the action⁹

$$S = -\frac{1}{2} \int d^4x \text{tr} (F_{\mu\nu} F^{\mu\nu}) \quad (6.177)$$

to begin canonical quantization, we identify the Lagrangian, which in a field theory is the spatial integral of the Lagrangian density,

$$L = \int d^3x \sum_{a=1}^{\dim g} \left(\frac{1}{2} (\mathbf{E}^a)^2 - \frac{1}{2} (\mathbf{B}^a)^2 \right) \quad (6.178)$$

where we have written the Yang–Mills field strengths in terms of electric and magnetic fields,

$$\mathbf{E}^a(\mathbf{x}) = \dot{\mathbf{A}}^a(\mathbf{x}) - \nabla A_0^a(\mathbf{x}) + f^{abc} \mathbf{A}^b(\mathbf{x}) A_0^c(\mathbf{x}) \quad (6.179)$$

$$B_i^a(\mathbf{x}) = \sum_{j,k=1}^3 \epsilon_{ijk} (\partial_j A_k^a(\mathbf{x}) + f^{abc} A_j^b(\mathbf{x}) A_k^c(\mathbf{x})) , \quad (6.180)$$

respectively. Here, we denote time derivatives by an over-dot. We consider all variables, A_μ as functions of the spatial coordinates \mathbf{x} with the time set to some fixed value, zero for convenience. In the Hamiltonian formulation, the time dependence of the fields will be recovered by solving the Hamilton equations of motion where time derivatives of the fields are found by taking their Poisson brackets with the Hamiltonian. The time derivatives of the fields

⁹ Here, for concreteness, we shall discuss 3+1 dimensional Yang–Mills theory. Generalization of the arguments that we present here to other dimensions is straightforward.

are supposed to be treated as independent variables, the so-called generalized velocities.

The next step is to identify the canonical momenta which are functional derivatives of the Lagrangian by the generalized velocities,

$$\begin{aligned}\Pi_0^a(\mathbf{x}) &= \frac{\delta L}{\delta \dot{A}_0^a(\mathbf{x})} = 0 \\ \Pi^a(\mathbf{x}) &= \frac{\delta L}{\delta \dot{\mathbf{A}}^a(\mathbf{x})} = \mathbf{E}^a(\mathbf{x}) .\end{aligned}\quad (6.181)$$

Since the Lagrangian does not depend on the time derivative of the temporal component of the gauge field, its canonical momentum vanishes. The fact that the canonical momentum conjugate to A_0 vanishes implies the existence of a constraint. In fact, the equation of motion which follows from the variation of A_0 in the action is an equation without time derivatives. Such an equation could not arise as a Hamilton equation of motion - it must therefore be an equation of constraint. To see how this constraint arises, we first form the Hamiltonian which is deduced by standard procedure

$$H = \int d^3x \left\{ \frac{1}{2}(\mathbf{E}^a)^2 + \frac{1}{2}(\mathbf{B}^a)^2 - \nabla A_0^a \cdot \mathbf{E}^a + f^{abc} A_0^a \mathbf{A}^b \cdot \mathbf{E}^c \right\} \quad (6.182)$$

and the canonical Poisson brackets are

$$\begin{aligned}\{A_i^a(\mathbf{x}), E_j^b(\mathbf{y})\}_{PB} &= \delta_{ij} \delta^{ab} \delta(\mathbf{x} - \mathbf{y}) \\ \{A_0^a(\mathbf{x}), \Pi_0^b(\mathbf{y})\}_{PB} &= \delta^{ab} \delta(\mathbf{x} - \mathbf{y}) .\end{aligned}\quad (6.183)$$

Here, A_0 plays the role of a Lagrange multiplier which enforces the constraint

$$G^a(\mathbf{x}) \equiv \nabla \cdot \mathbf{E}^a(\mathbf{x}) + f^{abc} \mathbf{A}^b \cdot \mathbf{E}^c \sim 0 , \quad (6.184)$$

which is one of Maxwell's equations, the so-called Gauss' law. Here, we use the symbol \sim to denote a constraint. Another way to obtain the constraint (6.184) is to require that the constraint

$$\Pi_0^a(\mathbf{x}) \sim 0 \quad (6.185)$$

is consistent with the time evolution in that it has vanishing Poisson bracket with the Hamiltonian. Indeed, the Poisson bracket of $\Pi_0(\mathbf{x})$ with the Hamiltonian is proportional to (6.184). Finally, the constraint (6.184) has vanishing bracket with the Hamiltonian and is therefore consistent with the time evolution of the system.

The Poisson brackets of the quantities in (6.184) are the gauge algebra,

$$\{G^a(\mathbf{x}), G^b(\mathbf{y})\} = f^{abc} G^c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \quad (6.186)$$

and their brackets with the degrees of freedom generate time-independent gauge transformations,

$$\begin{aligned} \left\{ \int d^3y \Omega^a(\mathbf{y}) G^a(\mathbf{y}), \mathbf{A}^b(\mathbf{x}) \right\} &= \nabla \Omega^b(\mathbf{x}) f^{bcd} \mathbf{A}^c(\mathbf{x}) \Omega^d(\mathbf{x}) \\ \left\{ \int d^3y \Omega^a(\mathbf{y}) G^a(\mathbf{x}), \mathbf{E}^b(\mathbf{x}) \right\} &= f^{bcd} \mathbf{E}^c(\mathbf{x}) \Omega^d(\mathbf{x}) . \end{aligned} \quad (6.187)$$

Let us consider quantization of this system. The Poisson brackets are replaced by commutator brackets by the standard prescription,

$$\{ , \} \rightarrow \frac{1}{i\hbar} [,] . \quad (6.188)$$

It is necessary to find representation of the algebra of commutator. We will use the functional Schrödinger picture where the states are wave-functionals of the gauge field configurations, $\Psi[A_\mu, t]$ and the canonical momenta are represented as functional derivative operators,

$$\Pi^\mu(\mathbf{x}) = \frac{1}{i} \frac{\delta}{\delta A_\mu(\mathbf{x})} . \quad (6.189)$$

The inner products of wave-functionals are computed by functional integration,

$$\langle \Psi_1 | \Psi_2 \rangle = \int \prod_{\mathbf{x}, \mu} [dA_\mu(\mathbf{x})] \Psi_1^\dagger[A, t] \Psi_2[A, t] . \quad (6.190)$$

Constraints are imposed by choosing a subspace of the set of all wave-functions which obey the physical state conditions,

$$\begin{aligned} \Pi_0^a(\mathbf{x}) \Psi_{\text{phys}}[A, t] &= 0 \\ (\nabla \cdot \mathbf{E}^a(\mathbf{x}) + f^{abc} \mathbf{A}^b(\mathbf{x}) \cdot \mathbf{E}^c(\mathbf{x})) \Psi_{\text{phys}}[A, t] &= 0 . \end{aligned} \quad (6.191)$$

The first of these conditions simply means that physical states do not depend on A_0 . This means that physical states are not normalizable since the integration over A_0 in the normalization integral will always diverge. This can be remedied by modifying the normalization integral measure when applied to physical states by imposing the “gauge fixing condition” $A_0(\mathbf{x}) = 0$,

$$\int \prod_{\mathbf{x}, \mu} [dA_\mu(\mathbf{x})] \delta(A_0(\mathbf{x})) . \quad (6.192)$$

A similar problem arises due to the other constraint (6.184). Since the operator generates time independent gauge transformations, the constraint implies that physical states are gauge invariant,¹⁰

$$\Psi_{\text{phys}}[\mathbf{A}, t] = \Psi_{\text{phys}}[\mathbf{A}^U, t] , \quad (6.193)$$

¹⁰ Here, we are ignoring the possible existence of large gauge transformations.

where

$$\mathbf{A}^U = U^{-1} \mathbf{A} U - i U^{-1} \nabla U . \quad (6.194)$$

This means that a physical state must be a constant on an orbit of the gauge group i.e. the set of all field configurations related to a given one by gauge transformations. The gauge orbits have infinite volume, leading to a divergence in the normalization integral for physical states. This divergence can be cured by a gauge fixing procedure applied to the normalization integral. Since, in the computation of an overlap integral for physical states, the integrand is invariant under time independent gauge transformations, the volume of the gauge orbit can be removed from the integration measure using a version of the Fadeev–Popov trick. We insert the following unit into the integral,

$$1 = \int \prod_x [dU(\mathbf{x})] \delta(F[\mathbf{A}^U]) \det \{ \delta F[\mathbf{A}^U]/\delta U \} . \quad (6.195)$$

Here, $F[\mathbf{A}]$ should be a functional of \mathbf{A} such that the determinant in the integrand above is non-zero. The integration over unitary matrices U is defined as the point-wise invariant Haar measure. For physical states,

$$\begin{aligned} \langle \Psi_1 | \Psi_2 \rangle &= \int \prod_{\mathbf{x}, \mu} [dA_\mu(\mathbf{x})] \delta(A_0(\mathbf{x})) \delta(F[\mathbf{A}]) \\ &\quad \times \det \{ \delta F[\mathbf{A}]/\delta U|_{U=1} \} \Psi_1^\dagger[\mathbf{A}, t] \Psi_2[\mathbf{A}, t]. \end{aligned} \quad (6.196)$$

Here, the gauge invariance of the physical states has allowed us to gauge transform the integration variable and then to remove a factor of the gauge group volume,

$$\int \prod_x [dU(\mathbf{x})] . \quad (6.197)$$

This procedure removes the singularity in the normalization integrals coming from the infinite volume of the gauge orbits. The divergences that remain come from the short distance singularities of the theory, rather than the gauge redundancy of the dynamical variables. They can be dealt with by a regularizations procedure. In particular, in following sections we will discuss a lattice regularizations.

For physical states, the Schrödinger equation is now

$$\begin{aligned} i \frac{\partial}{\partial t} \Psi[\mathbf{A}, t] \\ = \int d^3x \sum_a \left\{ -\frac{1}{2} \frac{\delta^2}{\delta \mathbf{A}^a(\mathbf{x})^2} + \frac{1}{2} (\nabla \times \mathbf{A}^a(\mathbf{x}) + f^{abc} \mathbf{A}^b \times \mathbf{A}^c)^2 \right\} \Psi[\mathbf{A}, t]. \end{aligned} \quad (6.198)$$

This equation should be solved for the wave-functional $\Psi[\mathbf{A}, t]$ and, from among the solutions, we should choose those which are gauge invariant,

$$\left(\nabla \cdot \frac{\delta}{\delta \mathbf{A}^a(\mathbf{x})} + f^{abc} \mathbf{A}^b \cdot \frac{\delta}{\delta \mathbf{A}^c(\mathbf{x})} \right) \Psi[\mathbf{A}, t] = 0 . \quad (6.199)$$

Exercises:

- Show that the operators

$$G^a(\mathbf{x}) = (\nabla \cdot \mathbf{E}^a(\mathbf{x}) + f^{abc} \mathbf{A}^b(\mathbf{x}) \cdot \mathbf{E}^c(\mathbf{x}))$$

obey the algebra

$$[G^a(\mathbf{x}), G^b(\mathbf{y})] = i f^{abc} G^c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) .$$

- In the discussion of Yang–Mills theory above, all non-linear parts of field equations, and non-quadratic parts of the Schrödinger equation and constraints contain the structure constant f^{abc} . These can be thought of as the coupling constant. The free, non-interacting theory would be the one obtained by setting $f^{abc} = 0$. Solve the Schrödinger equation and the gauge constraint for the ground state of the system in this case. Choose the gauge fixing condition $F[\mathbf{A}] = \nabla \cdot \mathbf{A}^a$ and compute the normalization of the ground state. Show that, in four dimensions, there are two physical polarizations for each vector field. For quantum electrodynamics, these would be the two polarizations of the photon.

6.4.1 Canonical Quantization of Electrodynamics

As another exercise, let us consider the Hamiltonian and constraints of electrodynamics coupled to a Dirac fermion. The action from which the classical coupled Maxwell–Dirac equations can be derived is

$$S = \int d^4x \left\{ \bar{\psi} (-i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu + m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} . \quad (6.200)$$

This dynamical system is described by the following Hamiltonian, constraints and Poisson brackets,

$$H = \int d^3x \left\{ \psi^\dagger (i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + e\boldsymbol{\alpha} \cdot \mathbf{A} + \beta m) \psi + \frac{1}{2} \mathbf{E}^2 + \frac{1}{2} \mathbf{B}^2 \right\} \quad (6.201)$$

$$\Pi_0 \sim 0 \quad (6.202)$$

$$\nabla \cdot \mathbf{E} + e\psi^\dagger \psi \sim 0 \quad (6.203)$$

$$\{A_i(\mathbf{x}), \Pi_j(\mathbf{y})\} = \delta_{ij} \delta(\mathbf{x} - \mathbf{y}) \quad (6.204)$$

$$\{A_0(\mathbf{x}), \Pi_0(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) \quad (6.205)$$

$$\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) . \quad (6.206)$$

In this case, we shall the constraints at the classical level and then quantize. In order to do that, we should choose gauge fixing conditions. The reason that we are allowed to do that is the following. Here, we have two constraints which are combinations of the dynamical variables and whose brackets with the Hamiltonian generate other constants. Furthermore, their brackets with each other vanish. The brackets of the constraints with dynamical variables thus generates an orbit on the space of variables, all of whose members are physically equivalent. Choosing gauge fixing conditions, one for each independent constraint, simply chooses representatives of the orbits. Thus, conjugate to the constraint (6.202) we choose the gauge fixing condition

$$A_0(\mathbf{x}) \sim 0 . \quad (6.207)$$

Conjugate to the constraint (6.203) we can choose, for example, the gauge fixing condition

$$\nabla \cdot \mathbf{A}(\mathbf{x}) \sim 0 . \quad (6.208)$$

The Poisson bracket of this gauge fixing condition with (6.203) is

$$\{\nabla \cdot \mathbf{A}(\mathbf{x}), \nabla \cdot \mathbf{E}(\mathbf{y}) + e\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\} = -\nabla^2\delta(\mathbf{x} - \mathbf{y}) . \quad (6.209)$$

Since this Poisson bracket is non-zero on the space of Fourier transformable functions, the gauge fixing condition chooses a unique representative of the gauge orbit of all gauge field configurations. Now, we can use the constraints and gauge conditions to eliminate the redundant variables. The remaining variables are the physical degrees of freedom. However, we must still be careful that this elimination of variables is consistent with the Poisson brackets of the remaining variables. This is taken care of by replacing the Poisson brackets by the so-called Dirac brackets of the remaining variables¹¹. The remaining variables are the fermion creation and annihilation operators, $\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})$ and the transverse components of the vector fields $(A_i{}^T, E_i{}^T)$ which obey the Dirac brackets

$$\{A_i{}^T(\mathbf{x}), E_j{}^T(\mathbf{x})\}_{DB} = \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) \delta(\mathbf{x} - \mathbf{y}) . \quad (6.210)$$

¹¹ The Poisson bracket of the remaining variables must be projected so that it is compatible with the constraints. This leads to the so-called Dirac bracket. In general, if we have a set of constraints $\{\chi^i\}$ which have the property that

$$\det \{\chi^i, \chi^j\} \neq 0$$

so that the inverse matrix $\{\chi, \chi\}^{-1}$ exists, then, the Dirac bracket for two of the remaining dynamical variables is defined as

$$\{x^i, x^j\}_{DB} = \{x^i, x^j\} - \sum \{x^i, \chi^k\} \{\chi, \chi\}_{kl}^{-1} \{\chi_l, x^k\} .$$

These replace the Poisson brackets for the remaining variables.

Note that this bracket is compatible with the transversality conditions $\nabla \cdot \mathbf{A}^T = 0$ and $\nabla \cdot \mathbf{E}^T = 0$. The Poisson brackets for the remaining matter fields are un-modified.

The Hamiltonian for the transverse variables becomes

$$\begin{aligned} H = & \int d^3x \left(\frac{1}{2}(\mathbf{E}^T)^2 + \frac{1}{4}(\nabla \times \mathbf{A}^T)^2 + \psi^* \frac{-\hbar^2}{2m} (\nabla - ie\mathbf{A}^T)^2 \psi \right) \\ & + \frac{e^2}{2} \int d^3x d^3y \rho(\mathbf{x})(\mathbf{x}| \frac{1}{-\nabla^2} | \mathbf{y}) \rho(\mathbf{y}) . \end{aligned} \quad (6.211)$$

The last term is the Coulomb energy which comes from using the solution of the constraint (6.203)

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}^T(\mathbf{x}) - \frac{\nabla}{\nabla^2} e \rho(\mathbf{x}) , \quad (6.212)$$

where $1/\nabla^2$ is a shorthand notation for the Green function of the three dimensional Laplace operator.

Quantization is now straightforward. The Dirac brackets are replaced by commutators and anti-commutators for the fermions. Our task would then be to solve the Schrödinger equation for the energy levels of the system. In quantum electrodynamics, the coupling constant is small, $e^2/4\pi \approx 1/137$, so perturbation theory in powers of the coupling is accurate. However, before such computations can be done, the ultraviolet divergences of the theory must be dealt with. In fact, this is a rather difficult to carry out in the Hamiltonian formulation that we have described in this section. The reason is that, usually one is interested in a Lorentz covariant regularizations of the theory, which is more readily done in a Lorentz covariant description. The path integral formulation that we review briefly in the following subsection is such a formulation. In later Sections, we will describe a Hamiltonian version of lattice gauge theory where strong coupling computations can be done.

Exercise:

- Find the ground state wave-function of electrodynamics when the coupling constant is set to zero, $e = 0$.

6.4.2 Path Integrals

Yang–Mills theory can also be quantized using path integrals, in direct analogy with the path integral quantization of a single particle in quantum mechanics. Let us consider that problem for the moment. Consider a quantum mechanical system with one degree of freedom with Hamiltonian

$$H(q, p) = \frac{1}{2m} p^2 + V(q) \quad (6.213)$$

and canonical commutator bracket

$$\langle q, p \rangle = i\hbar . \quad (6.214)$$

The transition amplitude which expresses the probability that the particle will propagate from a state localized at q_i to one at q_f in time T is given by

$$\begin{aligned} & \langle q_f | e^{-iHt} | q_i \rangle \\ &= \int \prod_{t \in [0, T]} [dq(t) dp(t)] \exp \left(i \int_0^T (p(t)\dot{q}(t) - H(q(t), p(t))) \right) , \end{aligned} \quad (6.215)$$

where the integration is over all functions $q(t)$ with the property that $q(0) = q_i$ and $q(T) = q_f$ and $p(t)$ has free boundary conditions. This integral can be presented in Lagrangian form by doing the Gaussian integral over the momenta to get

$$\langle q_f | e^{-iHt} | q_i \rangle = \int \prod_{t \in [0, T]} [dq(t)] \exp \left(i \int_0^T \left(\frac{m}{2} \dot{q}(t)^2 - V(q(t)) \right) \right) . \quad (6.216)$$

Typically, in field theory, one is interested in time ordered correlation functions of the variables in the ground state. For these, it is straightforward to derive the path integral formula

$$\begin{aligned} & \langle 0 | T q(t_1) \dots q(t_n) | 0 \rangle \\ &= \frac{\int \prod_t [dq(t)] q(t_1) \dots q(t_n) \exp \left(i \int_{-\infty}^{\infty} \left(\frac{m}{2} \dot{q}(t)^2 - V(q(t)) \right) \right)}{\int \prod_t [dq(t)] \exp \left(i \int_{-\infty}^{\infty} \left(\frac{m}{2} \dot{q}(t)^2 - V(q(t)) \right) \right)} , \end{aligned} \quad (6.217)$$

where the time ordering symbol T orders the operators that follow it so that the time arguments decrease from left to right.

