Conformal symmetry and holographic cosmology

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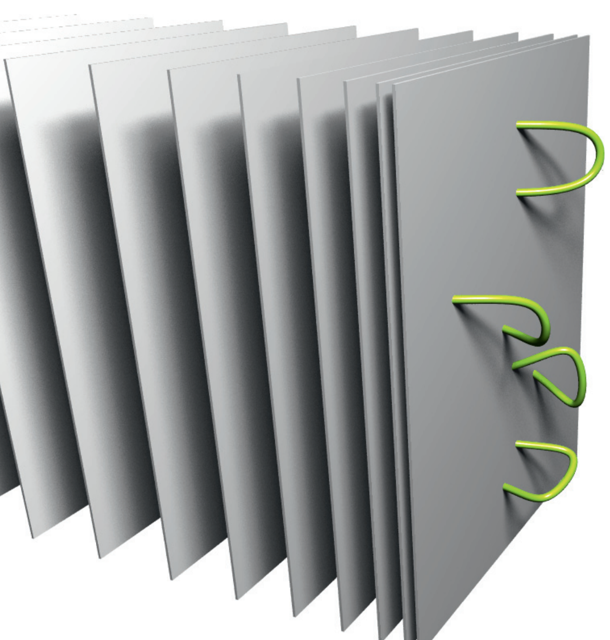
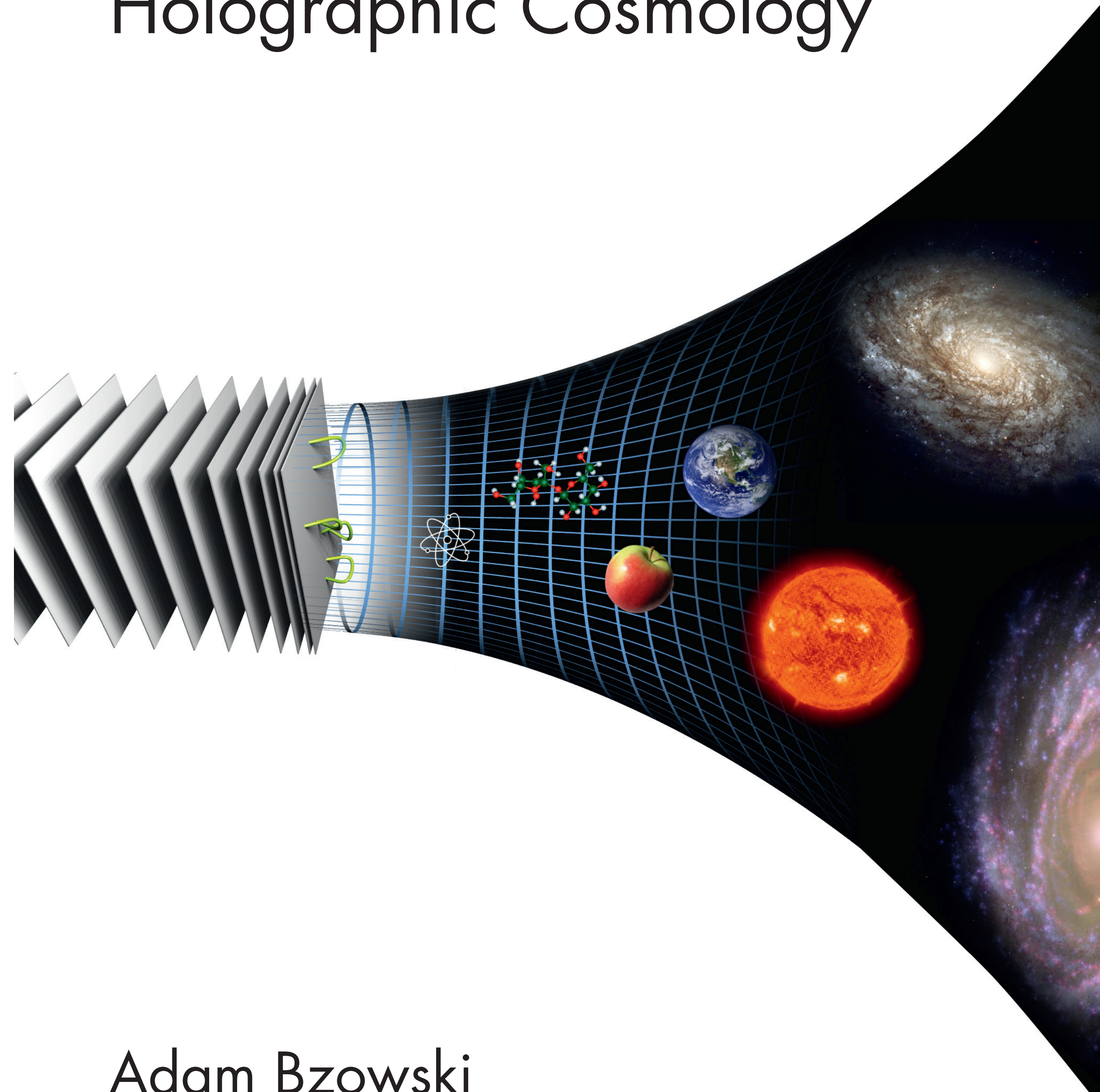
This thesis presents a novel approach to cosmology using gauge/gravity duality. Analysis of the implications of conformal invariance in field theories leads to quantitative cosmological predictions which are in agreement with current data. Furthermore, holographic cosmology extends the theory of inflation beyond classical gravity.



Conformal Symmetry and Holographic Cosmology



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Publications

This thesis is based on the following publications:

[1] A. Bzowski, P. McFadden and K. Skenderis
Holographic predictions for cosmological 3-point functions
JHEP **1203** (2012) 091, arXiv:1112.1967 [hep-th].

[2] A. Bzowski, P. McFadden and K. Skenderis
Holography for inflation using conformal perturbation theory
JHEP **1304** (2013) 047, arXiv:1211.4550 [hep-th].

[3] A. Bzowski, P. McFadden and K. Skenderis
Implications of conformal invariance in momentum space
Submitted to *JHEP*, arXiv:1304.7760 [hep-th].

Contents

Introduction

Summary

> There is geometry in the humming of the strings, there is music in the spacing of the spheres.
> *Pythagoras, 6th c. BC.*

> There is nothing new to be discovered in physics now. All that remains is more and more precise measurement.
> *Attributed to lord Kelvin, ca. 1900.*

> Whether you can observe a thing or not depends on the theory which you use. It is the theory which decides what can be observed.
> *Albert Einstein, 1926.*

> Nothing is real, everything is permitted.
> *From 'Alamut' by V. Bartol, 1938.*

Modern physics is a direct descendent of the natural philosophy and the ultimate aim of this thesis is the application for the degree of the *philosophiae doctor*. But physics is no longer a philosophy. Natural philosophy, in the sense laid out by Isaac Newton and his predecessors, describes the world as it is, perfect in nature, predictive, based on classical logic and objective observations.

For millennia people were convinced that the idealised concepts found in mathematics, geometry and other branches of science represent the world we live in and ultimately will allow us to answer the question 'what is the Universe like?'. With the great success of theories developed in the nineteenth century such as electromagnetism or thermodynamics, the answer to this question seemed to be closer than ever.

The infamous quotation attributed to lord Kelvin, however, could not possibly be voiced at a worse time. Following the year 1900, in less than half a century all the foundations of the natural philosophy were shattered to pieces. The revolution started with small, at first almost insignificant discrepancies. The classical thermodynamics predicted the spectrum of the black body radiation with peculiar features at very low temperatures. While at the time experimentally unmeasurable, analysis of this behaviour led Max Planck and his successors to the stunning discovery of quantum mechanics. Another irregularity was observed in the Maxwell equations of electromagnetism. Their strange, non-Galilean transformation properties, together with the measurements pointing towards a constant speed of light led Albert Einstein to the development of special relativity and subsequently the theory of general relativity.

Quantum mechanics and general relativity are the two theories at heart of the modern physics. Their predictions, tested with an unprecedented precision, have led to thousands of inventions from the microprocessor to the GPS navigation system. Despite their success, their 'philosophy' is in dire opposition to the rules of natural philosophy:

- The world is not deterministic. The laws of classical logic do not apply. Classical mechanics only emerges from quantum mechanics in some appropriate limit.
- Time and space are not absolute. Both are interconnected and both evolve in connection to the matter content of the Universe.
- Wave/particle duality. One cannot depict the 'nature' of quantum objects. They can be described equally well both as waves and particles although the usefulness of a description depends on a particular situation.

Modern theoretical physics does not try to answer the question 'what is the Universe like?' but rather develops mathematical models that predict the behaviour of actual physical objects. All successful models are subjected to some limitations and hold only in their regimes of validity. There may be – and there usually is – more than one model describing a given physical phenomenon. Using scientific method, one usually chooses the model that gives more accurate predictions, is simpler, requires a smaller number of parameters and has a vast region of validity.

Throughout the twentieth century quantum mechanics and special relativity evolved into the quantum field theory and subsequently the Standard Model of elementary particles. Its last missing element: the Higgs boson, theoretised forty years ago, was eventually spotted in the LHC in 2012. On the other hand, the consequences of general relativity were tested to a high precision and for the first time in history a reliable cosmological model of the evolution of the entire Universe was developed and found to be in agreement with astronomy, geology and evolutionary biology. Is physics complete again?

Perhaps not. One of the most fascinating puzzles in the high energy physics is the quantisation of gravity. The Standard Model of elementary particles does not include gravity and theoretical considerations lead to severe problems when one tries to unite Einstein's gravity with quantum field theories. For decades generations of physicists have struggled to quantise gravity. While the physics of all the other forces takes plays in the arena of space and time, gravity *is* space and time. Therefore, the quantisation of gravity requires a completely new understanding of the fabric of the Universe. Hints for the quantisation of gravity have followed from many directions including high energy physics, black hole mechanics and cosmology and led to the development of the stunning gauge/gravity duality in 1997. For the first time actual calculations in the true quantum gravity could be carried out.

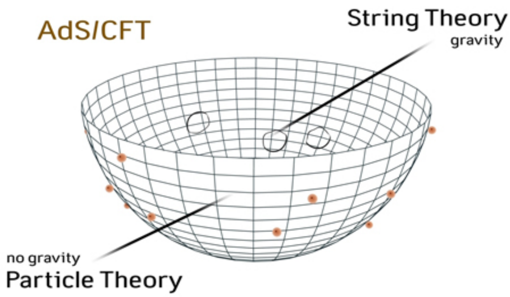


Figure 1: *The basic idea of gauge/gravity duality. The gravitational theory such as string theory living in the bulk is equivalent to the quantum field theory without gravity living on the boundary of the bulk.*

Gauge/gravity duality, also known as the *AdS/CFT correspondence* or *holography*, states that a quantum system containing gravity, matter and other forces is completely equivalent to another system described by a specific quantum field theory without gravity. One of the most amazing features of the duality is that the strongly coupled, non-perturbative regime of one theory, where actual computations are very difficult if not impossible to carry out, corresponds to a weakly coupled, perturbative regime of the dual theory, where physical predictions can be obtained. The physics is described equally well by both theories. It is because of our place and time in the Universe that we experience the Newton's gravity more often than the dynamics of the dual field theory in the same way as the wavelike nature of light was observed before the molecular one.

Gauge/gravity duality emerges from the analysis of string theory. For decades, string theory has been the most promising candidate for a theory of quantum gravity. Despite the lack of direct experimental evidence, the beauty of the theory made people believe that there must be some physical importance behind it. String theory is the unique predictive theory that unifies gravity, matter and other forces into a consistent quantum theory. In string theory all particles, including force carriers such as gravitons, emerge as various modes of vibrations of tiny strings.

The problem, however, was that the only known definition of string theory was based on the perturbative expansion, where the effects of quantum gravity are considered as small corrections to the classical solution. Gauge/gravity duality enabled an unprecedented possibility for the analysis of the strongly coupled regime of string theory, where the quantum gravity effects dominate.

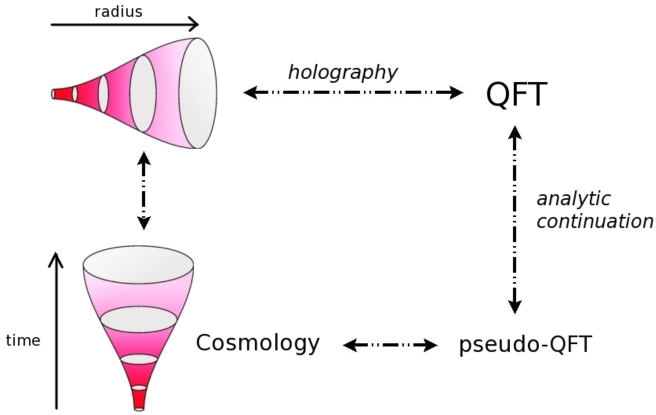


Figure 2: *Steps leading to the holographic description of cosmology. By the cosmology/domain wall correspondence, one can translate cosmological observables into the perturbations around some Euclidean domain wall spacetime. Then the gauge/gravity can be applied. The procedure leads to the holographic formulae, expressing cosmological observables in terms of the analytically continued correlation functions of the dual quantum field theory (QFT).*

The gauge/gravity duality has been successfully applied to various condensed matter systems such as the physics of superconductors, superfluids or hydrodynamics of quark-gluon plasma. Since all these phenomena are described by some strongly coupled quantum field theories, holography translates their dynamics into the equivalent dynamics of some gravitational systems such as black holes. In this thesis, however, we will use the gauge/gravity duality in the opposite direction. By carrying out specific calculations in a non-gravitational field theory, we will obtain measurable predictions pertaining to the very early phase of our Universe.

It is a currently accepted paradigm that during its very early phase the Universe rapidly expanded while the minute quantum fluctuations within were stretched to gigantic sizes creating the seeds of future stars and galaxies. We have good experimental evidence, mostly due to the measurements of the Cosmic Microwave Background: electromagnetic radiation filling the entire space and carrying valuable information about the very early Universe. It is also accepted that before inflation took place the Universe was dominated by strongly coupled, non-geometric quantum gravity. In such a regime space and time cease to exist, the geometrical description is no longer valid, nevertheless holography gives us some insight into this fascinating phase.

In this thesis we will show how one can access inflationary and pre-inflationary stage of the Universe by utilisation of gauge/gravity duality. We will show specific models that describe the very early phase of the Universe and we will compare their predictions with the currently available data. As we will see both models fit the data very well and become a serious alternative for the standard theory of inflation. Both models resolve the initial singularity of our Universe by reinterpreting the Big Bang as the exit from its non-geometric phase. Diagram 2 shows how the holographic models of cosmology are obtained.

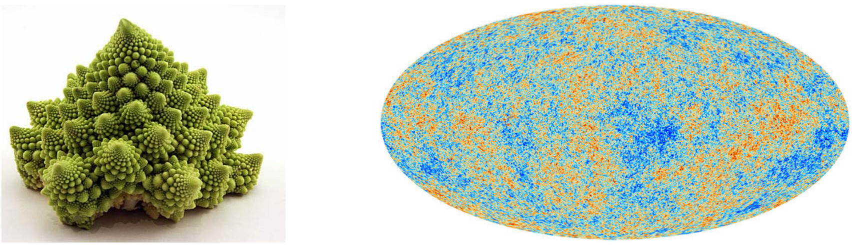


Figure 3: *Cauliflower is an example of an approximately scale invariant object. Another example of an almost scale invariant system is the Cosmic Microwave Background. The plot on the right shows thermal fluctuations of the electromagnetic radiation depending on the direction in the sky.*

In order to retrieve quantitative results following from our models, one must analyse the properties of the dual quantum field theory. In the case of gauge/gravity duality, the quantum field theory exhibits usually additional symmetries such as conformal invariance. Conformal symmetry is closely related to scale invariance: the situation where the physics is identical at various length or energy scales. However, most of the environment around us is not scale invariant. For example, the hydrodynamical description of water breaks down when the size of the wave is small enough for the dynamics of single molecules to become important. However, many systems are scale invariant within some approximation, see figure 3.

In the thesis we present a novel approach to the classification of correlation functions in conformal field theories directly in momentum space. Our approach speeds up and simplifies the holographic analysis significantly. Furthermore, our results are not limited to cosmology and they can be utilised in many applications of conformal field theories.

Organisation of the thesis

The thesis is divided into two parts: the *conformal field theory* part and the *holographic cosmology* part. In the first part we concentrate on the analysis of conformal field theories and their deformations from the momentum space point of view. In the first chapter we review basic properties of conformal field theories. We discuss the implications of conformal symmetry, introduce Ward identities and we analyse the structure of 2- and 3-point correlation functions in position space.

In the second chapter we analyse properties of 2- and 3-point functions in conformal field theories in momentum space. We develop a novel approach by a direct solution to the conformal Ward identities in momentum space. We prove that all 3-point functions of scalar operators, conserved currents and the stress-energy tensor in conformal field theories can be expressed in terms of the triple-K integrals: a certain class of integrals containing three Bessel functions K. We list all the results in chapter three.

In the fourth chapter we present three examples of calculations based on the momentum space analysis. We start with the example of the quantum field theory deformed by a single nearly marginal deforming operator. Next, we show how the usual computations based on Feynman diagrams can be simplified in terms of the triple-K integrals in odd spacetime dimensions. Finally, we utilise our momentum space approach to the calculations of 2- and 3-point functions of the stress-energy tensors in free theories in three spacetime dimensions.

In the second part of the thesis we focus on gravitational theories with the emphasis being on cosmology. In the first chapter of the second part of the thesis we review the fundamentals of cosmology. We discuss the cosmological 2- and 3-point functions of the primordial perturbations. In chapter six we introduce gauge/gravity duality both for conformal and non-conformal branes. We present an example of holographic renormalisation and we finish the chapter with a discussion of the cosmology/domain wall correspondence.

In the last chapter we use all the elements to build two models of holographic inflation. The first model is based on the deformation of the dual CFT by a relevant operator. We show that such a model corresponds to the usual hilltop inflation and we show that both the gravitational and holographic descriptions match exactly. The second model we discuss is based on a weakly-coupled QFT, which leads to a new type of cosmology with a strongly-coupled, non-geometric gravitational phase at early times. We show that despite the qualitative difference between this model and the standard inflationary models, the physical predictions match the data very well.

Part I

Conformal field theory

Chapter 1

Conformal invariance

1.1. Fundamentals

1.1.1. Conformal symmetry

Let M be a smooth manifold and let g and h be two Riemannian metrics on M. We say that g and h are *conformally equivalent* if there exists a smooth scalar function $\Omega : M \to \mathbb{R}$ such that $g(\boldsymbol{x}) = \Omega^2(\boldsymbol{x})h(\boldsymbol{x})$. This is an equivalence relation in space of Riemannian structures on M. Its classes of equivalence are called *conformal structures* on M and the pair $(M, [g])$ is a *conformal manifold*.

Morphisms in the category of conformal manifolds are smooth maps that preserve their conformal structures. To be precise a *conformal map* $F : (M, [g]) \to (N, [h])$ between conformal manifolds $(M, [g])$ and $(N, [h])$ is a smooth function $F : M \to N$ such that

$$F^* h = \Omega^2 g, \tag{1.1.1}$$

where $F^* h(X, Y) = h(F_* X, F_* Y)$ is a pull-back of h via F and X, Y are vector fields on M. In a local system of coordinates x^μ on M this equation reads

$$h_{ij}(F(\boldsymbol{x}))\partial_\mu F^i(\boldsymbol{x})\partial_\nu F^j(\boldsymbol{x}) = \Omega^2(\boldsymbol{x})g_{\mu\nu}(\boldsymbol{x}) \tag{1.1.2}$$

at each point $\boldsymbol{x} \in M$.

As an important example, consider a d-dimensional unit sphere $S^d \subseteq \mathbb{R}^{d+1}$, with the natural metric following from this embedding. Let $x_1, \ldots x_{d+1}$ denote the standard coordinates on S^d satisfying $\sum_{j=1}^{d+1} x_j^2 = 1$ and denote $N = (0, \ldots, 0, 1)$. Define a stereographic projection $X : S^d \backslash \{N\} \to \mathbb{R}^d$ by

$$(X_1, \ldots, X_d) = \frac{1}{1 - x_{d+1}}(x_1, \ldots, x_d). \tag{1.1.3}$$

The stereographic projection is a diffeomorphism and the pull-back of the metric on S^d via its inverse leads to the induced metric on $\mathbb{R}^d$ to be

$$\mathrm{d}s^2 = \frac{4}{\left(1+\sum_{j=1}^{d} X_j^2\right)^2} \sum_{j=1}^{d} \mathrm{d}X_j^2. \tag{1.1.4}$$

This metric is conformally equivalent to the flat metric $\sum_{j=1}^{d} \mathrm{d}X_j^2$. Therefore the standard metric on a sphere is conformally equivalent to the flat metric on $\mathbb{R}^d$.

In this work we are interested in flat Euclidean field theories, *i.e.*, field theories living on $\mathbb{R}^d$ with a metric $\delta_{\mu\nu}$. Therefore, to simplify further discussion and bring it closer to the physical point of view, we will study conformal maps between open subsets of $\mathbb{R}^d$ with a constant, flat metric $\delta_{\mu\nu}$ and small perturbations around it. It can be shown, [4], that our results remain valid for a general metric. The generalisation amounts to a change of coordinate derivatives into covariant ones.

One can classify all conformal maps $F : \mathbb{R}^d \supseteq M \to \mathbb{R}^d$ by looking at vector fields that generate them. In other words we expand (1.1.1) locally in an infinitesimal parameter ϵ,

$$x'^{\mu} = F^{\mu}(\boldsymbol{x}) = x^{\mu} + \epsilon\, \xi^{\mu} + O(\epsilon^2), \qquad \Omega(\boldsymbol{x}) = 1 + \epsilon\, \omega(\boldsymbol{x}) + O(\epsilon^2) \tag{1.1.5}$$

and use the fact that the variation of a metric under the change of coordinates is

$$\delta g_{\mu\nu} = -\epsilon(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \tag{1.1.6}$$

Therefore, the equation (1.1.1) leads to

$$-(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 2\omega g_{\mu\nu}. \tag{1.1.7}$$

By taking a trace of this equation one finds that ω is not arbitrary and must satisfy

$$\omega = -\frac{1}{d}(\partial \cdot \xi). \tag{1.1.8}$$

By applying some derivatives and contractions to (1.1.7) one retrieves a simple differential equation

$$\partial_\mu \partial_\nu \omega = 0. \tag{1.1.9}$$

See [5] for details. The most general solution is

$$\omega = \alpha + \beta_\mu x^\mu \tag{1.1.10}$$

for some number α and a vector β^μ. This means that the infinitesimal diffeomorphism ξ^μ is at most quadratic in x^μ. The detailed analysis leads to the conclusion, that the most general form of the infinitesimal conformal transformation is a combination of the following maps

$$\begin{array}{lll} \text{(i)} & \text{translations:} & x^\mu \mapsto x^\mu + \epsilon\xi^\mu, \\ \text{(ii)} & \text{rotations:} & x^\mu \mapsto x^\mu + \epsilon\omega^\mu{}_\alpha x^\alpha, \\ \text{(iii)} & \text{dilatations or scalings:} & x^\mu \mapsto x^\mu + \epsilon\lambda x^\mu, \\ \text{(iv)} & \text{special conformal:} & x^\mu \mapsto x^\mu + \epsilon\left[2(\boldsymbol{b}\cdot\boldsymbol{x})x^\mu - x^2 b^\mu\right], \end{array} \tag{1.1.11}$$

where $\lambda \in \mathbb{R}$ is a real number, $\boldsymbol{\xi}$ and $\boldsymbol{b}$ are arbitrary vectors and ω is an antisymmetric matrix of an infinitesimal rotation.

The first two transformations: translations and rotations are isometries. As such that have $\Omega = 1$ in (1.1.1). The dilatation is the expected scaling transformation. Conformal invariance implies scaling invariance as discussed in the introduction. It is more interesting that also the fourth class of transformation appears: the special conformal transformations. They are indeed in many respects very different than maps (i) – (iii), for example they are quadratic, rather than linear in $\boldsymbol{x}$. As we will see they are quite essential in the further analysis.

It is also possible to integrate the transformations in (1.1.11) or check, that the following maps reduce to them under when the transformation parameters are expanded in ϵ,

$$\begin{array}{lll} \text{(i)} & \text{translations:} & x^\mu \mapsto x^\mu + a^\mu, \\ \text{(ii)} & \text{rotations:} & x^\mu \mapsto \Lambda^\mu{}_\alpha x^\alpha, \\ \text{(iii)} & \text{dilatations or scalings:} & x^\mu \mapsto \lambda x^\mu, \\ \text{(iv)} & \text{special conformal:} & x^\mu \mapsto \dfrac{x^\mu - x^2 b^\mu}{1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2 x^2}, \end{array} \tag{1.1.12}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, $\lambda \in \mathbb{R}_+$ is a positive real number and Λ is a matrix of a rotation. Note that parameters $\boldsymbol{b}$ and λ here are different than ones in (1.1.11). Parameters in (1.1.11) are obtained by the first order expansion ϵ of the parameters in (1.1.12),

$$\begin{array}{ll} a^\mu \mapsto \epsilon\xi^\mu, & \Lambda^\mu{}_\alpha \mapsto \delta^\mu_\alpha + \tfrac{1}{2}\epsilon\omega^\mu{}_\alpha, \\ \lambda \mapsto 1 + \epsilon\lambda, & b^\mu \mapsto \epsilon b^\mu. \end{array} \tag{1.1.13}$$

From the context it should be clear whether we consider the generators of the transformations or finite transformations.

The special conformal transformations, unlike the maps (i) – (iii) in (1.1.12) are not defined in the entire space $\mathbb{R}^d$. Indeed, for $\boldsymbol{x} = \boldsymbol{b}/b^2$ the denominator of (iv) vanishes. There are two ways to solve the problem. The first one is to ignore it and stick with the infinitesimal form of the transformations. In physical context this is a reasonable procedure, since all transformation properties are local and therefore the local analysis is sufficient. The mathematical solution is to embed $\mathbb{R}^d$ into some compact manifold M in such a way that all transformations (1.1.12) are well-defined as functions $\mathbb{R}^d \mapsto M$ and extend to conformal diffeomorphisms $M \to M$. Such a construction can be found in [4].

An important property of the special conformal transformations (1.1.12) is that they can be written as a composition of two inversions $I^\mu(\boldsymbol{x})$ and a translation by a vector $\boldsymbol{b}$, *i.e.*,

$$I^\mu(\boldsymbol{x}) = \frac{x^\mu}{x^2}, \qquad \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu. \tag{1.1.14}$$

Therefore, in order to analyse the implications of the action of the special conformal transformation, in many cases it is enough to analyse the action of inversions. Also note that the distance of two point after the inversion is

$$|I(\boldsymbol{x}) - I(\boldsymbol{y})| = \frac{|\boldsymbol{x} - \boldsymbol{y}|}{xy}. \tag{1.1.15}$$

1.1.2. Conformal group

We would like to understand the group structure generated by the conformal transformations. Clearly, the d-dimensional Poincaré group consisting of translations and rotations as well as the multiplicative group $\mathbb{R}_+$ generated by dilatations are subgroups of the conformal group.

As usual in such cases, instead of looking at the whole group, one can look at its Lie algebra, *i.e.*, the algebra of vector fields/differential operators coming from the ϵ part of the infinitesimal transformations (1.1.11). The corresponding vector fields are

$$\begin{array}{lll} \text{(i)} & \text{translations:} & P_\mu = \partial_\mu, \\ \text{(ii)} & \text{rotations:} & L_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu, \\ \text{(iii)} & \text{dilatations:} & D = x^\alpha \partial_\alpha, \\ \text{(iv)} & \text{special conformal:} & K_\mu = 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu. \end{array} \tag{1.1.16}$$

The commutation relations of (1.1.16) can be easily worked out,

$$\begin{aligned} &[D, D] = 0, && [P_\mu, P_\nu] = 0, \\ &[D, P_\mu] = -P_\mu, && [P_\mu, K_\nu] = 2(\delta_{\mu\nu} D + L_{\mu\nu}), \\ &[D, K_\mu] = K_\mu, && [P_\rho, L_{\mu\nu}] = \delta_{\rho\nu} P_\mu - \delta_{\rho\mu} P_\nu, \\ &[D, L_{\mu\nu}] = 0, && [K_\mu, K_\nu] = 0, \\ &[K_\rho, L_{\mu\nu}] = \delta_{\rho\nu} K_\mu - \delta_{\rho\mu} K_\nu, \\ &[L_{\mu\nu} L_{\rho\sigma}] = \delta_{\mu\rho} L_{\nu\sigma} + \delta_{\nu\sigma} L_{\mu\rho} - \delta_{\nu\rho} L_{\mu\sigma} - \delta_{\mu\sigma} L_{\nu\rho}. \end{aligned} \tag{1.1.17}$$

In order to see the group structure behind the commutation relations (1.1.17) it is convenient to define a new set generators J_{AB} as follows

$$\begin{array}{ll} J_{\mu\nu} = L_{\mu\nu}, & J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu), \\ J_{-10} = D, & J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu), \end{array} \tag{1.1.18}$$

where $J_{AB} = -J_{BA}$ and $A, B \in \{-1, 0, 1, \ldots, d\}$. One can work out that the new generators satisfy

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} + \eta_{BD}J_{AC} - \eta_{AD}J_{BC} - \eta_{BC}J_{AD} \tag{1.1.19}$$

where $\eta = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. These are commutation relations of the group $SO(d+1, 1)$. Therefore, we have proved that the conformal group in d Euclidean dimensions is locally isomorphic to the symmetry group $SO(d+1, 1)$ of the D-dimensional hyperbolic space as we will discuss in section 5.1.3.

1.1.3. Conformal fields

A (classical) field theory is called a *conformal field theory* if all fields transform in some representation of the conformal group. Let us be more precise and give a general definition valid for a curved manifold. It will be useful when we couple the theory to gravity. If M is a Riemannian manifold than the conformal group G is defined by (1.1.1). Let $g \in G$ be a conformal transformation. By x^μ we denote coordinates on M in some chart and by $x'^\mu = g^\mu(\boldsymbol{x})$ the coordinates after the transformation g is applied. Consider a field ϕ on M with values in a vector space V. The field ϕ is conformal if there exists a family of representations $R^\phi(\boldsymbol{x})$ of the conformal group G at each $\boldsymbol{x} \in M$ such that components of the field in coordinates x^μ and x'^μ are related by

$$\phi'(\boldsymbol{x}') = R^\phi(\boldsymbol{x})(g)\phi(\boldsymbol{x}). \tag{1.1.20}$$

This definition is rather imprecise, since it does not say anything about the continuity/smoothness and is local in nature. From the mathematical point of view ϕ is a section of an associated vector bundle over M with the structure group being the conformal group. For the details, see [6, 7, 8, 9]. We will disregard mathematical subtleties here.

Infinitesimally, under the action of the infinitesimal symmetry δg, both spacetime coordinates and field transform,

$$x'^\mu = x^\mu + \delta_g x'^\mu, \qquad \phi'(\boldsymbol{x}') = \phi(\boldsymbol{x}) + \delta_g R^\phi(\boldsymbol{x}). \tag{1.1.21}$$

Then the infinitesimal transformation of the field ϕ at the same point $\boldsymbol{x}$ is

$$\phi'(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \partial_\mu\phi(\boldsymbol{x})\delta_g x'^\mu + \delta_g R^\phi(\boldsymbol{x})\phi(\boldsymbol{x}). \tag{1.1.22}$$

We define an *infinitesimal transformation* $\delta_g\phi$ of the field and a *generator* G^ϕ_g of the infinitesimal transformation δg as

$$\delta_g\phi(\boldsymbol{x}) = \phi'(\boldsymbol{x}) - \phi(\boldsymbol{x}) = -G^\phi_g\phi(\boldsymbol{x})\delta g. \tag{1.1.23}$$

From (1.1.22) we find

$$G_g^\phi \phi(\boldsymbol{x})\delta g = \partial_\mu \phi(\boldsymbol{x})\delta_g x'^\mu - \delta_g R^\phi(\boldsymbol{x}). \tag{1.1.24}$$

The first term in this expression is due to the change of coordinates only and is given exactly by the ϵ part of the infinitesimal transformations (1.1.11). The second term is determined by the choice of the representation on the target space V. Finally note that we can write the finite transformation R^ϕ in (1.1.20) as

$$R^\phi(\boldsymbol{x})(g) = e^{G_g^\phi(\boldsymbol{x})\delta g}. \tag{1.1.25}$$

For example, the generator of the translations for any field is just

$$P_\mu^\phi = \partial_\mu. \tag{1.1.26}$$

This follows directly from (1.1.24) and (1.1.11) since for the translations the representation R^ϕ is trivial for any field, so $\delta R^\phi/\delta g = 0$.

Since the conformal group has a group of regular rotations of $\mathbb{R}^d$ as its subgroup, the conformal fields carry a representation of rotations. In other words the conformal fields have a determined value of spin. We define a *spin operator* $\boldsymbol{S}_{\mu\nu}$ by the transformation properties of ϕ at $\boldsymbol{x} = 0$,

$$L_{\mu\nu}^\phi \phi(0) = \boldsymbol{S}_{\mu\nu}\phi(0), \tag{1.1.27}$$

where $\boldsymbol{S}_{\mu\nu}$ is a representation matrix of rotations on the target vector space V. In other words $\boldsymbol{S}_{\mu\nu}$ is one of the standard representations of the group of rotations, for example for vectors

$$(\boldsymbol{S}_{\mu\nu})^{\alpha\beta} = \delta_\mu^\beta \delta_\nu^\alpha - \delta_\mu^\alpha \delta_\nu^\beta. \tag{1.1.28}$$

This follows from the transformation property (1.1.24) applied to rotations at $\boldsymbol{x} = 0$ and the form of infinitesimal rotations (1.1.11). From this expression one can easily get representations for other tensors. Given a representation $\boldsymbol{S}_V$ on a vector space V, we have

$$\boldsymbol{S}_{V\otimes V} = \mathbf{1} \otimes \boldsymbol{S}_V + \boldsymbol{S}_V \otimes \mathbf{1}, \qquad \boldsymbol{S}_{V^*} = -\boldsymbol{S}_V, \tag{1.1.29}$$

where V^* denotes a dual of V. We will discuss representations for fermions in sections 2.7.5 and 4.3.4.

The action of the operator $L_{\mu\nu}^\phi$ can be extended to all points by means of the formula (1.1.25). Any generator $G_g(0)$ defined at $\boldsymbol{x} = 0$ can be extended to $G_g(\boldsymbol{x})$ via,

$$(G_g\phi)(\boldsymbol{x}) = e^{x^\alpha P_\alpha} G_g(0) e^{-x^\alpha P_\alpha} \phi(\boldsymbol{x}), \tag{1.1.30}$$

where we used the fact that $[G_g(0), P_\mu] = 0$. Then the right hand side can be evaluated by means of the Hausdorff formula

$$e^{x^\alpha P_\alpha} A e^{-x^\alpha P_\alpha} = A - x^\alpha [A, P_\alpha] + \frac{1}{2} x^\alpha x^\beta [[A, P_\alpha], P_\beta] - \frac{1}{3!} x^\alpha x^\beta x^\gamma [[[A, P_\alpha], P_\beta], P_\gamma] + \ldots \tag{1.1.31}$$

In this way, using the commutation relations (1.1.17) we find

$$L^\phi_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu + \boldsymbol{S}_{\mu\nu}. \tag{1.1.32}$$

Finally, we can analyse the consequences of the dilatations and special conformal transformations. As for the rotations, let us start with the rigid transformations at $\boldsymbol{x} = 0$. Since, according to (1.1.17), D commutes with all generators of rotations, then by Schur's lemma in an irreducible representation of rotations the representation matrix for dilatations is diagonal and takes form $\Delta \mathbf{1}$, where $\Delta \in \mathbb{R}$ is a real number called a *conformal dimension* of field ϕ and $\mathbf{1}$ is the identity matrix. Similarly, $[D, K_\mu] = K_\mu$, which implies that at $\boldsymbol{x} = 0$ we have $K_\mu \phi(0) = 0$. By means of (1.1.31) one finds the following actions of the generators of the conformal group

(i)	translations:	$P^\phi_\mu \phi(\boldsymbol{x}) = \partial_\mu \phi(\boldsymbol{x})$,
(ii)	rotations:	$L^\phi_{\mu\nu} \phi(\boldsymbol{x}) = (x_\nu \partial_\mu - x_\mu \partial_\nu) \phi(\boldsymbol{x}) + \boldsymbol{S}_{\mu\nu} \phi(\boldsymbol{x})$,
(iii)	dilatations:	$D^\phi \phi(\boldsymbol{x}) = (x^\alpha \partial_\alpha + \Delta) \phi(\boldsymbol{x})$,
(iv)	special conf.:	$K^\phi_\mu \phi(\boldsymbol{x}) = (2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu + 2\Delta x_\mu) \phi(\boldsymbol{x}) - 2x^\alpha \boldsymbol{S}_{\mu\alpha} \phi(\boldsymbol{x})$.

(1.1.33)

Note that the representation of the conformal group in $d \geqslant 3$ is uniquely determined by specification of the representation of the rotations $\boldsymbol{S}_{\mu\nu}$, *i.e.*, the spin of the representation and the conformal dimension Δ.

Observe that for a conformal field ϕ we have,

$$K^\phi \phi(0) = 0. \tag{1.1.34}$$

This can be viewed as a definition of the conformal field. Indeed, assume that some field ϕ with the spin operator $\boldsymbol{S}_{\mu\nu}$ satisfies (1.1.34). Using the Jacobi identity one can show that this implies

$$D^\phi \phi(0) = \Delta \phi(0) \tag{1.1.35}$$

for some number Δ. Then, the use of the Hausdorff formula (1.1.31) leads to the action (1.1.33).

Furthermore, observe that if a general field ϕ is an eigenfunction of D^ϕ with eigenvalue Δ, then by means of the commutation relations (1.1.17),

$$D^\phi P_\mu^\phi \phi(\boldsymbol{x}) = (\Delta+1) P_\mu^\phi \phi(\boldsymbol{x}), \tag{1.1.36}$$

$$D^\phi K_\mu^\phi \phi(\boldsymbol{x}) = (\Delta-1) K_\mu^\phi \phi(\boldsymbol{x}), \tag{1.1.37}$$

$$D^\phi L_{\mu\nu}^\phi \phi(\boldsymbol{x}) = \Delta L_{\mu\nu}^\phi \phi(\boldsymbol{x}). \tag{1.1.38}$$

Therefore P_μ^ϕ increases the conformal dimension by 1, K_μ^ϕ decreases by 1 and $L_{\mu\nu}^\phi$ does not change the conformal dimension of ϕ.

Finally, the infinitesimal variations (1.1.24) can be integrated out to the global transformation rule. For a conformal map $\boldsymbol{x} \mapsto \boldsymbol{x}'$ denote

$$J_\alpha^\mu = \frac{x'^\mu}{x^\alpha}, \qquad J = \det(J_\alpha^\mu) = \Omega^{\frac{d}{2}}, \tag{1.1.39}$$

where Ω is defined in (1.1.1). The finite conformal transformations for the tensor field $\phi^{\mu_1\ldots\mu_m}$ with all indices up and of conformal dimension Δ are

$$\phi'^{\mu_1\ldots\mu_m}(\boldsymbol{x}') = J^{-\frac{\Delta}{d}} J_{\alpha_1}^{\mu_1} \ldots J_{\alpha_m}^{\mu_m} \phi^{\alpha_1\ldots\alpha_m}(\boldsymbol{x}). \tag{1.1.40}$$

For forms with indices down one must replace J by J^{-1}. The values of the Jacobian appearing in the transformation property above for various conformal maps are

$$\begin{array}{lll} \text{(i)} & \text{translations:} & J = 1, \\ \text{(ii)} & \text{rotations:} & J = 1, \\ \text{(iii)} & \text{dilatations:} & J = \lambda^d, \\ \text{(iv)} & \text{special conformal:} & J = (1 + 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2 x^2)^{-d}, \\ & \text{inversions (1.1.14):} & J = x^{2d}. \end{array} \tag{1.1.41}$$

In particular for any $\lambda \in \mathbb{R}^+$,

$$\phi^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n}(\lambda\boldsymbol{x}) = \lambda^{-\Delta+m-n} \phi^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n}(\boldsymbol{x}). \tag{1.1.42}$$

The last entry in the list above shows the Jacobian for the inversion defined in (1.1.14).

1.1.4. Conserved currents

Assume a classical field theory is given on a flat background $\mathbb{R}^d$ by specifying its action S. By Noether theorem, each continuous symmetry of the action implies an existence of a conserved current. In particular, all theories we consider are transnationally invariant, which implies that the theory is invariant under $x^\mu \mapsto x^\mu + \epsilon\xi^\mu$, with a constant vector ξ^μ. The set of conserved currents is indexed by a choice of the vector ξ^μ, $j^\mu_{\boldsymbol{\xi}} = \xi^\nu \Theta^\mu{}_\nu$, where

$$\Theta^\mu{}_\nu = \frac{\delta L}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu L \tag{1.1.43}$$

is a canonical stress-energy tensor and L is a Lagrangian which integrates to the action S.

The canonical stress-energy tensor is not necessarily symmetric and therefore is not a stress-energy tensor that appears in Einstein equations of General Relativity. However, it is always possible to obtain a symmetric stress-energy tensor by an addition of a total derivative. Such a procedure does not change the conserved charges, which are obtained by the integration of the stress-energy tensor over appropriate hyperplanes. The symmetrized stress-energy tensor, called a *Belinfante stress-energy tensor* is

$$T_B^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{4}\partial_\alpha(B^{\alpha\mu\nu} + B^{\nu\alpha\mu} + B^{\mu\nu\alpha}), \tag{1.1.44}$$

where

$$B^{\alpha\mu\nu} = \frac{\delta L}{\delta(\partial_\mu \phi^I)}\boldsymbol{S}^{\nu\alpha}\phi. \tag{1.1.45}$$

Using the fact that $\boldsymbol{S}_{\mu\nu} = -\boldsymbol{S}_{\nu\mu}$ one can show that the Belinfante tensor is indeed symmetric.

Now we want to analyse the conservation laws following from scaling and special conformal transformations. The Noether theorem states, that the conserved current j^μ associated with an infinitesimal symmetry δg is [5],

$$j^\mu = \Theta^\mu{}_\alpha \delta_g x^\alpha - \frac{\delta L}{\delta(\partial_\mu \phi)}\delta_g R^\phi(\boldsymbol{x}) \tag{1.1.46}$$

where R^ϕ is defined by the transformation property (1.1.20). For special conformal transformations we find a set of currents $j_{\boldsymbol{b}}^\mu$ indexed by a choice of the vector $\boldsymbol{b}$ in (1.1.12). Therefore we can define the set of special conformal currents j_ν^μ by $j_{\boldsymbol{b}}^\mu = b^\nu j_\nu^\mu$. Using (1.1.46) and the transformations we found in the previous section we obtain [10],

$$j_D^\mu = \Theta^\mu{}_\alpha x^\alpha + l^\mu, \tag{1.1.47}$$

$$j_\nu^\mu = \Theta^\mu{}_\alpha(2x_\nu x^\alpha - x^2\delta_\nu^\mu) - 2x_\nu k^\mu - 2l^\mu{}_\nu, \tag{1.1.48}$$

for some functions of fields k_μ and $l_{\mu\nu}$. They can be evaluated explicitly, but they depend on the Lorentz structure of the fields. For a tensor field $\phi^{\mu_1\cdots\mu_m}_{\nu_1\ldots\nu_n}$ we have

$$k^\mu = (\Delta - m + n)\frac{\delta L}{\delta(\partial_\mu \phi)}\phi, \tag{1.1.49}$$

$$l^\mu{}_\nu = -\frac{\delta L}{\delta(\partial_\mu \phi)}\boldsymbol{S}_\nu^\mu \phi. \tag{1.1.50}$$

Applying the derivative to the currents, $0 = \partial_\mu j_D^\mu = \partial_\mu j_\nu^\mu$ we find that the conservation laws imply

$$\Theta^\mu{}_\mu = -\partial_\alpha k^\alpha, \qquad \partial_\alpha l_\nu^\alpha = -k_\nu. \tag{1.1.51}$$

These conditions give $\Theta^{\mu}{}_{\mu} = \partial_\alpha \partial_\beta l^{\alpha\beta}$ and the following tensor

$$T^{\mu\nu} = T_B^{\mu\nu} + \frac{1}{d-2}\Big[\delta^{\alpha\mu}\delta^{\beta\gamma}\delta^{\nu\delta} + \delta^{\alpha\nu}\delta^{\beta\gamma}\delta^{\mu\delta} - \delta^{\alpha\beta}\delta^{\mu\gamma}\delta^{\nu\delta} - \delta^{\mu\nu}\delta^{\alpha\gamma}\delta^{\beta\delta} + \frac{\delta^{\gamma\delta}}{d-1}\left(\delta^{\mu\nu}\delta^{\alpha\beta} - \delta^{\mu\alpha}\delta^{\nu\beta}\right)\Big]\partial_\alpha\partial_\beta l_{\gamma\delta}, \tag{1.1.52}$$

is a conserved, symmetric and *traceless* tensor that has the same charges as the canonical stress-energy tensor. By $T_B^{\mu\nu}$ we denoted the Belinfante tensor (1.1.44).

From now on we will refer to $T^{\mu\nu}$ as the *stress-energy tensor* of a conformal field theory. It satisfies

$$T^{\mu\nu} = T^{\nu\mu}, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad T = T^\mu_\mu = 0. \tag{1.1.53}$$

Having such a stress-energy tensor in the theory is equivalent to the conformal symmetry.

Note that if we assume the scaling symmetry only, then the conservation of the dilatation current would imply the first equality in (1.1.51) only. This equation is not enough to improve the canonical stress-energy tensor to the traceless one via (1.1.52). The classical theory invariant under scalings but not necesairly under the special conformal transformations is called *scale-invariant*.

The existence of scale invariance leads also to the conclusion that the Lagrangian of a conformally invariant theory cannot contain dimensionful coupling constants. Indeed, apart from its conformal dimension, any field has also its physical dimension in units of the length L. Assume a dimension of a coupling constant to be L^α and note that the physical dimension can be also assigned to derivatives as follows

	Conformal dim.	Physical dim.
fields	Δ	L^Δ
derivatives	$+1$	L
couplings	0	L^α

For the action to be well-defined, the Lagrangian must have both conformal and physical dimension equal to d. Therefore if a term in a Lagrangian contains a coupling constant, it must be dimensionless, $\alpha = 0$.

1.1.5. Weyl invariance

In the previous section we have shown that each conformal field theory possess a stress-energy tensor that is symmetric, conserved and traceless and has the same conserved charges as the canonical stress-energy tensor. There is another possibility for a construction of such a tensor by coupling the theory to a background metric and taking a functional derivative

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{1.1.54}$$

or equivalently

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{1.1.55}$$

To motivate it, note that for a symmetric and conserved stress-energy tensor we can write by Noether theorem

$$\begin{aligned} 0 = \delta_{\boldsymbol{\xi}} S &= \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\sqrt{g}T^{\mu\nu}(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu), \\ &= -\frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\sqrt{g}T^{\mu\nu}\delta g_{\mu\nu}, \end{aligned} \tag{1.1.56}$$

where in the second line we used (1.1.6). By applying the conformal transformations (1.1.40) and taking the flat space limit one shows that $T_{\mu\nu}$ is a conformal primary field of the conformal weight $\Delta = d$, where d is the dimension of spacetime.

Such a procedure follows yet from a different perspective, when the conserved currents can be obtained by gauging the symmetry and looking at the response of the action in the first order of perturbation of the background gauge field. For a stress-energy tensor, the symmetry to be gauged is a rotational symmetry (Lorentz symmetry) and the gauge field is a Cartan metric connection. By definition, the gauged system is invariant under arbitrary diffeomorphisms,

$$\boldsymbol{x} \mapsto \boldsymbol{x}', \qquad g_{\mu\nu} \mapsto \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}. \tag{1.1.57}$$

The initial Lorentz invariance is now encoded as the local Lorentz symmetry, which requires that under an infinitesimal diffeomorphism $x^\mu \mapsto x^\mu + \xi^\mu(\boldsymbol{x})$ the action is invariant.

A similar procedure can be applied to the scaling symmetry. By coupling the theory to gravity this symmetry extends to the local invariance under

$$g_{\mu\nu}(\boldsymbol{x}) \mapsto e^{2\sigma(\boldsymbol{x})}g_{\mu\nu}(\boldsymbol{x}), \tag{1.1.58}$$

for arbitrary *function* $\sigma(\boldsymbol{x})$. Infinitesimally,

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}. \tag{1.1.59}$$

Such an invariance is called a *Weyl invariance*. Weyl invariance follows from the full conformal symmetry since the variation of the action is

$$0 = \delta S = 2\int \mathrm{d}^d\boldsymbol{x}\sqrt{g}T^\mu_\mu\sigma, \tag{1.1.60}$$

which is equivalent to the tracelessness of the stress-energy tensor. The converse is also true: a local Lorentz and Weyl invariant theory is conformally invariant.

When a flat theory is extended to the theory valid for more general backgrounds, one can extend it in a Weyl invariant way. This may require addition to the action of some terms that vanish in a flat space limit. Such terms must be invariant under diffeomorphisms, and therefore must be built up from Riemann tensors, covariant derivatives and other generally covariant objects. As an example, consider a free scalar field on a flat background

$$S = \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\, \partial_\mu\phi\partial^\mu\phi. \tag{1.1.61}$$

If the conformal dimension Δ of the field ϕ is $\frac{d}{2}-1$, then this action is invariant under conformal transformations (1.1.12). The canonical stress-energy tensor is

$$\Theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi, \tag{1.1.62}$$

and while it is symmetric, it is not traceless. By means of the procedure described in section 1.1.4, one can modify it by the addition of an appropriate full derivative term in order to find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \frac{d-2}{4(d-1)}(\delta_{\mu\nu}\delta^{\alpha\beta} - \delta_\mu^\alpha\delta_\nu^\beta)\partial_\alpha\partial_\beta\phi^2. \tag{1.1.63}$$

This field is traceless when the equations of motion $\Box\phi = 0$ are used.

Now we would like to couple the system (1.1.61) to the metric in a Weyl invariant way, which limits to (1.1.61) upon substitution $g_{\mu\nu} = \delta_{\mu\nu}$. The correct action is

$$S = \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\mu\phi + \frac{d-2}{4(d-1)}R\phi^2\right], \tag{1.1.64}$$

where R is a Ricci scalar for a background metric $g_{\mu\nu}$. The Ricci scalar vanishes for the flat space and this action reproduces (1.1.61). The additional factor is necessary for the Weyl invariance, as one can see from the following Weyl transformations

$$\delta_\sigma\sqrt{g} = d\sigma\sqrt{g}, \qquad \delta_\sigma R = -2\sigma R - 2(d-1)\Box\sigma. \tag{1.1.65}$$

Applying the transformation both to the background metric and the dynamical field one finds indeed $\delta_\sigma S = 0$.

Finally note that the stress-energy stress-energy tensor of the Weyl invariant theory (1.1.64) is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + \frac{d-2}{4(d-1)}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\phi^2 \tag{1.1.66}$$

and its trace is

$$T = T^{\mu}_{\mu} = \frac{d-2}{2}\left[\phi\Box\phi - \frac{d-2}{4(d-1)}R\phi^2\right], \tag{1.1.67}$$

which vanishes by equations of motion. As we can see, (1.1.66) reduces to (1.1.63) on flat background $g_{\mu\nu} = \delta_{\mu\nu}$. The additional term in the action (1.1.64) is chosen in such a way that it reproduces the correction in (1.1.63) when the derivative with respect to the metric is taken.

1.2. Conformal quantum field theory

1.2.1. Definitions

A quantum field theory can be defined either by a set of quantum operators acting on a Hilbert space of states or by a set of time-ordered correlation functions. Since we want to work mostly in Euclidean signture, it will be much more convenient to use the formulation with correlation functions. Note that by means of the reconstruction theorems one can reinterpret all postulates in terms of quantum fields. The details of the mathematical constructions can be found in [11, 12, 13].

A conformal field theory is a quantum field theory which satisfies the following properties:

1. There exists a set of quantum fields $\{\mathcal{A}_j\}$, which in general is infinite and contains all derivatives of all fields.

2. There exists a subset $\{\mathcal{O}_j\} \subseteq \{\mathcal{A}_j\}$ of *primary fields* that transform covariantly under the action of the conformal group. To be more precise, each field $\mathcal{O}_j$ carries a representation of the conformal group, *i.e.*, it has a definite conformal dimension Δ_j and a representation of the group of rotations $\boldsymbol{S}_{\mu\nu}$. The transformation property for the correlation functions follows from the transformation property of the fields (1.1.20). For scalar conformal primaries one has

$$\langle\mathcal{O}_1(\boldsymbol{x}'_1)\dots\mathcal{O}_n(\boldsymbol{x}'_n)\rangle = \left|\frac{\partial\boldsymbol{x}'_1}{\partial\boldsymbol{x}_1}\right|^{-\Delta_1/d}\dots\left|\frac{\partial\boldsymbol{x}'_n}{\partial\boldsymbol{x}_n}\right|^{-\Delta_n/d}\langle\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_n(\boldsymbol{x}_n)\rangle. \tag{1.2.1}$$

   For fields of general spin one adds the appropriate transformation matrices as in (1.1.40). Every field in $\{\mathcal{A}_j\}$ can be expressed as linear combinations of the primary fields and their derivatives, called *descendent* fields.

3. The theory can be coupled to the background matrix $g_{\mu\nu}$ and has a well-defined symmetric stress-energy tensor, traceless and conserved on-shell, that generates conformal transformations.

4. The vaccum state $|0\rangle$ is invariant under conformal transformations.

Every conformal field theory is Weyl invariant, *i.e.* under the Weyl transformation of the metric $g_{\mu\nu}(\boldsymbol{x}) \mapsto e^{2\sigma(\boldsymbol{x})} g_{\mu\nu}(\boldsymbol{x})$,

$$\langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle_{e^{2\sigma} g_{\mu\nu}} = e^{-\sigma(\boldsymbol{x}_1)\Delta_1} \ldots e^{-\sigma(\boldsymbol{x}_n)\Delta_n} \langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle_{g_{\mu\nu}}, \tag{1.2.2}$$

for fields $\mathcal{O}_j$, $j = 1, \ldots, n$ of any tensor structure and conformal dimensions Δ_j. Due to the regularisation issues this statement is valid at non-coincident point only, *i.e.* for $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, $i \neq j$. The reason is that in order to define a quantum field theory one must use a regularisation scheme that may break some of the symmetries of the theory. After the regulator is removed, the broken symmetries may but do not have to be restored. Throughout this work we will use various types of the dimensional regularisations, which maintain Lorentz invariance, but break Weyl invariance. As we will see in specific examples, the Weyl invariance sometimes is not restored after the regulator is removed. In such cases the violation of Weyl invariance is local, *i.e.*, it affects the correlation functions at coincident points only. Therefore, the correlation functions remain Weyl invariant at non-coincident points, while the generating functional as a functional of a metric fails to be Weyl invariant. Such a behaviour is called a *trace* or *Weyl anomaly*.

In this work we assume that the conformal field theories we consider are invariant under the action of the entire conformal group. However, one can consider quantum field theories invariant under Poincaré group and scalings only. Such field theories are called *scale invariant*. Every conformal field theory is by definition scale invariant, but the opposite is true only for $d = 2$ [14, 10]. In other dimensions scale invariance does not imply conformal invariance. For counterexamples and the detailed discussion see [15, 16, 17].

Alternatively, one can define a theory by a set of axioms to be satisfied by quantum operators in Lorentzian signature. An element g of the Poincaré group can be uniquely decomposed as a pair $(\boldsymbol{a}, \Lambda)$, where $\boldsymbol{a}$ is a vector corresponding to the translation $\boldsymbol{x} \mapsto \boldsymbol{x}+\boldsymbol{a}$ and Λ is a matrix of rotation. In a standard quantum field theory one has given a unitary representation $(\boldsymbol{a}, \Lambda) \mapsto U(\boldsymbol{a}, \Lambda)$ of the Poincaré group on the Hilbert space of states such that

$$U(\boldsymbol{a}, \Lambda) \mathcal{O}(\boldsymbol{x}) U^{-1}(\boldsymbol{a}, \Lambda) = L(\Lambda^{-1}) \mathcal{O}(\Lambda \boldsymbol{x} + \boldsymbol{a}), \tag{1.2.3}$$

where $L(\Lambda^{-1})$ is a representation matrix for rotations. In conformal field theories we require an invariance under the full conformal group. Therefore, we need a representation $g \mapsto U(g)$ of the conformal group on the Hilbert space of states such that the field operators transform accordingly. For dilatations $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$ the transformation property following from (1.1.40) is

$$U(\lambda) \mathcal{O}^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(\boldsymbol{x}) U^{-1}(\lambda) = \lambda^{\Delta - m + n} \mathcal{O}^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(\lambda \boldsymbol{x}). \tag{1.2.4}$$

Infinitesimally, this leads to the adjoint representation of the generators of the conformal group on the space of quantum fields. Exact expressions are very similar to (1.1.33),

$$\begin{array}{lll} \text{(i)} & \text{translations:} & [P^{\mathcal{O}}_\mu, \mathcal{O}(\boldsymbol{x})] = \partial_\mu \mathcal{O}(\boldsymbol{x}), \\ \text{(ii)} & \text{rotations:} & [L^{\mathcal{O}}_{\mu\nu} \mathcal{O}(\boldsymbol{x})] = (x_\nu \partial_\mu - x_\mu \partial_\nu) \mathcal{O}(\boldsymbol{x}) + \boldsymbol{S}_{\mu\nu} \mathcal{O}(\boldsymbol{x}), \\ \text{(iii)} & \text{dilatations:} & [D^{\mathcal{O}}, \mathcal{O}(\boldsymbol{x})] = (x^\alpha \partial_\alpha + \Delta) \mathcal{O}(\boldsymbol{x}), \\ \text{(iv)} & \text{special conf.:} & [K^{\mathcal{O}}_\mu, \mathcal{O}(\boldsymbol{x})] = (2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu + 2\Delta x_\mu) \mathcal{O}(\boldsymbol{x}) \\ & & \qquad\qquad\qquad -2x^\alpha \boldsymbol{S}_{\mu\alpha} \mathcal{O}(\boldsymbol{x}). \end{array} \tag{1.2.5}$$

In particular the defining property of the conformal primary field is a quantum version of (1.1.34),

$$[K^{\mathcal{O}}_\mu, \mathcal{O}(0)] = 0. \tag{1.2.6}$$

1.2.2. Operator-state correspondence

In any quantum field theory the Hilbert space of states can be generated by the action of smeared field operators on the vaccum state $|0\rangle$,

$$\begin{aligned} |\psi\rangle = f_0|0\rangle &+ \int \mathrm{d}^d \boldsymbol{x}_1 f_1(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_1)|0\rangle \\ &+ \int \mathrm{d}^d \boldsymbol{x}_1 \mathrm{d}^d \boldsymbol{x}_2 f_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2)|0\rangle + \ldots \end{aligned} \tag{1.2.7}$$

Here f_j is the set of smearing functions that vanish at infinity faster than any polynomial. For the state $|\psi\rangle$ to be well-defined, only a finite number of the smearing functions should be non-zero. In principle, by changing smearing functions, one can generate a class of different states. In a conformal field theory the situation is more treatable. There is a one-to-one correspondence between states and local operators in a CFT. In other words for each state $|\psi\rangle$ there exists a unique *local* operator $\mathcal{O}_\psi$ such that

$$|\psi\rangle = \mathcal{O}_\psi(0)|0\rangle. \tag{1.2.8}$$

In order to prove the operator-state correspondence, consider a general state $|\psi\rangle$, (1.2.7). By linearity, we can consider a single term, and for simplicity consider only the second term on the right hand side and assume $\mathcal{O}$ is a scalar. By $U(\lambda)$ denote the unitary action of the scaling on the Hilbert space as in (1.2.4). By substituting $\boldsymbol{x}_1 = \lambda \boldsymbol{y}$ under the integral in (1.2.7) we have

$$|\psi\rangle = \int \mathrm{d}^d \boldsymbol{y} \lambda^d f_1(\lambda \boldsymbol{y}) \lambda^{-\Delta} U(\lambda) \mathcal{O}(\boldsymbol{y})|0\rangle. \tag{1.2.9}$$

Now take $\lambda \to \infty$ and notice that by the assumption on the support of the smearing functions

$$\lim_{\lambda\to\infty} f_1(\lambda \boldsymbol{y}) = \frac{c_1}{\lambda^d} \delta(\boldsymbol{y}), \tag{1.2.10}$$

where c_1 is some constant and $\delta(\boldsymbol{y})$ is Dirac delta. Therefore

$$|\psi\rangle = c_1 \lim_{\lambda\to\infty} \lambda^{-\Delta} U(\lambda)\mathcal{O}(0)|0\rangle, \tag{1.2.11}$$

which finishes the proof.

Let us elaborate on the correspondence. First observe that the identity operator is an operator of conformal dimension zero and via the operator-state correspondence it is mapped to the vacuum state $|0\rangle$. Next, if we assume that $\mathcal{O}$ is a primary field of dimension Δ, then the generated state $|\Delta\rangle = \mathcal{O}(0)|0\rangle$ is an eigenstate of the dilatation operator,

$$D^{\mathcal{O}}|\Delta\rangle = \Delta|\Delta\rangle, \tag{1.2.12}$$

where we use the definition of $|\Delta\rangle$ and (1.2.5).

1.2.3. Operator product expansion

A direct consequence of the operator-state correspondence is the existence of the operator product expansion. Consider a complete basis $|n\rangle$ in the Hilbert space of states. By the operator-state correspondence, each state in the basis corresponds to a local operator in the theory $\mathcal{O}_n$. Operator product expansion is a statement that a product of any two local operators $\mathcal{O}$ and $\mathcal{O}'$ can be expanded as

$$\mathcal{O}(\boldsymbol{x})\mathcal{O}'(\boldsymbol{0}) = \sum_n c_n(\boldsymbol{x})\mathcal{O}_n(\boldsymbol{0}), \tag{1.2.13}$$

under the expectation value. The coefficients $c_n(\boldsymbol{x})$ are functions depending on the operators $\mathcal{O}$, $\mathcal{O}'$ and a choice of the basis. Indeed, of we consider the state $\mathcal{O}'$ corresponding to the operator $\mathcal{O}'$, we can write

$$\mathcal{O}(\boldsymbol{x})|\mathcal{O}'\rangle = \sum_n c_n(\boldsymbol{x})|\mathcal{O}_n\rangle. \tag{1.2.14}$$

Then the second use of the operator-state correspondence leads to (1.2.13).

As we know operators in a CFT are ordered into conformal families. Therefore, one can group operators $\mathcal{O}_n$ into a set of primaries $\tilde{\mathcal{O}}_k$ and their descendants. Then the OPE (1.2.13) can be rewritten as

$$\mathcal{O}(\boldsymbol{x})\mathcal{O}'(\boldsymbol{0}) = \sum_k \left[c_{k0}(\boldsymbol{x})\tilde{\mathcal{O}}_k(\boldsymbol{0}) + c_{k1}^{\mu_1}(\boldsymbol{x})\partial_{\mu_1}\tilde{\mathcal{O}}_k(\boldsymbol{0}) + c_{k2}^{\mu_1\mu_2}(\boldsymbol{x})\partial_{\mu_1}\partial_{\mu_2}\tilde{\mathcal{O}}_k(\boldsymbol{0}) + \dots\right], \tag{1.2.15}$$

where the sum is taken over conformal families. If $\mathcal{O}$ and $\mathcal{O}'$ are conformal primaries of dimensions Δ and Δ' respectively, then

$$c_{kj}^{\mu_1\dots\mu_j}(\lambda\boldsymbol{x}) = \lambda^{\Delta_k-\Delta_1-\Delta_2+j}c_{kj}^{\mu_1\dots\mu_j}(\boldsymbol{x}). \tag{1.2.16}$$

This means that the expansion under the sum in (1.2.15) is in fact a Taylor expansion. This observation will allow to calculate the OPE coefficients for specific examples. We will present such an example in section 1.4.1.

1.2.4. Unitarity bounds

So far conformal dimensions were arbitrary real numbers. In a quantum field theory there exists a lower bound on all conformal dimensions, known as a *unitarity bound*. Any theory with operators of conformal dimensions violating the unitarity bound would be non-unitary in Lorentzian signature and non-positive in Eucildean one.

From section 1.2.2 we know that the identity operator corresponds to the vacuum state of a CFT. This is the 'lowest energy state' with respect to the 'hamiltonian' D, which exponentiates to the 'evolution operator' $U(\lambda)$. In other words $\Delta \geqslant 0$ for all states in the theory.

Using the operator-state correspondence (1.2.11) we can derive more strict positivity conditions. To do so, introduce polar coordinates in $\mathbb{R}^d$ by writing any point as $(r, \boldsymbol{e})$, where $\boldsymbol{e} \in S^{d-1}$ is a point on a unit sphere. Define a new 'radial' coordinate $\tau = \log r$, which we can view as a diffeomorphism between $\mathbb{R}^d \backslash \{0\}$ and the cylinder $\mathbb{R} \times S^{d-1}$ with τ parametrising $\mathbb{R}$ and $\boldsymbol{e}$ parametrising the sphere. Note that the metric induced on the cylinder is

$$\mathrm{d}s^2 = e^{2\tau}(\mathrm{d}\tau^2 + \mathrm{d}\Omega^2_{d-1}), \tag{1.2.17}$$

where $\mathrm{d}\Omega^2_{d-1}$ is a standard metric on a unit sphere S^{d-1}. This means that the conformal field theory on the cylinder parametrised by τ and $\boldsymbol{e}$ is Weyl equivalent to the flat space theory. The evolution in τ corresponds to the evolution in the radial direction r and therefore it is generated by the dilatation operator D. If one quantises the theory on the cylinder, (so called *radial quantisation*) then the usual reflection positivity of the Euclidean theory requires

$$\mathcal{O}^\dagger(\tau, \boldsymbol{e}) = \mathcal{O}(-\tau, \boldsymbol{e}), \tag{1.2.18}$$

so that the norm of the state (1.2.7) is non-negative, $\langle\psi|\psi\rangle \geqslant 0$. This means that in the radial quantisation,

$$\mathcal{O}^\dagger(r, \boldsymbol{e}) = \mathcal{O}^\dagger(e^\tau, \boldsymbol{e}) = \mathcal{O}(e^{-\tau}, \boldsymbol{e}) = r^{2\Delta}\mathcal{O}(r^{-1}, \boldsymbol{e}), \tag{1.2.19}$$

where Δ is the conformal dimension of $\mathcal{O}$. Therefore the state conjugate to $|\psi\rangle$ is the state corresponding to

$$\mathcal{O}^\dagger(\boldsymbol{x}) = x^{2\Delta}\mathcal{O}(I(\boldsymbol{x})), \tag{1.2.20}$$

where I is the inversion defined in (1.1.14). From this we can establish conjugation rules for the operators in the conformal algebra. Using the relation (1.1.14) and the fact that $I^{-1} = I$ we find

$$P^\dagger_\mu = IP_\mu I = K_\mu. \tag{1.2.21}$$

The procedure for the extraction of unitarity bounds is straightforward but complex in execution. One considers an expression

$$A_{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} = \langle\mathcal{O}|K_{\mu_1}\ldots K_{\mu_n}P_{\nu_1}\ldots P_{\nu_n}|\mathcal{O}\rangle, \tag{1.2.22}$$

which is positively definite due to the reflection positivity. By permuting all K_μ with P_ν one arrives at the expectation value with insertions of D and $L_{\mu\nu}$, due to the commutation relations (1.1.17). By means of the representation theory one is able to extract the bound on the lowest eigenvalue of such operators. In such a way the following unitarity bounds for fields of spin s can be obtained

$$\begin{aligned}
&\Delta \geqslant \frac{d}{2}-1, && s=0,\\
&\Delta \geqslant \frac{d-1}{2}, && s=\frac{1}{2},\\
&\Delta \geqslant d+s-2, && s\geqslant 1.
\end{aligned} \tag{1.2.23}$$

In general one can derive stringent bounds for fields in any representation of the group of rotations. For the detail of the procedure, see [18, 19].