Generalization of this formula to of Yang–Mills theory is straightforward. It leads to the functional integral for correlation functions of gauge invariant operators,

$$\langle F[A] \rangle = \frac{\int dA_\mu(x) e^{iS[A]} F[A]}{\int dA_\mu(x) e^{-S[A]}} \quad (6.218)$$

with the classical action,

$$S[A] = \int d^4x \frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) . \quad (6.219)$$

Also, of interest are the computation of correlations functions of operators in Euclidean time or, equivalently, in statistical mechanics,

$$\langle F[A] \rangle = \frac{\text{TR} (F[A] e^{-\beta H})}{\text{TR} e^{-\beta H}}, \quad (6.220)$$

where TR denotes the trace of operators over all physical quantum states. This correlation function is given by the path integral expression

$$\langle F[A] \rangle = \frac{\int dA_\mu(\mathbf{x}, \tau) e^{-S_E[A]} F[A]}{\int dA_\mu(\mathbf{x}, \tau) e^{-S_E[A]}}, \quad (6.221)$$

where the Euclidean action S_E is obtained from the Yang–Mills action by the replacements

$$t \rightarrow -i\tau \quad A_0(\mathbf{x}, t) \rightarrow iA_0(\mathbf{x}, \tau) \quad (6.222)$$

$$S_E = \frac{1}{2} \int d\mathbf{x} \int_0^\beta d\tau \text{tr} F_{\mu\nu} F_{\mu\nu} \quad (6.223)$$

and the integration is over all variables which are periodic in Euclidean time, $A_\mu(\mathbf{x}, \tau + \beta) = A_\mu(\mathbf{x}, \tau)$.

In the limit $\beta \rightarrow \infty$, only the ground state expectation value contributes to the sum on the right-hand-side of (6.220) and the correlators that are computed from the Euclidean path integral correspond to those which would be computed from (6.218) with the replacements (6.222). Since the integrand in the resulting path integral is damped exponentially for large times, and is therefore better defined, it is usually more convenient to study the behaviour of Euclidean correlators.

Since both the path integral measure and the action are gauge invariant, only the correlation functions of gauge invariant operators will be non-zero. There are two classes of such operators. First are local operators made out of invariant combinations of the field strengths and covariant derivatives, such as the Lagrangian density $F_{\mu\nu} F^{\mu\nu}$. Second are non-local operators such as the Wilson loop operator, defined as the path-ordered phase integral

$$W[\Gamma] = \text{tr} \left(\mathcal{P} \exp \oint_{\Gamma} A_\mu dl^\mu \right). \quad (6.224)$$

Here, Γ is a closed curve which is embedded in either Minkowski space or Euclidean space. The uses of the Wilson loop operator will be discussed in following Sections.

6.5 Quantum Spins and Strong Coupling in Gauge Theory

6.5.1 Overview of the Problem

Though it is far from the scaling regime, the strong coupling limit of lattice gauge theories is often used to study their qualitative properties [29].

Two important features of the spectrum of non-Abelian gauge theories appear there. The strong coupling limit exhibits confinement which is related in a rather natural way to the gauge symmetry and compactness of the non-Abelian gauge group. Furthermore, with certain numbers of flavours and colours of dynamical quarks, it is straightforward to show that strongly coupled gauge theories exhibit dynamical chiral symmetry breaking. This has motivated several quantitative investigations of gauge theories using strong coupling techniques and there have even been attempts to compute the mass spectrum of realistic models such as quantum chromodynamics (QCD) (e.g. [5]). In some of the following sections we will review some of similar calculations for the one and multi-flavour Schwinger model. Our point here is to emphasize the analogy of the chiral symmetry breaking problem with the phase structure of quantum spin systems.

The strong coupling expansion is an expansion in the coefficient of the kinetic term for the gauge fields in the action. It assumes that this coefficient is small. It is well known that, because of asymptotic freedom, the continuum limit of the lattice gauge theory is found where this coefficient is a large positive number, going to infinity as one approaches the scaling limit. The strong coupling expansion is therefore not expected to give accurate quantitative information about the continuum gauge theory. However, the expansion does have a finite radius of convergence [46,10] which indicates that its properties are shared by the model for a wide range of the coupling parameter and certain information can be obtained by analytic continuation to domains outside of the radius of convergence. Furthermore, there are thought to be no phase transitions in the pure gauge theory as the coupling constant is varied between infinity and zero. Thus, qualitative information which is gained from the strong coupling expansion, particularly the value of order parameters for symmetry breaking, etc. are likely shared by the phase which is obtained just above the weak coupling limit.

In a parallel line of research, it has been recognised for some time that the strong coupling limit of lattice gauge theory with dynamical fermions is related to certain quantum spin systems. This is particularly true in the Hamiltonian picture and had already appeared in some of the earliest analyses of chiral symmetry breaking in the strong coupling limit [52]. In that work, mean field theory for antiferromagnets was used to analyse the gauge theory spectrum in strong coupling. In particular, when chiral symmetry is broken spontaneously, the mapping of the strong coupling problem to a generalized antiferromagnets and spin-wave analysis gives a way to compute properties of the pion spectrum.

On the other hand, in subsequent literature [18], it has been noted that there are several formal similarities between some condensed matter systems with lattice fermions (particularly certain antiferromagnetic spin systems) and lattice gauge theory systems, usually in their strong coupling limit. For example, it is well known that the quantum spin 1/2 Heisenberg antiferro-

magnets is equivalent to the strong coupling limit of a $U(1)$ lattice gauge theory. It can also be written as a kind of $SU(2)$ lattice gauge theory.

Moreover, the staggered fermions which are used to put the Dirac operator on a lattice resemble ordinary lattice fermions used in tight binding models in condensed matter physics when the latter have a half-filled band and are placed in a background $U(1)$ magnetic field $\pi \pmod{2\pi}$ ($1/2$ of a flux quantum) through every plaquette of the lattice. This is an old result for $d = 2 + 1$ [20] where it was already recognised in the first work on the Azbel–Wannier–Hofstaeder problem and has since been discussed in the context of the so-called flux phases of the Hubbard model. It is actually true for all $d \geq 2 + 1$ [48], i.e. the staggered lattice fermion approximation of the relativistic Dirac field with the minimal number of flavours is identical to a condensed matter fermion problem with simple nearest neighbour hopping in a background magnetic field which has $1/2$ flux quanta per plaquette. In the condensed matter context, the magnetic flux can be produced by a condensate, as in the flux phases of the Heisenberg and Hubbard models [18]. As an external magnetic field, for ordinary lattice spacings it is as yet an experimentally inaccessible flux density. However, it could be achieved in analog experiments where macroscopic arrays of Josephson junctions, for example, take the place of atoms at lattice sites. These systems are described in mean field theory by tight binding electrons in a background $U(1)$ magnetic field with accuracy increasing with d [19]. For $d = 2$ their ground state is expected to be a flux phase [17].

This analogy between Dirac fermions and tight-binding condensed matter fermions was exploited to find an exact mapping of the strong coupling limit of a large class of lattice gauge theories with dynamical staggered fermions onto certain quantum antiferromagnets, both the conventional Heisenberg antiferromagnet with spin j and generalized antiferromagnets with spins taking values in Lie algebras other than $SU(2)$ [49,31]. The precise structure of the resulting antiferromagnet depends on the number of colours in the gauge group and also on the number of fermion flavours. There are four different cases.

i.) It was shown that a gauge theory with gauge group $U(N_C)$ and with N_L (an *even* number of) lattice flavours of staggered lattice fermions¹² is an $SU(N_L)$ antiferromagnet in the strong coupling limit [31]. The number of colours determined the representation of $SU(N_L)$ which lives at the sites of

¹² Due to fermion doubling this yields $2^{d-1-[d/2]}N_L$ continuum flavours of $2^{[d/2]}$ -component Dirac fermions in the weak coupling continuum limit, where $[d/2]$ is the largest integer less than or equal to $d/2$ and d is the spacetime dimension:

d	no. of flavours	no. of components
2	1	2
3	2	2
4	2	4

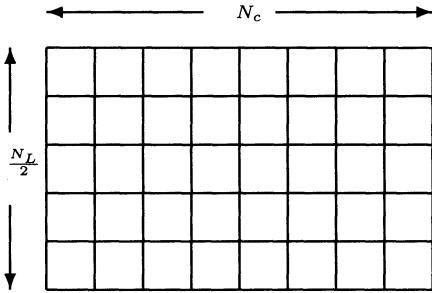


Fig. 6.1. Young tableau with N_c columns and $N_L/2$ rows

the antiferromagnet as being the one represented by the Young tableau with N_C columns and $N_L/2$ rows as in Fig. 6.1.

ii.) For a lattice gauge theory with colour group $SU(N_C)$ rather than $U(N_C)$ and $N_C \geq 3$, the strong coupling limit is a $U(N_L)$ antiferromagnet, the difference being that the representation of the $U(1)$ subgroup of $U(N_L)$ is not constrained as it was in the previous case where the local $U(1)$ charge had to vanish, and the representation of the $SU(N_L)$ subgroup of $U(N_L)$ can then take on a given site any of the representations depicted in Fig. 6.2, subject only to the constraint that the average fermion density is $N_L N_C/2$ (the number of fermions on a site is given by the number of boxes in the corresponding Young Tableau).

iii.) $SU(2)$ lattice gauge theories are special: in the strong coupling limit additional terms (as compared to $SU(N_C \geq 3)$) appear. Due to these, one obtains for N_L lattice flavours antiferromagnets associated with the symplectic group $Sp(2N_L)$, especially for $N_L = 1$ a spin 1/2 Heisenberg antiferromagnet [31].

iv.) For a gauge theory with colour gauge group $U(N_C)$ and an *odd* number of flavours of lattice fermions, the strong coupling limit is an $SU(N_L)$ antiferromagnet where the size of representations on neighbouring sites differs. On a bipartite lattice, we divide the sites into two sub-lattices such that the nearest neighbours of all of the sites of one sub-lattice are in the other sub-lattice. (The definition of bipartite lattice is one where this can be done.) Then on one of the sub-lattices (A) we have the representation of $SU(N_L)$ with Young Tableau with $(N_L + 1)/2$ rows and N_C columns and on the other sub-lattice (B) the representation with Young Tableau with $(N_L - 1)/2$ rows and N_C columns (see Fig. 6.3).

This case has a lower translational symmetry than the case of $U(N_C)$ gauge theory with an even number of lattice flavours. It breaks the symmetry of translation invariance by one site. In the other cases, this symmetry is broken by Néel order in the antiferromagnet, if indeed the ground state is ordered.

The most interesting case of $SU(N_C)$ lattice gauge theory turns out to be the most complicated, since at the sites the $SU(N_L)$ representation can take on any of those shown in Fig. 6.2, subject only to the constraint that the average number of rows in the Young tableaux is given by $N_L/2$. If N_L is

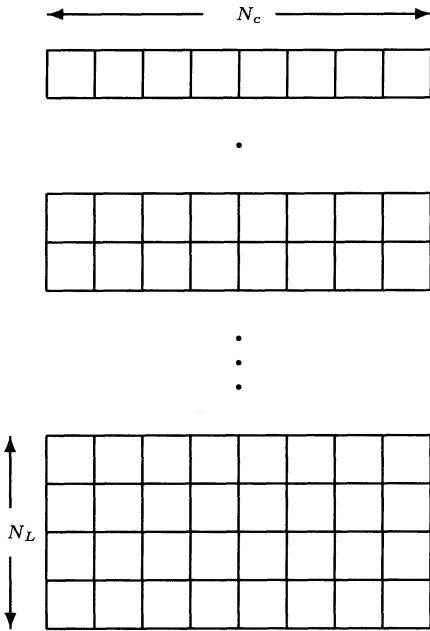


Fig. 6.2. Young tableaux representations allowed at a given site

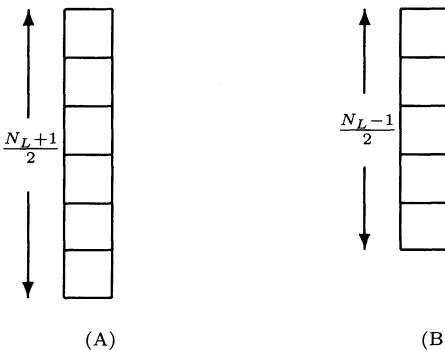


Fig. 6.3. Representations of $SU(N_L)$ on each site of the even sublattice (A) and the odd sublattice (B), when N_L is odd

odd, it is not possible to have the same representation everywhere since the representations only come with an integer number of rows. If N_L is even, the homogeneous state with $N_L/2$ rows everywhere can be achieved.

To determine the configuration of representations is a difficult and as yet unsolved problem. It can, on the other hand, be made more tractable by the simplifying assumption that the ground state has discrete rotation invariance and also translation invariance by two sites. On a cubic lattice (which is bipartite) we decompose the lattice into two sublattices A and B , one (A) with lattice sites $x = (x_1, \dots, x_{(d-1)})$ obeying $\sum_{p=1}^{d-1} x_p = \text{even}$ (we set the lattice spacing to 1, *viz.* the x_i are integers) and the other (B) with

$\sum_{p=1}^{d-1} x_p = \text{odd}$. The antiferromagnetic system we obtain has only $SU(N_L)$ representations with Young tableaux having n rows on A and $N_L - n$ rows on B , respectively, with $m = 1, \dots, N_L$. It is then necessary to find which value of n minimizes the ground state energy. In some cases this can be done. Also, mean field techniques could be used.

Quantum antiferromagnets can be studied in these representations using mean field theory or else other known results about their phase structure and spectra. What we obtain in general is a rather rich picture of the strong coupling limit, its possible phases and the possible influence of adding other operators.

In this and the following Sections we shall review the use of these mappings to prove that certain lattice gauge theories with staggered dynamical fermions break chiral symmetry in their strong coupling limit [49,31]. For staggered fermions the chiral symmetries are discrete and correspond to lattice translations by one site [25]. The mechanism for chiral symmetry breaking is the formation of charge or isospin density waves [52,49,31]. This is of course natural in an un-frustrated antiferromagnetic system. In the next Section, we shall review the staggered fermion method of putting relativistic fermions on a lattice.

6.6 Staggered Fermions

We shall use staggered fermions on a $(d - 1)$ -dimensional hyper-cubic lattice and continuum time which are obtained by spin-diagonalization [25] of the naively latticized Dirac Hamiltonian

$$\begin{aligned} H_f &= \frac{i}{2} \sum_{x,j} \left(\psi_\sigma^\dagger(x) \alpha_{\sigma\sigma'}^j \nabla_j \psi_{\sigma'}(x) - \nabla^j \psi_\sigma^\dagger(x) \alpha_{\sigma\sigma'}^j \psi_{\sigma'}(x) \right) \\ &= -\frac{i}{2} \sum_{x,j} \left(\psi_\sigma^\dagger(x + \hat{j}) \alpha_{\sigma\sigma'}^j \psi_{\sigma'}(x) - \psi_\sigma^\dagger(x) \alpha_{\sigma\sigma'}^j \psi_{\sigma'}(x + \hat{j}) \right), \end{aligned} \quad (6.225)$$

where $\psi_\sigma(x)$ are the fermion field operators, $\sigma = 1, 2, \dots, 2^{[d/2]}$ is the spin index, $x \equiv (x_1, x_2, \dots, x_{(d-1)}) = \sum_{j=1}^{d-1} x_j \hat{j}$ (the x_j are integers) refers to lattice sites, \hat{j} are unit vectors, ∇^j is the forward lattice difference operator,

$$\nabla_j f(x) = f(x + \hat{j}) - f(x) , \quad (6.226)$$

and α^j are the $2^{[d/2]} \times 2^{[d/2]}$ Hermitian Dirac matrices. The second form of the Hamiltonian in (6.225) describes a fermion hopping problem in a $U(2^{[d/2]})$ background gauge field given by the unitary¹³ matrices α^j . The crucial observation which allows spin diagonalization is that this background field has

¹³ Dirac's α -matrices are Hermitian, $\alpha^{i\dagger} = \alpha^i$, and, due to their anti-commutator algebra, $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$, they are also unitary.

only $U(1)$ curvature, i.e. if we consider the product around any plaquette

$$\alpha^j \alpha^k (\alpha^j)^\dagger (\alpha^k)^\dagger = -1 . \quad (6.227)$$

Hence the only information carried by the α -matrices which is invariant under a space dependent change of phase of the fermion fields is that a product of α 's around any plaquette is -1 . This allows diagonalization of the α 's using a gauge transformation. A gauge transformation which does the job is

$$\psi(x) \rightarrow (\alpha^1)^{x_1} (\alpha^2)^{x_2} \dots (\alpha^{(d-1)})^{x_{(d-1)}} \psi(x) \quad (6.228)$$

resulting in the Hamiltonian

$$H_f = -\frac{i}{2} \sum_{x,j} (-1)^{\sum_{p=1}^{j-1} x_p} (\psi^\dagger(x+j)\psi(x) - \psi^\dagger(x)\psi(x+j)) , \quad (6.229)$$

which describes $2^{[d/2]}$ species of lattice fermions with background $U(1)$ magnetic flux π through every plaquette of the lattice. This flux is contained in the background $U(1)$ field $U^0[x,j] = (-1)^{\sum_{p=1}^{j-1} x_p}$. Each species of fermion must have the same spectrum as the original one given by the Dirac Hamiltonian (6.225). This allows reduction of the fermion multiplicity by a factor of $2^{[d/2]}$ (by dropping the fermion spin index σ). The result resembles a condensed matter hopping problem with a single species of fermion where there is a background magnetic field π per plaquette.

Chiral symmetries are lattice translations by one site. This translation interchanges the even ($\sum_{p=1}^{d-1} x_p = \text{even}$) and odd ($\sum_{p=1}^{d-1} x_p = \text{odd}$) sublattices. The substitutions

$$\psi(x) \rightarrow (-1)^{\sum_{p=j+1}^{d-1} x_p} \psi(x+j) \quad (6.230)$$

for $j = 1, \dots, (d-1)$ leave the Hamiltonian in (6.229) invariant. A candidate for Dirac mass operator, which changes sign under the transformations in (6.230), is the staggered charge density operator

$$\mu = m \sum_x (-1)^{\sum_{p=1}^{d-1} x_p} \psi^\dagger(x) \psi(x) . \quad (6.231)$$

To obtain the continuum limit and the number of fermion species, we first divide the lattice into 2^{d-1} sublattices according to whether the components of their coordinates are even or odd. For example, when $(d-1) = 3$, we label 8 fermion species as

$$\begin{aligned} \psi(\text{even, even, even}) &\equiv \psi_1 , & \psi(\text{even, odd, odd}) &\equiv \psi_2 , \\ \psi(\text{odd, even, odd}) &\equiv \psi_3 , & \psi(\text{odd, odd, even}) &\equiv \psi_4 , \\ \psi(\text{even, even, odd}) &\equiv \psi_5 , & \psi(\text{even, odd, even}) &\equiv \psi_6 , \\ \psi(\text{odd, even, even}) &\equiv \psi_7 , & \psi(\text{odd, odd, odd}) &\equiv \psi_8 . \end{aligned} \quad (6.232)$$

Then, if we add the mass operator (6.231), in momentum space the Hamiltonian (6.229) has the form

$$H_f = \int_{\Omega_B} d^3k \psi^\dagger(k) (A^i \sin k_i + Bm) \psi(k), \quad (6.233)$$

with

$$\psi^\dagger(k) \equiv (\psi_1^\dagger(k), \dots, \psi_8^\dagger(k)) \quad (6.234)$$

and 8×8 matrices

$$A^i = \begin{pmatrix} 0 & \alpha^i \\ \alpha^i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \quad (6.235)$$

with σ^i the Pauli matrices; we have used the Fourier transform

$$\psi(x) = \int_{\Omega_B} \frac{d^3k}{(2\pi)^{3/2}} e^{-ik \cdot x} \psi(k), \quad (6.236)$$

where $\Omega_B = \{k = (k_1, k_2, k_3), -\pi/2 < k_i \leq \pi/2\}$ is the Brillouin zone of the (even, even, even) sublattice. The fermion spectrum is

$$\omega(k) = \sqrt{\sum_{i=1}^3 \sin^2 k_i + m^2}, \quad (6.237)$$

and only the region $k_i \sim 0$ is relevant to the continuum limit. We have normalized $\psi(k)$ so that

$$\{\psi(x), \psi^\dagger(y)\} = \delta(x - y), \quad \{\psi(k), \psi^\dagger(l)\} = \delta^3(k - l). \quad (6.238)$$

If we define $\beta = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$ and the unitary matrix

$$M = \frac{1}{2} \begin{pmatrix} 1 - \beta & 1 + \beta \\ 1 + \beta & 1 - \beta \end{pmatrix} \quad (6.239)$$

and

$$\psi = M\psi' \quad (6.240)$$

with

$$\psi' = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad (6.241)$$

the Hamiltonian is

$$H_f = \int_{\Omega_B} d^3k \begin{pmatrix} \psi_a^\dagger & \psi_b^\dagger \end{pmatrix} \begin{pmatrix} \alpha^i \sin k_i - \beta m & 0 \\ 0 & \alpha^i \sin k_i + \beta m \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \quad (6.242)$$

the low momentum limit, $\sin k_i \sim k_i$, with fermion density 1/2 per site so that the Fermi level is at the intersection point of the positive and negative energy bands, we obtain 2 continuum Dirac fermions. Furthermore, the staggered charge operator gives a Dirac mass of differing sign for the two species.

In general we shall consider $N_L N_C$ lattice species in d dimensions, i.e. fermions ψ_A^α with $\alpha = 1, \dots, N_L$ and $A = 1, \dots, N_C$ a lattice flavour and colour index, respectively. This yields $N_L N_C 2^{(d-1)} / 2^{[d/2]}$ continuum species of Dirac fermions where the lattice fermion density is $N_L N_C / 2$ per site. For the Dirac mass operator we may choose

$$\mu = \sum_x (-1)^{\sum_{p=1}^{d-1} x_p} \psi_A^{\alpha\dagger}(x) m_{\alpha\beta} \psi_B^\beta(x), \quad (6.243)$$

with $m_{\alpha\beta} = \text{diag}(m_1, \dots, m_{N_L})$, and this gives the fermion spectrum

$$\omega(k) = \sqrt{\sum_{i=1}^{d-1} \sin^2 k_i + m_\alpha}, \quad (6.244)$$

where $-\pi/2 \leq k_i < \pi/2$ and $\alpha = 1, \dots, N_L$.

We shall now consider the continuum limit in a general number of dimensions. A formalism which is similar but not exactly the same as the present one can be found in [25].

We shall begin with a slight generalization of (6.229),

$$H = -\frac{i}{2} \sum_{x,j} (-1)^{\sum_{p=1}^{j-1} x_p} (\psi^\dagger(x+j)\psi(x) - \psi^\dagger(x)\psi(x+j)). \quad (6.245)$$

We consider an elementary hypercube of the lattice with sides of length 1 and 2^d sites generated by taking a site all of whose coordinates are even and adding to it the vectors

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{d-1}) \quad \alpha_i = 0 \text{ or } 1. \quad (6.246)$$

This also decomposes the lattice into 2^{d-1} sublattices generated by taking a site of the elementary hypercube, (even, even, ...)+ $\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha}$ and translating it by all even multiples of lattice unit vectors, \hat{i} . We label the fermions which reside on the sublattice of each of the corners of the elementary hypercube as $\psi_{\alpha_1, \dots, \alpha_{d-1}}(x)$. In momentum space the Hamiltonian is

$$H = \int_{\tilde{\Omega}_B} \frac{d^{d-1}k}{(2\pi)^{d-1}} \times \sum_{i=1}^{d-1} \psi_{\alpha_1, \dots, \alpha_{d-1}}^\dagger(k) \Gamma_{\alpha_1 \dots \alpha_{d-1} \beta_1 \dots \beta_{d-1}}^i \sin k_i \psi_{\beta_1 \dots \beta_{d-1}}(k), \quad (6.247)$$

where $\tilde{\Omega}_B = \{k_i : -\pi/2 < k_i \leq \pi/2\}$ is the Brillouin zone of the (even, even, ...) sublattice and the Dirac tensors are

$$\begin{aligned} \Gamma_{\alpha_1 \dots \alpha_{d-1} \beta_1 \dots \beta_{d-1}}^i \\ = \delta_{\alpha_1 \beta_1} \dots \delta_{\alpha_{i-1} \beta_{i-1}} \sigma_{\alpha_i \beta_i}^1 \delta_{\alpha_{i+1} \beta_{i+1}} \dots \delta_{\alpha_{d-1} \beta_{d-1}} (-1)^{\sum_{p=1}^{i-1} \alpha_p} . \end{aligned} \quad (6.248)$$

They obey the Clifford algebra

$$\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta^{ij} . \quad (6.249)$$

The spectrum of the Dirac operator is $\omega(k) = \pm \sqrt{\sum_{i=1}^{d-1} \sin^2 k_i}$. Note that, to set the Fermi level of the fermions at the degeneracy point where the two branches of the spectrum meet, it is necessary that the fermion states are exactly half-filled. This is also required for charge conjugation invariance, or particle-hole symmetry of the vacuum state.

The staggered charge density operator is equivalent to a mass operator where

$$\Sigma = \int_{\tilde{\Omega}_B} \frac{d^{d-1}k}{(2\pi)^{d-1}} \psi_{\alpha_1 \dots \alpha_{d-1}}^\dagger(k) \Gamma_{\alpha_1 \dots \alpha_{d-1} \beta_1 \dots \beta_{d-1}}^0 \psi_{\beta_1 \dots \beta_{d-1}}(k) , \quad (6.250)$$

where

$$\Gamma_{\alpha_1 \dots \alpha_{d-1} \beta_1 \dots \beta_{d-1}}^0 = \delta_{\alpha_1 \beta_1} \dots \delta_{\alpha_{d-1} \beta_{d-1}} (-1)^{\sum_{p=1}^{d-1} x_p} . \quad (6.251)$$

Here, Γ^0 satisfies the algebra

$$\Gamma^0 \Gamma^i + \Gamma^i \Gamma^0 = 0 \quad (6.252)$$

and

$$\Gamma^0 \Gamma^0 = 1 . \quad (6.253)$$

Thus the spectrum of the operator $H_f + m\Sigma$ is $\omega(k) = \pm \sqrt{\sum_{i=1}^{d-1} \sin^2 k_i + m^2}$ which is the spectrum of a relativistic fermion in the limit $k \sim 0$.

Here, we count the number of flavours of fermions obtained in the continuum limit by noting that (6.247) describes a 2^{d-1} -component fermion. In d dimensions the Dirac matrices are $[d/2]$ dimensional, therefore the continuum limit of (6.247) describes $2^{d-1}/2^{[d/2]}$ species of Dirac fermions.

6.7 Strong Coupling Lattice Gauge Theory

6.7.1 Hamiltonian Formulation of Lattice Gauge Theory

We shall consider Hamiltonian lattice gauge theory [29] where time is continuous and space is a $(d-1)$ -dimensional hyper-cubic lattice with sites

$x = \sum_{i=1}^{d-1} x_i \hat{i}$ as above, oriented links $[x, i]$ labelled by their site of origin and their direction, $i = \pm 1, \dots, \pm(d-1)$, and oriented plaquettes $[x, i, j]$ with corners $x, x + \hat{i}, x + \hat{i} + \hat{j}, x + \hat{j}$ and orientation $\hat{i} \times \hat{j}$. The gauge field $U[x, i]$ associated with the link $[x, i]$ is a group element in the fundamental representation $[N_C]$ of $SU(N_C)$ and, if the colour group is $U(N_C)$, also carries a representation of $U(1)$. It has the property $U[x, -i] = U^\dagger[x - \hat{i}, i]$. The electric field operator $E^a[x, i]$ associated with link $[x, i]$ has the Lie algebra

$$[E^a[x, i], E^b[y, j]] = if^{abc} E^c[x, i] \delta([x, i] - [y, j]) \quad (6.254)$$

and

$$E[x, -i] = -U^\dagger[x - \hat{i}, i] E[x - \hat{i}, i] U[x - \hat{i}, i], \quad (6.255)$$

where

$$E[x, i] \equiv E^a[x, i] T^a \quad (6.256)$$

with $T^a = (T^a)^\dagger$, $a = 0 \dots, N_C^2 - 1$ the generators of the Lie algebra of $U(N_C)$ obeying

$$[T^a, T^b] = if^{abc} T^c, \quad (6.257)$$

$T^0 = \mathcal{I}$ the $N_C \times N_C$ unit matrix representing $U(1)$ and T^a , $a \neq 0$, in the representation $[N_C]$ of $SU(N_C)$. It generates the left-action of the Lie algebra on $U[x, i]$, i.e.,

$$[E^a[x, i], U[y, j]] = -T^a U[x, i] \delta([x, i] - [y, j]). \quad (6.258)$$

The Hamiltonian for lattice gauge theory with dynamical fermions and discrete chiral symmetry is

$$\begin{aligned} H = & \sum_{[x, i], a} \frac{g^2}{2} E^a[x, i]^2 \\ & + \left(\sum_{[x, i, j]} \frac{1}{2g^2} \text{Tr}(U[x, i] U[x + \hat{i}, j] U^\dagger[x + \hat{j}, i] U^\dagger[x, j]) + \text{h.c.} \right) \\ & + \left(\sum_{[x, i]} t_{[x, i]} \psi_A^{\alpha\dagger}(x + \hat{i}) U_{AB}[x, i] \psi_B^\alpha(x) + \text{h.c.} \right) \end{aligned} \quad (6.259)$$

($\text{Tr}(\cdot)$ the $N_C \times N_C$ matrix trace) with

$$t_{[x, i]} = t \frac{i}{2} (-)^{\sum_{p=1}^i x_p}, \quad t_{[x, -i]} = t_{[x - \hat{i}, i]}^* \quad (6.260)$$

($t > 0$) for $i > 0$ the Dirac ‘matrices’ for the staggered fermions, and the generators of static gauge transformations are

$$\mathcal{G}^a(x) = \sum_{i=-(d-1)}^{(d-1)} E^a[x, i] + \psi_A^{\alpha\dagger}(x) T_{AB}^a \psi_B^\alpha(x) \quad (6.261)$$

with $a = 1, \dots, N_C^2 - 1$. If the gauge group is $U(N_C)$ rather than $SU(N_C)$ there is also the $U(1)$ generator

$$\mathcal{G}^0(x) = \sum_{i=1}^{(d-1)} (E^0[x, i] - E^0[x - \hat{i}, i]) + \frac{1}{2} [\psi_A^{\alpha\dagger}(x), \psi_A^\alpha(x)] \quad (6.262)$$

with the fermionic $U(1)$ charge operator ordered so that, like E , it changes sign under time reversal. (Note that there is no ordering ambiguity for the \mathcal{G}^a , $a \neq 0$.) These generators obey the Lie algebra

$$[\mathcal{G}^a(x), \mathcal{G}^b(y)] = i f^{abc} \mathcal{G}^c(x) \delta(x - y) . \quad (6.263)$$

The Hamiltonian is gauge invariant,

$$[\mathcal{G}^a(x), H] = 0 . \quad (6.264)$$

The dynamical problem of lattice gauge theory is to find the eigenstates of the Hamiltonian operator (6.259) which are also gauge invariant, i.e. which obey the physical state condition

$$\mathcal{G}^a(x) |\psi_{\text{phys}}\rangle = 0 . \quad (6.265)$$

6.7.2 Strong Coupling

In order to do a strong-coupling expansion, we consider

$$H = H_0 + H_1 + H_2 ,$$

where $H_0 = \sum \frac{g^2}{2} E^2$, $H_1 = \sum (t\psi^\dagger U\psi + \text{h.c.})$ and $H_2 = \sum \frac{1}{2g^2} (\text{Tr} UUUU + \text{h.c.})$. Each term is gauge invariant,

$$[\mathcal{G}^a(x), H_i] = 0 .$$

Therefore, if we find a gauge invariant eigenstate of H_0 , perturbations in H_1 and H_2 remain gauge invariant. H_0 is the sum of group manifold Laplacians for each link. If $|0\rangle$ is a singlet of the algebra (3.1), i.e.

$$E^a[x, i] |0\rangle = 0 ,$$

then

$$H_0|0\rangle = 0$$

and

$$H_0 U[x, i]|0\rangle = \frac{g^2}{2} C_2(N_C) U[x, i]|0\rangle$$

with the Casimir operator

$$C_2(N_C)\mathcal{I} = \sum_a T^a T^a ,$$

where a runs either from 0 (for $U(N_C)$) or 1 (for $SU(N_C)$) to $N_C^2 - 1$.

Consider the empty vacuum which is a singlet of (3.1) and which has no fermions,

$$\psi_A^\alpha(x)|0\rangle = 0 , \quad \forall x, A, \alpha .$$

It is necessary to find the lowest energy (E_0 = eigenvalue of H_0) eigenstates $|\Psi\rangle$ of H_0 which are gauge invariant,

$$\mathcal{G}^a(x)|\Psi\rangle = 0 \quad \forall a, x ,$$

with the constraint that the fermion states are half-filled.

We shall first review the simplest case of $SU(N_C \geq 3)$ gauge theory with 1 lattice flavour ($N_L = 1$) and $2^d/2^{[d/2]}$ continuum flavours [5]. The density is $N_C/2$ fermions per site. The lowest energy eigenstates of H_0 with $E_0 = 0$ are the states which are singlets of the electric field algebra (3.1) and, since they are gauge invariant they must also be colour singlets, i.e. singlets of the algebra (3.7). Then, they must also be singlets of the of the algebra of the fermion currents $\psi^\dagger(x)T^a\psi(x)$. We can form a ‘baryon’ (=colour singlet) at a site by either leaving it unoccupied or putting N_C fermions with antisymmetrized singlet wave-function using the creation operator

$$S^\dagger(x) = \epsilon_{A_1 \dots A_{N_C}} \psi_{A_1}^\dagger(x) \dots \psi_{A_{N_C}}^\dagger(x) \quad (6.266)$$

($\epsilon_{A_1, \dots, A_{N_C}}$ is the usual antisymmetric tensor). Fermi statistics allows at most one singlet per site. Otherwise we can distribute the $V/2$ singlets arbitrarily (V is the total number of lattice sites). Thus, there are $V!/((V/2)!)^2$ degenerate ground states with a typical state being

$$|\{\rho_x\}_x\rangle = \prod_{\text{occupied } x} S^\dagger(x)|0\rangle .$$

We label them by the eigenvalues ρ_x of the local fermion number operators

$$\rho(x) = \frac{1}{2}[\psi^\dagger(x), \psi(x)] = \psi^\dagger(x)\psi(x) - N_C/2 .$$

The vacuum degeneracy must be resolved by diagonalizing the first non-trivial order in perturbation theory. In the following, $\langle \cdot, \cdot \rangle$ is the inner product in the total Hilbert space of the model,

$$\langle \cdot, \cdot \rangle = \left\{ \prod_{[x,i]} dU[x,i] \right\} (\cdot, \cdot) ,$$

where dU is the Haar measure on the gauge group manifold and (\cdot, \cdot) the inner product in the fermion Fock space.

All matrix elements in the vector space of degenerate vacua of the first order Hamiltonian vanish,

$$\langle \{\rho_x\}_x | H_1 | \{\rho'_x\}_x \rangle = 0$$

(this follows from the group integrals $\int dUU_{AB}^{(\dagger)} = 0$ [12]), so we must consider second order perturbation theory and diagonalize the matrix with elements

$$\langle \{\rho_x\}_x | H_2 | \{\rho'_x\}_x \rangle - \langle \{\rho_x\}_x | H_1 \frac{1}{H_0 - E_0} H_1 | \{\rho'_x\}_x \rangle .$$

Using group integrals

$$\int dUU_{AB}U_{CD}^\dagger = \frac{1}{N_C} \delta_{AD} \delta_{BC}$$

etc. [12] we see that the first term is zero and that diagonalizing the second term is equivalent to diagonalizing the four-fermion Hamiltonian

$$H_{\text{eff}} = -K \sum_{[x,i]} \psi_A^\dagger(x + i) \psi_B(x) \psi_B^\dagger(x) \psi_A(x + i) = K \sum_{[x,i]} \rho(x + i) \rho(x) \quad (6.267)$$

(up to an additive constant), with

$$K = \frac{t^2}{2g^2 N_C C_2(N_C)} > 0 , \quad (6.268)$$

in the space of pure fermion states where $\rho(x)$ has eigenvalues

$$\rho_x = \pm N_C / 2$$

and

$$\sum_x \rho_x = 0 .$$

This is the antiferromagnetic Ising model.

Now we consider arbitrary N_L . Then the density is $N_C N_L / 2$ fermions per site and the ‘baryon’ (= colour singlet) creation operator is

$$S_{\alpha_1 \dots \alpha_{N_C}}^\dagger(x) = \epsilon_{A_1 \dots A_{N_C}} \psi_{A_1}^{\alpha_1\dagger}(x) \dots \psi_{A_{N_C}}^{\alpha_{N_C}\dagger}(x). \quad (6.269)$$

Since this operator is symmetric in the lattice flavour indices $\alpha_1, \dots, \alpha_{N_C}$, it carries an irreducible representation of $SU(N_L)$ with Young Tableau with N_C columns and 1 row. Fermi statistics allows at most N_L singlets on a given site. Thus, the allowed representations of the flavour $SU(N_L)$ algebra at one site are the empty singlet and those with the Young Tableaux given in Fig. 6.1 of the Introduction and which are distinguished by the fermion numbers

$$\rho_x = N_C(2\nu - N_L)/2, \quad \nu = 0, 1, 2, \dots, N_L, \quad (6.270)$$

respectively. A ground state of H_0 is obtained by creating $V N_L / 2$ colour singlets. The states are labelled by the local fermion densities ρ_x and the vector in the corresponding $SU(N_L)$ representation at each site and are degenerate.

Again this degeneracy must be resolved by diagonalizing perturbations. The first-order perturbation to the vacuum energy vanishes and diagonalizing the second order perturbations is equivalent to diagonalizing the effective Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= -K \sum_{[x,i]} \psi_A^{\alpha\dagger}(x + i) \psi_B^\alpha(x) \psi_B^{\beta\dagger}(x) \psi_A^\beta(x + i) \\ &= K \sum_{[x,i]} J^{\alpha\beta}(x + i) J^{\beta\alpha}(x) \end{aligned} \quad (6.271)$$

(up to an additive constant) where the operators

$$J^{\alpha\beta}(x) = \frac{1}{2} [\psi_A^{\alpha\dagger}(x), \psi_A^\beta(x)] = \psi_A^{\alpha\dagger}(x) \psi_A^\beta(x) - \delta^{\alpha\beta} \frac{N_L}{2} = (J^{\beta\alpha}(x))^\dagger \quad (6.272)$$

are generators of the Lie algebra of $U(N_L)$,

$$[J^{\alpha\beta}(x), J^{\alpha'\beta'}(y)] = \delta(x - y) \left(\delta^{\beta\alpha'} J^{\alpha\beta'}(x) - \delta^{\beta'\alpha} J^{\alpha'\beta}(x) \right). \quad (6.273)$$

Thus, the effective Hamiltonian is that of a $U(N_L)$ quantum antiferromagnet with representations given in Fig. 6.2 and with the constraint $\sum_x \rho_x = 0$, and we have shown that it is equivalent to the strong coupling limit of $SU(N_C)$ lattice gauge theory with N_L lattice flavours of dynamical fermions.