As we have seen in the example in section 1.1.5 a free scalar field has its conformal dimension saturating the bound, $\Delta = \frac{d}{2}-1$. As it was shown in [20], in $d=4$, if the field transforming in the representation $(s,0)$ or $(0,s)$ of the complexification of the Lorentz group $sl_2\mathbb{C}\oplus sl_2\mathbb{C}$ saturates the unitarity bound, it is a free field with free field correlation functions.

1.3. Ward identities

Ward identities are quantum laws of conservation. The equations of conservation such as $\partial_\mu j^\mu = 0$ do not hold in quantum case in general. The reason is that in a classical theory $\partial_\mu j^\mu = 0$ holds on shell only. The quantum laws of conservation are stated in terms of Ward identities. These idenities express the n-point functions with the insertion of a divergence of a conserved current in terms of $(n-1)$-point correlation functions. In this section we will study the Ward identities for all conformal symmetries.

1.3.1. Canonical Ward identities

Invariance of the field theory under conformal transformations is expressed via (1.2.1) and its obvious generalisation to higher spin fields. Assume now that the correlation functions in the field theory are given by path integrals

$$\langle\mathcal{O}_1(\boldsymbol{x}_1)\ldots\mathcal{O}_n(\boldsymbol{x}_n)\rangle = \int\mathcal{D}\Phi\,\mathcal{O}_1(\boldsymbol{x}_1)\ldots\mathcal{O}_n(\boldsymbol{x}_n)e^{-S}, \tag{1.3.1}$$

where S is the action. In such case the infinitesimal version of the invariance means that

$$\begin{aligned}0 &= \delta_g \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle \\ &= -\langle \delta_g S \, \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle + \sum_{j=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_g \mathcal{O}_j(\boldsymbol{x}_j) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle. \end{aligned} \tag{1.3.2}$$

We assume here that the integration measure in the Feynman integral is invariant under g. If this is not the case, anomalies appear. If g is a classical symmetry of the system, then by Noether theorem

$$\delta_g S = -\int \mathrm{d}^d \boldsymbol{x} \, g(\boldsymbol{x}) \partial_\mu j^\mu(\boldsymbol{x}) \tag{1.3.3}$$

where j^μ is a Noether current. Using Dirac deltas we can write (1.3.2) as

$$\partial_\mu \langle j^\mu(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle = -\sum_{j=1}^{n} \delta(x - x_j) \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_g \mathcal{O}_j(\boldsymbol{x}) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle. \tag{1.3.4}$$

This form of the Ward identity is called *local*. It expresses the n-point function with the insertion of a divergence of a conserved current in terms of $(n-1)$-point correlation functions. One can integrate both sides of (1.3.4) over $\boldsymbol{x}$ to obtain the *global* Ward identity,

$$\begin{aligned}0 &= \sum_{j=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_g \mathcal{O}_j(\boldsymbol{x}_j) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle \\ &= \sum_{j=1}^{n} G_g(\boldsymbol{x}_j) \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle, \end{aligned} \tag{1.3.5}$$

where G_g is defined in (1.1.23).

In this section we will analyse the basic consequences of global Ward identities. This is due to the fact that it is much more convenient to analyse the local Ward identities in the context of background fields. We will analyse their properties in sections 1.3.3 and 1.3.4.

Let us see what is implied by the global Ward identities for the conformal symmetries (1.1.12). We can directly use (1.3.5) with the infinitesimal transformations of the fields given by (1.1.33) due to the definition (1.1.23). For example, for translations the global Ward identity is

$$0 = \sum_{j=1}^{n} \frac{\partial}{\partial x_j^\mu} \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle, \tag{1.3.6}$$

which implies that the correlation function depends on the differences $\boldsymbol{x}_i - \boldsymbol{x}_j$ only. Similarly, the rotational invariance implies that the correlation function depends on the distances $|\boldsymbol{x}_i - \boldsymbol{x}_j|$ only.

Let now $\mathcal{O}_1, \ldots, \mathcal{O}_n$ denote a set of conformal primaries of dimensions $\Delta_1, \ldots, \Delta_n$ and of arbitrary Lorentz structure. The dilatation Ward identity in position space is especially simple and reads

$$0 = \left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle. \tag{1.3.7}$$

This identity is easy to solve. Notice that for any constant c, the monomial $f(x) = cx^{-\alpha}$ is the most general solution to $(x\partial + \alpha) f(x) = 0$. This means that (1.3.7) fixes the degree of the correlation function to be $\Delta_t = \sum_{j=1}^{n} \Delta_j$. One can write the form of the n-point function as

$$\langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle = x_{12}^{-\Delta_t} F\left(\frac{x_{ij}}{x_{12}} \right), \tag{1.3.8}$$

where F is a function of dimensionless quantities. The distance x_{12} can be replaced by any other distance or combination of distances in such a way that the total degree remains equal to Δ_t.

By taking G_g to be K_μ in (1.3.5) we obtain the Ward identity associated with special conformal transformations. For the n-point function of *scalar* operators $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ it reads

$$0 = \left[\sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\kappa}} \right) \right] \langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle, \tag{1.3.9}$$

where κ is a free Lorentz index. For tensor operators one needs to add an additional term to the equation. This term depends on the Lorentz structure, and to write it down, we assume that the tensor $\mathcal{O}_j$ has r_j Lorentz indices, *i.e.*, $\mathcal{O}_j = \mathcal{O}_j^{\mu_{j1} \ldots \mu_{jr_j}}$, for $j = 1, 2, \ldots, n$. In this case, the contribution

$$\begin{aligned} 2 \sum_{j=1}^{n} \sum_{k=1}^{r_j} & \left[(x_j)_{\alpha_{jk}} \delta^{\kappa \mu_{jk}} - x_j^{\mu_{jk}} \delta^\kappa_{\alpha_{jk}} \right] \times \\ & \times \langle \mathcal{O}_1^{\mu_{11} \ldots \mu_{1n_1}}(\boldsymbol{x}_1) \ldots \mathcal{O}_j^{\mu_{j1} \ldots \alpha_{jk} \ldots \mu_{jr_j}}(\boldsymbol{x}_j) \ldots \mathcal{O}_n^{\mu_{n1} \ldots \mu_{nr_n}}(\boldsymbol{x}_n) \rangle \end{aligned} \tag{1.3.10}$$

must then be added to the right-hand side of (1.3.9). The Ward identities associated to the special conformal transformation impose very strong conditions on the correlation functions. We will solve them in section 1.4 in position space and in section 2.4 it will be our starting point for the analysis in momentum space.

Local Ward identities deliver a different kind of information. In classical field theory divergences of conserved currents vanish on-shell. In quantum field theory they do not vanish, but the expectation values of such operators can be simplified via (1.3.4). We could study the consequences of local Ward identities right now, but it will be much more convenient to discuss them in the context of background fields. The main reason is that the formalism utilising background fields is much more convenient, but the currents it produces usually differ slightly from the canonical ones.

1.3.2. Generating functional and coupling to gravity

In previous section, in order to write down the global Ward identities, we worked directly with correlation functions. From now on we will assume that all correlation functions can be packaged into a *generating functional*. Assume $\{\mathcal{O}_j\}$ is a set of conformal primaries of arbitrary Lorentz spin in a given CFT and the dynamics of the CFT is determined by the flat space action S_{CFT}. The generating functional is

$$Z[\phi_0^{(j)}] = \int \mathcal{D}\Phi \, \exp\left(-S_{CFT} - \sum_j \int \mathrm{d}^d\boldsymbol{x}\phi_0^{(j)}\mathcal{O}_j\right), \tag{1.3.11}$$

where $\{\phi_0^{(j)}\}$ is a set smooth functions rapidly vanishing at infinity known as *sources* or *background fields* for corresponding operators. Sources are not dynamical, *i.e.*, we do not integrate over them in the path integral.

Now by taking the functional derivatives with respect to the sources one gets correlation functions, *e.g.*,

$$\langle\mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\rangle = \frac{-\delta}{\delta\phi_0^{(1)}(\boldsymbol{x}_1)}\frac{-\delta}{\delta\phi_0^{(2)}(\boldsymbol{x}_2)}Z[\phi_0^{(j)}]\bigg|_{\text{All }\phi_0^{(j)}=0}, \tag{1.3.12}$$

We would like to express the invariance of the correlation functions using the generating functional. In this case we must find out the action of the symmetries on the background fields in the same way as we found the action of the conformal and Weyl transformations on the metric in section 1.1.5. From there we know that a CFT can be coupled to gravity and the stress-energy tensor can be obtained by a differentiation with respect to the metric. In other words, the source for stress-energy tensor is the metric. This means that the source for $T^{\mu\nu}$ can be removed from the explicit list in (1.3.11). Instead, the CFT action becomes a functional of the metric. In a similar fashion we can gauge every other symmetry group G leading to the conserved current. Then, by taking the derivatives with respect to the corresponding gauge connection, one obtains the correlation functions of the currents. Therefore we assume that every other current corresponding to a global symmetry can be covariantly coupled to a background gauge field via the change of derivatives into covariant derivatives,

$$\partial_\mu \;\longmapsto\; D_\mu^{IJ} = \delta^{IJ}\partial_\mu - \mathrm{i}A_\mu^a (T_R^a)^{IJ}. \tag{1.3.13}$$

Here we assume that each operator $\mathcal{O}$ in the theory transforms in some representation $R_{\mathcal{O}}$ given by a set of matrices T_R^a, $a = 1, \ldots, \dim G$. Our conventions follow [21], for generators of the group G we have

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \qquad\qquad [T^a, T^b] = \mathrm{i}f^{abc}T^c, \tag{1.3.14}$$

where f^{abc} are structure constants of the group and in the adjoint representation we have

$$(T_A^a)^{bc} = -\mathrm{i}f^{abc}. \tag{1.3.15}$$

In the remaining part of the thesis we will be interested in correlation functions involving stress-energy tensor, conserved currents and scalar primary operators only, therefore we will limit ourselves to the following form of the generating functional

$$Z[\phi_0^I, A_\mu^a, g^{\mu\nu}] = \int \mathcal{D}\Phi \, \exp\left(-S_{CFT}[A_\mu^a, g^{\mu\nu}] - \sum_j \int \mathrm{d}^d\boldsymbol{x}\sqrt{g}\phi_0^{(j)}\mathcal{O}_j\right). \tag{1.3.16}$$

Now the 1-point functions with sources turned on are given by the following functional derivatives,

$$\langle T_{\mu\nu}(\boldsymbol{x})\rangle = -\frac{2}{\sqrt{g(\boldsymbol{x})}}\frac{\delta}{\delta g^{\mu\nu}(\boldsymbol{x})}Z, \tag{1.3.17}$$

$$\langle J^{\mu a}(\boldsymbol{x})\rangle = -\frac{1}{\sqrt{g(\boldsymbol{x})}}\frac{\delta}{\delta A_\mu^a(\boldsymbol{x})}Z, \tag{1.3.18}$$

$$\langle \mathcal{O}_j(\boldsymbol{x})\rangle = -\frac{1}{\sqrt{g(\boldsymbol{x})}}\frac{\delta}{\delta \phi_0^{(j)}(\boldsymbol{x})}Z. \tag{1.3.19}$$

By taking more functional derivatives we can obtain higher-point correlation functions, *e.g.*,

$$\begin{aligned}
&\langle T_{\mu_1\nu_1}(\boldsymbol{x}_1)T_{\mu_2\nu_2}(\boldsymbol{x}_2)T_{\mu_3\nu_3}(\boldsymbol{x}_3)\rangle = \\
&\quad= \frac{-2}{\sqrt{g(\boldsymbol{x}_3)}}\frac{\delta}{\delta g^{\mu_3\nu_3}(\boldsymbol{x}_3)}\frac{-2}{\sqrt{g(\boldsymbol{x}_2)}}\frac{\delta}{\delta g^{\mu_2\nu_2}(\boldsymbol{x}_2)}\frac{-2}{\sqrt{g(\boldsymbol{x}_1)}}\frac{\delta}{\delta g^{\mu_1\nu_1}(\boldsymbol{x}_1)}Z[g^{\mu\nu}] \\
&\qquad + \; 2\langle\frac{\delta T_{\mu_1\nu_1}(\boldsymbol{x}_1)}{\delta g^{\mu_2\nu_2}(\boldsymbol{x}_2)}T_{\mu_3\nu_3}(\boldsymbol{x}_3)\rangle + 2\langle\frac{\delta T_{\mu_2\nu_2}(\boldsymbol{x}_2)}{\delta g^{\mu_3\nu_3}(\boldsymbol{x}_3)}T_{\mu_1\nu_1}(\boldsymbol{x}_1)\rangle \\
&\qquad\qquad + \; 2\langle\frac{\delta T_{\mu_3\nu_3}(\boldsymbol{x}_3)}{\delta g^{\mu_1\nu_1}(\boldsymbol{x}_1)}T_{\mu_2\nu_2}(\boldsymbol{x}_2)\rangle.
\end{aligned} \tag{1.3.20}$$

Note that here we define the 3-point function of the stress-energy tensor to be the correlator of three separate stress-energy tensor insertions (and similarly for other correlators involving conserved currents), rather than the correlator obtained by functionally differentiating the generating functional with respect to the metric three times. While the latter definition is used in [22, 23, 24, 25], our definition here is simpler for direct QFT computations. To convert between the two definitions simply requires the addition or subtraction of the semi-local terms in the formula above.

Furthermore note that we always include the source terms in the definition of the stress-energy tensor. We have

$$T_{\mu\nu}(\boldsymbol{x}) = T_{CFT\mu\nu}(\boldsymbol{x}) - g_{\mu\nu} \sum_j \phi_0^{(j)}(\boldsymbol{x}) \mathcal{O}_j(\boldsymbol{x}), \tag{1.3.21}$$

where $T_{CFT\mu\nu}$ is the stress-energy tensor following form the action S_{CFT}.

Now one would expect that the invariance of the correlation functions under a given symmetry group G is

$$\delta_g Z[g_{\mu\nu}, A_\mu, \phi_0^{(j)}] = 0, \tag{1.3.22}$$

for any $g \in G$, provided we know the transformation properties of the background fields. We will give these in the following section, however, before we do it, we must point out that (1.3.22) may not be valid for all symmetries in quantum case. The reason is that in order to define a quantum field theory one must usually use a regularisation scheme that may break some of the symmetries of the theory. After the regulator is removed, the broken symmetries may but do not have to be restored. Throughout this work we will use various types of the dimensional regularisations, which maintain Poincaré invariance. Therefore (1.3.22) is valid for translations and rotations. In general, (1.3.22) receives an *anomalous contribution* on its right hand side. Since in case of the conformal symmetry, dilatations and special conformal transformations can be realised by Weyl transformation, the anomaly manifests itself when the Weyl transformation $g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu}$ is taken,

$$\delta_\sigma Z[g_{\mu\nu}, A_\mu, \phi_0^{(j)}] = \mathcal{W}_\sigma[g_{\mu\nu}, A_\mu, \phi_0^{(j)}], \tag{1.3.23}$$

where $\mathcal{W}$ is a computable, theory dependent functional. In flat space theory one can also find this anomaly by considering the scalings. It turns out that the the integration measure in the path integral is not invariant under the scalings, which leads to the Weyl anomaly.

1.3.3. Transverse Ward identities

By coupling the system to the background fields, we can obtain the conserved currents by a functional differentiation with respect to the background field, rather than by considering the action of the symmetry on the dynamical fields. We have three local symmetries to consider,

1. Diffeomorphism symmetry that follows from the gauging of the Lorentz symmetry. This symmetry is related to the divergence of the stress-energy tensor $\partial_\mu T^{\mu\nu}$.

2. Other gauge symmetries that follow from global symmetries of the flat space theory. These symmetries are related to the divergence of Noether currents $\partial_\mu j^\mu$.

3. Weyl symmetry. This symmetry is related to the trace of the stress-energy tensor $T = T^\mu_\mu$.

In this section we will discuss the first two symmetries and the Weyl symmetry will be analysed in the following section.

Under a diffeomorphism ξ^μ the sources transform as

$$\delta g^{\mu\nu} = -(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu), \tag{1.3.24}$$

$$\delta A^a_\mu = \xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu \cdot A^a_\nu, \tag{1.3.25}$$

$$\delta \phi^I_0 = \xi^\nu \partial_\nu \phi^I_0, \tag{1.3.26}$$

where ∇ is a Levi-Civita connection.

Under a gauge symmetry transformation with parameter α^a the sources transform as

$$\delta g^{\mu\nu} = 0, \tag{1.3.27}$$

$$\delta A^a_\mu = -D^{ac}_\mu \alpha^c = -\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c, \tag{1.3.28}$$

$$\delta \phi^I_0 = -\mathrm{i}\alpha^a (T^a_R)^{IJ} \phi^J_0, \tag{1.3.29}$$

where T^a_R are matrices of a representation R and f^{abc} are structure constants of the group G. The gauge field transforms in the adjoint representation while ϕ^I may transform in any representation R. The covariant derivative is $D^{IJ}_\mu = \delta^{IJ}\partial_\mu - \mathrm{i}A^a_\mu (T^a_R)^{IJ}$.

Ward identities follow from the requirement that the generating functional (1.3.16) is invariant under the variations

$$\delta_{\boldsymbol{\xi}} = \int \mathrm{d}^d \boldsymbol{x} \left[-(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) \frac{\delta}{\delta g^{\mu\nu}} + (\xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu \cdot A^a_\nu) \frac{\delta}{\delta A^a_\mu} + \xi^\mu \partial_\mu \phi^I_0 \frac{\delta}{\delta \phi^I_0} \right], \tag{1.3.30}$$

$$\delta_\alpha = -\int \mathrm{d}^d \boldsymbol{x} \left[(\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c) \frac{\delta}{\delta A^a_\mu} + \mathrm{i}\alpha^a (T^a_R)^{IJ} \phi^J_0 \frac{\delta}{\delta \phi^I_0} \right], \tag{1.3.31}$$

so that the transverse Ward identities are

$$\delta_{\boldsymbol{\xi}} Z = 0, \qquad \delta_\alpha Z = 0. \tag{1.3.32}$$

Using definitions (1.3.17) - (1.3.19) these lead to the following equations with sources turned on,

$$\begin{aligned} 0 &= D^{ac}_\mu \langle J^{\mu a} \rangle - \mathrm{i}(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}^I \rangle \\ &= \nabla_\mu \langle J^{\mu a} \rangle + f^{abc} A^b_\mu \langle J^{\mu c} \rangle - \mathrm{i}(T^a_R)^{IJ} \phi^J_0 \langle \mathcal{O}^I \rangle, \end{aligned} \tag{1.3.33}$$

$$\begin{aligned} 0 &= \nabla^\mu \langle T_{\mu\nu} \rangle + \nabla_\nu A^a_\mu \langle J^{\mu a} \rangle - \nabla_\mu A^a_\nu \langle J^{\mu a} \rangle + \partial_\nu \phi^I_0 \langle \mathcal{O}^I \rangle - A^a_\nu \nabla_\mu \langle J^{\mu a} \rangle \\ &= \nabla^\mu \langle T_{\mu\nu} \rangle - F^a_{\mu\nu} \langle J^{\mu a} \rangle + D^{IJ}_\nu \phi^J_0 \langle \mathcal{O}^I \rangle, \end{aligned} \tag{1.3.34}$$

These equations may then be differentiated with respect to the sources to obtain the corresponding Ward identities for higher point functions.

1.3.4. Trace Ward identities

In section 1.1.3 we showed that the Lagrangian of a conformally invariant theory cannot contain dimensionful coupling constants. This constraint can be circumvented if we allow 'position-dependent couplings', *i.e.*, background fields and if we prescribe the correct transformation properties under Weyl transformation. Assume the operator $\mathcal{O}$ has a conformal dimension Δ. For the term

$$\int \mathrm{d}^d \boldsymbol{x} \mathcal{O}^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} \phi^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m} \tag{1.3.35}$$

to be invariant under the scalings, we must have

$$\phi^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}(\lambda \boldsymbol{x}) = \lambda^{-(d-\Delta+m-n)} \phi^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}(\boldsymbol{x}). \tag{1.3.36}$$

Therefore under the Weyl transformation parametrised by σ a general source transforms as

$$\delta_\sigma \phi^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m} = (d - \Delta + m - n) \phi^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m} \sigma. \tag{1.3.37}$$

For metric, gauge field and scalar source we have familiar transformation rules,

$$\delta_\sigma g_{\mu\nu} = 2 g_{\mu\nu} \sigma, \tag{1.3.38}$$

$$\delta_\sigma A^a_\mu = 0, \tag{1.3.39}$$

$$\delta_\sigma \phi_0 = (d - \Delta) \phi_0 \sigma. \tag{1.3.40}$$

Let us first consider the case when (1.3.16) is Weyl anomaly free, *i.e.*, $\delta_\sigma Z = 0$. The variation of the generating functional is realised by the following operator,

$$\delta_\sigma = \int \mathrm{d}^d \boldsymbol{x} \sigma \left[2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + (d - \Delta) \phi_0 \frac{\delta}{\delta \phi_0} \right]. \tag{1.3.41}$$

In this case we find the following trace (or Weyl) Ward identity in the presence of the sources is

$$\langle T(\boldsymbol{x})\rangle = (\Delta - d)\phi_0^I(\boldsymbol{x})\langle \mathcal{O}^I(\boldsymbol{x})\rangle, \tag{1.3.42}$$

where $T = T^{\mu}_{\mu}$. Functionally differentiating with respect to the sources then yields trace Ward identities for n-point functions, *e.g.*,

$$\begin{aligned}\langle T(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle = -\Delta\,[&\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_3)\rangle \\ &+ \delta(\boldsymbol{x}_1 - \boldsymbol{x}_3)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle]\,,\end{aligned} \tag{1.3.43}$$

$$\langle T(\boldsymbol{x}_1)T_{\mu\nu}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle\rangle = 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{\mu\nu}(\boldsymbol{x}_2)}\mathcal{O}(\boldsymbol{x}_3)\rangle, \tag{1.3.44}$$

$$\begin{aligned}\langle T(\boldsymbol{x}_1)T_{\mu\nu}(\boldsymbol{x}_2)T_{\rho\sigma}(\boldsymbol{x}_3)\rangle = {}& 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{\mu\nu}(\boldsymbol{x}_2)}T_{\rho\sigma}(\boldsymbol{x}_3)\rangle + 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{\rho\sigma}(\boldsymbol{x}_3)}T_{\mu\nu}(\boldsymbol{x}_2)\rangle \\ &+ 2\langle T(\boldsymbol{x}_1)\frac{\delta T_{\mu\nu}(\boldsymbol{x}_2)}{\delta g^{\rho\sigma}(\boldsymbol{x}_3)}\rangle.\end{aligned} \tag{1.3.45}$$

If the generating functional (1.3.16) is anomalous (1.3.23) then we have

$$\langle T(\boldsymbol{x})\rangle = (\Delta - d)\phi_0^I(\boldsymbol{x})\langle \mathcal{O}^I(\boldsymbol{x})\rangle + \mathcal{A}[g_{\mu\nu}, A_\mu, \phi_0^{(j)}], \tag{1.3.46}$$

where

$$\mathcal{A}[g_{\mu\nu}, A_\mu, \phi_0^{(j)}] = \frac{\delta}{\delta\sigma}\mathcal{W}_\sigma[g_{\mu\nu}, A_\mu, \phi_0^{(j)}]\bigg|_{\sigma=0}. \tag{1.3.47}$$

is called the *Weyl/scaling/trace anomaly*. We will show how anomalies arise in the regularisation procedure in section 2.8.

1.4. Correlation functions

Conformal symmetry imposes very strong constraints on the structure of correlation functions. In particular the form of 1-, 2- and 3-point functions is uniquely fixed up to a small collection of numbers.

Due to the scaling symmetry the most general form of a 1-point function of a primary operator of dimension Δ is

$$\langle \mathcal{O}(\boldsymbol{x})\rangle = \frac{c_\Delta}{x^\Delta}, \tag{1.4.1}$$

where c_Δ is a constant. By the application of the special conformal tranformation,

$$K_\mu\langle \mathcal{O}(\boldsymbol{x})\rangle = 0 \tag{1.4.2}$$

we find that $c_\Delta = 0$ if $\Delta \neq 0$. In section 1.2.4 we discussed the unitarity bounds which require that in a unitary CFT all operators, apart from the identity operator, have strictly positive conformal dimensions, therefore in a fully conformal theory

$$\langle \mathcal{O}(\boldsymbol{x})\rangle = 0, \tag{1.4.3}$$

assuming $\mathcal{O}$ is not proportional to the identity operator.

1.4.1. Scalar operators

Let us analyse the most general form of a 2-point function of two scalar operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of dimensions Δ_1 and Δ_2. Poincaré invariance requires that the two point function takes form

$$\langle \mathcal{O}_1(\boldsymbol{x})\mathcal{O}_2(\boldsymbol{y})\rangle = F(|\boldsymbol{x}-\boldsymbol{y}|), \tag{1.4.4}$$

for some function F. Scale invariance requires that

$$F(\lambda r) = \lambda^{-\Delta_1-\Delta_2} F(r), \tag{1.4.5}$$

which uniquely solves to

$$F(r) = C_{12} r^{-\Delta_1-\Delta_2}, \tag{1.4.6}$$

where C_{12} is an undetermined constant. Finally, the transformation properties under the special conformal transformations can be analysed. However, as explained in section 1.1.1, it is enough to analyse the transformations under the inversion (1.1.14). Using (1.1.15) we find that the 2-point function (1.4.4) can be non-vanishing only if $\Delta_1 = \Delta_2$. Therefore we have found,

$$\langle \mathcal{O}_1(\boldsymbol{x})\mathcal{O}_2(\boldsymbol{y})\rangle = \frac{C_{12}}{|\boldsymbol{x}-\boldsymbol{y}|^{2\Delta_1}}\delta_{\Delta_1\Delta_2}, \tag{1.4.7}$$

where C_{12} is a constant.

A similar method can be applied to the 3-point functions, which leads to their most general form [26, 5],

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{C_{123}}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\boldsymbol{x}_2-\boldsymbol{x}_3|^{\Delta_2+\Delta_3-\Delta_1}|\boldsymbol{x}_3-\boldsymbol{x}_1|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{1.4.8}$$

where C_{123} is a single undetermined constant and Δ_j, $j = 1,2,3$ are conformal dimensions of operators $\mathcal{O}_j$.

At this point we can return to the problem of determination of the OPE coefficients in (1.2.15). For simplicity assume all operators $\mathcal{O}_j = \mathcal{O}$, $j = 1,2,3$ have the same dimension Δ. The OPE is then

$$\mathcal{O}(\boldsymbol{x})\mathcal{O}(0) = \frac{C_{12}}{x^{2\Delta}} + \frac{C}{x^{\Delta}}\mathcal{O}(0) + \text{descendants} + \text{other operators}. \tag{1.4.9}$$

The first term is determined by the 2-point function. We see that the constant C_{12} is exactly the normalisation constant in (1.4.7). The constant C is the OPE coefficient of $\mathcal{O}\mathcal{O} \mapsto \mathcal{O}$. It turns out that it is determined by the C_{123} coefficient in (1.4.8). If $\boldsymbol{x}$ is close to zero, then we have $|\boldsymbol{x}-\boldsymbol{y}| \sim y$ and we can expand the 3-point as

$$\langle \mathcal{O}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = \frac{C_{123}}{x^{\Delta}y^{2\Delta}}[1+O(x)]. \tag{1.4.10}$$

On the other hand the OPE gives

$$\begin{aligned}\langle \mathcal{O}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle &= \frac{C}{x^{\Delta}}\langle \mathcal{O}(\boldsymbol{y})\mathcal{O}(0)\rangle + \text{descendants}\\ &= \frac{CC_{12}}{x^{\Delta}y^{2\Delta}} + \text{descendants}.\end{aligned} \tag{1.4.11}$$

Therefore we find

$$C_{123} = CC_{12}. \tag{1.4.12}$$

In this way one can calculate any OPE coefficients, since - as we will see in the following sections - the form of all 3-point functions is fixed.

The strong constraints on 2- and 3-point functions we have obtained can be understood from the point of view of representation theory. Assume one wants to build a scalar conformal invariants from some set $\{\boldsymbol{x}_j\}$ of distinct points. As long as Poincaré invariance is considered only, such invariants are the distances $|\boldsymbol{x}_i - \boldsymbol{x}_j|$. For further simplicity denote

$$\boldsymbol{x}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j, \qquad x_{ij} = |\boldsymbol{x}_{ij}|. \tag{1.4.13}$$

The distances, however, are not scale invariant as they scale with λ when all points are scaled by λ. Instead, one could try taking ratios of two distances. Such an object is scale invariant but is not invariant under inversions when (1.1.15) is used. Therefore the simplest objects that are conformally invariant are

$$u_{ijkl} = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}, \qquad v_{ijkl} = \frac{x_{ij}x_{kl}}{x_{il}x_{jk}}, \tag{1.4.14}$$

known as *conformal ratios*. For the conformal ratios to be well-defined and non-zero, one needs four distinct points. This leads to the conclusion that the 2- and 3-point functions in any CFT must be uniquely determined up to some number of constants. 4- and higher-point functions, on the other hand, are determined up to an arbitrary function of conformal ratios, for example

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\mathcal{O}_3(\boldsymbol{x}_3)\mathcal{O}_4(\boldsymbol{x}_4)\rangle = F(u_{1234}, v_{1234}) \prod_{1\leqslant i<j\leqslant 4} x_{ij}^{\frac{\Delta_t}{3}-\Delta_i-\Delta_j}, \tag{1.4.15}$$

where $\Delta_t = \sum_{j=1}^4 \Delta_j$ and F is an undetermined function. In case of 4-point functions there are two independent conformal ratios, in case of n-point functions it follows from a simple combinatorics [5] that there are $n(n-3)/2$ independent conformal ratios.

1.4.2. Embedding formalism

Similar considerations as in the case of correlation functions of scalar operators can be applied to general tensor operators. Historically, the form of correlation functions of scalar operators in a CFT appeared in [26] and was quickly generalised to the 3-point function of currents for $d = 4$ in [27]. A complete analysis of all 3-point functions of scalars and tensors of spin one and two, and in general dimension, was carried out in [22, 23], The analysis is based on the fact that any special conformal transformation can be decomposed into translations and inversions. Therefore, in order to impose the special conformal invariance, it is enough to analyse the transformation properties of the correlation function under inversions only. Using such an approach, the analysis was extended to arbitrary operators in [28]. For a sample of more recent work on this topic see also [29, 30, 31, 32].

In this thesis we will use a different elegant approach: *the embedding formalism.* As we have shown in section 1.1.2, the conformal group is locally isomorphic to $SO(d+1,1)$ and therefore it naturally acts via isometries on the space $\mathbb{R}^{d+1,1}$ with the Minkowski metric. The idea is to embed the *physical space* $\mathbb{R}^d$ into the *embedding space* $\mathbb{R}^{d+1,1}$ in such a way that the natural action of $SO(d+1,1)$ on $\mathbb{R}^{d+1,1}$ restricts to the conformal transformations on the physical $\mathbb{R}^d$. Such an approach dates back to Dirac, [33], and its applications to the correlation functions were developed in [34, 35, 36, 37].

Let us introduce the light-cone coordinates in $\mathbb{R}^{d+1,1}$,

$$X^A = (X^+, X^-, X^a) \in \mathbb{R}^{d+1,1}, \tag{1.4.16}$$

where $a = 1, \dots, d$ and the metric is

$$\eta_{AB}\mathrm{d}X^A\mathrm{d}X^B = -\mathrm{d}X^+\mathrm{d}X^- + \delta_{ab}\mathrm{d}X^a\mathrm{d}X^b. \tag{1.4.17}$$

Choose an embedding $X : \mathbb{R}^d \mapsto \mathbb{R}^{d+1,1}$,

$$X(\boldsymbol{x}) = (1, x^2, x^\mu) \tag{1.4.18}$$

in light-cone coordinates, known as the *Poincaré section.* Note that the metric induced on $\mathbb{R}^d$ via X is a flat space metric so the embedding is isometric. Denote the image of the embedding as $P = X(\mathbb{R}^d) \subseteq \mathbb{R}^{d+1,1}$. Notice that $X^2(\boldsymbol{x}) = 0$ in the Minkowski metric and $X^+(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in \mathbb{R}^d$. This means that the physical subspace P is invariant under the natural action of the conformal group $SO(d+1,1)$. With little algebra one can find that this action is precisely the action (1.1.12) in the standard coordinates on $\mathbb{R}^d$. Finally, for two points $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ in P, denote $\boldsymbol{X}_{ij} = \boldsymbol{X}_i - \boldsymbol{X}_j$ and observe that

$$X_{ij}^2 = -2\boldsymbol{X}_i \cdot \boldsymbol{X}_j = (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 = x_{ij}^2. \tag{1.4.19}$$

We used the fact that $\boldsymbol{X}^2 = 0$ on the physical space P.

The remaining piece of information is how to extend field living on $\mathbb{R}^d$ to the embedding space. It turns out that the extension is unique,

$$\mathcal{O}(\lambda X^A) = \lambda^{-\Delta}\mathcal{O}(X^A). \tag{1.4.20}$$

Let us only sketch the argument, for details see [36]. It turns out that a change of the Poincaré section (1.4.18) to a different section leads to a change of the metric on the physical space to a different metric within the same conformal class. The physical space P can be viewed as a space of null rays in $\mathbb{R}^{d+1,1}$ with a conformal structure and the extension (1.4.20) is the only extension consistent with it.

The method for finding the most general form of the correlation function in the embedding space formalism is as follows. First find the most general correlation function of the extended field in the embedding space. Such a correlation function is an invariant of the Lorentz group $SO(d+1,1)$ and scales according to (1.4.20). Then substitute (1.4.18) and use (1.4.19) in order to obtain the usual position space expression. For the 2-point function of operators of dimension Δ we find

$$\langle \mathcal{O}(\boldsymbol{X}_1)\mathcal{O}(\boldsymbol{X}_2)\rangle = \frac{C_{12}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{\Delta}}. \tag{1.4.21}$$

Note that $\boldsymbol{X}_1\cdot\boldsymbol{X}_2$ is the only Lorentz invariant built up from two points, since on P $\boldsymbol{X}_j^2 = 0$. Also note that we do not have translational invariance in the embedding space. Now, using (1.4.19), we recover (1.4.7).

1.4.3. Embedding formalism for tensors

In this section we will extend the embedding formalism to tensor operators. We will consider totally symmetric and traceless tensors only. This is because such tensors transform in the most common irreducible representation of the rotation group. However, it is possible to extend the discussion to other irreducible representations, [36].

In order to analyse the tensor structure, one must understand the push-forward of the tangent bundle of the physical $\mathbb{R}^d$ via X defined in (1.4.18). Since $X^2 = 0$ on the physical space $P = X(\mathbb{R}^d) \subseteq \mathbb{R}^{d+1,1}$, by differentiation we have $X\cdot X_* v = 0$ for any tangent vector v in $\mathbb{R}^d$. Therefore for an extension of any tensor field $\mathcal{O}_{a_1\ldots a_n}$ to the embedding space field $\mathcal{O}_{A_1\ldots A_n}$ we have

$$X^{A_j}\mathcal{O}_{A_1\ldots A_n}(X) = 0, \quad j = 1,\ldots n \tag{1.4.22}$$

on P. Even with this condition there remains a redundancy in the extension of the fields to the embedding space. Indeed, since $\boldsymbol{X}^2 = 0$, one can add an arbitrary function $\Lambda_{A_2\ldots A_n}$,

$$\mathcal{O}_{A_1\ldots A_n}(X) \longmapsto \mathcal{O}_{A_1\ldots A_n}(X) + X_{(A_1}\Lambda_{A_2\ldots A_n)} \tag{1.4.23}$$

and the redefined field is still transverse, *i.e.*, (1.4.22) holds. Because of this redundancy, one can always add appropriate terms to the correlation function in the embedding space so that the tensor structure depends on the differences $\boldsymbol{X}_i - \boldsymbol{X}_j$ only.

Now one can look for the most general Lorentz covariant expression for the 2- and 3-point function which satisfies:

(i) Lorentz covariance in the embedding space,

(ii) scaling as in (1.4.20),

(iii) transversness condition (1.4.22),

(iv) total symmetry and tracelessness in any pair of indices for each operator.

up to the redefinition (1.4.23).

Let us write the most general form of the two-point of two operators with spin one and dimension Δ. According to the rules above we have

$$\langle \mathcal{O}^A(\boldsymbol{X}_1)\mathcal{O}^B(\boldsymbol{X}_2)\rangle = \frac{C_{12}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{\boldsymbol{X}_1\cdot\boldsymbol{X}_2}\right]. \tag{1.4.24}$$

The tracelessness condition fixes $\alpha = -1$. Note that terms proportional to X_1^A or X_2^B are redundant according to (1.4.23). Observe that they can be added at will so when the projection onto the physical space is carried out, one can apply the following rules

$$\begin{aligned} &X_1^B \mapsto x_1^\nu - x_2^\nu, && X_2^A \mapsto x_2^\mu - x_1^\mu, \\ &\eta^{AB} \mapsto \delta^{\mu\nu}, && \boldsymbol{X}_1\cdot\boldsymbol{X}_2 \mapsto -\tfrac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = -\tfrac{1}{2}x_{12}^2. \end{aligned} \tag{1.4.25}$$

Eventually we find

$$\langle \mathcal{O}^\mu(\boldsymbol{x}_1)\mathcal{O}^\nu(\boldsymbol{x}_2)\rangle = \frac{C_{12}I^{\mu\nu}(\boldsymbol{x}_{12})}{x_{12}^{2\Delta}}, \tag{1.4.26}$$

where

$$I^{\mu\nu}(\boldsymbol{x}) = \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2} \tag{1.4.27}$$

and C_{12} is an undetermined constant.

The $I^{\mu\nu}$ tensor appearing in (1.4.26) is an important ingredient of the conformal structure in physical space. It arises as a derivative of the inversion (1.1.14),

$$\frac{\partial I^\mu(\boldsymbol{x})}{\partial x^\nu} = \frac{1}{x^2}I^{\mu\nu}(\boldsymbol{x}) \tag{1.4.28}$$

and it is immediate to check that,

$$I^{\mu\nu}[I(\boldsymbol{x}) - I(\boldsymbol{y})] = I^\mu_\alpha(\boldsymbol{x})I^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y})I^\nu_\beta(\boldsymbol{y}) \tag{1.4.29}$$

and moreover

$$I^{\mu\alpha}(\boldsymbol{x})I^{\nu}_{\alpha}(\boldsymbol{x}) = \delta^{\mu\nu}. \tag{1.4.30}$$

Therefore the operator $I^{\mu\nu}$ can be used to represent inversion on the space of fields and will eventually appear in all correlation functions of tensor operators in position space as found in [22, 23, 28].

Similar considerations can be applied to higher spin correlation functions. For a traceless spin-2 operator of dimension Δ we find the most general form of the 2-point function,

$$\langle \mathcal{O}^{\mu\nu}(\boldsymbol{x}_1)\mathcal{O}^{\rho\sigma}(\boldsymbol{x}_2)\rangle = \frac{C_{12}}{x_{12}^{2\Delta}}\left[I^{\mu\rho}(\boldsymbol{x}_{12})I^{\nu\sigma}(\boldsymbol{x}_{12}) + I^{\mu\sigma}(\boldsymbol{x}_{12})I^{\nu\rho}(\boldsymbol{x}_{12}) - \frac{2}{d}\delta^{\mu\nu}\delta^{\rho\sigma}\right]. \tag{1.4.31}$$

The expressions (1.4.26) and (1.4.31) are symmetric and traceless but they do not vanish when the divergence is taken. Instead, for conserved currents we have

$$\partial_\mu\langle J^\mu J^\nu\rangle = 0, \qquad \partial_\mu\langle T^{\mu\nu}T^{\rho\sigma}\rangle = 0. \tag{1.4.32}$$

This follows from the transverse Ward identities (1.3.33) and (1.3.34) by a differentiation with respect to A_μ and $g_{\mu\nu}$ respectively and the utilisation of the fact that 1-point functions in any CFT vanish, (1.4.3). Using the expressions (1.4.26) and (1.4.31), by direct calculations one finds that J^μ and $T^{\mu\nu}$ are conserved for specific conformal dimensions only. The conclusion is that:

1. a conformal primary operator of spin-1 is conserved if and only if its dimension is $\Delta = d - 1$,

2. a conformal primary operator of spin-2 is conserved if and only if its dimension is $\Delta = d$.

This result assures us that the dimensions of J^μ and $T_{\mu\nu}$ are protected in any CFT from any quantum corrections.

1.4.4. 3-point functions

For 3-point functions we follow the formalism described in the previous sections. The problem can be greatly simplified by finding all possible tensor structures that can appear in any correlation function. Consider the most general correlation function of n tensor operators $\mathcal{O}_j$, each being a tensor of rank r_j,

$$\langle \mathcal{O}_1^{A_{11}A_{12}\ldots A_{1r_1}}(\boldsymbol{X}_1)\mathcal{O}_2^{A_{21}A_{22}\ldots A_{2r_2}}(\boldsymbol{X}_2)\ldots\mathcal{O}_n^{A_{n1}A_{n2}\ldots A_{nr_n}}(\boldsymbol{X}_n)\rangle. \tag{1.4.33}$$

Then, for the conditions (i) - (iii) of listed in the previous section to be satisfied, the correlation function can depend on the following variables only, [35],

$$H_{IJ}^{A_{Ii}A_{Jj}} = \boldsymbol{X}_I \cdot \boldsymbol{X}_J \eta^{A_{Ii}A_{Jj}} - X_I^{A_{Jj}} X_J^{A_{Ii}}, \tag{1.4.34}$$

$$\begin{aligned} V_{I,JK}^{A_{Ii}} &= \frac{1}{\boldsymbol{X}_J \cdot \boldsymbol{X}_K} H_{IJ}^{A_{Ii}B_{Jj}} \boldsymbol{X}_{K\ B_{Jj}} \\ &= \frac{\boldsymbol{X}_I \cdot \boldsymbol{X}_J\, X_K^{A_{Ii}} - \boldsymbol{X}_I \cdot \boldsymbol{X}_K\, X_J^{A_{Ii}}}{\boldsymbol{X}_J \cdot \boldsymbol{X}_K}, \end{aligned} \tag{1.4.35}$$

where $I, J, K \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, r_I\}$, $j \in \{1, \ldots, r_J\}$. To make it more eligible, consider a representative example of the $\langle T^{A_1B_1}T^{A_2B_2}\mathcal{O}\rangle$ correlation function, where T^{AB} is a symmetric tensor of conformal dimension d and $\mathcal{O}$ is a scalar of conformal dimension Δ. The most general form of the correlation function is,

$$\begin{aligned} \langle T^{A_1B_1}T^{A_2B_2}\mathcal{O}\rangle &= \frac{1}{(\boldsymbol{X}_1 \cdot \boldsymbol{X}_3)^{\frac{\Delta+2}{2}} (\boldsymbol{X}_2 \cdot \boldsymbol{X}_3)^{\frac{\Delta+2}{2}} (\boldsymbol{X}_1 \cdot \boldsymbol{X}_2)^{\frac{2d-\Delta}{2}}} \times \\ &\times \left[C_1 H_{12}^{(A_1(A_2} H_{12}^{B_1)B_2)} + C_2 H_{12}^{(A_1(A_2} V_{1,23}^{B_1)} V_{2,31}^{B_2)} + C_3 V_{1,23}^{A_1} V_{1,23}^{B_1} V_{2,31}^{A_2} V_{2,31}^{B_2} \right] \end{aligned} \tag{1.4.36}$$

where C_j, $j = 1, 2, 3$ are three undetermined numerical constants. In the determination of the powers in the denominator we used the fact that $H(\lambda\boldsymbol{X}) = \lambda^2 H(\boldsymbol{X})$ and $V(\lambda\boldsymbol{X}) = \lambda V(\boldsymbol{X})$ for H and V functions in (1.4.34) and (1.4.35). Note also that we still did not impose tracelessness in appropriate indices.

Now one can project (1.4.36) to the physical space. This leads to the substitutions

$$\begin{array}{lll} \boldsymbol{X}_1 \mapsto \boldsymbol{x}_1 - \boldsymbol{x}_2, & \boldsymbol{X}_2 \mapsto \boldsymbol{x}_2 - \boldsymbol{x}_3, & \boldsymbol{X}_3 \mapsto \boldsymbol{x}_3 - \boldsymbol{x}_1, \\ A_j, B_j \mapsto \mu_j, \nu_j & \eta \mapsto \delta, & \boldsymbol{X}_i \cdot \boldsymbol{X}_j \mapsto -\frac{1}{2} x_{ij}^2. \end{array} \tag{1.4.37}$$

We used the freedom expressed by the equation (1.4.23) in order to obtain the result that depends on the differences of points only. Finally, if one requires that the $T^{\mu\nu}$ operators are traceless, then the following projector

$$\mathcal{E}_{\alpha\beta}^{\mu\nu} = \frac{1}{2}(\delta_\alpha^\mu \delta_\beta^\nu + \delta_\beta^\mu \delta_\alpha^\nu) - \frac{1}{d}\delta_{\mu\nu}\delta_{\alpha\beta} \tag{1.4.38}$$

should be applied. After a lot of algebra one finds the following position space expression for the $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}\mathcal{O}\rangle$ correlation function with $T^{\mu\nu}$ being symmetric and traceless of conformal dimension d,

$$\begin{aligned} \langle T^{\mu_1\nu_1}(\boldsymbol{x}_1) T^{\mu_2\nu_2}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{x}_3)\rangle &= \mathcal{E}_{\alpha_1\beta_1}^{\mu_1\nu_1} \mathcal{E}_{\alpha_2\beta_2}^{\mu_2\nu_2} I_{\kappa_1}^{\alpha_1}(\boldsymbol{x}_{13}) I_{\lambda_1}^{\beta_1}(\boldsymbol{x}_{13}) I_{\kappa_2}^{\alpha_2}(\boldsymbol{x}_{23}) I_{\lambda_2}^{\beta_2}(\boldsymbol{x}_{23}) \times \\ &\times \frac{1}{x_{12}^{2d-\Delta} x_{23}^{\Delta} x_{31}^{\Delta}} \left[c_1 X_{12}^{\lambda_1} X_{12}^{\kappa_1} X_{12}^{\lambda_2} X_{12}^{\kappa_2} + 4c_2 \delta^{\lambda_1\lambda_2} X_{12}^{\kappa_1} X_{12}^{\kappa_2} + 2c_3 \delta^{\lambda_1\lambda_2}\delta^{\kappa_1\kappa_2} \right], \end{aligned} \tag{1.4.39}$$

Correlation function	Tensors	Constants	Ward	Local Ward
$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$	1	1	-	-
$\langle J^{\mu_1}\mathcal{O}\mathcal{O} \rangle$	1	1	1	0
$\langle J^{\mu_1}J^{\mu_2}\mathcal{O} \rangle$	2	2	1	1
$\langle J^{\mu_1}J^{\mu_2}J^{\mu_3} \rangle$	3	2	1	1
$\langle T^{\mu_1\nu_1}\mathcal{O}\mathcal{O} \rangle$	1	1	1	0
$\langle T^{\mu_1\nu_1}J^{\mu_2}\mathcal{O} \rangle$	2	2	0	0
$\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3} \rangle$	5	4	2	1
$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}\mathcal{O} \rangle$	3	3	1	1
$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J^{\mu_3} \rangle$	7	5	0	0
$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3} \rangle$	11	5	3	2

Table 1.1: *The list of undetermined coefficients at each level of the analysis for the 3-point functions in generic cases in dimensions $d \geqslant 3$. We assume that $T^{\mu\nu}$ are symmetric and traceless. The conservation is not assumed neither for $T^{\mu\nu}$ nor for J^{μ}. The first column lists the number of tensor structures following from the equation (1.4.41). This is a total number of the tensor structures in the decomposition of the 3-point functions. The second column lists the number of undetermined constants. This is a number of tensor structures modulo symmetries of the correlation function. The third column lists the number of independent constants after the imposition of the conservation equations at non-coincident points, following from transverse Ward identities (1.3.33, 1.3.34). The last column lists the number of the independent constants after the imposition of the conservation equations at coincident and non-coincident points. This will be discussed in section 2.5.3.*

where tensors $I^{\mu\nu}$ are defined in (1.4.27), c_j, $j = 1, 2, 3$ are constants proportional to C_j in (1.4.36) and

$$X_{12}^{\mu} = \frac{x_{23}^2 x_{13}^{\mu} - x_{13}^2 x_{23}^{\mu}}{x_{12}x_{23}x_{31}}. \tag{1.4.40}$$

The expression (1.4.39) looks a little complicated, but the main point is that all 3-point functions in a CFT are determined up to a small number of numerical constants. In this case we found three undetermined constants. In general, using the embedding formalism one can find the number of the independent tensor structures building any correlation function exactly. If one orders spins of the operators as $s_1 \leqslant s_2 \leqslant s_3$ and defines $p = \max(0, s_1 + s_2 - s_3)$, then this number is

$$N(s_1, s_2, s_3) = \frac{(s_1+1)(s_1+2)(3s_2 - s_1 + 3)}{6} - \frac{p(p+2)(2p+5)}{24} - \frac{1-(-1)^p}{16}. \tag{1.4.41}$$

Each tensor structure is multiplied by a numerical constant. The number of independent constants, however, may be smaller than the number of tensors if some symmetry properties are exploited. For example in case of $s_1 = s_2 = s_3 = 2$, there is 11 tensor structures, but symmetries between them reduce the number of independent constants down to 5.

Finally, one can impose the conservation conditions on the currents and the stress-energy tensor. For non-coincident points one just has

$$\partial_{j\mu_j} \langle T^{\mu_1\nu_1}(\boldsymbol{x}_1) T^{\mu_2\nu_2}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{x}_3) \rangle = 0, \quad j = 1, 2 \tag{1.4.42}$$

following from the Ward identity (1.3.34). When applied to (1.4.39) one finds two relations between the constants,

$$c_1 + 4c_2 - \tfrac{1}{2}(d - \Delta)(d - 1)(c_1 + 4c_2) - d\Delta c_2 = 0, \tag{1.4.43}$$

$$c_1 + 4c_2 + d(d - \Delta)c_2 + d(2d - \Delta)c_3 = 0, \tag{1.4.44}$$

so the $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$ correlation function is unique up to one numerical constant.

Finally, one can consider the transverse Ward identities following from (1.3.33) and (1.3.34) more carefully, including the local terms coming from 2-point functions. Such a procedure may restrict the form of the 3-point function further and express the coefficients undetermined so far in terms of normalisations of 2-point functions. We will postpone the analysis of the local terms until the discussion of the momentum space expressions for 3-point functions.

Table 1.1 lists the numbers of undetermined coefficients at each level of the analysis in generic cases in dimensions $d \geqslant 3$.

Chapter 2

Implications of conformal invariance in momentum space

In section 1.4 we have presented a method that allows to write down the most general expressions for all 2- and 3-point functions in a conformal field theory. The method was based on the embedding formalism and the analysis was carried out in position space. The purpose of this part of the thesis is to present the analogous set of results in momentum space.

In principle, the results in momentum space can be obtained from those in position space by Fourier transform. Typically, however, the position space expressions such as (1.4.7) and (1.4.8) are only valid at separated points, and do not possess a Fourier transform prior to renormalisation. Therefore, before the Fourier transform could be applied, the regularisation procedure is necessary. In the first section of this chapter we will introduce a convinient regularisation scheme known as differential regularisation. However, even after renormalisation, it is technically rather difficult to carry out explicitly the Fourier transforms, see for example [24]. Here we will present a complete analysis from first principles of the constraints due to conformal symmetry directly in momentum space. We believe such an analysis gives considerably more insight into the results and is interesting in its own right.

A momentum space analysis is natural from the perspective of Feynman diagram computations, which are usually performed in momentum space. Furthermore, a number of recent works have exemplified the need for CFT results in momentum space. Our original motivation for analysing this question was the requirement for these results in our work on holographic cosmology [38, 39, 40, 1, 2], and similar applications of the conformal/de Sitter symmetry in cosmology have been discussed in [41, 29, 42, 43, 44, 45, 46]. Other recent works that contain explicit computations of CFT correlation functions in momentum space include [47, 48, 49, 24, 50]. Our results may also be useful in the context of work on an a-theorem in diverse dimensions, see [25] for a relevant discussion in $d = 4$.

There are two main issues that complicate the analysis of the implications of conformal invariance in momentum space. While conformal transformations act naturally in position space, they lead to differential operators in momentum space. Dilatations, $\delta x^\mu = \lambda x^\mu$, being linear in x^μ lead to a Ward identity (1.3.7) that is a first-order differential equation, and as such, it is easy to solve in complete generality. Special conformal transformations however are non-linear, so after Fourier transform we obtain a Ward identity that is a second-order differential equation, which makes the analysis more complicated.

The second main issue is the complicated tensorial decomposition required for correlators involving vectors and tensors. Lorentz invariance implies that the tensor structure will be carried by tensors constructed from the momenta p^μ and the metric $\delta_{\mu\nu}$. The standard procedure consists of writing down all possible such independent tensor structures and expressing the correlators as a sum of these structures, each multiplied by scalar form factor. In the case of correlators involving conserved currents and/or stress-energy tensors one then imposes the restrictions enforced by conservation (and tracelessness of the stress-energy tensor in the case of CFTs). Recent works discussing such a tensor decomposition include [49, 25, 48, 47, 24]. This methodology is in principle straightforward, but an inefficient parametrisation can produce unwieldy expressions. Here we present a new parametrisation that appears to yield a minimal number of form factors.

In this work we assume that the underlying theory is parity-invariant. Additional parity-violating terms can appear in the tensorial decomposition of the various correlators and it would be interesting to incorporate them in our analysis.

In the course of the analysis we will show that the counting of independent OPE coefficients presented in table 1.1 is recovered by the independent momentum space analysis. The finital solution for all 3-point functions will be given in form of so called *triple-K integrals*, containing three Bessel K functions. Such integrals usually cannot be expressed by elementary functions. However, in case of correlation functions of conserved currents and stress-energy tensor in odd dimensions, these triple-K integrals reduce to elementary integrals. In such cases we will find a novel way of calculating one-loop Feynman integrals, without any need for the recursion schemes typically used, *e.g.*, [51, 52, 53].

2.1. 2-point functions in momentum space

We will start with the discussion of the form of 2-point functions in momentum space. Due to the momentum conservation any correlation function in position space contains the Dirac delta and one can defined the reduced correlation function as

$$\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\rangle = (2\pi)^d \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(-\boldsymbol{p}_1)\rangle\!\rangle. \tag{2.1.1}$$

We will often use the symbol $\langle\!\langle - \rangle\!\rangle$ and its generalisations to any correlation functions.

2.1.1. Dimensional regularisation

The expressions for 2- and 3-point functions we have found in section 1.4 are valid at non-coincident points only. When two points coincide, the correlation function becomes singular. In Lorentzian signature correlation functions should be well-defined distributions, in particular they should have well-defined Fourier transforms. Therefore, we can analyse the coincident limit and regularise the position space correlation functions by requiring that the Fourier transform exists.

Consider the 2-point function of scalar operators first,

$$\langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = \frac{1}{x^{2\Delta}}, \tag{2.1.2}$$

where we put $C_{12} = 1$, as this is insignificant information here. The Fourier transform of this 2-point function can be done explicitly, since

$$\int \mathrm{d}^d\boldsymbol{x}\, e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{x^{2\Delta}} = \frac{\pi^{d/2} 2^{d-2\Delta}\Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d}. \tag{2.1.3}$$

The integral converges for $0 < 2\Delta < d$. By unitarity bounds $\Delta > 0$, but usually the upper bound on Δ is violated. However, the right hand side of (2.1.3) is well-defined analytic function of Δ. Therefore one can extend it to any $2\Delta \neq d + 2n$, where n is a non-negative integer. If, however, $2\Delta = d + 2n$, a non-trivial regularisation is required.

The analytic continuation in Δ is similar to the analytic continuation performed in *dimensional regularisation*, where the spacetime dimension d is continued away from its physical value. In standard dimensional regularisation in position space one keeps Δ fixed and substitutes $d \mapsto d-\epsilon$ into (2.1.3). The result can be expanded in ϵ and the relevant part is

$$\Gamma\left(-n - \frac{\epsilon}{2}\right) p^{2n+\epsilon} = (-p^2)^n \left[-\frac{2}{n!\epsilon} + \frac{H_n - \gamma_E}{n!} \right] - \frac{1}{n!} p^{2n} \log p^2 + O(\epsilon) \tag{2.1.4}$$

where $H_n = \sum_{j=1}^n j^{-1}$ is n-th harmonic number and γ_E is the Euler gamma constant. As expected, the expansion is singular. The singularity should be subtracted by the addition of local counterterms to the action. For example, if one redefines the action in (1.3.11) as follows

$$S = S_{CFT} + \int \mathrm{d}^{d-\epsilon}\boldsymbol{x}\, \Lambda^{-\epsilon}\phi_0 \mathcal{O} + \frac{c_{d,\Delta}}{\epsilon} \int \mathrm{d}^{d-\epsilon}\boldsymbol{x}\, \Lambda^{-\epsilon}\Box^n \phi_0^2, \tag{2.1.5}$$

then the divergent contribution in (2.1.3) cancels against the divergent counterterm if the constant $c_{d,\Delta}$ is chosen appropriately. After this the $\epsilon \to 0$ limit can be taken. The additional term $\Lambda^{-\epsilon}$ where Λ has a dimension of length is introduced since the total dimension of spacetime was reduced by ϵ.

The described procedure leads to the interesting phenomena of the renormalisation group. Notice that (2.1.5) contains new parameters: the scale Λ and the subleading corrections to $c_{d,\Delta}$ in ϵ. One can see that both parameters are related and they can only contribute to the finite part without the logarithm in (2.1.4). Such a freedom is called a *scheme dependence* and for the physical theory to be predictive, one must fix the value of Λ by a direct measurement of one physical quantity. Then, the theory should not depend on Λ any more. We will discuss general features of renormalisation in section 4.1.2.

2.1.2. Differential regularisation

There exists another elegant procedure to regularise position space expressions in such a way that the result is automatically finite. The method of *differential regularisation* was developed in [54, 55] and is based on the 'pulling derivatives out' trick. Consider the Fourier transform (2.1.3). Since

$$\frac{1}{x^{2\Delta}} = \frac{1}{(2\Delta - 2)(2\Delta - d)} \Box \frac{1}{x^{2\Delta - 2}}, \tag{2.1.6}$$

one can pull a number of boxes so that the power of x fits into the range $0 < 2\Delta < d$. The Fourier transform of $\Box$ is $-p^2$ and one can show that such a procedure leads to (2.1.3) for any $2\Delta \neq d + 2n$.

For $2\Delta = d + 2n$, however, the procedure breaks down as (2.1.6) is not valid since

$$\Box \frac{1}{x^{d-2}} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \delta(\boldsymbol{x}). \tag{2.1.7}$$

This is the reason why (2.1.3) cannot be analytically extended to these values of Δ. In this case one can exchange the expression (2.1.2) for another one that differs only at $\boldsymbol{x} = 0$. It is easy to see that

$$\frac{1}{x^d} = \frac{1}{2(2-d)} \Box \frac{\log(x^2 M^2)}{x^{d-2}} \tag{2.1.8}$$

for $\boldsymbol{x} \neq 0$, where M is an arbitrary parameter. Now one can use (2.1.6) in order to extend it to all $\Delta > 0$. Finally the Fourier transform of (2.1.8) is finite, since

$$\int \mathrm{d}^d \boldsymbol{x}\, e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{\log(x^2 M^2)}{x^\alpha} = -\frac{\pi^{d/2} 2^{d-\alpha} \Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \frac{1}{p^{d-\alpha}} \log\left(\frac{p^2}{\Lambda^2}\right), \tag{2.1.9}$$

with

$$\Lambda^2 = 4M^2 \exp\left(\psi\left(\frac{d-\alpha}{2}\right) + \psi\left(\frac{\alpha}{2}\right)\right), \tag{2.1.10}$$

where ψ is digamma function and $\alpha \neq d + 2n$, with n a non-negative integer.

As we can see, since M is arbitrary, we recovered the same kind of the scheme dependence as in the dimensional regularisation. It was shown [56, 57] that dimensional regularisation is equivalent to differential regularisation up to local terms, *i.e.*, they differ by terms that can be adjusted by fixing the regularisation scheme. The differential regularisation is convenient if one is interested in calculations of Fourier transforms, since it automatically takes care of the divergence.

2.1.3. Scalar 2-point function

Summarising the previous sections, the 2-point function of the scalar conformal primary operator of conformal dimension Δ is given by (2.1.3) and (2.1.9),

$$\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = c_{\mathcal{O}} \times \begin{cases} p^{2\Delta-d} & \text{if } 2\Delta \neq d+2n, \\ p^{2\Delta-d}\left(-\log p^2 + \text{local}\right) & \text{if } 2\Delta = d+2n, \end{cases} \tag{2.1.11}$$

where n is a non-negative integer and $c_{\mathcal{O}}$ is an undetermined normalisation constant. In a unitary theory $c_{\mathcal{O}} > 0$. The 'local' terms are terms of the form $c_M p^{2\Delta-d}$, where c_M is a scheme-dependent constant depending on the regularisation scheme.

The value of $c_{\mathcal{O}}$ is proportional to C_{12} in the position space expression (1.4.7). For example if $\Delta > 0$ and $2\Delta - d \neq 2n$, where n is a non-negative integer, then

$$c_{\mathcal{O}} = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} C_{12}. \tag{2.1.12}$$

2.1.4. Tensorial 2-point functions

Here we will consider the most general form of the 2-point functions of conserved currents J^μ of spin-1 and the stress-energy tensor $T^{\mu\nu}$. In principle one could start with the correlation functions (1.4.26) and (1.4.31) and apply the Fourier transform. However, it is convenient to choose a different approach where the structure of the correlation functions in momentum space is much clearer.

Observe that the operator

$$\pi^\mu_\alpha(\boldsymbol{p}) = \delta^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2} \tag{2.1.13}$$

is a projector onto tensors transverse to $\boldsymbol{p}$, *i.e.*, $p_\mu \pi^\mu_\alpha(\boldsymbol{p}) = 0$. Similarly, in d dimensions, the operator

$$\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = \frac{1}{2}\left[\pi^\mu_\alpha(\boldsymbol{p})\pi^\nu_\beta(\boldsymbol{p}) + \pi^\mu_\beta(\boldsymbol{p})\pi^\nu_\alpha(\boldsymbol{p})\right] - \frac{1}{d-1}\pi^{\mu\nu}(\boldsymbol{p})\pi_{\alpha\beta}(\boldsymbol{p}) \tag{2.1.14}$$

is a projector onto transverse to $\boldsymbol{p}$, traceless, symmetric tensors of rank two. In particular

$$p_\mu \pi^\mu_\alpha(\boldsymbol{p}) = 0, \qquad \delta_{\mu\nu}\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = 0, \qquad p_\mu \Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = 0. \tag{2.1.15}$$

Therefore, any transverse to $\boldsymbol{p}$, traceless, symmetric tensor $t^{\mu\nu}$ of rank two may be written as $t^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p})X^{\alpha\beta}$, where $X^{\alpha\beta}$ is an arbitrary tensor. More properties of the projectors are listed in appendix 2.A.8.

Due to the Ward identities (1.3.33) and (1.3.34) the divergence of any 2-point function of conserved currents is proportional to 1-point functions. Assuming these vanish, we may write the most general decompositions

$$\langle\!\langle T^{\mu\nu}(\boldsymbol{p})T^{\rho\sigma}(-\boldsymbol{p})\rangle\!\rangle = \Pi^{\mu\nu\rho\sigma}(\boldsymbol{p})A(p) + \pi^{\mu\nu}(\boldsymbol{p})\pi^{\rho\sigma}(\boldsymbol{p})B(p), \tag{2.1.16}$$

$$\langle\!\langle J^\mu(\boldsymbol{p})J^\nu(-\boldsymbol{p})\rangle\!\rangle = \pi^{\mu\nu}(\boldsymbol{p})C(p), \tag{2.1.17}$$

where, due to the Lorentz invariance, A, B, C are arbitrary functions of the amplitude of the momentum $\boldsymbol{p}$. These expressions are general, valid in any quantum field theory with $\langle J^\mu\rangle = \langle T^{\mu\nu}\rangle = 0$.

Let us now move to conformal theories, where 1-point functions vanish and the trace Ward identity (1.3.42) implies that $\langle\!\langle T T^{\rho\sigma}\rangle\!\rangle = 0$. This requires $B = 0$ in (2.1.16). Furthermore conformal dimensions of $T^{\mu\nu}$ and J^μ are fixed to d and $d-1$ respectively. Therefore, in position space the most general expressions for the 2-point functions following from (2.1.16) with $B = 0$ and (2.1.17) are

$$\langle T^{\mu\nu}(\boldsymbol{x})T^{\rho\sigma}(0)\rangle = \hat{\Pi}^{\mu\nu\rho\sigma}(\boldsymbol{x})\frac{C_T}{x^{2d}}, \tag{2.1.18}$$

$$\langle J^\mu(\boldsymbol{x})J^\nu(0)\rangle = \hat{\pi}^{\mu\nu}(\boldsymbol{x})\frac{C_J}{x^{2(d-1)}}, \tag{2.1.19}$$

where

$$\hat{\pi}^{\mu\nu} = \delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}, \tag{2.1.20}$$

$$\hat{\Pi}^{\mu\nu\rho\sigma} = \frac{1}{2}\left[\hat{\pi}^{\mu\rho}\hat{\pi}^{\nu\sigma} + \hat{\pi}^{\mu\sigma}\hat{\pi}^{\nu\rho}\right] - \frac{1}{d-1}\hat{\pi}^{\mu\nu}\hat{\pi}^{\rho\sigma} \tag{2.1.21}$$

and C_T and C_J are undetermined constants. In order to Fourier transform (2.1.18) and (2.1.19) one must Fourier transform the factors x^{-2d} and $x^{-2(d-1)}$ by the method presented in the previous sections. The result is

$$\langle\!\langle T^{\mu\nu}(\boldsymbol{p})T^{\rho\sigma}(-\boldsymbol{p})\rangle\!\rangle = c_T \Pi^{\mu\nu\rho\sigma}(\boldsymbol{p}) \times \begin{cases} p^d & \text{if } d = 3, 5, 7, \dots \\ p^d\left(-\log p^2 + \text{local}\right) & \text{if } d = 4, 6, 8, \dots \end{cases} \tag{2.1.22}$$

$$\langle\langle J^\mu(\boldsymbol{p}) J^\nu(-\boldsymbol{p})\rangle\rangle = c_J \pi^{\mu\nu}(\boldsymbol{p}) \times \begin{cases} p^{d-2} & \text{if } d = 3,5,7,\dots \\ p^{d-2}\left(-\log p^2 + \text{local}\right) & \text{if } d = 4,6,8,\dots \end{cases} \tag{2.1.23}$$

where c_T and c_J are two undetermined normalisation constants. In unitary CFTs both must by positive.

In the analysis of the 2-point functions we did not use constraints following from the special conformal transformations. However, one can show that they do not impose any additional conditions on the 2-point functions we have already found. A quick argument is that all momentum space expressions possess a single undetermined constant, exactly as we found in the position space.

2.2. 3-point function of scalar operators

2.2.1. From position to momentum space

Let us start with the analysis of the 3-point function of scalar primary operators. The position space expression is given by (1.4.8). This expression, in principle, can be Fourier transformed in order to obtain the result in momentum space. Extracting the overall Dirac delta function encoding momentum conservation, we define the reduced matrix element, denoted with double brackets,

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = (2\pi)^d \delta(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle. \tag{2.2.1}$$

Assuming $d \geqslant 3$, since $\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3 = 0$ there are two independent momenta. We define

$$\Delta_t = \Delta_1+\Delta_2+\Delta_3, \qquad \delta_j = \frac{d-\Delta_t}{2} + \Delta_j, \quad j = 1,2,3. \tag{2.2.2}$$

A useful representation of the Fourier transform of the position space expression (1.4.8) is

$$\begin{aligned}
&\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle = \\
&\quad = C_{123}\pi^{\frac{3d}{2}} 2^{3d-\Delta_t} \prod_{j=1}^{3} \frac{\Gamma(\delta_j)}{\Gamma\left(\frac{d}{2}-\delta_j\right)} \cdot \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{1}{|\boldsymbol{k}|^{2\delta_3}|\boldsymbol{p}_1-\boldsymbol{k}|^{2\delta_2}|\boldsymbol{p}_2+\boldsymbol{k}|^{2\delta_1}} \\
&\quad = \frac{c_{123}\pi^d 2^{4+\frac{3d}{2}-\Delta_t}}{\Gamma\left(\frac{\Delta_t-d}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\right)} \times \\
&\qquad \times p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty \mathrm{d}x\, x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x),
\end{aligned} \tag{2.2.3}$$

where $K_\nu(z)$ is a Bessel K function, *i.e.*, a modified Bessel function of the second kind. A derivation of this representation may be found in section 4.2.1. As mentioned in the introduction, we will generally refer to integrals of the form above featuring three Bessel K functions and a power as *triple-K* integrals. This form of the 3-point function is familiar in the context of the AdS/CFT correspondence, where every bulk-to-boundary propagator for the field dual to the conformal operator $\mathcal{O}_j$ contains one Bessel K function [58].

The expression (2.2.3) may be severely divergent and requires a regularisation. This stems from the fact that the original position space expression (1.4.8) is valid at non-coincident points only and itself requires a regularisation. A simple solution is to analytically continue (2.2.3) to a function of d and Δ_j, with a regularisation in these parameters then yielding a finite result.

To illustrate this, consider the case of three operators of dimension one in $d = 3$, *i.e.*, $\Delta_j = 1$, $j = 1, 2, 3$. In this case, the Bessel functions can be expressed in terms of elementary functions (see (2.A.24)) and the integral in (2.2.3) has a logarithmic divergence. To regularise the result, we then substitute

$$d \mapsto d + 2\epsilon, \qquad\qquad \Delta_j \mapsto \Delta_j + \epsilon. \tag{2.2.4}$$

This regularisation scheme is extremely useful in context of triple-K integrals since it preserves the indices of Bessel functions in (2.2.3). In the present case, we find

$$\begin{aligned}\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle &= \frac{16 c_{123}\pi^3}{\Gamma\left(\frac{\epsilon}{2}\right) p_1 p_2 p_3} \cdot \int_0^\infty \mathrm{d}x\, x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)} \\ &= \frac{(2\pi)^3 c_{123}}{p_1 p_2 p_3} + O(\epsilon).\end{aligned} \tag{2.2.5}$$

This result can be confirmed by direct calculation using the fact that

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{|\boldsymbol{k}|^2|\boldsymbol{p}_1 - \boldsymbol{k}|^2|\boldsymbol{p}_2 + \boldsymbol{k}|^2} = \frac{1}{8 p_1 p_2 p_3}, \tag{2.2.6}$$

as follows from the substitution $\tilde{\boldsymbol{k}} = \boldsymbol{k}/k^2$.

In summary then, the Fourier transform of the position space expression (1.4.8) for the 3-point function of scalar operators in any CFT may be expressed, at least formally, and up to an overall multiplicative constant, in terms of the triple-K integral (2.2.3). In the next section we will show that this representation in terms of a triple-K integral is very natural in the context of the conformal Ward identities. In fact, we will be able to re-derive the expression (2.2.3) by solving the conformal Ward identities directly in momentum space, without any reference to position space.

2.2.2. Conformal Ward identities

The conformal Ward identities (CWIs) in position space may be found in any standard reference text, *e.g.*, [5]. In momentum space, the Ward identities for scalar operators have been partially analysed in [29, 42], and we will use these results here before generalising them in the following sections. First, observe that due to Lorentz invariance any 3-point function may be expressed in terms of the magnitudes of the momenta,

$$p_j = |\boldsymbol{p}_j| = \sqrt{\boldsymbol{p}_j^2}, \qquad j = 1, 2, 3. \tag{2.2.7}$$

The expression (2.2.3) is in accord with this fact. Regarding the 3-point function as a function of the momentum magnitudes, the dilatation Ward identity then reads

$$0 = \left[2d + \sum_{j=1}^{3} \left(p_j \frac{\partial}{\partial p_j} - \Delta_j \right) \right] \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle\!\rangle. \tag{2.2.8}$$

Similarly, the Ward identity associated with special conformal transformations is

$$0 = \sum_{j=1}^{3} p_j^\kappa \left[\frac{\partial^2}{\partial p_j^2} + \frac{d + 1 - 2\Delta_j}{p_j} \frac{\partial}{\partial p_j} \right] \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle\!\rangle, \tag{2.2.9}$$

where κ is a free Lorentz index. These two equations are direct Fourier transforms of the position space Ward identities (1.3.7) and (1.3.9). We will discuss them in more detail in section 2.4.1.

Choosing $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as independent momenta, we may split this vector equation into two independent scalar equations

$$0 = \left[\left(\frac{\partial^2}{\partial p_1^2} + \frac{d + 1 - 2\Delta_1}{p_1} \frac{\partial}{\partial p_1} \right) - \left(\frac{\partial^2}{\partial p_3^2} + \frac{d + 1 - 2\Delta_3}{p_3} \frac{\partial}{\partial p_3} \right) \right] \langle\!\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle\!\rangle, \tag{2.2.10}$$

$$0 = \left[\left(\frac{\partial^2}{\partial p_2^2} + \frac{d + 1 - 2\Delta_2}{p_2} \frac{\partial}{\partial p_2} \right) - \left(\frac{\partial^2}{\partial p_3^2} + \frac{d + 1 - 2\Delta_3}{p_3} \frac{\partial}{\partial p_3} \right) \right] \langle\!\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle\!\rangle, \tag{2.2.11}$$

where $\langle\!\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle\!\rangle = \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle\!\rangle$ for short. As an immediate check, we may verify that the expression (2.2.3) satisfies (2.2.10, 2.2.11) using the well-known Bessel function relations [59]

$$\frac{\partial}{\partial a} \left[a^\nu K_\nu(ax) \right] = -x a^\nu K_{\nu-1}(ax), \tag{2.2.12}$$

$$K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) = K_{\nu+1}(x). \tag{2.2.13}$$

As we will see shortly, equations of the form (2.2.10, 2.2.11) also arise in the case of 3-point correlators of general tensor operators.

2.2.3. Uniqueness of the solution

To frame our analysis purely in momentum space, we need to show that there is a unique physically acceptable solution, up to an overall multiplicative constant, of the system (2.2.8, 2.2.10, 2.2.11) of dilatation and special CWIs. To accomplish this, it suffices to transform these equations into generalised hypergeometric form by writing

$$\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\!\rangle = p_3^{\Delta_t-2d}\left(\frac{p_1^2}{p_3^2}\right)^{\mu}\left(\frac{p_2^2}{p_3^2}\right)^{\lambda} F\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right), \tag{2.2.14}$$

where the overall power of momenta on the right-hand side is fixed by the dilatation Ward identity (2.2.8), and we have chosen to multiply the arbitrary function F by the prefactor
$(p_1^2/p_3^2)^{\mu}(p_2^2/p_3^2)^{\lambda}$, where μ and λ are arbitrary constants. Substituting this parametrisation into (2.2.10, 2.2.11) then yields a pair of differential equations satisfied by F. Taking μ and λ to be any of the four combinations obtainable from the values

$$\mu = 0,\ \Delta_1 - \frac{d}{2}, \qquad\qquad \lambda = 0,\ \Delta_2 - \frac{d}{2}, \tag{2.2.15}$$

these equations for F read

$$\begin{aligned} 0 = \Bigg[&\xi(1-\xi)\frac{\partial^2}{\partial\xi^2} - \eta^2\frac{\partial^2}{\partial\eta^2} - 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} \\ &+ (\gamma-(\alpha+\beta+1)\xi)\frac{\partial}{\partial\xi} - (\alpha+\beta+1)\eta\frac{\partial}{\partial\eta} - \alpha\beta\Bigg] F(\xi,\eta), \end{aligned} \tag{2.2.16}$$

$$\begin{aligned} 0 = \Bigg[&\eta(1-\eta)\frac{\partial^2}{\partial\eta^2} - \xi^2\frac{\partial^2}{\partial\xi^2} - 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} \\ &+ (\gamma'-(\alpha+\beta+1)\eta)\frac{\partial}{\partial\eta} - (\alpha+\beta+1)\xi\frac{\partial}{\partial\xi} - \alpha\beta\Bigg] F(\xi,\eta), \end{aligned} \tag{2.2.17}$$

where

$$\xi = \frac{p_1^2}{p_3^2}, \qquad\qquad \eta = \frac{p_2^2}{p_3^2}, \tag{2.2.18}$$

and the values of the parameters $\alpha,\beta,\gamma,\gamma'$ depend on the choice of μ and λ. Specifically, parametrising the four choices for μ and λ by two variables $\epsilon_1,\epsilon_2 \in \{-1,+1\}$ according to

$$\mu = \frac{1}{2}(\Delta_1 - \frac{d}{2})(\epsilon_1+1), \qquad \lambda = \frac{1}{2}(\Delta_2 - \frac{d}{2})(\epsilon_2+1), \tag{2.2.19}$$

we have

$$\begin{aligned} &\alpha = \frac{1}{2}\left[\epsilon_1\left(\Delta_1-\frac{d}{2}\right)+\epsilon_2\left(\Delta_2-\frac{d}{2}\right)+\Delta_3\right], \qquad \beta = \alpha - \left(\Delta_3-\frac{d}{2}\right), \\ &\gamma = 1+\epsilon_1\left(\Delta_1-\frac{d}{2}\right), \qquad \gamma' = 1+\epsilon_2\left(\Delta_2-\frac{d}{2}\right). \end{aligned} \tag{2.2.20}$$

The system of equations (2.2.16, 2.2.17) defines the generalised hypergeometric function of two variables Appell F_4. This function has been extensively studied by mathematicians (see, *e.g.*, [60, 61]), and its important properties are summarised in appendix 2.A.4. In particular, the system of equations (2.2.16, 2.2.17) has at most four linearly independent solutions, each of which may be expressed in terms of the F_4 function [61, 62]. The four possible choices for μ and λ reproduce these four solutions exactly.

In a physical context only one linear combination of these four solutions is acceptable: all the others contain divergences for collinear momentum configurations, for example when $p_1 + p_2 = p_3$. To see this, consider the integral representation [63]

$$F_4\left(\alpha, \beta; \gamma, \gamma'; \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{2^{\alpha+\beta-\gamma-\gamma'}\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{p_3^{\alpha+\beta}}{p_1^{\gamma-1}p_2^{\gamma'-1}} \times \\ \times \int_0^\infty \mathrm{d}x x^{\alpha+\beta-\gamma-\gamma'+1} I_{\gamma-1}(p_1 x) I_{\gamma'-1}(p_2 x) K_{\beta-\alpha}(p_3 x), \tag{2.2.21}$$

where $I_\nu(x)$ is the Bessel I function. This expression is formal in the sense that the integral converges only for α, β, γ, γ' in certain ranges, see appendix 2.A.4 for details. For the remaining parameter values the integral is defined by the analytic continuation (2.2.4). Using (2.2.20), one can then write the four solutions for the 3-point functions in the form

$$p_1^{\Delta_1 - \frac{d}{2}} p_2^{\Delta_2 - \frac{d}{2}} p_3^{\Delta_3 - \frac{d}{2}} \int_0^\infty \mathrm{d}x\, x^{\frac{d}{2}-1} I_{\pm(\Delta_1 - \frac{d}{2})}(p_1 x) I_{\pm(\Delta_2 - \frac{d}{2})}(p_2 x) K_{\Delta_3 - \frac{d}{2}}(p_3 x). \tag{2.2.22}$$

For large x we have the asymptotic expansions

$$I_\nu(x) = \frac{1}{\sqrt{2\pi}} \frac{e^x}{\sqrt{x}} + \ldots, \qquad K_\nu(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} + \ldots, \tag{2.2.23}$$

from which we see that the integral (2.2.22) converges at infinite x only for non-triangle (*i.e.*, unphysical) momentum configurations where $p_1 + p_2 < p_3$. Moreover, for the physical collinear momentum configuration $p_1 + p_2 = p_3$, the integral diverges for dimensions $d \geqslant 3$. However, the 3-point function itself is a linear combination of these four solutions and may be continued to the physical regime by choosing the linear combination for which the collinear divergences cancel. This may be accomplished by subtracting two integrals with the same asymptotics, *i.e.*, we sum the four terms of the form (2.2.22) with signs chosen so as to obtain Bessel K functions

$$K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)} \left[I_\nu(x) - I_{-\nu}(x)\right]. \tag{2.2.24}$$

Therefore we arrive at the unique solution

$$\begin{aligned}\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle &= C_{123}\cdot p_1^{\Delta_1-\frac{d}{2}}p_2^{\Delta_2-\frac{d}{2}}p_3^{\Delta_3-\frac{d}{2}}\times\\ &\times\int_0^\infty \mathrm{d}x\, x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(p_1x)K_{\Delta_2-\frac{d}{2}}(p_2x)K_{\Delta_3-\frac{d}{2}}(p_3x), \end{aligned}\tag{2.2.25}$$

where C_{123} is an overall undetermined constant. From the asymptotic expansion (2.2.23), it is clear that this triple-K integral converges at infinite x for physical momentum configurations $p_1+p_2+p_3>0$. Depending on the values of the parameters Δ_j and d, however, the triple-K integral may still diverge at $x=0$. This divergence may be regularised using (2.2.4) as we will discuss in the next section.

In summary then, we have shown that the conformal Ward identities may be solved directly in momentum space leading to a unique result (2.2.25). As we will see shortly, a similar procedure also holds for tensorial correlation functions: solving the momentum space Ward identities will lead to a unique solution for 3-point correlators without any reference to the position space analysis.

2.2.4. Region of validity and anomalies

In this section, we now discuss the regularisation of the potential divergence of the triple-K integral at $x=0$. In general, assuming all parameters and variables are real, the triple-K integral (2.2.25) converges for [63]

$$\frac{d}{2} > \sum_{j=1}^{3}\left|\Delta_j-\frac{d}{2}\right|+2, \qquad\qquad p_1,p_2,p_3>0. \tag{2.2.26}$$

If the parameters in the integral do not satisfy this inequality, however, the integral may be defined via analytic continuation in d and Δ_j. If for some set of parameters the integral exhibits a singularity, then a regularisation is necessary and the scheme (2.2.4) can be used.

Let us consider a concrete example, also discussed in [29]. We set $d=3$ and consider three scalar operators of dimensions $\Delta_j=2$, $j=1,2,3$. The triple-K integral is then logarithmically divergent and a regularisation is necessary. We obtain

$$\begin{aligned}\langle\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\rangle &= C_{123}\left(\frac{\pi}{2}\right)^{3/2}\cdot\int_0^\infty \mathrm{d}x\, x^{-1+\epsilon}e^{-x(p_1+p_2+p_3)}\\ &= C_{123}\left(\frac{\pi}{2}\right)^{3/2}\Gamma(\epsilon)(p_1+p_2+p_3)^{-\epsilon}\\ &= C_{123}\left(\frac{\pi}{2}\right)^{3/2}\left[\frac{1}{\epsilon}-(\gamma_E+\log(p_1+p_2+p_3))+O(\epsilon)\right]. \end{aligned}\tag{2.2.27}$$

The first two terms are proportional to the Fourier transform of $\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)$ and may be removed by adding local counterterms. In the regularisation scheme (2.2.4), such a counterterm has the form

$$S_{ct} = c_\epsilon \int \mathrm{d}^{3+2\epsilon}\boldsymbol{x}\, \phi_0^3 \mu^{-\epsilon}, \tag{2.2.28}$$

where μ is a scale that we introduce (as usual) so that the action is dimensionless. By taking three functional derivatives with respect to the source, we find the contribution of this term to the 3-point function is

$$3! c_\epsilon \mu^{-\epsilon} \delta(\boldsymbol{x}_1 - \boldsymbol{x}_3)\delta(\boldsymbol{x}_2 - \boldsymbol{x}_3). \tag{2.2.29}$$

Therefore, in order to cancel the divergence in (2.2.27), we need to choose

$$c_\epsilon = -\frac{C_{123}}{3!\epsilon}\left(\frac{\pi}{2}\right)^{3/2} + O(\epsilon^0). \tag{2.2.30}$$

Let us make a few comments. First, we can choose the subleading terms in c_ϵ in such a way that the constant in (2.2.27) can attain an arbitrary value. This is just scheme dependence. Secondly, notice that the same value of c_ϵ and the same factor of $\mu^{-\epsilon}$ in (2.2.28) are obtained by standard dimensional regularisation $d \mapsto d - \epsilon$. This observation will be a key point in a relation between regularisations of 2- and 3-point functions discussed in section 2.5.3.

With the renormalisation procedure carried out, we are then left with the finite result

$$\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = -C_{123}\left(\frac{\pi}{2}\right)^{3/2} \log\frac{p_1 + p_2 + p_3}{\tilde{\mu}}, \tag{2.2.31}$$

where

$$\tilde{\mu} = \mu \exp\left[\frac{c_\epsilon^{(0)}}{C_{123}}\left(\frac{2}{\pi}\right)^{3/2}\right] \tag{2.2.32}$$

and $c_\epsilon^{(0)}$ denotes the first subleading term in c_ϵ. Note that this 3-point function is anomalous, *i.e.*, it does not satisfy the dilatation Ward identity (2.2.8) and the scale dependence can be extracted from the μ dependence of the counterterm (2.2.29),

$$\mu\frac{\partial}{\partial\mu}\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = C_{123}\left(\frac{\pi}{2}\right)^{3/2} + O(\epsilon). \tag{2.2.33}$$

Of course, this can also be obtained directly from (2.2.31).

The violation of scale invariance means that the trace Ward identity is anomalous,

$$\langle T \rangle = -\phi_0\langle\mathcal{O}\rangle + \frac{1}{3!}C_{123}\left(\frac{\pi}{2}\right)^{3/2}\phi_0^3. \tag{2.2.34}$$

This in turn implies that the 4-point function of the stress-energy tensor with three $\mathcal{O}$s contains a specific ultra-local term that is not scheme-dependent because it is fixed by the anomalous Ward identity (2.2.34).

2.3. Decomposition of tensors

In this section, we present a natural decomposition of tensorial correlation functions. Correlation functions of conserved currents are transverse and/or traceless – up to local terms – and we would like to find a decomposition which reflects these properties. At this point, we will not yet impose conformal invariance.

The problem of decomposition has already been tackled in a number of papers, see for example [64, 25, 47, 48, 49, 24]. The usual approach consists of writing down the most general tensor structure before imposing the constraints following from symmetries and Ward identities. Here we refine this approach to take account of the permutation symmetries of operator insertions inside correlators, obtaining a convenient and natural decomposition applicable for any correlation function. In particular, our decomposition contains the minimal number of tensor structures, leading to the simplest form for the conformal Ward identities.

We remind the reader we will always be working in d-dimensional Euclidean field theory with a flat metric $\delta_{\mu\nu}$ for which indices are raised and lowered trivially.

2.3.1. Representations of tensor structures

As we are interested in correlation functions, we must consider tensor functions that depend on a number of momenta. Let $\mathcal{O}_j^{\mu_{j1}\mu_{j2}\ldots\mu_{jr_j}}$, $j = 1, 2, \ldots, n$ be a given set of QFT operators. Due to the momentum conservation, the n-point function contains a delta function which may be written explicitly by introducing the reduced matrix element which we denote with double brackets $\langle\!\langle \ldots \rangle\!\rangle$,

$$\begin{aligned}&\langle \mathcal{O}_1^{\mu_{11}\mu_{12}\ldots\mu_{1r_1}}(\boldsymbol{p}_1)\mathcal{O}_2^{\mu_{21}\mu_{22}\ldots\mu_{2r_2}}(\boldsymbol{p}_2)\ldots\mathcal{O}_n^{\mu_{n1}\mu_{n2}\ldots\mu_{nr_n}}(\boldsymbol{p}_n)\rangle \\ &= (2\pi)^d \delta\left(\sum_{k=1}^n \boldsymbol{p}_k\right) \langle\!\langle \mathcal{O}_1^{\mu_{11}\mu_{12}\ldots\mu_{1r_1}}(\boldsymbol{p}_1)\mathcal{O}_2^{\mu_{21}\mu_{22}\ldots\mu_{2r_2}}(\boldsymbol{p}_2)\ldots\mathcal{O}_n^{\mu_{n1}\mu_{n2}\ldots\mu_{nr_n}}(\boldsymbol{p}_n)\rangle\!\rangle.\end{aligned} \tag{2.3.1}$$

The n-point function thus depends on at most $n-1$ vectors, say $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-1}$. If $n - 1 < d$, then all $n-1$ momenta are independent. If $n-1 \geqslant d$, however, then only d generic momenta are independent. In this case we can write [53]

$$\delta^{\mu\nu} = \sum_{j,k=1}^{d} p_j^\mu p_k^\nu (Z^{-1})_{kj}, \tag{2.3.2}$$

where Z is the Gram matrix, $Z = [\boldsymbol{p}_k \cdot \boldsymbol{p}_l]_{k,l=1}^d$, hence the metric $\delta^{\mu\nu}$ is no longer an independent tensor.

From now on we assume $d \geqslant 3$. Since we are primarily interested in 3-point functions, the degeneracy does not occur. Nevertheless, the case $d = 3$ is still special since the existence of the cross-product allows the metric tensor to be re-expressed purely in terms of the momenta. This degeneracy serves to reduce the number of independent form factors for certain correlators, as we discuss in appendix 2.A.2. In the following discussion we will ignore this degeneracy however and concentrate on the general case. We will therefore choose two out of the three $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ as independent momenta, and treat the metric $\delta^{\mu\nu}$ as an independent tensor.

As an example consider a 3-point function of two transverse, traceless, symmetric rank two operators $t^{\mu\nu}$ and a scalar operator $\mathcal{O}$. Using the projectors (2.1.14) one can write the most general form

$$\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) X^{\alpha_1\beta_1\alpha_2\beta_2}, \tag{2.3.3}$$

where $X^{\alpha_1\beta_1\alpha_2\beta_2}$ is a general tensor of rank four built from the metric and momenta. Usually one chooses two independent momenta once and for all. On the other hand, there is no obstacle to choosing different independent momenta for different Lorentz indices. In the thesis we always choose

$$\boldsymbol{p}_1, \boldsymbol{p}_2 \text{ for } \mu_1, \nu_1; \ \boldsymbol{p}_2, \boldsymbol{p}_3 \text{ for } \mu_2, \nu_2 \text{ and } \boldsymbol{p}_3, \boldsymbol{p}_1 \text{ for } \mu_3, \nu_3. \tag{2.3.4}$$

Such a choice enhances the symmetry properties of the decomposition, as we will discuss at length in the next section.

Let us now enumerate all possible tensors that can appear in $X^{\alpha_1\beta_1\alpha_2\beta_2}$. Observe that whenever a simple tensor contains at least one of the following tensors

$$\delta^{\alpha_1\beta_1}, \delta^{\alpha_2\beta_2}, p_1^{\alpha_1}, p_1^{\beta_1}, p_2^{\alpha_2}, p_2^{\beta_2}, \tag{2.3.5}$$

then the contraction with the projectors in (2.3.3) vanishes. Therefore, in accordance with the choice (2.3.4), the only tensors giving a non-zero result after contraction with the projectors are

$$\delta^{\alpha_1\alpha_2}, \delta^{\alpha_1\beta_2}, \delta^{\beta_1\alpha_2}, \delta^{\beta_1\beta_2}, p_2^{\alpha_1}, p_2^{\beta_1}, p_3^{\alpha_2}, p_3^{\beta_2}. \tag{2.3.6}$$

Since the projector (2.1.14) is symmetric in $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$, the most general form of our 3-point function is then

$$\begin{aligned} \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \Big[& A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} \\ & + A_2 p_2^{\alpha_1} p_3^{\alpha_2} \delta^{\beta_1\beta_2} + A_3 \delta^{\alpha_1\alpha_2} \delta^{\beta_1\beta_2} \Big], \end{aligned} \tag{2.3.7}$$

where the coefficients A_1, A_2 and A_3 are scalar functions of momenta. We will refer to the coefficients A_j, and their analogous counterparts in more general correlation functions, as *form factors*. By Lorentz invariance, these form factors are functions of the momentum magnitudes

$$p_j = |\boldsymbol{p}_j| = \sqrt{\boldsymbol{p}_j^2}, \qquad j = 1, 2, 3, \tag{2.3.8}$$

i.e., $A_j = A_j(p_1, p_2, p_3)$. In particular, any scalar product of two momenta can be written as a combination of momentum magnitudes, for example

$$\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = \frac{1}{2}(p_3^2 - p_1^2 - p_2^2). \tag{2.3.9}$$

For brevity, we will henceforth suppress the dependence of form factors on the momentum magnitudes, writing $A_j(p_1, p_2, p_3)$ as simply A_j.

Note that the correlation function on the left-hand side of (2.3.7) is symmetric under a transposition $(\boldsymbol{p}_1, \mu_1, \nu_1) \leftrightarrow (\boldsymbol{p}_2, \mu_2, \nu_2)$. One can apply this symmetry to the right-hand side to find that all form factors A_1, A_2 and A_3 are symmetric under $p_1 \leftrightarrow p_2$. To prove this, observe that one has, for example, $\pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1) p_3^{\alpha_1} = -\pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1) p_2^{\alpha_1}$. Therefore $\boldsymbol{p}_2$ and $-\boldsymbol{p}_3$ can be exchanged under both $\pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1)$ and $\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)$, and similarly for other momenta.

For any form factor A_j we define an associated non-negative integer N_j, the *tensorial dimension* of A_j, similar to that defined in [25]. Specifically, the tensorial dimension N_j is the number of momenta that appear in the tensorial expression multiplying A_j (excluding those in the transverse-traceless projectors) in the decomposition of the correlation function. As we will see later, this quantity will appear explicitly in the conformal Ward identities. In the example (2.3.7), we have the following tensorial dimensions: $N_1 = 4$, $N_2 = 2$ and $N_3 = 0$.

Decompositions for other correlation functions may be found in chapter 3. Observe that in each case the form factor A_1 stands in front of the unique tensor containing momenta only. The tensorial dimension N_1 is therefore always equal to the number of Lorentz indices in the 3-point function, and tensorial dimensions of all remaining form factors are smaller than N_1.

2.3.2. Decomposition of $\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t^{\mu_3\nu_3} \rangle$

In the previous section we introduced a natural decomposition of tensor structures. Rather than fixing two independent momenta (as is done for example in [64, 25, 47, 48, 49, 24]) we chose a different set of independent momenta for different Lorentz indices according to (2.3.4). Such a choice respects all symmetries of the correlation function, as we now discuss.

In [25], it was shown that the transverse-traceless correlation function $\langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t^{\mu_3\nu_3} \rangle\rangle$ can be decomposed into eight tensor structures plus their $\boldsymbol{p}_1 \leftrightarrow \boldsymbol{p}_2$ symmetric versions. In our method, however, we arrive at only five tensor structures (for the general case $d \geqslant 3$, see appendix 2.A.2 for the case $d = 3$) according to the following decomposition

$$\langle\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) t^{\mu_3\nu_3}(\boldsymbol{p}_3) \rangle\rangle$$
$$= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3) \Big[A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} +$$

$$
\begin{aligned}
&+\ A_2\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3}+A_2(p_1\leftrightarrow p_3)\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3}\\
&\qquad +\ A_2(p_2\leftrightarrow p_3)\delta^{\beta_1\beta_3}p_2^{\alpha_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}\\
&+\ A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3}+A_3(p_1\leftrightarrow p_3)\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}\\
&\qquad +\ A_3(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}\delta^{\beta_1\beta_3}p_3^{\alpha_2}p_3^{\beta_2}\\
&+\ A_4\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_3^{\beta_2}+A_4(p_1\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}p_3^{\beta_2}p_1^{\beta_3}\\
&\qquad +\ A_4(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_2}p_2^{\beta_1}p_1^{\beta_3}\\
&+A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\Big]\,.
\end{aligned}\tag{2.3.10}
$$

By $p_1 \leftrightarrow p_3$ we denote the exchange of the arguments p_1 and p_3, $A_2(p_1 \leftrightarrow p_3) = A_2(p_3, p_2, p_1)$. If no arguments are specified, then the standard ordering is assumed, *i.e.*, $A_2 = A_2(p_1, p_2, p_3)$.

First observe that this decomposition is manifestly invariant under the permutation group S_3 of the set $\{1, 2, 3\}$, *i.e.*, for any $\sigma \in S_3$,

$$
\begin{aligned}
&\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle =\\
&\quad = \langle\!\langle t^{\mu_{\sigma(1)}\nu_{\sigma(1)}}(\boldsymbol{p}_{\sigma(1)})t^{\mu_{\sigma(2)}\nu_{\sigma(2)}}(\boldsymbol{p}_{\sigma(2)})t^{\mu_{\sigma(3)}\nu_{\sigma(3)}}(\boldsymbol{p}_{\sigma(3)})\rangle\!\rangle.
\end{aligned}\tag{2.3.11}
$$

In particular, the form factors A_1 and A_5 are S_3-invariant,

$$
A_j(p_1, p_2, p_3) = A_j(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}), \quad j \in \{1, 5\}, \tag{2.3.12}
$$

since the tensors they multiply are S_3-invariant. The action of the symmetry group on the remaining terms is then clearly visible. As an example, let us concentrate on the third line of (2.3.10) with the A_2 form factor. The $(\boldsymbol{p}_1, \mu_1, \nu_1) \leftrightarrow (\boldsymbol{p}_2, \mu_2, \nu_2)$ permutation leaves the tensor in the first term invariant, therefore the A_2 factor exhibits the $p_1 \leftrightarrow p_2$ symmetry. On the other hand, the $(\boldsymbol{p}_1, \mu_1, \nu_1) \leftrightarrow (\boldsymbol{p}_3, \mu_3, \nu_3)$ permutation swaps tensor structures of the first and the second term in the third line. This requires that the form factor of the second term is related to the form factor of the first term by the $p_1 \leftrightarrow p_3$ permutation, as indicated. Working out the remaining lines of (2.3.10) one finds that both remaining factors A_3 and A_4 are $p_1 \leftrightarrow p_2$ symmetric.

Let us comment then on the apparent disagreement between the number of tensor structures between (2.3.10) and the results of [25]. As already mentioned, the mismatch follows from the choice of two independent momenta in [25] to be $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, in our notation. Such a choice breaks the S_3 symmetry down to the $(\boldsymbol{p}_1, \mu_1, \nu_1) \leftrightarrow (\boldsymbol{p}_2, \mu_2, \nu_2)$ symmetry. One can easily recover eight tensor structures from (2.3.10) by substituting $\boldsymbol{p}_3 = -\boldsymbol{p}_1 - \boldsymbol{p}_2$ and writing the decomposition in terms of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ only. One finds

$$\begin{aligned}
&\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)\left[\frac{1}{2}A_1 p_2^{\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3}\right.\\
&\qquad - \frac{1}{2}A_2\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_1^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3} - A_2(p_1\leftrightarrow p_3)\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\alpha_3}\\
&\qquad + \frac{1}{2}A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_3(p_1\leftrightarrow p_3)\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}\\
&\qquad - \frac{1}{2}A_4\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_1^{\beta_2} - A_4(p_1\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}p_1^{\beta_2}p_1^{\beta_3}\\
&\qquad + \frac{1}{2}A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\\
&\qquad \left. + \text{everything with } (\boldsymbol{p}_1,\alpha_1,\beta_1)\leftrightarrow(\boldsymbol{p}_2,\alpha_2,\beta_2)\right]. \qquad (2.3.13)
\end{aligned}$$

As we can see, the number of tensor structures increases to exactly eight, as the symmetry group is diminished.

Summarising, our decomposition method based on (2.3.4) gives the minimal number of tensor structures obeying the symmetries of the correlation function. It is an improvement over the standard method with two independent momenta fixed, since such a choice breaks symmetries and leads therefore to the larger number of tensor structures.

Finally, we should comment on the fact that we decompose the transverse-traceless part of the correlation function only. This is because the difference between the full 3-point function and its transverse-traceless part is semi-local, and hence may be entirely reconstructed from the Ward identities. We will discuss this method for recovering the full correlation function from its transverse-traceless part in the next section.

Let us note in passing that the decomposition method described here may also be used for correlation functions in non-conformal theories. For example, in cases where the stress-energy tensor is transverse but no longer traceless one should use the π^{μ}_{α} projectors (2.1.13) in place of $\Pi^{\mu\nu}_{\alpha\beta}$ in (2.3.10). In this way one obtains ten tensor structures, five of which have nonzero trace. This decomposition is given in appendix 2.A.1.

2.3.3. Finding the form factors

We would like to apply the results of the previous section to spin-1 and spin-2 conserved currents J^{μ} and a stress-energy tensor $T^{\mu\nu}$. These quantum operators, however, are only transverse and traceless on-shell, and in the quantum case, we need to analyse Ward identities. To proceed, we define transverse, transverse-traceless and local parts of J^μ and $T^{\mu\nu}$ by

$$j^\mu \equiv \pi^\mu_\alpha J^\alpha, \qquad j^\mu_{\text{loc}} \equiv J^\mu - j^\mu, \tag{2.3.14}$$
$$t^{\mu\nu} \equiv \Pi^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}, \qquad t^{\mu\nu}_{\text{loc}} \equiv T^{\mu\nu} - t^{\mu\nu}, \tag{2.3.15}$$

as well as longitudinal and trace parts

$$r = p_\mu J^\mu, \qquad R^\nu = p_\mu T^{\mu\nu}, \qquad R = p_\nu R^\nu, \qquad T = T^\mu_\mu. \tag{2.3.16}$$

From these definitions, we then have

$$j^\mu_{\text{loc}} = \frac{p^\mu}{p^2} r, \tag{2.3.17}$$
$$\begin{aligned} t^{\mu\nu}_{\text{loc}} &= \frac{p^\mu}{p^2} R^\nu + \frac{p^\nu}{p^2} R^\mu - \frac{p^\mu p^\nu}{p^4} R + \frac{1}{d-1} \pi^{\mu\nu} \left(T - \frac{R}{p^2} \right) \\ &= \mathscr{T}^{\mu\nu}_\alpha R^\alpha + \frac{\pi^{\mu\nu}}{d-1} T, \end{aligned} \tag{2.3.18}$$

where the operator

$$\mathscr{T}^{\mu\nu}_\alpha(\boldsymbol{p}) = \frac{1}{p^2} \left[2 p^{(\mu} \delta^{\nu)}_\alpha - \frac{p_\alpha}{d-1} \left(\delta^{\mu\nu} + (d-2) \frac{p^\mu p^\nu}{p^2} \right) \right]. \tag{2.3.19}$$

In the following, we will also use $\mathscr{T}^{\mu\nu\alpha} = \delta^{\alpha\beta} \mathscr{T}^{\mu\nu}_\beta$.

We now observe that in a CFT, all terms in (2.3.17) and (2.3.18) are computable by means of the transverse and trace Ward identities. We can therefore divide a 3-point function into two parts: the *transverse-traceless part* represented as in section 2.3.1, and the *semi-local part* (indicated by the subscript *loc*) expressible through the transverse Ward identities. For simplicity we will use the term ‘transverse-traceless part’ in all cases, even if the correlation function does not contain the stress-energy tensor.

As an example, consider

$$\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \langle\!\langle T^{\alpha_1\beta_1}(\boldsymbol{p}_1) T^{\alpha_2\beta_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle. \tag{2.3.20}$$

One can recover the full 3-point function by writing

$$\begin{aligned} &\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle = \\ &\quad = \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2}_{\text{loc}} \mathcal{O} \rangle\!\rangle + \langle\!\langle t^{\mu_1\nu_1}_{\text{loc}} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \langle\!\langle t^{\mu_1\nu_1}_{\text{loc}} t^{\mu_2\nu_2}_{\text{loc}} \mathcal{O} \rangle\!\rangle \\ &\quad = \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle - \langle\!\langle T^{\mu_1\nu_1} t^{\mu_2\nu_2}_{\text{loc}} \mathcal{O} \rangle\!\rangle - \langle\!\langle t^{\mu_1\nu_1}_{\text{loc}} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \langle\!\langle t^{\mu_1\nu_1}_{\text{loc}} t^{\mu_2\nu_2}_{\text{loc}} \mathcal{O} \rangle\!\rangle \end{aligned} \tag{2.3.21}$$

All terms on the right-hand side apart from the first may be computed by means of Ward identities. The exact form of the Ward identities depends on the exact definition of the operators involved, but more importantly, all these terms depend on 2-point functions only.

Due to the complicated nature of contractions of the projectors (2.1.13) and (2.1.14) one might fear that it is very difficult to calculate the form factors by means of Feynman rules, given some particular QFT. Reassuringly, this is not the case, as we can see in the following example. First, we decompose the full 3-point function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ into simple tensors using the choice of momenta (2.3.4) and denote

$$\langle\!\langle T^{\alpha_1\beta_1} T^{\alpha_2\beta_2} \mathcal{O} \rangle\!\rangle = \tilde{A}_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} + \tilde{A}_2 p_2^{\alpha_1} p_3^{\alpha_2} \delta^{\beta_1\beta_2} + \tilde{A}_3 \delta^{\alpha_1\alpha_2} \delta^{\beta_1\beta_2} + \ldots, \tag{2.3.22}$$

where the omitted terms do not contain the tensors we have listed explicitly. Next, we apply the projectors (2.1.14). Obverse, for example, that the projector $\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)$ is constructed from the metric and the momentum $\boldsymbol{p}_1$ only, and therefore when applied to the 3-point function it cannot change the coefficient of any tensor containing $p_2^{\alpha_1} p_2^{\beta_1}$, *i.e.*,

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) p_2^{\alpha_1} p_2^{\beta_1} = p_2^{\mu_1} p_2^{\nu_1} + \ldots, \tag{2.3.23}$$

where the omitted terms do not contain $p_2^{\mu_1} p_2^{\nu_1}$. Using the same argument for $\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)$, we see that the coefficients of $p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2}$ in (2.3.22) and $p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2}$ in $\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle$ in (2.3.20) are equal, *i.e.*, $A_1 = \tilde{A}_1$. Similarly, we find that

$$\begin{aligned} &\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \langle\!\langle T^{\alpha_1\beta_1}(\boldsymbol{p}_1) T^{\alpha_2\beta_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle = \\ &\qquad = \tilde{A}_1 p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2} + \frac{1}{4} \tilde{A}_2 p_2^{\mu_1} p_3^{\mu_2} \delta^{\nu_1\nu_2} + \frac{1}{2} \tilde{A}_3 \delta^{\mu_1\mu_2} \delta^{\nu_1\nu_2} + \ldots, \end{aligned} \tag{2.3.24}$$

where the omitted terms do not contain the tensors we have listed explicitly. We therefore have

$$\begin{aligned} A_1 &= \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2} \text{ in } \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle, \\ A_2 &= 4 \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} \delta^{\nu_1\nu_2} \text{ in } \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle, \\ A_3 &= 2 \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \delta^{\nu_1\nu_2} \text{ in } \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle. \end{aligned} \tag{2.3.25}$$

We list the analogous formulae for all other 3-point functions in chapter 3.

2.3.4. Example

Let us consider a conformally coupled free scalar free massless field ϕ in d Euclidean dimensions given by the action (1.1.64). In the presence of a non-trivial source $g^{\mu\nu}$ for the stress-energy tensor, the stress-energy tensor in given by (1.1.66). In this CFT, $\mathcal{O}(x) = \phi^2(x)$ is a conformal primary operator of dimension $\Delta_3 = d - 2$.

For later use we quote the result for the form factors of $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ in this theory. Writing down the expression for $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ using the regular

Feynman rules, from (2.3.25) we may then read off expressions for the form factors. Explicitly evaluating these integrals for the case $d = 3$, we find

$$\begin{aligned} A_1 &= \frac{3(p_1^2 + p_2^2) + p_3^2 + 12p_1p_2 + 4p_3(p_1 + p_2)}{48\, p_3(p_1 + p_2 + p_3)^4}, \\ A_2 &= \frac{2(p_1^2 + p_2^2) + p_3^2 + 6p_1p_2 + 3p_3(p_1 + p_2)}{12\, (p_1 + p_2 + p_3)^3}, \\ A_3 &= -\frac{2(p_1^2p_2 + p_1p_2^2 + p_1p_3^2 + p_3p_1^2 + p_2p_3^2 + p_3p_2^2) + 2p_1p_2p_3 + p_1^3 + p_2^3 + p_3^3}{24\, (p_1 + p_2 + p_3)^2}, \end{aligned} \tag{2.3.26}$$

in agreement with the direct evaluation of this correlator given in [1].

2.4. Conformal Ward identities in momentum space

In section 2.2.2 we wrote down the Ward identities associated with dilatations and special conformal transformation for the case of correlators involving three scalars. In this section, we discuss the corresponding Ward identities for 3-point correlators involving insertions of the stress-energy tensor and conserved currents. First, in section 2.4.1, we obtain the dilatation and special conformal Ward identities in momentum space by Fourier transforming the well-known position space expressions; in sections 2.4.2 and 2.4.3 we then reduce these identities to a set of simple scalar equations using the tensor decomposition introduced in section 2.3.1. Finally, in section 2.4.4 we write down the transverse and trace Ward identities.

2.4.1. From position space to momentum space

Now we want to write down the Ward identities associated with dilatations and special conformal transformations in momentum space. We start from position space equations (1.3.7), (1.3.9) and (1.3.10) and we calculate their Fourier transforms in a similar manner to that discussed in [65]. Due to the translation invariance the position space correlators depend only on the differences $\boldsymbol{x}_j - \boldsymbol{x}_n$. Therefore, we can set $\boldsymbol{x}_n = 0$ and take

$$\boldsymbol{p}_n = -\sum_{j=1}^{n-1} \boldsymbol{p}_j. \tag{2.4.1}$$

The Ward identities (1.3.7) and (1.3.9) then transform to

$$0 = \left[\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle\langle \mathcal{O}_1(\boldsymbol{p}_1) \ldots \mathcal{O}_n(\boldsymbol{p}_n) \rangle\rangle, \tag{2.4.2}$$

$$0=\left[\sum_{j=1}^{n-1}\left(2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa}+(p_j)_\kappa\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{j\alpha}}\right)\right]\langle\!\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle\!\rangle, \tag{2.4.3}$$

while the additional contribution (1.3.10) transforms to

$$2\sum_{j=1}^{n}\sum_{k=1}^{n_j}\left(\delta^{\mu_{jk}\kappa}\frac{\partial}{\partial p_j^{\alpha_{jk}}}-\delta^{\kappa}_{\alpha_{jk}}\frac{\partial}{\partial p_{j\mu_{jk}}}\right)\times$$
$$\times\ \langle\!\langle\mathcal{O}_1^{\mu_{11}\ldots\mu_{1r_1}}(\boldsymbol{p}_1)\ldots\mathcal{O}_j^{\mu_{j1}\ldots\alpha_{jk}\ldots\mu_{jr_j}}(\boldsymbol{p}_j)\ldots\mathcal{O}_n^{\mu_{n1}\ldots\mu_{nr_n}}(\boldsymbol{p}_n),\rangle\!\rangle \tag{2.4.4}$$

and once again must be added to the right-hand side of (2.4.3). It will be useful to denote the differential operator obtained by summing the right-hand side of (2.4.3) and (2.4.4) as $\mathcal{K}^\kappa$, so that the CWIs may be compactly expressed as

$$\mathcal{K}^\kappa\langle\!\langle\mathcal{O}_1(\boldsymbol{p}_1)\ldots\mathcal{O}_n(\boldsymbol{p}_n)\rangle\!\rangle=0. \tag{2.4.5}$$

In view of (2.4.4), note that $\mathcal{K}^\kappa$ acts non-trivially on Lorentz indices and so in fact is really of the form

$$\mathcal{K}^\kappa=\mathcal{K}^{\mu_{11}\ldots\mu_{nr_n},\kappa}_{\alpha_{11}\ldots\alpha_{nr_n}}, \tag{2.4.6}$$

however for simplicity we will omit the tensor indices on $\mathcal{K}^\kappa$.

In the following analysis we will focus specifically on 3-point functions. The idea will be to take the tensor decomposition the 3-point function described in section 2.3.1, then apply the operators (2.4.3) and (2.4.4) yielding differential equations for the form factors. Since by Lorentz invariance the form factors are purely functions of the momentum magnitudes, the action of momentum derivatives on form factors may be obtained using the chain rule,

$$\begin{aligned}\frac{\partial}{\partial p_{1\mu}}&=\frac{\partial p_1}{\partial p_{1\mu}}\frac{\partial}{\partial p_1}+\frac{\partial p_2}{\partial p_{1\mu}}\frac{\partial}{\partial p_2}+\frac{\partial p_3}{\partial p_{1\mu}}\frac{\partial}{\partial p_3}\\&=\frac{p_1^\mu}{p_1}\frac{\partial}{\partial p_1}+\frac{p_1^\mu+p_2^\mu}{p_3}\frac{\partial}{\partial p_3},\end{aligned} \tag{2.4.7}$$

noting that $\boldsymbol{p}_3$ is fixed via (2.4.1). We may express derivatives with respect to $\boldsymbol{p}_2$ similarly, and the final results may then be re-expressed purely in terms of the momentum magnitudes.

2.4.2. Dilatation Ward identity

Using (2.4.7), it is simple to rewrite the dilatation Ward identity (2.4.2) for a 3-point function of three conformal primary operator of any tensor structure in terms of its form factors as

$$0=\left[2d+N_n+\sum_{j=1}^{3}\left(p_j\frac{\partial}{\partial p_j}-\Delta_j\right)\right]A_n(p_1,p_2,p_3), \tag{2.4.8}$$

where N_n is the tensorial dimension of A_n, *i.e.*, the number of momenta in the tensor multiplying the form factor A_n and the transverse-traceless projectors. As previously, Δ_j, $j = 1, 2, 3$ denote the conformal dimensions of the operators $\mathcal{O}_j$ in the 3-point function: for a conserved current we thus have $\Delta = d - 1$ while for a stress-energy tensor $\Delta = d$.

The dilatation Ward identity determines the total degree of the 3-point function and hence of its form factors. In general, if a function A satisfies

$$0 = \left[-D + \sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j} \right] A(p_1, p_2, p_3) \tag{2.4.9}$$

for some constant D then we will refer to D as the *degree* of A, denoted $\deg(A) = D$. (A homogeneous polynomial in momenta of degree D has dilatation degree D.) Therefore (2.4.8) implies that the form factor A_n has degree

$$\deg(A_n) = \Delta_t - 2d - N_n, \tag{2.4.10}$$

where $\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$.

2.4.3. Special conformal Ward identities

In this section, we now extract scalar equations for the form factors by inserting our tensor decomposition for the transverse-traceless part of the 3-point functions into the special conformal Ward identities. While the details are somewhat involved, the procedure is nonetheless conceptually straightforward. We will outline the method using as an example the 3-point function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$, which captures all the important features.

Consider then the action of the CWI operator $\mathcal{K}^\kappa$ defined in (2.4.5) on the decomposition (2.3.21),

$$\begin{aligned} 0 &= \mathcal{K}^\kappa \langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle \qquad\qquad (2.4.11) \\ &= \mathcal{K}^\kappa \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \mathcal{K}^\kappa \langle\!\langle t^{\mu_1\nu_1} t_{\rm loc}^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \mathcal{K}^\kappa \langle\!\langle t_{\rm loc}^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle + \mathcal{K}^\kappa \langle\!\langle t_{\rm loc}^{\mu_1\nu_1} t_{\rm loc}^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle, \end{aligned}$$

recalling that our notation for $\mathcal{K}^\kappa$ suppresses Lorentz indices so that in reality, *e.g.*,

$$\mathcal{K}^\kappa \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle = \mathcal{K}^{\mu_1\nu_1\mu_2\nu_2,\kappa}_{\alpha_1\beta_1\alpha_2\beta_2} \langle\!\langle t^{\alpha_1\beta_1} t^{\alpha_2\beta_2} \mathcal{O} \rangle\!\rangle. \tag{2.4.12}$$

Through a direct but lengthy calculation we find that the first term on the right-hand side of (2.4.11), $\mathcal{K}^\kappa \langle\!\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$, is transverse-traceless in μ_1, ν_1 and μ_2, ν_2

with respect to the corresponding momenta,

$$\begin{aligned}
0 &= \delta_{\mu_1\nu_1}\left[\mathcal{K}^\kappa\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right],\\
0 &= \delta_{\mu_2\nu_2}\left[\mathcal{K}^\kappa\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right],\\
0 &= p_{1\mu_1}\left[\mathcal{K}^\kappa\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right],\\
0 &= p_{2\mu_2}\left[\mathcal{K}^\kappa\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right],
\end{aligned} \tag{2.4.13}$$

where we used the definitions (2.4.3) and (2.4.4) for $\mathcal{K}^\kappa$ and the identities given in appendix 2.A.8. For correlators involving conserved currents, we find that the analogue of (2.4.13) similarly applies.

We are now free to apply transverse-traceless projectors (2.1.13) and (2.1.14) to (2.4.11), in order to isolate equations for the form factors appearing in the decomposition of $\langle\!\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}\mathcal{O}\rangle\!\rangle$. Evaluating the action of $\mathcal{K}^\kappa$ on the semi-local terms in (2.4.11) via the formulae in appendix 2.A.8, we find

$$\pi^\mu_\alpha\mathcal{K}^\kappa j^\alpha_{\text{loc}} = \frac{2(d-2)}{p^2}\pi^{\kappa\mu}r, \tag{2.4.14}$$

$$\Pi^{\mu\nu}_{\alpha\beta}\mathcal{K}^\kappa t^{\alpha\beta}_{\text{loc}} = \frac{4d}{p^2}\Pi^{\mu\nu\kappa}{}_\alpha R^\alpha, \tag{2.4.15}$$

$$\pi^\mu_\alpha j^\alpha_{\text{loc}} = \Pi^{\mu\nu}_{\alpha\beta}t^{\alpha\beta}_{\text{loc}} = 0. \tag{2.4.16}$$

The last equation implies that any correlation function with more than one insertion of $t^{\mu\nu}_{\text{loc}}$ or j^μ_{loc} vanishes when the CWI operator $\mathcal{K}^\kappa$ and the projectors (2.1.13) and (2.1.14) are applied. This is because the CWI operator $\mathcal{K}^\kappa$ can be written as a sum of two terms

$$\mathcal{K}^\kappa = \mathcal{K}^\kappa_1\left(\frac{\partial}{\partial p_1^\mu}\right) + \mathcal{K}^\kappa_2\left(\frac{\partial}{\partial p_2^\mu}\right), \tag{2.4.17}$$

each depending only on derivatives with respect to the appropriate momenta, hence

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{K}^\kappa\langle\!\langle t^{\alpha_1\beta_1}_{\text{loc}}(\boldsymbol{p}_1)t^{\alpha_2\beta_2}_{\text{loc}}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = 0. \tag{2.4.18}$$

Substituting all results into (2.4.11), we find

$$\begin{aligned}
0 &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{K}^\kappa\langle\!\langle t^{\alpha_1\beta_1}(\boldsymbol{p}_1)t^{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\\
&\quad+\frac{4d}{p_1^2}\Pi^{\mu_1\nu_1\kappa}{}_{\alpha_1}(\boldsymbol{p}_1)\left[p_{1\beta_1}\langle\!\langle T^{\alpha_1\beta_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right]\\
&\quad+\frac{4d}{p_2^2}\Pi^{\mu_2\nu_2\kappa}{}_{\alpha_2}(\boldsymbol{p}_2)\left[p_{2\beta_2}\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle\right].
\end{aligned} \tag{2.4.19}$$

Two last terms are semi-local and may be re-expressed in terms of 2-point functions via the transverse Ward identities. The remaining task is then to rewrite the first term of (2.4.19) in terms of form factors and extract the CWIs. Via the method of section 2.3.1, we can write the most general form of the result as

$$\begin{aligned}
\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{K}^\kappa\langle\langle t^{\alpha_1\beta_1}(\boldsymbol{p}_1)t^{\alpha_2\beta_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\times \\
\times \Big[p_1^\kappa \left(C_{11}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + C_{12}p_2^{\alpha_1}p_3^{\alpha_2}\delta^{\beta_1\beta_2} + C_{13}\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\right) \\
+ p_2^\kappa \left(C_{21}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + C_{22}p_2^{\alpha_1}p_3^{\alpha_2}\delta^{\beta_1\beta_2} + C_{23}\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\right) \\
+ \Big(C_{31}\delta^{\kappa\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + C_{32}\delta^{\kappa\alpha_1}\delta^{\alpha_2\beta_1}p_3^{\beta_2} \\
+ C_{41}\delta^{\kappa\alpha_2}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\beta_2} + C_{42}\delta^{\kappa\alpha_2}\delta^{\alpha_1\beta_2}p_2^{\beta_1}\Big)\Big]. \qquad (2.4.20)
\end{aligned}$$

In this expression, the coefficients C_{jk} are differential equations involving the form factors A_1, A_2 and A_3 of (2.3.7). Each CWI can then be presented in terms of the momentum magnitudes $p_j = |\boldsymbol{p}_j|$.

As we can see, there are ten coefficients C_{jk} in (2.4.20), so there are at most ten equations to consider. Usually not all of the CWIs, however, are independent. In this example, the $\boldsymbol{p}_1 \leftrightarrow \boldsymbol{p}_2$ symmetry implies that the equations following from C_{1j} and C_{2j}, as well as C_{3j} and C_{4j}, are pairwise equivalent.

For any 3-point function, the resulting equations can be divided into two groups: the *primary* and the *secondary* CWIs. The primary CWIs are second-order differential equations and appear as the coefficients of transverse or transverse-traceless tensors containing p_1^κ or p_2^κ, where κ is the 'special' index in the CWI operator $\mathcal{K}^\kappa$. In the expression (2.4.20) above, the primary CWIs are equivalent to the vanishing of the coefficients C_{1j} and C_{2j}. The remaining equations, following from all other transverse or transverse-traceless terms, are then the secondary CWIs and are first-order differential equations. In the expression (2.4.20), the secondary CWIs are equivalent to the vanishing of the coefficients C_{3j} and C_{4j}.

In the next two subsections we will examine the general form of the primary and secondary CWIs and discuss some of their properties. In section 2.5, we will return to analyse their solution for the form factors. In outline our strategy will be, first, to solve each of the primary CWIs up to an overall multiplicative constant, then second, to insert these solutions into the secondary CWIs typically allowing the number of undetermined constants to be further reduced. In the case of the correlator $\langle\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}\mathcal{O}\rangle\rangle$, for example, we will find that the final result is then uniquely determined up to one numerical constant, in agreement with the position space analysis of [22].

Primary conformal Ward identities

It turns out that in all cases the primary CWIs look very similar to the CWIs (2.2.10) for scalar operators. In order to write the primary CWIs in a simple way, we define the following fundamental differential operators

$$\mathrm{K}_j = \frac{\partial^2}{\partial p_j^2} + \frac{d+1-2\Delta_j}{p_j}\frac{\partial}{\partial p_j}, \quad j=1,2,3, \tag{2.4.21}$$

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j, \quad j=1,2,3, \tag{2.4.22}$$

where Δ_j is the conformal dimension of the j-th operator in the 3-point function under consideration. (Observe that this same operator appeared earlier in (2.2.10, 2.2.11).)

In the case of our example $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$, the primary CWIs for the form factors defined in (2.3.7) read

$$\begin{aligned} &\mathrm{K}_{13}\, A_1 = 0, && \mathrm{K}_{13}\, A_2 = 8A_1, && \mathrm{K}_{13}\, A_3 = 2A_2, \\ &\mathrm{K}_{23}\, A_1 = 0, && \mathrm{K}_{23}\, A_2 = 8A_1, && \mathrm{K}_{23}\, A_3 = 2A_2, \end{aligned} \tag{2.4.23}$$

Note that, from the definition (2.4.22), we have

$$\mathrm{K}_{ii} = 0, \qquad \mathrm{K}_{ji} = -\mathrm{K}_{ij}, \qquad \mathrm{K}_{ij} + \mathrm{K}_{jk} = \mathrm{K}_{ik}, \tag{2.4.24}$$

for any $i, j, k \in \{1,2,3\}$. One can therefore subtract corresponding pairs of equations and obtain the following system of independent partial differential equations

$$\begin{aligned} &\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{12}\, A_2 = 0, && \mathrm{K}_{12}\, A_3 = 0, \\ &\mathrm{K}_{13}\, A_1 = 0, && \mathrm{K}_{13}\, A_2 = 8A_1, && \mathrm{K}_{13}\, A_3 = 2A_2. \end{aligned} \tag{2.4.25}$$

As we will prove, each equation has a unique solution up to one numerical constant. This means that at this point the 3-point function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ is determined by three numerical constants. After the application of the secondary CWIs this number will decrease further.

The primary CWIs for all 3-point functions are listed explicitly in chapter 3. They share the following properties:

1. All primary CWIs are second-order linear differential equations.
2. The equations for the coefficient A_1 are always homogeneous and given by (2.2.10, 2.2.11) exactly, *i.e.*,

$$\mathrm{K}_{12}\, A_1 = 0, \qquad\qquad \mathrm{K}_{13}\, A_1 = 0. \tag{2.4.26}$$

3. The equations for the remaining form factors are similar to (2.2.10, 2.2.11), but they may contain a linear inhomogeneous part. For a form factor A_n multiplying a tensor of tensorial dimension N_n, the only form factors A_j which can appear in the inhomogeneous part are those with $N_j = N_n + 2$. It is therefore always possible to solve the primary CWIs recursively, starting with A_1.

   In the case of our example, the recursive structure of the equations (2.4.25) is clearly visible.

4. There is no semi-local contribution to any primary CWI. In our example, last two terms in (2.4.19) do not contribute to the primary CWIs. This conclusion is valid in general and can be checked explicitly in all cases.

5. The solution to each pair of primary CWIs is unique up to one numerical constant, as we will prove in section 2.5.

Secondary conformal Ward identities

The secondary CWIs are first-order partial differential equations and in principle involve the semi-local information contained in j^μ_{loc} and $t^{\mu\nu}_{\text{loc}}$. In order to write them compactly, we define the two differential operators

$$\begin{aligned} \mathrm{L}_{s,N} &= p_1(p_1^2 + p_2^2 - p_3^2)\frac{\partial}{\partial p_1} + 2p_1^2 p_2 \frac{\partial}{\partial p_2} \\ &\quad + \left[(2d - \Delta_1 - 2\Delta_2 + N + s)p_1^2 + (\Delta_1 - 2 + s)(p_3^2 - p_2^2)\right], \end{aligned} \tag{2.4.27}$$

$$\mathrm{R}_s = p_1 \frac{\partial}{\partial p_1} - (\Delta_1 - 2 + s), \tag{2.4.28}$$

as well as their symmetric versions

$$\mathrm{L}'_{s,N} = \mathrm{L}_{s,N} \text{ with } p_1 \leftrightarrow p_2 \text{ and } \Delta_1 \leftrightarrow \Delta_2, \tag{2.4.29}$$
$$\mathrm{R}'_s = \mathrm{R}_s \text{ with } p_1 \mapsto p_2 \text{ and } \Delta_1 \mapsto \Delta_2. \tag{2.4.30}$$

These operators depend on two parameters N and s determined by the Ward identity in question. Physically, while N has no clear interpretation, the parameter s denotes the spin of the first operator insertion in the 3-point function (or the second in the case of $\mathrm{L}'_{s,N}$ and R'_s). For conserved currents, we therefore have $s = 1$ while for the stress-energy tensor $s = 2$.

In our example (2.3.7) one finds two independent secondary CWIs following from the coefficients C_{31} and C_{32} in (2.4.20), namely

$$\begin{aligned} &\mathrm{L}_{2,0}\, A_1 + \mathrm{R}_2\, A_2 = \\ &\qquad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle, \end{aligned} \tag{2.4.31}$$
$$\begin{aligned} &\mathrm{L}_{2,0}\, A_2 + 4\,\mathrm{R}_2\, A_3 = \\ &\qquad = 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\mu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle, \end{aligned} \tag{2.4.32}$$

with $\Delta_1 = \Delta_2 = d$. Note that in order to correctly extract the coefficient of a tensor, the rule (2.3.4) regarding the momenta associated with a given Lorentz index must be observed. The semi-local terms on the right-hand sides may be computed by means of transverse Ward identities, to which we now turn our attention.

2.4.4. Transverse and trace Ward identities

In this section we review briefly the transverse (diffeomorphism) and trace (Weyl) Ward identities in momentum space. These can be obtaioned by a direct Fourier transform of the position space expressions obtained in sections 1.3.3 and 1.3.4. In particular we will need the precise form of all semi-local terms that appear in these Ward identities since these terms are required for the explicit evaluation of the right-hand sides of the secondary CWIs such as (2.4.31, 2.4.32).

Explicit expressions for all the higher-point transverse and trace Ward identities we need are listed in chapter 3. In obtaining these expressions we have used the assumptions:

1. $\mathcal{O}^I$ is independent of the sources, *i.e.*,
$$\frac{\delta \mathcal{O}^I}{\delta \phi_0^J} = 0, \qquad \frac{\delta \mathcal{O}^I}{\delta A_\mu^a} = 0, \qquad \frac{\delta \mathcal{O}^I}{\delta g^{\mu\nu}} = 0. \tag{2.4.33}$$

2. The source ϕ_0^I appears only as in (1.3.16), so that
$$\frac{\delta T_{\mu\nu}(\boldsymbol{x})}{\delta \phi_0^I(\boldsymbol{y})} = -g_{\mu\nu}(\boldsymbol{x})\mathcal{O}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad \frac{\delta J^{\mu a}}{\delta \phi_0^I} = 0. \tag{2.4.34}$$

3. The gauge field A_μ^a couples either through covariant derivatives or acts as an external source for the current in the form of $A_\mu^a J^{\mu a}$. This means there are no kinetic terms for A_μ^a, *i.e.*, no derivatives acting on A_μ^a in the action, hence
$$\frac{\delta T_{\mu\nu}(\boldsymbol{x})}{\delta A_\rho^a(\boldsymbol{y})} = F_{\mu\nu}^{\rho a}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad \frac{\delta J^{\mu a}(\boldsymbol{x})}{\delta A_\nu^b(\boldsymbol{y})} = G^{\mu\nu ab}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.4.35}$$
where F and G are functions of the CFT fields.

Of course, it may happen that renormalisation requires us to add counterterms violating one or more of the assumptions above, in which case the relevant Ward identities would need to be modified accordingly.

As a specific illustration of the general discussion above, let us consider the transverse Ward identity for $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ for a matter content consisting of conformal scalars, as defined in section 2.3.4. We will take the operator $\mathcal{O} = \phi^2$. The relevant Ward identity is
$$p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 2 p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{2.4.36}$$

where $\delta T_{\mu_1\nu_1}/\delta g^{\mu_2\nu_2}$ denotes taking the functional derivative of the stress-energy tensor with respect to the metric, after which we restore the metric to its background value $g_{\mu\nu} = \delta_{\mu\nu}$. Evaluating this functional derivative explicitly using

(1.1.66), we find [1]

$$\begin{aligned}
\frac{\delta T_{\mu\nu}(\boldsymbol{x})}{\delta g^{\rho\sigma}(\boldsymbol{y})} = &-\frac{1}{2}\left[\delta_{\mu\nu}\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} + 2\delta_{\mu(\rho}\delta_{\sigma)\nu} - \delta_{\mu\nu}\delta_{\rho\sigma}\delta^{\alpha\beta}\right]\delta(\boldsymbol{x}-\boldsymbol{y})T_{\alpha\beta}(\boldsymbol{x}) \\
&+ \frac{1}{16}\Big[C^{(1)\alpha\beta}_{\mu\nu\rho\sigma}\delta(\boldsymbol{x}-\boldsymbol{y})\partial_\alpha\partial_\beta + C^{(2)\alpha\beta}_{\mu\nu\rho\sigma}\partial_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\partial_\beta \\
&\qquad + C^{(3)\alpha\beta}_{\mu\nu\rho\sigma}\partial_\alpha\partial_\beta\delta(\boldsymbol{x}-\boldsymbol{y})\Big]\mathcal{O}(\boldsymbol{x}),
\end{aligned} \tag{2.4.37}$$

where partial derivatives are taken with respect to $\boldsymbol{x}$ and the prefactors are

$$\begin{aligned}
C^{(1)\alpha\beta}_{\mu\nu\rho\sigma} &= \delta_{\mu\nu}\delta^{(\alpha}_{\rho}\delta^{\beta)}_{\sigma} + 2\delta_{\mu(\rho}\delta_{\sigma)\nu}\delta^{\alpha\beta} - \delta_{\mu\nu}\delta_{\rho\sigma}\delta^{\alpha\beta}, \\
C^{(2)\alpha\beta}_{\mu\nu\rho\sigma} &= 2\delta_{\mu\nu}\delta^{(\alpha}_{\rho}\delta^{\beta)}_{\sigma} + \delta_{\mu(\rho}\delta_{\sigma)\nu}\delta^{\alpha\beta} - \delta_{\mu\nu}\delta_{\rho\sigma}\delta^{\alpha\beta} - 2\delta^{\alpha}_{(\mu}\delta_{\nu)(\rho}\delta^{\beta}_{\sigma)}, \\
C^{(3)\alpha\beta}_{\mu\nu\rho\sigma} &= \delta_{\mu\nu}\delta^{(\alpha}_{\rho}\delta^{\beta)}_{\sigma} + \delta_{\mu(\rho}\delta_{\sigma)\nu}\delta^{\alpha\beta} - \delta_{\mu\nu}\delta_{\rho\sigma}\delta^{\alpha\beta} - 2\delta^{\alpha}_{(\mu}\delta_{\nu)(\rho}\delta^{\beta}_{\sigma)} + \delta_{\rho\sigma}\delta^{(\alpha}_{\mu}\delta^{\beta)}_{\nu}.
\end{aligned} \tag{2.4.38}$$

After Fourier transforming and using the result for the 2-point function

$$\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = \frac{1}{4p} \tag{2.4.39}$$

we obtain

$$p_{1\nu_1}\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{32\,p_3}p_2^{\mu_1}p_3^{\mu_2}p_3^{\nu_2} + \frac{p_3}{32}\delta^{\mu_1\mu_2}p_3^{\nu_2} + \ldots, \tag{2.4.40}$$

where we have retained only the terms appearing in the right-hand sides of the secondary CWIs (2.4.31) and (2.4.32). The omitted terms do not contain the tensors listed explicitly and will play no further role in our analysis. As usual, we use the convention (2.3.4) for the Lorentz indices.

A similar considerations can be applied to the trace Ward identity. By Fourier transforming identities (1.3.43) and (1.3.44) we arrive at the following moemntum space expressions,

$$\begin{aligned}
\langle\!\langle T(\boldsymbol{p}_1)\mathcal{O}^I(\boldsymbol{p}_2)\mathcal{O}^J(\boldsymbol{p}_3)\rangle\!\rangle &= -\Delta\left[\langle\!\langle \mathcal{O}^I(\boldsymbol{p}_2)\mathcal{O}^J(-\boldsymbol{p}_2)\rangle\!\rangle + \langle\!\langle \mathcal{O}^I(\boldsymbol{p}_3)\mathcal{O}^J(-\boldsymbol{p}_3)\rangle\!\rangle\right], \\
\langle\!\langle T(\boldsymbol{p}_1)T_{\mu\nu}(\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle &= 2\langle\!\langle \frac{\delta T}{\delta g^{\mu\nu}}(\boldsymbol{p}_1,\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle.
\end{aligned} \tag{2.4.41}$$

A complete list of all trace Ward identities is given in chapter 3.