The peculiarity of the $SU(2)$ colour group mentioned in the Introduction can be traced back to the fact that for 2 colours the singlet creation and annihilation operators involve only 2 fermion field operators, and in second

order perturbation theory terms are generated describing the hopping of these singlets [31]. Mathematically these arise from the special group integrals

$$\int_{SU(2)} U_{AB} U_{CD} = \frac{1}{2} \epsilon_{AC} \epsilon_{BD}$$

(this is zero for all other colour groups!) [12]. Due to this (6.271) is not the right effective Hamiltonian but

$$\begin{aligned} & - K \sum_{[x,i]} \left(\psi_A^{\alpha\dagger}(x) \psi_B^\alpha(x + i) \psi_B^{\beta\dagger}(x + i) \psi_A^\beta(x) \right. \\ & \quad \left. - \epsilon_{AC} \epsilon_{BD} \psi_A^{\alpha\dagger}(x + i) \psi_B^\alpha(x) \psi_C^{\beta\dagger}(x + i) \psi_D^\beta(x) \right) \\ & = H_{\text{eff}} = K \sum_{[x,i]} \left(J^{\alpha\beta}(x) J^{\beta\alpha}(x + i) + S_+^{\alpha\beta}(x) S_-^{\beta\alpha}(x + i) \right) \end{aligned} \quad (6.274)$$

(up to a constant), with

$$S_+^{\alpha\beta}(x) = \epsilon_{AB} \psi_A^{\alpha\dagger}(x) \psi_B^{\beta\dagger}(x) , \quad (6.275)$$

$$S_-^{\alpha\beta}(x) = \epsilon_{AB} \psi_A^\alpha(x) \psi_B^\beta(x) = (S_+^{\beta\alpha}(x))^\dagger , \quad (6.276)$$

and the $J^{\alpha\beta}(x)$ as above. One can check that the operators $J^{\alpha\beta}(x)$ and $S_\pm^{\alpha\beta}(x)$ obey the relations of the Lie algebra of the symplectic group $\text{Sp}(2N_L)$, i.e. Eqs. (6.273) and

$$\begin{aligned} & \left[S_+^{\alpha\beta}(x), S_-^{\alpha'\beta'}(y) \right] = \\ & \delta(x-y) \left(\delta^{\beta\alpha'} J^{\alpha\beta'}(x) + \delta^{\alpha\beta'} J^{\beta\alpha'}(x) + \delta^{\alpha\alpha'} J^{\beta\beta'}(x) + \delta^{\beta\beta'} J^{\alpha\alpha'}(x) \right) , \end{aligned} \quad (6.277)$$

$$\left[J^{\alpha\beta}(x), S_+^{\alpha'\beta'}(y) \right] = \delta(x-y) \left(\delta^{\beta\alpha'} S_+^{\alpha\beta'}(x) + \delta^{\beta\beta'} S_+^{\alpha\alpha'}(x) \right) , \quad (6.278)$$

$$\left[J^{\alpha\beta}(x), S_-^{\alpha'\beta'}(y) \right] = -\delta(x-y) \left(\delta^{\alpha\alpha'} S_-^{\beta\beta'}(x) + \delta^{\alpha\beta'} S_+^{\beta\alpha'}(x) \right) , \quad (6.279)$$

hence (6.274) is a $\text{Sp}(2N_L)$ antiferromagnet. Especially for $N_L = 1$ it is the spin 1/2 Heisenberg antiferromagnet as originally found in [30].

If, instead of $SU(N_C)$ we had the gauge group $U(N_C)$, gauge invariance would require that we impose the extra constraint

$$\mathcal{G}^0(x)|\Psi\rangle = 0 , \quad \forall x .$$

This can be fulfilled without electric fields, i.e. with $\rho_x = 0 \ \forall x$, only when N_L is even. Then the allowed $SU(N_C)$ representation at each site is the one with the Young Tableau given in Fig. 6.2 in the Introduction, and, with the constraint

$$\sum_\alpha J^{\alpha\alpha}(x) = 0 ,$$

$J^{\alpha\beta}(x)$ generate the $SU(N_L)$ algebra in this representation. Thus, $U(N_C)$ gauge theory is equivalent to an $SU(N_L)$ quantum antiferromagnet in the representation (6.2). This actually is true for all $N_C \geq 1$ ($N_C = 2$ is *not* special here as no hopping of colour singlets is allowed for $U(N_C)$ color groups).

As a concrete example, let us consider the case $N_L = 2$ which gives 4 continuum flavours in $d = 4$. The $SU(N_C \leq 3)$ gauge theory is equivalent to a $U(2)$ antiferromagnet. Using the identity

$$2\delta_{\beta\gamma}\delta_{\alpha\epsilon} = \boldsymbol{\sigma}_{\alpha\beta} \cdot \boldsymbol{\sigma}_{\gamma\epsilon} + \delta_{\alpha\beta}\delta_{\gamma\epsilon} \quad (6.280)$$

(with $\boldsymbol{\sigma}$ the Pauli matrices) we can change basis in the $U(2)$ Lie algebra to find the effective strong coupling Hamiltonian

$$H_{\text{eff}} = \frac{K}{2} \sum_{[x,i]} (\psi^\dagger(x)\boldsymbol{\sigma}\psi(x) \cdot \psi^\dagger(x+\hat{i})\boldsymbol{\sigma}\psi(x+\hat{i}) + \rho(x)\rho(x+\hat{i})) , \quad (6.281)$$

where

$$(\psi^\dagger \boldsymbol{\sigma} \psi \equiv \psi_A^{\alpha\dagger} \boldsymbol{\sigma}_{\alpha\beta} \psi_A^\beta) \quad (6.282)$$

and the representation at a given site has either $\rho_x = 0$ and the spin $j = N_C/2$ representation of $SU(2)$ or $\rho_x = \pm N_C$ and the spin $j = 0$.

If the gauge group is $U(N_C)$ then the density is constrained, $\rho_x = 0$ and each site is occupied by the $j = N_C/2$ representation of $SU(2)$ and (6.281) is the Hamiltonian of the quantum Heisenberg antiferromagnet.

The most subtle case is that of a $U(N_C)$ lattice gauge theory with *odd* N_L . There we first solve the problem of minimizing the contribution of $U(1)$ electric fields to the energy. This is done by solving the auxiliary problem of minimizing the energy function

$$\frac{1}{2g^2} \sum_{[x,i]} (E^0[x,i])^2$$

with the constraint

$$\sum_i (E^0[x,i] - E^0[x-\hat{i},i]) + \rho(x) \sim 0 , \quad (6.283)$$

where

$$\rho(x) = \psi_A^{\alpha\dagger}(x)\psi_A^\alpha(x) - N_C N_L / 2 .$$

If $N_C N_L$ is odd, the charge density $\rho(x)$ has only half-odd-integer eigenvalues ρ_x . Since they have no zero eigenvalue, the constraint (3.24) cannot be satisfied unless the $U(1)$ electric field is non-zero. This is also true of $SU(N_C)$ colour singlet states, since they always contain an integral multiple of N_C

particles. Then the operator $\psi^\dagger \psi$ has eigenvalue which is an integer multiple of N_C and when N_L is odd, the charge density $\rho(x)$ again has no nonzero eigenvalues. The configuration which minimizes the $U(1)$ energy is the most symmetric one,

$$E^0[x, \pm i] = \pm \frac{N_C}{4(d-1)} (-1)^{\sum_{p=1}^{d-1} x_p}$$

with the accompanying eigenvalues of the charge density

$$\rho_x = \pm \frac{N_C}{2} (-1)^{\sum_{p=1}^{d-1} x_p} .$$

If, instead of $U(1)$ electric fields, we had colour non-singlets, the energy would be higher. We have found a simple proof of this fact and will present it in a future publication.

6.8 Results on Strong Coupling

Using the mapping of the last section, known properties of antiferromagnets can be translated into nontrivial statements about the corresponding gauge theories. In the following we show how rigorous results about the former can be used to prove spontaneous broken (discrete) chiral symmetry in certain strong coupling gauge theories.

For $N_L = 1$ and $SU(N_C \geq 3)$ gauge group we obtained the antiferromagnetic Ising Hamiltonian (6.267). This Hamiltonian has two degenerate ground states,

$$\rho_x = \pm \frac{N_C}{2} (-1)^{\sum_{p=1}^{d-1} x_p} , \quad (6.284)$$

the antiferromagnetic phases. This is true in $(d-1) \geq 1$ and indicates chiral symmetry breaking. In fact, if we add a bare mass term to the Hamiltonian,

$$m \sum_x (-1)^{\sum_{p=1}^{d-1} x_p} \psi^\dagger(x) \psi(x) \quad (6.285)$$

then ($\langle \dots \rangle$ is the ground state expectation value)

$$\lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} \left\langle \frac{1}{V} \sum_x (-1)^{\sum_{p=1}^{d-1} x_p} \psi^\dagger(x) \psi(x) \right\rangle = \pm N_C/2 . \quad (6.286)$$

For $N_L = 2$ lattice flavours, the fermion mass term is

$$\sum_x (-1)^{\sum_{p=1}^{d-1} x_p} \psi^\dagger(x) \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \psi(x) \quad (6.287)$$

with m_1 and m_2 the masses for the two fermion lattice flavours. For $m_1 = -m_2 = m$ this term is identical with the order parameter for a Néel ordered ground state of the corresponding antiferromagnet [24]. For gauge group $U(N_C)$, we have the spin- j Heisenberg antiferromagnet (6.281) ($\rho(x) \sim 0$) with $j = N_C/2$. In $(d-1) \geq 2$ its ground state is known to have long-range order for $N_C/2 \geq 1$ and in $(d-1) \geq 3$ for $N_C/2 = 1/2$ [24]. (It is likely also true in $(d-1) = 2$ for $N_C/2 = 1/2$.) This implies Néel order [24], hence breaking of chiral symmetry.

If the gauge group is $SU(N_C)$ the strong coupling limit is the $U(2)$ antiferromagnet (6.281). To show whether its ground state has Néel order we must decide on the configuration of the representations of $U(2)$ as well as the spin states in the ground state. We are presently only able to do this with the assumption of translation symmetry by 2 sites. Then, by checking explicit configurations in 2+1 and 3+1 dimensions we see that the ground state of the $U(2)$ antiferromagnet coincides with that of the $SU(2)$ antiferromagnet. The reason is that the links which connect different representations have a higher bond energy than the links connecting sites with the same representation. Then, the ground state must be described either by the vacuum of the Heisenberg model ($j = N_C/2$, $\rho = 0$ everywhere) or the Ising model ($j = 0$, $\rho = \pm N_C$ everywhere). The Heisenberg model has the lower ground state energy and is therefore preferred. It is interesting that the Ising ground state is stable to fluctuations and could form meta-stable domains in a phase transition. It contains number density waves rather than spin density waves, breaks chiral symmetry but does not break the flavour $SU(2)$ symmetry.

Lower dimensional $SU(N)$ antiferromagnets have recently been studied by several authors [43,44]. Aside from antiferromagnetic phases, they have disordered phases with the critical behaviour of the second order phase transition represented by a nonlinear sigma model. It would be interesting to study whether these had an interpretation in strong coupling gauge theory. For example, it is well established that there is chiral symmetry breaking in 2+1 dimensional electrodynamics [2] and in that particular dimension there is a critical value of N below which the symmetry breaking sets in [3,4]. The result, which is obtained using Schwinger Dyson equations, has a striking similarity with the result of Read and Sachdev; chiral symmetry breaking disappears when the number of flavours is too large, the latter being also in the strong coupling limit. It is interesting that these results are obtained in mean field theory in the antiferromagnet picture, whereas in the field theory picture their derivation is much more complicated. It would be interesting if we could address questions about chiral symmetry in realistic gauge theories where, for example, chiral symmetry breaking has been suggested as a mechanism for generating Higgs in the standard model [6,37].

Salmhofer and Seiler proved that chiral symmetry breaking occurs in Euclidean lattice $U(N_C \leq 4)$ lattice gauge theory with latticized time and with 1 lattice flavour of staggered fermions in $d \geq 4$ [45]. To compare their results

with ours we observe that latticizing time leads to one additional fermion species doubling, so 1 lattice flavour on a spacetime lattice should be compared with 2 flavours on a continuum time, spatial lattice. Our present work agrees with their result and extends it to all N_C and to $SU(N_C)$ theories with the assumption of translation invariance by 2 sites.

Actually, for the minimal gauge theory that we discuss here, the mass term which is the order parameter in the $N_L = 2$ gauge theory and signals the presence of a symmetry breaking condensate is an iso-vector, rather than an iso-scalar operator. This means that for a realistic theory like QCD we obtain a different symmetry breaking pattern from that which is seen in the low energy spectrum whereas according to PCAC the condensate is that of an iso-scalar operator. This is because, among the ground states which are invariant under translations by two sites, the $j = N_C/2$ Heisenberg antiferromagnet ground state wins the competition for lower ground state energy. The other possible state, which looks like the Ising model, and which would be preferred from a physical point of view since it would have an iso-scalar condensate, has a higher ground state energy. Of course, this situation could be remedied by adding operators to the Hamiltonian which lower the energy of the iso-scalar ground state compared to the iso-vector ground state. One possibility is the four-fermion interaction

$$H_{\text{int}} = \frac{\text{const.}}{g^2} \sum_{x,i} \rho(x) \rho(x + \hat{i}) , \quad (6.288)$$

where the constant in front should be negative and at least as large in magnitude as the difference between the classical and quantum ground state energies of the quantum Heisenberg antiferromagnet. We conjecture that, if the lattice QCD with staggered fermions which we formulate is to reproduce the continuum QCD of the real world, it is necessary to add the term (6.288) at the tree level, that is in order to produce the correct symmetry breaking pattern in the strong coupling limit.

We finally note that a correspondence of $U(4N_L)$ antiferromagnets to strong coupling $SU(N_C)$ lattice gauge theories with N_L lattice flavours of naive fermions in $d = 4$ ($= 8N_L$ continuum flavours) was found already by Smit [52] (the factor 4 is due to additional doubling of the naive fermions). To our knowledge no rigorous results implying chiral symmetry breaking exist for these models. Also, from our point of view the identification of chiral symmetry is incorrect in that paper. We would interpret their chiral symmetry as a flavour symmetry and the discrete translation invariance by one site as chiral symmetry. This then makes the identification of all other properties of the symmetry breaking condensate different. In fact the generation of a spin density wave, which seems to be the most favourable mode for condensation, gives a condensate of a different sort from the one considered by [52].

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7 Quantum Chaos and Transport in Mesoscopic Systems

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7.1 The Dimensionless Conductance g

The so-called *Anderson localization* [1,2] is basically the fact that, because of the coherent scattering with the impurities, the electron wave functions in a metal are localized in space: as a result, the diffusion constant \mathcal{D} and the Drude conductivity $\sigma_D = e^2 \nu \mathcal{D}$ (where ν is the density of states) vanish and the system is an insulator. In fact, the diffusion constant is defined as¹

$$\mathcal{D} \equiv \frac{v_F l}{d}, \quad (7.1)$$

where d is the number of the spatial dimensions, v_F is the Fermi velocity and l is the electron mean free path between two impurity scattering events. If the impurity concentration is sufficiently high, l tends to zero, forcing \mathcal{D} to vanish.

Following the Thouless approach to the weak localization, we introduce now the conductance \mathcal{G} . We can use the Landauer–Büttiker formula [3]:

$$\mathcal{G} \equiv \sum_{\text{channels}} \mathcal{T}, \quad (7.2)$$

where \mathcal{T} is the transmission coefficient of the single channel. Using this expression, in a sample of dimension L , we have

$$\sigma = \mathcal{G} L^{2-d}. \quad (7.3)$$

The parameter \mathcal{G} is directly connected with the quantum nature of the transport inside the sample. In fact, let be δ the mean level spacing of the single-particle energy levels in the sample: then we have

$$\mathcal{G} = e^2 \nu \mathcal{D} L^{d-2} = \frac{e^2}{\hbar} (\nu L^d) \left(\frac{\mathcal{D} \hbar}{L^2} \right) = \frac{e^2}{\hbar} \frac{E_c}{\delta}, \quad (7.4)$$

¹ Note that the dimensions of \mathcal{D} are

$$[\mathcal{D}] = \frac{[\text{Length}]^2}{[\text{Time}]}$$

as follows from the Brownian motion theory.

where

$$E_c = \frac{\hbar D}{L^2} \quad (7.5)$$

is the Thouless energy which is defined as the inverse electron diffusion time throughout the sample. e^2/\hbar is a universal value, independent of the system, approximately equal to $(25 k\Omega)^{-1}$. Then, it becomes natural to introduce a *dimensionless* quantity, called *dimensionless conductance*:

$$g \equiv \frac{G}{e^2/\hbar} \quad (7.6)$$

simply equal to the ratio between the crucial energies E_c and δ or between the crucial times in the problem, the Heisenberg time $\tau_H = \hbar/\delta$ and the diffusion time (or Thouless time) $\tau_c = \hbar/E_c$. Consider now a wire with cross-sectional area A : then we can write

$$g = \frac{\nu D}{L} A \propto m p_F v_F l \frac{A}{L} = \frac{l}{L} (p_F^2 A) ,$$

where we can recognize the fact that $1/p_F^2$ is the area “occupied” by an electron involved in the transport (which has a momentum near the Fermi momentum): then, the ratio $A/(1/p_F^2)$ is the number of channels available for the conductance. We can observe that, for a long wire, the system is an insulator with a *localization length*

$$\xi = l (p_F^2 A) \quad (7.7)$$

(for a typical metallic wire with $\phi = 2$ mm, we obtain $\xi \approx 4 \cdot 10^7$ m, i.e., the Earth circumference...). In fact, if $L > \xi$, it results $g < 1$.

It is worth to stress the fact that the dimensionless conductance g is not related to the transport outside of the system: in other words, it is not necessary to connect the system (i.e., by leads) with the environment to define g which is the ratio between the Heisenberg time and the time to diffuse across the system (and not the time to leave the system).

7.2 Quantum Chaos

In Classical Mechanics, with the expression “Chaotic System” one usually indicates a physical system without continuous dependence on the initial conditions. In Quantum Mechanics, due to the Heisenberg indeterminacy, the evolution of the state of the system is not determined by the initial conditions but by the eigenvalue energy spectrum obtained by solving the Schrödinger equation. One possibility is to define a quantum system as chaotic, if its limiting classical dynamics is chaotic. However, one can enquire if there are quantum chaotic systems which have no easy classical limit as the heavy atomic nuclei.

We have one general suggestion from this definition: because in the chaotic classical dynamics we can recognize only one prime integer of the motion, i.e., the total energy, the quantum chaotic systems can have only one good quantum number, i.e., the total energy itself. To study the quantum chaos, we are forced to analyze the features of the energy spectrum, as the level distribution.

Then the questions are:

- A:** *How does the classical chaotic behaviour manifest itself in the quantum energy spectrum?*
- B:** *Are there special statistical properties in the energy spectrum which are signature of chaotic quantum behaviour?*

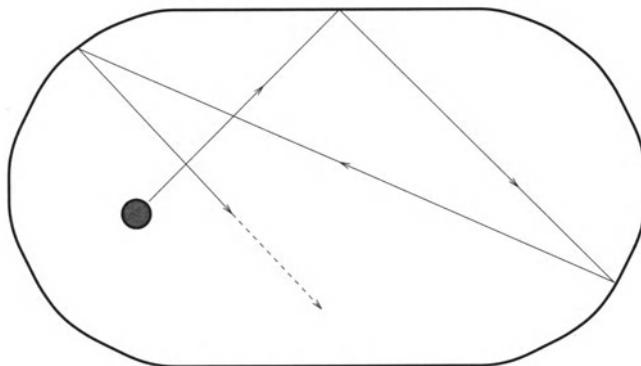


Fig. 7.1. Bunimovich Stadium: this is typical example of a classical chaotic system with a corrispective quantum spectrum described by the Wigner-Dyson Statistics (see below)

An example is the Bunimovich Stadium: it can be obtained adding two straight segments to a circular billiard, which is an integrable system. The classical dynamics of the Bunimovich Stadium is chaotic because of the two straight segments, which destroys the separability between different components of the momentum.

There are many physical examples of this kind, as the heavy atomic nuclei, the hydrogenic atoms in high magnetic field, isolated quantum dots etc. . . .

In that case, the energy spectrum of the quantum system is discrete and one can study the level distribution in the energy space: this distribution can be described by a function $\rho(s)$ where s is the nearest neighbours level spacing $s_i = \epsilon_{i+1} - \epsilon_i$. Further, we can define the function $\rho(s)$ as

$$\rho(s) = \sum_i \delta(s - s_i) \quad (7.8)$$

and the *mean level spacing* δ as

$$\delta = \frac{\int ds s \rho(s)}{\int ds \rho(s)} . \quad (7.9)$$

In the following, we rescale s as the dimensionless quantity

$$s \rightarrow \frac{s}{\delta} \quad (7.10)$$

and we will assume that the function $\rho(s)$ is continuous. At this point the question [B] becomes:

B1: *Is there a special spectral distribution for the nearest neighbours level spacing s in the case of quantum chaotic systems?*

The Random Matrix Theory [4] reproduces features of the spectral distributions of quantum systems whose classical analogs are chaotic. This theory associates at each system an ensemble of random Hamiltonians, satisfying the same symmetries, with a finite number of states, say N , or better an ensemble of $N \times N$ matrices, possibly dependent on an external parameter x , of the form

$$\hat{H}(x) = \hat{H}_0 + x\delta\hat{H}, \quad (7.11)$$

where \hat{H}_0 and $\delta\hat{H}$ are matrices with random elements. Finally, $\delta\hat{H}$ can be related to an external perturbation as the magnetic field in the case of highly excited hydrogenic atoms. x is the strength of the perturbation. The constraints on the form of $\delta\hat{H}$ are $\text{Tr}\delta\hat{H} = 0$ and the fact that this matrix has to satisfy the same symmetries of \hat{H}_0 . At the end of the calculation, one has to take the limit $N \rightarrow \infty$. To each realization of the Hamiltonian a probability is attached of the form

$$\rho(\hat{H}) = c \exp \left[-\beta \text{Tr} \mathcal{V}(\hat{H}) \right], \quad (7.12)$$

where c is a normalization constant. If $\mathcal{V}(\hat{H}) \propto \hat{H}^2$, the ensemble is called Gaussian. Wigner studied this ensemble because of the independently distributed matrix elements. In the $N \rightarrow \infty$ limit, \mathcal{V} becomes largely independent of the form of the spectrum (far from the edge of the spectrum). The index β depends on the symmetry of the Hamiltonians and counts the number of degrees of freedom in the matrix elements; we can have three kinds of Gaussian ensembles, whose names are related to the properties of the matrices that they contain: the orthogonal ensemble ($\beta = 1$) contains real symmetric matrices, the unitary one ($\beta = 2$) complex hermitean matrices and the symplectic one ($\beta = 4$) contains real quaternionic matrices. Physically, the case $\beta = 2$ applies to the case of time-reversal symmetry breaking (for example,

by an external magnetic field); in presence of time-reversal symmetry, $\beta = 1$ corresponds to the conservation of the spin and $\beta = 4$ to the spin-rotation symmetry breaking (by spin-orbit scattering). The distribution (7.12) can be transformed in the eigenvalue distribution

$$\rho(\{E_n\}) = c \prod_{i < j} |E_i - E_j|^\beta \prod_k \exp[-\beta V(E_k)] , \quad (7.13)$$

where we can recognize in the first term (the Van der Monde determinant [3]) a sort of “level repulsion”, in the sense that the probability to find two levels much closer than δ is suppressed.

We can resume our definitions in the following table:

ensemble	orthogonal	unitary	symplectic
β	1	2	4
matrix elements	real	complex	quaternionic
T – invariance	yes	no	yes
realizations	scattering potential	+ magnetic field	+spin – orbit scattering

(7.14)

Recalling the question [B1], there is an historic answer, resumed in the *Bohigas–Giannoni–Schmidt conjecture* [5]:

The spectrum fluctuation of the mean level spacing of quantum time-reversal invariant systems, whose classical analogs are strongly chaotic, have the Gaussian orthogonal ensemble pattern, which is a very particular case of our discussion.

The principal quantity needed to analyze the energy level statistics is the two-level correlation function

$$\mathcal{R}_2(s) = \left\langle \sum_{i,j} \delta(\epsilon - \epsilon_i) \delta(\epsilon + s - \epsilon_j) \right\rangle , \quad (7.15)$$

where i and j labels the eigenstates and the energies are measured in units of δ . In the presence of an external parameter x , as the magnetic field for the Hydrogen atoms, we have

$$\mathcal{R}_2(x, s) = \langle \epsilon(x) \epsilon(0) \rangle = \left\langle \sum_{i,j} \delta[\epsilon - \epsilon_i(x_0)] \delta[\epsilon + s - \epsilon_j(x_0 + x)] \right\rangle . \quad (7.16)$$

All these quantities are understood to be dimensionless, i.e. measured in units of δ . In the simplest unitary case, we obtain [6]:

$$\mathcal{R}_2(0, s) = \delta(s) - \left[\frac{\sin(\pi s)}{\pi s} \right]^2 . \quad (7.17)$$

The signature of the Random Matrix Theory behaviour, the so-called Wigner–Dyson statistics, is the rigidity of the energy spectrum [8], which corresponds to the following phenomena:

- the probability to find two levels separated by $s \ll 1$ vanishes as $s \rightarrow 0$;
- the level number variance $(\delta N)^2$ in an energy strip of width $N\delta$ is proportional to $\log N$ rather than N with proportionality factor dependent on $1/\beta$ and then $\delta N \ll \sqrt{N}$;
- oscillations in $\mathcal{R}_2(0, s)$ decay only algebraically.

The Fourier transform of $\mathcal{R}_2(0, s)$, the *form factor* or the *two-level structure factor* $S(0, \tau)$, presents a rapid growth for small arguments, i.e. for small times compared to the *Heisenberg time* (see Fig. 7.2) τ_H and then a plateau for bigger values of τ/τ_H : this means that at low energies or long times the levels are ONLY self-correlated. The singularity at $\tau = 2\pi$ is due exactly to the algebraic decay of the oscillations in $\mathcal{R}_2(0, s)$, while the vanishing of $S(0, \tau)$ when $\tau \rightarrow 0$ is due to the reduced fluctuations of the levels. These features are common for all the Gaussian ensembles, but only the limit $\tau \rightarrow 0$ can be studied by means of a semiclassical approach [9]; the other limit is not accessible with such an approximation. In summary, the comparison between numerical and analytical (RMT) results strongly suggests that the chaotic systems have a spectrum which follows the Wigner–Dyson statistics in the so-called *Universal Limit*, i.e. $g \rightarrow \infty$ or $E_c \gg \delta$: this limit is achieved when

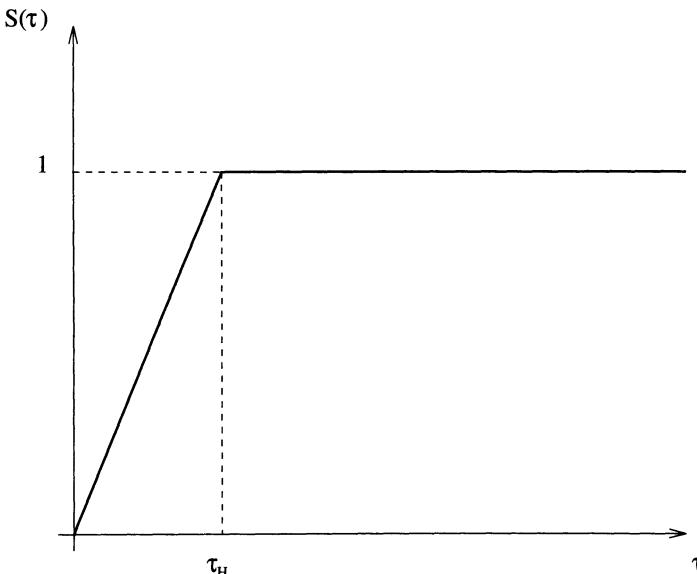


Fig. 7.2. The Form Factor $S(0, \tau)$ for the unitary ensemble. The linear part can be obtained by a semiclassical approximation, while the singularity in $\tau = \tau_H$ is completely nonperturbative

we are studying level correlations for $\tau \gg 1/E_c = \tau_c$, i.e. for times much larger than the ergodic time (see below).

Let consider now a free particle in a system with size L : now we are faced to a diffusion problem, in which we have to solve the diffusion equation, i.e., the states which describe the system are the eigenstates of the operator

$$\hat{D} = \hbar (\partial_\tau - \mathcal{D} \nabla^2) . \quad (7.18)$$

For the stationary case, the equation is separable and the spatial part becomes an eigenvalue equation for the local charge density:

$$\hbar \mathcal{D} \nabla^2 \rho_n = \gamma_n \rho_n , \quad (7.19)$$

where, for a box with dimension L , the eigenvalues γ_n are

$$\gamma_n = \frac{\hbar \mathcal{D}}{L^2} (\pi n)^2 = E_c (\pi n)^2 = g \delta (\pi n)^2 . \quad (7.20)$$

In this system, the Thouless energy acquires a deeper meaning: it is the inverse of the ergodic time, i.e. the time scale at which a wavepacket has become uniformly spread throughout the available phase space. Finally, we can recognize that the two questions, [A] and [B1], are strictly related but the answer to the second one allows one to define the quantum chaotic systems in a very general way, independently from their classical analogs. In conclusion, **a quantum system in the Universal Limit $g \rightarrow \infty$ can be defined as chaotic when its spectral fluctuations have the Wigner–Dyson pattern.**

But a better understanding is required in order to relate in a direct manner the Universal Regime and the Wigner–Dyson statistics from a complete analytical approach.

7.2.1 Beyond Random Matrix Theory

In the following we consider only quantum chaotic systems with $g \neq 0$. The behaviour of the quantum chaotic systems for times $t \gg \tau_c$ or energies $\epsilon \ll E_c$ is universal in the sense that it can be described by the Wigner–Dyson statistics without any non-universal parameter (Fig. 7.3). In fact, if $\epsilon \ll E_c$, the Thouless energy loses completely its importance and the system has only one parameter, δ ; measuring the energies in units of δ , we obtain a universal theory. For larger energies (i.e. smaller times), one can observe sample specific features. For frequencies much smaller than the Thouless energy ($s \ll g$), the statistics are close to the universal ones and then the perturbation theory works well; in fact, one can find that [10] the corrections goes as $(s/g)^2$. In this case, and for $s \gg 1$, the monotonic part of the two-level correlator can be obtained perturbatively [11], using the fact that

$$\nu(\epsilon) \equiv \frac{1}{\pi V} \int d\mathbf{r} \text{Im} \mathcal{G}^A(\mathbf{r}, \mathbf{r}, \epsilon) \quad (7.21)$$

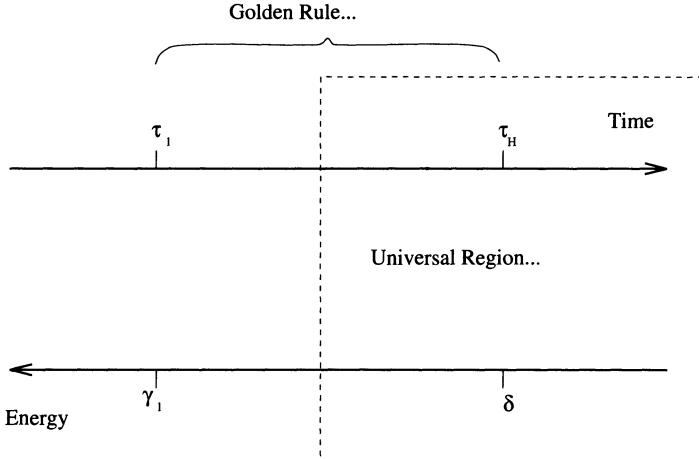


Fig. 7.3. Energy and time scales: the Universal region is shadowed

(\mathcal{G}^A is the advanced Green's function of the diffusion operator) and (7.16), as

$$\mathcal{R}_2^P(x, s) = \frac{1}{\alpha\pi^2} \text{Re} \sum_n \frac{1}{-is + x^2 + \gamma_n}, \quad (7.22)$$

where $\alpha = 1$ for the unitary ensemble and $\alpha = 2$ for the orthogonal and the symplectic ones [7]. We underline that (7.22) expresses \mathcal{R}_2^P as the real part of the Green's function of the diffusion operator. Of course, the perturbation theory can determine $S(0, \tau)$ only for small values of τ and since the oscillatory “tail” of $\mathcal{R}(0, s)$ is nonanalytic in $1/s$, it cannot be obtained perturbatively.

On the contrary, in order to obtain at least the leading contribution in $s \gg 1$ to the asymptotics of the two-level correlator, retaining the oscillatory tails, it is necessary to use the nonperturbative approach of [12] that is valid for any value of the ratio s/g . The oscillatory part $\mathcal{R}_2^{\text{osc}} \equiv \mathcal{R}_2 - \mathcal{R}_2^P$ is equal to

$$\mathcal{R}_2^{\text{osc}}(x, s) = \begin{cases} \frac{\cos(2\pi s)}{2\pi^2 |y|^2} \mathcal{D}(x, s), & \beta = 2 \\ \frac{\cos(2\pi s)}{2\pi^4 |y|^4} \mathcal{D}^2(x, s), & \beta = 1 \\ \frac{\cos(\pi s)}{2|y|} \mathcal{D}(x, s) - \frac{\cos(2\pi s)}{2\pi^4 |y|^2} \mathcal{D}^2(x, s) & \beta = 4, \end{cases} \quad (7.23)$$

where [7]

$$\mathcal{D}(s, x) = \prod_{n, \gamma_n \neq 0} \frac{1}{\left(\frac{s}{\gamma_n}\right)^2 + \left(1 + \frac{x^2}{\gamma_n}\right)^2} \quad (7.24)$$

and

$$y = x^2 - is. \quad (7.25)$$

Now, following the fact that (7.22) relates \mathcal{R}_2^p and the Green's function, we can argue that \mathcal{R}_2^p and \mathcal{R}_2^{osc} are related, regardless of the spectrum γ_n :

$$\mathcal{R}_2^p(x, s) = \operatorname{Re} \frac{1}{\alpha \pi^2 y^2} - \frac{1}{2\alpha \pi^2} \frac{\partial^2 \log \mathcal{D}(x, s)}{\partial s^2}. \quad (7.26)$$

From (7.24) it follows that $\mathcal{D}(x, s)$ decays exponentially for $s \gg g$: as a result, the long tails of the Wigner–Dyson two-level correlator becomes exponentially depressed for high frequencies and the singularity around the Heisenberg time of the form factor $S(x, \tau)$ is washed out on a scale $1/g$. In the opposite limit $1 \ll s \ll g$ the total correlator (the sum of (7.22) and (7.23)) gives the leading asymptotics of the universal limit². In the case of a closed cubic box with side L , (7.24) becomes:

$$\mathcal{D}(s, x) = \prod_{\mathbf{n} \neq 0} \frac{1}{\left[\frac{s}{g\pi^2(l^2 + m^2 + n^2)}\right]^2 + \left[1 + \frac{x^2}{g\pi^2(l^2 + m^2 + n^2)}\right]^2}, \quad (7.27)$$

where $\mathbf{n} = (l, m, n)$ and $l, m, n \in \mathcal{N}$; for $s \gg g$ and $d < 4$ we have

$$\mathcal{D}(0, s) \approx \frac{\exp\{-\pi(s/g\pi)^{d/2}\}}{d\Gamma(d/2) \sin \frac{\pi d}{4}}. \quad (7.28)$$

This kind of extrapolation allows one to assume that the first of the equations (7.23) together with (7.22) are the correct $g \gg 1$ asymptotics of the unitary ensemble at arbitrary frequency. The calculation is quite cumbersome and makes use of the supersymmetry formalism presented in [12]. In the quasi-1d case for $x = 0$ the spectral determinant can be evaluated exactly and, for $\beta = 2$,

$$\mathcal{R}_{2,1d}^{osc}(0, s) = \frac{1}{2g\pi^2 s^2} \frac{\cos(2\pi s)}{\sinh^2\left(\sqrt{\frac{s}{2g}}\right) + \sin^2\left(\sqrt{\frac{s}{2g}}\right)} \quad (7.29)$$

² In fact, when $g = E_c/\delta \rightarrow \infty$, the coefficients γ_n diverges apart γ_0 , then $\mathcal{D}(s) \rightarrow 1$.

from which one can obtain the form factor

$$\mathcal{S}_{1d}(0, 2\pi + \tau) = -\frac{|t|}{4\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\pi^2 n^2 g|t|}}{\pi^2 g n \sinh(\pi n)}. \quad (7.30)$$

This expression is regular for $\tau = 0$. In [7], the authors estimate also $\mathcal{S}(0, 2\pi)$ for any dimension:

$$\mathcal{S}(0, 2\pi) = \frac{1}{4\pi^4 g} \int_{-\infty+in}^{\infty+in} \frac{dz}{z^2} \prod_{n \neq 0} \left[1 + \left(\frac{z\gamma_1}{\gamma_n} \right)^2 \right]^{-1}. \quad (7.31)$$

We resume, to conclude the section, the conclusions drawn by the authors of [7]:

- Equation (7.23) describes the deviation from the universal behaviour of the level statistics in a disordered metallic grain; the deviation is controlled by the diffusion operator, which is a classical operator: it seems plausible that the nonuniversal part of any quantum chaotic system can be expressed as a function of the spectral determinant of some classical (system-specific) operator. In that case, (7.26) should be universally correct.
- The formalism used in [7] should be applicable even to the systems weakly connected with the environment, as long as the level broadening is smaller than $\min \{ \delta x^2, E_c \}$.
- The classification of physical systems into three universality classes may be an oversimplification: in fact, a number of crossovers from a class to another is known, as the case of a system subjected to a magnetic field which remains orthogonal for short times and has unitary long time behaviour.
- The $1/g$ -corrections to the level statistics are substantially different from Random Matrix Theory in preferred basis [13] and then from the finite temperature behaviour of the Calogero–Sutherland model [14].

7.2.2 From Disordered to Chaotic Systems

In the last section, we summarize the results about the $1/g$ -corrections to the universal level statistics in a disordered metallic grain. Now, we will try to get a deeper insight to the first of the three conclusions drawn in the last section.