As is well known, due to renormalisation the trace Ward identity may acquire an anomalous contribution. The exact contribution depends strongly on the specifics of the theory, but its form is universal. In this work we assume no anomalies in the transverse Ward identities (1.3.33) and (1.3.34) can appear. The anomalous contributions are therefore still transverse.

In section 2.8 we will consider in detail the divergences and anomalies in the correlation functions $\langle\!\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle\!\rangle$ and $\langle\!\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle\!\rangle$ for the case $d=4$.

Here, as an example, we consider the most general form of the trace anomalies in the correlation functions considered above, which is

$$\begin{aligned}
&\langle\!\langle T(\boldsymbol{p}_1)\mathcal{O}^I(\boldsymbol{p}_2)\mathcal{O}^J(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = B_1^{IJ},\\
&\langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)p_3^{\alpha_2}p_3^{\beta_2}B_1^I + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2)B_2^I, \quad (2.4.42)
\end{aligned}$$

where the form factors B_1^{IJ}, B_1^I and B_2^I are functions of the momentum magnitudes specific to the theory in question. For example, the contribution to $\langle\!\langle T^{\mu_1\nu_1}\mathcal{O}^I\mathcal{O}^J\rangle\!\rangle_{\text{anomaly}}$ follows from the counterterm, [22, 66]

$$\frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\sqrt{g}P^{IJ}\phi_0^I\Box^{\Delta-\frac{d}{2}}\phi_0^J, \tag{2.4.43}$$

where P^{IJ} are numerical coefficients and we assume that $\Delta - \frac{d}{2}$ is a non-negative integer. In this case we then find

$$B_1^{IJ} = (-p_1^2)^{\Delta-\frac{d}{2}}P^{IJ}. \tag{2.4.44}$$

In section 2.8 we provide a short worked example of the renormalisation procedure in the case of the $\langle\!\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle\!\rangle$ and $\langle\!\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle\!\rangle$ correlators and show how the anomalies arise.

2.5. Solutions to conformal Ward identities

It is a rather pleasant fact that all the primary CWIs can be solved in terms of the triple-K integrals similar to (2.2.25). We will start by analysing some properties of the triple-K integrals before proceeding to show how this class of integrals solves the primary CWIs. In particular, we will find that the solution to each primary CWI is unique up to one numerical constant. Finally, we will analyse the structure and implications of the secondary CWIs.

2.5.1. Triple-K integrals

All primary CWIs can be solved in terms of the general triple-K integral

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1,p_2,p_3) = \int_0^\infty \mathrm{d}x\, x^\alpha \prod_{j=1}^3 p_j^{\beta_j}K_{\beta_j}(p_jx), \tag{2.5.1}$$

where K_ν is a Bessel K function. This integral depends on four parameters, namely the power α of the integration variable x, and the three Bessel function indices β_j. Its arguments, p_1, p_2, p_3, are magnitudes of momenta $p_j = |\boldsymbol{p}_j|$, $j = 1, 2, 3$. In the following we will generically refer to these as α and β parameters respectively.

It will be useful to define a reduced version $J_{N\{k_1k_2k_3\}}$ of the triple-K integral by substituting

$$\alpha = \frac{d}{2} - 1 + N, \qquad \beta_j = \Delta_j - \frac{d}{2} + k_j, \quad j = 1, 2, 3. \tag{2.5.2}$$

Here we assume that we concentrate on some particular 3-point function and the conformal dimensions Δ_j, $j = 1, 2, 3$ are therefore fixed. In other words we define

$$J_{N\{k_j\}} = I_{\frac{d}{2}-1+N\{\Delta_j-\frac{d}{2}+k_j\}}, \tag{2.5.3}$$

where we use a shortened notation $\{k_j\} = \{k_1k_2k_3\}$, etc. Finally we define

$$\Delta_t = \Delta_1 + \Delta_2 + \Delta_3, \qquad \beta_t = \beta_1 + \beta_2 + \beta_3, \qquad k_t = k_1 + k_2 + k_3. \tag{2.5.4}$$

The main point of this section is to present relations showing that all primary CWIs for a given 3-point function can be solved by the triple-K integrals (2.5.1). The representation (2.5.3) turns out to be extremely useful, as the parameters N and k_j are fixed by the primary CWIs and have no dependence on either Δ_j or d. If desired, these triple-K integrals may also be re-expressed in terms of other familiar integrals such as Feynman or Schwinger parametrised integrals, as discussed in appendix 4.2.1.

Region of validity, regularisation and renormalisation

We assume all parameters and variables in the triple-K integral (2.5.1) are real. From the asymptotic expansion (2.A.28) the integral converges at large x, however in general there may still be a divergence at $x = 0$. From the series expansion (2.A.21) and the definition (2.A.22), we see the triple-K integral only converges if [61]

$$\alpha > \sum_{j=1}^{3} |\beta_j| + 1, \qquad p_1, p_2, p_3 > 0. \tag{2.5.5}$$

If α does not satisfy this inequality, we can regard the triple-K integral (2.5.1) as a function of α

$$\alpha \mapsto I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1, p_2, p_3). \tag{2.5.6}$$

with the other parameters and momenta fixed and use analytic continuation. The scheme (2.2.4) is very convenient here as it does not change the indices of the Bessel functions. In terms of the parameters α and β_j this corresponds to the substitutions

$$\alpha \mapsto \alpha + \epsilon, \qquad \beta_j \text{ does not change, } j = 1, 2, 3 \tag{2.5.7}$$

in (2.5.1). Generically one finds that the limit $\epsilon \to 0$ then exists, except for cases where

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2n, \tag{2.5.8}$$

for some non-negative n. In these cases we find poles in ϵ. This can be seen as follows. The only source of the singularity is the divergence of the integral at $x = 0$. Therefore we can expand the integrand about $x = 0$ and look for possible divergences.

First, assume that the condition (2.5.8) is met but all $\beta_j \notin \mathbb{Z}$. We will show that in such a case we should expect single poles in ϵ. Indeed, equations (2.A.21) and (2.A.22) show that the expansion contains power terms x^a for various $a \in \mathbb{R}$ only. Note that the function (2.5.6) is singular only if there exists a term x^a in the expansion of the integrand with $a = -1$. Indeed, the integral of such a term near $x = 0$ is

$$\int_0 x^a \mathrm{d}x = \frac{\text{const}}{a+1}, \tag{2.5.9}$$

which has a single pole at $a = -1$ only. Note that while the triple-K integral as it stands will diverge if the expansion contains any x^a terms with $a < -1$, the analytic continuation will exist due to (2.5.9). Therefore, the function (2.5.6) is well-defined as long as the expansion of the integrand near $x = 0$ does not contain a $1/x$ term, which is exactly the condition (2.5.8).

Now observe that if some $\beta_j \in \mathbb{Z}$, then the series expansion of the integrand in the triple-K integral about $x = 0$ contains logarithms due to (2.A.27). Since, for example

$$\int_0 x^a \log x \, \mathrm{d}x = -\frac{\text{const}_1}{(a+1)^2} + \frac{\text{const}_2}{a+1}, \tag{2.5.10}$$

we may expect a double pole in α in (2.5.6) when some of the β_j are integer. While the order of the pole may increase, its position remains at $a = -1$. Using the series expansions (2.A.21) and (2.A.27), we can see that the positions of the poles in any triple-K integral are given by (2.5.8).

Basic properties

Having defined triple-K integrals, we want to show that they solve the primary CWIs. We will therefore now analyse the basic properties of the triple-K integrals. The most obvious of these is the permutation symmetry

$$I_{\alpha\{\beta_{\sigma(1)}\beta_{\sigma(2)}\beta_{\sigma(3)}\}}(p_1, p_2, p_3) = I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, p_{\sigma^{-1}(3)}), \tag{2.5.11}$$

where σ is any permutation of the set $\{1, 2, 3\}$. We also have the relations

$$\frac{\partial}{\partial p_n} I_{\alpha\{\beta_j\}} = -p_n I_{\alpha+1\{\beta_j - \delta_{jn}\}}, \tag{2.5.12}$$

$$I_{\alpha\{\beta_j + \delta_{jn}\}} = p_n^2 I_{\alpha\{\beta_j - \delta_{jn}\}} + 2\beta_n I_{\alpha-1\{\beta_j\}}, \tag{2.5.13}$$

$$I_{\alpha\{\beta_1\beta_2,-\beta_3\}} = p_3^{-2\beta_3} I_{\alpha\{\beta_1\beta_2\beta_3\}}, \tag{2.5.14}$$

for any $n = 1, 2, 3$, as follows from the basic Bessel function relations

$$\frac{\partial}{\partial a}\left[a^\nu K_\nu(ax)\right] = -x a^\nu K_{\nu-1}(ax), \tag{2.5.15}$$

$$K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) = K_{\nu+1}(x), \tag{2.5.16}$$

$$K_{-\nu}(x) = K_\nu(x). \tag{2.5.17}$$

Some additional properties of Bessel functions and triple-K integrals are listed in appendix 2.A.3.

Dilatation degree of the triple-K integral

As the triple-K integral solves CWIs, it should also solve the dilatation Ward identity (2.4.9). In other words it should have a definite dilatation dimension. Using (2.5.15) and (2.5.12) we can write

$$\begin{aligned}\int_0^\infty \mathrm{d}x \frac{\partial}{\partial x}\left(x^{\alpha+1} \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x)\right) &= \\ &= (\alpha+1-\beta_t) I_{\alpha\{\beta_k\}} - \sum_{j=1}^3 p_j^2 I_{\alpha+1\{\beta_k - \delta_{jk}\}} \\ &= (\alpha+1-\beta_t) I_{\alpha\{\beta_k\}} + \sum_{j=1}^3 p_j \frac{\partial}{\partial p_j} I_{\alpha\{\beta_k\}}.\end{aligned} \tag{2.5.18}$$

The expression on the left-hand side leads to a boundary term at $x = 0$. In the region of convergence (2.5.5) all integrals in this expression are well-defined and the boundary term vanishes. Now we can use the analytic continuation (2.5.7) in order to argue that the analytically continued left-hand side vanishes, except in the case where (2.5.8) is satisfied. Indeed, if we regard both sides of (2.5.18) as analytic functions of α with other parameters and momenta fixed, then the validity of (2.5.18) in the region (2.5.5) implies its validity in the entire domain of analyticity. Therefore, we have shown that

$$\deg I_{\alpha\{\beta_j\}} = \beta_t - \alpha - 1, \qquad \deg J_{N\{k_j\}} = \Delta_t + k_t - 2d - N \tag{2.5.19}$$

provided $\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2n$ for some non-negative n and independent choice of signs.

Finally we can argue that, if (2.5.8) holds, we should expect scaling anomalies in $I_{\alpha\{\beta_j\}}$. Using the power series expansion (2.A.21) of the Bessel I functions one can see that the series expansion of the boundary term $x^{\alpha+1}\prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x)$ in (2.5.18) about $x = 0$ contains a constant piece exactly when (2.5.8) holds. This indicates that the dilatation Ward identity for the $I_{\alpha\{\beta_j\}}$ is not satisfied in such a case. Note that this is not a strict argument since the regulator cannot be removed from the integrals appearing in (2.5.18). One should instead expand both sides in the regulator ϵ and match terms order by order.

2.5.2. Solutions to the primary conformal Ward identities

In the previous section we defined the triple-K integral and analysed its basic properties. We now want to use this knowledge in order to write a solution to the CWIs. For this we need the following fundamental identity. For any $m, n = 1, 2, 3$,

$$\mathrm{K}_{mn}\, J_{N\{k_j\}} = -2k_m J_{N+1\{k_j - \delta_{jm}\}} + 2k_n J_{N+1\{k_j - \delta_{jn}\}}, \tag{2.5.20}$$

for $k_1, k_2, k_3, N \in \mathbb{R}$. The operator K_{mn} is the CWI operator defined in (2.4.22). This relation is a direct consequence of the identities (2.5.12) and (2.5.13).

Let us first consider the pair of primary CWIs for the form factor A_1. As discussed in section 2.4.3, such CWIs are always homogeneous and take the form (2.4.26). Observe that if we set all $k_j = 0$ in (2.5.20) then $A_1 = \alpha_1 J_{N\{000\}}$ is a solution for arbitrary $N \in \mathbb{R}$ and an integration constant $\alpha_1 \in \mathbb{R}$. Furthermore, observe that, if we impose only one homogeneous equation, say $\mathrm{K}_{12}\, A = 0$, then the most general solution in terms of the triple-K integrals is $\alpha J_{N\{00k_3\}}$ for any $\alpha, N, k_3 \in \mathbb{R}$. In general the equation (2.5.20) is sufficient to write down solutions to all primary CWIs.

The remaining piece of information is the value of N. In general, if $A_n = \alpha_n J_{N\{k_j\}}$ is a form factor of tensorial dimension N_n, then (2.4.10) and (2.5.19) determine the value of $N = N(A_n)$ to be

$$N(A_n) = N_n + k_t. \tag{2.5.21}$$

Let us see how the procedure works for our example $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$. The primary CWIs are given by (2.4.25) and, in particular,

$$\begin{aligned}
&\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0, \\
&\mathrm{K}_{12}\, A_2 = 0, && \mathrm{K}_{12}\,\mathrm{K}_{13}\, A_2 = 0, && \mathrm{K}_{13}^2\, A_2 = 0, \\
&\mathrm{K}_{12}\, A_3 = 0, && \mathrm{K}_{12}\,\mathrm{K}_{13}\, A_3 = 0, && \mathrm{K}_{12}\,\mathrm{K}_{13}^2\, A_3 = 0, && \mathrm{K}_{13}^3\, A_3 = 0,
\end{aligned} \tag{2.5.22}$$

Therefore, using (2.5.20) and (2.5.21), we can write the most general solution given in terms of the triple-K integrals,

$$\begin{aligned}
A_1 &= \alpha_1 J_{4\{000\}}, \\
A_2 &= \alpha_{21} J_{3\{001\}} + \alpha_2 J_{2\{000\}}, \\
A_3 &= \alpha_{31} J_{2\{002\}} + \alpha_{32} J_{1\{001\}} + \alpha_3 J_{0\{000\}},
\end{aligned} \tag{2.5.23}$$

where all the α are numerical constants. Finally, the inhomogeneous parts of (2.4.25) fix some of these constants. When the solution above is substituted into the primary CWIs, (2.5.20) requires that

$$\alpha_{21} = 4\alpha_1, \qquad \alpha_{31} = 2\alpha_1, \qquad \alpha_{32} = \alpha_2. \tag{2.5.24}$$

The three remaining undetermined constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ multiply integrals of the form $J_{N\{000\}}$. Such integrals solve the homogeneous parts of the CWIs. Therefore the remaining constants, undetermined by the primary CWIs, will be called *primary constants*.

Let us summarise our results. We have analysed the primary CWIs for the $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ correlation function and we found a solution

$$\begin{aligned}
A_1 &= \alpha_1 J_{4\{000\}}, \\
A_2 &= 4\alpha_1 J_{3\{001\}} + \alpha_2 J_{2\{000\}}, \\
A_3 &= 2\alpha_1 J_{2\{002\}} + \alpha_2 J_{1\{001\}} + \alpha_3 J_{0\{000\}},
\end{aligned} \tag{2.5.25}$$

with three undetermined constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. We will show shortly that this solution to the primary CWIs is indeed unique, specifying $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ in momentum space up to three constants. Following application of the secondary CWIs, we will find that the number of undetermined constants is reduced to just one. The method we have described is based purely in momentum space and is applicable to all 3-point functions. Explicit solutions for all primary CWIs are listed in chapter 3.

The triple-K integrals we discuss here also arise in AdS/CFT calculations of momentum space 3-point functions using a dual gravitational theory (recent papers include, *e.g.*, [67, 68, 50]). As such, these calculations apply only to the specific CFTs dual to particular gravitational theories. In contrast, our approach here is completely general, showing that all 3-point functions of conserved currents, stress-energy tensors and scalar operators in *any* CFT can be expressed in terms of triple-K integrals.

Finally, let us return to the issue of regularisation. In some cases, despite the finiteness of the final result, the regularisation procedure described in section 2.5.1 may still be necessary. Furthermore, it can happen that single triple-K integrals may diverge on their own while the form factor they build remains finite. It is therefore essential to keep track of the expansion in the regulator ϵ carefully. In particular one must consider the primary constants as functions of the regulator and take the $\epsilon \to 0$ limit only after the substitution into the final expression for the form factors. We will show an example of this behaviour in the next section.

More on $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$

In this section we wish to illustrate that the solution to the primary CWIs in terms of the triple-K integrals can be evaluated explicitly with ease. A systematic discussion of the evaluation of the triple-K integrals will be given in section 2.6. For concreteness, consider $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$ with a scalar operator $\mathcal{O}$ of dimension $\Delta_3 = 1$ in $d = 3$ dimensional CFT. The solution to the primary CWIs is given by (2.5.25) with constants fixed according to (2.5.24). In order to write the solution explicitly, we can use expressions (2.A.24) and (2.A.25), after which all integrals turn out to be elementary. The following integrals converge and can be easily computed

$$\begin{aligned} J_{4\{000\}} &= I_{\frac{9}{2}\{\frac{3}{2}\frac{3}{2},-\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{3(p_1^2+p_2^2)+p_3^2+12p_1p_2+4p_3(p_1+p_2)}{p_3(p_1+p_2+p_3)^4}, \\ J_{3\{001\}} &= I_{\frac{7}{2}\{\frac{3}{2}\frac{3}{2}\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{2(p_1^2+p_2^2)+p_3^2+6p_1p_2+3p_3(p_1+p_2)}{(p_1+p_2+p_3)^3}, \end{aligned} \tag{2.5.26}$$

assuming $\Delta_1 = \Delta_2 = 3$ and $\Delta_3 = 1$. The remaining integrals diverge and require a regularisation. As discussed in section 2.5.1, we consider the integrals $J_{N+\epsilon\{k_j\}}$ and we expand the result in ϵ. In this manner, we find

$$\begin{aligned} J_{2+\epsilon\{002\}} &= I_{\frac{5}{2}+\epsilon\{\frac{3}{2}\frac{3}{2}\frac{3}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \frac{1}{(p_1+p_2+p_3)^2} \big[2p_1p_2p_3 + p_1^3 + p_2^3 + p_3^3 \\ &\qquad + 2(p_1^2p_2 + p_1p_2^2 + p_1^2p_3 + p_1p_3^2 + p_2^2p_3 + p_2p_3^2)\big] + O(\epsilon), \\ J_{2+\epsilon\{000\}} &= I_{\frac{5}{2}+\epsilon\{\frac{3}{2}\frac{3}{2},-\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{1}{p_3\epsilon} + O(\epsilon^0), \\ J_{1+\epsilon\{001\}} &= I_{\frac{3}{2}+\epsilon\{\frac{3}{2}\frac{3}{2}\frac{1}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{p_3}{\epsilon} + O(\epsilon^0), \\ J_{0+\epsilon\{000\}} &= I_{\frac{1}{2}+\epsilon\{\frac{3}{2}\frac{3}{2},-\frac{1}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \frac{p_1^2+p_2^2-p_3^2}{2p_3\epsilon} + O(\epsilon^0). \end{aligned} \tag{2.5.27}$$

As we will see the omitted terms make no contribution in our subsequent analysis. In order to further constrain the primary constants $\alpha_1, \alpha_2, \alpha_3$ we must consider the secondary CWIs. We will return to this task in section 2.5.3.

At this point we can compare the result given by (2.5.25) with the direct calculations of the 3-point function for the free scalar field carried out in section 2.3.4. We see that the form of the integrals $J_{4\{000\}}$, $J_{3\{001\}}$ and $J_{2\{002\}}$ match the form factors A_1, A_2 and A_3 in the equations (2.3.26). Therefore one finds the primary constants for this particular model to be

$$\alpha_1 = \frac{1}{48}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}, \qquad \alpha_2 = 0, \qquad \alpha_3 = 0. \tag{2.5.28}$$

Note that the relations (2.5.24) provide a cross-check on our solution. Later, we will see that the secondary Ward identities impose two additional constraints on the primary constants that are not yet visible.

Uniqueness of the solution

In the previous sections we argued that all CWIs may be solved in terms of triple-K integrals (2.5.1). A case-by-case analysis confirms this and the list of complete solutions is given in chapter 3. Here we want to establish that these solutions are unique. To be more precise, we want to argue that each pair of the primary CWIs determines a form factor A_n uniquely up to one numerical constant. This may be achieved by essentially the same reasoning as in section 2.2.3.

First, assume that A_n satisfies a pair of homogeneous primary CWIs

$$\mathrm{K}_{12}\, A_n = 0, \qquad \mathrm{K}_{13}\, A_n = 0, \tag{2.5.29}$$

together with the dilatation Ward identity (2.4.8) with tensorial dimension N_n. We can then use the substitution

$$A_n(p_1, p_2, p_3) = p_3^{\Delta_t - 2d - N_n} \left(\frac{p_1^2}{p_3^2} \right)^{\mu} \left(\frac{p_2^2}{p_3^2} \right)^{\lambda} F\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right) \tag{2.5.30}$$

and proceed with the analysis analogous to that following equation (2.2.14). The substitution leads to the system of equations (2.2.16, 2.2.17) with four possible choices of parameters

$$\alpha = \frac{1}{2}\left[N_n + \epsilon_1 \left(\Delta_1 - \frac{d}{2} \right) + \epsilon_2 \left(\Delta_2 - \frac{d}{2} \right) + \Delta_3 \right], \qquad \beta = \alpha - \left(\Delta_3 - \frac{d}{2} \right),$$
$$\gamma = 1 + \epsilon_1 \left(\Delta_1 - \frac{d}{2} \right), \qquad \gamma' = 1 + \epsilon_2 \left(\Delta_2 - \frac{d}{2} \right), \tag{2.5.31}$$

parametrised by $\epsilon_1, \epsilon_2 = \pm 1$. We can now use equation (2.2.21) and the analysis that follows. This leads to the conclusion that the only physically acceptable solution to the homogeneous part of the CWIs is given by the triple-K integral $\alpha_n J_{N_n\{000\}}(p_1, p_2, p_3)$, where α_n is a single undetermined constant.

In general, the primary CWIs for a form factor A_n contain inhomogeneous parts. The recursive nature of the primary CWIs discussed in section 2.4.3 then allows us to solve these equations one-by-one. Since the inhomogeneous part is linear in the other form factors, every two solutions to a given pair of equations differ by a solution to the homogeneous part of the equation. The full solution to the pair of primary CWIs and the dilatation Ward identity is therefore unique up to one numerical constant.

It is important to emphasise that while the solution to each pair of primary CWIs is unique up to one primary constant, the representation in terms of triple-K integrals may not be. For example, for generic parameter values the equation (2.5.18) shows that

$$(\alpha + \beta_t) I_{\alpha - 1\{\beta_1 \beta_2 \beta_3\}} = I_{\alpha\{\beta_1 + 1, \beta_2, \beta_3\}} + I_{\alpha\{\beta_1, \beta_2 + 1, \beta_3\}} + I_{\alpha\{\beta_1, \beta_2, \beta_3 + 1\}}. \tag{2.5.32}$$

One can therefore rewrite one triple-K integral as a combination of others hence the representation is not unique.

2.5.3. Solutions to the secondary conformal Ward identities

In this section we will finalise our theoretical considerations by solving the secondary CWIs. In general, the secondary CWIs lead to linear algebraic equations between the various primary constants appearing in solutions to the primary CWIs. The precise form of the secondary CWIs depends on the semi-local information provided by transverse Ward identities, which may be written in terms of the 2-point functions.

We will first return to our example from section 2.5.2 and show how the two secondary CWIs (2.4.31, 2.4.32) constrain the values of the three primary constants appearing in the solution (2.5.25) to the primary CWIs. As expected, we will find two algebraic linear equations between the three primary constants. From this we may conclude that the 3-point function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ depends on a single undetermined primary constant.

Next, we will discuss how the secondary CWIs in the general case lead into a set of algebraic equations for the primary constants. The discussion is sensitive on whether or not the regulator can be thoroughly removed from all triple-K integrals building a given 3-point function. In either cases the procedure is based on taking a zero-momentum limit. In this limit the triple-K integrals simplify and the secondary CWIs can be shown to lead to a set of equations between primary constants.

The procedure is considerably simpler in the case where the regulator can be removed from all triple-K integrals, while in the case one needs to keep the regulator special care must be taken when regulating the 2-point functions that appear in the right-hand side of the secondary CWIs.

$\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$ for free scalars

Let us begin by discussing our example correlation function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$. We derived the secondary CWIs earlier in (2.4.31) and (2.4.32), where the terms on the right-hand side of these equations are given by the transverse Ward identity (2.4.40). We now want to show that these data fix two out of three primary constants in the solution (2.5.25) of the primary CWIs. To fix the final remaining constant then requires additional physical input in the form of the specific field content.

Since the regulator ϵ in the triple-K integrals (2.5.27) cannot be removed, we must assume that the primary constants α_2 and α_3 depend on the regulator ϵ as well. As remarked earlier in section 2.5.1, while each individual component may depend on the regulator, the full expression for the form factors A_j cannot. Let us therefore define the power series expansions

$$\alpha_j = \sum_{n=-\infty}^{\infty} \alpha_j^{(n)} \epsilon^n, \quad j = 2, 3. \tag{2.5.33}$$

Since the integral $J_{4\{000\}}$ is finite, we can assume that the constant α_1 does not depend on the regulator, *i.e.*, $\alpha_1 = \alpha_1^{(0)}$.

We start by substituting first two solutions (2.5.25) together with the series expansions (2.5.33) into the secondary CWI (2.4.31), with right-hand side given by (2.4.40). Organising the equations according to powers of ϵ, upon sending $\epsilon \to 0$ all equations associated with negative powers of ϵ must vanish. In the present case, this yields $\alpha_2^{(n)} = 0$ for all $n \leqslant 0$. The equation coming from the ϵ^0 terms then reads

$$-\frac{3}{p_3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}} (\alpha_2^{(1)} + 3\alpha_1^{(0)}) = -\frac{3}{16 p_3}. \tag{2.5.34}$$

The same procedure may now be applied to the remaining secondary CWI (2.4.32), yielding $\alpha_3^{(n)} = 0$ for all $n \leqslant 0$ and

$$\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \left[2\alpha_3^{(1)} + 3\alpha_1^{(0)} - \frac{1}{16}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\right] \frac{p_1^2 + 3p_2^2 - 3p_3^2}{p_3} + \frac{3}{4}p_3 = \frac{3}{4}p_3. \tag{2.5.35}$$

Putting everything together, we have

$$\alpha_2 = \left[-3\alpha_1 + \frac{1}{16}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\right]\epsilon + O(\epsilon^2), \tag{2.5.36}$$

$$\alpha_3 = \frac{1}{2}\left[-3\alpha_1 + \frac{1}{16}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\right]\epsilon + O(\epsilon^2), \tag{2.5.37}$$

where the constant α_1 remains undetermined by Ward identities. When we take the limit $\epsilon \to 0$, the leading terms of order ϵ in these expressions then multiply $1/\epsilon$ poles in the $J_{2+\epsilon\{000\}}$, $J_{1+\epsilon\{001\}}$ and $J_{0+\epsilon\{000\}}$ integrals yielding the correct finite result. The omitted higher order terms in (2.5.36) and(2.5.37) make no contribution.

Finally, we can check the results of this section against the result (2.5.28) for the specific theory discussed in section 2.5.2. Inserting the value of α_1 from (2.5.28) into (2.5.36) and (2.5.37) we indeed recover the correct result $\alpha_2 = \alpha_3 = 0$ up to insignificant $O(\epsilon^2)$ terms.

Simplifications in the generic case

In the previous section we substituted the full solutions to the primary CWIs into the secondary CWIs in order to extract more information about the primary constants. At first sight this procedure might appear hard to carry out in general since the triple-K integrals usually cannot be expressed in terms of elementary functions. It turns out, however, that examining the zero-momentum limit leads to simple algebraic equations for the primary constants.

In this section, for reasons of simplicity, we will assume that each triple-K integral in a solution to the primary CWIs can be defined by an analytic continuation and the regulator can be completely removed. We will refer to this as the 'generic case' in the present and following sections. We will then analyse the remaining cases later.

In the zero-momentum limit

$$\boldsymbol{p}_3 \to 0, \qquad \boldsymbol{p}_1 = \boldsymbol{p}_2 = \boldsymbol{p}, \tag{2.5.38}$$

we have

$$\begin{aligned} p_3^{\beta_3} K_{\beta_3}(p_3 x) = &\left[\frac{2^{\beta_3 - 1}\Gamma(\beta_3)}{x^{\beta_3}} + O(p_3^2) \right] \\ &+ p_3^{2\beta_3} \left[2^{-\beta_3 - 1}\Gamma(-\beta_3) x^{\beta_3} + O(p_3^2) \right], \end{aligned} \tag{2.5.39}$$

for $\beta_3 \notin \mathbb{Z}$ and

$$K_0(p_3 x) = -\log p_3 - \log x + \log 2 - \gamma_E + O(p_3^2), \tag{2.5.40}$$

$$\begin{aligned} p_3^n K_n(p_3 x) = &\left[\frac{2^{n-1}\Gamma(n)}{x^n} + O(p_3^2) \right] \\ &+ p_3^{2n} \left[\frac{(-1)^{n+1}}{2^n \Gamma(n+1)} x^n \log p_3 + \text{ultralocal} + O(p_3^2) \right], \end{aligned} \tag{2.5.41}$$

for $n = 1, 2, 3, \ldots$. From these expressions one can see that the zero momentum limit of $p^{\beta_3} K_{\beta_3}(p_3 x)$ exists if $\beta_3 > 0$. Since for any correlation function and any form factor $\beta_3 = \Delta_3 - \frac{d}{2} + k_3$ with non-negative k_3, this condition is satisfied if $\Delta_3 > \frac{d}{2}$. (For conserved currents and for the stress-energy tensor we thus have $\beta_3 > 0$ automatically.) We will return to discuss the case where $\beta_3 \leqslant 0$ later in the text.

Assuming $\beta_3 > 0$ then, we can calculate the limit of the triple-K integrals in the generic case

$$\lim_{p_3 \to 0} I_{\alpha\{\beta_j\}}(p, p, p_3) = l_{\alpha\{\beta_j\}} \cdot p^{\beta_t - \alpha - 1}, \tag{2.5.42}$$

where, using the result (2.A.49), we find

$$l_{\alpha\{\beta_k\}} = \frac{2^{\alpha-3}\Gamma(\beta_3)}{\Gamma(\alpha - \beta_3 + 1)} \prod_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} \Gamma\left(\frac{\alpha - \beta_3 + 1 + \epsilon_1 \beta_1 + \epsilon_2 \beta_2}{2} \right), \tag{2.5.43}$$

which is valid away from poles of the gamma function. Since the derivatives in the L and R operators defined in (2.4.27) and (2.4.28) acting on triple-K integrals can also be expressed via (2.5.12) in terms of triple-K integrals, this procedure leads to algebraic constraints on the primary constants.

Derivation of the equations in the generic case

Let us illustrate the considerations above in the case of the correlator $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$. The secondary CWIs are given by (2.4.31) and (2.4.32) with $\Delta_1 = \Delta_2 = d$ and L and R defined by (2.4.27) and (2.4.28). The right-hand sides are semi-local and can be expressed in terms of 2-point functions by means of the transverse Ward identities. In section 2.4.4 we found the Ward identity (2.4.36), which reads

$$p_1^{\nu_1} \langle\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\rangle = 2 p_1^{\nu_1} \langle\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\rangle. \tag{2.5.44}$$

We omit the group index on $\mathcal{O}$ as we consider only one scalar operator. First, we want to argue that the right-hand side of (2.5.44) vanishes if $\beta_3 > 0$, unless some specific conditions on conformal dimensions are met. Therefore, in this section we will assume that the right-hand sides of (2.4.31) and (2.4.32) vanish, leaving a discussion of the various special cases to the following sections. Indeed, the only possibility for a non-vanishing right-hand side of (2.5.44) is if the functional derivative $\delta T_{\mu_1\nu_1} / \delta g^{\mu_2\nu_2}$ contains the operator $\mathcal{O}$ or its descendants. Since the dilatation degree of $\delta T_{\mu_1\nu_1} / \delta g^{\mu_2\nu_2}$ is equal to d, this requires $d = \Delta_3 + 2n$ where n is a non-negative integer. Consider first the case $\Delta_3 = d$. In this case, we can write the most general form of $\delta T_{\mu_1\nu_1} / \delta g^{\mu_2\nu_2}$ which contains $\mathcal{O}$ as

$$\frac{\delta T_{\mu_1\nu_1}(\boldsymbol{x})}{\delta g^{\mu_2\nu_2}(\boldsymbol{y})} = \left[c_1 \delta_{\mu_1\nu_1} \delta_{\mu_2\nu_2} + c_2 \delta_{(\mu_1(\mu_2} \delta_{\nu_2)\nu_1)} \right] \delta(\boldsymbol{x} - \boldsymbol{y}) \mathcal{O}(\boldsymbol{x}) + \ldots \tag{2.5.45}$$

where c_1 and c_2 are numerical constants. If, on the other hand, $d = \Delta_3 + 2n$ with $n > 0$ then derivatives acting on both $\mathcal{O}$ and $\delta(\boldsymbol{x} - \boldsymbol{y})$ may also appear. In all cases, the Fourier transform reads

$$\langle\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\rangle = P_{\mu_1\nu_1\mu_2\nu_2}(p_1^2, p_2^2, p_3^2) \langle\langle \mathcal{O}(\boldsymbol{p}_3) \mathcal{O}(-\boldsymbol{p}_3) \rangle\rangle, \tag{2.5.46}$$

where P is some polynomial built from momenta and the metric $\delta_{\mu\nu}$, with kinematic dependence on squares of momenta only. This form arises from the Fourier transform of the position space expression containing derivatives acting on delta functions and on the 2-point function. Since $\langle\langle \mathcal{O}(\boldsymbol{p}_3) \mathcal{O}(-\boldsymbol{p}_3) \rangle\rangle \sim p_3^{2\beta_3}$, the expression vanishes in the $p_3 \to 0$ limit as long as $\beta_3 > 0$.

We now substitute the solutions of the primary CWIs (2.5.25) into the left-hand side of (2.4.31) and take the zero-momentum limit. Assuming the regulator can be removed (see section 2.5.3 if not), the result is

$$-\frac{l_{\frac{d}{2}+1\{\frac{d}{2},\frac{d}{2},\Delta_3-\frac{d}{2}\}}}{2} p^{\Delta_3-2} (2+2d-\Delta_3) \left[\alpha_2 + \alpha_1 (\Delta_3+2)(\Delta_3-d+2) \right] = 0, \tag{2.5.47}$$

which leads to

$$\alpha_2 = -(\Delta_3+2)(\Delta_3+2-d)\alpha_1. \tag{2.5.48}$$

The same reasoning as above applied to (2.4.32) leads to the equation

$$\alpha_3 = \frac{1}{4}\Delta_3(\Delta_3+2)(\Delta_3-d)(\Delta_3+2-d)\alpha_1. \tag{2.5.49}$$

Summarising, in this and the previous section we presented a method for extracting algebraic dependencies between the primary constants following from the secondary CWIs. The analysis was performed in the generic case, where the regulator can be removed from all triple-K integrals involved. Note that the results (2.5.48) and (2.5.49) agree with our example (2.5.36) and (2.5.37) in the leading term in ϵ only, *i.e.*, they correctly predict $\alpha_2 = \alpha_3 = 0 + O(\epsilon)$. This is due to the fact that in our example the regulator cannot be removed from each triple-K integral separately. Therefore, it does not satisfy the assumption of this section. Note, however, that the analysis of the generic case is sufficient if one is merely interested in finding the solution up to semi-local terms. This is because the possible non-generic cases arise due to the regularisation procedure, correcting the generic solution by at most semi-local terms.

Triple-K integrals and 2-point functions

Before we discuss the general procedure applicable to all cases, we first need to analyse the possible singularities associated with the 2-point functions. This is because the secondary CWIs connect triple-K integrals with semi-local terms expressible in terms of 2-point functions. Therefore, if the regulator is kept explicitly, the singular terms in triple-K integrals must match the singularities appearing in the 2-point functions.

An initial obstacle is that our convenient regularisation scheme (2.2.4) does not work for 2-point functions. The Fourier transform of the position space expression for a generic 2-point function is given by (2.1.3) where Δ is a conformal dimension of a scalar operator $\mathcal{O}$. The singularity occurs if $2\Delta = d + 2n$ for a non-negative integer n and is not regularised by the scheme (2.2.4).

Let us now try to find a different scheme which regularises 2-point functions and it yields results equivalent to (2.2.4, 2.5.7) when applied to the expressions entering the left-hand side of the secondary CWIs. It turns out that standard dimensional regularization has this property. We explicitly checked this in all cases discussed in the thesis. To motivate this choice consider the following integral

$$\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{k^{\mu_1}\dots k^{\mu_r}}{\boldsymbol{k}^{2\delta_3}|\boldsymbol{k}-\boldsymbol{p}_1|^{2\delta_2}|\boldsymbol{k}+\boldsymbol{p}_2|^{2\delta_1}} \tag{2.5.50}$$

regulated by shifting the space-time dimension while keeping the δ_j parameters fixed. In terms of the α and β parameters in (2.5.1), this corresponds to $\alpha \mapsto \alpha - \frac{\epsilon}{2}$ and $\beta_j \mapsto \beta_j - \frac{\epsilon}{2}$ as can be seen from (2.2.3) and (2.5.2). We can evaluate the integral (2.5.50) both by the triple-K integrals or the usual Feynman parametrised integrals. This leads to equation (4.2.23), namely

$$I_{\alpha\{\beta_1\beta_2\beta_3\}} = 2^{\alpha-3}\Gamma\left(\frac{\alpha-\beta_t+1}{2}\right)\Gamma\left(\frac{\alpha+\beta_t+1}{2}\right)\times$$
$$\times\int_{[0,1]^3} \mathrm{d}X\, D^{\frac{1}{2}(\beta_t-\alpha-1)}\prod_{j=1}^{3} x_j^{\frac{1}{2}(\alpha-1-\beta_t)+\beta_j}, \tag{2.5.51}$$

where

$$\mathrm{d}X = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(x_1+x_2+x_3-1), \tag{2.5.52}$$
$$D = p_1^2x_2x_3 + p_2^2x_1x_3 + p_3^2x_1x_2. \tag{2.5.53}$$

The leading divergence comes from a possible zero under the first gamma function and both schemes regularise $\alpha-\beta_t+1$ in the same way, shifting it by $\epsilon/2$.

A detailed analysis shows that all singularities and the finite part (modulo terms analytic in all momenta) in the expressions appearing in the left-hand side of the secondary CWIs match between the two schemes (*i.e.*, dimensional regularisation, $d \mapsto d-\epsilon$, and the scheme (2.2.4, 2.5.7)). Thus we can consistently take into account the contributions of the 2-point functions by computing them using dimensional regularization.

In the renormalised theory the divergent terms can be removed. Therefore we define constants $c_{\mathcal{O}}$, c_J and c_T by

$$\langle\!\langle \mathcal{O}^I(\boldsymbol{p})\mathcal{O}^J(-\boldsymbol{p})\rangle\!\rangle = c_{\mathcal{O}}\delta^{IJ}\times\begin{cases} p^{2\Delta-d} & \text{if } 2\Delta \neq d+2n,\\ p^{2\Delta-d}\left(-\log p^2+\text{local}\right) & \text{if } 2\Delta = d+2n,\end{cases} \tag{2.5.54}$$

$$\langle\!\langle J^{\mu a}(\boldsymbol{p})J^{\nu b}(-\boldsymbol{p})\rangle\!\rangle = c_J\delta^{ab}\pi^{\mu\nu}(\boldsymbol{p})\times\begin{cases} p^{d-2} & \text{if } d=3,5,7,\ldots\\ p^{d-2}\left(-\log p^2+\text{local}\right) & \text{if } d=4,6,8,\ldots\end{cases} \tag{2.5.55}$$

$$\langle\!\langle T^{\mu\nu}(\boldsymbol{p})T^{\rho\sigma}(-\boldsymbol{p})\rangle\!\rangle = c_T\Pi^{\mu\nu\rho\sigma}(\boldsymbol{p})\times\begin{cases} p^{d} & \text{if } d=3,5,7,\ldots\\ p^{d}\left(-\log p^2+\text{local}\right) & \text{if } d=4,6,8,\ldots\end{cases} \tag{2.5.56}$$

where n is a non-negative integer and all $\mathcal{O}^I$ operators have the same conformal dimension Δ and the dimensions of $J^{\mu a}$ and $T^{\mu\nu}$ are $d-1$ and d respectively. In general the normalisation constants $c_{\mathcal{O}}$ and c_J carry group indices. For simplicity we assume that the Killing form is trivial, *i.e.*,

$$c_{\mathcal{O}}^{IJ} = c_{\mathcal{O}}\delta^{IJ}, \qquad\qquad c_J^{ab} = c_J\delta^{ab}. \tag{2.5.57}$$

Secondary conformal Ward identities in all cases

Let us now return to the discussion of the secondary CWIs in the case where the regulator cannot be removed in certain triple-K integrals. In principle, the procedure is simple. One must keep the explicit dependence on ϵ, both in the triple-K integrals as well as in the primary constants, and carry out the analysis of sections 2.5.3 and 2.5.3 order by order in the regulator. Note that if the index of a Bessel function is integral, then the expansions (2.5.40) and (2.5.41) should be used instead of (2.5.39).

The only difference with section 2.5.3 is that looking at the zero-momentum limit may not be enough. We should look at both terms following from the first and second brackets in (2.5.39), *i.e.*, the coefficients of p_3^0 and $p_3^{2\beta_3}$ in the expansion in powers of p_3 with $p_1 = p_2 = p$. If the Bessel index is integral, then we should use (2.5.40) and (2.5.41) and look for the coefficients of p_3^0 and $p_3^{2\beta_3} \log p_3$. This procedure will provide a set of algebraic equations relating the primary constants.

Let us now explain why this procedure is valid for $\beta_3 < 0$ and why the remaining terms in the expansions (2.5.39) - (2.5.41) are irrelevant. First, the unitarity bound requires $-1 \leqslant \beta_3$. The unitarity bound can only be saturated by a non-composite scalar operator in a free field theory, [29] and [20]. We can therefore assume $-1 < \beta_3 < 0$. It turns out that the considerations encountered in the case $\beta_3 > 0$ remain valid here. Since the zero-momentum limit does not exist in this case, we are going to look for the coefficient of p_3^0 in the expansion in p_3. The key observation is that on the left-hand sides of the secondary CWIs such as (2.4.31, 2.4.32), the differential operators L and R defined by (2.4.27) and (2.4.28) do not contain derivatives with respect to p_3, and can only increase powers of p_3 by two. Therefore, the coefficient of p_3^0 in the series expansion in p_3 remains unaltered provided $-1 < \beta_3$. A similar analysis applies to the right-hand sides of the secondary CWIs.

Let us now examine why it is sufficient to look at the leading coefficients in (2.5.39) - (2.5.41) only. From (2.A.21), we know that in each successive term the power of the integration variable x increases by two. After taking the zero-momentum limit, the integral (2.A.49) therefore leads to essentially the same expression as (2.5.43) with $\beta_3 \mapsto \beta_3 + 2n$ for a non-negative integer n plus some finite pre-factor following from the series expansion of the Bessel K function. Since the singularities manifest themselves as poles of the gamma functions, we see that the result cannot be more singular than the original $l_{\alpha\{\beta_j\}}$.

Back to the example

Finally, let us see how the general consideration of the previous section work in the case of the $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$ correlation function. First, we carry out the same analysis as in sections 2.5.3 and 2.5.3 but keep the regulator explicitly. Instead of (2.5.47), we then find

$$\begin{aligned} -\frac{1}{2} l_{\frac{d}{2}+1+\epsilon\{\frac{d}{2},\frac{d}{2},\Delta_3-\frac{d}{2}\}} p^{\Delta_3-2+\epsilon} (2+2d-\Delta_3+\epsilon) \times \\ \times \left[a_2 + a_1((\Delta_3-\epsilon)^2 - d(\Delta+2-\epsilon) + 4(1+\Delta+\epsilon))\right] = 0. \qquad (2.5.58) \end{aligned}$$

If the $\epsilon \to 0$ limit exists, we recover (2.5.47). The limit does not exist, however, if Δ_3 satisfies some non-generic relations. From the definition of $l_{\frac{d}{2}+1+\epsilon\{\frac{d}{2},\frac{d}{2},\Delta_3-\frac{d}{2}\}}$ in (2.5.43), we see that this happens if at least one of the following conditions are satisfied:

- $\Delta_3 = 2 + 2n_1$,
- $\Delta_3 = d + 2 + 2n_2$,
- $\Delta_3 = d + 4 + 2n_3$,
- $\Delta_3 = 2d + 4 + 2n_4$,

where $n_1, \ldots, n_4$ are non-negative integers. The order of the singularity increases if more than one of these conditions is satisfied. Since there exists a choice of Δ_3, d and the n_j constants such that all conditions are satisfied, we might expect a pole of order four in ϵ. Note however that there is another gamma function in the denominator of (2.5.43) which becomes singular if the numerator has a pole of order four. The maximal order of the pole is therefore only three.

The discussion above leads to the conclusion that we should expand the primary constant α_2 up to third order in ϵ as in (2.5.33), hence

$$\begin{aligned}\alpha_2 J_{2\{000\}} = {} & \frac{1}{\epsilon^3}\alpha_2^{(0)} J_{2\{000\}}^{(-3)} + \frac{1}{\epsilon^2}\left[\alpha_2^{(0)} J_{2\{000\}}^{(-2)} + \alpha_2^{(1)} J_{2\{000\}}^{(-3)}\right] \\ & + \frac{1}{\epsilon}\left[\alpha_2^{(0)} J_{2\{000\}}^{(-1)} + \alpha_2^{(1)} J_{2\{000\}}^{(-2)} + \alpha_3^{(2)} J_{2\{000\}}^{(-3)}\right] \\ & + \left[\alpha_2^{(0)} J_{2\{000\}}^{(0)} + \alpha_2^{(1)} J_{2\{000\}}^{(-1)} + \alpha_2^{(2)} J_{2\{000\}}^{(-2)} + \alpha_2^{(3)} J_{2\{000\}}^{(-3)}\right] + O(\epsilon).\end{aligned} \tag{2.5.59}$$

With the appropriate choice of primary constants we will now see that the singular terms in the various triple-K integrals building a given form factor cancel out, leaving ultralocal singular terms at most.

Let us now extract the primary constants from the secondary CWI (2.4.31). We substitute (2.5.59) into (2.5.58) and expand the result in ϵ in any possible combination of the cases itemised above. One can subsequently solve the equations starting from the most singular one. In this way, one finds

$$\begin{aligned} \alpha_2^{(0)} &= -(\Delta_3+2)(\Delta_3+2-d)\alpha_1, & \alpha_2^{(2)} &= -\alpha_1, \\ \alpha_2^{(1)} &= (2\Delta_3 - d + 4)\alpha_1, & \alpha_2^{(3)} &= 0. \end{aligned} \tag{2.5.60}$$

We assume here that α_1 is a true constant, *i.e.*, $\alpha_1 = \alpha_1^{(0)}$. These solutions are valid for the special cases listed at the beginning of this section. Notice that they still do not cover all special cases, for example they do not cover the case of the example we studied in section 2.5.3. This is because so far we have considered the equations following from the first parts of the expansions (2.5.39) - (2.5.41) only:

we must now turn our attention to the equations following from the second parts. In many cases the equations to follow will agree with (2.5.60), but in some special cases new contributions will arise.

The equation (2.5.39) and the integral (2.A.49) lead to the equation

$$-\frac{l_{\frac{d}{2}+1+\epsilon\{\frac{d}{2},\frac{d}{2},\frac{d}{2}-\Delta_3\}}}{2}(2+d+\Delta_3+\epsilon)\left[a_2+a_1(\Delta_3+2+\epsilon)(\Delta_3-d+2+\epsilon)\right]=$$
$$= 4d\cdot\text{coefficient of } p_2^{\mu_1}p_3^{\mu_2}p_3^{\nu_2}\cdot p_3^{2\Delta_3-d}p^{d-\Delta_3-2} \text{ in } p_2^{\nu_1}\langle\!\langle\frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1,\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle \tag{2.5.61}$$

following from the coefficient of $p_3^{2\beta_3}$ in the series expansion (2.5.39). First note that if the $\epsilon\to 0$ limit exists, then the left-hand side vanishes when the solution (2.5.48) is substituted. On the other hand, we know from section 2.5.3 that the right-hand side can be non-zero only if $\Delta_3=d-2-2n$ for some non-negative integer n. Therefore, in such a case we expect the left-hand side to be more singular than the right-hand side, so that the solution (2.5.48) cancels the leading order singularity while the sub-leading terms match the right-hand side. Indeed, the left-hand side is singular if $l_{\frac{d}{2}+1\{\frac{d}{2},\frac{d}{2},\frac{d}{2}-\Delta_3\}}$ is singular. Analysing the expression (2.5.43), we see that this can happen only if $\Delta_3=d-2-2n$, where n is a non-negative integer. Note that our example analysed in section 2.5.3 is of this kind as there $d=3$, $\Delta_3=1$.

Finally, we would like to extract the sub-leading equations in the case where $\Delta_3=d-2-2n$. In order to do this, we write $\alpha_2=\alpha_2^{(0)}+\epsilon\alpha_2^{(1)}$ in (2.5.61) and expand the result in ϵ. At zeroth order we recover (2.5.48), after which we find

$$C\cdot\Gamma\left(\frac{d}{2}-\Delta_3\right)\left[(2\Delta_3-d+4)\alpha_1+\alpha_2^{(1)}\right]=4d\cdot c_1^I c_{\mathcal{O}}, \tag{2.5.62}$$

where

$$C=\frac{(-1)^{\frac{d-\Delta_3}{2}}2^{\frac{d}{2}-\Delta_3-2}\sqrt{\pi}\,\Gamma\left(\frac{\Delta_3+2}{2}\right)\Gamma\left(\frac{\Delta_3+d+4}{2}\right)}{\Gamma\left(\frac{d-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_3+3}{2}\right)} \tag{2.5.63}$$

and the constant c_1^K is defined as

$$p_1^{\nu_1}\langle\!\langle\frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1,\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p})_3\rangle\!\rangle\Bigg|_{p_1=p_2=p}=c_1^K p_2^{\mu_1}p_3^{\mu_2}p_3^{\nu_2}p^{d-\Delta_3-2}\langle\!\langle\mathcal{O}^I(\boldsymbol{p}_3)\mathcal{O}^K(-\boldsymbol{p}_3)\rangle\!\rangle$$
$$+\ c_2^K\delta^{\mu_1\mu_2}p_3^{\nu_2}p^{d-\Delta_3}\langle\!\langle\mathcal{O}^I(\boldsymbol{p}_3)\mathcal{O}^K(-\boldsymbol{p}_3)\rangle\!\rangle+\ldots \tag{2.5.64}$$

By $p_1=p_2=p$, we mean here the following procedure: First, the correlation function on the left-hand side is expanded in terms of simple tensors according to the convention (2.3.4), then secondly, one applies $p_1=p_2=p$ to each coefficient separately.

In order to derive (2.5.62) we used (2.5.39). If $\Delta_3 = d-2-2n$ and $2\Delta_3 = d+2m$ for some non-negative integers m and n, *i.e.*, if β_3 is an integer, then one must use instead (2.5.41). The procedure remains identical and the final result is

$$-\frac{C}{\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right)} \left[(2\Delta_3 - d + 4)\alpha_1 + \alpha_2^{(1)}\right] = 4d \cdot c_1^I c_{\mathcal{O}}. \tag{2.5.65}$$

In total, we have that

$$\alpha_2^{(0)} = -(\Delta_3 + 2)(\Delta_3 + 2 - d)\alpha_1 \tag{2.5.66}$$

in all cases while

$$\alpha_2^{(1)} = (2\Delta_3 - d + 4)\alpha_1, \qquad \alpha_2^{(2)} = -\alpha_1 \tag{2.5.67}$$

if $\Delta_3 \neq d - 2 - 2n$, where n is a non-negative integer and

$$\begin{aligned} 4d \cdot c_1^I c_{\mathcal{O}} = & \left[(2\Delta_3 - d + 4)\alpha_1 + \alpha_2^{(1)}\right] \times C \times \\ & \times \begin{cases} \Gamma\left(\frac{d}{2} - \Delta_3\right) & \text{if } 2\Delta_3 \neq d + 2, \\ -\dfrac{1}{\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right)} & \text{if } 2\Delta_3 = d + 2m, \end{cases} \end{aligned} \tag{2.5.68}$$

if $\Delta_3 = d - 2 - 2n$, where m and n are non-negative integers.

A similar analysis may be carried out for the second CWI, (2.4.32). Putting all the ingredients together, we can now write the most general form of the correlator $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O}^I \rangle\!\rangle$ for $d = 3$ and $\Delta_2 = \Delta_3 = 1$. Using the results for the triple-K integrals from section 2.5.2 we have

$$A_1^I = \frac{\alpha_1^I}{p_3 a_{123}^4} \left[p_3^2 + 4p_3 a_{12} + 3(a_{12}^2 + 2b_{12})\right], \tag{2.5.69}$$

$$A_2^I = \frac{\alpha_1^I}{p_3 a_{123}^3} \left[p_3^3 + 3p_2^2 a_{12} + p_3(-a_{12}^2 + 8b_{12}) - 3a_{12}^3\right] - \frac{4c_1^I c_{\mathcal{O}}}{p_3}, \tag{2.5.70}$$

$$\begin{aligned} A_3^I = & \frac{\alpha_1^I (a_{12} - p_3)}{4p_3 a_{123}^2} \left[-p_3^3 - 3p_3^2 a_{12} + p_3(a_{12}^2 - 10b_{12}) + 3a_{12}(a_{12}^2 - 2b_{12})\right] \\ & + \frac{c_{\mathcal{O}}}{p_3} \left[(c_1^I - 3c_2^I)(p_1^2 + p_2^2) + 3(c_1^I + c_2^I)p_3^2\right], \end{aligned} \tag{2.5.71}$$

where we have defined the symmetric polynomials in momentum magnitudes

$$\begin{aligned} a_{123} = p_1 + p_2 + p_3, \qquad b_{123} = p_1 p_2 + p_1 p_3 + p_2 p_3, \qquad c_{123} = p_1 p_2 p_3, \\ a_{ij} = p_i + p_j, \qquad b_{ij} = p_i p_j, \end{aligned} \tag{2.5.72}$$

where $i, j = 1, 2, 3$. The solution for this correlator is thus uniquely determined up to one numerical constant α_1^I. The remaining constants in the solution, namely

$c_{\mathcal{O}}$, c_1^I and c_2^I, are determined by the 2-point function normalisations: $c_{\mathcal{O}}$ is given in (2.5.54) while c_1^I and c_2^I are given in (2.5.64).

One can check this result against our example in section 2.5.3. From (2.4.39) and (2.4.40), the solution for the parameters is

$$c_{\mathcal{O}} = \frac{1}{4}, \qquad c_1^I = -\frac{1}{16}, \qquad c_2^I = 0. \tag{2.5.73}$$

2.6. Evaluation of triple-K integrals

In the preceding sections we have developed a method for calculating all 3-point functions of the stress-energy tensor, conserved currents and scalar operators in any CFT. The method operates purely in momentum space, and is based on a direct solution of the conformal Ward identities. The results we obtain are expressed in the form of triple-K integrals.

In the present section we now turn to discuss the evaluation of these triple-K integrals. The ease with which this may be accomplished depends on the dimensionality of the space d. In an odd number of dimensions, it turns out that all 3-point functions of conserved currents and the stress-energy tensor are expressible in terms of triple-K integrals in which the Bessel function indices are half-integer. In this case, the Bessel functions reduce to elementary functions (see appendix 2.A.3) and the triple-K integral may be straightforwardly evaluated. When the spatial dimension d is even, we obtain instead triple-K integrals in which the Bessel function indices are integer. In this case, the triple-K integrals we encounter may be evaluated in terms of a single master integral through the use of a reduction scheme. In the following we will focus primarily on the case $d = 4$, but method we present extends straightforwardly to higher even dimensions.

Our reduction scheme is based on the observation that, given a triple-K integral $I_{\alpha\{\beta_1\beta_2\beta_3\}}$, through differentiation we may easily obtain the integrals $I_{\alpha+1\{\beta_1+1,\beta_2,\beta_3\}}$ and $I_{\alpha+1\{\beta_1-1,\beta_2,\beta_3\}}$, and similarly for β_2 and β_3 by permutation. We may then start with a known integral with a sufficiently small value of α and obtain integrals with larger α through repeated differentiation. In some cases a relation for lowering α also exists. For 3-point functions of conserved currents and stress-energy tensors in $d = 4$ the necessary triple-K integrals are:

$$\begin{aligned}
&I_{4\{111\}} \mapsto I_{5\{211\}}, I_{6\{221\}}, I_{7\{222\}},\\
&I_{2\{111\}} \mapsto I_{3\{211\}}, I_{4\{221\}}, I_{4\{311\}}, I_{5\{222\}}, I_{5\{321\}}, I_{6\{322\}},\\
&I_{0\{111\}} \mapsto I_{1\{211\}}, I_{2\{221\}}, I_{3\{222\}}, I_{3\{321\}}, I_{4\{322\}}, I_{5\{422\}}, I_{5\{332\}},\\
&I_{1\{222\}} \mapsto I_{3\{332\}}, I_{4\{333\}}.
\end{aligned} \tag{2.6.1}$$

The integrals are organised into four families, each with constant $\alpha - \beta_t$, where each integral within a given family may be obtained from the corresponding integral on the leftmost end. The four leftmost integrals can then be derived from the single master integral, $I_{0\{111\}}$, as we will show in section 2.6.4.

2.6.1. Reduction scheme

The elementary properties of Bessel functions imply

$$I_{\alpha\{\beta_{\sigma(1)}\beta_{\sigma(2)}\beta_{\sigma(3)}\}}(p_1,p_2,p_3) = I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_{\sigma^{-1}(1)},p_{\sigma^{-1}(2)},p_{\sigma^{-1}(3)}), \tag{2.6.2}$$

$$I_{\alpha\{\beta_1\beta_2\beta_3\}} = -\frac{1}{p_1}\frac{\partial}{\partial p_1} I_{\alpha-1\{\beta_1+1,\beta_2\beta_3\}}, \tag{2.6.3}$$

$$I_{\alpha\{\beta_1\beta_2,-\beta_3\}} = p_3^{-2\beta_3} I_{\alpha\{\beta_1\beta_2\beta_3\}}, \tag{2.6.4}$$

$$\begin{aligned}(\alpha-\beta_t) I_{\alpha-1\{\beta_1\beta_2\beta_3\}} &= p_1^2 I_{\alpha\{\beta_1-1,\beta_2,\beta_3\}} + p_2^2 I_{\alpha\{\beta_1,\beta_2-1,\beta_3\}} \\ &\quad + p_3^2 I_{\alpha\{\beta_1,\beta_2,\beta_3-1\}}, \qquad \alpha-\beta_t \neq -2n,\end{aligned} \tag{2.6.5}$$

where n is a non-negative integer and the triple-K integral $I_{\alpha\{\beta_1\beta_2\beta_3\}}$ was defined in (2.5.1). The first of these equations appeared previously as (2.5.11) and simply expresses the fact that the triple symmetry under permutation, with σ representing a permutation of the set $\{1,2,3\}$. The second and third of these equations appeared earlier as (2.5.13) and (2.5.14), while the last follows from the second line of (2.5.18). Starting from these relations, we may now set up our reduction scheme as follows.

First, assume we are given an integral $I_{\alpha\{\beta_1\beta_2\beta_3\}}$. Applying (2.6.3) in the form

$$I_{\alpha+1\{\beta_1-1,\beta_2,\beta_3\}} = -\frac{1}{p_1}\frac{\partial}{\partial p_1} I_{\alpha\{\beta_1\beta_2\beta_3\}}, \tag{2.6.6}$$

we increase α by one while decreasing β_1 by one. Equivalently, this operation increases the difference $\alpha-\beta_t$ by two but keeps the sum $\alpha+\beta_t$ fixed.

If instead we first use equation (2.6.4), followed by (2.6.3) then (2.6.4) again, we find

$$\begin{aligned} I_{\alpha+1\{\beta_1+1,\beta_2,\beta_3\}} &= -p_1^{2\beta_1+1}\frac{\partial}{\partial p_1}\left[p_1^{-2\beta_1} I_{\alpha\{\beta_1\beta_2\beta_3\}}\right] \\ &= \left(2\beta_1 - p_1\frac{\partial}{\partial p_1}\right) I_{\alpha\{\beta_1\beta_2\beta_3\}}, \end{aligned} \tag{2.6.7}$$

where the sum $\alpha+\beta_t$ increases by two but the difference $\alpha-\beta_t$ remains fixed. Similarly, applying this operation and its permutations repeatedly we obtain

$$I_{\alpha+n\{\beta_j+k_j\}} = (-1)^{k_t}\left[\prod_{j=1}^{3} p_j^{2(\beta_j+k_j)}\left(\frac{1}{p_j}\frac{\partial}{\partial p_j}\right)^{k_j}\right]\left[p_1^{-2\beta_1}p_2^{-2\beta_2}p_3^{-2\beta_3} I_{\alpha\{\beta_j\}}\right]. \tag{2.6.8}$$

where $k_t = \sum_j k_j$ and the k_j are non-negative integers.

Now, both reduction relations (2.6.3) and (2.6.4) obtained by differentiation happen to increase α by one. To reduce α we may instead use the (2.6.5) in the form

$$I_{\alpha-1\{\beta_1\beta_2\beta_3\}} = \frac{1}{\alpha-\beta_t}\left[p_1^2 I_{\alpha\{\beta_1-1,\beta_2,\beta_3\}} + p_2^2 I_{\alpha\{\beta_1,\beta_2-1,\beta_3\}} + p_3^2 I_{\alpha\{\beta_1,\beta_2,\beta_3-1\}}\right], \tag{2.6.9}$$

assuming $\alpha-\beta_t \neq -2n$, where n is a non-negative integer. This equation is closely related to Davydychev's recursion relation (3.4) introduced in [52]. Indeed, using

$$\begin{tikzcd}[column sep=large, row sep=large]
\boxed{I_{0\{111\}}} \arrow[r, "\left(2-p_1\frac{\partial}{\partial p_1}\right)\cdot"] & I_{1\{211\}} \arrow[r, "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"] \arrow[d, "\left(2-p_2\frac{\partial}{\partial p_2}\right)\cdot"] & I_{2\{111\}} \arrow[d, "\left(2-p_1\frac{\partial}{\partial p_1}\right)\cdot"] & \\
I_{1\{222\}} \arrow[r, "-\frac{1}{p_3}\frac{\partial}{\partial p_3}"] \arrow[d, "\left(4-p_1\frac{\partial}{\partial p_1}\right)\cdot"] & I_{2\{221\}} \arrow[r, "-\frac{1}{p_2}\frac{\partial}{\partial p_2}"] \arrow[l, dotted, bend left] \arrow[d, "\begin{array}{c}\left(2-p_3\frac{\partial}{\partial p_3}\right)\cdot\\ \left(4-p_1\frac{\partial}{\partial p_1}\right)\cdot\end{array}"] & I_{3\{211\}} \arrow[r, "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"] \arrow[d, "\begin{array}{c}\left(2-p_2\frac{\partial}{\partial p_2}\right)\cdot\\ \left(4-p_1\frac{\partial}{\partial p_1}\right)\cdot\end{array}"] & I_{4\{111\}} \arrow[d, "\left(2-p_1\frac{\partial}{\partial p_1}\right)\cdot"] \\
I_{2\{322\}} \arrow[r, "-\frac{1}{p_1}\frac{\partial}{\partial p_1}", "-\frac{1}{p_3}\frac{\partial}{\partial p_3}"'] \arrow[d, "\left(4-p_2\frac{\partial}{\partial p_2}\right)\cdot"] & \begin{matrix} I_{3\{222\}} \\ I_{3\{321\}} \end{matrix} \arrow[r, "-\frac{1}{p_3}\frac{\partial}{\partial p_3}", "-\frac{1}{p_2}\frac{\partial}{\partial p_2}"'] \arrow[d, "\begin{array}{c}\left(4-p_1\frac{\partial}{\partial p_1}\right)\cdot\\ \left(2-p_3\frac{\partial}{\partial p_3}\right)\cdot\end{array}"] & \begin{matrix} I_{4\{221\}} \\ I_{4\{311\}} \end{matrix} \arrow[r, "-\frac{1}{p_2}\frac{\partial}{\partial p_2}", "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"'] \arrow[d, "\begin{array}{c}\left(2-p_3\frac{\partial}{\partial p_3}\right)\cdot\\ \left(2-p_2\frac{\partial}{\partial p_2}\right)\cdot\end{array}"] & I_{5\{211\}} \arrow[d, "\left(2-p_2\frac{\partial}{\partial p_2}\right)\cdot"] \\
I_{3\{332\}} \arrow[r, "-\frac{1}{p_2}\frac{\partial}{\partial p_2}"] \arrow[d, "\begin{array}{c}\left(4-p_3\frac{\partial}{\partial p_3}\right)\cdot\\ \left(6-p_1\frac{\partial}{\partial p_1}\right)\cdot\end{array}"] & I_{4\{322\}} \arrow[r, "-\frac{1}{p_1}\frac{\partial}{\partial p_1}", "-\frac{1}{p_3}\frac{\partial}{\partial p_3}"'] \arrow[d, "\begin{array}{c}\left(4-p_2\frac{\partial}{\partial p_2}\right)\cdot\\ \left(6-p_1\frac{\partial}{\partial p_1}\right)\cdot\end{array}"] & \begin{matrix} I_{5\{222\}} \\ I_{5\{321\}} \end{matrix} \arrow[r, "-\frac{1}{p_3}\frac{\partial}{\partial p_3}", "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"'] \arrow[d, "\begin{array}{c}\left(4-p_1\frac{\partial}{\partial p_1}\right)\cdot\\ \left(2-p_3\frac{\partial}{\partial p_3}\right)\cdot\end{array}"] & I_{6\{221\}} \arrow[d, "\left(2-p_3\frac{\partial}{\partial p_3}\right)\cdot"] \\
\begin{matrix} I_{4\{333\}} \\ I_{4\{432\}} \end{matrix} \arrow[r, "-\frac{1}{p_3}\frac{\partial}{\partial p_3}", "-\frac{1}{p_2}\frac{\partial}{\partial p_2}"'] & \begin{matrix} I_{5\{332\}} \\ I_{5\{422\}} \end{matrix} \arrow[r, "-\frac{1}{p_2}\frac{\partial}{\partial p_2}", "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"'] & I_{6\{322\}} \arrow[r, "-\frac{1}{p_1}\frac{\partial}{\partial p_1}"] & I_{7\{222\}}
\end{tikzcd}$$

Table 2.1: *Reduction scheme for the integrals required in the calculations of 3-point functions of conserved currents and stress-energy tensor in $d = 4$. All integrals can be obtained as indicated from a single master integral $I_{0\{111\}}$ given by* (2.A.57)*. The dotted line indicates the use of* (2.6.9)*.*

(4.2.17) one can rewrite Davydychev's J integral defined in (2.1) of [52] as

$$J(\delta_1, \delta_2, \delta_3) = \frac{4\pi^2}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)\Gamma(4-\delta_t)} I_{1\{2-\delta_2-\delta_3, 2-\delta_1-\delta_3, 2-\delta_1-\delta_2\}}. \quad (2.6.10)$$

Note that the rather complicated form of equation (3.4) in [52] is a consequence of the specific index structure in the triple-K integral above.