We will start noticing that (7.22) for $x = 0$ can be written as

$$\mathcal{R}_2^P(s) \equiv -\frac{1}{2\alpha\pi^2} \frac{\partial^2}{\partial s^2} \log \mathcal{P}(s), \quad (7.32)$$

where α has been introduced above and [15]

$$\mathcal{P}(s) = \frac{\mathcal{D}(s)}{s^2} = \prod_n \frac{A(\gamma_\mu)}{s^2 + \gamma_\mu^2} \quad (7.33)$$

with

$$A(\gamma_\mu) = \begin{cases} \gamma_\mu^2 & \text{if } \gamma_\mu \neq 0 \\ 1 & \text{if } \gamma_\mu = 0. \end{cases} \quad (7.34)$$

The determinant $\mathcal{P}(s)$ differs from (7.24) by the insertion of the zero mode contribution s^{-2} . From now on, we will refer as spectral determinant to the definition (7.33). Using this new definition, we will try to find a semiclassical approach to the quantum chaos, relating the behaviour of the level statistics and the spectral determinant of some classical operator. The semiclassical analysis starts from the Gutzwiller's trace formula [16] which expresses the density of states as a sum over classical periodic orbits:

$$\nu(\epsilon) = \frac{1}{\delta} + \frac{1}{\pi\hbar} \operatorname{Re} \sum_u \left\{ T_u \sum_r \frac{\exp \left[i \frac{S_u(\epsilon)}{\hbar} - i\varphi_u r \right]}{\sqrt{|\det(M_u^r - I)|}} \right\}. \quad (7.35)$$

Here u labels the periodic orbits with period T_u , action $S_u(E)$ and Maslov phase φ_u ; r is the number of the repetitions of the orbit and M_u^r is the monodromy matrix associated with the linearized dynamics on the Poincaré section perpendicular to the orbit. Using the formula (7.35), one can express the two-level correlator as a double sum over periodic orbits, whose diagonal part will be the perturbative \mathcal{R}_2^p . Expanding $S_u(\epsilon + s)$ up to the first order in s , $S_u(\epsilon + s) \approx S_u(\epsilon) + T_u s$, we obtain

$$\mathcal{R}_2^p(s) = \frac{1}{2\pi^2} \operatorname{Re} \sum_u T_u^2 \left[\sum_{r=1}^{\infty} \frac{e^{isT_u r}}{|\det(M_u^r - I)|} \right] \quad (7.36)$$

($\hbar = 1$). The traditional way to evaluate the sum is to approximate it by an integral

$$\sum_u \frac{f(T_u)}{|\det(M_u - I)|} \approx \int \frac{dt}{t} f(t) \quad (7.37)$$

for any sufficiently smooth function $f(t)$. This approximation [17] holds in the limit $t \rightarrow \infty$ where long periodic orbits which explore the whole energy shell are considered. To calculate $\mathcal{R}_2^p(s)$, the time t have to satisfy two opposite limits: $\tau_c \ll t \ll \tau_H$, i.e. $\delta \ll \epsilon \ll E_c$. In this way, it is possible to get only the smooth part of the two-level correlator. In order to improve this result, in [15], the authors analyse carefully the sum in (7.36). Let Λ_u (with $|\Lambda_u| > 1$) the eigenvalue of the matrix M_u . The *area preserving property* of the latter implies that the second eigenvalue is $1/\Lambda_u$ and then

$$|\det(M_u^r - I)| = \frac{1}{|\Lambda_u|^r} \sum_{k=0}^{\infty} \left(\frac{k+1}{\Lambda_u^{rk}} \right) \quad (7.38)$$

and

$$\mathcal{R}_2^p = -\frac{1}{2\pi^2} \frac{\partial^2}{\partial s^2} \text{Re} \sum_{u,k} \left[(k+1) \sum_{r=1}^{\infty} \frac{(t_{uk})^r}{r^2} \right], \quad (7.39)$$

where

$$t_{uk} = \frac{e^{isT_u}}{|\Lambda_u| \Lambda_u^k}. \quad (7.40)$$

Using the new definition of the spectral determinant, we can write, up to a normalization constant

$$\mathcal{D}(s) = \left| \mathcal{N} \tilde{Z}(is) \right|^2, \quad (7.41)$$

where

$$\begin{aligned} \frac{1}{\tilde{Z}(is)} &= \prod_u \prod_{k=0}^{\infty} e^{[(k+1)\phi(t_{uk})]}, \\ \phi(x) &= \int_0^x \frac{dt}{t} \log(1-t). \end{aligned} \quad (7.42)$$

Since the normalization constant is not essential in the determination of the two-level correlator, we will discuss it later. Suppose now that all the periodic orbits are very unstable, and then that the system is strongly chaotic, namely $|\Lambda_u| \gg 1$ for all the u 's. In this case, $t_{uk} \rightarrow 0$, $\phi(t_{uk}) \rightarrow -t_{uk}$ and finally the $\tilde{Z}(z)$ reduces to the dynamical zeta function [18]:

$$\frac{1}{Z(z)} = \prod_u \prod_{k=0}^{\infty} \left(1 - \frac{e^{zT_u}}{|\Lambda_u| \Lambda_u^k} \right)^{k+1}. \quad (7.43)$$

This function is the spectral determinant associated with the Perron–Frobenius operator \mathcal{L}^t [19], which is the classical evolution operator which propagates the phase space density for a time $t > 0$ with kernel:

$$\mathcal{L}^t(\mathbf{y}, \mathbf{x}) = \delta[\mathbf{y} - \mathbf{u}(\mathbf{x}; t)], \quad (7.44)$$

where \mathbf{v} is a phase space vector with $2d$ components and $\mathbf{u}(\mathbf{x}; t)$ is the phase space point at which a particle that starts its motion in \mathbf{x} arrives after a time t . The eigenvalues of the Perron–Frobenius operator have the form $\exp(-\gamma_n t)$, where the γ_n appear in complex conjugate pairs with $\text{Re}\gamma_n \geq 0$: they are associated with the decaying modes of a perturbation in the density of classical particles with chaotic dynamics, analogous to the diffusion modes in a disordered system, with the difference that the eigenvalues in the latter case are real numbers (see the last section). The only real eigenvalue of \mathcal{L}^t is the leading one, $\gamma_0 = 0$, which corresponds to the particle conservation. The

dynamical zeta function can be written, apart for a normalization constant, as

$$\frac{1}{Z(z)} = \prod_n B_n (z - \gamma_n) , \quad (7.45)$$

where B_n are suitable regularization factors. The periodic orbit expansion, i.e. the product in (7.43), of the general expression (7.45) is exact. Of course, this form of the dynamical zeta function is useless and we need a resummed formula: this can be obtained expanding the product over the periodic orbits and ordering the various terms in a way that leads to the maximal cancellation among them: this is the so-called *cycle expansion* [20], whose principal features is the fact that the long periodic orbits may be approximated by linear combinations of few short ones. Using this approximation, one finds

$$\mathcal{R}_2^p(s) = \frac{1}{2\pi^2} \operatorname{Re} \sum_n \frac{1}{(-is + \gamma_n)^2} \quad (7.46)$$

in complete analogy with the result (7.22) and (7.26) of the last section. The universal part of the two-level correlator corresponds simply to the first term of this sum ($\gamma_0 = 0$) while the other terms are sample specific.

Now we can discuss the normalization constant \mathcal{N} . Assuming that the system is ergodic, γ_0 has unit multiplicity. We obtain

$$\frac{1}{\mathcal{N}} = \lim_{z \rightarrow 0} z Z(z) . \quad (7.47)$$

At this point, we can use a formalism typical of the field theory (for later developments) and express the semiclassical density of states as the logarithmic derivative of the Selberg zeta function [21,22], that is defined as the spectral determinant associated with the semiclassical energy spectrum of the system under consideration:

$$\zeta_S(\epsilon) \equiv \prod_j b_j (\epsilon - \epsilon_j) = \prod_u \prod_{k=0}^{\infty} \left(1 - \frac{e^{iS_u(\epsilon) - i\nu_u}}{\sqrt{|A_u| A_u^k}} \right) , \quad (7.48)$$

where the b_j are regularization factors and ϵ_j are the semiclassical energy levels of the system. The second equality is written in the case of two-dimensional systems. One can show that the two-level correlator can be written as

$$\mathcal{R}(s) = -\frac{1}{\pi^2} \frac{\partial^2}{\partial s^2} \langle \operatorname{Im} \log [\zeta_S(\epsilon + s)] \operatorname{Im} \log [\zeta_S(\epsilon)] \rangle , \quad (7.49)$$

where the angular brackets indicate an averaging which retains only the diagonal elements in the double sum. Again, the diagonal approximation of the (7.49) gives the perturbative term. In this formalism, one can also write down the spectral determinant as

$$\mathcal{P}(s) = \mathcal{N}^2 \exp \{ 2 \operatorname{Re} \langle \log [\zeta_S(\epsilon + s)] \log [\zeta_S^*(\epsilon)] \rangle_d \} , \quad (7.50)$$

where the subscript d indicates the diagonal approximation. Assuming that all the eigenvalues have unit multiplicity, we can write also the form factor

$$\mathcal{S}(\tau) = \frac{|\tau|}{2\pi} \sum_n e^{-\gamma_n |\tau|} \quad (7.51)$$

and again the universal limit corresponds to neglect all the eigenvalues but the leading one $\gamma_0 = 0$. The higher eigenvalues contribute corrections which are in general oscillatory and exponentially decreasing; the oscillatory part of the form factor can be written as

$$\mathcal{S}_{\text{osc}}(\tau) = -\frac{|\tau|}{2\pi} - \sum_{n \neq 0} \left[1 + \left(\frac{s}{\gamma_n} \right)^2 \right] \frac{\mathcal{P}(i\gamma_n)}{2\pi\gamma_n} e^{-\gamma_n |\tau|}. \quad (7.52)$$

In general, one can expect that the leading contribution comes only from the lowest eigenvalues of the Perron–Frobenius operator. To discuss the results, we will assume in the following that the dynamics is dominated by a pair of eigenvalues γ_1 and that $\text{Re}\gamma_n \gg \text{Re}\gamma_1$ for all $n > 1$. In this case, the authors of [15] can identify five regions of the parameter τ :

1. $\tau \approx \tau_c$: here $\mathcal{S}(\tau)$ is composed by several δ -function peaks located at the periods of the short orbits and weighted by their instability;
2. $\tau_c < \tau < 1/\text{Re}\gamma_1$: in this region, one can observe deviations from universalities due to the contribution of many short time orbits, with oscillation period $1/\text{Im}\gamma_1 \approx \tau_c$
3. $1/\text{Re}\gamma_1 < \tau < 2\pi - 1/\text{Re}\gamma_1$: this is the universal perturbative regime where the form factor is linear in τ ;
4. $2\pi - 1/\text{Re}\gamma_1 < \tau < 2\pi + 1/\text{Re}\gamma_1$: this is the nonperturbative region around the Heisenberg time. The deviations, exponentially decreasing oscillations, from the universality are strikingly similar to those close to τ_c , but their amplitudes and/or phases can be different. The singularity typical of the Random Matrix Theory will be in general smeared by these deviations;
5. $\tau > 2\pi + \text{Re}\gamma_1$: again the universality results, with $\mathcal{S}(\tau) \equiv 1$.

The generalization to orthogonal and symplectic ensembles can be achieved using the correct (7.23) with the substitution $\mathcal{D}(s) \rightarrow \mathcal{P}(s)/s^2$.

In conclusion, the behaviour of the structure factor in the vicinity of the Heisenberg time is a manifestation of a kind of ricorsione effect in the Gutzwiller's formula: the tail of the series, the long periodic orbit contribution, is encoded in its "head", i.e. in the short periodic orbits contribution. As a result, the form factor close to τ_H is determined by the same short periodic orbits as around τ_c . Thus, the dynamics of a quantum chaotic system for very long times is determined by the short time dynamics of the classical system. In [15], the authors established a close relation between the diffusion operator in disordered grains and the Perron–Frobenius operator in the general case

and then they associated the spectral determinant of the diffusion equation and the dynamical zeta function. The extension to more general cases has been tempted in [22] and [23].

7.3 Quantum Dots

Present nano-fabrication technologies allow the realization of microscopic systems in which the motion of electrons is limited in one or more dimensions. If the confinement is made on 1 or 2 dimensions one has a to 2D quantum gas and a quantum wire respectively. Three-dimensional confinement produces a quantum dot (QD), a 0D system which is expected to have an atomic-like spectrum with a discrete DoS $\propto \sum_i \delta(E - E_i)$.

The confinement of the QD electrons can be made through depletion potentials (horizontal QD) or using several heterostructures, one on the top of the other (vertical QD) [24].

The discrete QD spectrum can be determined in several ways and is qualitatively very different from that of a system with a continuous DoS.

By coupling two or more QDs by interposing thin gates, through which electrons can tunnel, one can produce QD molecules [25] and, in the case of QD arrays, it is also possible to observe the band structure characteristics of a solid [24].

7.3.1 Spectroscopy

Several techniques, corresponding to different kinds of spectroscopy, can be used in order to determine the spectrum of a QD. When use of a QD array is made, as in the conventional capacitance spectroscopy or in the far infrared spectroscopy, the energy levels turn out to be broadened because of the inhomogeneities of the QDs which make up the array. On the contrary, the discrete spectrum can be directly observed if it is possible to investigate the properties of a single QD.

This last case is possible for instance when using Resonance Tunneling Spectroscopy (RTS) [26–30] or Capacitance Spectroscopy (CP) of a single QD [31–33]. The spectrum is here mainly influenced by two factors:

- The wave nature of electrons, which determines the atomic-like quantum level structure of the QD (energy quantization).
- The discrete value of the electric charge (charge quantization), which gives rise to a charging energy threshold $\approx e^2/C$, where C is the capacitance of the QD, and to the consequent Coulomb Blockade (CB) effects [34].

RTS is based on the measurement of the current I versus the applied voltage V , while CS considers the dependence of the (differential) capacitance

$C(dC/dV)$ on the applied voltage V . In any case one obtains resonance peaks from which one can infer the energy levels of the QD.

Intraband spectroscopy [35] (i.e. intraband far infrared spectroscopy) is made on QD arrays and studies the dependence of the radiation transmittance on the magnetic field or frequency. The resonances observed are related to the energy spectrum and to the number of electrons of the QD.

Photoluminescence Spectroscopy [36] can be performed both on QD arrays and on single QDs. On the contrary of intraband spectroscopy, it strongly depends also on the many-body effects due to the electron-electron interaction, besides providing information on relaxation mechanisms [37].

Thermopower spectroscopy [38] is based on the measurement of the QD thermopower $S = \nabla V / \nabla T$, which is a quantity very sensible to all the QD characteristics and, in particular, to the many-body effects. In the regime of single-electron charging, S presents oscillations analogous to the Coulomb blockade oscillation of electrical conductance.

All these techniques have provided evidence of the atomic-like nature of microscopic three-dimensionally confined systems and on their peculiar electronic properties.

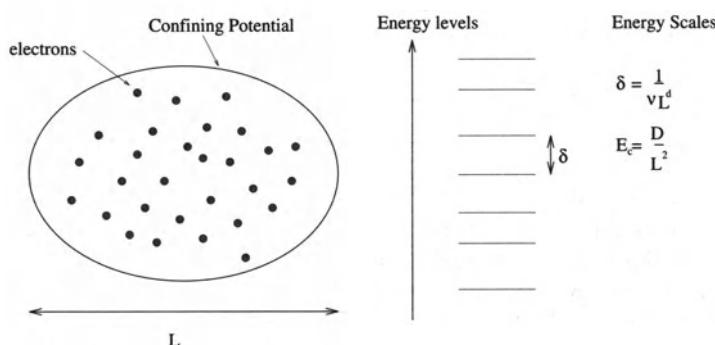


Fig. 7.4. Schematic representation of a Quantum Dot

In Fig. 7.4 we report a schematic representation of a Quantum Dot which can be viewed as a small conducting island in which one can confine a limited number of electrons, using physical (insulating walls) and/or electrostatic obstacles (suitable *ddp*). From a more general point of view, also the heavy atomic nuclei can be thought of a Quantum Dot (the confined particles being the nucleons and the confining potential being the nuclear potential) or an heavy atom (electrons and Coulomb potential).

Of course, the particles inside the Dot can interact one with each other with energies which are smaller than the confining potential. A first approximation is to consider the particles as independent and interacting only with the confining potential and with an overall potential due to the entire ensem-

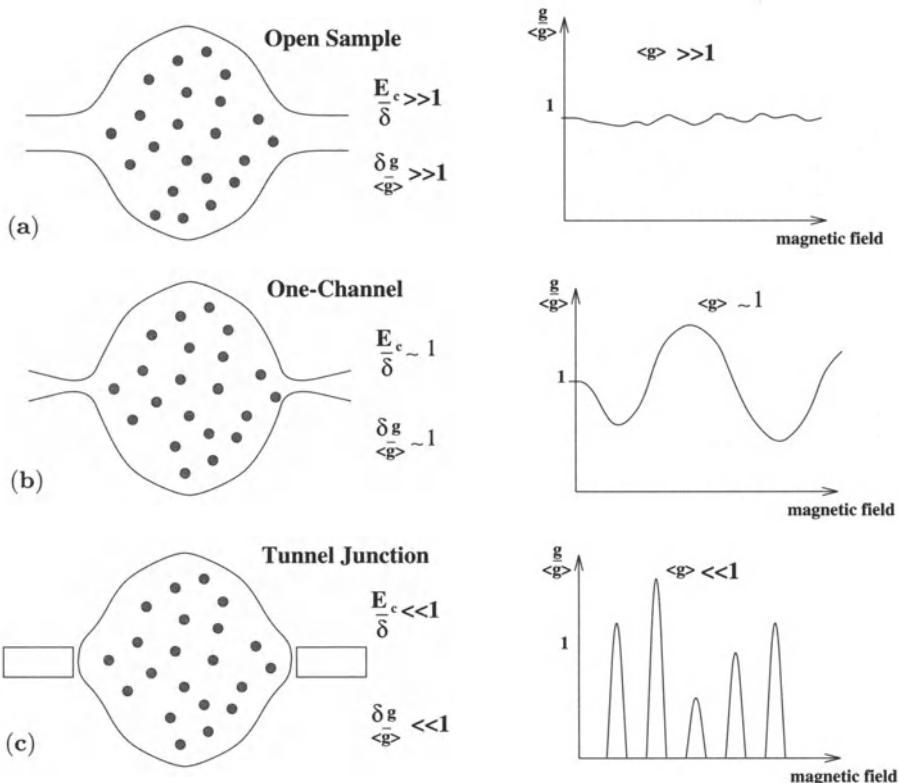


Fig. 7.5a–c. Open, one-channel and closed (tunnel junction) systems. On the right, a schematic picture of the relative dimensionless conductance is reported, together with the related mesoscopic fluctuations. Note that the fluctuations are of the order of the average only where the theory is a two-parameter theory (E_c and δ), while when the theory is a one-parameter theory can be considered universal

ble of the other particles: in this case, the peaks in Fig. 7.5c (which can be obtained measuring the differential conductance $G = dI/dV_{SD}$ varying V_{SD} (Fig. 7.6), are spaced by the mean level spacing and the fluctuations in the amplitude are of the order of the mean value. These are exactly the characteristics of the Universal Regime: the spectral statistics can be obtained from the Random Matrix Theory.

7.3.2 Orthodox Theory

The typical approach to the physics of Quantum Dots is based on the hypothesis that every electron can be considered interacting only with a Mean Field potential due to the confining potential and to the charging effect related to the Coulomb interaction with the other electrons. This charging effect is re-

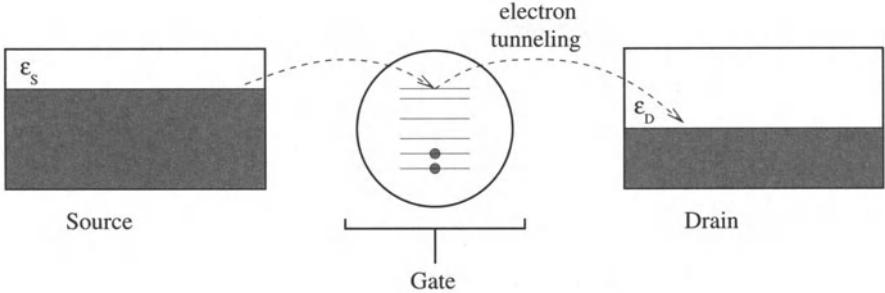


Fig. 7.6. Schematic energy diagram for a tunnel junction between a Quantum Dot and two external leads, “Source” and “Drain”, with a ddp V_{SD} . On the top of the Dot, a gate voltage is applied to explore the dependence of the conductance from the energy spectrum of the Dot itself. A schematic electron tunneling, with a “bounce” on a resonant level on the Dot, is drawn

lated to the Dot capacitance³. In such a way, the Hamiltonian of the system results

$$\hat{H} = \sum_{\alpha} \hat{n}_{\alpha} \epsilon_{\alpha} + \left(\hat{N} - N_0 \right)^2 \frac{e^2}{2C} = \hat{H}_0 + \left(\hat{N} - N_0 \right)^2 \frac{e^2}{2C}, \quad (7.53)$$

where

$$\begin{aligned} \hat{N} &\equiv \sum_{\alpha} \hat{n}_{\alpha} \\ \hat{n}_{\alpha} &\equiv \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha} \end{aligned} \quad (7.54)$$

and \hat{a} is the electron annihilation operator. Defining

$$\begin{aligned} \delta &= \langle \epsilon_{\alpha+1} - \epsilon_{\alpha} \rangle \quad (\text{mean level spacing}) \\ E_{ch} &= \frac{e^2}{2C} \quad (\text{charging energy}) \end{aligned} \quad (7.55)$$

we find, in real systems,

$$E_{ch} \gg \delta, \quad (7.56)$$

i.e. $E_{ch}/\delta \approx 2 \div 20$. This inequality ensures the fact that the important quantity in the problem is the number N of electrons into the Dot: to change N one needs a large energy E_{ch} and this corresponds to a big rearrangement of the electron levels.

A heuristic analysis gives immediately the conceptual explanation for the phenomenon of Coulomb Blockade. In fact, the energy of the system can be

³ The Dot capacitance is defined as the capacitance needed to produce the electrostatic field outside the Dot, when the Dot itself is isolated; because the Dot cannot be isolated, this is a sort of “formal” definition.

estimated as

$$\mathcal{E}_{\text{el}}(Q) \approx \frac{Q^2}{2C} - V_g Q , \quad (7.57)$$

where Q is the charge on the Dot, while V_g is the gate potential. Q is discrete:

$$Q = eN \quad (7.58)$$

(e is the electronic charge and N the total number of electrons). Because the charging energy is much larger than the mean level spacing, the fundamental quantity in the problem is just the electron number N . Then, we can identify two different situations:

A: N is odd: the transition from the N electrons state and the $N \pm 1$ electrons state requires a finite energy.

B: N is even: the states with N and $N \pm 1$ are almost degenerate.

The situation [A] corresponds to the Coulomb Blockade (we cannot have a finite current for low voltages) and the situation [B] corresponds to peaks of the conductivity.

The important features can be understood in a mean field scheme. One can ask if there is something of interest *beyond* this sort of “Standard Model” for quantum dots. For example, we can introduce the electron-electron interaction beyond the Mean Field Theory:

$$\hat{H} = \hat{H}_0 + \left(\hat{N} - N_0 \right)^2 \frac{e^2}{2C} + \hat{H}_{\text{int}} , \quad (7.59)$$

where

$$\hat{H}_{\text{int}} = \sum_{\alpha\beta\gamma\delta,\sigma\sigma'} M_{\alpha\beta\gamma\delta} \hat{a}_{\alpha\sigma}^\dagger \hat{a}_{\beta\sigma'}^\dagger \hat{a}_{\gamma\sigma'} \hat{a}_{\delta\sigma} \quad (7.60)$$

(α is the orbital index while $\sigma = (\uparrow, \downarrow)$ is the spin index) and

$$M_{\alpha\beta\gamma\delta} = \int d\mathbf{r}_1 d\mathbf{r}_2 \times [\psi_\alpha^*(\mathbf{r}_1) \psi_\beta^*(\mathbf{r}_2) \psi_\gamma(\mathbf{r}_1) \psi_\delta^*(\mathbf{r}_2) U(\mathbf{r}_2 - \mathbf{r}_1)] ; \\ U(\mathbf{r}) = \lambda \delta V \delta(\mathbf{r}) \quad (7.61)$$

(λ is a dimensionless interaction constant, δ is the mean level spacing and $V = L^d$ is the sample volume).

The $M_{\alpha\beta\gamma\delta}$ tensor depends on the overlap of the electronic wave functions: in a second quantized form

$$M_{\alpha\beta\gamma\delta} = \lambda \delta N \sum_i \psi_\alpha^*(\mathbf{i}) \psi_\beta^*(\mathbf{i}) \psi_\gamma(\mathbf{i}) \psi_\delta(\mathbf{i}) , \quad (7.62)$$

where, because of the orthogonality of the electronic wave functions

$$\langle \psi_\alpha^*(\mathbf{i}) \psi_\beta(\mathbf{j}) \rangle = \frac{\delta_{\alpha\beta}\delta_{ij}}{N}. \quad (7.63)$$

We will call “double diagonal elements” those with $\alpha = \beta = \gamma = \delta$ and “diagonal elements” they such that $\alpha = \beta$ and $\gamma = \delta$ or $\alpha = \gamma$ and $\beta = \delta$ or $\alpha = \delta$ and $\beta = \gamma$. Otherwise, we will speak of off-diagonal elements. The universal limit corresponds to the fact that the off-diagonal elements vanishes and the diagonal elements are of the order of the mean level spacing (and *do not fluctuate*).

7.3.3 0d Hamiltonian

We begin with the Hamiltonian (7.59). Now we want investigate what happens in a quantum dot with large but finite g . We know that [39] the corrections to the diagonal elements are of the order of $\frac{1}{g} \log L$ and the fluctuations of these elements are also of the order of $1/g$. Furthermore, the off-diagonal elements comes out to be $\mathcal{O}\left(\frac{1}{g}\right)$. Our goal is to understand what happens to the single-particle spectrum under these conditions. In summary, we have

$$\begin{aligned} g &= \frac{E_c}{\delta} \gg 1 \\ M_{\text{offdiag}} &\approx \frac{\delta}{g} \quad (\text{random and small}) \\ \langle M_{\text{diag}} \rangle &\approx \delta \left[1 + \mathcal{O}\left(\frac{1}{g}\right) \right] \\ \sqrt{\langle M_{\text{diag}}^2 \rangle - \langle M_{\text{diag}} \rangle^2} &\approx \frac{\delta}{g} \end{aligned} \quad (7.64)$$

and for a finite 2d system:

$$\langle M_{\text{diag}} \rangle = \delta \left[1 + \# \left(\frac{1}{g} \right) \log L \right]. \quad (7.65)$$

Superconductivity. We know that, in the universal regime, the Anderson Theorem holds:

THM: The superconductive gap Δ and the transition temperature T_c does not depend on g .

In this section, we want to analyse the possible corrections to this statement in a quantum dot. Let suppose that we can neglect the off-diagonal elements of the interaction tensor. Then the interaction Hamiltonian can be written as

$$\hat{H}_{\text{int}} = \delta \sum_{\alpha\beta} \left\{ \lambda_1 \hat{n}_\alpha \hat{n}_\beta + \lambda_2 \hat{S}_\alpha \hat{S}_\beta + \lambda_3 \left(\hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\downarrow}^\dagger \right) \left(\hat{a}_{\beta\uparrow} \hat{a}_{\beta\downarrow} \right) \right\}, \quad (7.66)$$

where

$$\begin{aligned}\hat{n}_\alpha &\equiv \hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\uparrow} + \hat{a}_{\alpha\downarrow}^\dagger \hat{a}_{\alpha\downarrow} \\ \hat{S}_\alpha &= \hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\uparrow} - \hat{a}_{\alpha\downarrow}^\dagger \hat{a}_{\alpha\downarrow}.\end{aligned}\quad (7.67)$$

The first contribution, measured by λ_1 , to the (7.66) represents a correction to the charging energy due to the coupling between electrons in different electronic orbitals (Hartree contribution). The second term, proportional to λ_2 , is directly related to the spin-exchange interaction between electrons while the third term (λ_3) is the usual BCS interaction. The last term is dominant in a superconductive system: so, we can suppose that all the other terms can be included into the eigenenergies ϵ_α of the hamiltonian \hat{H}_0 (7.59). Then, we obtain a new hamiltonian

$$\hat{H} = \sum_{\alpha,\sigma} \epsilon_\alpha \hat{a}_{\alpha\sigma}^\dagger \hat{a}_{\alpha\sigma} + \lambda \delta \sum_{\alpha\beta} \left(\hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\downarrow}^\dagger \right) \left(\hat{a}_{\beta\uparrow} \hat{a}_{\beta\downarrow} \right). \quad (7.68)$$

With the help of this model, assuming the eigenenergies ϵ_α randomly distributed with mean level spacing δ , we can study the possibility that the superconductive gap depends on disorder (i.e., on g), i.e. the corrections to the Anderson Theorem. The correction to the theorem in a system with a large but finite g is due to Ovchinnikov [39] and Fukuyama *et al.* [40]:

$$\frac{\delta T_c}{T_c} \approx -\frac{1}{g} (\log T_c \tau)^3, \quad (7.69)$$

where τ is the impurity scattering time (the quasiparticles lifetime in the Fermi liquid). These are just the corrections to $\langle M_{\text{diag}} \rangle$.

Defining

$$\begin{aligned}\hat{S}_\alpha^z &= \hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\uparrow} + \hat{a}_{\alpha\downarrow}^\dagger \hat{a}_{\alpha\downarrow} \\ \hat{S}_\alpha^+ &= \hat{a}_{\alpha\uparrow}^\dagger \hat{a}_{\alpha\downarrow}^\dagger \\ \hat{S}_\alpha^- &= \hat{a}_{\alpha\uparrow} \hat{a}_{\alpha\downarrow}\end{aligned}\quad (7.70)$$

we obtain the hamiltonian of a $S = 1/2$ chain of an XY model with long-range λ -interaction in a transverse magnetic field whose intensity grows with the distance from the Fermi level:

$$\frac{\hat{H}}{\delta} = \frac{\lambda}{2} \sum_{\alpha\beta} \left(\hat{S}_\alpha^x \hat{S}_\beta^x + \hat{S}_\alpha^y \hat{S}_\beta^y \right) + \sum_\alpha \alpha \hat{S}_\alpha^z. \quad (7.71)$$

Obviously, this model presents an ill-defined thermodynamic limit but one can notice that both terms in the (7.71) are proportional to N^2 , and then, to recover a correct thermodynamic limit, it is sufficient to multiply by $1/N^2$. Because of the long-range interaction, this 1d-model can undergo a $T = 0$ phase transition, varying the dimensionless coupling λ or adding randomness to the energies ϵ_α .

Furthermore, using the hamiltonian (7.68) without disorder, we can analyse the tunneling density of states (DoS) in a superconducting system driven by Zeeman splitting E_Z into the paramagnetic phase. One can consider an ultrasmall superconducting grain, obtained in a $2d$ electron gas, placed in a magnetic field parallel to the plane of the film. It is known that the size of a Cooper pair is geometrically restricted and a flux through this pair reaches a flux quantum for fields higher than H_{c2} . We assume that the superconductivity is already destroyed by the Zeeman splitting $E_Z = g_L \mu_B H$ when it happens (g_L is the Landé factor and μ_B is the Bohr magneton).

One might expect that the Cooper pairing is irrelevant far from the “critical” splitting ($E_Z \simeq \sqrt{2}\Delta$) in the paramagnetic phase. Instead, this is not the case. In fact, in this situation, the eigenenergies ϵ_α depend on the electron spin direction: we have $\epsilon_{\alpha\uparrow} = \epsilon_\alpha - E_Z/2$ and $\epsilon_{\alpha\downarrow} = \epsilon_\alpha + E_Z/2$. The structure of the ground state, in absence of a superconductive gap, is similar to that *without* interaction: the orbitals with energies $|\epsilon_\alpha| > E_Z/2$ are spinless (the positive-energy orbitals are empty and the negative-energy ones are doubly occupied) while orbitals in the strip $|\epsilon_\alpha| < E_Z/2$ are spin polarized along the external field (spin up)⁴. The interaction term mixes only spinless orbitals, leaving unaffected the spin polarized ones. But the spinless orbitals are well separated in energy by a gap E_Z , then the mixing is perturbative and, with a strong enough external field, the ground state will be not qualitatively changed.

But the spectrum of the excitations (the tunneling DoS) changes drastically when a spin-down electron with energy $0 < \epsilon < E_Z/2$ tunnels into the Dot: this electron can be put only in an orbital occupied by a spin-up electron, producing a spinless orbital. It necessarily has to interact with the other spinless orbitals and the mixing results resonant at an energy $\epsilon = \epsilon^*$: it requires a nonperturbative treatment. Rigorous calculation requires to write down the correct propagator for the superconductive fluctuations, but we can understand qualitatively the phenomenon with a heuristic analysis: we consider only two electrons, then the single-electron orbitals for this pair are restricted to the orbitals with energies $\epsilon_\alpha > E_Z/2$ and with ϵ_0 (the other orbitals being filled by the rest of electrons, which have only the role to restrict the Hilbert space for a given electron pair) [42].

The interaction involves only spin singlet orbitals. Thus, the orbital wave function can be labelled by a single orbital index and it obeys to the Schrödinger equation $\epsilon\psi_\alpha = 2\epsilon_\alpha\psi_\alpha + \lambda\delta\sum_\beta\psi_\beta$ and the eigenenergies can be found from

$$\frac{\delta}{2\epsilon_0 - \epsilon} + \sum_{\epsilon_\beta > E_Z/2} \frac{\delta}{2\epsilon_\beta - \epsilon} = \frac{1}{|\lambda|} \quad (7.72)$$

(the interaction has to be attractive, thus $\lambda < 0$). Assuming low-lying eigenstates $\epsilon < E_Z$, one can substitute the sum with an integration with a Debye-

⁴ The energies are measured from the Fermi level.

like cut-off $\bar{\omega} \gg \epsilon$ and for small mean level spacing $\delta \ll E_Z, \bar{\omega}e^{-2/|\lambda|}$ the solution is easily found as

$$\epsilon = \epsilon_0 + \frac{\Omega}{2} \pm \left[\left(\frac{\Omega}{2} - \epsilon_0 \right)^2 + 2\delta e^{-2/|\lambda|} \right]^{1/2}; \quad \Omega = E_Z - \bar{\omega}e^{-2/|\lambda|}. \quad (7.73)$$

Prior the spin-down electron tunnels in, the energy of the spin-up electron on the orbital ϵ_0 was equal to $\epsilon_{\uparrow} = \epsilon_0 - E_Z/2$ and thus the energy of the one-electron excitation is given by $E = \epsilon - \epsilon_{\uparrow}$:

$$E = E_{\text{gap}} \pm \left[\left(\frac{\Omega}{2} - \epsilon_0 \right)^2 + 2\delta e^{-2/|\lambda|} \right]^{1/2}; \quad E_{\text{gap}} = \frac{E_Z + \Omega}{2}. \quad (7.74)$$

Note that a hard gap in the tunneling DoS is formed: there is no one spin-down electron excitation in the energy strip $\left| E - \frac{E_Z + \Omega}{2} \right| < \sqrt{2\delta\bar{\omega}e^{-2/|\lambda|}}$, due to the repulsion between the state formed immediately after an electron tunnels into the system and the bound states of the Cooper pairs. It is important to notice that the width of the gap $\sqrt{8\delta\bar{\omega}e^{-2/|\lambda|}}$ significantly exceeds the single-electron mean level spacing and this singularity persists even far from the critical splitting. It is also noteworthy that nothing happens if a spin-up electron tunnels into the Dot: so, the tunneling DoS does not vanish but shows a suppression by a factor 2.

The same argument can be used to justify the DoS suppression when a spin-up electron with energy $-E_Z < \epsilon < 0$ tunnels out from the Dot, while the spin-down electron tunneling is not affected.

These qualitative arguments shows correctly the physics behind the phenomenon but fails completely to describe it quantitatively. To obtain the exact results, we need to use the fluctuation theory. In this way, one can verify that

$$E_{\text{gap}} = \frac{E_Z + \sqrt{E_Z^2 - \Delta^2}}{2}, \quad (7.75)$$

where $\Delta = \bar{\omega}e^{-1/|\lambda|}$ is the superconducting BCS gap.

7.4 Quasiparticle Lifetime in a Finite System

A quasiparticle (*qp*) in a Fermi liquid is not an eigenstate of the Hamiltonian [43]: it decays into two *qp*'s and a hole. In an infinite clean system, by using the Golden Rule (GR), *qp* decay rate has been estimated as

$$\gamma_{ee} \approx \frac{\epsilon^2}{\epsilon_F}, \quad (7.76)$$

where ϵ is the qp energy and ϵ_F is the Fermi energy [44]. In this case, qp's may be viewed as wave packets constructed of such states, the packet width being determined by the lifetime in an infinite system $\delta\epsilon \approx \gamma_{ee}$. A number of papers have been devoted to clarify the relation between qp's and many-particle states in a finite system [45–48], and we will try to review these works in this section. Conventionally, qp's are defined provided that $\gamma_{ee} \ll \epsilon$ and this relation is verified in pure metals in any spatial dimension [44]. In a finite system, another energy scale appears, the mean level spacing δ between qp states, i.e. the mean level spacing between Hartree–Fock energy levels. This energy scale is always larger than the lifetime of the qp's, if only diagonal elements of the interaction tensor (7.61) are considered. Thus, we have a more stringent condition to “resolve” qp's in a mesoscopic system:

$$\gamma_{ee} < \delta. \quad (7.77)$$

As an example, consider qp peaks in the tunneling conductance of a quantum dot [46,49]: the peaks observed in the nonlinear conductance at certain bias are interpreted as the qp tunneling Density of States (DoS). Each peak corresponds to a “quasiparticle state”, and its width measures the lifetime of the state. Below we will consider an isolated Fermi liquid, ignoring any contributions to the qp lifetime due to finite escape rate, phonons, etc. [50].

In a finite metal, in the presence of a random distribution of static defects, one has a contribution to γ_{ee} from small momentum transfers in elastic scattering with impurities [51,53]; in the large g limit, one can find, making use of the Golden Rule:

$$\gamma_{ee} \approx \delta \left(\frac{\epsilon}{E_c} \right)^{d/2} = \delta \left(\frac{L}{L_\epsilon} \right)^d, \quad (7.78)$$

where E_c is the Thouless energy, L is the sample dimension, $L_\epsilon = \sqrt{\mathcal{D}/\epsilon}$ is the energy relaxation length of a qp with energy ϵ and δ is the mean level spacing between “Hartree–Fock levels” in the system. As one can easily see, this result depends strongly on d :

$$\frac{\gamma_{ee}}{\epsilon} = \begin{cases} \frac{1}{g} \sqrt{\frac{\epsilon}{E_c}}; & d = 3 \\ \frac{1}{g}; & d = 2 \\ \sqrt{\frac{\delta}{g\epsilon}} \equiv \frac{L_\epsilon}{\xi}; & d = 1, \end{cases} \quad (7.79)$$

where

$$\xi = \frac{\mathcal{D}\hbar}{\delta}$$

is the localization length in 1 dimension (we are assuming $g \gg 1$, i.e. $L \ll \xi$). From (7.79), one can draw the following conclusion: in the large g limit, the decay rate condition $\gamma_{ee} \ll \epsilon$ is always fulfilled *but* the relation (7.77) is violated if $L < L_\epsilon$. The last inequality defines an energy region $\epsilon > E_c$, in which the qp's are not distinguishable and the spectrum can be considered as effectively continuous.

The meaning of quasiparticle lifetime needs some clarifications in a finite system; strictly speaking, since a quantum dot is a finite system, any many-particle eigenstate is related to an infinitely narrow peak and cannot decay. However, we will see that only a small fraction of those states overlap with a single-particle excitation strongly enough to be detected by a finite sensitivity measurement. Under certain conditions, these strong peaks group into clusters of the width of the order of γ_{ee} that can be interpreted as qp peaks.

Before discussing the possible regimes, let us review the Golden Rule approach [47,51]: the quantity to be evaluated is the phase-breaking time τ_φ [52] due to the Coulomb interaction with the rest of the Fermi sea. Of course, the condition (7.77) becomes, in this picture,

$$\delta\tau_\varphi > 1 \quad (7.80)$$

and in this region the spectrum is effectively discrete. Writing down the (7.80), we assume that $\gamma_{ee} \propto \tau_\varphi^{-1}$. In general this is not true and in fact we can have $\tau_\varphi \ll \tau_{ee}$ and then $\gamma_{ee} \ll \tau_\varphi^{-1}$. Here we are assuming that the dephasing can occur only via an inelastic process i.e., an energy relaxation process. In the following, we will abandon this strong assumption. However, as we have seen above (see (7.79)), for large enough energies, the dephasing time becomes sufficiently short to violate the last condition⁵. Then one can imagine the qp spectrum as a limited number of discrete states close to the Fermi energy and an effectively continuous spectrum elsewhere.