2.6.2. The case $d = 4$

Let us now apply the reduction relations above to the integrals listed in (2.6.1). Integrals within a given family have constant $\alpha - \beta_t$ and hence are connected by (2.6.7). Similarly, integrals belonging to different families but with the same sum $\alpha + \beta_t$ may be connected using (2.6.6). We summarise in table 2.1 the dependencies between all integrals necessary for the evaluation of 3-point functions of conserved currents and the stress-energy tensor in $d = 4$.

Integrals within a single row (except for the first) are connected via equation (2.6.6), while integrals within a single column are related by (2.6.7). The difference $\alpha - \beta_t$ is thus constant within each column while the sum $\alpha + \beta_t$ is constant along each row. Similarly, the index α is constant along diagonals from top-right to bottom-left, while β_t is constant along diagonals from the top-left to the bottom-right. In cases where there are two integrals in a given table entry, the arrows entering and leaving this entry have two labels, with the upper label referring to the upper integral and the lower label referring to the lower integral, thus, *e.g.*,

$$I_{3\{222\}} = -\frac{1}{p_1}\frac{\partial}{\partial p_1} I_{2\{322\}}, \qquad I_{3\{321\}} = -\frac{1}{p_3}\frac{\partial}{\partial p_3} I_{2\{322\}}. \quad (2.6.11)$$

We assume all integrals are regularised in our standard scheme (2.5.7). In this case, the operations assigned to the arrows should be applied order by order in the regulator. Note that if one uses a different regularisation scheme which changes the values of the β parameters, one cannot apply the operators on the vertical lines order by order in ϵ. This is because (2.6.7) contains a β parameter which would become a function of the regulator.

In section 2.6.4 we will evaluate the master integral $I_{0\{111\}}$. However, as the entries on the left side of the table are generally more singular (and more complicated) than the entries on the right, in practice it is more convenient to move in columns rather than in rows. The required expressions for the three integrals $I_{0\{111\}}$, $I_{2\{111\}}$ and $I_{1\{000\}}$, which generate all the integrals in the three rightmost columns are given in appendix 2.A.5.

Starting from $I_{0+\epsilon\{111\}}$ we can follow the arrows in table 2.1 to obtain $I_{2+\epsilon\{221\}}$.

Using (2.6.2) and (2.6.9) we find

$$\begin{aligned}
I_{1+\epsilon\{222\}} &= \frac{1}{\epsilon-4}\left[p_1^2 I_{2+\epsilon\{122\}} + p_2^2 I_{2+\epsilon,\{212\}} + p_3^2 I_{2+\epsilon\{221\}}\right] \\
&= \frac{p_1^4+p_2^4+p_3^4}{2\epsilon^2} \\
&\quad + \frac{1}{8}\left(\frac{4}{\epsilon}+1\right) \times \left[\left(p_1^2 p_2^2 + (\tfrac{3}{4} - \gamma_E + \log 2) p_3^4 - p_3^4 \log p_3\right)\right. \\
&\qquad\qquad \left. + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3)\right] \\
&\quad - \frac{1}{4}\left[p_3^2\left(2 - p_1 \frac{\partial}{\partial p_1}\right)\left(2 - p_2 \frac{\partial}{\partial p_2}\right) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3)\right] I^{(0)}_{0\{111\}} \\
&\quad + O(\epsilon),
\end{aligned} \tag{2.6.12}$$

where $I^{(0)}_{0\{111\}}$ denotes the ϵ^0 term in the series expansion in ϵ of $I_{0+\epsilon\{111\}}$. Thus all integrals in the table can be obtained from $I_{0\{111\}}$.

2.6.3. Integrals in even dimensions $d \geqslant 4$

In the previous section we presented a comprehensive procedure for the evaluation of all triple-K integrals appearing in 3-point functions of conserved currents and the stress-energy tensor in $d = 4$. In this section we want to extend our analysis to all even dimensions $d \geqslant 4$. We will present a recursive procedure which yields all required integrals from the master integral $I_{0\{111\}}$.

Let us start with the case $d = 6$. Looking at the solutions to the primary CWIs, one sees that new integrals arise only in the form factor A_5 of the $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\!\rangle$ correlator. One of these new integrals, $J_{0\{000\}}$, is equal to $I_{2\{333\}}$ when $d = \Delta_j = 6$, $j = 1, 2, 3$ is used in (2.5.3). All remaining integrals in the form factor A_5 then satisfy the relation $\alpha - \beta_t = -7$ meaning they can be obtained from $I_{2\{333\}}$ using (2.6.7). Finally, notice that by (2.6.6),

$$-\frac{1}{p_3}\frac{\partial}{\partial p_3} I_{2\{333\}} = I_{3\{332\}}. \tag{2.6.13}$$

Thus to obtain all required integrals in $d = 6$ we need to add another column to the left side of table 2.1.

This discussion generalises for general even d. Assume first that we have all integrals required for the evaluation of the 3-point functions of conserved currents and the stress-energy tensor in a given even dimension. The leftmost integrals in table 2.1 have the lowest value of $\alpha - \beta_t$ and it is these integrals which appear in the A_5 form factor in the correlator of three stress-energy tensors. For these integrals $\alpha - \beta_t = -(d+1)$, and the lowest value of $\alpha + \beta_t$ is attained for the $J_{0\{000\}}$ integral. For the correlator of three stress-energy tensors in d dimensions

$$J_{0\{000\}} = I_{\frac{d}{2}-1\{\frac{d}{2}\frac{d}{2}\frac{d}{2}\}} \tag{2.6.14}$$

and all remaining integrals in the form factor A_5 can be obtained from this one by means of (2.6.7). Therefore, if one knows the integrals in a dimension $d-2$, in order to obtain the missing integrals in dimension d, one needs to reduce the value of $\alpha - \beta_t$. This can be achieved by (2.6.9),

$$I_{\frac{d}{2}-1+\epsilon\{\frac{d}{2}\frac{d}{2}\frac{d}{2}\}} = -\frac{1}{d-\epsilon}\left[p_1^2 I_{\frac{d}{2}+\epsilon\{\frac{d}{2}-1,\frac{d}{2},\frac{d}{2}\}} + p_2^2 I_{\frac{d}{2}+\epsilon\{\frac{d}{2},\frac{d}{2}-1,\frac{d}{2}\}} + p_3^2 I_{\frac{d}{2}+\epsilon\{\frac{d}{2},\frac{d}{2},\frac{d}{2}-1\}}\right]. \tag{2.6.15}$$

If one denotes $d = d' + 2$, then the integrals featuring in this expression can be obtained from the integral (2.6.14) in dimension d' as follows

$$\begin{aligned} I_{\frac{d}{2}+\epsilon\{\frac{d}{2}-1,\frac{d}{2},\frac{d}{2}\}} &= I_{\frac{d'}{2}+1+\epsilon\{\frac{d'}{2},\frac{d'}{2}+1,\frac{d'}{2}+1\}} \\ &= \left(d' - p_2 \frac{\partial}{\partial p_2}\right)\left(d' - p_3 \frac{\partial}{\partial p_3}\right) I_{\frac{d'}{2}-1+\epsilon\{\frac{d'}{2}\frac{d'}{2}\frac{d'}{2}\}}. \end{aligned} \tag{2.6.16}$$

This shows that the recursive use of (2.6.16) and (2.6.7) allows analytic expressions to be found for all triple-K integrals required for the evaluation of 3-point functions of conserved currents and the stress-energy tensor in arbitrary even dimension $d \geqslant 4$. Visually, the procedure adds new columns to the left side of table 2.1 with lower values of $\alpha - \beta_t$. One can move down the column through the repeated use of (2.6.7), and the starting entry in each column is (2.6.14). In summary, one can find all required integrals starting from a single master integral $I_{0\{111\}}$ which we will evaluate analytically in the following section.

2.6.4. Evaluation of the master integral

In this section we present a method to evaluate integrals of the form $I_{\nu+1\{\nu\nu\nu\}}$, $\nu \in \mathbb{R}$. In particular, if we choose $\nu = -1$ to evaluate $I_{0\{-1-1-1\}}$, using (2.6.4) then gives

$$I_{0\{111\}} = p_1^2 p_2^2 p_3^2 I_{0\{-1-1-1\}} \tag{2.6.17}$$

which is the master integral we will need for the analysis of conserved currents and stress tensors.

In appendix 2.A.5 we also present expressions for $I_{2\{111\}}$ and $I_{1\{000\}}$. In particular, $I_{1\{000\}}$ is convergent and finite and is known in the literature, *e.g.*, [52, 69].

Let us now evaluate the master integral $I_{0\{111\}}$. We will start with the more general problem of evaluating integrals of the form $I_{\nu+1\{\nu\nu\nu\}}$, $\nu \in \mathbb{R}$. To write the results in compact form, we introduce the following variables. First, we define

$$\begin{aligned} J^2 &= (p_1+p_2-p_3)(p_1-p_2+p_3)(-p_1+p_2+p_3)(p_1+p_2+p_3) \\ &= -p_1^4 - p_2^4 - p_3^4 + 2p_1^2p_2^2 + 2p_1^2p_3^2 + 2p_2^2p_3^2 \\ &= 4\left[p_1^2p_2^2 - (\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\right] = 4\cdot\mathrm{Gram}(\boldsymbol{p}_1,\boldsymbol{p}_2), \end{aligned} \tag{2.6.18}$$

where Gram is the Gram determinant. For physical momentum configurations obeying the triangle inequalities we have $J^2 \geqslant 0$, with $J^2 = 0$ holding if and only if the momenta are collinear. Next, we define

$$X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2 p_3}, \qquad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1 p_3},$$
$$Z = \frac{p_3^2 - p_1^2 - p_2^2 - \sqrt{-J^2}}{2p_1 p_2}. \tag{2.6.19}$$

Note that X and Y are symmetric under $p_1 \leftrightarrow p_2$, while the sign in front of the square root in Z is different to that in X and Y. For physical momentum configurations these variables are complex, although the entire expression for a given triple-K integral remains real.

Starting with the representation (2.A.46) of the triple-K integral, one can use the reduction formulae (2.A.40) - (2.A.43) in order to find

$$I_{\nu+1\{\nu\nu\nu\}} = \frac{2^{\nu-2}\Gamma(\nu)\pi}{\sin(\pi\nu)} \left[\frac{p_3^{2\nu}}{p_1 p_2} Z \cdot F_\nu(Z^2) + \frac{p_2^{2\nu}}{p_1 p_3} Y \cdot F_\nu\left(Z\frac{Y}{X}\right) + \frac{p_1^{2\nu}}{p_2 p_3} X \cdot F_\nu\left(Z\frac{X}{Y}\right)\right]$$
$$+ \frac{2^{3\nu-2}\pi^{\frac{3}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}{\sin^2(\pi\nu)} (p_1p_2p_3)^{2\nu} (\sqrt{-J^2})^{-(2\nu+1)}. \tag{2.6.20}$$

where

$$F_\nu(x) = {}_2F_1(1, \nu+1; 1-\nu; x) \tag{2.6.21}$$

and the X, Y, Z variables are defined in (2.6.19) while J^2 is given by (2.6.18). Note that this particular combination of parameters in the hypergeometric function appears in Legendre functions.

For generic values of ν the expression (2.6.20) is finite. In order to evaluate $I_{0\{111\}}$, however, we require $\nu = -1$ (see (2.6.17)) where (2.6.20) has singularities. In cases such as this, (2.6.20) may be series expanded in ν. The relevant expansion of the hypergeometric function here is

$$F_{-1+\epsilon}(x) = 1 + F_{-1}^{(1)}(x)\epsilon + F_{-1}^{(2)}(x)\epsilon^2 + O(\epsilon^3), \tag{2.6.22}$$

where

$$F_{-1}^{(1)}(x) = 1 - \left(1 - \frac{1}{x}\right)\log(1-x), \tag{2.6.23}$$
$$F_{-1}^{(2)}(x) = 2 + \left(1 - \frac{1}{x}\right)\left[-\log(1-x) + \log^2(1-x) + \mathrm{Li}_2\, x\right]. \tag{2.6.24}$$

Combining everything we obtain an analytic expression for the integral $I_{0+\epsilon\{-1+\epsilon,-1+\epsilon,-1+\epsilon\}}$. Note however that this result is given in a different regularisation scheme than our usual (2.5.7). We can easily change the regularisation scheme through a comparison of the local terms in the triple-K integrals in two regularisation schemes. This can be done without actually evaluating the integrals: we simply use (2.A.21, 2.A.22) and/or (2.A.27) to expand out two of the three Bessel K functions in a given triple-K integral then use the formulae (2.A.51, 2.A.53, 2.A.55). In each case, we find only a finite number of terms leading to singularities.

To illustrate this, consider for example the integral $I_{2\{111\}}$. The calculations for $I_{0\{111\}}$ are essentially identical, however, due to the fact that $I_{0+\epsilon\{111\}}$ has a double pole in ϵ, the resulting expressions are much longer. The integrals in the two regularization schemes are given by

$$I_{2+\epsilon\{111\}} = \int_0^\infty \mathrm{d}x\, x^\epsilon \left[1 + O(x^2)\right] p_3 K_1(p_3 x) \tag{2.6.25}$$
$$I_{2+\epsilon\{1+\epsilon,1+\epsilon,1+\epsilon\}} = \int_0^\infty \mathrm{d}x\, x^{-\epsilon} \left[(4p_3)^\epsilon \Gamma^2(1+\epsilon) + O(x^2)\right] p_3^{1+\epsilon} K_{1+\epsilon}(p_3 x)$$

Using the expansion of the Bessel K functions we find that the two integrals differ by local terms:

$$I_{2+\epsilon\{111\}} = I_{2+\epsilon\{1+\epsilon,1+\epsilon,1+\epsilon\}} + \frac{3}{2\epsilon} + \frac{3}{2}(-\gamma_E + \log 2) + O(\epsilon). \tag{2.6.26}$$

We can obtain $I_{0+\epsilon\{-1-1-1\}}$ from $I_{0+\epsilon\{-1+\epsilon,-1+\epsilon,-1+\epsilon\}}$ in a similar way, and then use (2.6.17). The exact analytic expression for $I_{0\{111\}}$ is given in appendix 2.A.5. The only special functions appearing in the result are dilogarithms.

2.7. Worked example: $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$

Now that our general method is complete, in this section we present a full worked example, the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ correlation function. Here we will take J^μ to be a conserved $U(1)$ current; more general results are listed in chapter 3. This correlator provides a useful test case as, while more complex than the $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$ correlator we used to illustrate the method in earlier sections, it is nonetheless simpler than correlators with more stress-energy tensors.

We will also discuss the complete evaluation of all integrals in both $d = 3$ and $d = 4$ and present a concrete model, free fermions, where these correlators can be explicitly computed by standard Feynman diagrams. These results provide a nontrivial consistency check on our method.

2.7.1. Primary conformal Ward identities

We start with the analysis of primary CWIs for the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ correlation function in general Euclidean dimension d. For the decomposition of the transverse-traceless part of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ we follow the analysis of section 2.3.1. The decomposition consists of four form factors,

$$\begin{aligned} &\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2}(\boldsymbol{p}_2) j^{\mu_3}(\boldsymbol{p}_3) \rangle\!\rangle \\ &\quad = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \Big[A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + A_2 \delta^{\alpha_2\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} \\ &\qquad + A_3 \delta^{\alpha_1\alpha_2} p_2^{\beta_1} p_1^{\alpha_3} + A_3(p_2 \leftrightarrow p_3) \delta^{\alpha_1\alpha_3} p_2^{\beta_1} p_3^{\alpha_2} \\ &\qquad + A_4 \delta^{\alpha_1\alpha_3} \delta^{\alpha_2\beta_1} \Big] . \end{aligned} \tag{2.7.1}$$

Here, $p_2 \leftrightarrow p_3$ denotes exchange of the arguments p_2 and p_3, *i.e.*, $A_3(p_2 \leftrightarrow p_3) = A_3(p_1, p_3, p_2)$. If on the other hand no arguments are given then the standard ordering is assumed, *i.e.*, $A_3 = A_3(p_1, p_2, p_3)$. Note that the form factors A_1, A_2 and A_4 are symmetric under $p_2 \leftrightarrow p_3$,

$$A_j(p_1, p_3, p_2) = A_j(p_1, p_2, p_3), \qquad j \in \{1, 2, 4\}, \tag{2.7.2}$$

while the form factor A_3 does not exhibit any symmetry properties.

Next, the primary CWIs can be extracted by means of the procedure described in section 2.4.3. These CWIs are

$$\begin{aligned} &\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0, \\ &\mathrm{K}_{12}\, A_2 = -2A_1, && \mathrm{K}_{13}\, A_2 = -2A_1, \\ &\mathrm{K}_{12}\, A_3 = 0, && \mathrm{K}_{13}\, A_3 = 4A_1, \\ &\mathrm{K}_{12}\, A_4 = 2A_3, && \mathrm{K}_{13}\, A_4 = 2A_3(p_2 \leftrightarrow p_3). \end{aligned} \tag{2.7.3}$$

The solution follows from the analysis of section 2.5.2,

$$\begin{aligned} A_1 &= \alpha_1 J_{4\{000\}}, \\ A_2 &= \alpha_1 J_{3\{100\}} + \alpha_2 J_{2\{000\}}, \\ A_3 &= 2\alpha_1 J_{3\{001\}} + \alpha_3 J_{2\{000\}}, \\ A_4 &= 2\alpha_1 J_{2\{011\}} + \alpha_3 \left(J_{1\{010\}} + J_{1\{001\}} \right) + \alpha_4 J_{0\{000\}}. \end{aligned} \tag{2.7.4}$$

2.7.2. Evaluation of secondary conformal Ward identities

The independent secondary CWIs for $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ are listed in chapter 3 and read

$$\begin{aligned} &(*)\ \mathrm{L}_{2,2}\, A_1 + \mathrm{R}_2 \left[A_3 - A_3(p_2 \leftrightarrow p_3) \right] \\ &\qquad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\!\rangle, \quad (2.7.5) \\ &\mathrm{L}'_{1,2}\, A_1 + 2\,\mathrm{R}'_1 \left[A_3 - A_2 \right] \\ &\qquad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_1^{\mu_3} \text{ in } p_{2\mu_2} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\!\rangle, \quad (2.7.6) \\ &\mathrm{L}_{2,0}\, A_2 - p_1^2 \left[A_3 - A_3(p_2 \leftrightarrow p_3) \right] \\ &\qquad = 2d \cdot \text{coefficient of } \delta^{\mu_2\mu_3} p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\!\rangle, \quad (2.7.7) \end{aligned}$$

$$\begin{aligned} &\mathrm{L}_{2,2}\, A_3 - 2\,\mathrm{R}_2\, A_4 \\ &\qquad = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{2.7.8}$$

where L and R operators are defined in (2.4.27) and (2.4.28). They can be obtained by the procedure outlined in section 2.4.3. Note that there are four primary constants and four secondary CWIs. As some of the secondary CWIs are trivially satisfied, however, not all four of the primary constants are fixed, as we expect from the position space analysis [22]. Secondary CWIs that are trivially satisfied are denoted by asterisks in chapter 3 (for example (2.7.5) above is of this type).

Before solving the secondary CWIs, we must simplify the semi-local terms appearing on their right-hand sides. Differentiating (1.3.33, 1.3.34, 1.3.42) we find the following transverse and trace Ward identities,

$$\begin{aligned} &p_1^{\nu_1} \langle\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = \\ &\quad = p_1^{\nu_1} \langle\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2}(\boldsymbol{p}_2) \rangle\rangle + p_1^{\nu_1} \langle\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle \\ &\qquad - p_3^{\mu_1} \langle\langle J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(-\boldsymbol{p}_2) \rangle\rangle - p_2^{\mu_1} \langle\langle J^{\mu_2}(\boldsymbol{p}_3) J^{\mu_3}(-\boldsymbol{p}_3) \rangle\rangle \\ &\qquad + \delta^{\mu_3}_{\mu_1} p_{3\alpha} \langle\langle J^{\mu_2}(\boldsymbol{p}_2) J^{\alpha}(-\boldsymbol{p}_2) \rangle\rangle + \delta^{\mu_2}_{\mu_1} p_{2\alpha} \langle\langle J^{\alpha}(\boldsymbol{p}_3) J^{\mu_3}(-\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{2.7.9}$$

$$\begin{aligned} &p_{2\mu_2} \langle\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = \\ &\quad = 2 p_{2\mu_2} \langle\langle \frac{\delta J^{\mu_2}}{\delta g^{\mu_1\nu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle + \delta_{\mu_1\nu_1} p_{1\alpha} \langle\langle J^{\alpha}(\boldsymbol{p}_3) J^{\mu_3}(-\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{2.7.10}$$

$$\langle\langle T(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = \langle\langle \frac{\delta T}{\delta A_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle + \langle\langle \frac{\delta T}{\delta A_{\mu_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2}(\boldsymbol{p}_2) \rangle\rangle. \tag{2.7.11}$$

In the next section we will extract algebraic equations between the primary constants by taking the zero-momentum limit $p_3 \to 0$. The details of this procedure are described in section 2.5.3. We will find that in the zero-momentum limit the right-hand sides of the secondary CWIs (2.7.5) - (2.7.7) are given by

$$\lim_{\substack{p_3 \to 0 \\ p_1 = p_2 = p}} \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_1^{\mu_3} \text{ in } p_{2\mu_2} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = 0, \tag{2.7.12}$$

$$\lim_{\substack{p_3 \to 0 \\ p_1 = p_2 = p}} \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = 0, \tag{2.7.13}$$

$$\begin{aligned} &\lim_{\substack{p_3 \to 0 \\ p_1 = p_2 = p}} \text{coefficient of } \delta^{\mu_2\mu_3} p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle = \\ &\qquad\qquad = \text{coefficient of } \delta^{\mu_2\mu_3} \text{ in } \langle\langle J^{\mu_2}(\boldsymbol{p}) J^{\mu_3}(-\boldsymbol{p}) \rangle\rangle. \end{aligned} \tag{2.7.14}$$

Let us start with the first result (2.7.12). Due to conformal invariance, the only operators in $\delta J^{\mu_2} / \delta g^{\mu_1\nu_1}$ that can give a non-vanishing result under the expectation value with the current is another current J^{μ}. In general, the descendants of the current can also give a non-vanishing 2-point function with another current. In this case, however, the dilatation degree of $\delta J^{\mu_2}/\delta g^{\mu_1\nu_1}$ is $d-1$, and so descendants cannot appear. The most general form of the functional derivative term is therefore

$$\frac{\delta J^{\mu_2}}{\delta g^{\mu_1\nu_1}} = c_1 \delta_{\mu_1\nu_1} J^{\mu_2} + c_2 \delta^{\mu_2}_{(\mu_1} J_{\nu_1)} + \ldots \tag{2.7.15}$$

where c_1 and c_2 are numerical constants and the omitted terms may contain operators from different conformal families to that of J^μ. The 2-point function then reads

$$\langle\!\langle \frac{\delta J^{\mu_2}}{\delta g^{\mu_1\nu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) J^{\mu_3}(\boldsymbol{p}_3)\rangle\!\rangle = \left[c_1 \delta_{\mu_1\nu_1}\delta^{\mu_2}_{\alpha} + c_2 \delta^{\mu_2}_{(\mu_1}\delta_{\nu_1)\alpha} \right] \langle\!\langle J^\alpha(\boldsymbol{p}_3) J^{\mu_3}(-\boldsymbol{p}_3)\rangle\!\rangle. \tag{2.7.16}$$

In the limit $p_3 \to 0$, however, the 2-point function vanishes, since it behaves as p_3^{d-2} and $d > 2$. The same argument works for the second term in (2.7.10) and so (2.7.12) also vanishes.

Let us now establish the remaining formulae (2.7.13) and (2.7.14). Following the same argument for the limit $p_3 \to 0$, we can restrict consideration to the following terms in (2.7.9)

$$\begin{aligned} p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2}(\boldsymbol{p}_2)\rangle\!\rangle &- p_3^{\mu_1} \langle\!\langle J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(-\boldsymbol{p}_2)\rangle\!\rangle \\ &+ \delta^{\mu_3}_{\mu_1} p_{3\alpha} \langle\!\langle J^{\mu_2}(\boldsymbol{p}_2) J^{\alpha}(-\boldsymbol{p}_2)\rangle\!\rangle. \end{aligned} \tag{2.7.17}$$

Using the representation (2.5.55) it is straightforward to expand the last two terms. As usual, we must use the convention (2.3.4) for the momenta associated with Lorentz indices, leading to the right-hand sides of (2.7.13) and (2.7.14). The remaining task is then to show that there are no contributions from the first term with the functional derivative.

Since the dimension of the stress-energy tensor is d, while that of the conserved current is $d-1$ and that of the source A_μ is 1, the only possible contributions to the first term in (2.7.9) are

$$T_{\mu\nu} = c_3 \left[A_\mu J_\nu + A_\nu J_\mu \right] + \ldots \tag{2.7.18}$$

where c_3 is a numerical constant and the omitted terms do not contain the current or its descendants. This definition of c_3 applies if the J^μ operator is the unique spin-1 conserved current in theory. If not, we can instead define the constant c_3 through the 2-point function

$$\langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3)\rangle\!\rangle = 2 c_3 \delta^{\mu_2}_{(\mu_1} \langle\!\langle J_{\nu_1)}(\boldsymbol{p}_3) J^{\mu_3}(-\boldsymbol{p}_3)\rangle\!\rangle. \tag{2.7.19}$$

After taking the functional derivative one finds that tensors $p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3}$ and $\delta^{\mu_2\mu_3}$ are absent in (2.7.9).

Finally, with the definition of the c_3 constant as in (2.7.19), the same method can be applied to work out the zero-momentum limit of the right-hand side of the final secondary CWI (2.7.8), yielding the result

$$\lim_{\substack{p_3 \to 0 \\ p_1 = p_2 = p}} \text{coefficient of } \delta^{\mu_1\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\!\rangle =$$
$$= c_3 \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \text{ in } \langle\!\langle J^{\mu_1}(\boldsymbol{p}) J^{\mu_2}(-\boldsymbol{p}) \rangle\!\rangle. \qquad (2.7.20)$$

2.7.3. Solutions to secondary conformal Ward identities

Our goal now is to analyse the additional constraints imposed by the secondary CWIs (2.7.5) - (2.7.8) on the solution (2.7.4) of the primary CWIs. We proceed as in sections 2.5.3 and 2.5.3. First, we use solutions (2.7.4) and take the zero-momentum limit $p_3 \to 0$. From (2.7.6) and (2.7.7), we then derive the following equations for the primary constants

$$\alpha_2 = \alpha_3, \qquad (2.7.21)$$

$$d\alpha_1 + \alpha_2 = \frac{2^{3-\frac{d}{2}} s_d c_J}{\Gamma\left(\frac{d}{2}\right)}, \qquad (2.7.22)$$

where

$$s_d = \begin{cases} \frac{1}{\pi}(-1)^{\frac{d-1}{2}} & \text{if } d = 3, 5, 7, \dots, \\ (-1)^{\frac{d}{2}-1} & \text{if } d = 4, 6, 8, \dots \end{cases} \qquad (2.7.23)$$

and c_J encodes the normalisation of the 2-point function as given in (2.5.55). For the right-hand sides we used (2.7.12) - (2.7.14)

The situation is more interesting for the last of the secondary CWIs (2.7.8). Assuming d to be odd, we find this equation exhibits a singularity and the $\epsilon \to 0$ limit cannot be taken. Expanding in powers of ϵ, the zero-momentum limit of the left-hand side is

$$\left(\frac{2}{\epsilon} - \log p^2 \right) \cdot \frac{2^{\frac{d}{2}-3} d p^{d-2} \pi}{\sin\left(\frac{d\pi}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)} \left[\alpha_4 + (-2 + d)\alpha_3 \right] + \dots \qquad (2.7.24)$$

where the omitted terms are of order ϵ^0. Since the right-hand side is finite and does not contain logarithms, (2.7.8) forces

$$\alpha_4 = (2 - d)\alpha_3 + O(\epsilon). \qquad (2.7.25)$$

With this value of α_4, the left-hand side of (2.7.8) is finite, but does not necessarily match the right-hand side. To solve this problem we must consider a first-order correction to α_4, *i.e.*, we write

$$\alpha_4 = (2 - d)\alpha_3 + \epsilon \alpha_4^{(1)}. \qquad (2.7.26)$$

Using (2.7.20), the equation for $\alpha_4^{(1)}$ leads to

$$2d(d-2)\alpha_1 + d\alpha_3 + 2\alpha_4^{(1)} = \frac{2^{5-\frac{d}{2}} s_d c_3 c_J}{\Gamma\left(\frac{d}{2}-1\right)}. \tag{2.7.27}$$

This equation is valid for even d as well: in this case we find that the left-hand side of (2.7.8) contains a $1/\epsilon^2$ singularity. As there is no corresponding singularity on the right-hand side we recover the same conditions (2.7.25) and (2.7.27) as above.

Summarising, we found that the primary constants in the solution to the primary CWIs (2.7.4) satisfy

$$\begin{aligned}
&\alpha_3 = \alpha_2, && \alpha_4 = -(d-2)\alpha_2 \\
&d\alpha_1 + \alpha_2 = \frac{2^{3-\frac{d}{2}} s_d c_J}{\Gamma\left(\frac{d}{2}\right)}, && \\
&2d(d-2)\alpha_1 + d\alpha_2 + 2\alpha_4^{(1)} = \frac{2^{5-\frac{d}{2}} s_d c_3 c_J}{\Gamma\left(\frac{d}{2}-1\right)}. &&
\end{aligned} \tag{2.7.28}$$

Through an analysis similar to that of section 2.5.3, we find there are no further constraints on the primary constants.

Our solution of the primary and secondary CWIs above depends on one undetermined primary constant as well as two different 2-point function normalisations. This result is in fact consistent with the position space result of [22] (which involves only a single 2-point function normalisation) by virtue of our different definition for the 3-point function, namely

$$\begin{aligned}
&\langle T_{\mu_1\nu_1}(\boldsymbol{x}_1) J^{\mu_2}(\boldsymbol{x}_2) J^{\mu_3}(\boldsymbol{x}_3)\rangle = \\
&\quad = \frac{-1}{\sqrt{g(\boldsymbol{x}_3)}} \frac{\delta}{\delta A_{\mu_3}(\boldsymbol{x}_3)} \frac{-1}{\sqrt{g(\boldsymbol{x}_2)}} \frac{\delta}{\delta A_{\mu_2}(\boldsymbol{x}_2)} \frac{-2}{\sqrt{g(\boldsymbol{x}_1)}} \frac{\delta}{\delta g^{\mu_1\nu_1}(\boldsymbol{x}_1)} Z[g^{\mu\nu}, A_\rho] \\
&\qquad + \langle \frac{\delta T_{\mu_1\nu_1}(\boldsymbol{x}_1)}{\delta A_{\mu_2}(\boldsymbol{x}_2)} J^{\mu_3}(\boldsymbol{x}_3)\rangle + \langle \frac{\delta T_{\mu_1\nu_1}(\boldsymbol{x}_1)}{\delta A_{\mu_3}(\boldsymbol{x}_3)} J^{\mu_2}(\boldsymbol{x}_2)\rangle.
\end{aligned} \tag{2.7.29}$$

In [22] (and similarly [47, 49]) the semi-local terms on the right-hand side of this formula are absorbed into the definition of the $\langle T_{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle$ correlator: it is these semi-local terms that are responsible, via (2.7.19), for the dependence of our solution on the additional normalisation constant c_3.

2.7.4. General form of $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle$ in $d = 3$

Let us now focus on the special case of $d = 3$. Examining the form of the solution (2.7.4) to the primary CWIs, we find that all triple-K integrals can be evaluated in terms of elementary integrals using (2.A.24). If an integral diverges, we use the regularisation (2.5.7). In this way, we find

$$\begin{aligned}
J_{4\{000\}} &= I_{\frac{9}{2}\{\frac{3}{2}\frac{1}{2}\frac{1}{2}\}} = 2\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{4p_1+p_2+p_3}{(p_1+p_2+p_3)^4},\\
J_{3\{100\}} &= I_{\frac{7}{2}\{\frac{5}{2}\frac{1}{2}\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{9(p_1p_2+p_1p_3)+6p_2p_3+8p_1^2+3(p_2^2+p_3^2)}{(p_1+p_2+p_3)^3},\\
J_{2\{000\}} &= I_{\frac{5}{2}\{\frac{3}{2}\frac{1}{2}\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{2p_1+p_2+p_3}{(p_1+p_2+p_3)^2},\\
J_{2\{011\}} &= I_{\frac{5}{2}\{\frac{3}{2}\frac{3}{2}\frac{3}{2}\}} = -\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{1}{(p_1+p_2+p_3)^2}\left[2p_1p_2p_3+p_1^3+p_2^3+p_3^3\right.\\
&\qquad\left.+2(p_1^2p_2+p_1p_2^2+p_1p_3^2+p_3p_1^2+p_2p_3^2+p_3p_2^2)\right],\\
J_{1+\epsilon\{010\}} &= I_{\frac{3}{2}+\epsilon\{\frac{3}{2}\frac{3}{2}\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\left[-\frac{p_3}{\epsilon}+p_3\log(p_1+p_2+p_3)\right.\\
&\quad\left.+\frac{-p_1p_2+(\gamma_E-2)(p_1p_3+p_2p_3)-p_1^2-p_2^2+(\gamma_E-1)p_3^2}{p_1+p_2+p_3}+O(\epsilon)\right],\\
J_{0+\epsilon\{000\}} &= I_{\frac{1}{2}+\epsilon\{\frac{3}{2}\frac{1}{2}\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\left[-\frac{p_2+p_3}{\epsilon}+(p_2+p_3)\log(p_1+p_2+p_3)\right.\\
&\qquad\left.+(\gamma_E-1)(p_2+p_3)-p_1+O(\epsilon)\right],
\end{aligned} \tag{2.7.30}$$

with similar integrals following from the permutation formula (2.5.11).

Applying the secondary CWIs (2.7.28) we then obtain the final result

$$\begin{aligned}
A_1 &= \alpha_1\frac{2(4p_1+p_2+p_3)}{(p_1+p_2+p_3)^4},\\
A_2 &= \frac{2\alpha_1p_1^2}{(p_1+p_2+p_3)^3}-\frac{2(2p_1+p_2+p_3)}{(p_1+p_2+p_3)^2}c_J,\\
A_3 &= \frac{\alpha_1}{(p_1+p_2+p_3)^3}\left[-2p_1^2-p_2^2+p_3^2-3p_1p_2+3p_1p_3\right]\\
&\quad-\frac{2(2p_1+p_2+p_3)}{(p_1+p_2+p_3)^2}c_J,\\
A_4 &= \alpha_1\frac{(p_1+p_2-p_3)(p_1-p_2+p_3)(2p_1+p_2+p_3)}{2(p_1+p_2+p_3)^2}\\
&\qquad+\left(\frac{2p_1^2}{p_1+p_2+p_3}-p_2-p_3\right)c_J+2(p_2+p_3)c_3c_J.
\end{aligned} \tag{2.7.31}$$

In these results we rescaled the coefficient α_1 according to $\alpha_1(\pi/2)^{3/2}\mapsto\alpha_1$, so as to remove the awkward factor of $(\pi/2)^{3/2}$.

The form factors build the transverse-traceless part of the correlation function according to (2.7.1). The full correlation function can then be recovered by means of (2.3.17) and (2.3.18). Using the transverse and trace Ward identities (2.7.9, 2.7.10, 2.7.11), we find

$$\begin{aligned}
&\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3)\rangle\!\rangle = \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2}(\boldsymbol{p}_2) j^{\mu_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad + \left[2\mathscr{T}^{\mu_1\nu_1}_{\alpha}(\boldsymbol{p}_1)\pi^{\mu_3[\alpha}(\boldsymbol{p}_3) p_3^{\beta]} + \frac{p_3^{\mu_3}}{p_3^2}\delta^{\mu_1\nu_1} p_{1\beta} \right] \langle\!\langle J^{\mu_2}(\boldsymbol{p}_2) J_\beta(-\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1}\delta^{\alpha_1\beta_1} \right] \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta A_{\alpha_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2}(\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \frac{2 p_3^{\mu_3} p_{3\alpha_3}}{p_3^2}\delta^{\mu_1\alpha_1}\delta^{\nu_1\beta_1} \langle\!\langle \frac{\delta J^{\alpha_3}}{\delta g^{\alpha_1\beta_1}}(\boldsymbol{p}_3, \boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \text{everything with } (\boldsymbol{p}_2, \mu_2) \leftrightarrow (\boldsymbol{p}_3, \mu_3), \qquad (2.7.32)
\end{aligned}$$

where $\mathscr{T}^{\mu\nu}_{\alpha}$ was given in (2.3.19). Here we assume no scale anomalies are present: if anomalies occur, the additional ultralocal contribution (2.8.16) should be added to (2.7.32), as we will discuss in section 2.8.

The result (2.7.32) is the most general explicit expression for the $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\!\rangle$ correlation function in the momentum space. As we can see, it depends on one undetermined primary constant plus the normalisations of the 2-point functions.

2.7.5. Free fermions in $d = 3$

As a cross-check on our calculations we now consider free fermions in $d = 3$ Euclidean dimensions given by the action

$$S = \int \mathrm{d}^3\boldsymbol{x}\, e \left[\bar{\psi} e^{\mu}_{a} \gamma^a \overleftrightarrow{D}_\mu \psi \right], \qquad (2.7.33)$$

where

$$D_\mu = \nabla_\mu - \mathrm{i} A_\mu, \qquad \nabla_\mu = \partial_\mu - \frac{\mathrm{i}}{2}\omega^{ab}_{\mu}\Sigma_{ab}, \qquad (2.7.34)$$

and ω^{ab}_μ is the spin connection

$$\omega^{ab}_{\mu} = e^a_\nu \partial_\mu e^{\nu b} + e^a_\nu e^{\sigma b}\Gamma^{\nu}{}_{\sigma\mu}, \qquad \Sigma^{ab} = \frac{\mathrm{i}}{4}[\gamma^a, \gamma^b]. \qquad (2.7.35)$$

Here $\Gamma^{\nu}{}_{\sigma\mu}$ is the Christoffel symbol associated with the metric $g_{\mu\nu}$, while e^a_μ are vielbeins satisfying $e^a_\mu e_{\nu a} = g_{\mu\nu}$ and the gamma matrices γ^a satisfy $\gamma^\mu = e^\mu_a \gamma^a$. On flat space, we then have $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$. In $d = 3$, the spin-$\frac{1}{2}$ representation of the group $SO(3)$ is 2-dimensional and $\mathrm{Tr}(\gamma^a\gamma^b) = -2\delta^{ab}$.

Notice that the gauge field A_μ is treated as a source for the conserved current and is not a degree of freedom. The stress-energy tensor and the conserved current in the presence of the sources are

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \bar{\psi}\gamma_{(\mu} \overleftrightarrow{D}_{\nu)} \psi - g_{\mu\nu}\bar{\psi}\gamma^{\alpha} \overleftrightarrow{D}_{\alpha} \psi, \tag{2.7.36}$$

$$J^{\mu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A^{\mu}} = -\mathrm{i}\bar{\psi}\gamma^{\mu}\psi. \tag{2.7.37}$$

In this case the current is associated with the $U(1)$ symmetry, therefore we omit the group indices on J^{μ}. By direct calculation we find

$$\langle\!\langle J^{\mu}(\boldsymbol{p}) J^{\nu}(-\boldsymbol{p})\rangle\!\rangle = -\frac{1}{16} p \pi^{\mu\nu}(\boldsymbol{p}), \tag{2.7.38}$$

$$\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}) T^{\mu_2\nu_2}(-\boldsymbol{p})\rangle\!\rangle = \frac{1}{128} p^3 \Pi^{\mu_1\nu_1\mu_2\nu_2}(\boldsymbol{p}). \tag{2.7.39}$$

The transverse Ward identities can be obtained by differentiation of the equations (1.3.33, 1.3.34) and are listed in chapter 3. Some terms of the terms involve functional derivatives and may be evaluated directly from expressions (2.7.36, 2.7.37),

$$\frac{\delta T_{\mu\nu}(\boldsymbol{x})}{\delta A_{\rho}(\boldsymbol{y})} = \frac{1}{2}[J_{\mu}\delta^{\rho}_{\nu} + J_{\nu}\delta^{\rho}_{\mu} - 2J^{\rho}\delta_{\mu\nu}]\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{2.7.40}$$

$$\frac{\delta J^{\mu}(\boldsymbol{x})}{\delta g^{\alpha\beta}(\boldsymbol{y})} = \frac{1}{4}[J_{\beta}\delta^{\mu}_{\alpha} + J_{\alpha}\delta^{\mu}_{\beta}]\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{2.7.41}$$

where the sources are turned off after the derivative is taken. All together, for this particular CFT we find

$$c_J = -\frac{1}{16}, \qquad c_T = \frac{1}{128}, \qquad c_3 = \frac{1}{2}. \tag{2.7.42}$$

where the 2-point function normalisations c_J and c_T, and the constant c_3, are as defined in (2.5.55), (2.5.56) and (2.7.19) respectively.

The 3-point function can be calculated by the usual Feynman rules. Using the results of section 2.3.3, one finds

$$\begin{aligned}
A_1 &= -\frac{4p_1 + p_2 + p_3}{12\,(p_1+p_2+p_3)^4}, \\
A_2 &= \frac{9(p_1p_2 + p_1p_3) + 6p_2p_3 + 4p_1^2 + 3(p_2^2 + p_3^2)}{24\,(p_1+p_2+p_3)^3}, \\
A_3 &= \frac{6p_1p_2 + 3p_1p_3 + 3p_2p_3 + 4p_1^2 + 2p_2^2 + p_3^2}{12\,(p_1+p_2+p_3)^3}, \\
A_4 &= -\frac{4p_1p_2p_3 + 7(p_1^2p_2 + p_1^2p_3) - 2(p_1p_2^2 + p_1p_3^2) + p_2p_3^2 + p_3p_2^2 + 8p_1^3 - (p_2^3 + p_3^3)}{48\,(p_1+p_2+p_3)^2}.
\end{aligned} \tag{2.7.43}$$

The form factors A_j are defined in the decomposition (2.7.1).

We can compare this result directly with the solution (2.7.31). Since we know the 2-point function normalisations (2.7.42) there is only one undetermined constant, α_1. The solution (2.7.43) then fits perfectly with $\alpha_1 = -\frac{1}{24}$. In fact, the secondary Ward identities provide quite a robust check on the standard QFT calculation of the 3-point function: for example, a mistake leading to the overall rescaling of all form factors in (2.7.43) by some factor would immediately lead to an inconsistency with the 2-point function normalisation constants (2.7.42).

2.7.6. General form of $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$ in $d = 4$

Using the reduction scheme of section 2.6, we can write down the most general form of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ in $d = 4$. Starting from the solutions (2.7.4) and (2.7.28) for the primary and secondary CWIs, using the regularisation scheme (2.5.7) and the relations in table 2.1, page 94, after removing divergences we find

$$\begin{aligned}
A_1 &= \alpha_1 I_{5\{211\}}, \\
A_2 &= -\left(2c_J + \alpha_1 p_1 \frac{\partial}{\partial p_1} \right) I^{(0)}_{3\{211\}}, \\
A_3 &= -2 \left(c_J + \alpha_1 p_3 \frac{\partial}{\partial p_3} \right) I^{(0)}_{3\{211\}}, \\
A_4 &= 2c_J \left[-2 + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} \right] I^{(0)}_{1\{211\}} + 2\alpha_1 p_2 p_3 \frac{\partial^2}{\partial p_2 \partial p_3} I^{(0)}_{1\{211\}} \\
&\quad + 4(c_J - c_3 c_J) \left[p_2^2 \log p_2 + p_3^2 \log p_3 - (p_2^2 + p_3^2) \left(\tfrac{1}{2} - \gamma_E + \log 2 \right) - \tfrac{1}{2} p_1^2 \right],
\end{aligned} \tag{2.7.44}$$

where $I^{(0)}_{\alpha\{\beta_j\}}$ denotes the the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$ in ϵ. This is the exact, finite and fully renormalised result for the transverse-traceless part of the correlation function. The triple-K integrals appearing can be straightforwardly evaluated using the reduction scheme in table 2.1 on page 94 starting from the master integral $I_{0\{111\}}$ given in appendix 2.A.5. (We have nonetheless retained the above form for its compactness.) The transverse-traceless part of the correlation function can be recovered using (2.7.1), as in the case of $d = 3$. The full 3-point function can then be reconstructed by means of (2.7.32). For $d = 4$ an anomalous contribution appears, however, due to the addition of counterterms required to render the 3-point function finite that are not of the form assumed in section 2.4.4. We will return to the treatment of anomalies shortly, in section 2.8.

2.7.7. Free fermions in $d = 4$

The momentum space $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ correlation function for free fermions in $d = 4$ dimensions was discussed in [47, 49]. In this section we will show how to simplify these calculations considerably using our method. We already know that the solution to the primary CWIs is given by (2.7.4). We therefore need to calculate explicitly only one primary constant, say α_1, since the remaining constants are determined by the secondary CWIs (2.7.28).

To evaluate α_1 we can use the standard Feynman parametrisation, which gives

$$\text{coeff. of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_1) \rangle\!\rangle = -\frac{2}{\pi^2} \int_{[0,1]^3} \mathrm{d}X \, \frac{x_1 x_2 x_3^2}{D}, \tag{2.7.45}$$

where

$$\mathrm{d}X = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \delta(x_1 + x_2 + x_3 - 1), \tag{2.7.46}$$

$$D = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2. \tag{2.7.47}$$

On the other hand, using (4.2.23), we find

$$J_{4\{000\}} = I_{5\{211\}} = 96 \cdot \int_{[0,1]^3} \mathrm{d}X \, \frac{x_1 x_2 x_3^2}{D} \tag{2.7.48}$$

and hence

$$\alpha_1 = -\frac{1}{48\pi^2}. \tag{2.7.49}$$

The 2-point function normalisation c_J and the constant c_3 are

$$c_J = -\frac{1}{12\pi^2}, \qquad c_3 = \frac{1}{2}, \tag{2.7.50}$$

and so, from (2.7.28), we find

$$\alpha_2 = \alpha_3 = \frac{1}{4\pi^2}, \qquad \alpha_4 = -\frac{1}{2\pi^2} - \frac{\epsilon}{6\pi^2}. \tag{2.7.51}$$

With these primary constants the expressions (2.7.44) represent a complete and concise solution to the transverse-traceless part of the $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ correlation function for free fermions in $d = 4$. The full 3-point function can be recovered as in the $d = 3$ case via (2.7.1) and (2.7.32).

The above solution can be confirmed by direct calculations. For free field theory we computed the entire $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ correlator using Passarino-Veltman reduction [51]. The coefficients of the appropriate tensors were then extracted according to section 2.3.3 and the result compared with the Feynman parametrised integrals. Exact agreement was found, both for the functional form of (2.7.4) and for the constants in (2.7.51).

Our result can also be compared with those of [49]. Up to a multiplicative factor, we find that

$$\text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_1^{\mu_3} \sim \int_{[0,1]^3} \mathrm{d}X \, \frac{c_9(x_1, x_2) - 2c_8(x_1, x_2) + c_7(x_1, x_2)}{D}$$
$$= \int_{[0,1]^3} \mathrm{d}X \, \frac{4x_1 x_2 x_3^2}{D}, \tag{2.7.52}$$

where the c_j polynomials are defined in the table 3 of [49]. This expression agrees with (2.7.48) and indeed represents the form factor A_1.

2.8. Divergences and anomalies in $d = 4$

We discussed in previous sections the solution of the conformal Ward identities and we have seen that in certain cases the triple-K integrals diverge and need to be regularised and renormalised. These infinities should be removed by means of local covariant counterterms. It turns out, however, that in some cases the counterterms break some of the symmetries and this leads to anomalies. We have already encountered this issue when discussing the 3-point function of scalar operators in section 2.2. There, we saw the triple-K integral corresponding to the 3-point function of a dimension two operator in $d = 3$ diverges and the infinity can be removed by adding a local counterterm that is cubic in the source of the operator. The counterterm however is not scale invariant and this implies a trace anomaly. The anomaly then implies that the 4-point function of the stress-energy tensor with three scalar operators contains an ultra-local term that is not scheme-dependent because it is fixed unambiguously by the the anomalous Ward identity. Recall also that finite local counterterms are related to/parametrise scheme-dependence. In this case the same counterterm but with a finite coefficient is related to scheme-dependence.

In this section we would like to like to extend this discussion to correlators of the stress-energy tensor and of symmetry currents. For concreteness, we will focus on the case $d = 4$ but the discussion generalises to all dimensions and/or other operators. In particular, we will analyse the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ and $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\rangle$ correlators. We will show that the divergences in the solutions for the transverse-traceless parts are cancelled by counterterms. These counterterms break scale invariance and lead to trace anomalies. The anomalies that originate from infinities in 2-point functions lead to scheme-independent ultra-local terms in 3-point functions, while those originating in 3-point functions lead to ulta-local scheme-independent terms in 4-point functions. As in the discussion above, the counterterms but with finite coefficients are related to/parametrise scheme-dependence of these correlators.

2.8.1. Counterterms and anomalies

In $d = 4$ the following counterterms can be introduced

$$S = S_{\mathrm{CFT}}[g^{\mu\nu}, A^a_\mu] + \int \mathrm{d}^4\boldsymbol{x}\sqrt{g}\left[\frac{\kappa_0}{4} F^a_{\mu\nu}F^{\mu\nu a} + a_0 E_4 + c_0 W^2 + b_0 R^2 + \tilde{b}_0 \Box R\right], \tag{2.8.1}$$

where E_4 is Euler density and W^2 is the square of Weyl tensor for the metric $g_{\mu\nu}$,

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{2.8.2}$$
$$W^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2. \tag{2.8.3}$$

The regularisation scheme is a dimensional regularisation with $d = 4 - \epsilon$. Our conventions for the Riemann and Ricci tensors follow [70], see (5.1.1) and (5.1.2). The constants in the counterterm action are functions of ϵ and are typically divergent. These divergences are required to cancel the corresponding singularities in the regularised solutions of the primary and secondary CWIs so that a finite limit exists as we send $\epsilon \to 0$. The counterterms (2.8.1) also necessarily contribute finite pieces, however, leading to trace anomalies. By taking a functional derivative and using the results of [71] one finds the following anomalous contribution to the trace Ward identity (1.3.42)

$$\langle T \rangle = \epsilon\left[\frac{\kappa_0}{4}F^a_{\mu\nu}F^{\mu\nu a} + a_0\left(E_4 + \frac{2}{3}\Box R\right) + c_0 W^2 + 12 b_0\sqrt{g}\Box R\right]. \tag{2.8.4}$$

With the usual representation for the anomalous trace of the stress-energy tensor

$$\langle T \rangle = \frac{\kappa}{4}F^a_{\mu\nu}F^{\mu\nu a} + aE_4 + cW^2 + b\Box R, \tag{2.8.5}$$

we find the anomalies

$$\kappa = \kappa_0^{(-1)}, \qquad a = a_0^{(-1)}, \qquad c = c_0^{(-1)}, \tag{2.8.6}$$

where $(\ldots)^{(-1)}$ denotes the term of order ϵ^{-1} in the series expansion of the coupling constant in the regulator.

2.8.2. $\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle$

For the $\langle\!\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle\!\rangle$ correlator only the second term in (2.8.1) contributes. The value of the constant κ_0 is fixed by a renormalisation of the 2-point function $\langle\!\langle J^\mu J^\nu\rangle\!\rangle$. Using the dimensional regularisation $d = 4 - \epsilon$, we find the 2-point function

$$\langle\!\langle J^\mu(\boldsymbol{p})J^\nu(-\boldsymbol{p})\rangle\!\rangle_{\mathrm{reg}} = \pi^{\mu\nu}(\boldsymbol{p})p^2\left(\frac{2c_J}{\epsilon} - c_J\log p^2 - \kappa_0\right). \tag{2.8.7}$$

We therefore need

$$\kappa_0 = \frac{2c_J}{\epsilon} + O(\epsilon^0) \tag{2.8.8}$$

fixing the anomaly coefficient in (2.8.5) to be

$$\kappa = \kappa_0^{(-1)} = 2c_J. \tag{2.8.9}$$

With the value of κ_0 fixed, we can check that all divergences in the 3-point function now cancel as well. This is a non-trivial check on our solutions and the singularities of the triple-K integrals. We find that the counterterm contribution to the solution (2.7.4) is

$$\begin{aligned}
A_1^{\text{anomaly}} &= O(\epsilon^0), \\
A_2^{\text{anomaly}} &= 2\kappa_0 = \frac{4c_J}{\epsilon} + O(\epsilon^0), \\
A_3^{\text{anomaly}} &= 2\kappa_0 = \frac{4c_J}{\epsilon} + O(\epsilon^0), \\
A_4^{\text{anomaly}} &= -\kappa_0(p_1^2 - p_2^2 - p_3^2) - 2c_3\kappa_0(p_2^2 + p_3^2) \\
&= -\frac{2c_J}{\epsilon}(p_1^2 - p_2^2 - p_3^2) - \frac{4c_3c_J}{\epsilon}(p_2^2 + p_3^2) + O(\epsilon^0),
\end{aligned} \tag{2.8.10}$$

where the constant c_J is defined in (2.5.55) and c_3 is defined in (2.7.19). The appearance of c_3 is due to the our definition of the 3-point function (2.7.29).

The result (2.8.10) is very constraining: in particular, the coefficients of the singular terms in the regularised solution must be multiples of c_J and not the undetermined primary constant α_1.

The divergences of the triple-K integrals entering the regularised solution can be evaluated using the method outlined in section 2.6.4, yielding

$$\begin{aligned}
I_{5\{211\}} &= \text{finite}, \\
I_{3+\epsilon\{211\}} &= \frac{2}{\epsilon} + O(\epsilon^0), \\
I_{1+\epsilon\{211\}} &= -\frac{p_2^2 + p_3^2}{\epsilon^2} + \frac{1}{\epsilon}\left[p_2^2 \log p_2 + p_3^2 \log p_3 \right. \\
&\qquad \left. + (p_2^2 + p_3^2)\left(-\tfrac{1}{2} + \gamma_E - \log 2\right) - \tfrac{1}{2}p_1^2\right].
\end{aligned} \tag{2.8.11}$$

Substituting everything into the solution (2.7.4), one can check that all singularities cancel leaving a finite result. The scheme-dependent terms arise due to the $O(\epsilon^0)$ ambiguity in the definition of κ_0 in (2.8.8). Looking at (2.8.10), we see that the scheme-dependence may only change the transverse-traceless part according to

$$\begin{aligned}
A_2 &\mapsto A_2 + 2\kappa_0^{(0)}, \\
A_3 &\mapsto A_3 + 2\kappa_0^{(0)}, \\
A_4 &\mapsto A_4 - \kappa_0^{(0)}(p_1^2 - p_2^2 - p_3^2) - 2c_3\kappa_0^{(0)}(p_2^2 + p_3^2),
\end{aligned} \tag{2.8.12}$$

where $\kappa_0^{(0)}$ is the $O(\epsilon^0)$ part of κ_0.

Finally, one can look for anomalies in the trace of the stress-energy tensor. Due to the dimensional regularisation $d = 4 - \epsilon$, the trace of the stress-energy tensor acquires a contribution from the counterterm

$$\langle T \rangle = \frac{\kappa}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \kappa = \kappa_0^{(-1)} = 2c_J. \tag{2.8.13}$$

The anomalous term contributes to the trace Ward identity, which acquires a contribution

$$\langle\langle T(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle_{\text{anomaly}} = \kappa \left[p_3^{\mu_2} p_2^{\mu_3} - \tfrac{1}{2}(p_1^2 - p_2^2 - p_3^2) \delta^{\mu_2\mu_3} \right] \tag{2.8.14}$$

modifying the form of the total 3-point function. In arbitrary dimension, the form of the anomaly is

$$\langle\langle T(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle_{\text{anomaly}} = \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \left[B_1 p_3^{\alpha_2} p_1^{\alpha_3} + B_2 \delta^{\alpha_2\alpha_3} \right] \tag{2.8.15}$$

where the form factors B_1 and B_2 are functions of the momentum magnitudes. The contribution to the full 3-point function can then be recovered using (2.7.32), yielding

$$\begin{aligned} &\langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2}(\boldsymbol{p}_2) J^{\mu_3}(\boldsymbol{p}_3) \rangle\rangle_{\text{anomaly}} = \\ &\quad = \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \left[B_1 p_3^{\alpha_2} p_1^{\alpha_3} + B_2 \delta^{\alpha_2\alpha_3} \right]. \end{aligned} \tag{2.8.16}$$

In our example (2.8.14), in $d = 4$ we find

$$B_1 = -\kappa, \qquad B_2 = -\frac{\kappa}{2}(p_1^2 - p_2^2 - p_3^2). \tag{2.8.17}$$

A list of all anomalies and their contributions to all correlators is given in appendix 2.A.7.

2.8.3. $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle$

The situation for $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\rangle$ is very similar to that for $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$. The $\Box R$ and R^2 in the action (2.8.1) can be omitted from the analysis since the former is a total derivative while the latter does not contribute to the transverse-traceless part of the correlators, and can instead be fixed by the renormalisation of the trace part.

The 2-point function in the presence of the counterterms is

$$\langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}) T^{\mu_2\nu_2}(-\boldsymbol{p}) \rangle\rangle_{\text{reg}} = \Pi^{\mu_1\nu_1\mu_2\nu_2}(\boldsymbol{p}) p^4 \left[\frac{2c_T}{\epsilon} - c_T \log p^2 - 4c_0 + O(\epsilon) \right], \tag{2.8.18}$$

where we assume that the trace part was removed by adding the appropriate R^2 term. As we can see, the transverse-traceless part depends on c_0 only and requires

$$c_0 = \frac{c_T}{2\epsilon} + O(\epsilon^0). \tag{2.8.19}$$

The anomalous contributions to the transverse-traceless part of the 3-point function are then

$$\begin{aligned}
A_1^{\text{anomaly}} &= 0, \\
A_2^{\text{anomaly}} &= -16(a_0 + c_0), \\
A_3^{\text{anomaly}} &= 8\left[c_0(p_1^2 + p_2^2) - a_0 p_3^2\right], \\
A_4^{\text{anomaly}} &= 8\left[c_0(p_1^2 + p_2^2 + 3p_3^2) - a_0(p_1^2 + p_2^2 - p_3^2)\right], \\
A_5^{\text{anomaly}} &= -4\left[a_0 J^2 + c_0(p_1^2 + p_2^2 + p_3^2)^2 + 8c_g c_0(p_1^4 + p_2^4 + p_3^4)\right],
\end{aligned} \tag{2.8.20}$$

where we used the definition of the 3-point function (1.3.20). The constant c_g is defined, in the case where $T_{\mu\nu}$ is the unique spin-2 conserved current, as

$$\frac{T_{\mu_1\nu_1}(\boldsymbol{x})}{g^{\mu_2\nu_2}(\boldsymbol{y})} = 4c_g \delta_{(\mu_1(\mu_2} T_{\nu_1)\nu_2)}(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{y}) + \ldots \tag{2.8.21}$$

or more generally

$$\langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle = 4c_g \delta_{(\mu_1(\mu_2} \langle\!\langle T_{\nu_1)\nu_2)}(\boldsymbol{p}_3) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle + \ldots \tag{2.8.22}$$

where the omitted terms do not contain tensors we have listed explicitly.

As in the case of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\!\rangle$, we can find the divergences in the regularised form factors coming from the divergences in the triple-K integrals, giving

$$\begin{aligned}
A_1^{\text{reg}} &= O(\epsilon^0), \\
A_2^{\text{reg}} &= \frac{8}{\epsilon}(16\alpha_1 + \alpha_2) + O(\epsilon^0), \\
A_3^{\text{reg}} &= -\frac{4}{\epsilon}\left[(p_1^2 + p_2^2)c_T + p_3^2(c_T - 16\alpha_1 - \alpha_2)\right] + O(\epsilon^0), \\
A_4^{\text{reg}} &= \frac{4}{\epsilon}\left[(p_1^2 + p_2^2 - p_3^2)(16\alpha_1 + \alpha_2 - 2c_T) - 4c_T p_3^2\right] + O(\epsilon^0), \\
A_5^{\text{reg}} &= \frac{2}{\epsilon}\left[(16\alpha_1 + \alpha_2)J^2 + 2c_T(1 + 4c_g)(p_1^4 + p_2^4 + p_3^4)\right] + O(\epsilon^0),
\end{aligned} \tag{2.8.23}$$

Since the value of c_0 is already fixed by (2.8.19), we can use one of the form factors, say A_2, to find

$$a_0 = \frac{1}{2\epsilon}(16\alpha_1 + \alpha_2 - c_T) + O(\epsilon^0). \tag{2.8.24}$$

The immediate conclusion is that all singularities must appear with coefficients that are multiples of $16\alpha_1 + \alpha_2$ or c_T. As we can see, this is indeed the case. Substituting a_0 and c_0 as given by (2.8.24) and (2.8.19) into the remaining equations, we obtain an exact cancellation of divergences. The anomaly coefficients in (2.8.5) are thus

$$a = \frac{1}{2}(16\alpha_1 + \alpha_2 - c_T), \qquad c = \frac{c_T}{2}. \tag{2.8.25}$$

Finally, from (2.8.20), we find the scheme-dependent contributions take the form

$$\begin{aligned}
&A_2 \mapsto A_2 - 16(a_0^{(0)} + c_0^{(0)}),\\
&A_3 \mapsto A_3 + 8\left[c_0^{(0)}(p_1^2 + p_2^2) - a_0^{(0)} p_3^2\right],\\
&A_4 \mapsto A_4 + 8\left[c_0^{(0)}(p_1^2 + p_2^2 + 3p_3^2) - a_0^{(0)}(p_1^2 + p_2^2 - p_3^2)\right],\\
&A_5 \mapsto A_5 - 4\left[a_0^{(0)} J^2 + c_0^{(0)}(p_1^2 + p_2^2 + p_3^2)^2 + 8c_g c_0^{(0)}(p_1^4 + p_2^4 + p_3^4)\right],
\end{aligned} \tag{2.8.26}$$

where $a_0^{(0)}$ and $c_0^{(0)}$ are arbitrary terms of order ϵ^0 in a_0 and c_0.

2.9. Helicity formalism

In this section we will work entirely in $d = 3$ spacetime dimensions.

2.9.1. Definitions

Consider conserved vector field j^μ and the transverse-traceless, symmetric tensor $t_{\mu\nu}$ of rank two in $d = 3$ dimensions. It is easy to count that both these objects contain 2 independent degrees of freedom. One can exploit this fact by rewritting the projectors (2.1.13) and (2.1.14) as

$$\pi_{\mu\nu}(\boldsymbol{p}) = \sum_{s=\pm 1} \xi_\mu^{(s)}(\boldsymbol{p}) \bar{\xi}_\nu^{(s)}(\boldsymbol{p}), \tag{2.9.1}$$

$$\Pi_{\mu\nu\rho\sigma}(\boldsymbol{p}) = \frac{1}{2} \sum_{s=\pm 1} \epsilon_{\mu\nu}^{(s)}(\boldsymbol{p}) \bar{\epsilon}_{\rho\sigma}^{(s)}(\boldsymbol{p}), \tag{2.9.2}$$

where $\xi^{(s)}$ and $\epsilon_{\mu\nu}^{(s)}$ are polarisation vectors and tensors satisfying

$$\begin{aligned}
&p^\mu \xi_\mu^{(s)}(\boldsymbol{p}) = p^\mu \epsilon_{\mu\nu}^{(s)}(\boldsymbol{p}) = 0, \qquad \epsilon_{\mu\nu}^{(s)} = \epsilon_{\nu\mu}^{(s)}, \qquad \delta^{\mu\nu} \epsilon_{\mu\nu}^{(s)} = 0,\\
&\bar{\epsilon}_{\mu\nu}^{(s)}(\boldsymbol{p}) = \epsilon_{\mu\nu}^{(s)}(-\boldsymbol{p}), \qquad \bar{\xi}_\mu^{(s)}(\boldsymbol{p}) = \xi_\mu^{(s)}(-\boldsymbol{p}).
\end{aligned} \tag{2.9.3}$$

The parameter of s takes two values $s = \pm 1$ known as *helicities*. The bar over a symbol denotes complex conjugation. By using the fact that in $d = 3$

$$\pi_{\mu\nu} \pi^{\mu\nu} = 2, \qquad \Pi_{\mu\nu\rho\sigma} \Pi^{\mu\nu\rho\sigma} = 2, \tag{2.9.4}$$

we can find

$$\xi_\mu^{(s)} \xi^{(s')\mu} = \delta^{ss'}, \qquad \epsilon_{\mu\nu}^{(s)} \epsilon^{(s')\mu\nu} = 2\delta^{ss'}. \tag{2.9.5}$$

Furthermore we define the helicity projected operators,

$$T(\boldsymbol{p}) = \delta^{\mu\nu} T_{\mu\nu}(\boldsymbol{p}), \tag{2.9.6}$$

$$T^{(s)}(\boldsymbol{p}) = \frac{1}{2}\epsilon^{(s)}_{\mu\nu}(-\boldsymbol{p}) T_{\mu\nu}(\boldsymbol{p}), \tag{2.9.7}$$

$$\Upsilon(\boldsymbol{p}_1, \boldsymbol{p}_2) = \delta^{\mu\nu}\delta^{\rho\sigma}\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2), \tag{2.9.8}$$

$$\Upsilon^{(s)}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{1}{2}\delta^{\mu\nu}\epsilon^{(s)\rho\sigma}(-\boldsymbol{p}_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2), \tag{2.9.9}$$

$$\Upsilon^{(s_1 s_2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{1}{4}\epsilon^{(s_1)\mu\nu}(-\boldsymbol{p}_1)\epsilon^{(s_2)\rho\sigma}(-\boldsymbol{p}_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2). \tag{2.9.10}$$

Finally, the expressions for the polarisation tensor can be found explicitly. To do so, we first observe that the momenta $\boldsymbol{p}_i$, $i = 1, 2, 3$ lie in a single plane due to momentum conservation. Taking this plane to be the (x_1, x_3) plane, we may then write

$$\boldsymbol{p}_i = p_i(\sin\theta_i,\, 0,\, \cos\theta_i) \tag{2.9.11}$$

where without loss of generality we may choose $\theta_1 = 0$, $0 \leqslant \theta_2 \leqslant \pi$ and $\pi \leqslant \theta_3 \leqslant 2\pi$ so that

$$\cos\theta_2 = \frac{(p_3^2 - p_1^2 - p_2^2)}{2p_1 p_2}, \qquad \sin\theta_2 = \frac{J}{2p_1 p_2}, \tag{2.9.12}$$

$$\cos\theta_3 = \frac{(p_2^2 - p_1^2 - p_3^2)}{2p_1 p_3}, \qquad \sin\theta_3 = -\frac{J}{2p_1 p_3}, \tag{2.9.13}$$

with J as given in (2.6.18). The required helicity tensors then follow by rotation in the (x_1, x_3) plane:

$$\epsilon^{(s_i)}(\boldsymbol{p}_i) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos^2\theta_i & is_i\cos\theta_i & -\sin\theta_i\cos\theta_i \\ is_i\cos\theta_i & -1 & -is_i\sin\theta_i \\ -\sin\theta_i\cos\theta_i & -is_i\sin\theta_i & \sin^2\theta_i \end{pmatrix}. \tag{2.9.14}$$

2.9.2. Correlation functions

Correlation functions of the helicity-projected operators can easily be obtained from the transverse-traceless parts of the correlators. First observe that the semi-local parts of any correlation function vanish when contracted with polarisation tensors. Indeed, equations (2.9.1) and (2.9.2) imply that

$$\pi_\mu^\nu \bar{\xi}^{(s)}_\nu = \bar{\xi}^{(s)}_\mu, \qquad \Pi^{\rho\sigma}_{\mu\nu}\bar{\epsilon}^{(s)}_{\rho\sigma} = \bar{\epsilon}^{(s)}_{\mu\nu}. \tag{2.9.15}$$

Then, using equation (2.4.16) we can write

$$\bar{\xi}^{(s)}_\mu j^\mu_{\text{loc}} = \bar{\xi}^{(s)}_\nu \pi^\nu_\mu j^\mu_{\text{loc}} = 0 \tag{2.9.16}$$

and similarly $\bar{\epsilon}^{(s)}_{\mu\nu} t^{\mu\nu}_{\text{loc}} = 0$. To obtain correlation functions in the helicity formalism, one can therefore apply helicity projectors to the transverse-traceless parts of correlators only. Due to (2.9.15), the projectors (2.1.13) and (2.1.14) can then be removed as well. Finally, one needs to compute a small number of contractions of the helicity projectors with momenta and with the metric.

As an example, consider the $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$ correlation function. Applying first the helicity projectors to its decomposition (2.3.7), we find

$$\begin{aligned}
\langle\langle T^{(s_1)}(\boldsymbol{p}_1) T^{(s_2)}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\rangle &= \frac{1}{4} \bar{\epsilon}^{(s_1)}_{\mu_1\nu_1}(\boldsymbol{p}_1) \bar{\epsilon}^{(s_2)}_{\mu_2\nu_2}(\boldsymbol{p}_2) \langle\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\rangle \\
= \frac{1}{4} \Big[A_1 \bar{\epsilon}^{(s_1)}_{\mu_1\nu_1}(\boldsymbol{p}_1) p_2^{\mu_1} p_2^{\nu_1} \bar{\epsilon}^{(s_2)}_{\mu_2\nu_2}(\boldsymbol{p}_2) p_3^{\mu_2} p_3^{\nu_2} &+ A_2 \bar{\epsilon}^{(s_1)}_{\mu_1\alpha}(\boldsymbol{p}_1) \bar{\epsilon}^{(s_2)\alpha}_{\mu_2}(\boldsymbol{p}_2) p_2^{\mu_1} p_3^{\mu_2} \\
&+ A_3 \bar{\epsilon}^{(s_1)}_{\alpha\beta}(\boldsymbol{p}_1) \bar{\epsilon}^{(s_2)\alpha\beta}(\boldsymbol{p}_2) \Big] . \qquad (2.9.17)
\end{aligned}$$

The contractions with helicity tensors depend on the precise definition of the latter and also the overall dimension. For case of $d = 3$ the required contractions can be found in appendix 2.A.9 based on [1]. Using (2.5.69) - (2.5.71) for the form factors, the most general solution is

$$\begin{aligned}
\langle\langle T^{(s_1)}(\boldsymbol{p}_1) T^{(s_2)}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\rangle &= \alpha_1^I \frac{3 p_1 p_2}{4 p_3} \left(\frac{p_1 + p_2 - p_3}{p_1 + p_2 + p_3} \right)^2 \delta^{s_1 s_2} \qquad (2.9.18) \\
&+ \frac{c_{\mathcal{O}} S_3^{(s_1 s_2)}}{32 p_1^2 p_2^2 p_3} \left[-2 c_1^I J^2 + \left((c_1^I - 3 c_2^I)(p_1^2 + p_2^2) + 3 (c_1^I + c_2^I) p_3^2 \right) S_3^{(s_1 s_2)} \right] .
\end{aligned}$$

The constants c_1^J and c_2^J are defined in (2.5.64) and $c_{\mathcal{O}}$ is the normalisation constant of the 2-point function $\langle\langle \mathcal{O}^J \mathcal{O}^I \rangle\rangle$ defined in (2.5.54).

As a check on our results in (2.7.4), we compared our solution with that obtained in [50] for the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ correlator of free scalars and fermions finding perfect agreement.

2.9.3. $\langle T^{(s_1)} T^{(s_2)} T^{(s_3)} \rangle$

The application of the helicity formalism in $d = 3$ to the correlation function of three stress-energy tensors in $d = 3$ given by (3.11.21) - (3.11.25) leads to the following result

$$\begin{aligned}
\langle\langle T^{(+)}(\boldsymbol{p}_1) T^{(+)}(\boldsymbol{p}_2) T^{(+)}(\boldsymbol{p}_3) \rangle\rangle &= \frac{30\sqrt{2} \alpha_1 \lambda^2 p_1 p_2 p_3}{a_{123}^4} \\
&- c_T \frac{\lambda^2 a_{123}^2}{16\sqrt{2} c_{123}^2} \left[\left(3 a_{123}^3 - 7 a_{123} b_{123} + 5 c_{123} \right) + 8 (p_1^3 + p_2^3 + p_3^3) c_g \right] , \qquad (2.9.19)
\end{aligned}$$

$$
\begin{aligned}
\langle\!\langle T^{(+)}(\boldsymbol{p}_1)T^{(+)}(\boldsymbol{p}_2)T^{(-)}(\boldsymbol{p}_3)\rangle\!\rangle &= -c_T \frac{\lambda^2(p_1+p_2-p_3)^2}{16\sqrt{2}c_{123}^2} \times \\
\times \Bigg[\frac{1}{a_{123}^2} \big(3p_3^5 + 4p_3^4 a_{12} &+ p_3^3(a_{12}^2 - b_{12}) + p_3 a_{12}(p_3 + 4a_{12})(a_{12}^2 - 3b_{12}) \\
&+ a_{12}^3(3a_{12}^2 - 7b_{12}) \big) + 8(p_1^3+p_2^3+p_3^3)c_g \Bigg], \qquad (2.9.20)
\end{aligned}
$$

where λ^2 is defined in (2.6.18) and all remaining variables are symmetric polynomials in magnitudes of momenta defined in (3.1.2). Notice that this solution does not depend on the primary constant α_2, which features in the solution (3.11.21) - (3.11.25). The reason is that in $d=3$, in position space there is one independent conformal structure less than for $d>3$ [22]. The same result can be obtained directly in the momentum space, as presented in appendix 2.A.2.

Note also that the $\langle\!\langle T^{(+)}T^{(+)}T^{(-)}\rangle\!\rangle$ part of the correlation function does not depend on α_1, hence it is determined uniquely in terms of the 2-point function.

2.A. Appendix

2.A.1. Decomposition of $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle$ in non-conformal case

In this section we present the decomposition of the stress-energy tensor 3-point function for a general quantum field theory. As the stress-energy tensor in a general theory is no longer traceless, our arguments in the main text need some minor modifications. First, we discuss how to reconstruct the full correlation function from the purely transverse part, making use of the transverse Ward identities in a similar fashion to section 2.3.3. We then proceed to construct the general tensor decomposition of this transverse part in terms of ten independent form factors.

As in the main text, we will denote the transverse-traceless part of the stress-energy tensor by $t^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}T^{\alpha\beta}$. Here, we will also make use of the purely transverse part, $t_T^{\mu\nu} = \pi^\mu_\alpha \pi^\nu_\beta T^{\alpha\beta}$, which includes a nonvanishing trace part $(t_T)^\mu_\mu$. The difference between the stress-energy tensor and its transverse part can then be written $\tilde{t}^{\mu\nu}_{\text{loc}} = T^{\mu\nu} - t_T^{\mu\nu}$, *i.e.*,

$$\tilde{t}^{\mu\nu}_{\text{loc}} = \left(\frac{p^\mu}{p^2}\delta^\nu_\alpha + \frac{p^\nu}{p^2}\delta^\mu_\alpha - \frac{p^\mu p^\nu p_\alpha}{p^4} \right) p_\beta T^{\alpha\beta}. \qquad (2.A.1)$$

To obtain the reconstruction formula, we use the Ward identity (3.11.1) to re-express $p_\beta T^{\alpha\beta}$ in terms of 2-point functions when the expectation value of $\tilde{t}^{\mu\nu}_{\text{loc}}$

with other operators is taken. Defining the operator

$$\tilde{\mathcal{L}}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3) = \frac{1}{p_1^2}\left(2p_1^{(\mu_1}\delta^{\nu_1)}_{\alpha_1} - \frac{p_1^{\mu_1}p_1^{\nu_1}p_{1\alpha_1}}{p_1^2}\right)\times$$
$$\times\Bigg[2\delta^{\mu_3\alpha_3}\delta^{\nu_3\alpha_3}p_1^{\beta_1}\langle\langle\frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_3\beta_3}}(\boldsymbol{p}_1,\boldsymbol{p}_3)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\rangle\rangle$$
$$+\left(\delta^{\beta_3\alpha_1}(2p_1^{(\mu_3}\delta^{\nu_3)\alpha_3} + p_3^{\alpha_3}\delta^{\mu_3\nu_3}) - p_3^{\alpha_1}\delta^{\alpha_3\mu_3}\delta^{\beta_3\nu_3}\right)\times$$
$$\times\langle\langle T_{\alpha_3\beta_3}(\boldsymbol{p}_2)T^{\mu_2\nu_2}(-\boldsymbol{p}_2)\rangle\rangle\Bigg], \tag{2.A.2}$$

the reconstruction formula takes the form

$$\langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle = \langle\langle t_T^{\mu_1\nu_1}(\boldsymbol{p}_1)t_T^{\mu_2\nu_2}(\boldsymbol{p}_2)t_T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle$$
$$+\sum_\sigma \tilde{\mathcal{L}}^{\mu_{\sigma(1)}\nu_{\sigma(1)}\mu_{\sigma(2)}\nu_{\sigma(2)}\mu_{\sigma(3)}\nu_{\sigma(3)}}(\boldsymbol{p}_{\sigma(1)},\boldsymbol{p}_{\sigma(2)},\boldsymbol{p}_{\sigma(3)})$$
$$-\frac{1}{p_3^2}\left(2p_3^{(\mu_3}\delta^{\nu_3)}_{\alpha_3} - \frac{p_3^{\mu_3}p_3^{\nu_3}p_{3\alpha_3}}{p_3^2}\right)p_{3\beta_3}\tilde{\mathcal{L}}^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$$
$$-[(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)]$$
$$-[(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)], \tag{2.A.3}$$

where the sum is taken over all six permutations σ of the set $\{1,2,3\}$. Note the similarity between these expression and (3.11.3, 3.11.4).

We turn now to the tensor decomposition of the purely transverse part of the 3-point function. The most general form of this is

$$\langle\langle t_T^{\mu_1\nu_1}(\boldsymbol{p}_1)t_T^{\mu_2\nu_2}(\boldsymbol{p}_2)t_T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle = \tag{2.A.4}$$
$$= \pi^{\mu_1}_{(\alpha_1}(\boldsymbol{p}_1)\pi^{\nu_1}_{\beta_1)}(\boldsymbol{p}_1)\pi^{\mu_2}_{(\alpha_2}(\boldsymbol{p}_2)\pi^{\nu_2}_{\beta_2)}(\boldsymbol{p}_2)\pi^{\mu_3}_{(\alpha_3}(\boldsymbol{p}_3)\pi^{\nu_3}_{\beta_3)}(\boldsymbol{p}_3)X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3},$$

where $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ is a general tensor built from the metric $\delta^{\mu\nu}$ and two independent momenta, with a kinematic dependence on the momentum magnitudes p_1, p_2 and p_3. Note, however, that if $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ contains $p_j^{\alpha_j}$ or $p_j^{\beta_j}$ for $j\in\{1,2,3\}$ then the contractions with the corresponding transverse projectors vanish. We will assume that $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ is symmetric under $\alpha_j\leftrightarrow\beta_j$ and we use the convention (2.3.4) (explained in detail in section 2.3.1) for the momenta appearing under the various Lorentz indices:

$$\boldsymbol{p}_1,\boldsymbol{p}_2 \text{ for } \mu_1,\nu_1;\ \boldsymbol{p}_2,\boldsymbol{p}_3 \text{ for } \mu_2,\nu_2 \text{ and } \boldsymbol{p}_3,\boldsymbol{p}_1 \text{ for } \mu_3,\nu_3. \tag{2.A.5}$$

The following table lists all 24 simple tensors from which $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ may be built.

Contracting each tensor in the table with the transverse projectors we obtain 24 transverse tensors denoted by P_a, $a=1,2,\ldots,24$. Each tensor P_a can then be

	$p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3}$	
$\delta^{\beta_1\beta_2} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3}$	$\delta^{\beta_2\beta_3} p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3}$	$\delta^{\beta_1\beta_3} p_2^{\alpha_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3}$
$\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2} p_1^{\alpha_3} p_1^{\beta_3}$	$\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3} p_2^{\alpha_1} p_2^{\beta_1}$	$\delta^{\alpha_1\alpha_3}\delta^{\beta_1\beta_3} p_3^{\alpha_2} p_3^{\beta_2}$
$\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3} p_2^{\beta_1} p_3^{\beta_2}$	$\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1} p_3^{\beta_2} p_1^{\beta_3}$	$\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_2} p_2^{\beta_1} p_1^{\beta_3}$
	$\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}$	
$\delta^{\alpha_3\beta_3} p_3^{\alpha_1} p_3^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2}$	$\delta^{\alpha_1\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3}$	$\delta^{\alpha_2\beta_2} p_3^{\alpha_1} p_3^{\beta_1} p_1^{\alpha_3} p_1^{\beta_3}$
$\delta^{\alpha_3\beta_3}\delta^{\beta_1\beta_2} p_2^{\alpha_1} p_3^{\alpha_2}$	$\delta^{\alpha_1\beta_1}\delta^{\beta_2\beta_3} p_3^{\alpha_2} p_1^{\alpha_3}$	$\delta^{\alpha_2\beta_2}\delta^{\beta_1\beta_3} p_2^{\alpha_1} p_1^{\alpha_3}$
$\delta^{\alpha_3\beta_3}\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}$	$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}$	$\delta^{\alpha_2\beta_2}\delta^{\alpha_1\alpha_3}\delta^{\beta_1\beta_3}$
$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\beta_2} p_1^{\alpha_3} p_1^{\beta_3}$	$\delta^{\alpha_2\beta_2}\delta^{\alpha_3\beta_3} p_2^{\alpha_1} p_2^{\beta_1}$	$\delta^{\alpha_1\beta_1}\delta^{\alpha_3\beta_3} p_3^{\alpha_2} p_3^{\beta_2}$
	$\delta^{\alpha_1\beta_1}\delta^{\alpha_2\beta_2}\delta^{\alpha_3\beta_3}$	

Table 2.2: *When contracted with the transverse projectors, this table presents all* 24 *tensor structures in the decomposition of the transverse part of* $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\!\rangle$. *Tensors are divided into* 10 *orbits of the action of the symmetry group* S_3*, after the contractions with the transverse projectors are taken.*

multiplied by a form factor B_a to obtain the decomposition

$$\langle\!\langle t_T^{\mu_1\nu_1}(\boldsymbol{p}_1) t_T^{\mu_2\nu_2}(\boldsymbol{p}_2) t_T^{\mu_3\nu_3}(\boldsymbol{p}_3) \rangle\!\rangle = \sum_{a=1}^{24} B_a(p_1, p_2, p_3) P_a^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \tag{2.A.6}$$

However, the number of independent form factors may be reduced by looking at the symmetry properties. If we denote the permutation group of the set $\{1, 2, 3\}$ by S_3, then the 3-point function is S_3-invariant, *i.e.*, for any $\sigma \in S_3$,

$$\langle\!\langle t_T^{\mu_1\nu_1}(\boldsymbol{p}_1) t_T^{\mu_2\nu_2}(\boldsymbol{p}_2) t_T^{\mu_3\nu_3}(\boldsymbol{p}_3) \rangle\!\rangle = \langle\!\langle t_T^{\mu_{\sigma(1)}\nu_{\sigma(1)}}(\boldsymbol{p}_{\sigma(1)}) t_T^{\mu_{\sigma(2)}\nu_{\sigma(2)}}(\boldsymbol{p}_{\sigma(2)}) t_T^{\mu_{\sigma(3)}\nu_{\sigma(3)}}(\boldsymbol{p}_{\sigma(3)}) \rangle\!\rangle. \tag{2.A.7}$$

When contracted with the transverse projectors, the tensors at the first, fifth and the last row of the table lead to the S_3-invariant tensors. Therefore, corresponding form factors are invariant under any permutation of their arguments, for example

$$B_1(p_1, p_2, p_3) = B_1(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) \tag{2.A.8}$$

for any $\sigma \in S_3$. The remaining tensors transform non-trivially under the action of S_3. For concreteness, consider the second line of the table, *i.e.*, the part of the decomposition

$$B_2(p_1, p_2, p_3) P_2^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_3(p_1, p_2, p_3) P_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_4(p_1, p_2, p_3) P_4^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \tag{2.A.9}$$

Under the action of the symmetry group the tensors P_2, P_3, P_4 shuffle among each other. For example, under the action of the transposition $(\boldsymbol{p}_1, \mu_1, \nu_1) \leftrightarrow (\boldsymbol{p}_3, \mu_3, \nu_3)$

we obtain

$$B_2(p_3,p_2,p_1)P_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}+B_3(p_3,p_2,p_1)P_2^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}+B_4(p_3,p_2,p_1)P_4^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \tag{2.A.10}$$

Since the entire 3-point function is S_3-invariant, this implies that (2.A.9) and (2.A.10) are equal. Since all tensor structures P_a are independent, we find

$$B_3(p_1,p_2,p_3) = B_2(p_3,p_2,p_1), \qquad B_4(p_1,p_2,p_3) = B_4(p_3,p_2,p_1). \tag{2.A.11}$$

By analysing other symmetries we find that (2.A.9) depends on one form factor only, say B_2,

$$B_2(p_1,p_2,p_3)P_2^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}+B_2(p_1\leftrightarrow p_3)P_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}+B_2(p_2\leftrightarrow p_3)P_4^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \tag{2.A.12}$$

Moreover, $B_2(p_1,p_2,p_3) = B_2(p_1 \leftrightarrow p_2)$.

The described procedure reduces the number of independent form factors from 24 down to 10. The same procedure applied to the transverse-traceless part of the 3-point function reduces the number of independent tensors from 11 down to 5. In this case the decomposition is given by (2.3.10).

2.A.2. Degeneracy of $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle$ in $d=3$

In dimension $d = 3$, a special degeneracy occurs which allows the transverse-traceless part of $\langle\!\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle\!\rangle$ to be decomposed in terms of only two form factors rather than five.[1]

To see this, we first define the cross-product

$$\boldsymbol{n} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = \boldsymbol{p}_2 \times \boldsymbol{p}_3 = \boldsymbol{p}_3 \times \boldsymbol{p}_1 \tag{2.A.13}$$

and note that $n^2 = J^2/4$, where J^2 is defined in (2.6.18). Using (2.3.2) we find

$$\delta^{\mu\nu} = \frac{4}{J^2}\left[p_i^2 p_j^\mu p_j^\nu + p_j^2 p_i^\mu p_i^\nu - \boldsymbol{p}_i\cdot\boldsymbol{p}_j(p_i^\mu p_j^\nu + p_j^\mu p_i^\nu) + n^\mu n^\nu\right] \tag{2.A.14}$$

for any $i,j = 1,2,3$ and $i \neq j$. From the fact that $\delta^{\alpha\beta}\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}_j) = 0$, we find

$$\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}_j)n^\alpha n^\beta = -p_j^2\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}_j)\, p^\alpha_{(j+1)\bmod 3}\, p^\beta_{(j+1)\bmod 3}, \qquad j=1,2,3. \tag{2.A.15}$$

We can now go back to the decomposition of the transverse-traceless part of $\langle\!\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle\!\rangle$, equation (3.11.5), and exchange all $\delta^{\alpha\beta}$ for (2.A.14). However, if one transverse-traceless projector is contracted with two vectors $\boldsymbol{n}$, then, according to (2.A.15), we can replace such a contraction with a contraction of two momenta with appropriate prefactors. Therefore, the only terms surviving in

[1] From the point of view of the helicity formalsim, see the end of section (4.1.9).

(3.11.5) are terms with either zero or two vectors $\boldsymbol{n}$. Hence we find only two tensor structures in the decomposition of $\langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t^{\mu_3\nu_3}\rangle\rangle$,

$$\begin{aligned}
&\langle\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\rangle \\
&\quad = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)\Big[B_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \\
&\qquad + B_2 n^{\beta_1} n^{\beta_2} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3} + B_2(p_1 \leftrightarrow p_3) n^{\beta_2} n^{\beta_3} p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} \\
&\qquad\qquad + B_2(p_2 \leftrightarrow p_3) n^{\beta_1} n^{\beta_3} p_2^{\alpha_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3}\Big]. \qquad (2.A.16)
\end{aligned}$$

The new form factors B_j are functions of the momentum magnitudes. As usual, if no arguments are specified then the standard ordering is assumed, $B_j = B_j(p_1, p_2, p_3)$, while by $p_i \leftrightarrow p_j$ we denote the exchange of the two momenta, *e.g.*, $B_1(p_1 \leftrightarrow p_3) = B_2(p_3, p_2, p_1)$.

We can now express the new form factors B_j in terms of the old ones, A_j, defined in (3.11.5). Using equation (2.A.15), we write the explicit form of the contraction of two transverse-traceless projectors with a metric as

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\delta^{\beta_1\beta_2} = \frac{4}{J^2}\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\left[n^{\beta_1} n^{\beta_2} + \frac{1}{2}(p_3^2 - p_1^2 - p_2^2) p_2^{\beta_1} p_3^{\beta_2}\right], \qquad (2.A.17)$$

from which we find

$$\begin{aligned}
B_1 &= A_1 + \frac{2}{J^2}\left[(p_3^2 - p_1^2 - p_2^2) A_2(p_1, p_2, p_3) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3)\right] \\
&\quad + \frac{4}{J^4}\left[\big((8p_1^2 p_2^2 - J^2) A_3 + (p_3^4 - (p_1^2 - p_2^2)^2) A_4\big) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3)\right] \\
&\quad - \frac{8}{J^4}(p_1^2 + p_2^2 + p_3^2) A_5, \\
B_2 &= \frac{4}{J^2} A_2 + \frac{16}{J^4}\Big[(p_3^2 - p_1^2 - p_2^2) A_3 - p_3^2 A_4 + \\
&\qquad + \frac{1}{2}(p_2^2 - p_1^2 - p_3^2) A_4(p_1 \leftrightarrow p_3) + \frac{1}{2}(p_1^2 - p_2^2 - p_3^2) A_4(p_2 \leftrightarrow p_3)\Big] \\
&\quad + \frac{16}{J^4} A_5. \qquad (2.A.18)
\end{aligned}$$

Using the general expressions (3.11.21) - (3.11.25) for the form factors in $d = 3$, we arrive at the final result

$$\begin{aligned}
B_1 &= 1920\alpha_1 \frac{c_{123}^3}{J^4 a_{123}^4} - \frac{8c_T}{J^4 a_{123}^2}\Big[(3 + 8c_g) a_{123}^5 (a_{123}^2 - 5b_{123}) + 24(1 + 2c_g) a_{123}^3 b_{123}^2 \\
&\qquad - 8a_{123} b_{123}^3 + \big(3(8c_g - 1) a_{123}^4 - 48 c_g a_{123}^2 b_{123} - 8b_{123}^2\big) c_{123} + 8a_{123} c_{123}^2\Big], \qquad (2.A.19)
\end{aligned}$$

$$\begin{aligned}
B_2 = -1920\alpha_1 \frac{c_{123}^2 p_3}{J^4 a_{123}^4} + \frac{64 c_T}{J^4 a_{123}^2} \big[& 2(1+c_g)p_3^4(p_3+2a_{12}) \\
& + p_3^3\left(2(2+c_g)a_{12}^2 - 4b_{12}\right) + p_3^2 a_{12}\left((3+2c_g)a_{12}^2 - (5+6c_g)b_{12}\right) \\
& + 2p_3\left((1+2c_g)a_{12}^2(a_{12}^2-3b_{12}) + b_{12}^2\right) \\
& - 3(1+2c_g)a_{12}^3 b_{12} + a_{12}b_{12}^2 + (1+2c_g)a_{12}^5\big] . \qquad (2.A.20)
\end{aligned}$$

The variables used in this expression are symmetric polynomials of the momentum magnitudes as defined in (3.1.2). Note that this expression has no dependence on the primary constant α_2. Therefore, in $d = 3$, the most general form of the correlation function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\!\rangle$ depends on only one undetermined primary constant and on two 2-point function normalisations c_T and c_g. This is in agreement with [22], noting that the normalisation constant c_g arises through our definition of the 3-point function in (1.3.20).

Finally, while similar considerations hold for other 3-point correlators in $d = 3$ involving the stress-energy tensor, in these cases it turns out that the use of equation (2.A.14) does not reduce the number of independent primary constants in the final result.

2.A.3. Properties of triple-K integrals

In this appendix we list some properties of modified Bessel functions used in the main text. For further references, see *e.g.*, [59].

The Bessel function I (modified Bessel function of the first kind) is given by the series

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(\nu+j+1)} \left(\frac{x}{2}\right)^{\nu+2j}, \quad \nu \neq -1, -2, -3, \ldots \qquad (2.A.21)$$

The Bessel function K (modified Bessel function of the second kind) is defined by

$$K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)} \left[I_{-\nu}(x) - I_\nu(x) \right], \quad \nu \notin \mathbb{Z}, \qquad (2.A.22)$$

$$K_n(x) = \lim_{\epsilon \to 0} K_{n+\epsilon}(x), \quad n \in \mathbb{Z}. \qquad (2.A.23)$$

The finite pointwise limit for $x > 0$ exists for any integer n. K_ν is an even function of ν, *i.e.*, $K_{-\nu}(x) = K_\nu(x)$ for any $\nu \in \mathbb{R}$. If $\nu = \frac{1}{2} + n$, for an integer n, the Bessel function reduces to elementary functions

$$K_\nu(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \sum_{j=0}^{\lfloor |\nu| - \frac{1}{2} \rfloor} \frac{\left(|\nu| - \frac{1}{2} + j\right)!}{j! \left(|\nu| - \frac{1}{2} - j\right)!} \frac{1}{(2x)^j}, \quad \nu + \frac{1}{2} \in \mathbb{Z}, \qquad (2.A.24)$$

and in particular

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{1}{2}}}, \qquad K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{3}{2}}} (1+x),$$
$$K_{\frac{5}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{5}{2}}} (x^2+3x+3), \qquad K_{\frac{7}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{7}{2}}} (x^3+6x^2+15x+5). \tag{2.A.25}$$

The series expansion of the Bessel function K_ν for $\nu \notin \mathbb{Z}$ is given directly in terms of the expansion (2.A.21) via the definition (2.A.22). In particular

$$K_\nu(x) = \left[\Gamma(-\nu) 2^{-\nu-1} x^\nu + O(x^{2-\nu}) \right] + \left[\frac{\Gamma(\nu) 2^{\nu-1}}{x^\nu} + O(x^{2+\nu}) \right], \quad \nu \notin \mathbb{Z}. \tag{2.A.26}$$

For non-negative integer index n, the expansion reads

$$\begin{aligned} K_n(x) = &\ \frac{1}{2} \left(\frac{x}{2} \right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} (-1)^j \left(\frac{x}{2} \right)^{2j} \\ &+ (-1)^{n+1} \log \left(\frac{x}{2} \right) I_n(x) \\ &+ (-1)^n \frac{1}{2} \left(\frac{x}{2} \right)^n \sum_{j=0}^{\infty} \frac{\psi(j+1) + \psi(n+j+1)}{j!(n+j)!} \left(\frac{x}{2} \right)^{2j}, \end{aligned} \tag{2.A.27}$$

where ψ is the digamma function. At large x, the Bessel functions have the asymptotic expansions

$$I_\nu(x) = \frac{1}{\sqrt{2\pi}} \frac{e^x}{\sqrt{x}} + \ldots, \qquad K_\nu(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} + \ldots, \qquad \nu \in \mathbb{R}. \tag{2.A.28}$$

For any index $\nu \in \mathbb{R}$, the Bessel function K satisfies the following identities

$$\frac{\partial}{\partial a} \left[a^\nu K_\nu(ax) \right] = - x a^\nu K_{\nu-1}(ax), \tag{2.A.29}$$
$$K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) = K_{\nu+1}(x), \tag{2.A.30}$$
$$K_{-\nu}(x) = K_\nu(x). \tag{2.A.31}$$

2.A.4. Appell's F_4 function

Appell's F_4 function can be defined by the following double series [60, 61]

$$F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta) = \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j} (\beta)_{i+j}}{(\gamma)_i (\gamma')_j i! j!} \xi^i \eta^j, \quad \sqrt{|\xi|} + \sqrt{|\eta|} < 1, \tag{2.A.32}$$

where $(\alpha)_i$ is a Pochhammer symbol. Notice that

$$F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta) = F_4(\beta, \alpha; \gamma, \gamma'; \xi, \eta) = F_4(\alpha, \beta; \gamma', \gamma; \eta, \xi). \tag{2.A.33}$$

The series representation, however, is not very useful as in our case $\xi = \frac{p_1^2}{p_3^2}$ and $\eta = \frac{p_2^2}{p_3^2}$, so the series converges when $p_3 > p_1 + p_2$, which is opposite to the triangle inequality.

As in the case of ordinary hypergeometric functions, the F_4 function satisfies certain differential equations. Let $\alpha, \beta, \gamma, \gamma'$ be fixed numbers. The following system of equations

$$0 = \left[\xi(1-\xi)\frac{\partial^2}{\partial\xi^2} - \eta^2\frac{\partial^2}{\partial\eta^2} - 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} + (\gamma - (\alpha+\beta+1)\xi)\frac{\partial}{\partial\xi} - (\alpha+\beta+1)\eta\frac{\partial}{\partial\eta} - \alpha\beta\right] F(\xi,\eta), \tag{2.A.34}$$

$$0 = \left[\eta(1-\eta)\frac{\partial^2}{\partial\eta^2} - \xi^2\frac{\partial^2}{\partial\xi^2} - 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} + (\gamma' - (\alpha+\beta+1)\eta)\frac{\partial}{\partial\eta} - (\alpha+\beta+1)\xi\frac{\partial}{\partial\xi} - \alpha\beta\right] F(\xi,\eta), \tag{2.A.35}$$

has exactly four solutions given by [61, 62]

$$F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta), \tag{2.A.36}$$

$$\xi^{1-\gamma} F_4(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma, \gamma'; \xi, \eta), \tag{2.A.37}$$

$$\eta^{1-\gamma'} F_4(\alpha + 1 - \gamma', \beta + 1 - \gamma'; \gamma, 2 - \gamma'; \xi, \eta), \tag{2.A.38}$$

$$\xi^{1-\gamma}\eta^{1-\gamma'} F_4(\alpha + 2 - \gamma - \gamma', \beta + 2 - \gamma - \gamma'; 2 - \gamma, 2 - \gamma'; \xi, \eta). \tag{2.A.39}$$

The following reduction formulae can be found in [63] or [61]

$$F_4\left(\alpha, \beta; \alpha, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = \frac{(1-x)^\beta (1-y)^\alpha}{1-xy}, \tag{2.A.40}$$

$$F_4\left(\alpha, \beta; \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = (1-x)^\alpha (1-y)^\alpha {}_2F_1(\alpha, 1+\alpha-\beta; \beta; xy), \tag{2.A.41}$$

$$F_4\left(\alpha, \beta; 1+\alpha-\beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = (1-y)^\alpha {}_2F_1\left(\alpha, \beta; 1+\alpha-\beta; -\frac{x(1-y)}{1-x}\right), \tag{2.A.42}$$

$${}_2F_1(2\nu - 1, \nu; \nu; x) = (1-x)^{1-2\nu}. \tag{2.A.43}$$

Integrals

Here we present the list of integrals we use in the thesis, which may be found in [63].

(i)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} I_\lambda(ax) I_\mu(bx) K_\nu(cx) = \tag{2.A.44}$$

$$= \frac{2^{\alpha-2}\Gamma\left(\frac{\alpha+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha+\lambda+\mu+\nu}{2}\right)}{\Gamma(\lambda+1)\Gamma(\mu+1)} \cdot \frac{a^\lambda b^\mu}{c^{\alpha+\lambda+\mu}} \times$$

$$\times F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2}, \frac{\alpha+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right),$$

valid for

$$\operatorname{Re}(\alpha+\lambda+\mu) > |\operatorname{Re}\nu|, \qquad |c| > |a| + |b|, \qquad \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|. \tag{2.A.45}$$

(ii)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} K_\lambda(ax) K_\mu(bx) K_\nu(cx) =$$

$$= \frac{2^{\alpha-4}}{c^\alpha}\left[A(\lambda,\mu) + A(\lambda,-\mu) + A(-\lambda,\mu) + A(-\lambda,-\mu)\right], \tag{2.A.46}$$

where

$$A(\lambda,\mu) = \left(\frac{a}{c}\right)^\lambda \left(\frac{b}{c}\right)^\mu \Gamma\left(\frac{\alpha+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha+\lambda+\mu+\nu}{2}\right)\Gamma(-\lambda)\Gamma(-\mu)\times$$

$$\times F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2}, \frac{\alpha+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right), \tag{2.A.47}$$

valid for

$$\operatorname{Re}\alpha > |\operatorname{Re}\lambda| + |\operatorname{Re}\mu| + |\operatorname{Re}\nu|, \qquad \operatorname{Re}(a+b+c) > 0. \tag{2.A.48}$$

(iii)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} K_\mu(cx) K_\nu(cx) = \tag{2.A.49}$$

$$= \frac{2^{\alpha-3}}{\Gamma(\alpha)c^\alpha}\Gamma\left(\frac{\alpha+\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha-\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha-\mu-\nu}{2}\right),$$

valid for

$$\operatorname{Re}\alpha > |\operatorname{Re}\mu| + |\operatorname{Re}\nu|, \qquad \operatorname{Re} c > 0. \tag{2.A.50}$$

(iv)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} K_\nu(cx) = \frac{2^{\alpha-2}}{c^\alpha}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right) \tag{2.A.51}$$

valid for

$$\operatorname{Re}\alpha > |\operatorname{Re}\nu|, \qquad \operatorname{Re} c > 0. \tag{2.A.52}$$

(v)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} \log x K_\nu(cx) = \frac{2^{\alpha-3}}{c^\alpha}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right)\times$$
$$\times\left[\psi\left(\frac{\alpha+\nu}{2}\right)+\psi\left(\frac{\alpha-\nu}{2}\right)-2\log\frac{c}{2}\right], \tag{2.A.53}$$

valid for

$$\mathrm{Re}\,\alpha > |\,\mathrm{Re}\,\nu|, \qquad \mathrm{Re}\,c > 0. \tag{2.A.54}$$

(vi)

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} \log^2 x K_\nu(cx) = \frac{2^{\alpha-4}}{c^\alpha}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right)\times$$
$$\times\left[\left(\psi\left(\frac{\alpha+\nu}{2}\right)+\psi\left(\frac{\alpha-\nu}{2}\right)\right)\cdot\left(\psi\left(\frac{\alpha+\nu}{2}\right)+\psi\left(\frac{\alpha-\nu}{2}\right)-4\log\frac{c}{2}\right)\right.$$
$$\left.+\psi'\left(\frac{\alpha+\nu}{2}\right)+\psi'\left(\frac{\alpha-\nu}{2}\right)+4\log^2\frac{c}{2}\right], \tag{2.A.55}$$

valid for

$$\mathrm{Re}\,\alpha > |\,\mathrm{Re}\,\nu|, \qquad \mathrm{Re}\,c > 0. \tag{2.A.56}$$

2.A.5. Master integral

The master integral is

$$\begin{aligned}
I_{0+\epsilon\{111\}} &= -\frac{p_1^2+p_2^2+p_3^2}{2\,\epsilon^2} \\
&+\ \frac{1}{2\epsilon}\left[-h_{1/2}(p_1^2+p_2^2+p_3^2)+(p_1^2\log p_1+p_2^2\log p_2+p_3^2\log p_3)\right] \\
&+\ \left[\frac{p_1p_2Z}{16}\left(\tfrac{1}{2}F^{(2)}(Z^2)+2(h_1+2\log p_3)F^{(1)}(Z^2)+(h_1+2\log^2 p_3)^2+1+\tfrac{1}{2}\pi^2\right)\right. \\
&\quad\left.+\left(p_1\leftrightarrow p_3, Z\mapsto X \text{ but } Z^2\mapsto Z\frac{X}{Y}\right)+\left(p_2\leftrightarrow p_3, Z\mapsto Y \text{ but } Z^2\mapsto Z\frac{Y}{X}\right)\right] \\
&\quad-\ \frac{\sqrt{-J^2}}{8}\left[\log^2(p_1p_2p_3)+h_2\log(p_1p_2p_3)+\tfrac{1}{4}h_2^2+1+\tfrac{7}{24}\pi^2\right. \\
&\qquad\left.-(h_2+2\log(p_1p_2p_3))\log\sqrt{-J^2}+\log^2\sqrt{-J^2}\right] \\
&\quad+\ \frac{1}{8}\left[\left(p_3^2(-\tfrac{7}{4}h_{2/7}^2+\tfrac{8}{7}+\tfrac{1}{8}\pi^2)+(h_2(p_1^2+p_2^2)+3h_0p_3^2)\log p_3\right.\right. \\
&\qquad\left.\left.+(p_1^2+p_2^2-p_3^2)(\log^2 p_3-2\log p_1\log p_2)\right)+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)\right] \\
&+\ O(\epsilon),
\end{aligned} \tag{2.A.57}$$

where

$$F^{(1)}(x) = 1 - \left(1 - \frac{1}{x}\right) \log(1-x), \tag{2.A.58}$$

$$F^{(2)}(x) = 2 + \left(1 - \frac{1}{x}\right) \left[- \log(1-x) + \log^2(1-x) + \operatorname{Li}_2 x\right] \tag{2.A.59}$$

are coefficients of the series expansion

$${}_2F_1(1, \epsilon; 2-\epsilon; x) = 1 + F^{(1)}(x)\epsilon + F^{(2)}(x)\epsilon^2 + O(\epsilon^3). \tag{2.A.60}$$

We have also defined

$$J^2 = (p_1+p_2-p_3)(p_1-p_2+p_3)(-p_1+p_2+p_3)(p_1+p_2+p_3), \tag{2.A.61}$$

$$X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2p_3}, \qquad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1p_3}, \tag{2.A.62}$$

$$Z = \frac{p_3^2 - p_1^2 - p_2^2 - \sqrt{-J^2}}{2p_1p_2} \tag{2.A.63}$$

and the constant

$$h_\alpha = \alpha - \gamma_E + \log 2. \tag{2.A.64}$$

The master integral can be evaluated by the method presented in section 2.6.4. All remaining integrals required for the calculations of 3-point functions of conserved currents and the stress-energy tensor in any even dimension $d \geqslant 4$ can be obtained from the master integral (2.A.57) by the reduction scheme. For $d = 4$, the reduction scheme is presented in table 2.1 on page 94.

Below we present the expressions for two other integrals, $I_{1\{222\}}$ and $I_{1\{000\}}$, appearing in table 2.1 on page 94. Both integrals can be obtained by the method described in section 2.6.4, but the resulting expressions are simpler than the master integral $I_{0\{111\}}$. Therefore, whenever possible, it is convenient to use them as a starting point in the reduction scheme. In particular the integral $I_{1\{000\}}$ is convergent and known in the literature, *e.g.*, [52, 69]

$$I_{1\{000\}} = \frac{1}{2\sqrt{-J^2}} \left[\frac{\pi^2}{6} - 2 \log \frac{p_1}{p_3} \log \frac{p_2}{p_3} + \log\left(-X \frac{p_2}{p_3}\right) \log\left(-Y \frac{p_1}{p_3}\right) \right.$$
$$\left. - \operatorname{Li}_2\left(-X \frac{p_2}{p_3}\right) - \operatorname{Li}_2\left(-Y \frac{p_1}{p_3}\right) \right]. \tag{2.A.65}$$

From this integral one can find

$$I_{4\{111\}} = -p_1p_2p_3 \frac{\partial^3}{\partial p_1 \partial p_2 \partial p_3} I_{1\{000\}}, \tag{2.A.66}$$

which is the top rightmost entry in table 2.1, page 94.

The second integral is

$$
\begin{aligned}
I_{2+\epsilon\{111\}} = \frac{1}{\epsilon} - \Bigg[& \frac{p_3^2 Z}{2p_1p_2}\left(\tilde{F}^{(1)}(Z^2) + (h_0 + 2\log p_3)\tilde{F}^{(0)}(Z^2)\right.\\
& \left. + (2\log^2 p_3 + 2h_0 \log p_3 + \tfrac{1}{2}h_0^2 + \tfrac{1}{4}\pi^2)\tilde{F}^{(-1)}(Z^2)\right)\\
+ & \left(p_1 \leftrightarrow p_3, Z \mapsto X \text{ but } Z^2 \mapsto Z\frac{X}{Y}\right) + \left(p_2 \leftrightarrow p_3, Z \mapsto Y \text{ but } Z^2 \mapsto Z\frac{Y}{X}\right)\Bigg]\\
& + \frac{2p_1^2p_2^2p_3^2}{(\sqrt{-J^2})^3}\Big[\log^2(p_1p_2p_3) + h_2\log(p_1p_2p_3) + \tfrac{1}{4}h_2^2 - 1 + \tfrac{7}{24}\pi^2\\
& \quad - (h_2 + 2\log(p_1p_2p_3))\log\sqrt{-J^2} + \log^2\sqrt{-J^2}\Big] + \frac{3}{2}h_0\\
& + O(\epsilon),
\end{aligned}
\tag{2.A.67}
$$

where

$$
\tilde{F}^{(-1)}(x) = \frac{2x}{(x-1)^3}, \tag{2.A.68}
$$

$$
\tilde{F}^{(0)}(x) = \frac{1}{(x-1)^3}\left[-1 + x(4+x) - 4x\log(1-x)\right], \tag{2.A.69}
$$

$$
\tilde{F}^{(1)}(x) = \frac{4x}{(x-1)^3}\left[-2\log(1-x) + \log^2(1-x) + \operatorname{Li}_2 x\right] \tag{2.A.70}
$$

are coefficients of the series expansion

$$
{}_2F_1(1, 2+\epsilon; -\epsilon; x) = \frac{\tilde{F}^{(-1)}(x)}{\epsilon} + \tilde{F}^{(0)}(x) + \tilde{F}^{(1)}(x)\epsilon + O(\epsilon^2). \tag{2.A.71}
$$

Note that both $I_{1\{000\}}$ and $I_{1\{222\}}$ can be obtained from $I_{0\{111\}}$,

$$
I_{1\{000\}} = \frac{1}{2p_1p_2p_3}\left[p_1\frac{\partial^2}{\partial p_2 \partial p_3} + p_2\frac{\partial^2}{\partial p_1 \partial p_3} + p_3\frac{\partial^2}{\partial p_1 \partial p_2}\right] I_{0+\epsilon\{111\}}, \tag{2.A.72}
$$

$$
I_{2+\epsilon\{111\}} = \left[\frac{\partial^2}{\partial p_1^2} - \frac{1}{p_1}\frac{\partial}{\partial p_1}\right] I_{0+\epsilon\{111\}}. \tag{2.A.73}
$$

2.A.6. Triviality of $\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O}\rangle$

As our analysis shows, for any $d \geqslant 3$ and Δ_3 satisfying unitarity bound the correlation functions $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O}\rangle\!\rangle$ and $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3}\rangle\!\rangle$ are trivial, *i.e.*, they are at most semi-local. The triviality of $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3}\rangle\!\rangle$ was proved in [36] through a position space analysis. Our results independently confirm this through calculations in momentum space. In this section we discuss the triviality of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O}\rangle\!\rangle$ as an example.

The tensor decomposition of the transverse-traceless part of the $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O}\rangle\!\rangle$ correlator, the primary and secondary CWIs, and the transverse Ward identities are presented along with all 3-point functions in chapter 3. Let us rewrite here the important data. The solution in terms of triple-K integrals is

$$A_1^{aI} = \alpha_1^{aI} J_{3\{000\}}, \tag{2.A.74}$$
$$A_2^{aI} = 2\alpha_1^{aI} J_{2\{001\}} + \alpha_2^{aI} J_{1\{000\}}. \tag{2.A.75}$$

The independent secondary CWIs are

$$\mathrm{L}_{2,0}\, A_1^{aI} + \mathrm{R}_2\, A_2^{aI} = $$
$$= 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{2.A.76}$$
$$\mathrm{L}'_{1,0}\, A_1^{aI} + 2\,\mathrm{R}'_1\, A_2^{aI} \tag{2.A.77}$$
$$= -2(d-2) \cdot \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} \text{ in } p_{2\mu_2} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle,$$
$$\mathrm{L}_{2,0}\, A_2^{aI} = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \tag{2.A.78}$$

and the transverse Ward identities are

$$p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A^a_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{2.A.79}$$
$$p_{2\mu_2} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 2 p_{2\mu_2} \langle\!\langle \frac{\delta J^{\mu_2 a}}{\delta g^{\mu_1\nu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle. \tag{2.A.80}$$

If $\beta_3 = \Delta_3 - \frac{d}{2} > 0$, the same reasoning as in section 2.7.2 shows that the right-hand sides of (2.A.79, 2.A.80) vanish in the zero-momentum limit. Then, by the results of section 2.5.3, in the remaining cases the coefficient of p_3^0 in the series expansion of the right-hand sides of (2.A.79, 2.A.80) is at most ultralocal. The secondary CWIs lead to the following equations

$$0 = \frac{(\Delta_3 - 2d - \epsilon)(\Delta_3 - d - 2 - \epsilon)}{2(\Delta_3 - d - 1 - \epsilon)} l_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}-1, \Delta_3 - \frac{d}{2}\}} \times$$
$$\times \left[\left(\Delta_3^2 - \Delta_3(d - 2 + 2\epsilon) + \epsilon(2 + d + \epsilon) \right) \alpha_1^{aI} + \alpha_2^{aI} \right], \tag{2.A.81}$$
$$0 = \frac{(\Delta_3 - 2d - \epsilon)(\Delta_3 - d - \epsilon)}{2(\Delta_3 - d - 1 - \epsilon)} l_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}-1, \Delta_3 - \frac{d}{2}\}} \times \tag{2.A.82}$$
$$\times \left[\left((\Delta_3 - \epsilon)^2 - d(\Delta_3 + 2 - \epsilon) + 4(\Delta_3 + 1 + \epsilon) \right) \alpha_1^{aI} + 2\alpha_2^{aI} \right],$$
$$0 = (\Delta_3 - d - \epsilon) l_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}-1, \Delta_3 - \frac{d}{2}\}} \left[2(2\Delta_3 - d)\alpha_1^{aI} + \alpha_2^{aI} \right]. \tag{2.A.83}$$

If the $\epsilon \to 0$ limit exists, we find three equations whose only solutions are either $\alpha_1^{aI} = \alpha_2^{aI} = 0$, or else $\alpha_2^{aI} = 2(d-4)\alpha_1^{aI}$ and $\Delta_3 = 2$. The second solution, however, is not valid since $\Delta_3 = 2$ is a special case where the regulator cannot be removed. Instead one must analyse all the special cases when the regulator cannot be removed from $l_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}-1, \Delta_3 - \frac{d}{2}\}}$. The analysis is identical to that of section 2.5.3 and leads to the conclusion that $\alpha_1^{aI} = \alpha_2^{aI} = 0$.

Unlike in section 2.5.3, there are no additional conditions following from the coefficients of $p_3^{2\beta_3}$ or $p_3^{2\beta_3}\log p_3$ in the series expansion in p_3 of the secondary CWIs (2.A.81) - (2.A.83). Recall that such additional constraints arise when the equations following from the coefficients of $p_3^{2\beta_3}$ or $p_3^{2\beta_3}\log p_3$ are more singular than the equations following from the zero-momentum limit. In our case, it turns out that all coefficients of $p_3^{2\beta_3}$ or $p_3^{2\beta_3}\log p_3$ on the left-hand sides of (2.A.81) - (2.A.83) can be written in terms of $l_{\frac{d}{2}+\epsilon\{\frac{d}{2},\frac{d}{2}-1,-\Delta_3+\frac{d}{2}\}}$, accounting for all possible singularities. One can check that $l_{\frac{d}{2}+\epsilon\{\frac{d}{2},\frac{d}{2}-1,-\Delta_3+\frac{d}{2}\}}$ cannot be more singular than $l_{\frac{d}{2}+\epsilon\{\frac{d}{2},\frac{d}{2}-1,\Delta_3-\frac{d}{2}\}}$ assuming the unitarity bound $\Delta_3 \geqslant \frac{d}{2}-1$.

2.A.7. Form of scaling anomalies

In this appendix we list the most most general form for the anomalous part of the trace Ward identities, and evaluate the resulting contribution to the reconstruction formulae. Following the discussion at the end of section 2.8.2, the most general anomalous contribution to the trace Ward identities takes the form

$$
\begin{aligned}
&\langle\!\langle T(\boldsymbol{p}_1)\mathcal{O}^I(\boldsymbol{p}_2)\mathcal{O}^J(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = B_1^{IJ},\\
&\langle\!\langle T(\boldsymbol{p}_1)J^{\mu_2 a}(\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)p_3^{\alpha_2}\cdot B_1^{aI},\\
&\langle\!\langle T(\boldsymbol{p}_1)J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\left[B_1^{a_2a_3}p_3^{\alpha_2}p_1^{\alpha_3} + B_2^{a_2a_3}\delta^{\alpha_2\alpha_3}\right],\\
&\langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)p_3^{\alpha_2}p_3^{\beta_2}B_1^I + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2)B_2^I,\\
&\langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\left[B_1^a p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3} + B_2^a\delta^{\alpha_2\alpha_3}p_3^{\beta_2}\right]\\
&\qquad + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)p_1^{\alpha_3}B_3^a,\\
&\langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)\Big[B_1 p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3}\\
&\qquad + B_2\delta^{\beta_2\beta_3}p_3^{\alpha_2}p_1^{\alpha_3} + B_3\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}\Big]\\
&\qquad + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)p_1^{\alpha_3}p_1^{\beta_3}B_4 + \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)p_3^{\alpha_2}p_3^{\beta_2}B_4(p_2\leftrightarrow p_3)\\
&\qquad + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2)\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)B_5,
\end{aligned}
\tag{2.A.84}
$$

where the form factors B_j are polynomials in momenta squared p_j^2, $j=1,2,3$. In particular B_1^{IJ} can be non-zero only if $2\Delta = d+2n$ for some non-negative integer n.

The anomaly in the trace of the stress-energy tensor leads to specific anomalous contributions to the full correlation functions. These anomalous contributions can be recovered from the expressions above by the reconstruction formulae using the results of the section 2.3.3 and equation (2.3.18). In the presence of the trace anomaly, the following expressions should be added to the right-hand sides of the corresponding reconstruction equations, such as (3.11.4), listed in chapter 3.

$$
\begin{aligned}
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)\mathcal{O}^I(\boldsymbol{p}_2)\mathcal{O}^J(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} B_1^{IJ}, \\
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)J^a(\boldsymbol{p}_1)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) p_3^{\alpha_2} B_1^{aI}, \\
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \\
&= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \left[B_1^{a_2 a_3} p_3^{\alpha_2} p_1^{\alpha_3} + B_2^{a_2 a_3} \delta^{\alpha_2\alpha_3} \right], \\
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\mathcal{O}^I(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \left[\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) p_3^{\alpha_2} p_3^{\beta_2} B_1^I + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2) B_2^I \right] \\
&\quad + \left[(\boldsymbol{p}_1,\mu_1,\nu_1) \leftrightarrow (\boldsymbol{p}_2,\mu_2,\nu_2) \right] - \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1} B_2^I, \\
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)J^a(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} \\
&\quad + \left[(\boldsymbol{p}_1,\mu_1,\nu_1) \leftrightarrow (\boldsymbol{p}_2,\mu_2,\nu_2) \right] \\
&\quad - \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1} \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) p_1^{\alpha_3} B_3^a, \\
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} \\
&\quad \left[(\boldsymbol{p}_1,\mu_1,\nu_1) \leftrightarrow (\boldsymbol{p}_2,\mu_2,\nu_2) \right] + \left[(\boldsymbol{p}_1,\mu_1,\nu_1) \leftrightarrow (\boldsymbol{p}_3,\mu_3,\nu_3) \right] \\
&\quad - \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1} \left[\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3) B_4(p_1,p_2,p_3) + \pi^{\mu_3\nu_3}(\boldsymbol{p}_3) B_5(p_1,p_2,p_3) \right] \\
&\quad \left[(\boldsymbol{p}_1,\mu_1,\nu_1) \leftrightarrow (\boldsymbol{p}_3,\mu_3,\nu_3) \right] + \left[(\boldsymbol{p}_2,\mu_2,\nu_2) \leftrightarrow (\boldsymbol{p}_3,\mu_3,\nu_3) \right] \\
&\quad + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)}{d-1} B_5.
\end{aligned}
\tag{2.A.85}
$$

If present, such contributions should be added to the reconstruction formulae for the corresponding 3-point functions presented in chapter 3.

In particular, in $d = 4$ the anomalous trace Ward identity is

$$
\langle T \rangle = \frac{1}{2} P^{IJ} \phi_0^I \Box^{\Delta-2} \phi_0^J + \frac{1}{4} \kappa F^a_{\mu\nu} F^{\mu\nu a} + a E_4 + c W^2, \tag{2.A.86}
$$

where W^2 is the square of the Weyl tensor and E_4 is the Euler density, (2.8.3). From this we find

$$
\begin{aligned}
& B_1^{IJ} = (-p_1^2)^{\Delta-2} P^{IJ}, && B_1^{aI} = 0, \\
& B_1^{a_2 a_3} = -\kappa \delta^{a_2 a_3}, && B_2^{a_2 a_3} = \tfrac{1}{2}(p_2^2 + p_3^2 - p_1^2)\kappa \delta^{a_2 a_3}, \\
& B_1^I = 0, \\
& B_1^a = B_2^a = B_3^a = 0.
\end{aligned}
\tag{2.A.87}
$$

For $\langle\!\langle T(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3) \rangle\!\rangle$,

$$
\begin{aligned}
B_1 &= 8(a+c), & B_2 &= 8(p_1^2 - p_2^2 - p_3^2)(a+c), \\
B_3 &= -2J^2(a+c) + 4p_2^2 p_3^2 c, & B_4 &= -\frac{8}{3} a p_3^2, \\
B_5 &= \frac{4}{9} a J^2.
\end{aligned}
\tag{2.A.88}
$$

These results are in agreement with the results of sections 2.8.2 and 2.8.3.

2.A.8. Identities with projectors

The projectors are defined as

$$
\pi^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} - \frac{p^\mu p_\alpha}{p^2}, \tag{2.A.89}
$$

$$
\Pi^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\pi^{\mu}_{\alpha}\pi^{\nu}_{\beta} + \pi^{\mu}_{\beta}\pi^{\nu}_{\alpha} \right) - \frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}, \tag{2.A.90}
$$

$$
\Pi^{\mu\nu\rho\sigma} = \delta^{\rho\alpha}\delta^{\sigma\beta}\Pi^{\mu\nu}_{\alpha\beta}, \tag{2.A.91}
$$

One can find the following identities:

$$
\begin{aligned}
p_\mu \pi^{\mu\nu} = p_\mu \Pi^{\mu\nu\rho\sigma} &= 0, \\
\delta_{\mu\nu}\pi^{\mu\nu} = \pi^{\mu}_{\mu} &= d-1, \\
\Pi^{\mu\nu\rho}{}_{\rho} = \delta_{\rho\sigma}\Pi^{\mu\nu\rho\sigma} = \pi_{\rho\sigma}\Pi^{\mu\nu\rho\sigma} &= 0, \\
\Pi^{\mu\rho\nu}{}_{\rho} = \delta_{\rho\sigma}\Pi^{\mu\rho\nu\sigma} = \pi_{\rho\sigma}\Pi^{\mu\rho\nu\sigma} &= \frac{(d+1)(d-2)}{2(d-1)}\pi^{\mu\nu}, \\
\Pi^{\mu\nu\rho\sigma}\Pi_{\mu\nu\rho\sigma} &= \frac{1}{2}(d+1)(d-2), \\
\pi^{\mu}_{\alpha}\pi^{\alpha}_{\nu} &= \pi^{\mu}_{\nu}, \\
\Pi^{\mu\nu}_{\alpha\beta}\Pi^{\alpha\beta}_{\rho\sigma} &= \Pi^{\mu\nu}_{\rho\sigma}, \\
\Pi^{\mu\nu\rho}{}_{\alpha}\pi^{\alpha\sigma} &= \Pi^{\mu\nu\rho\sigma}, \\
\Pi^{\mu\nu}_{\alpha\beta}\Pi^{\alpha\rho\beta\sigma} &= \frac{d-3}{2(d-1)}\Pi^{\mu\nu\rho\sigma}.
\end{aligned}
\tag{2.A.92}
$$

Basic derivatives can be calculated directly. Denoting $\partial_\mu = \frac{\partial}{\partial p^\mu}$ we find

$$
\begin{aligned}
\partial_\kappa \pi_{\mu\nu} &= -\frac{p_\mu}{p^2}\pi_{\nu\kappa} - \frac{p_\nu}{p^2}\pi_{\mu\kappa}, \\
\partial_\kappa \Pi_{\mu\nu\rho\sigma} &= -\frac{p_\mu}{p^2}\Pi_{\kappa\nu\rho\sigma} - \frac{p_\nu}{p^2}\Pi_{\mu\kappa\rho\sigma} - \frac{p_\rho}{p^2}\Pi_{\mu\nu\kappa\sigma} - \frac{p_\sigma}{p^2}\Pi_{\mu\nu\rho\kappa}, \\
\pi^{\mu}_{\alpha}\partial_\kappa \pi^{\alpha}_{\nu} &= -\frac{p_\nu}{p^2}\pi^{\mu}_{\kappa}, \\
\pi^{\mu\kappa}\partial_\alpha \pi^{\alpha}_{\nu} - \pi^{\mu\alpha}\partial_\alpha \pi^{\kappa}_{\nu} &= -(d-2)\frac{p_\nu}{p^2}\pi^{\mu\kappa} + \frac{p^\kappa}{p^2}\pi^{\mu}_{\nu},
\end{aligned}
\tag{2.A.93}
$$

$$\begin{aligned}
\Pi^{\mu\nu}_{\alpha\beta}\partial_\kappa \Pi^{\alpha\beta}_{\rho\sigma} &= -\frac{p_\rho}{p^2}\Pi^{\mu\nu}_{\kappa\sigma} - \frac{p_\sigma}{p^2}\Pi^{\mu\nu}_{\rho\kappa}, \\
\Pi^{\mu\nu}_{\kappa\beta}\partial_\alpha \Pi^{\alpha\beta}_{\rho\sigma} - \Pi^{\mu\nu\alpha}{}_{\beta}\partial_\alpha \Pi^{\kappa\beta}_{\rho\sigma} &= -\frac{1}{2}\frac{d-1}{p^2}\left[p_\rho \Pi^{\mu\nu}_{\kappa\sigma} + p_\sigma \Pi^{\mu\nu}_{\rho\kappa}\right] + \frac{p_\kappa}{p^2}\Pi^{\mu\nu}_{\rho\sigma}. \qquad (2.A.94)
\end{aligned}$$

Analogous expressions with two derivatives are

$$\begin{aligned}
\pi^\mu_\alpha \partial^2 \pi^\alpha_\nu &= -\frac{2}{p^2}\pi^\mu_\nu, \\
p^\alpha \pi^\mu_\beta \partial_\alpha \partial_\kappa \pi^\beta_\nu &= \frac{p_\nu}{p^2}\pi^\mu_\kappa, \\
\Pi^{\mu\nu}_{\alpha\beta}\partial^2 \Pi^{\alpha\beta}_{\rho\sigma} &= -\frac{4}{p^2}\Pi^{\mu\nu}_{\rho\sigma}, \\
p^\gamma \Pi^{\mu\nu}_{\alpha\beta}\partial_\gamma \partial_\kappa \Pi^{\alpha\beta}_{\rho\sigma} &= \frac{p_\rho}{p^2}\Pi^{\mu\nu}_{\kappa\sigma} + \frac{p_\sigma}{p^2}\Pi^{\mu\nu}_{\rho\kappa}. \qquad (2.A.95)
\end{aligned}$$

For the semi-local operators defined in (2.3.14) and (2.3.15) we find

$$\begin{aligned}
\pi^\mu_\alpha \partial_\kappa j^\alpha_{\text{loc}} &= \frac{1}{p^2}\pi^\mu_\kappa r, \\
\pi^{\mu\kappa}\partial_\alpha j^\alpha_{\text{loc}} - \pi^{\mu\alpha}\partial_\alpha j^\kappa_{\text{loc}} &= \frac{d-3}{p^2}\pi^{\mu\kappa} r + \frac{1}{p^2}\pi^{\mu\kappa}p^\alpha \partial_\alpha r - \frac{p^\kappa}{p^2}\pi^{\mu\alpha}\partial_\alpha r, \\
\pi^\mu_\alpha \partial^2 j^\alpha_{\text{loc}} &= \frac{2}{p^2}\pi^{\mu\alpha}\partial_\alpha r, \\
p^\alpha \pi^\mu_\beta \partial_\alpha \partial_\kappa j^\beta_{\text{loc}} &= -\frac{2}{p^2}\pi^\mu_\kappa r + \frac{1}{p^2}\pi^\mu_\kappa p^\alpha \partial_\alpha r, \\
\Pi^{\mu\nu}_{\alpha\beta}\partial_\kappa t^{\alpha\beta}_{\text{loc}} &= \frac{2}{p^2}\Pi^{\mu\nu}_{\alpha\kappa}R^\alpha, \\
\Pi^{\mu\nu\kappa}{}_{\beta}\partial_\alpha t^{\alpha\beta}_{\text{loc}} - \Pi^{\mu\nu\alpha}{}_{\beta}\partial_\alpha t^{\kappa\beta}_{\text{loc}} &= \frac{d-2}{p^2}\Pi^{\mu\nu\kappa}{}_{\alpha}R^\alpha + \frac{p^\beta}{p^2}\Pi^{\mu\nu\kappa}{}_{\alpha}\partial_\beta R^\alpha - \frac{p^\kappa}{p^2}\Pi^{\mu\nu\alpha}{}_{\beta}\partial_\alpha R^\beta, \\
\Pi^{\mu\nu}_{\alpha\beta}\partial^2 t^{\alpha\beta}_{\text{loc}} &= \frac{4}{p^2}\Pi^{\mu\nu\alpha}{}_{\beta}\partial_\alpha R^\beta, \\
p^\gamma \Pi^{\mu\nu}_{\alpha\beta}\partial_\gamma \partial_\kappa t^{\alpha\beta}_{\text{loc}} &= -\frac{4}{p^2}\Pi^{\mu\nu}_{\alpha\kappa}R^\alpha + \frac{2}{p^2}\Pi^{\mu\nu}_{\alpha\kappa}p^\beta \partial_\beta R^\alpha. \qquad (2.A.96)
\end{aligned}$$

2.A.9. Contractions with helicity tensors

This appendix summarises our notation for the various contractions of helicity tensors. In $d = 3$ the helicity tensors satisfy

$$\Pi_{\mu\nu\rho\sigma}(\boldsymbol{p}) = \frac{1}{2}\sum_{s=\pm 1}\epsilon^{(s)}_{\mu\nu}(\boldsymbol{p})\bar{\epsilon}^{(s)}_{\rho\sigma}(\boldsymbol{p}), \qquad (2.A.97)$$

$$\epsilon^{(s)}_{\mu\nu}(\boldsymbol{p})\epsilon^{(s')\mu\nu}(\boldsymbol{p}) = 2\delta^{ss'}, \qquad (2.A.98)$$

$$\bar{\epsilon}^{(s)}_{\mu\nu}(\boldsymbol{p}) = \epsilon^{(s)}_{\mu\nu}(-\boldsymbol{p}), \qquad (2.A.99)$$

where overbar denotes the complex conjugation.

We use the following symbols,

$$\theta^{(s_3)}(p_i) = \epsilon^{(s_3)}_{\alpha\beta}(-\boldsymbol{p}_3)p_1^\alpha p_1^\beta = \epsilon^{(s_3)}_{\alpha\beta}(-\boldsymbol{p}_3)p_2^\alpha p_2^\beta, \tag{2.A.100}$$

$$\begin{aligned}\theta^{(s_2 s_3)}(p_i) &= \epsilon^{(s_2)}_{\alpha_1\beta_1}(-\boldsymbol{p}_2)\epsilon^{(s_3)}_{\alpha_2\beta_2}(-\boldsymbol{p}_3)\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\\ &= \epsilon^{(s_2)}_{\alpha\beta}(-\boldsymbol{p}_2)\epsilon^{(s_3)\alpha\beta}(-\boldsymbol{p}_3),\end{aligned} \tag{2.A.101}$$

$$\theta^{(s_1 s_2 s_3)}(p_i) = \epsilon^{(s_1)}_{\alpha_1\beta_1}(-\boldsymbol{p}_1)\epsilon^{(s_2)}_{\alpha_2\beta_2}(-\boldsymbol{p}_2)\epsilon^{(s_3)}_{\alpha_3\beta_3}(-\boldsymbol{p}_3)t^{\alpha_1\alpha_2\alpha_3}t^{\beta_1\beta_2\beta_3}, \tag{2.A.102}$$

where $t_{\alpha_1\alpha_2\alpha_3} = \delta_{\alpha_1\alpha_2}p_{1\alpha_3} + \delta_{\alpha_2\alpha_3}p_{2\alpha_1} + \delta_{\alpha_3\alpha_1}q_{3\alpha_2}$. In addition, the following contractions arise in the holographic analysis

$$\Theta^{(s_3)}_n(p_i) = \pi^{\alpha\beta}(\boldsymbol{p}_n)\epsilon^{(s_3)}_{\alpha\beta}(-\boldsymbol{p}_3) = -\theta^{(s_3)}_1(p_i), \quad n = 1,2,3, \tag{2.A.103}$$

$$\begin{aligned}\Theta^{(s_2 s_3)}(p_i) &= \pi^{\alpha\beta}(p_1)\epsilon^{(s_2)}_{\alpha\gamma}(-\boldsymbol{p}_2)\epsilon^{(s_3)\gamma}_{\beta}(-\boldsymbol{p}_3)\\ &= \theta^{(s_2 s_3)}(p_i) - \frac{p_1^\alpha p_1^\beta}{p_1^2}\epsilon^{(s_2)}_{\alpha\gamma}(-\boldsymbol{p}_2)\epsilon^{(s_3)\gamma}_{\beta}(-\boldsymbol{p}_3),\end{aligned} \tag{2.A.104}$$

$$\Theta^{(s_1 s_2 s_3)}(p_i) = \delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\epsilon^{(s_1)}_{\alpha_1\beta_1}(-\boldsymbol{p}_1)\epsilon^{(s_2)}_{\alpha_2\beta_2}(-\boldsymbol{p}_2)\epsilon^{(s_3)}_{a_3\beta_3}(-\boldsymbol{p}_3). \tag{2.A.105}$$

All the symbols depend on magnitudes of momenta only.