The starting point of this calculation is the observation [54,55] that the phase uncertainty accumulated by an electron interacting with an environment is also given by twice the probability that the environment changes its initial state due to this interaction. Calculating this probability in the second order perturbation theory and putting it arbitrarily of order 1, one can estimate the phase breaking time. In [47], following the tracks of [56,57] the authors try to extend the formalism for an infinite system to a finite one; they make the following assumptions:

- the dot is overall neutral and then the $q = 0$ mode of the Coulomb potential does not contribute to the interaction;
- the density-density correlation function [57] is translationally invariant also in a quantum dot, because the relevant correlation lengths (the mean free path $l \ll L$ for diffusive dots and the screening length for ballistic ones) are much shorter than the sample dimension L ;

⁵ It is worth noticing the fact that this transition occurs within the validity range of the Fermi liquid.

- it is allowed to use the bulk expression for the imaginary part of the inverse dielectric function: here it is necessary that the levels are broadened beyond the average level spacing and the system have to be *ergodic* (the spectrum must be continuous), to apply the Golden Rule.

The rigorous calculation are quite cumbersome and we will skip its description. The important result can be obtained by direct estimation on the approximate formula [47]

$$P_1 \approx \frac{1}{g} \int_0^{\tau_\varphi} dt_1 \int_0^{t_1} dt_2 \int_0^\epsilon d\omega (\omega e^{-E_c t_2}) , \quad (7.81)$$

from which, in the limit $\epsilon \ll E_c$, one can get

$$P_1^{(0d)} \approx 1 \approx \delta \tau_\varphi \left(\frac{\epsilon}{E_c} \right)^2 \rightarrow \frac{\gamma_{ee}^{(0d)}}{\epsilon} \approx \frac{1}{g} \epsilon \quad (7.82)$$

while, in the opposite limit $\epsilon \gg E_c$, substituting to the integration over t_2 and to t_2 itself in the exponential their typical values $1/\epsilon$ and taking into account that the number of important terms is proportional to $\sqrt{\epsilon/E_c}$, we find

$$P_1^{(3d)} \approx 1 \approx \delta \tau_\varphi \left(\frac{\epsilon}{E_c} \right)^{3/2} \rightarrow \frac{\gamma_{ee}^{(3d)}}{\epsilon} \approx \frac{1}{g} \sqrt{\frac{\epsilon}{E_c}} . \quad (7.83)$$

As we already pointed out, this approach can be used only if the decay rate is larger than the average level spacing of the final unperturbed states

$$\gamma_{ee}^{GR} > \delta_3 , \quad (7.84)$$

where δ_3 is the level spacing of a three qp's state with total energy ϵ : the density of such states is

$$\nu_3 = \frac{\epsilon^2}{2\delta^3} , \quad (7.85)$$

so that a new energy scale appears determined by the inequality (7.84) and by the fact that $\gamma_{ee}^{GR} \approx \delta/g$ (see below [45]), i.e.

$$\epsilon > \epsilon^* \approx \delta \sqrt{g} = \sqrt{\delta E_c} : \quad (7.86)$$

if $\epsilon < \epsilon^*$, GR gives only a first order correction to the energy of the state and not the lifetime: the qp's *cannot decay*, because there are not "suitable" states to occupy after the decay. Note that, since $g \gg 1$, $\epsilon^* \gg \delta$ and there are many states whose lifetime can not be obtained by the GR. If $\epsilon \ll \epsilon^*$, the matrix elements of the interaction are much smaller than δ_3 and qp's cannot decay at all: they are just *slightly* perturbed single-particle peaks; in this situation one can observe narrow conductance peaks that can have weak satellites due

to the small but finite coupling with many-particle states. When $\epsilon \rightarrow \epsilon^*$ from below, the number of satellites increase and for $\epsilon \gg \epsilon^*$ they merge to generate finite width peaks described by GR: to explore the vicinity of ϵ^* and in general the energy range $0 < \epsilon < E_c$, in [48] an analogy with Anderson localization has been used. In fact, one can imagine the problem of the qp decay as a localization problem for many-particle states.

7.4.1 Transition Region

The model that we are studying is described by the hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} , \quad (7.87)$$

where

$$\begin{cases} \hat{H}_0 = \sum_{\alpha} \epsilon_{\alpha} \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha} & \text{single - particle hamiltonian} \\ \hat{H}_{\text{int}} = \sum_{\alpha\beta\gamma\delta} M_{\gamma\delta}^{\alpha\beta} \hat{a}_{\gamma}^{\dagger} \hat{a}_{\delta}^{\dagger} \hat{a}_{\alpha} \hat{a}_{\beta} & \text{interaction hamiltonian} \end{cases} \quad (7.88)$$

and, accordingly to the (7.61)

$$M_{\gamma\delta}^{\alpha\beta} = \int d\mathbf{r}_1 d\mathbf{r}_2 U(\mathbf{r}_2 - \mathbf{r}_1) \phi_{\alpha}(\mathbf{r}_1) \phi_{\gamma}^*(\mathbf{r}_1) \phi_{\beta}(\mathbf{r}_2) \phi_{\delta}^*(\mathbf{r}_2) . \quad (7.89)$$

Let us define a *distance* in the Fock space: the states with m particles and m holes are constructed from the Slater determinants starting from the N -particle Fermi vacuum

$$\Psi_N = \hat{a}_{\alpha_{2m}}^{\dagger} \dots \hat{a}_{\alpha_{m+1}}^{\dagger} \hat{a}_{\alpha_m} \hat{a}_{\alpha_1} |N\rangle . \quad (7.90)$$

These states can be represented as strings with entries 0 and 1. Let us define the distance between two states in Fock space as the number of entries in which the corresponding strings differ. In this way we can see that the interaction matrix elements $\langle \Psi_N | \hat{H}_{\text{int}} | \Psi'_N \rangle$ vanishes apart when the distance between $|\Psi_N\rangle$ and $|\Psi'_N\rangle$ is 0, 2 or 4. We can now build a “lattice” in the Fock space connecting all the $|\Psi_N\rangle$ which are at a distance 2 from each other and study the localization problem on this lattice.

First of all, we have to understand what is the meaning of the localization on this lattice: if a state is localized, it can be described simply by the “orbital” $|\Psi_N\rangle$. Instead, if it is delocalized, it has to be described using a complicated superposition of “orbitals”. These two limits show that localization means that the localized state is practically identical to a single-particle excitation (or to the superposition of very few quasiparticle states): in this case the energy of each individual quasiparticles constituting the state is a good quantum number whereas in the extended states only the total energy is conserved. The transition is of the Anderson type because the interaction

Hamiltonian is local in the Fock space and can couple only orbitals with similar quasiparticles content (“nearest neighbours orbitals”).

Now we turn to the hopping on the Fock lattice: of course, it is due to the off-diagonal elements of the tensor $M_{\alpha\beta\gamma\delta}$ in (7.89) which are not taken into account in writing down the Slater determinants in the Hartree–Fock approximation. In a disordered metal, the wave functions overlap weakly, hence the interaction tensor have large diagonal (and double diagonal) elements of the order of δ and small off-diagonal matrix elements of the order of $\delta/\sqrt{N} \approx \delta/g$. For $\alpha \neq \beta \neq \gamma \neq \delta$, the elements of the tensor are completely random and with zero average being the effect of the fluctuations. The root-mean-square of these elements can be obtained calculating the diagram in Fig. 7.7: the resulting M_{rms} is

$$M_{\text{rms}} = \lambda b_d \frac{\delta}{g}; \quad b_d^2 \equiv 2 \left(\frac{E_c}{\pi} \right)^2 \sum_{m \neq 0} \gamma_m^{-2}, \quad (7.91)$$

where γ_m are the eigenvalues of the diffusion operator⁶ (by definition, $E_c = \gamma_1$). This result is valid for small qp energies in comparison with the Thouless energy $\epsilon_{\alpha(\beta,\gamma,\delta)} < E_c$; the amplitude of M_{rms} decreases algebraically for larger energies and we shall neglect this contribution.

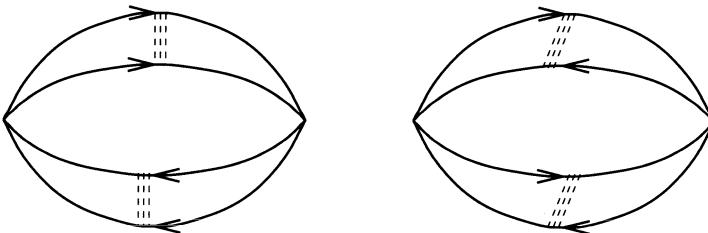


Fig. 7.7. Diagrams used to evaluate $\langle M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta} \rangle^{1/2}$

We can see now how to build directly the lattice on the Fock space: let $|N - 1\rangle$ the ground state of $N - 1$ particles. The N states $\Upsilon_\alpha \equiv \hat{a}_\alpha^\dagger |N - 1\rangle$ representing one particle added in the state α form the first “generation” of the Fock lattice. The second step (generation 3) is builded up from the first adding a particle and a hole $\Upsilon_{\alpha\beta}^\gamma \equiv \hat{a}_\beta^\dagger \hat{a}_\gamma \Upsilon_\alpha = \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma |N - 1\rangle$ and so on. The interaction hamiltonian \hat{H}_{int} couples only states of near generations, so that any given state from the generation $2n - 1$ (n -th step) is connected only to states from generations $2n + 1$, $2n - 1$ and $2n - 3$: in this sense, we can say that \hat{H}_{int} connects only nearest neighbours sites.

⁶ It is worth to stress that in the universal limit, $g \rightarrow \infty$, we have $b_d \equiv 0$ and then $M_{\text{rms}} \equiv 0$ as expected.

Consider now a state in the generation 1 with energy ϵ . The Density of States (DoS) in generation 3 accessible by hopping and having the same energy is given by the perturbation theory (if the hopping is sufficiently weak). In fact, we have

$$\phi_{\gamma}^{\alpha\beta} = \sum_{\delta} \frac{M_{\gamma\delta}^{\alpha\beta} \phi_{\delta}}{\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\delta}} + \dots , \quad (7.92)$$

where ϕ_{α} is a state in first generation and $\phi_{\alpha}^{\beta\gamma}$ is a state of third generation. For the first generation, the DoS coincides with (7.85)

$$\nu_3 = \frac{\epsilon^2}{2\delta^3} , \quad (7.93)$$

while, for higher order generations, the DoS grows very rapidly: for the $(2n+1)$ -th generation⁷ we have

$$\nu_n \propto \frac{1}{(2n)!} \left(\frac{\epsilon}{\delta}\right)^{2n} \nu_1 . \quad (7.94)$$

But the states from the $(2n+1)$ -th which are directly accessible (via the interaction hamiltonian) for a given state from the $(2n-1)$ -th generation have a much smaller DoS:

$$\nu_{2n+1} = \frac{\nu_3}{n} . \quad (7.95)$$

We note that from a state in the $(2n-1)$ -th generation, it is also possible to hop onto some states on the $(2n-1)$ -th or the $(2n-3)$ -th generation but with DoS parametrically smaller, for $n \ll n_{\max}$, than the other. We thus obtain a picture which is similar to that of the Cayley tree: each site on the $(2n+1)$ -th generation branches out to K_n sites of the next generation (Fig. 7.8). One can obtain the branching ratio simply integrating out the DoS until the maximum energy E_c

$$K_n \approx \int_0^{E_c} d\epsilon \frac{\nu_3(\epsilon)}{n} = g^3/6n . \quad (7.96)$$

Because the branching number decreases with n , the lattice can be thought as effectively finite. In order to simplify the considerations, we will consider

⁷ Note that, because ϵ is fixed, we have a maximum number of generations allowed by the fact that the energy of a state cannot be smaller than δ :

$$n < n_{\max} \approx \sqrt{\frac{\epsilon}{\delta}} .$$

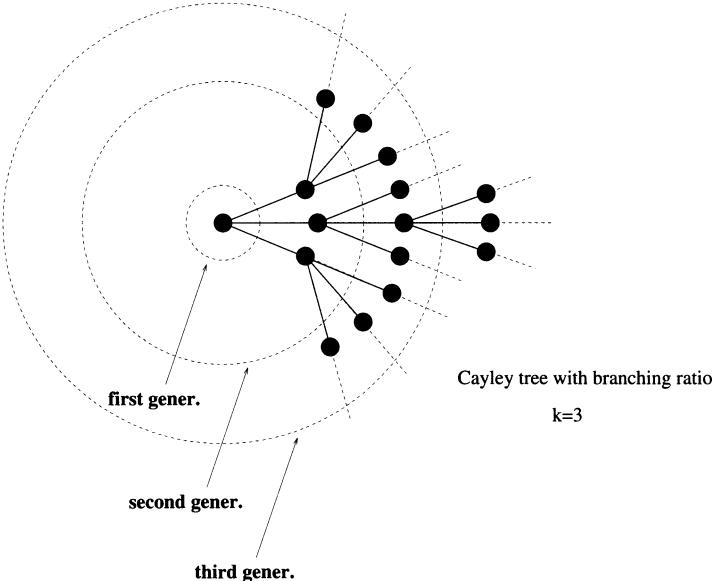


Fig. 7.8. A schematic representation of a Cayley tree: the shaded circles represent the various generations while the bonds are the possible quasiparticle decays. In the physical model, the tree is finite, because there is a maximum number of levels n_{\max} , but for sake of simplicity we will present here an analysis of a model on a infinite tree, with constant branching ratio

an infinite Cayley tree⁸ with constant branching number $K \equiv K_1 = g^3/6$. This model was solved in [58], where the authors studied an Anderson model on a Cayley lattice with on-site energies uniformly distributed in the interval $[-W, W]$, constant hopping amplitude M and branching ratio K ; they found that the delocalization occurs at

$$M_c \approx \frac{W}{K \ln K} \quad (7.97)$$

in apparent contrast with the Anderson criterion $M_c \approx W/K$ obtained by comparing the hopping amplitude with the spacing of the on-site energies in the nearest neighbour shell. Now, we have to understand the meaning of the log term: we use the same approach of [48] instead of the exact solution. A state from the $(2n+1)$ -th generation can be written in terms of a state from the first generation, in the lowest order in the hopping, as

$$\Phi_{\alpha,2n+1} = \prod_{i=1}^n \frac{M}{\epsilon - \epsilon_i} \Phi_{\alpha} \quad (7.98)$$

⁸ This implies that any phase transition will be smeared to a crossover in the finite system.

(for sake of simplicity, we put $M = M_{\text{rms}}$) and then the hopping amplitude is given by

$$\mathcal{A}_n = \prod_{i=1}^n \frac{M}{\epsilon - \epsilon_i}; \quad (7.99)$$

the probability that this amplitude is significant (namely, that $|\mathcal{A}_n| > C$ where C is a finite constant) can be calculated starting from the distribution function of the \mathcal{A}_n 's. In fact, let be

$$\ln |\mathcal{A}_n| = n \ln \frac{M}{W} + Y_n, \quad (7.100)$$

where

$$Y_n = \sum_{i=1}^n y_i = \sum_{i=1}^n \ln \left| \frac{W}{\epsilon - \epsilon_i} \right|. \quad (7.101)$$

Now we know that, from the distribution of the Y_n , we can obtain the distribution of the \mathcal{A}_n : in fact, from the fact that $\epsilon - \epsilon_i$ is uniformly distributed in the interval $[-W, W]$, we obtain $P(y_i) \propto \exp(-y_i)$. Then, Fourier transforming $P(y_i)$, taking the n^{th} power and transforming back, we obtain the distribution of the Y_n and finally, using the (7.101), we have

$$P(|\mathcal{A}_n|) = \frac{1}{(n-1)! |\mathcal{A}_n|^2} \left(\frac{M}{W} \right)^n \left\{ \ln \left[|\mathcal{A}_n| \left(\frac{W}{M} \right)^n \right] \right\}^{n-1}. \quad (7.102)$$

Thus, the probability

$$\begin{aligned} p(n, C) &\equiv \int_C^\infty d\mathcal{A} P(\mathcal{A}) \\ &\approx \frac{1}{(n-1)!} \frac{1}{C \ln \left[C \left(\frac{W}{M} \right)^n \right]} \left\{ \frac{M}{W} \ln \left[C \left(\frac{W}{M} \right)^n \right] \right\}^n. \end{aligned} \quad (7.103)$$

The probability that none of the K^n trajectories connecting a site in the first generation to sites in the $(2n+1)$ -th generation carries a large amplitude is of course

$$(1 - p(n, C))^{K^n} \equiv \exp(-f_n), \quad (7.104)$$

where for $p(n, C) \ll 1$, one has $f_n \approx K^n p(n, C)$. From (7.103) for $n \gg 1$, one obtains

$$f_n \approx \frac{1}{\sqrt{2\pi n} C} \frac{1}{\ln \frac{W}{M}} \left[\frac{K M e}{W} \ln \frac{W}{M} \right]^n. \quad (7.105)$$

If f_n increases with n , then at higher generations one has $f_n \gg 1$ i.e., strong coupling to the generation 1. A transition to the localized phase takes place

when the expression in the square brackets in the (7.105) assumes the value 1, from which we obtain the criterion (7.97) with logarithmic accuracy.

To apply this result to our problem, we replace $M_{\alpha\beta\gamma\delta}$ by M and approximate the density of states in the generation $(2n+1)$ accessible from a state in the generation $(2n-1)$ with energy ϵ to be uniform in the interval $[\epsilon - W, \epsilon + W]$ for $M < W < \epsilon$ and equal to $\nu_3 = \frac{K}{2W}$. Then we find that the transition occurs at the energy

$$\epsilon^{**} \approx \frac{1}{\sqrt{2e\lambda b_d}} \sqrt{\frac{\delta E_c}{\ln g}} . \quad (7.106)$$

Of course, this result is independent of C . At energies just above ϵ^{**} , the first generation is not very well connected with the next few generations ($f_n < 1$ for small n). The condition for *all* generations to be well connected with the first one is $f_1 = \frac{KM}{WC} > 1$. This coincides, using a naive implementation of the Thouless criterion, to a second energy scale

$$\epsilon^* \approx \sqrt{\frac{\lambda b_d}{C}} \sqrt{\delta E_c} = \delta \sqrt{g \frac{\lambda b_d}{C}} , \quad (7.107)$$

from which we can write

$$f_n(\epsilon) \approx \frac{1}{\sqrt{n}} \left(\frac{\epsilon^{**}}{\epsilon^*} \right)^2 \left(\frac{\epsilon}{\epsilon^{**}} \right)^{2n} . \quad (7.108)$$

Let us discuss the physical meaning of the various regimes. In the localized phase ($\epsilon < \epsilon^{**}$), the first generation is weakly connected with the rest of the network: at such energies, the exact many-body states are close to Slater determinants or to superpositions of few determinants. A mathematical description of a single particle injection into a dot involves projecting a single-particle state onto exact eigenstates of the system. In the localized phase each single-particle state will have a significant overlap with one or few exact eigenstates, producing a few resolved δ -function peaks in the spectrum of the single-particle DoS. This is a striking prediction, i.e. that qp's can be virtually stable states of the system. Other theoretical works has been devoted to this topic [61–63], and they substantially confirm the conclusion of [48]. At $\epsilon > \epsilon^*$, all generations are well connected. Due to the huge density of multiparticle states, the states of the first generation can be thought as effectively well coupled to the continuum. Each single particle peak associated with the generation 1 is replaced by a cluster of a large number of many-particle peaks, altogether forming a Lorentzian envelope. For $\epsilon < E_c$, the width of the envelope is less than δ and thus the quasiparticle states can be resolved, e.g., in transport measurements. For intermediate energies $\epsilon^{**} < \epsilon < \epsilon^*$ there are still many peaks in a cluster. However the probability that a particular generation is represented in a given cluster is small. As a result, the widths of the clusters as well as the shapes of their envelopes will strongly fluctuate from peak to peak.

Then, as long as quasiparticle peaks can be resolved, the many-particle spectrum is not truly chaotic in the Wigner–Dyson sense. Another important observation is that an extended state in the many-body problem is not necessarily ergodic but can be extended over a small fraction of the whole space (in our case, a subtree of the Cayley tree).

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