Define

$$\begin{aligned}S_1 &= -p_1^2 + (s_2 p_2 + s_3 p_3)^2,\\ S_2 &= -p_2^2 + (s_3 p_3 + s_1 p_1)^2,\\ S_3 &= -p_3^2 + (s_1 p_1 + s_2 p_2)^2.\end{aligned} \tag{2.A.106}$$

Using the exact presentation (2.9.14) one can find the following expressions for the defined symbols,

$$\theta^{(s_3)}(p_i) = \frac{J^2}{4\sqrt{2}p_3^2}, \tag{2.A.107}$$

$$\theta^{(s_2 s_3)}(p_i) = \frac{1}{8p_2^2 p_3^2}S_1^2, \tag{2.A.108}$$

$$\begin{aligned}\theta^{(s_1 s_2 s_3)}(p_i) &= \frac{J^2}{32\sqrt{2}p_1^2p_2^2p_3^2}\,(S_1+S_2+S_3)^2\\ &= \frac{J^2}{32\sqrt{2}p_1^2p_2^2p_3^2}\,(s_1p_1+s_2p_2+s_3p_3)^4,\end{aligned} \tag{2.A.109}$$

and similarly,

$$\Theta^{(s_3)}_n(p_i) = -\frac{J^2}{4\sqrt{2}p_n^2 p_3^2}, \quad n = 1,2, \qquad \Theta^{(s_3)}_3(p_i) = 0, \tag{2.A.110}$$

$$\Theta^{(s_2 s_3)}(p_i) = \frac{1}{8p_2^2 p_3^2} S_1^2 - \frac{J^2}{16 p_1^2 p_2^2 p_3^2} S_1, \tag{2.A.111}$$

$$\frac{p_1^\alpha}{p_1^\beta} p_1^2 \epsilon^{(s_2)}_{\alpha\gamma}(-\boldsymbol{p}_2)\epsilon^{(s_3)\gamma}_{\beta}(-\boldsymbol{p}_3) = \frac{J^2}{16 p_1^2 p_2^2 p_3^2} S_1, \tag{2.A.112}$$

$$\Theta^{(s_1 s_2 s_3)}(p_i) = -\frac{1}{16\sqrt{2} p_1^2 p_2^2 p_3^2} S_1 S_2 S_3, \tag{2.A.113}$$

where

$$J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3) \tag{2.A.114}$$

is defined in (2.6.18).

We define the helicity projected operators as

$$T(\boldsymbol{p}) = \delta^{\mu\nu} T_{\mu\nu}(\boldsymbol{p}), \tag{2.A.115}$$

$$T^{(s)}(\boldsymbol{p}) = \frac{1}{2}\epsilon^{(s)\mu\nu}(-\boldsymbol{p}) T_{\mu\nu}(\boldsymbol{p}), \tag{2.A.116}$$

$$\Upsilon(\boldsymbol{p}_1, \boldsymbol{p}_2) = \delta^{\mu\nu}\delta^{\rho\sigma}\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2), \tag{2.A.117}$$

$$\Upsilon^{(s)}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{1}{2}\delta^{\mu\nu}\epsilon^{(s)\rho\sigma}(-\boldsymbol{p}_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2), \tag{2.A.118}$$

$$\Upsilon^{(s_1 s_2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{1}{4}\epsilon^{(s_1)\mu\nu}(-\boldsymbol{p}_1)\epsilon^{(s_2)\rho\sigma}(-\boldsymbol{p}_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(\boldsymbol{p}_1, \boldsymbol{p}_2). \tag{2.A.119}$$

Chapter 3

List of results

3.1. Definitions

Here we collect together the necessary definitions and notation required to present our main results; further details may be found in chapter 2.

3.1.1. Basic conventions

We assume $d \geqslant 3$ Euclidean dimensions. Vectors are denoted by bold letters, *e.g.*, $\boldsymbol{p}_1$, but all results will be expressed in terms of the magnitudes of the momenta

$$p_j = |\boldsymbol{p}_j| = \sqrt{\boldsymbol{p}_j^2}, \qquad j = 1, 2, 3. \tag{3.1.1}$$

In particular, the form factors $A_j = A_j(p_1, p_2, p_3)$ are functions of the momentum magnitudes. Arrows denote the exchange of arguments, *e.g.*, $A_j(p_1 \leftrightarrow p_2) = A_j(p_2, p_1, p_3)$. If no arguments are given for a particular form factor then the standard ordering is assumed, $A_j(p_1, p_2, p_3)$.

To write the results in compact form, we frequently make use of the following symmetric polynomials in the momentum magnitudes

$$\begin{aligned} a_{123} &= p_1 + p_2 + p_3, \qquad b_{123} = p_1 p_2 + p_1 p_3 + p_2 p_3, \qquad c_{123} = p_1 p_2 p_3, \\ a_{ij} &= p_i + p_j, \qquad\qquad b_{ij} = p_i p_j, \end{aligned} \tag{3.1.2}$$

where $i, j = 1, 2, 3$. We also define the useful constant

$$s_d = \left\{ \begin{array}{ll} \frac{1}{\pi}(-1)^{\frac{d-1}{2}} & \text{if } d = 3, 5, 7, \dots \\ (-1)^{\frac{d}{2}-1} & \text{if } d = 4, 6, 8, \dots \end{array} \right. \tag{3.1.3}$$

3.1.2. Conformal Ward identities (CWIs) and triple-K integrals

The CWI operators K_j and K_{ij}, $i,j=1,2,3$ are defined in (2.4.21) and (2.4.22) by

$$\mathrm{K}_j = \frac{\partial^2}{\partial p_j^2} + \frac{d+1-2\Delta_j}{p_j}\frac{\partial}{\partial p_j}, \tag{3.1.4}$$

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j\,. \tag{3.1.5}$$

By Δ_j, $j=1,2,3$ we denote the conformal dimension of the j-th operator in a given 3-point function. For example in $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\!\rangle$ we have $\Delta_1 = d$ and $\Delta_2 = \Delta_3 = d-1$.

The triple-K integral (2.5.1) and its reduced version (2.5.3) are

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1,p_2,p_3) = \int_0^\infty \mathrm{d}x\, x^\alpha \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x), \tag{3.1.6}$$

$$J_{N\{k_j\}} = I_{\frac{d}{2}-1+N\{\Delta_j-\frac{d}{2}+k_j\}}, \tag{3.1.7}$$

where K_ν is the Bessel function K (modified Bessel function of the second kind) and we use a shortened notation $\{k_j\} = \{k_1k_2k_3\}$.

Triple-K integrals with half-integer β coefficients can be evaluated directly, while triple-K integrals with integer indices can be evaluated by means of the reduction scheme. These cases are sufficient to write the 3-point functions of conserved currents and stress-energy tensors in any dimension $d \geqslant 3$. In even dimensions the required integrals can all be evaluated from a single master integral $I_{0\{111\}}$. The result for this master integral is given in appendix 2.A.5 and the reduction scheme for other necessary integrals is presented in table 2.1 on page 94.

Solutions to the primary CWIs, see section 2.5.2, are given as linear combinations of reduced triple-K integrals multiplied by constants, denoted by α_j, $j=1,2,\ldots$ and called primary constants. If a primary constant is not restricted by means of the secondary CWIs, then it is a free parameter depending on the details of the theory.

If a triple-K integral diverges, then it can be regularised by

$$\alpha \mapsto \alpha + \epsilon, \qquad \beta_j \text{ does not change, } j=1,2,3 \tag{3.1.8}$$

At leading order in ϵ, this scheme is equivalent (up to ultralocal terms) to the momentum space dimensional regularisation $d \mapsto d-\epsilon$, see section 2.5.3. If the regulator ϵ cannot be removed, then both triple-K integrals and primary constants are power series in ϵ. By $x^{(n)}$ we denote the coefficient of ϵ^n in the series expansion of x about $\epsilon = 0$, where the regularisation $\alpha \mapsto \alpha+\epsilon$ is assumed.

The differential operators appearing in the secondary CWIs are defined by (2.4.27) and (2.4.28) and read

$$\begin{aligned}\mathrm{L}_{s,N} &= p_1(p_1^2+p_2^2-p_3^2)\frac{\partial}{\partial p_1} + 2p_1^2p_2\frac{\partial}{\partial p_2} \\ &\quad + \left[(2d-\Delta_1-2\Delta_2+s+N)p_1^2 + (\Delta_1-2+s)(p_3^2-p_2^2)\right],\end{aligned} \tag{3.1.9}$$

$$\mathrm{R}_s = p_1\frac{\partial}{\partial p_1} - (\Delta_1-2+s), \tag{3.1.10}$$

and their symmetric versions

$$\mathrm{L}'_{s,N} = \mathrm{L}_{s,N} \;\text{ with } (p_1 \leftrightarrow p_2) \text{ and } (\Delta_1 \leftrightarrow \Delta_2), \tag{3.1.11}$$

$$\mathrm{R}'_s = \mathrm{R}_s \;\text{ with } (p_1 \to p_2) \text{ and } (\Delta_1 \to \Delta_2). \tag{3.1.12}$$

The secondary CWIs denoted by an asterisk are redundant, *i.e.*, they do not impose any additional constraints on primary constants, see section 2.7.2.

3.1.3. Tensor decomposition

Our standard conventions for Lorentz indices are discussed in section 2.3.1 (see in particular (2.3.4)) and read

$$\boldsymbol{p}_1, \boldsymbol{p}_2 \text{ for } \mu_1, \nu_1; \;\; \boldsymbol{p}_2, \boldsymbol{p}_3 \text{ for } \mu_2, \nu_2 \text{ and } \boldsymbol{p}_3, \boldsymbol{p}_1 \text{ for } \mu_3, \nu_3. \tag{3.1.13}$$

The transverse and transverse-traceless projectors (2.1.13) and (2.1.14) are

$$\pi^{\mu}_{\alpha}(\boldsymbol{p}) = \delta^{\mu}_{\alpha} - \frac{p^{\mu} p_{\alpha}}{p^2}, \tag{3.1.14}$$

$$\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = \frac{1}{2}\left(\pi^{\mu}_{\alpha}(\boldsymbol{p})\pi^{\nu}_{\beta}(\boldsymbol{p}) + \pi^{\mu}_{\beta}(\boldsymbol{p})\pi^{\nu}_{\alpha}(\boldsymbol{p})\right) - \frac{1}{d-1}\pi^{\mu\nu}(\boldsymbol{p})\pi_{\alpha\beta}(\boldsymbol{p}). \tag{3.1.15}$$

The transverse(-traceless) and semi-local parts of the conserved current J^{μ} the stress-energy tensor $T^{\mu\nu}$ are given by (2.3.14) and (2.3.15) and read

$$j^{\mu} \equiv \pi^{\mu}_{\alpha} J^{\alpha}, \qquad\qquad j^{\mu}_{\text{loc}} \equiv J^{\mu} - j^{\mu}, \tag{3.1.16}$$

$$t^{\mu\nu} \equiv \Pi^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}, \qquad\qquad t^{\mu\nu}_{\text{loc}} \equiv T^{\mu\nu} - t^{\mu\nu}. \tag{3.1.17}$$

The semi-local parts (denoted with the subscript 'loc') can also be expressed as

$$j^{\mu}_{\text{loc}} = \frac{p^{\mu}}{p^2} r, \qquad t^{\mu\nu}_{\text{loc}} = \frac{p^{\mu}}{p^2} R^{\nu} + \frac{p^{\nu}}{p^2} R^{\mu} - \frac{p^{\mu} p^{\nu}}{p^4} R + \frac{1}{d-1}\pi^{\mu\nu}\left(T - \frac{R}{p^2}\right), \tag{3.1.18}$$

where longitudinal and trace parts are

$$r = p_{\mu} J^{\mu}, \qquad R^{\nu} = p_{\mu} T^{\mu\nu}, \qquad R = p_{\nu} R^{\nu}, \qquad T = T^{\mu}_{\mu}. \tag{3.1.19}$$

It will also be useful to define the operator $\mathscr{T}^{\mu\nu}_{\alpha}$ as in (2.3.19), namely

$$\mathscr{T}^{\mu\nu}_{\alpha}(\boldsymbol{p}) = \frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_{\alpha} - \frac{p_{\alpha}}{d-1}\left(\delta^{\mu\nu} + (d-2)\frac{p^{\mu} p^{\nu}}{p^2}\right)\right]. \tag{3.1.20}$$

We also denote $\mathscr{T}^{\mu\nu\alpha} = \delta^{\alpha\beta}\mathscr{T}^{\mu\nu}_{\beta}$.

3.1.4. Operators in the theory

We assume the CFT contains the following data:

- A symmetry group G. The conserved current $J^{\mu a}$, $a = 1, \ldots, \dim G$, is then the Noether current associated with the symmetry and is sourced by a potential A_μ^a. Currents transform in the adjoint representation and we denote the structure constants as f^{abc}. We assume the Killing form is diagonal, $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$, where T^a are generators of the group.
- Scalar primary operators $\mathcal{O}^I$ all of the same dimension Δ. They are sourced by ϕ_0^I and transform in a representation R of the symmetry group. The representation matrices are denoted by $(T_R^a)^{IJ}$.
- A stress-energy tensor $T_{\mu\nu}$ sourced by a metric $g^{\mu\nu}$.

The relevant Ward identities in the CFT are discussed in section 4; in particular the transverse Ward identities (and our assumptions for the terms in these containing functional derivatives) are given in section 2.4.4.

The normalisation constants $c_\mathcal{O}$, c_J, c_T of 2-point functions are

$$\langle\langle \mathcal{O}^I(\boldsymbol{p}) \mathcal{O}^J(-\boldsymbol{p}) \rangle\rangle = c_\mathcal{O} \delta^{IJ} \times \begin{cases} p^{2\Delta-d} & \text{if } 2\Delta \neq d+2n, \\ p^{2\Delta-d}\left(-\log p^2 + \text{local}\right) & \text{if } 2\Delta = d+2n, \end{cases} \tag{3.1.21}$$

$$\langle\langle J^{\mu a}(\boldsymbol{p}) J^{\nu b}(-\boldsymbol{p}) \rangle\rangle = c_J \delta^{ab} \times \begin{cases} \pi^{\mu\nu}(\boldsymbol{p}) p^{d-2} & \text{if } d = 3,5,7,\ldots \\ \pi^{\mu\nu}(\boldsymbol{p}) p^{d-2}\left(-\log p^2 + \text{local}\right) & \text{if } d = 4,6,8,\ldots \end{cases} \tag{3.1.22}$$

$$\langle\langle T^{\mu\nu}(\boldsymbol{p}) T^{\rho\sigma}(-\boldsymbol{p}) \rangle\rangle = c_T \times \begin{cases} \Pi^{\mu\nu\rho\sigma}(\boldsymbol{p}) p^{d} & \text{if } d = 3,5,7,\ldots \\ \Pi^{\mu\nu\rho\sigma}(\boldsymbol{p}) p^{d}\left(-\log p^2 + \text{local}\right) & \text{if } d = 4,6,8,\ldots \end{cases} \tag{3.1.23}$$

where n is anon-negative integer.

In the following, we will illustrate our general results with specific examples in $d = 3$, 4 and 5 dimensions. We consider for these purposes scalar operators both with dimensions $\Delta = d - 2$ and with dimension $\Delta = d$. The former may be constructed as $\mathcal{O} = \phi^2$ in a theory of free scalars, where ϕ is the fundamental field, while the latter presents an interesting case being marginal.

3.2. $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$

$$\langle\langle \mathcal{O}^{I_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\rangle = A_1^{I_1 I_2 I_3}(p_1, p_2, p_3), \tag{3.2.1}$$

The primary CWIs are

$$\mathrm{K}_{ij}\, A_1^{I_1 I_2 I_3} = 0, \qquad i, j = 1, 2, 3, \tag{3.2.2}$$

The solution in terms of triple-K integrals (3.1.7) is

$$A_1^{I_1 I_2 I_3} = \alpha_1^{I_1 I_2 I_3} J_{0\{000\}}, \tag{3.2.3}$$

where $\alpha_1^{I_1 I_2 I_3}$ is an arbitrary constant. (Note that primary constants inherit the group structure of the correlation function.) For any permutation σ of the set $\{1, 2, 3\}$ the A_1 form factor satisfies

$$A_1^{I_{\sigma(1)} I_{\sigma(2)} I_{\sigma(3)}}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = A_1^{I_1 I_2 I_3}(p_1, p_2, p_3). \tag{3.2.4}$$

3.3. $\langle J^{\mu_1}\mathcal{O}\mathcal{O}\rangle$

Ward identities. The transverse Ward identity is

$$\begin{aligned}
&p_{1\mu_1}\langle\!\langle J^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle = \\
&\quad = -(T^a_R)^{KI_3}\langle\!\langle\mathcal{O}^K(\boldsymbol{p}_2)\mathcal{O}^{I_2}(-\boldsymbol{p}_2)\rangle\!\rangle - (T^a_R)^{KI_2}\langle\!\langle\mathcal{O}^K(\boldsymbol{p}_3)\mathcal{O}^{I_3}(-\boldsymbol{p}_3)\rangle\!\rangle.
\end{aligned} \tag{3.3.1}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned}
&\langle\!\langle J^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle = \langle\!\langle j^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad - \frac{p_1^{\mu_1}}{p_1^2}\left[(T^a_R)^{KI_3}\langle\!\langle\mathcal{O}^K(\boldsymbol{p}_2)\mathcal{O}^{I_2}(-\boldsymbol{p}_2)\rangle\!\rangle + (T^a_R)^{KI_2}\langle\!\langle\mathcal{O}^K(\boldsymbol{p}_3)\mathcal{O}^{I_3}(-\boldsymbol{p}_3)\rangle\!\rangle\right].
\end{aligned} \tag{3.3.2}$$

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\langle\!\langle j^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle = \pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1)\cdot A_1^{aI_2I_3} p_2^{\alpha_1}, \tag{3.3.3}$$

where the form factor A_1 depends on the momentum magnitudes. This form factor is symmetric under $(p_2, I_2) \leftrightarrow (p_3, I_3)$, *i.e.*,

$$A_1^{aI_3I_2}(p_1, p_3, p_2) = A_1^{aI_2I_3}(p_1, p_2, p_3). \tag{3.3.4}$$

This form factor is given by

$$A_1^{aI_2I_3} = \text{coefficient of } p_2^{\mu_1} \text{ in } \langle\!\langle J^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle. \tag{3.3.5}$$

Primary conformal Ward identities. The primary CWIs are

$$\mathrm{K}_{ij}\, A_1^{aI_2I_3} = 0, \qquad i, j = 1, 2, 3, \tag{3.3.6}$$

The solution in terms of triple-K integrals (3.1.7) is

$$A_1^{aI_2I_3} = \alpha_1^{aI_2I_3} J_{1\{000\}}, \tag{3.3.7}$$

where $\alpha_1^{aI_2I_3}$ is a constant. In particular $\alpha_1^{aI_3I_2} = \alpha_1^{aI_2I_3}$. If the integral diverges, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWI is

$$\mathrm{L}_{1,0}\, A_1^{aI_2I_3} = 2(d-2)\left[p_{1\mu_1}\langle\!\langle J^{\mu_1 a}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle\right], \tag{3.3.8}$$

where $\mathrm{L}_{s,N}$ is given by (3.1.9). Assuming the unitarity bound $\Delta_2 = \Delta_3 = \Delta \geqslant \frac{d}{2} - 1$ for the dimensions of the scalar operators we find

$$\alpha_1^{aI_2I_3} = -\frac{2^{4-\frac{d}{2}}(T^a_R)^{I_2I_3} c_{\mathcal{O}}}{\Gamma\left(\frac{d}{2}-1\right)}\left(1 - \frac{1}{2}\delta_{\Delta,d+n}\right) \times C, \tag{3.3.9}$$

where

$$C = \begin{cases} \frac{1}{\pi} \sin\left(\frac{\pi}{2}(d - 2\Delta)\right) & \text{if } 2\Delta \neq d + 2n, \\ (-1)^{\Delta - \frac{d}{2}} & \text{if } 2\Delta = d + 2n, \end{cases} \tag{3.3.10}$$

where n is a non-negative integer and $c_{\mathcal{O}}$ represents the normalisation of the 2-point function (3.1.21). The 3-point function $\langle\langle J^{\mu_1} \mathcal{O}\mathcal{O} \rangle\rangle$ is therefore completely determined in terms of this normalisation.

Examples

For $d = 3$ and $\Delta_2 = \Delta_3 = 1$ we find

$$A_1^{aI_2I_3} = -\frac{2(T_R^a)^{I_2I_3} c_{\mathcal{O}}}{b_{23} a_{123}}. \tag{3.3.11}$$

For $d = 3$ and $\Delta_2 = \Delta_3 = 3$ we find

$$A_1^{aI_2I_3} = -p_1 (T_R^a)^{I_2I_3} c_{\mathcal{O}} \log a_{123} + \frac{c_{\mathcal{O}} (T_R^a)^{I_2I_3}}{a_{123}} \left[p_2^2 + p_3^2 - p_1^2 + p_2 p_3 + p_1 a_{123} (2 - \gamma_E) \right]. \tag{3.3.12}$$

For $d = 4$ and $\Delta_2 = \Delta_3 = 2$ we find

$$A_1^{aI_2I_3} = 4 p_1 (T_R^a)^{I_2I_3} c_{\mathcal{O}} \frac{\partial}{\partial p_1} I_{1\{000\}}, \tag{3.3.13}$$

where the $I_{1\{000\}}$ integral is given by (2.A.65).

For $d = 4$ and $\Delta_2 = \Delta_3 = 4$ we find

$$A_1^{aI_2I_3} = -2 (T_R^a)^{I_2I_3} c_{\mathcal{O}} I^{(0)}_{2\{122\}}, \tag{3.3.14}$$

where $I^{(0)}_{2\{122\}}$ denotes the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{2+\epsilon\{122\}}$. The integral can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 5$ and $\Delta_2 = \Delta_3 = 3$ we find

$$A_1^{aI_2I_3} = 2 (T_R^a)^{I_2I_3} c_{\mathcal{O}} \frac{p_1 + a_{123}}{a_{123}^2}. \tag{3.3.15}$$

For $d = 5$ and $\Delta_2 = \Delta_3 = 5$ we find

$$\begin{aligned} A_1^{aI_2I_3} = &-3 p_1^3 (T_R^a)^{I_2I_3} c_{\mathcal{O}} \log a_{123} + \frac{(T_R^a)^{I_2I_3} c_{\mathcal{O}}}{a_{123}^2} \Big[p_1^5 (4 - 3\gamma_E) + p_1^4 (11 - 6\gamma_E) a_{23} \\ &- p_1^3 \left((-10 + 3\gamma_E) a_{23}^2 + 3 b_{23} \right) + 2 a_{23} p_1^2 (2 a_{23}^2 - 3 b_{23}) \\ &+ (2 p_1 + a_{23})(a_{23}^4 - 3 a_{23}^2 b_{23} + b_{23}^2) \Big]. \end{aligned} \tag{3.3.16}$$

3.4. $\langle J^{\mu_1} J^{\mu_2} \mathcal{O} \rangle$

Ward identities. The transverse Ward identity is

$$p_{1\mu_1} \langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = p_{1\mu_1} \langle\!\langle \frac{\delta J^{\mu_1 a_1}}{\delta A^{a_2}_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle. \tag{3.4.1}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned} \langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle &= \langle\!\langle j^{\mu_1 a_1}(\boldsymbol{p}_1) j^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\ &+ \frac{p_1^{\mu_1} p_{1\alpha}}{p_1^2} \langle\!\langle \frac{\delta J^{\alpha a_1}}{\delta A^{a_2}_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle + \frac{p_2^{\mu_2} p_{2\beta}}{p_2^2} \langle\!\langle \frac{\delta J^{\beta a_2}}{\delta A^{a_1}_{\mu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\ &- \frac{p_1^{\mu_1} p_2^{\mu_2} p_{1\alpha} p_{2\beta}}{p_1^2 p_2^2} \langle\!\langle \frac{\delta J^{\alpha a_1}}{\delta A^{a_2}_{\beta}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle. \end{aligned} \tag{3.4.2}$$

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\langle\!\langle j^{\mu_1 a_1}(\boldsymbol{p}_1) j^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = \pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1) \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \left[A_1^{a_1 a_2 I} p_2^{\alpha_1} p_3^{\alpha_2} + A_2^{a_1 a_2 I} \delta^{\alpha_1 \alpha_2} \right]. \tag{3.4.3}$$

The form factors A_1 and A_2 are functions of the momentum magnitudes. Both form factors are symmetric under $(p_1, a_1) \leftrightarrow (p_2, a_2)$, *i.e.*, they satisfy

$$A_j^{a_2 a_1 I}(p_2, p_1, p_3) = A_j^{a_1 a_2 I}(p_1, p_2, p_3), \qquad j = 1, 2. \tag{3.4.4}$$

These form factors can be calculated as follows

$$\begin{aligned} A_1^{a_1 a_2 I} &= \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2}, \\ A_2^{a_1 a_2 I} &= \text{coefficient of } \delta^{\mu_1 \mu_2} \end{aligned} \tag{3.4.5}$$

in $\langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned} &\mathrm{K}_{12}\, A_1^{a_1 a_2 I} = 0, && \mathrm{K}_{13}\, A_1^{a_1 a_2 I} = 0, \\ &\mathrm{K}_{12}\, A_2^{a_1 a_2 I} = 0, && \mathrm{K}_{13}\, A_2^{a_1 a_2 I} = 2 A_1^{a_1 a_2 I}, \end{aligned} \tag{3.4.6}$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned} A_1^{a_1 a_2 I} &= \alpha_1^{a_1 a_2 I} J_{2\{000\}}, \\ A_2^{a_1 a_2 I} &= \alpha_1^{a_1 a_2 I} J_{1\{001\}} + \alpha_2^{a_1 a_2 I} J_{0\{000\}}, \end{aligned} \tag{3.4.7}$$

where $\alpha_j^{a I_2 I_3}$, $j = 1, 2$ are constants. In particular $\alpha_j^{a_2 a_1 I} = \alpha_j^{a_1 a_2 I}$ for $j = 1, 2$. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWI is

$$\begin{aligned} &\mathrm{L}_{1,0}\, A_1^{a_1 a_2 I} + 2\,\mathrm{R}_1\, A_2^{a_1 a_2 I} = \\ &\qquad = 2(d-2) \cdot \text{coefficient of } p_3^{\mu_2} \text{ in } p_{1\mu_1} \langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \end{aligned} \tag{3.4.8}$$

where L and R operators are given by (3.1.9) and (3.1.10). This leads to

$$\alpha_2^{(0) a_1 a_2 I} = \frac{1}{2} \Delta_3 (d - \Delta_3 - 2) \alpha_1^{a_1 a_2 I}, \tag{3.4.9}$$

$$\alpha_2^{(1) a_1 a_2 I} = \begin{cases} \frac{1}{2}(2\Delta_3 - d - 2)\alpha_1^{a_1 a_2 I} & \text{if } \Delta_3 \neq d - 2 - 2n, \\ -\frac{1}{2}(2\Delta_3 - d + 2)\alpha_1 + C c^{a_1 a_2 I} c_{\mathcal{O}} & \text{if } \Delta_3 = d - 2 - 2n, \end{cases} \tag{3.4.10}$$

$$\alpha_2^{(2) a_1 a_2 I} = -\frac{1}{2} \alpha_1^{a_1 a_2 I}, \tag{3.4.11}$$

where n is a non-negative integer and Δ_3 is the conformal dimension of the scalar operator satisfying the unitarity bound and $c_{\mathcal{O}}$ is the normalisation of the 2-point function (3.1.21). By $\alpha_j^{(n)}$ we mean the coefficient of ϵ^n in the series expansion of α_j, and we assume that the constant α_1 does not depend on the regulator, *i.e.*, $\alpha_1^{a_1 a_2 I} = \alpha_1^{(0) a_1 a_2 I}$. We denote

$$C = \frac{(-1)^{\frac{d-\Delta_3}{2}} 2^{2+\Delta_3-\frac{d}{2}} (d-2) \Gamma\left(\frac{d-\Delta_3}{2}\right) \Gamma\left(\frac{\Delta_3+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\Delta_3}{2}\right) \Gamma\left(\frac{d+\Delta_3}{2}\right)} \times \begin{cases} \frac{1}{\Gamma\left(\frac{d-2\Delta_3}{2}\right)} & \text{if } 2\Delta_3 \neq d + 2n, \\ -\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right) & \text{if } 2\Delta_3 = d + 2n, \end{cases} \tag{3.4.12}$$

and we define the $c^{a_1 a_2 K}$ constant as

$$p_{1\mu_1} \langle\!\langle \frac{\delta J^{\mu_1 a_1}}{\delta A_{\mu_2}^{a_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \bigg|_{p_1 = p_2 = p} = p_3^{\mu_2} \cdot c^{a_1 a_2 K} p^{d - \Delta_3 - 2} \langle\!\langle \mathcal{O}^K(\boldsymbol{p}_3) \mathcal{O}^I(-\boldsymbol{p}_3) \rangle\!\rangle + \ldots \tag{3.4.13}$$

What we mean here is the following: we first write down the most general tensor decomposition of $p_{1\mu_1} \langle\!\langle \frac{\delta J^{\mu_1 a_1}}{\delta A_{\mu_2}^{a_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle$ and take the coefficient of $p_3^{\mu_2}$. We then set $p_1 = p_2 = p$ in the expression for this coefficient. If the regulator is present, the limit $\epsilon \to 0$ can be taken after the primary constants and triple-K integrals are substituted into the primary CWIs (3.4.7).

To summarise, the 3-point function $\langle\!\langle J^{\mu_1} J^{\mu_2} \mathcal{O} \rangle\!\rangle$ depends on the 2-point function normalisations $c_{\mathcal{O}}$ and $c^{a_1 a_2 K}$, and on one undetermined primary constant $\alpha_1^{a_1 a_2 I}$.

Examples

For $d = 3$ and $\Delta_3 = 1$ we find

$$A_1^{a_1 a_2 I} = \frac{\alpha_1^{a_1 a_2 I}}{p_3 a_{123}^2}, \tag{3.4.14}$$

$$A_2^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} \left(\frac{1}{a_{123}} - \frac{1}{2p_3} \right) - \frac{c^{a_1 a_2 I} c_{\mathcal{O}}}{p_3}. \tag{3.4.15}$$

For $d = 3$ and $\Delta_3 = 3$ we find

$$A_1^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} \frac{a_{123} + p_3}{a_{123}^2}, \tag{3.4.16}$$

$$A_2^{a_1 a_2 I} = -\alpha_1^{a_1 a_2 I} \frac{-2p_3^2 + p_3 a_{12} + a_{12}^2}{2a_{123}}. \tag{3.4.17}$$

For $d = 4$ and $\Delta_3 = 2$ we find

$$A_1^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} p_1 p_2 \frac{\partial^2}{\partial p_1 p_2} I_{1\{000\}}, \tag{3.4.18}$$

$$A_2^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} I_{2\{111\}}^{(0)} + (\alpha_1^{a_1 a_2 I} - 2c^{a_1 a_2 I} c_{\mathcal{O}})(\log p_3 - \log 2 + \gamma_E), \tag{3.4.19}$$

where $I_{\alpha\{\beta_j\}}^{(0)}$ denotes the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$. Both $I_{1\{000\}}$ and $I_{2\{111\}}$ integrals are given in appendix (2.A.5).

For $d = 4$ and $\Delta_3 = 4$ we find

$$A_1^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} I_{3\{112\}}^{(0)}, \tag{3.4.20}$$

$$\begin{aligned} A_2^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} \Big[& -p_3 \frac{\partial}{\partial p_3} I_{1\{112\}}^{(0)} + p_1^2 \log p_1 + p_2^2 \log p_2 \\ & - (p_1^2 + p_2^2) \left(\tfrac{1}{2} - \gamma_E + \log 2 \right) - \tfrac{1}{2} p_3^2 \Big], \end{aligned} \tag{3.4.21}$$

where $I_{\alpha\{\beta_j\}}^{(0)}$ denotes the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$. The integrals can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 5$ and $\Delta_3 = 3$ we find

$$A_1^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} \frac{p_3^2 + 3p_3 a_{12} + 2(a_{12}^2 + b_{12})}{a_{123}^3}, \tag{3.4.22}$$

$$A_2^{a_1 a_2 I} = \alpha_1^{a_1 a_2 I} \frac{p_3^3 + 2p_3^2 a_{12} + p_3(-a_{12}^2 + 4b_{12}) + 2a_{13}(-a_{12}^2 + b_{12})}{2a_{123}^2} - p_3 c^{a_1 a_2 I} c_{\mathcal{O}}.$$

For $d = 5$ and $\Delta_3 = 5$ we find

$$\begin{aligned} A_1^{a_1 a_2 I} &= \frac{\alpha_1^{a_1 a_2 I}}{a_{123}^3} \left[-2p_3^4 - 6p_3^3 a_{12} - 2p_3^2(5a_{12}^2 - 4b_{12}) - 3a_{12}(a_{12} + 3p_3)(a_{12}^2 - b_{12}) \right], \\ A_2^{a_1 a_2 I} &= \frac{\alpha_1^{a_1 a_2 I}}{2a_{123}^2} \left[-2p_3^5 - 4p_3^4 a_{12} - 4p_3^3(a_{12}^2 - b_{12}) + p_3^2 a_{12}(a_{12}^2 - 7b_{12}) \right. \\ & \qquad \left. + 3a_{12}^2(2p_3 + a_{12})(a_{12}^2 - 3b_{12}) \right]. \end{aligned} \tag{3.4.23}$$

3.5. $\langle J^{\mu_1} J^{\mu_2} J^{\mu_3} \rangle$

Ward identities. The transverse Ward identity is

$$\begin{aligned} & p_{1\mu_1} \langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle = \\ & \quad = \mathrm{i} f^{a_1 b a_3} \langle\!\langle J^{\mu_3 b}(\boldsymbol{p}_2) J^{\mu_2 a_2}(-\boldsymbol{p}_2) \rangle\!\rangle - \mathrm{i} f^{a_1 a_2 b} \langle\!\langle J^{\mu_2 b}(\boldsymbol{p}_3) J^{\mu_3 a_3}(-\boldsymbol{p}_3) \rangle\!\rangle, \end{aligned} \tag{3.5.1}$$

where f^{abc} are the structure constants of the symmetry group.

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned}
\langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle j^{\mu_1 a_1}(\boldsymbol{p}_1) j^{\mu_2 a_2}(\boldsymbol{p}_2) j^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&+ \left[\frac{p_1^{\mu_1}}{p_1^2}\left(\mathrm{i} f^{a_1 b a_3}\langle\!\langle J^{\mu_3 b}(\boldsymbol{p}_2) J^{\mu_2 a_2}(-\boldsymbol{p}_2)\rangle\!\rangle - \mathrm{i} f^{a_1 a_2 b}\langle\!\langle J^{\mu_2 b}(\boldsymbol{p}_3) J^{\mu_3 a_3}(-\boldsymbol{p}_3)\rangle\!\rangle\right)\right] \\
&\quad + [(\mu_1, a_1, \boldsymbol{p}_1) \leftrightarrow (\mu_2, b_2, \boldsymbol{p}_2)] + [(\mu_1, a_1, \boldsymbol{p}_1) \leftrightarrow (\mu_3, a_3, \boldsymbol{p}_3)] \\
&+ \left[\frac{p_1^{\mu_1} p_2^{\mu_2}}{p_1^2 p_2^2}\mathrm{i} f^{a_1 a_2 b} p_{2\alpha}\langle\!\langle J^{\alpha b}(\boldsymbol{p}_3) J^{\mu_3 a_3}(-\boldsymbol{p}_3)\rangle\!\rangle\right] \\
&\quad + [(\mu_1, a_1, \boldsymbol{p}_1) \leftrightarrow (\mu_3, a_3, \boldsymbol{p}_3)] + [(\mu_2, a_2, \boldsymbol{p}_2) \leftrightarrow (\mu_3, a_3, \boldsymbol{p}_3)]. \qquad (3.5.2)
\end{aligned}$$

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\begin{aligned}
\langle\!\langle j^{\mu_1 a_1}(\boldsymbol{p}_1) j^{\mu_2 a_2}(\boldsymbol{p}_2) j^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle &= \pi^{\mu_1}_{\alpha_1}(\boldsymbol{p}_1)\pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\,[A_1^{a_1 a_2 a_3} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} \\
&+ A_2^{a_1 a_2 a_3}\delta^{\alpha_1\alpha_2} p_1^{\alpha_3} + A_2^{a_1 a_3 a_2}(p_2 \leftrightarrow p_3)\delta^{\alpha_1\alpha_3} p_3^{\alpha_2} \\
&+ A_2^{a_3 a_2 a_1}(p_1 \leftrightarrow p_3)\delta^{\alpha_2\alpha_3} p_2^{\alpha_1}]\,. \qquad (3.5.3)
\end{aligned}$$

The form factors A_1 and A_2 are functions of the momentum magnitudes. If no arguments are given, then we assume the standard ordering, $A_j = A_j(p_1, p_2, p_3)$, while by $p_i \leftrightarrow p_j$ we denote the exchange of the two momenta, *e.g.*, $A_2(p_1 \leftrightarrow p_3) = A_2(p_3, p_2, p_1)$.

The A_1 factor is completely antisymmetric, *i.e.*, for any permutation σ of the set $\{1, 2, 3\}$ it satisfies

$$A_1^{a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)}}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = (-1)^\sigma A_1^{a_1 a_2 a_3}(p_1, p_2, p_3), \qquad (3.5.4)$$

where $(-1)^\sigma$ denotes the sign of the permutation σ. The form factors A_2 is antisymmetric under $(p_1, a_1) \leftrightarrow (p_2, a_2)$, *i.e.*,

$$A_2^{a_2 a_1 a_3}(p_2, p_1, p_3) = -A_2^{a_1 a_2 a_3}(p_1, p_2, p_3). \qquad (3.5.5)$$

Note that the group structure of the form factors requires an existence of a tensor of the form $t^{a_1 a_2 a_3}$. As argued in [22], the correlation function vanishes if the symmetry group is Abelian.

The form factors can be calculated as follows

$$\begin{aligned}
A_1^{a_1 a_2 a_3} &= \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3}, \\
A_2^{a_1 a_2 a_3} &= \text{coefficient of } \delta^{\mu_1\mu_2} p_1^{\mu_3} \qquad (3.5.6)
\end{aligned}$$

in $\langle\!\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned}
&\mathrm{K}_{12}\, A_1^{a_1 a_2 a_3} = 0, && \mathrm{K}_{13}\, A_1^{a_1 a_2 a_3} = 0, \\
&\mathrm{K}_{12}\, A_2^{a_1 a_2 a_3} = 0, && \mathrm{K}_{13}\, A_2^{a_1 a_2 a_3} = 2 A_1^{a_1 a_2 a_3}. \qquad (3.5.7)
\end{aligned}$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned} A_1^{a_1 a_2 a_3} &= \alpha_1^{a_1 a_2 a_3} J_{3\{000\}}, \\ A_2^{a_1 a_2 a_3} &= \alpha_1^{a_1 a_2 a_3} J_{2\{001\}} + \alpha_2^{a_1 a_2 a_3} J_{1\{000\}}, \end{aligned} \tag{3.5.8}$$

where $\alpha_j^{a_1 a_2 a_3}$, $j = 1, 2$ are constants. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned} (*)\ \ &\mathrm{L}_{1,2}\, A_1^{a_1 a_2 a_3} + 2\,\mathrm{R}_1\, [A_2^{a_1 a_2 a_3} - A_2^{a_1 a_3 a_2}(p_2 \leftrightarrow p_3)] = \\ &\quad = 2(d-2) \cdot \text{coefficient of } p_3^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\mu_1} \langle\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{3.5.9}$$

$$\begin{aligned} &\mathrm{L}_{1,0}\, [A_2^{a_3 a_2 a_1}(p_1 \leftrightarrow p_3)] + 2p_1^2\, [A_2^{a_1 a_3 a_2}(p_2 \leftrightarrow p_3) - A_2^{a_1 a_2 a_3}] = \\ &\quad = 2(d-2) \cdot \text{coefficient of } \delta^{\mu_2 \mu_3} \text{ in } p_{1\mu_1} \langle\langle J^{\mu_1 a_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{3.5.10}$$

where L and R operators are given by (2.4.27) and (2.4.28). The identity denoted by the asterisk is redundant, *i.e.*, it is trivially satisfied in all cases and does not impose any additional conditions on primary constants. The secondary CWIs lead to

$$\alpha_2^{a_1 a_2 a_3} + (d-2)\alpha_1^{a_1 a_2 a_3} = \frac{2^{4-\frac{d}{2}} s_d \cdot \mathrm{i} f^{a_1 a_2 a_3} c_J}{\Gamma\left(\frac{d}{2} - 1\right)}, \tag{3.5.11}$$

where c_J is the 2-point function normalisation (3.1.22) and the constant s_d is defined in (3.1.3). If the regulator is present, the limit $\epsilon \to 0$ can be taken after the primary constants and triple-K integrals are substituted into the primary CWIs (3.5.8).

The 3-point function $\langle\langle J^{\mu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ therefore depends on the 2-point function normalisation c_J and on one undetermined primary constant $\alpha_1^{a_1 a_2 a_3}$.

Examples

For $d = 3$ we find

$$A_1^{a_1 a_2 a_3} = \frac{2\alpha_1^{a_1 a_2 a_3}}{a_{123}^3}, \tag{3.5.12}$$

$$A_2^{a_1 a_2 a_3} = \alpha_1^{a_1 a_2 a_3} \frac{p_3}{a_{123}^2} - \frac{2\mathrm{i} f^{a_1 a_2 a_3} c_J}{a_{123}}. \tag{3.5.13}$$

For $d = 4$ we find

$$A_1^{a_1 a_2 a_3} = \alpha_1^{a_1 a_2 a_3} I_{4\{111\}}, \tag{3.5.14}$$

$$A_2^{a_1 a_2 a_3} = -\left(\alpha_1^{a_1 a_2 a_3} p_3 \frac{\partial}{\partial p_3} + 4\mathrm{i} f^{a_1 a_2 a_3} c_J \right) I_{2\{111\}}^{(0)}, \tag{3.5.15}$$

where $I_{2\{111\}}^{(0)}$ is the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{2+\epsilon\{111\}}$. The integrals can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 5$ we find

$$A_1^{a_1 a_2 a_3} = \frac{2\alpha_1^{a_1 a_2 a_3}}{a_{123}^4} \left[a_{123}^3 + a_{123} b_{123} + 3 c_{123} \right], \tag{3.5.16}$$

$$\begin{aligned} A_2^{a_1 a_2 a_3} &= \frac{\alpha_1^{a_1 a_2 a_3} p_3^2}{a_{123}^3} \left[p_3^2 + 3 p_3 a_{12} + 2(a_{12}^2 + b_{12}) \right] \\ &\qquad - \frac{2\mathrm{i} f^{a_1 a_2 a_3} c_J}{a_{123}^2} \left[a_{123}^3 - a_{123} b_{123} - c_{123} \right]. \end{aligned} \tag{3.5.17}$$

3.6. $\langle T^{\mu_1 \nu_1} \mathcal{O} \mathcal{O} \rangle$

Ward identities. The transverse and trace Ward identities are

$$\begin{aligned} & p_1^{\nu_1} \langle\!\langle T_{\mu_1 \nu_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\!\rangle = \\ & \qquad = p_{3\mu_1} \langle\!\langle \mathcal{O}^{I_2}(\boldsymbol{p}_3) \mathcal{O}^{I_3}(-\boldsymbol{p}_3) \rangle\!\rangle + p_{2\mu_1} \langle\!\langle \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(-\boldsymbol{p}_2) \rangle\!\rangle, \end{aligned} \tag{3.6.1}$$

$$\langle\!\langle T(\boldsymbol{p}_1) \mathcal{O}^{I_2} \mathcal{O}^{I_3} \rangle\!\rangle = -\Delta_3 \left[\langle\!\langle \mathcal{O}^{I_2}(\boldsymbol{p}_3) \mathcal{O}^{I_3}(-\boldsymbol{p}_3) \rangle\!\rangle + \langle\!\langle \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(-\boldsymbol{p}_2) \rangle\!\rangle \right]. \tag{3.6.2}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned} \langle\!\langle T^{\mu_1 \nu_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\!\rangle &= \langle\!\langle t^{\mu_1 \nu_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\!\rangle \\ &+ \left[p_2^\alpha \mathscr{T}_\alpha^{\mu_1 \nu_1}(\boldsymbol{p}_1) - \frac{\Delta_3}{d-1} \pi^{\mu_1 \nu_1}(\boldsymbol{p}_1) \right] \langle\!\langle \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(-\boldsymbol{p}_2) \rangle\!\rangle + [\boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_3], \end{aligned} \tag{3.6.3}$$

where $\mathscr{T}_\alpha^{\mu_1 \nu_1}$ is defined in (3.1.20).

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\langle\!\langle t^{\mu_1 \nu_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\!\rangle = \Pi^{\mu_1 \nu_1}_{\alpha_1 \beta_1}(\boldsymbol{p}_1) \cdot A_1^{I_2 I_3} p_2^{\alpha_1} p_2^{\beta_1}, \tag{3.6.4}$$

where A_1 is a form factor depending on the momentum magnitudes. This form factor is symmetric under $(p_2, I_2) \leftrightarrow (p_3, I_3)$, *i.e.*,

$$A_1^{I_3 I_2}(p_1, p_3, p_2) = A_1^{I_2 I_3}(p_1, p_2, p_3) \tag{3.6.5}$$

and may be calculated as

$$A_1^{I_2 I_3} = \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} \text{ in } \langle\!\langle T^{\mu_1 \nu_1}(\boldsymbol{p}_1) \mathcal{O}^{I_2}(\boldsymbol{p}_2) \mathcal{O}^{I_3}(\boldsymbol{p}_3) \rangle\!\rangle. \tag{3.6.6}$$

Primary conformal Ward identities. The primary CWIs are

$$\mathrm{K}_{ij} A_1^{I_2 I_3} = 0, \qquad i, j = 1, 2, 3. \tag{3.6.7}$$

The solution in terms of triple-K integrals (3.1.7) is

$$A_1^{I_2 I_3} = \alpha_1^{I_2 I_3} J_{2\{000\}}, \tag{3.6.8}$$

where $\alpha_1^{I_2 I_3}$ is a constant (note $\alpha_1^{I_3 I_2} = \alpha_1^{I_2 I_3}$). If the integral diverges, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWI is

$$\mathrm{L}_{2,0}\, A_1^{I_2 I_3} = 2d \cdot \text{coefficient of } p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)\mathcal{O}^{I_2}(\boldsymbol{p}_2)\mathcal{O}^{I_3}(\boldsymbol{p}_3)\rangle\!\rangle, \tag{3.6.9}$$

where $\mathrm{L}_{s,N}$ is defined in (3.1.9). Assuming the unitarity bound for the conformal dimension of the scalar operator $\Delta_2 = \Delta_3 = \Delta \geqslant \frac{d}{2} - 1$ we find

$$\alpha_1^{I_2 I_3} = \frac{2^{3-\frac{d}{2}} c_{\mathcal{O}} \delta^{I_2 I_3}}{\Gamma\left(\frac{d}{2}\right)} \left(1 - \frac{1}{2}\delta_{\Delta, d+n+1}\right) \times C, \tag{3.6.10}$$

where

$$C = \begin{cases} \frac{1}{\pi}\sin\left(\frac{\pi}{2}(d - 2\Delta)\right) & \text{if } 2\Delta \neq d + 2n, \\ (-1)^{\Delta - \frac{d}{2}} & \text{if } 2\Delta = d + 2n, \end{cases} \tag{3.6.11}$$

n is a non-negative integer and $c_{\mathcal{O}}$ is the normalisation of the 2-point function (2.5.54).

The 3-point function $\langle\!\langle T^{\mu_1\nu_1}\mathcal{O}\mathcal{O}\rangle\!\rangle$ is thus uniquely determined in terms of the 2-point function normalisation $c_{\mathcal{O}}$.

Examples

For $\boldsymbol{d = 3}$ and $\boldsymbol{\Delta_2 = \Delta_3 = 1}$ we find

$$A_1^{I_2 I_3} = -2 c_{\mathcal{O}} \delta^{I_2 I_3} \frac{p_1 + a_{123}}{b_{23} a_{123}^2}. \tag{3.6.12}$$

For $\boldsymbol{d = 3}$ and $\boldsymbol{\Delta_2 = \Delta_3 = 3}$ we find

$$A_1^{I_2 I_3} = 2 c_{\mathcal{O}} \delta^{I_2 I_3} \frac{a_{123}^3 - a_{123} b_{123} - c_{123}}{a_{123}^2}. \tag{3.6.13}$$

For $\boldsymbol{d = 4}$ and $\boldsymbol{\Delta_2 = \Delta_3 = 2}$ we find

$$A_1^{I_2 I_3} = -2 c_{\mathcal{O}} \delta^{I_2 I_3} \left(p_1^2 \frac{\partial^2}{\partial p_1^2} - p_1 \frac{\partial}{\partial p_1} \right) I_{1\{000\}}, \tag{3.6.14}$$

where the $I_{1\{000\}}$ integral is given by (2.A.65).

For $\boldsymbol{d = 4}$ and $\boldsymbol{\Delta_2 = \Delta_3 = 4}$ we find

$$A_1^{I_2 I_3} = -2 c_{\mathcal{O}} \delta^{I_2 I_3} I_{3\{222\}}^{(0)}, \tag{3.6.15}$$

where $I_{3\{222\}}^{(0)}$ is the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{3+\epsilon\{222\}}$. The integral can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 5$ and $\Delta_2 = \Delta_3 = 3$ we find

$$A_1^{I_2 I_3} = \tfrac{2}{3} c_{\mathcal{O}} \delta^{I_2 I_3} \frac{8p_1^2 + 9p_1 a_{23} + 3a_{23}^2}{a_{123}^3}. \tag{3.6.16}$$

For $d = 5$ and $\Delta_2 = \Delta_3 = 5$ we find

$$A_1^{I_2 I_3} = \frac{2c_{\mathcal{O}} \delta^{I_2 I_3}}{3a_{123}^3} \left[3a_{123}^6 - 9a_{123}^4 b_{123} + 3a_{123}^2 b_{123}^2 + 3a_{123}^3 c_{123} + 3a_{123} b_{123} c_{123} + 2c_{123}^2 \right]. \tag{3.6.17}$$

3.7. $\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O} \rangle$

The transverse-traceless part of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O} \rangle\!\rangle$ vanishes. We present the detailed analysis of this case in appendix 2.A.6.

Ward identities. The transverse and trace Ward identities are

$$p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A^a_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{3.7.1}$$

$$p_{2\mu_2} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 2p_{2\mu_2} \langle\!\langle \frac{\delta J^{\mu_2 a}}{\delta g^{\mu_1\nu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{3.7.2}$$

$$\langle\!\langle T(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 0. \tag{3.7.3}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle &= \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\ &+ \frac{\pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) p_1^{\beta_1}}{p_1^2} \mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta A^a_{\alpha_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\ &+ \frac{2p_2^{\mu_2} p_{2\alpha_2}}{p_2^2} \delta^{\mu_1\alpha_1} \delta^{\nu_1\beta_1} \langle\!\langle \frac{\delta J^{\alpha_2 a}}{\delta g^{\alpha_1\beta_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \end{aligned} \tag{3.7.4}$$

where $\mathscr{T}^{\mu_1\nu_1}_{\alpha}$ is defined in (3.1.20).

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) j^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \left[A_1^{aI} p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} + A_2^{aI} \delta^{\alpha_1\alpha_2} p_2^{\beta_1} \right]. \tag{3.7.5}$$

The form factors A_1 and A_2 depend on the momentum magnitudes, and may be calculated as follows

$$\begin{aligned} A_1^{aI} &= \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2}, \\ A_2^{aI} &= 2 \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_2^{\nu_1} \end{aligned} \tag{3.7.6}$$

in $\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle$. These form factors do not exhibit any symmetry properties.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned} &\mathrm{K}_{12}\, A_1^{aI} = 0, && \mathrm{K}_{13}\, A_1^{aI} = 0, \\ &\mathrm{K}_{12}\, A_2^{aI} = 0, && \mathrm{K}_{13}\, A_2^{aI} = 4 A_1^{aI}. \end{aligned} \tag{3.7.7}$$

The solution in terms of triple-K integrals (3.1.7) is

$$A_1^{aI} = \alpha_1^{aI} J_{3\{000\}}, \tag{3.7.8}$$

$$A_2^{aI} = 2\alpha_1^{aI} J_{2\{001\}} + \alpha_2^{aI} J_{1\{000\}}, \tag{3.7.9}$$

where α_j^{aI}, $j = 1, 2$ are constants. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned} &\mathrm{L}_{2,0}\, A_1^{aI} + \mathrm{R}_2\, A_2^{aI} = \\ &= 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, && (3.7.10) \\ &\mathrm{L}'_{1,0}\, A_1^{aI} + 2\,\mathrm{R}'_1\, A_2^{aI} = \\ &\qquad = -2(d-2) \cdot \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} \text{ in } p_{2\mu_2} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, && (3.7.11) \\ &\mathrm{L}_{2,0}\, A_2^{aI} = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, && (3.7.12) \end{aligned}$$

where the L and R operators are given by (3.1.9) and (3.1.10). This leads to the trivial solution

$$\alpha_1^{aI} = \alpha_2^{aI} = 0. \tag{3.7.13}$$

3.8. $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$

Ward identities. The transverse and trace Ward identities are

$$\begin{aligned} &p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle = && (3.8.1) \\ &\qquad = p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_3}^{a_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2 a_2}(\boldsymbol{p}_2) \rangle\!\rangle + p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_2}^{a_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle \\ &\qquad\qquad + \ 2\delta^{\mu_3[\mu_1} p_3^{\alpha]} \langle\!\langle J^{\mu_2 a_2}(\boldsymbol{p}_2) J_\alpha^{a_3}(-\boldsymbol{p}_2) \rangle\!\rangle + 2\delta^{\mu_2[\mu_1} p_2^{\alpha]} \langle\!\langle J_\alpha^{a_2}(\boldsymbol{p}_3) J^{\mu_3 a_3}(-\boldsymbol{p}_3) \rangle\!\rangle, \\ &p_{2\mu_2} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle = \\ &\qquad = 2 p_{2\mu_2} \langle\!\langle \frac{\delta J^{\mu_2 a_2}}{\delta g^{\mu_1\nu_1}}(\boldsymbol{p}_2, \boldsymbol{p}_1) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle + \delta_{\mu_1\nu_1} p_{1\alpha} \langle\!\langle J^{\alpha a_2}(\boldsymbol{p}_3) J^{\mu_3 a_3}(-\boldsymbol{p}_3) \rangle\!\rangle, && (3.8.2) \\ &\langle\!\langle T(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle = \\ &\qquad = \langle\!\langle \frac{\delta T}{\delta A_{\mu_2}^{a_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle + \langle\!\langle \frac{\delta T}{\delta A_{\mu_3}^{a_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) J^{\mu_2 a_2}(\boldsymbol{p}_2) \rangle\!\rangle. && (3.8.3) \end{aligned}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$
\begin{aligned}
&\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle = \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)j^{\mu_2 a_2}(\boldsymbol{p}_2)j^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad + \left[2\mathscr{T}^{\mu_1\nu_1}_{\alpha}(\boldsymbol{p}_1)\pi^{\mu_3[\alpha}(\boldsymbol{p}_3)p_3^{\beta]} + \frac{p_3^{\mu_3}}{p_3^2}\delta^{\mu_1\nu_1}p_{1\beta}\right]\langle\!\langle J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{a_3}_{\beta}(-\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1)p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1}\delta^{\alpha_1\beta_1}\right]\langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta A^{a_3}_{\alpha_3}}(\boldsymbol{p}_1,\boldsymbol{p}_3)J^{\mu_2 a_2}(\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \frac{2p_3^{\mu_3}p_{3\alpha_3}}{p_3^2}\delta^{\mu_1\alpha_1}\delta^{\nu_1\beta_1}\langle\!\langle \frac{\delta J^{\alpha_3 a_3}}{\delta g^{\alpha_1\beta_1}}(\boldsymbol{p}_3,\boldsymbol{p}_1)J^{\mu_2 a_2}(\boldsymbol{p}_2)\rangle\!\rangle \\
&\quad + \text{everything with } (\boldsymbol{p}_2, a_2, \mu_2) \leftrightarrow (\boldsymbol{p}_3, a_3, \mu_3),
\end{aligned}
\tag{3.8.4}
$$

where $\mathscr{T}^{\mu_1\nu_1}_{\alpha}$ is defined in (3.1.20).

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$
\begin{aligned}
&\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)j^{\mu_2 a_2}(\boldsymbol{p}_2)j^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\Big[A_1^{a_2a_3}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3} + A_2^{a_2a_3}\delta^{\alpha_2\alpha_3}p_2^{\alpha_1}p_2^{\beta_1} \\
&\qquad + A_3^{a_2a_3}\delta^{\alpha_1\alpha_2}p_2^{\beta_1}p_1^{\alpha_3} + A_3^{a_3a_2}(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}p_2^{\beta_1}p_3^{\alpha_2} \\
&\qquad + A_4^{a_2a_3}\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\Big].
\end{aligned}
\tag{3.8.5}
$$

The form factors A_j, $j = 1,2,3,4$ are functions of the momentum magnitudes. If no arguments are specified then the standard ordering is assumed, $A_j = A_j(p_1,p_2,p_3)$, while by $p_i \leftrightarrow p_j$ we denote the exchange of the two momenta, *e.g.*, $A_3(p_2 \leftrightarrow p_3) = A_3(p_1,p_3,p_2)$.

The form factors A_1, A_2 and A_4 are symmetric under $(p_2,a_2) \leftrightarrow (p_3,a_3)$, *i.e.*, they satisfy,

$$
A_j^{a_3a_2}(p_1,p_3,p_2) = A_j^{a_2a_3}(p_1,p_2,p_3), \qquad j \in \{1,2,4\},
\tag{3.8.6}
$$

while the form factor A_3 does not exhibit any symmetry properties.

The form factors may be determined as follows

$$
\begin{aligned}
A_1^{a_2a_3} &= \text{coefficient of } p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}p_1^{\mu_3}, \\
A_2^{a_2a_3} &= \text{coefficient of } \delta^{\mu_2\mu_3}p_2^{\mu_1}p_2^{\nu_1}, \\
A_3^{a_2a_3} &= 2\cdot\text{coefficient of } \delta^{\mu_1\mu_2}p_2^{\nu_1}p_1^{\mu_3}, \\
A_4^{a_2a_3} &= 2\cdot\text{coefficient of } \delta^{\mu_1\mu_2}\delta^{\mu_3\nu_1},
\end{aligned}
\tag{3.8.7}
$$

in $\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)J^{\mu_2 a_2}(\boldsymbol{p}_2)J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$
\begin{aligned}
&\mathrm{K}_{12}\, A_1^{a_2a_3} = 0, && \mathrm{K}_{13}\, A_1^{a_2a_3} = 0, \\
&\mathrm{K}_{12}\, A_2^{a_2a_3} = -2A_1^{a_2a_3}, && \mathrm{K}_{13}\, A_2^{a_2a_3} = -2A_1^{a_2a_3}, \\
&\mathrm{K}_{12}\, A_3^{a_2a_3} = 0, && \mathrm{K}_{13}\, A_3^{a_2a_3} = 4A_1^{a_2a_3}, \\
&\mathrm{K}_{12}\, A_4^{a_2a_3} = 2A_3^{a_2a_3}, && \mathrm{K}_{13}\, A_4^{a_2a_3} = 2A_3^{a_3a_2}(p_2\leftrightarrow p_3),
\end{aligned}
\tag{3.8.8}
$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned}
A_1^{a_2a_3} &= \alpha_1^{a_2a_3} J_{4\{000\}}, \\
A_2^{a_2a_3} &= \alpha_1^{a_2a_3} J_{3\{100\}} + \alpha_2^{a_2a_3} J_{2\{000\}}, \\
A_3^{a_2a_3} &= 2\alpha_1^{a_2a_3} J_{3\{001\}} + \alpha_3^{a_2a_3} J_{2\{000\}}, \\
A_4^{a_2a_3} &= 2\alpha_1^{a_2a_3} J_{2\{011\}} + \alpha_3^{a_2a_3} \left(J_{1\{010\}} + J_{1\{001\}} \right) + \alpha_4^{a_2a_3} J_{0\{000\}},
\end{aligned} \tag{3.8.9}$$

where $\alpha_j^{a_2a_3}$, $j = 1,2,3,4$ are constants. In particular all constants are symmetric in the group indices, $\alpha_j^{a_3a_2} = \alpha_j^{a_2a_3}$, $j = 1,2,3,4$. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned}
(*)\ \ & \mathrm{L}_{2,2}\, A_1^{a_2a_3} + \mathrm{R}_2 \left[A_3^{a_2a_3} - A_3^{a_2a_3}(p_2 \leftrightarrow p_3) \right] = \\
& \quad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle, \qquad (3.8.10) \\
& \mathrm{L}'_{1,2}\, A_1^{a_2a_3} + 2\,\mathrm{R}'_1 \left[A_3^{a_2a_3} - A_2^{a_2a_3} \right] = \\
& \quad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_1^{\mu_3} \text{ in } p_{2\mu_2} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle, \qquad (3.8.11) \\
& \mathrm{L}_{2,0}\, A_2^{a_2a_3} - p_1^2 \left[A_3^{a_2a_3} - A_3^{a_2a_3}(p_2 \leftrightarrow p_3) \right] = \\
& \quad = 2d \cdot \text{coefficient of } \delta^{\mu_2\mu_3} p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3) \rangle\!\rangle, \qquad (3.8.12) \\
& \mathrm{L}_{2,2}\, A_3^{a_2a_3} - 2\,\mathrm{R}_2\, A_4^{a_2a_3} = \\
& \quad = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3)) \rangle\!\rangle, \qquad (3.8.13)
\end{aligned}$$

where L and R are given by (3.1.9) and (3.1.10). The identity denoted by the asterisk is redundant, *i.e.*, it is trivially satisfied in all cases and does not impose any additional conditions on primary constants. The secondary CWIs lead to the following relations,

$$\alpha_3^{a_2a_3} = \alpha_2^{a_2a_3}, \tag{3.8.14}$$

$$d\alpha_1^{a_2a_3} + \alpha_2^{a_2a_3} = \frac{2^{3-\frac{d}{2}} s_d c_J \delta^{a_2a_3}}{\Gamma\left(\frac{d}{2}\right)}, \tag{3.8.15}$$

$$\alpha_4^{(0)a_2a_3} = -(d-2)\alpha_2^{a_2a_3}, \tag{3.8.16}$$

$$2d(d-2)\alpha_1^{a_2a_3} + d\alpha_3^{a_2a_3} + 2\alpha_4^{(1)a_2a_3} = \frac{2^{5-\frac{d}{2}} s_d c^{a_2a_3} c_J}{\Gamma\left(\frac{d}{2} - 1\right)}, \tag{3.8.17}$$

where c_J is the normalisation of the 2-point function (3.1.22), the constant s_d is defined in (3.1.3) while the constant c^{ab} is defined as

$$\langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta A^a_{\mu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 b}(\boldsymbol{p}_3) \rangle\!\rangle = 2c^{ab} \delta^{\mu_2}_{(\mu_1} \langle\!\langle J^a_{\nu_1)}(\boldsymbol{p}_3) J^{\mu_3 b}(-\boldsymbol{p}_3) \rangle\!\rangle. \tag{3.8.18}$$

By $\alpha_4^{(n)a_2a_3}$ we mean the coefficient of ϵ^n in the series expansion of $\alpha_4^{a_2a_3}$ in the regulator ϵ. We assume that the α_1 coefficient is independent of ϵ, *i.e.*, $\alpha_1^{a_2a_3} = \alpha_1^{(0)a_2a_3}$. If the regulator is present, the limit $\epsilon \to 0$ can be taken after the primary constants and triple-K integrals are substituted into the primary CWIs (3.8.9).

The 3-point function $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\!\rangle$ thus depends on the 2-point function normalisations c_J and c^{ab} and one undetermined primary constant $\alpha_1^{a_2 a_3}$. The dependence of this correlator on two 2-point function normalisations rather than the one found in [22] is related to our definition of this correlator, as discussed above (2.7.29).

Examples

For $d = 3$ we find

$$A_1^{a_2a_3} = \alpha_1^{a_2a_3} \frac{2(4p_1 + a_{23})}{a_{123}^4}, \tag{3.8.19}$$

$$A_2^{a_2a_3} = \alpha_1^{a_2a_3} \frac{2p_1^2}{a_{123}^3} - \frac{2(2p_1 + a_{23})}{a_{123}^2} c_J \delta^{a_2a_3}, \tag{3.8.20}$$

$$A_3^{a_2a_3} = \frac{\alpha_1^{a_2a_3}}{a_{123}^3} \left[-2p_1^2 - p_2^2 + p_3^2 - 3p_1p_2 + 3p_1p_3 \right] - \frac{2(2p_1 + a_{23})}{a_{123}^2} c_J \delta^{a_2a_3}, \tag{3.8.21}$$

$$\begin{aligned} A_4^{a_2a_3} &= \alpha_1^{a_2a_3} \frac{(2p_1 + a_{23})(p_1^2 - a_{23}^2 + 4b_{23})}{2a_{123}^2} \\ &\quad + \left(\frac{2p_1^2}{a_{123}} - a_{23} \right) c_J \delta^{a_2a_3} + 2a_{23} c^{a_3a_2} c_J. \end{aligned} \tag{3.8.22}$$

For $d = 4$ we find

$$A_1^{a_2a_3} = \alpha_1^{a_2a_3} I_{5\{211\}}, \tag{3.8.23}$$

$$A_2^{a_2a_3} = - \left(2c_J \delta^{a_2a_3} + \alpha_1^{a_2a_3} p_1 \frac{\partial}{\partial p_1} \right) I^{(0)}_{3\{211\}}, \tag{3.8.24}$$

$$A_3^{a_2a_3} = -2 \left(c_J \delta^{a_2a_3} + \alpha_1^{a_2a_3} p_3 \frac{\partial}{\partial p_3} \right) I^{(0)}_{3\{211\}}, \tag{3.8.25}$$

$$\begin{aligned} A_4^{a_2a_3} &= 2c_J \delta^{a_2a_3} \left[-2 + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} \right] I^{(0)}_{1\{211\}} + 2\alpha_1^{a_2a_3} p_2 p_3 \frac{\partial^2}{\partial p_2 \partial p_3} I^{(0)}_{1\{211\}} \\ &\quad + 4(c_J \delta^{a_2a_3} - c^{a_3a_2} c_J) \left[p_2^2 \log p_2 + p_3^2 \log p_3 \right. \\ &\quad \left. - (p_2^2 + p_3^2) \left(\tfrac{1}{2} - \gamma_E + \log 2 \right) - \tfrac{1}{2} p_1^2 \right], \end{aligned} \tag{3.8.26}$$

where $I^{(0)}_{\alpha\{\beta_j\}}$ denotes the the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$ in ϵ. The integrals can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

In case of $d = 4$ the trace Ward identity is anomalous,

$$\langle T \rangle = \frac{\kappa}{4} F^a_{\mu\nu} F^{\mu\nu a}, \qquad \kappa = 2c_J. \tag{3.8.27}$$

It leads to the anomalous contribution,

$$\langle\!\langle T(\mathbf{p}_1) J^{\mu_2 a_2}(\mathbf{p}_2) J^{\mu_3 a_3}(\mathbf{p}_3) \rangle\!\rangle_{\text{anomaly}} = -\kappa \delta^{a_2a_3} \left[p_3^{\mu_2} p_2^{\mu_3} + \tfrac{1}{2}(p_1^2 - p_2^2 - p_3^2) \delta^{\mu_2\mu_3} \right]. \tag{3.8.28}$$

Then the anomalous contribution to the full 3-point function is

$$\begin{aligned}\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) J^{\mu_2 a_2}(\boldsymbol{p}_2) J^{\mu_3 a_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} &= \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \pi^{\mu_2}_{\alpha_2}(\boldsymbol{p}_2) \pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3) \delta^{a_2 a_3} \times \\ &\times\ \kappa \left[-p_3^{\alpha_2} p_1^{\alpha_3} - \tfrac{1}{2}(p_1^2 - p_2^2 - p_3^2)\delta^{\alpha_2\alpha_3} \right] \end{aligned} \tag{3.8.29}$$

and should be added to the right hand side of (3.8.4).

The scheme dependence of the solution (3.8.23) - (3.8.26) due to the counterterm is

$$\begin{aligned} A_2^{a_2 a_3} &\mapsto A_2^{a_2 a_3} + 2\kappa_0 \delta^{a_2 a_3}, \\ A_3^{a_2 a_3} &\mapsto A_3^{a_2 a_3} + 2\kappa_0 \delta^{a_2 a_3}, \\ A_4^{a_2 a_3} &\mapsto A_4^{a_2 a_3} - \kappa_0 (p_1^2 - p_2^2 - p_3^2)\delta^{a_2 a_3} - 2 c^{a_3 a_2} \kappa_0 (p_2^2 + p_3^2), \end{aligned} \tag{3.8.30}$$

where κ_0 is an arbitrary constant. See section 2.8.2 for details.

For $d = 5$ we find

$$\begin{aligned} A_1^{a_2 a_3} &= \frac{2\alpha_1^{a_2 a_3}}{a_{123}^5} \left[4p_1^4 + 20 p_1^3 a_{23} + 4p_1^2 (7a_{23}^2 + 6b_{23}) \right. \\ &\qquad \left. + 15 p_1 a_{23}(a_{23}^2 + b_{23}) + 3a_{23}^2 (a_{23}^2 + b_{23}) \right], \end{aligned} \tag{3.8.31}$$

$$\begin{aligned} A_2^{a_2 a_3} &= \frac{2\alpha_1^{a_2 a_3} p_1^2}{a_{123}^4} \left[a_{123}^3 + a_{123} b_{123} + 3c_{123} \right] \\ &\quad - \frac{2 c_J \delta^{a_2 a_3}}{3 a_{123}^3} \left[2p_1^4 + 6 p_1^3 a_{23} + 2p_1^2 (5a_{23}^2 - 4b_{23}) \right. \\ &\qquad \left. + 9 p_1 (a_{23}^3 - a_{23} b_{23}) + 3 a_{23}^2 (a_{23}^2 - b_{23}) \right], \end{aligned} \tag{3.8.32}$$

$$\begin{aligned} A_3^{a_2 a_3} &= \frac{\alpha_1^{a_2 a_3}}{a_{123}^4} \left[-2p_1^5 - 8p_1^4 (p_2 + p_3) - 8 p_1^3 p_2 (2p_2 + 3p_3) \right. \\ &\qquad + p_1^2 (-19 p_2^3 - 40 p_2^2 p_3 + 24 p_2 p_3^2 + 15 p_3^3) \\ &\qquad \left. - 3(4p_1 + p_2 + p_3)(p_2^2 - p_3^2)(p_2^2 + 3p_2 p_3 + p_3^2) \right] \\ &\quad - \frac{2 c_J \delta^{a_2 a_3}}{3 a_{123}^3} \left[2p_1^4 + 6p_1^3 a_{23} + 2p_1^2 (5 a_{23}^2 - 4 b_{23}) \right. \\ &\qquad \left. + 9 p_1 a_{23} (a_{23}^2 - b_{23}) + 3 a_{23}^2 (a_{23}^2 - b_{23}) \right], \end{aligned} \tag{3.8.33}$$

$$\begin{aligned} A_4^{a_2 a_3} &= \frac{\alpha_1^{a_2 a_3}}{2 a_{123}^3} \left[2p_1^6 + 6p_1^5 a_{23} + 4p_1^4 (2a_{23}^2 - b_{23}) + p_1^2 (a_{23}^2 + b_{23})(3 p_1 a_{23} - 7a_{23}^2 + 32 b_{23}) \right. \\ &\qquad \left. - 3 a_{23} (3p_1 + a_{23})(a_{23}^2 - 4b_{23})(a_{23}^2 + b_{23}) \right] \\ &\quad + \frac{c_J \delta^{a_2 a_3}}{3 a_{123}^2} \left[2p_1^5 + 4p_1^4 a_{23} + 4p_1^3 (a_{23}^2 - b_{23}) - p_1^2 a_{23} (a_{23}^2 - 7b_{23}) \right. \\ &\qquad \left. - 6 p_1 a_{23}^2 (a_{23}^2 - 3b_{23}) - 3 a_{23}^3 (a_{23}^2 - 3b_{23}) \right] \\ &\quad + 6(p_2^3 + p_3^3) c^{a_3 a_2} c_J. \end{aligned} \tag{3.8.34}$$

3.9. $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$

Ward identities. The transverse and trace Ward identities are

$$p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 2 p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle, \tag{3.9.1}$$

$$\langle\!\langle T(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = 2 \langle\!\langle \frac{\delta T}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle. \tag{3.9.2}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned}
&\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle = \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\
&\quad + 2 \left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \delta^{\alpha_1\beta_1} \right] \delta^{\mu_2\alpha_2} \delta^{\nu_2\beta_2} \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle \\
&\quad + 2[(\mu_1, \nu_1, \boldsymbol{p}_1) \leftrightarrow (\mu_2, \nu_2, \boldsymbol{p}_2)] \\
&\quad - 4 \left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \delta^{\alpha_1\beta_1} \right] \left[\mathscr{T}^{\mu_2\nu_2\alpha_2}(\boldsymbol{p}_2) p_2^{\beta_2} + \frac{\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1} \delta^{\alpha_2\beta_2} \right] \times \\
&\qquad\qquad \times \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle,
\end{aligned} \tag{3.9.3}$$

where $\mathscr{T}^{\mu\nu}_{\alpha}$ is defined in (3.1.20).

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\begin{aligned}
\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(p_3) \rangle\!\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \left[A_1^I p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} \right. \\
&\quad \left. + A_2^I \delta^{\alpha_1\alpha_2} p_2^{\beta_1} p_3^{\beta_2} + A_3^I \delta^{\alpha_1\alpha_2} \delta^{\beta_1\beta_2} \right].
\end{aligned} \tag{3.9.4}$$

The form factors A_j, $j = 1, 2, 3$ are functions of the momentum magnitudes. All form factors are symmetric under $p_1 \leftrightarrow p_2$, *i.e.*, they satisfy

$$A_j^I(p_2, p_1, p_3) = A_j^I(p_1, p_2, p_3), \qquad j = 1, 2, 3. \tag{3.9.5}$$

These form factors may be calculated using

$$\begin{aligned}
A_1^I &= \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2}, \\
A_2^I &= 4 \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_2^{\nu_1} p_3^{\nu_2}, \\
A_3^I &= 2 \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \delta^{\nu_1\nu_2},
\end{aligned} \tag{3.9.6}$$

in $\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\!\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned}
&\mathrm{K}_{12} A_1^I = 0, && \mathrm{K}_{13} A_1^I = 0, \\
&\mathrm{K}_{12} A_2^I = 0, && \mathrm{K}_{13} A_2^I = 8 A_1^I, \\
&\mathrm{K}_{12} A_3^I = 0, && \mathrm{K}_{13} A_3^I = 2 A_2^I,
\end{aligned} \tag{3.9.7}$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned} A_1^I &= \alpha_1^I J_{4\{000\}}, \\ A_2^I &= 4\alpha_1^I J_{3\{001\}} + \alpha_2^I J_{2\{000\}}, \\ A_3^I &= 2\alpha_1^I J_{2\{002\}} + \alpha_2^I J_{1\{001\}} + \alpha_3^I J_{0\{000\}}, \end{aligned} \tag{3.9.8}$$

where α_j^I, $j = 1,2,3$ are constants. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned} &\mathrm{L}_{2,0}\, A_1^I + \mathrm{R}_2\, A_2^I = \\ &\quad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\rangle, \end{aligned} \tag{3.9.9}$$

$$\begin{aligned} &\mathrm{L}_{2,0}\, A_2^I + 4\,\mathrm{R}_2\, A_3^I = \\ &\quad = 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\mu_2} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) \mathcal{O}^I(\boldsymbol{p}_3) \rangle\rangle. \end{aligned} \tag{3.9.10}$$

where L and R are defined in (3.1.9) and (3.1.10). They lead to the following relations

$$\alpha_2^{(0)I} = (\Delta_3 + 2)(d - \Delta_3 - 2)\alpha_1^I, \tag{3.9.11}$$

$$\alpha_2^{(1)I} = \begin{cases} (2\Delta_3 - d - 4)\alpha_1^I & \text{if } \Delta_3 \neq d - 2 - 2n, \\ -(2\Delta_3 - d + 4)\alpha_1^I + C_1 c_1^I c_{\mathcal{O}} & \text{if } \Delta_3 \neq d - 2 - 2n, \end{cases} \tag{3.9.12}$$

$$\alpha_2^{(2)I} = -\alpha_1^I, \tag{3.9.13}$$

$$\alpha_3^{(0)I} = \frac{1}{4}\Delta_3(\Delta_3 + 2)(d - \Delta_3)(d - \Delta_3 - 2)\alpha_1^I, \tag{3.9.14}$$

$$\alpha_3^{(1)I} = \begin{cases} -\frac{1}{2}(6d + d^2 - 12\Delta_3 - 2d\Delta_3 + d^2\Delta_3 + 2\Delta_3^2 - 3d\Delta_3^2 + 2\Delta_3^3)\alpha_1^I \\ \qquad\qquad \text{if } \Delta_3 \neq d - 2n, \\ \frac{1}{4}\left[(2\Delta_3 - d)(\Delta_3 - d + 2)(\Delta_3 + 2)\alpha_1^I - \Delta_3(\Delta_3 - d)\alpha_2^{(1)}\right] + C_2 c_2^I c_{\mathcal{O}} \\ \qquad\qquad \text{if } \Delta_3 = d - 2n, \end{cases} \tag{3.9.15}$$

$$\alpha_3^{(2)I} = \frac{1}{4}(4 + 2d + d^2 - 4\Delta_3 - 6d\Delta_3 + 6\Delta_3^2)\alpha_1^I. \tag{3.9.16}$$

Here, n is a non-negative integer and $c_{\mathcal{O}}$ denotes the normalisation of the 2-point function (2.5.54) and the constants C_1 and C_2 are

$$\begin{aligned} C_1 = {} & \frac{(-1)^{\frac{d-\Delta_3}{2}} 2^{4+\Delta_3-\frac{d}{2}} d\, \Gamma\left(\frac{d-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_3+3}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta_3+2}{2}\right)\Gamma\left(\frac{\Delta_3+d+4}{2}\right)} \times \\ & \times \begin{cases} \frac{1}{\Gamma\left(\frac{d-2\Delta_3}{2}\right)} & \text{if } 2\Delta_3 \neq d + 2n, \\ -\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right) & \text{if } 2\Delta_3 = d + 2n, \end{cases} \end{aligned} \tag{3.9.17}$$

$$\begin{aligned} C_2 = {} & \frac{(-1)^{\frac{d-\Delta_3+2}{2}} 2^{4+\Delta_3-\frac{d}{2}} d\, \Gamma\left(\frac{d-\Delta_3+2}{2}\right)\Gamma\left(\frac{\Delta_3+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_3+d+2}{2}\right)} \times \\ & \times \begin{cases} \frac{1}{\Gamma\left(\frac{d-2\Delta_3}{2}\right)} & \text{if } 2\Delta_3 \neq d + 2n, \\ -\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right) & \text{if } 2\Delta_3 = d + 2n, \end{cases} \end{aligned} \tag{3.9.18}$$

where n is a non-negative integer. Moreover the c_1^K and c_2^K constants are defined as

$$p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^K(\boldsymbol{p})_3 \rangle\!\rangle \Bigg|_{p_1=p_2=p} = c_1^K p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p^{d-\Delta_3-2} \langle\!\langle \mathcal{O}^I(\boldsymbol{p}_3) \mathcal{O}^K(-\boldsymbol{p}_3) \rangle\!\rangle$$
$$+ \; c_2^K \delta^{\mu_1\mu_2} p_3^{\nu_2} p^{d-\Delta_3} \langle\!\langle \mathcal{O}^I(\boldsymbol{p}_3) \mathcal{O}^K(-\boldsymbol{p}_3) \rangle\!\rangle + \ldots \tag{3.9.19}$$

Here, we mean the following: first, write down the most general tensor decomposition of $p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}^K(\boldsymbol{p})_3 \rangle\!\rangle$ and take the coefficient of $p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2}$. We then set $p_1 = p_2 = p$ in the expression for this coefficient.

The primary constants are function of the regulator ϵ and by $\alpha_j^{(n)}$ we denote the coefficient of ϵ^n in the series expansion of α_j in ϵ. We assume that the constant α_1 does not depend on the regulator. If the regulator is present, the limit $\epsilon \to 0$ can be taken after the primary constants and triple-K integrals are substituted into the primary CWIs (3.9.8).

The 3-point function $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\!\rangle$ thus depends on the 2-point function normalisations $c_{\mathcal{O}}$, c_1^K and c_2^K and one undetermined primary constant α_1^I.

Examples

For $\boldsymbol{d = 3}$ and $\boldsymbol{\Delta_3 = 1}$ we find

$$A_1^I = \frac{\alpha_1^I}{p_3 a_{123}^4} \left[p_3^2 + 4 p_3 a_{12} + 3(a_{12}^2 + 2b_{12}) \right], \tag{3.9.20}$$

$$A_2^I = \frac{\alpha_1^I}{p_3 a_{123}^3} \left[p_3^3 + 3p_2^2 a_{12} + p_3(-a_{12}^2 + 8b_{12}) - 3a_{12}^3 \right] - \frac{4c_1^I c_{\mathcal{O}}}{p_3}, \tag{3.9.21}$$

$$A_3^I = \frac{\alpha_1^I (a_{12} - p_3)}{4 p_3 a_{123}^2} \left[-p_3^3 - 3p_3^2 a_{12} + p_3(a_{12}^2 - 10 b_{12}) + 3a_{12}(a_{12}^2 - 2b_{12}) \right]$$
$$+ \frac{c_{\mathcal{O}}}{p_3} \left[(c_1^I - 3c_2^I)(p_1^2 + p_2^2) + 3(c_1^I + c_2^I) p_3^2 \right]. \tag{3.9.22}$$

For $\boldsymbol{d = 3}$ and $\boldsymbol{\Delta_3 = 3}$ we find

$$A_1^I = \frac{2\alpha_1^I}{a_{123}^4} \left[a_{123}^3 + a_{123} b_{123} + 3c_{123} \right], \tag{3.9.23}$$

$$A_2^I = \frac{2\alpha_1^I}{a_{123}^3} \left[p_3^4 + 3p_3^3 a_{12} + 6p_3^2 b_{12} + 3p_3 a_{12}(-a_{12}^2 + b_{12}) + a_{12}^2(b_{12} - a_{12}^2) \right], \tag{3.9.24}$$

$$A_3^I = \frac{\alpha_1^I}{2a_{123}^2} \left[p_3^5 + 2p_3^4 a_{12} + p_3^3(-3a_{12}^2 + 8b_{12}) - p_3^2(a_{12}^3 + 5a_{12} b_{12}) \right.$$
$$\left. + 6p_3(a_{12}^4 - 3a_{12}^2 b_{12}) + 3a_{12}^3(a_{12}^2 - 3b_{12}) \right]$$
$$- \; 4c_2^I c_{\mathcal{O}}(p_1^3 + p_2^3 + p_3^3). \tag{3.9.25}$$

For $\boldsymbol{d = 4}$ and $\boldsymbol{\Delta_3 = 2}$ we find

$$A_1^I = \alpha_1^I I_{5\{220\}}, \tag{3.9.26}$$

$$A_2^I = 4\alpha_1^I I_{4\{221\}}^{(0)} + 8(2\alpha_1^I - c_1^I c_{\mathcal{O}})(\log p_3 + \gamma_E - \log 2), \tag{3.9.27}$$

$$
\begin{aligned}
A_3^I &= 2\alpha_1^I I^{(0)}_{3\{222\}} + 2(-2\alpha_1^I + c_1^I c_{\mathcal{O}}) \left[(p_1^2+p_2^2+p_3^2)\log p_3 \right. \\
&\qquad \left. - (p_1^2+p_2^2+p_3^2)(1-\gamma_E+\log 2) + p_3^2 \right] \\
&\quad + \tfrac{16}{3} c_2^I c_{\mathcal{O}} \left[(p_3^2-p_1^2-p_2^2)\log p_3 + (p_3^2-p_1^2-p_2^2)(\gamma_E - \log 2) - p_3^2 \right], \qquad (3.9.28)
\end{aligned}
$$

where $I^{(0)}_{\alpha\{\beta_j\}}$ is the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$. The integral can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 4$ and $\Delta_3 = 4$ we find

$$
A_1^I = \alpha_1^I I^{(0)}_{5\{222\}}, \qquad (3.9.29)
$$

$$
A_2^I = 4\alpha_1^I \left(1 - p_3 \frac{\partial}{\partial p_3} \right) I^{(0)}_{3\{222\}}, \qquad (3.9.30)
$$

$$
\begin{aligned}
A_3^I &= -2\alpha_1^I p_3 \frac{\partial}{\partial p_3} I^{(0)}_{2\{223\}} - 2(3\alpha_1^I + 4c_2^I c_{\mathcal{O}}) \left[\left(p_1^4 \log p_1 - p_2^2 p_3^2 - p_1^4 (\tfrac{3}{4} - \gamma_E + \log 2) \right) \right. \\
&\qquad \left. + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_3) \right], \qquad (3.9.31)
\end{aligned}
$$

where $I^{(0)}_{\alpha\{\beta_j\}}$ is the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$. The integral can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

For $d = 5$ and $\Delta_3 = 3$ we find

$$
\begin{aligned}
A_1^I &= \frac{3\alpha_1^I}{a_{123}^5} \left[3p_3^4 + 15p_3^3 a_{12} + p_3^2(29a_{12}^2 + 2b_{12}) + 5p_3 a_{12}(5a_{12}^2 + 2b_{12}) \right. \\
&\qquad \left. + 8(a_{12}^4 + a_{12}^2 b_{12} + b_{12}^2) \right], \qquad (3.9.32) \\
A_2^I &= \frac{\alpha_1^I}{a_{123}^4} \left[9p_3^5 + 36p_3^4 a_{12} + 6p_3^3(7a_{12}^2 + 4b_{12}) - 12p_3^2 a_{12}(a_{12}^2 - 8b_{12}) \right. \\
&\qquad \left. + p_3(-51a_{12}^4 + 96a_{12}^2 b_{12} + 32b_{12}^2) + 8a_{12}(-3a_{12}^4 + 3a_{12}^2 b_{12} + b_{12}^2) \right] \\
&\quad - 4p_3 c_1^I c_{\mathcal{O}}, \qquad (3.9.33) \\
A_3^I &= \frac{\alpha_1^I}{4a_{123}^3} \left[9p_3^6 + 27p_3^5 a_{12} + 6p_3^4(a_{12}^2 + 7b_{12}) + 18p_3^3 a_{12}(-3a_{12}^2 + 7b_{12}) \right. \\
&\qquad + p_3^2(-39a_{12}^4 + 30a_{12}^2 b_{12} + 64b_{12}^2) + 9p_3 a_{12}(a_{12}^2 - 4b_{12})(3a_{12}^2 - 2b_{12}) \\
&\qquad \left. + 24a_{12}^2(a_{12}^4 - 3a_{12}^2 b_{12} + b_{12}^2) \right] \\
&\quad + \frac{p_3 c_{\mathcal{O}}}{6} \left[3(2c_1^I - 5c_2^I)(p_1^2 + p_2^2) + (2c_1^I + 15c_2^I)p_3^2 \right]. \qquad (3.9.34)
\end{aligned}
$$

For $d = 5$ and $\Delta_3 = 5$ we find

$$
A_1^I = \frac{6\alpha_1^I}{a_{123}^5} \left[-3a_{123}^6 + 3a_{123}^4 b_{123} + a_{123}^2 b_{123}^2 + a_{123}^3 c_{123} + 3a_{123} b_{123} c_{123} + 4c_{123}^2 \right], \qquad (3.9.35)
$$

$$A_2^I = \frac{2\alpha_1^I}{a_{123}^4} \big[-9p_3^7 - 36p_3^6 a_{12} - 3p_3^5(17a_{12}^2 + 2b_{12}) - 24p_3^4 a_{12}(a_{12}^2 + b_{12}) + 8p_3^3(3a_{12}^4 - 12a_{12}^2 b_{12} + 5b_{12}^2) + p_3^2(51a_{12}^5 - 159a_{12}^3 b_{12} + 55a_{12}b_{12}^2) + 9a_{12}^2(a_{12} + 4p_3)(a_{12}^4 - 3a_{12}^2 b_{12} + b_{12}^2) \big], \tag{3.9.36}$$

$$A_3^I = \frac{\alpha_1^I}{2a_{123}^3} \big[-9p_3^8 - 27p_3^7 a_{12} - 3p_3^6(5a_{12}^2 + 8b_{12}) + 9p_3^5(3a_{12}^3 - 8a_{12}b_{12}) + 8p_3^4(6a_{12}^4 - 15a_{12}^2 b_{12} + 4b_{12}^2) + 9p_3^3(a_{12}^5 + 3a_{12}^3 b_{12} - 11a_{12}b_{12}^2) + p_3^2(-69a_{12}^6 + 369a_{12}^4 b_{12} - 393a_{12}^2 b_{12}^2) - 27a_{12}^3(a_{12} + 3p_3)(a_{12}^4 - 5a_{12}^2 b_{12} + 5b_{12}^2) \big] - 4c_2^I c_{\mathcal{O}}(p_1^5 + p_2^5 + p_3^5). \tag{3.9.37}$$

3.10. $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3} \rangle$

This correlation function is at most semi-local, as was proved in [36] through a position space analysis. Our result confirms the triviality of this correlator through independent calculations in momentum space. In appendix 2.A.6 we discuss the triviality of $\langle\!\langle T^{\mu_1\nu_1} J^{\mu_2} \mathcal{O} \rangle\!\rangle$, which is very similar to $\langle\!\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3} \rangle\!\rangle$.

Ward identities. The transverse and trace Ward identities are

$$p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle = 2p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle, \tag{3.10.1}$$

$$p_{3\mu_3} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{3.10.2}$$

$$\langle\!\langle T(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle = 2 \langle\!\langle \frac{\delta T}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle. \tag{3.10.3}$$

Reconstruction formula. The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned}
&\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle = \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1) t^{\mu_2\nu_2}(\boldsymbol{p}_2) j^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle \\
&\quad + 2 \left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1} \delta_{\alpha_1\beta_1} \right] \delta^{\mu_2\alpha_2} \delta^{\nu_2\beta_2} \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle \\
&\quad + 2[(\mu_1, \nu_1, \boldsymbol{p}_1) \leftrightarrow (\mu_2, \nu_2, \boldsymbol{p}_2)] \\
&\quad - 4 \left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1) p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_2}(\boldsymbol{p}_1)}{d-1} \delta_{\alpha_1\beta_1} \right] \left[\mathscr{T}^{\mu_2\nu_2\alpha_2}(\boldsymbol{p}_2) p_2^{\beta_2} + \frac{\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1} \delta_{\alpha_2\beta_2} \right] \times \\
&\qquad\qquad \times \langle\!\langle \frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3) \rangle\!\rangle,
\end{aligned} \tag{3.10.4}$$

where $\mathscr{T}^{\mu\nu\alpha}$ is defined in (3.1.20).

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\begin{aligned}\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)j^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\pi^{\mu_3}_{\alpha_3}(\boldsymbol{p}_3)\left[A^a_1 p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}\right.\\
&+ A^a_2\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}\\
&+ A^a_3\delta^{\alpha_1\alpha_3}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2} - A^a_3(p_1\leftrightarrow p_2)\delta^{\alpha_2\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\beta_2}\\
&+ A^a_4\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}\\
&+ A^a_5\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_2}p_2^{\beta_1} - A_5(p_1\leftrightarrow p_2)\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_1}p_3^{\beta_2}\left.\right].\end{aligned} \tag{3.10.5}$$

The form factors A_j, $j=1,\ldots,5$ are functions of the momentum magnitudes. If no arguments are specified then the standard ordering is assumed, $A_j = A_j(p_1,p_2,p_3)$, while by $p_i \leftrightarrow p_j$ we denote the exchange of the two momenta, *e.g.*, $A_3(p_1\leftrightarrow p_2) = A_3(p_2,p_1,p_3)$.

The form factors A_1, A_2 and A_4 are antisymmetric under $p_1\leftrightarrow p_2$, *i.e.*, they satisfy

$$A^a_j(p_2,p_1,p_3) = -A^a_j(p_1,p_2,p_3), \qquad j\in\{1,2,4\}. \tag{3.10.6}$$

The remaining form factors A_3 and A_5 do not exhibit any symmetry properties.

The form factors can be calculated as follows

$$\begin{aligned}A^a_1 &= \text{coefficient of } p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}p_3^{\nu_2}p_1^{\mu_3},\\
A^a_2 &= 4\cdot\text{coefficient of } \delta^{\nu_1\nu_2}p_2^{\mu_1}p_3^{\mu_2}p_1^{\mu_3},\\
A^a_3 &= 2\cdot\text{coefficient of } \delta^{\mu_1\mu_3}p_2^{\nu_1}p_3^{\mu_2}p_3^{\nu_2},\\
A^a_4 &= 2\cdot\text{coefficient of } \delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}p_1^{\mu_3},\\
A^a_5 &= 4\cdot\text{coefficient of } \delta^{\mu_1\mu_2}\delta^{\mu_3\nu_2}p_2^{\nu_1}\end{aligned} \tag{3.10.7}$$

in $\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned}&\mathrm{K}_{12}\,A^a_1 = 0, && \mathrm{K}_{13}\,A^a_1 = 0,\\
&\mathrm{K}_{12}\,A^a_2 = 0, && \mathrm{K}_{13}\,A^a_2 = 2A^a_1,\\
&\mathrm{K}_{12}\,A^a_3 = 2A^a_1, && \mathrm{K}_{13}\,A^a_3 = 0,\\
&\mathrm{K}_{12}\,A^a_4 = 0, && \mathrm{K}_{13}\,A^a_4 = 4A^a_2,\\
&\mathrm{K}_{12}\,A^a_5 = -2A^a_2, && \mathrm{K}_{13}\,A^a_5 = -2\left[A^a_2 + A^a_3(p_1\leftrightarrow p_2)\right].\end{aligned} \tag{3.10.8}$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned}A^a_1 &= \alpha^a_1 J_{5\{000\}},\\
A^a_2 &= \alpha^a_1 J_{4\{001\}} + \alpha^a_2 J_{3\{000\}},\\
A^a_3 &= \alpha^a_1 J_{4\{010\}} + \alpha^a_3 J_{3\{000\}},\\
A^a_4 &= \alpha^a_1 J_{3\{002\}} + 2\alpha^a_2 J_{2\{001\}} + \alpha^a_4 I_{1\{000\}},\\
A^a_5 &= \alpha^a_1 J_{3\{101\}} + \alpha^a_2 J_{2\{100\}} + \alpha^a_3 I_{2\{001\}} + \alpha^a_5 J_{1\{000\}},\end{aligned} \tag{3.10.9}$$

where α_j^a, $j = 1, \ldots, 5$ are constants. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$\begin{aligned}
&\mathrm{L}_{2,2}\, A_1^a + \mathrm{R}_2\,[A_2^a - A_3^a] = \\
&\qquad = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle, && (3.10.10)\\
&\mathrm{L}_{2,2}\, A_2^a + 2\,\mathrm{R}_2\,[2A_4^a + A_5^a(p_1 \leftrightarrow p_2)] = \\
&\qquad = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_3} p_3^{\mu_2} p_3^{\nu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle, && (3.10.11)\\
&\mathrm{L}_{2,-2}\, A_3^a - 2\,\mathrm{R}_2\,[A_5^a(p_1 \leftrightarrow p_2)] = \\
&\qquad = 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\nu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle, && (3.10.12)\\
&\mathrm{L}_{2,0}\, A_5^a - 2p_1^2\,[2A_4^a + A_5^a(p_1 \leftrightarrow p_2)] = \\
&\qquad = 8d \cdot \text{coefficient of } \delta^{\mu_2\mu_3}\delta^{\mu_1\nu_2} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle, && (3.10.13)
\end{aligned}$$

where L and R are given by (3.1.10) and (3.1.10). They lead to

$$\alpha_1^a = \alpha_2^a = \alpha_3^a = \alpha_4^a = \alpha_5^{(0)a} = 0, \tag{3.10.14}$$

$$\alpha_5^{(1)a} = -\frac{2^{4-\frac{d}{2}} c^a s_d}{\Gamma\left(\frac{d}{2}\right)}, \tag{3.10.15}$$

where

$$c^a = \text{coefficient of } \delta^{\mu_1\mu_3} p_3^{\mu_2} p_3^{\nu_2} \left\{ \begin{array}{ll} p_3^{d-2} & \text{for odd } d \\ -p_3^{d-2}\log p_3^2 & \text{for even } d \end{array} \right\}$$
$$\text{in } p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) J^{\mu_3 a}(\boldsymbol{p}_3)\rangle\!\rangle. \tag{3.10.16}$$

By $\alpha_5^{(n)a}$ we denote the coefficient of ϵ^n in the series expansion of α_5^a in the regulator ϵ. We assume that the α_1 coefficient is independent of ϵ.

3.11. $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle$

Ward identities. The transverse and trace Ward identities are

$$\begin{aligned}
&p_1^{\nu_1} \langle\!\langle T_{\mu_1\nu_1}(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle = \\
&\qquad = 2p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_3\nu_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) T_{\mu_2\nu_2}(\boldsymbol{p}_2)\rangle\!\rangle + 2p_1^{\nu_1} \langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\qquad + 2p_{1(\mu_3} \langle\!\langle T_{\nu_3)\mu_1}(\boldsymbol{p}_2) T_{\mu_2\nu_2}(-\boldsymbol{p}_2)\rangle\!\rangle + 2p_{1(\mu_2} \langle\!\langle T_{\nu_2)\mu_1}(\boldsymbol{p}_3) T_{\mu_3\nu_3}(-\boldsymbol{p}_3)\rangle\!\rangle \\
&\qquad + \delta_{\mu_3\nu_3} p_3^{\alpha} \langle\!\langle T_{\alpha\mu_1}(\boldsymbol{p}_2) T_{\mu_2\nu_2}(-\boldsymbol{p}_2)\rangle\!\rangle + \delta_{\mu_2\nu_2} p_2^{\alpha} \langle\!\langle T_{\alpha\mu_1}(\boldsymbol{p}_3) T_{\mu_3\nu_3}(-\boldsymbol{p}_3)\rangle\!\rangle \\
&\qquad - p_{3\mu_1} \langle\!\langle T_{\mu_2\nu_2}(\boldsymbol{p}_2) T_{\mu_3\nu_3}(-\boldsymbol{p}_2)\rangle\!\rangle - p_{2\mu_1} \langle\!\langle T_{\mu_2\nu_2}(\boldsymbol{p}_3) T_{\mu_3\nu_3}(-\boldsymbol{p}_3)\rangle\!\rangle, && (3.11.1)\\
&\langle\!\langle T(\boldsymbol{p}_1) T_{\mu_2\nu_2}(\boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle = \\
&\qquad = 2\langle\!\langle \frac{\delta T}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle + 2\langle\!\langle \frac{\delta T}{\delta g^{\mu_3\nu_3}}(\boldsymbol{p}_1, \boldsymbol{p}_3) T_{\mu_2\nu_2}(\boldsymbol{p}_2)\rangle\!\rangle. && (3.11.2)
\end{aligned}$$

Reconstruction formula. Define

$$\begin{aligned}
&\mathcal{L}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3) = \\
&\quad = 2\left[\mathscr{T}^{\mu_1\nu_1\alpha_1}(\boldsymbol{p}_1)p_1^{\beta_1} + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1}\delta^{\alpha_1\beta_1}\right]\delta^{\mu_3\alpha_3}\delta^{\nu_3\beta_3}\langle\!\langle\frac{\delta T_{\alpha_1\beta_1}}{\delta g^{\alpha_3\beta_3}}(\boldsymbol{p}_1,\boldsymbol{p}_3)T^{\mu_2\nu_2}(\boldsymbol{p}_2)\rangle\!\rangle \\
&\qquad + \left[\mathscr{T}^{\mu_1\nu_1\beta_3}(\boldsymbol{p}_1)(2p_1^{(\mu_3}\delta^{\nu_3)\alpha_3} + p_3^{\alpha_3}\delta^{\mu_3\nu_3}) - p_3^{\alpha}\mathscr{T}_{\alpha}^{\mu_1\nu_1}(\boldsymbol{p}_1)\delta^{\mu_3\alpha_3}\delta^{\nu_3\beta_3}\right. \\
&\qquad\qquad \left. + \frac{2\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1}\delta^{\mu_3\alpha_3}\delta^{\nu_3\beta_3}\right]\langle\!\langle T_{\alpha_3\beta_3}(\boldsymbol{p}_2)T^{\mu_2\nu_2}(-\boldsymbol{p}_2)\rangle\!\rangle,
\end{aligned} \tag{3.11.3}$$

where $\mathscr{T}^{\mu\nu\alpha}$ is defined in (3.1.20). The full 3-point function can be reconstructed from the transverse-traceless part as

$$\begin{aligned}
&\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle = \langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad + \sum_{\sigma}\mathcal{L}^{\mu_{\sigma(1)}\nu_{\sigma(1)}\mu_{\sigma(2)}\nu_{\sigma(2)}\mu_{\sigma(3)}\nu_{\sigma(3)}}(\boldsymbol{p}_{\sigma(1)},\boldsymbol{p}_{\sigma(2)},\boldsymbol{p}_{\sigma(3)}) \\
&\quad - \left[\mathscr{T}_{\alpha_3}^{\mu_3\nu_3}(\boldsymbol{p}_3)p_{3\beta_3} + \frac{\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)}{d-1}\delta_{\alpha_3\beta_3}\right]\mathcal{L}^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3) \\
&\quad - [(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)] \\
&\quad - [(\mu_1,\nu_1,\boldsymbol{p}_1)\mapsto(\mu_3,\nu_3,\boldsymbol{p}_3)\mapsto(\mu_2,\nu_2,\boldsymbol{p}_2)\mapsto(\mu_1,\nu_1,\boldsymbol{p}_1)],
\end{aligned} \tag{3.11.4}$$

where the sum is taken over all six permutations σ of the set $\{1,2,3\}$.

Decomposition of the 3-point function. The tensor decomposition of the transverse-traceless part is

$$\begin{aligned}
&\langle\!\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)t^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle \\
&\quad = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)\,\big[A_1 p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3} \\
&\qquad + A_2\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3} + A_2(p_1\leftrightarrow p_3)\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3} \\
&\qquad\qquad + A_2(p_2\leftrightarrow p_3)\delta^{\beta_1\beta_3}p_2^{\alpha_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3} \\
&\qquad + A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_3(p_1\leftrightarrow p_3)\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1} \\
&\qquad\qquad + A_3(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}\delta^{\beta_1\beta_3}p_3^{\alpha_2}p_3^{\beta_2} \\
&\qquad + A_4\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_3^{\beta_2} + A_4(p_1\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}p_3^{\beta_2}p_1^{\beta_3} \\
&\qquad\qquad + A_4(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\beta_2}p_2^{\beta_1}p_1^{\beta_3} \\
&\qquad + A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\big].
\end{aligned} \tag{3.11.5}$$

The form factors A_j, $j=1,\ldots,5$ are functions of the momentum magnitudes. If no arguments are specified then the standard ordering is assumed, $A_j = A_j(p_1,p_2,p_3)$, while by $p_i \leftrightarrow p_j$ we denote the exchange of the two momenta, *e.g.*, $A_1(p_1\leftrightarrow p_3) = A_2(p_3,p_2,p_1)$.

The form factors A_1 and A_5 are symmetric under any permutation of momenta, *i.e.*, for any permutation σ of the set $\{1, 2, 3\}$,

$$A_j(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = A_j(p_1, p_2, p_3), \qquad j \in \{1, 5\}. \tag{3.11.6}$$

The remaining form factors are symmetric under $p_1 \leftrightarrow p_2$, *i.e.*, they satisfy

$$A_j(p_2, p_1, p_3) = A_j(p_1, p_2, p_3), \qquad j \in \{2, 3, 4\}. \tag{3.11.7}$$

The form factors can be calculated as

$$\begin{aligned}
A_1 &= \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3}, \\
A_2 &= 4 \cdot \text{coefficient of } \delta^{\nu_1 \nu_2} p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} p_1^{\nu_3}, \\
A_3 &= 2 \cdot \text{coefficient of } \delta^{\mu_1 \mu_2} \delta^{\nu_1 \nu_2} p_1^{\mu_3} p_1^{\nu_3}, \\
A_4 &= 8 \cdot \text{coefficient of } \delta^{\mu_1 \mu_3} \delta^{\mu_2 \nu_3} p_2^{\nu_1} p_3^{\nu_2}, \\
A_5 &= 8 \cdot \text{coefficient of } \delta^{\mu_1 \nu_2} \delta^{\mu_2 \nu_3} \delta^{\mu_3 \nu_1},
\end{aligned} \tag{3.11.8}$$

in $\langle\langle T^{\mu_1 \nu_1}(\boldsymbol{p}_1) T^{\mu_2 \nu_2}(\boldsymbol{p}_2) T^{\mu_3 \nu_3}(\boldsymbol{p}_3) \rangle\rangle$.

Primary conformal Ward identities. The primary CWIs are

$$\begin{aligned}
&\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0, \\
&\mathrm{K}_{12}\, A_2 = 0, && \mathrm{K}_{13}\, A_2 = 8A_1, \\
&\mathrm{K}_{12}\, A_3 = 0, && \mathrm{K}_{13}\, A_3 = 2A_2, \\
&\mathrm{K}_{12}\, A_4 = 4\left[A_2(p_1 \leftrightarrow p_3) - A_2(p_2 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_4 = -4A_2(p_2 \leftrightarrow p_3), \\
&\mathrm{K}_{12}\, A_5 = 2\left[A_4(p_2 \leftrightarrow p_3) - A_4(p_1 \leftrightarrow p_3)\right], && \mathrm{K}_{13}\, A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right].
\end{aligned} \tag{3.11.9}$$

The solution in terms of triple-K integrals (3.1.7) is

$$\begin{aligned}
A_1 &= \alpha_1 J_{6\{000\}}, \\
A_2 &= 4\alpha_1 J_{5\{001\}} + \alpha_2 J_{4\{000\}}, \\
A_3 &= 2\alpha_1 J_{4\{002\}} + \alpha_2 J_{3\{001\}} + \alpha_3 J_{2\{000\}}, \\
A_4 &= 8\alpha_1 J_{4\{110\}} - 2\alpha_2 J_{3\{001\}} + \alpha_4 J_{2\{000\}}, \\
A_5 &= 8\alpha_1 J_{3\{111\}} + 2\alpha_2 \left(J_{2\{110\}} + J_{2\{101\}} + J_{2\{011\}}\right) + \alpha_5 J_{0\{000\}},
\end{aligned} \tag{3.11.10}$$

where α_j, $j = 1, \ldots, 5$ are constants. If the integrals diverge, the regularisation (3.1.8) should be used.

Secondary conformal Ward identities. The independent secondary CWIs are

$$
\begin{aligned}
(*)\;\; &\mathrm{L}_{2,4}\, A_1 + \mathrm{R}_2 \left[A_2 - A_2(p_2 \leftrightarrow p_3)\right] = && (3.11.11)\\
&\quad = 2d \cdot \text{coeff. of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, \\
&\mathrm{L}_{2,4}\, A_2 + 2\,\mathrm{R}_2 \left[2A_3 - A_4(p_1 \leftrightarrow p_3)\right] = && (3.11.12)\\
&\quad = 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, \\
(*)\;\; &\mathrm{L}_{2,2} \left[A_2(p_1 \leftrightarrow p_3)\right] + \mathrm{R}_2 \left[A_4(p_2 \leftrightarrow p_3) - A_4\right] + 2p_1^2 \left[A_2(p_2 \leftrightarrow p_3) - A_2\right] = && (3.11.13)\\
&\quad = 8d \cdot \text{coefficient of } \delta^{\mu_2\mu_3} p_2^{\mu_1} p_3^{\nu_2} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, \\
&\mathrm{L}_{2,2} \left[A_4(p_2 \leftrightarrow p_3)\right] - 2\,\mathrm{R}_2\, A_5 + 2p_1^2 \left[A_4(p_1 \leftrightarrow p_3) - 4A_3\right] = && (3.11.14)\\
&\quad = 16d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \delta^{\mu_3\nu_2} p_1^{\nu_3} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle, \\
&\mathrm{L}_{2,0} \left[A_3(p_1 \leftrightarrow p_3)\right] + p_1^2 \left[A_4 - A_4(p_2 \leftrightarrow p_3)\right] = && (3.11.15)\\
&\quad = 4d \cdot \text{coefficient of } \delta^{\mu_2\mu_3} \delta^{\nu_2\nu_3} p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle,
\end{aligned}
$$

where the operators L and R are defined in (3.1.9) and (3.1.10). The identities denoted by asterisks are redundant, *i.e.*, they are trivially satisfied in all cases and do not impose any additional conditions on primary constants. The secondary CWIs lead to which lead to

$$\alpha_4 = (3d+2)\alpha_2 + 2\alpha_3, \tag{3.11.16}$$

$$\alpha_5^{(0)} = -2d^2 \alpha_2, \tag{3.11.17}$$

$$2d(d+2)\alpha_1 + d\alpha_2 + \alpha_3 = -\frac{2^{3-\frac{d}{2}} s_d c_T}{\Gamma\left(\frac{d}{2}\right)}, \tag{3.11.18}$$

$$d(d+2)(8d\alpha_1 + 3\alpha_2) + 2d\alpha_3 + 2\alpha_5^{(1)} = -\frac{2^{5-\frac{d}{2}} s_d d}{\Gamma\left(\frac{d}{2}\right)} \left[c_T + 2c_g c_T\right]. \tag{3.11.19}$$

By $\alpha_5^{(n)}$ we mean the coefficient of ϵ^n in the series expansion of α_5 in the regulator ϵ. We assume that the α_1 coefficient is independent of ϵ, *i.e.*, $\alpha_1 = \alpha_1^{(0)}$. The constant c_T is the 2-point function normalisation (3.1.23), while s_d is defined in (3.1.3) and the constant c_g is defined as

$$\langle\!\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g^{\mu_2\nu_2}}(\boldsymbol{p}_1, \boldsymbol{p}_2) T_{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle = 4c_g \delta_{(\mu_1(\mu_2} \langle\!\langle T_{\nu_1)\nu_2)}(\boldsymbol{p}_3) T_{\mu_3\nu_3}(-\boldsymbol{p}_3)\rangle\!\rangle + \ldots \tag{3.11.20}$$

The omitted terms do not contain the tensor structures listed explicitly. If the regulator is present, the limit $\epsilon \to 0$ can be taken after the primary constants and triple-K integrals are substituted into the primary CWIs (3.11.10).

The 3-point function $\langle\!\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle\!\rangle$ therefore depends on the 2-point function normalisations c_T and c_g and two undetermined primary constants α_1 and α_2. The dependence of this correlator on two 2-point function normalisations rather than only one as found in [22] is related to the definition (1.3.20) we adopt for this correlator. Our definition differs by the semi-local terms on the right-hand side of (1.3.20), and it is these terms that produce the dependence of our solution on c_g through (3.11.20). (Similar considerations also apply for $\langle\!\langle T_{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle\!\rangle$ as discussed above (2.7.29).)

An additional effect in dimension $d = 3$ is that the tensor decomposition becomes degenerate meaning there are only two instead of the usual five form factors. In consequence, the stress-energy tensor 3-point function in $d = 3$ only depends on the primary constant α_1, rather than on both α_1 and α_2. Along with the two 2-point function normalisations, this makes three parameters in total (or two using the definition of the 3-point function in [22]). We present a discussion of this degeneracy in appendix 2.A.2, the results of which we make use of below.

Examples

For $d = 3$ we find

$$A_1 = \frac{8\alpha_1}{a_{123}^6} \left[a_{123}^3 + 3a_{123}b_{123} + 15c_{123} \right], \tag{3.11.21}$$

$$\begin{aligned} A_2 = {} & \frac{8\alpha_1}{a_{123}^5} \left[4p_3^4 + 20p_3^3 a_{12} + 4p_3^2(7a_{12}^2 + 6b_{12}) + 15p_3 a_{12}(a_{12}^2 + b_{12}) + 3a_{12}^2(a_{12}^2 + b_{12}) \right] \\ & + \frac{2\alpha_2}{a_{123}^4} \left[a_{123}^3 + a_{123}b_{123} + 3c_{123} \right], \end{aligned} \tag{3.11.22}$$

$$\begin{aligned} A_3 = {} & \frac{2\alpha_1 p_3^2}{a_{123}^4} \left[7p_3^3 + 28p_3^2 a_{12} + 3p_3(11a_{12}^2 + 6b_{12}) + 12a_{12}(a_{12}^2 + b_{12}) \right] \\ & + \frac{\alpha_2 p_3^2}{a_{123}^3} \left[p_3^2 + 3p_3 a_{12} + 2(a_{12}^2 + b_{12}) \right] - \frac{2c_T}{a_{123}^2} \left[a_{123}^3 - a_{123}b_{123} - c_{123} \right], \end{aligned} \tag{3.11.23}$$

$$\begin{aligned} A_4 = {} & \frac{4\alpha_1}{a_{123}^4} \left[-3p_3^5 - 12p_3^4 a_{12} - 9p_3^3(a_{12}^2 + 2b_{12}) + 9p_3^2 a_{12}(a_{12}^2 - 3b_{12}) \right. \\ & \quad \left. + (4p_3 + a_{12})(3a_{12}^4 - 3a_{12}^2 b_{12} + 4b_{12}^2) \right] \\ & + \frac{\alpha_2}{a_{123}^3} \left[-p_3^4 - 3p_3^3 a_{12} - 6p_3^2 b_{12} + a_{12}(a_{12}^2 - b_{12})(3p_3 + a_{12}) \right] \\ & - \frac{4c_T}{a_{123}^2} \left[a_{123}^3 - a_{123}b_{123} - c_{123} \right], \end{aligned} \tag{3.11.24}$$

$$\begin{aligned} A_5 = {} & \frac{2\alpha_1}{a_{123}^3} \left[-3a_{123}^6 + 9a_{123}^4 b_{123} + 12a_{123}^2 b_{123}^2 - 33a_{123}^3 c_{123} + 12a_{123}b_{123}c_{123} + 8c_{123}^2 \right] \\ & + \frac{\alpha_2}{2a_{123}^2} \left[-a_{123}^5 + 3a_{123}^3 b_{123} + 4a_{123}b_{123}^2 - 11a_{123}^2 c_{123} + 4b_{123}c_{123} \right] \\ & + 2(c_T + 4c_g c_T)(p_1^3 + p_2^3 + p_3^3). \end{aligned} \tag{3.11.25}$$

For $d = 4$ we find

$$A_1 = \alpha_1 I_{7\{222\}}, \tag{3.11.26}$$

$$A_2 = \left(16\alpha_1 + \alpha_2 - 4\alpha_1 p_1 \frac{\partial}{\partial p_1} \right) I_{5\{222\}}^{(0)}, \tag{3.11.27}$$

$$A_3 = \left(2c_T - (18\alpha_1 + \alpha_2) p_3 \frac{\partial}{\partial p_3} + 2\alpha_1 p_3^2 \frac{\partial^2}{\partial p_3^2} \right) I^{(0)}_{3\{222\}}, \tag{3.11.28}$$

$$\begin{aligned} A_4 = 2 \Bigg(& 16\alpha_1 - \alpha_2 + 2c_T - 16\alpha_1 p_1 \frac{\partial}{\partial p_1} - 16\alpha_1 p_2 \frac{\partial}{\partial p_2} + \alpha_2 p_3 \frac{\partial}{\partial p_3} \\ & + 4\alpha_1 p_1 p_2 \frac{\partial^2}{\partial p_1 \partial p_2} \Bigg) I^{(0)}_{3\{222\}}, \end{aligned} \tag{3.11.29}$$

$$\begin{aligned} A_5 = 2 \Bigg[& 32(8\alpha_1 + \alpha_2) - 8(8\alpha_1 + \alpha_2) \left(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} \right) \\ & + (16\alpha_1 + \alpha_2) \left(p_1 p_2 \frac{\partial^2}{\partial p_1 \partial p_2} + p_1 p_3 \frac{\partial^2}{\partial p_1 \partial p_3} + p_2 p_3 \frac{\partial^2}{\partial p_2 \partial p_3} \right) \\ & - 4\alpha_1 p_1 p_2 p_3 \frac{\partial^3}{\partial p_1 \partial p_2 \partial p_3} \Bigg] I^{(0)}_{1\{222\}} \\ & + 2 \left[48\alpha_1 + 5\alpha_2 - 2(c_T + 4c_g c_T) \right] \times \left[\left(p_1^4 \log p_1 - p_2^2 p_3^2 - p_1^4 (\tfrac{3}{4} - \gamma_E + \log 2) \right) \right. \\ & \left. + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_3) \right], \end{aligned} \tag{3.11.30}$$

where $I^{(0)}_{\alpha\{\beta_j\}}$ is the coefficient of ϵ^0 in the series expansion of the regulated integral $I_{\alpha+\epsilon\{\beta_j\}}$. The integrals can be obtained from the master integral (2.A.57) via the reduction scheme in table 2.1, page 94.

In case of $d = 4$ the trace Ward identity is anomalous,

$$\langle T \rangle = a E_4 + c W^2, \tag{3.11.31}$$

where E_4 is the Euler density and W^2 square of the Weyl tensor. It leads to the anomalous contribution,

$$\begin{aligned} \langle\!\langle T(\boldsymbol{p}_1) T^{\mu_2\nu_2}(\boldsymbol{p}_2) T^{\mu_3\nu_3}(\boldsymbol{p}_3) \rangle\!\rangle_{\text{anomaly}} = {} & \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3) \left[B_1 p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \right. \\ & \left. + B_2 \delta^{\beta_2\beta_3} p_3^{\alpha_2} p_1^{\alpha_3} + B_3 \delta^{\alpha_2\alpha_3} \delta^{\beta_2\beta_3} \right] \\ & + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2) \Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3) p_1^{\alpha_3} p_1^{\beta_3} B_4 + \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \pi^{\mu_3\nu_3}(\boldsymbol{p}_3) p_3^{\alpha_2} p_3^{\beta_2} B_4(p_2 \leftrightarrow p_3) \\ & + \pi^{\mu_2\nu_2}(\boldsymbol{p}_2) \pi^{\mu_3\nu_3}(\boldsymbol{p}_3) B_5, \end{aligned} \tag{3.11.32}$$

where

$$\begin{aligned} B_1 &= 8(a + c), \\ B_2 &= 8(p_1^2 - p_2^2 - p_3^2)(a + c), \\ B_3 &= -2J^2(a + c) + 4p_2^2 p_3^2 c, \\ B_4 &= -\frac{8}{3} a p_3^2, \\ B_5 &= \frac{4}{9} a J^2 \end{aligned} \tag{3.11.33}$$

and J^2 is defined (2.6.18). The anomalous contribution to the full 3-point function is then

$$
\begin{aligned}
\langle\!\langle T^{\mu_1\nu_1}(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} = {} & \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)}{d-1}\langle\!\langle T(\boldsymbol{p}_1)T^{\mu_2\nu_2}(\boldsymbol{p}_2)T^{\mu_3\nu_3}(\boldsymbol{p}_3)\rangle\!\rangle_{\text{anomaly}} \\
& [(\boldsymbol{p}_1,\mu_1,\nu_1)\leftrightarrow(\boldsymbol{p}_2,\mu_2,\nu_2)] + [(\boldsymbol{p}_1,\mu_1,\nu_1)\leftrightarrow(\boldsymbol{p}_3,\mu_3,\nu_3)] \\
& - \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)}{d-1}\left[\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(\boldsymbol{p}_3)B_4(p_1,p_2,p_3) + \pi^{\mu_3\nu_3}(\boldsymbol{p}_3)B_5(p_1,p_2,p_3)\right] \\
& [(\boldsymbol{p}_1,\mu_1,\nu_1)\leftrightarrow(\boldsymbol{p}_3,\mu_3,\nu_3)] + [(\boldsymbol{p}_2,\mu_2,\nu_2)\leftrightarrow(\boldsymbol{p}_3,\mu_3,\nu_3)] \\
& + \frac{\pi^{\mu_1\nu_1}(\boldsymbol{p}_1)\pi^{\mu_2\nu_2}(\boldsymbol{p}_2)\pi^{\mu_3\nu_3}(\boldsymbol{p}_3)}{d-1}B_5. \qquad (3.11.34)
\end{aligned}
$$

and should be added to the right hand side of (3.11.4). The anomalies are equal

$$
a = \frac{1}{2}(16\alpha_1 + \alpha_2 - c_T), \qquad c = \frac{c_T}{2}, \qquad (3.11.35)
$$

which allows us to rewrite the solution (3.11.26) - (3.11.30) as

$$
\begin{aligned}
A_1 &= \alpha_1 I_{7\{222\}}, && (3.11.36)\\
A_2 &= 2\left(a + c - 2\alpha_1 p_1 \frac{\partial}{\partial p_1}\right) I^{(0)}_{5\{222\}}, && (3.11.37)\\
A_3 &= 2\left(2c - (a + c + \alpha_1)p_3\frac{\partial}{\partial p_3} + \alpha_1 p_3^2 \frac{\partial^2}{\partial p_3^2}\right) I^{(0)}_{3\{222\}}, && (3.11.38)\\
A_4 &= 4\left(16\alpha_1 - a - 8\alpha_1 p_1\frac{\partial}{\partial p_1} - 8\alpha_1 p_2 \frac{\partial}{\partial p_2} + (a + c - 8\alpha_1)p_3\frac{\partial}{\partial p_3}\right. \\
&\qquad \left. + 2\alpha_1 p_1 p_2 \frac{\partial^2}{\partial p_1 \partial p_2}\right) I^{(0)}_{3\{222\}}, && (3.11.39)\\
A_5 &= 4\left[32(a + c - 4\alpha_1) - 8(a + c - 4\alpha_1)\left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3}\right)\right. \\
&\qquad + (a+c)\left(p_1p_2\frac{\partial^2}{\partial p_1\partial p_2} + p_1p_3\frac{\partial^2}{\partial p_1\partial p_3} + p_2p_3\frac{\partial^2}{\partial p_2\partial p_3}\right) \\
&\qquad \left. - 2\alpha_1 p_1p_2p_3\frac{\partial^3}{\partial p_1\partial p_2\partial p_3}\right] I^{(0)}_{1\{222\}} \\
&\quad + 4\left[5a - 16\alpha_1 + 3c - 8c_g c\right] \times \left[\left(p_1^4 \log p_1 - p_2^2p_3^2 - p_1^4\left(\tfrac{3}{4} - \gamma_E + \log 2\right)\right)\right. \\
&\qquad \left. + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_3)\right]. && (3.11.40)
\end{aligned}
$$

The scheme dependence of the solution due to the counterterms is

$$
\begin{aligned}
A_2 &\mapsto A_2 - 16(a_0 + c_0), \\
A_3 &\mapsto A_3 + 8\left[c_0(p_1^2 + p_2^2) - a_0 p_3^2\right], \\
A_4 &\mapsto A_4 + 8\left[c_0(p_1^2 + p_2^2 + 3p_3^2) - a_0(p_1^2 + p_2^2 - p_3^2)\right], \\
A_5 &\mapsto A_5 - 4\left[a_0 J^2 + c_0(p_1^2 + p_2^2 + p_3^2)^2 + 8c_g c_0(p_1^4 + p_2^4 + p_3^4)\right], \qquad (3.11.41)
\end{aligned}
$$

where a_0 and c_0 are arbitrary constants. See section 2.8.3 for details.

For $d = 5$ we find

$$A_1 = \frac{72\alpha_1}{a_{123}^7}\left[a_{123}^2(a_{123}^4 + a_{123}^2 b_{123} + b_{123}^2) + a_{123}(a_{123}^2 + 5b_{123})c_{123} + 10c_{123}^2\right], \quad (3.11.42)$$

$$\begin{aligned}
A_2 = {} & \frac{24\alpha_1}{a_{123}^6}\left[-12p_3^7 - 72p_3^6 a_{12} + 24p_3^5(-8a_{12}^2 + b_{12}) + 24p_3^4 a_{12}(-13a_{12}^2 + 6b_{12})\right.\\
& + 8p_3^3(-42a_{12}^4 + 33a_{12}^2 b_{12} + 8b_{12}^2) + 3p_3^2 a_{12}(-77a_{12}^4 + 73a_{12}^2 b_{12} + 23b_{12}^2)\\
& \left. + 30p_3 a_{12}^2(-3a_{12}^4 + 3a_{12}^2 b_{12} + b_{12}^2) + 5a_{12}^3(-3a_{12}^4 + 3a_{12}^2 b_{12} + b_{12}^2)\right]\\
& + \frac{6\alpha_2}{a_{123}^5}\left[-3a_{123}^6 + 3a_{123}^4 b_{123} + a_{123}^2 b_{123}^2 + a_{123}^3 c_{123} + 3a_{123} b_{123} c_{123} + 4c_{123}^2\right],
\end{aligned} \quad (3.11.43)$$

$$\begin{aligned}
A_3 = {} & \frac{2\alpha_1 p_3^2}{a_{123}^5}\left[-81p_3^6 - 405p_3^5 a_{12} - 3p_3^4(281a_{12}^2 - 22b_{12}) - 15p_3^3 a_{12}(65a_{12}^2 - 22b_{12})\right.\\
& - 8p_3^2(87a_{12}^4 - 63a_{12}^2 b_{12} - 13b_{12}^2) - 100p_3 a_{12}(3a_{12}^4 - 3a_{12}^2 b_{12} - b_{12}^2)\\
& \left. - 20a_{12}^2(3a_{12}^4 - 3a_{12}^2 b_{12} - b_{12}^2)\right]\\
& + \frac{a_2 p_3^2}{a_{123}^4}\left[-9p_3^5 - 36p_3^4 a_{12} - 3p_3^3(19a_{12}^2 - 2b_{12}) - 24p_3^2 a_{12}(2a_{12}^2 - b_{12})\right.\\
& \left. - 8p_3(3a_{12}^4 - 3a_{12}^2 b_{12} - b_{12}^2) - 2a_{12}(3a_{12}^4 - 3a_{12}^2 b_{12} - b_{12}^2)\right]\\
& - \frac{2c_T}{3a_{123}^3}\left[3a_{123}^2(a_{123}^4 - 3a_{123}^2 b_{123} + b_{123}^2) + 3a_{123}(a_{123}^2 + b_{123})c_{123} + 2c_{123}^2\right],
\end{aligned} \quad (3.11.44)$$

$$\begin{aligned}
A_4 = {} & \frac{4\alpha_1}{a_{123}^5}\left[45p_3^8 + 225p_3^7 a_{12} + 15p_3^6(29a_{12}^2 + 2b_{12}) + 75p_3^5 a_{12}(5a_{12}^2 + 2b_{12})\right.\\
& + 8p_3^4(75a_{12}^2 - 23b_{12})b_{12} - 5p_3^3 a_{12}(75a_{12}^4 - 255a_{12}^2 b_{12} + 79b_{12}^2)\\
& - p_3^2(435a_{12}^6 - 1335a_{12}^4 b_{12} + 343a_{12}^2 b_{12}^2 - 96b_{12}^3)\\
& \left. - 3a_{12}(5p_3 + a_{12})(15a_{12}^6 - 45a_{12}^4 b_{12} + 11a_{12}^2 b_{12}^2 - 4b_{12}^3)\right]\\
& + \frac{\alpha_2}{a_{123}^4}\left[9p_3^7 + 36p_3^6 a_{12} + 3p_3^5(17a_{12}^2 + 2b_{12}) + 24p_3^4 a_{12}(a_{12}^2 + b_{12})\right.\\
& - 8p_3^3(3a_{12}^4 - 12a_{12}^2 b_{12} + 5b_{12}^2) - p_3^2 a_{12}(51a_{12}^4 - 159a_{12}^2 b_{12} + 55b_{12}^2)\\
& \left. - 9a_{12}^2(4p_3 + a_{12})(a_{12}^4 - 3a_{12}^2 b_{12} + b_{12}^2)\right]\\
& - \frac{4c_T}{3a_{123}^3}\left[3a_{123}^2(a_{123}^4 - 3a_{123}^2 b_{123} + b_{123}^2) + 3a_{123}(a_{123}^2 + b_{123})c_{123} + 2c_{123}^2\right],
\end{aligned} \quad (3.11.45)$$

$$\begin{aligned}
A_5 = {} & \frac{6\alpha_1}{(p_1 + p_2 + p_3)^4}\left[5a_{123}^3(a_{123}^2 - 4b_{123})(3a_{123}^4 - 3a_{123}^2 b_{123} - b_{123}^2)\right.\\
& \left. + 5a_{123}^2(23a_{123}^4 - 23a_{123}^2 b_{123} + 4b_{123}^2)c_{123} - 4a_{123}(a_{123}^2 - 4b_{123})c_{123}^2 + 8c_{123}^3\right]\\
& + \frac{\alpha_2}{2(p_1 + p_2 + p_3)^3}\left[3a_{123}^2(a_{123}^2 - 4b_{123})(3a_{123}^4 - 3a_{123}^2 b_{123} - b_{123}^2)\right.\\
& \left. + 3a_{123}(23a_{123}^4 - 23a_{123}^2 b_{123} + 4b_{123}^2)c_{123} - 4(a_{123}^2 - 2b_{123})c_{123}^2\right]\\
& + 2(c_T + 4c_g c_T)(p_1^5 + p_2^5 + p_3^5).
\end{aligned} \quad (3.11.46)$$

Chapter 4

Examples

4.1. Conformal perturbation theory

Once we are given a CFT, we can deform its action by turning a source for some operator(s). Usually, we assume that the source turned on does not depend on time or space, so that we obtain a new theory with a specific coupling. Such a new theory, with the flat space action

$$S = S_{CFT} + g_0 \int \mathrm{d}^d \boldsymbol{x}\, \mathcal{O}(\boldsymbol{x}) \tag{4.1.1}$$

is usually not conformal any more. In this section we want to discuss basic properties of the deformed theories.

By $\langle \ldots \rangle_0$ we denote the expectation value in the pure CFT and by $\langle \ldots \rangle$ the expectation value in the deformed theory, i.e.,

$$\langle \ldots \rangle = \langle \ldots e^{-g_0 \int \mathrm{d}^d \boldsymbol{x} \mathcal{O}(\boldsymbol{x})} \rangle_0. \tag{4.1.2}$$

4.1.1. Stress-energy tensor

We can interpret (4.1.1) as the action,

$$S[g_{ij}, A_i, \phi_0] = S_{CFT}[g_{ij}, A_i] + \int \mathrm{d}^d \boldsymbol{x} \sqrt{g} \phi_0 \mathcal{O}, \tag{4.1.3}$$

with the source ϕ_0 for the operator $\mathcal{O}$ turned on. After the functional derivatives are applied, one sets $\phi_0 = g_0$. For further simplicity we will assume that $\mathcal{O}$ is a scalar and a conformal primary of dimension Δ. The stress-energy tensor of the full theory with the source ϕ_0 turned on is

$$T_{ij}[g_{ab}, \phi_0] = T_{ij}[g_{ab}, \phi_0 = 0] - \phi_0 g_{ij} \mathcal{O}(\boldsymbol{x}). \tag{4.1.4}$$

By turning the sources off we find

$$T_{ij} = \mathcal{T}_{ij} - \phi_0 g_{ij} \mathcal{O}(\boldsymbol{x}) \tag{4.1.5}$$

where $\mathcal{T}_{ij}$ denotes the stress-energy tensor of the underlying CFT. Furthermore we assume the operator $\mathcal{O}$ does not depend on the metric, *i.e.*, (2.4.33) and (2.4.34) hold. Furthermore,

$$\frac{\delta T_{ij}(\boldsymbol{x}_1)}{\delta \phi_0(\boldsymbol{x}_2)} = -g_{ij}\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_1). \tag{4.1.6}$$

The stress-energy tensor of the deformed theory is no longer traceless. However, one can still use the trace Ward identities (1.3.42) in order to obtain correlation functions of the trace of the stress-energy tensor. The only difference with the conformal case is that now one fixes $\phi_0 = g_0$ after the derivatives are taken. In this case we obtain a set of Ward identities in the deformed theory. Define parameter λ by

$$\Delta = d - \lambda \tag{4.1.7}$$

where Δ is the conformal dimension of $\mathcal{O}$. Noting that all 1-point functions vanish, for the 2-point functions we find

$$\langle T(\boldsymbol{x}_1)\rangle = -\lambda\phi_0(\boldsymbol{x}_1)\langle \mathcal{O}(\boldsymbol{x}_1)\rangle, \tag{4.1.8}$$

$$\langle T(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = -\lambda\phi_0(\boldsymbol{x}_1)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle, \tag{4.1.9}$$

$$\langle T(\boldsymbol{x}_1)T_{ij}(\boldsymbol{x}_2)\rangle = -\lambda\phi_0(\boldsymbol{x}_1)\langle \mathcal{O}(\boldsymbol{x}_1)T_{ij}(\boldsymbol{x}_2)\rangle, \tag{4.1.10}$$

$$\langle T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)\rangle = \lambda^2\phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle. \tag{4.1.11}$$

For 3-point functions we find

$$\begin{aligned}\langle T(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle &= -\lambda\phi_0(\boldsymbol{x}_1)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle \\ &- \Delta\left[\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_3)\rangle + \delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle\right], \end{aligned}\tag{4.1.12}$$

$$\begin{aligned}\langle T(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle &= -\lambda\phi_0(\boldsymbol{x}_1)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle + 2\langle \frac{\delta T(\boldsymbol{x}_1)}{\delta g^{kl}(\boldsymbol{x}_3)}\mathcal{O}(\boldsymbol{x}_2)\rangle \\ &- \Delta\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)T_{kl}(\boldsymbol{x}_3)\rangle, \end{aligned}\tag{4.1.13}$$

$$\begin{aligned}\langle T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle &= \lambda^2\phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle \\ &+ \lambda\Delta\left[\phi_0(\boldsymbol{x}_1)\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_3)\rangle + \phi_0(\boldsymbol{x}_2)\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle\right. \\ &\left.+ \phi_0(\boldsymbol{x}_3)\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\langle \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle\right] + 2\delta^{ij}\langle \frac{\delta T(\boldsymbol{x}_1)}{\delta g^{ij}(\boldsymbol{x}_2)}\mathcal{O}(\boldsymbol{x}_3)\rangle, \end{aligned}\tag{4.1.14}$$

$$\begin{aligned}\langle T(\boldsymbol{x}_1)T_{ij}(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle &= -\lambda\phi_0(\boldsymbol{x}_1)\langle\mathcal{O}(\boldsymbol{x}_1)T_{ij}(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle\\ &+ 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{ij}(\boldsymbol{x}_2)}T_{kl}(\boldsymbol{x}_3)\rangle + 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{kl}(\boldsymbol{x}_3)}T_{ij}(\boldsymbol{x}_2)\rangle + 2\langle\frac{\delta T_{ij}(\boldsymbol{x}_2)}{\delta g^{kl}(\boldsymbol{x}_3)}T(\boldsymbol{x}_1)\rangle,\\ &+ 2\lambda\phi_0(\boldsymbol{x}_1)\langle\mathcal{O}(\boldsymbol{x}_1)\frac{\delta T_{ij}(\boldsymbol{x}_2)}{\delta g^{kl}(\boldsymbol{x}_3)}\rangle, \end{aligned} \tag{4.1.15}$$

$$\begin{aligned}\langle T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle &= \lambda^2\phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_2)\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)T_{kl}(\boldsymbol{x}_3)\rangle\\ &+ 2\delta^{ij}\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{ij}(\boldsymbol{x}_2)}T_{kl}(\boldsymbol{x}_3)\rangle + 2\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{kl}(\boldsymbol{x}_3)}T(\boldsymbol{x}_2)\rangle + 2\langle\frac{\delta T(\boldsymbol{x}_2)}{\delta g^{kl}(\boldsymbol{x}_3)}T(\boldsymbol{x}_1)\rangle\\ &+ \lambda\Delta\phi_0(\boldsymbol{x}_2)\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\langle\mathcal{O}(\boldsymbol{x}_1)T_{kl}(\boldsymbol{x}_3)\rangle,\\ \langle T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)T(\boldsymbol{x}_3)\rangle &= -\lambda^3\phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_2)\phi_0(\boldsymbol{x}_3)\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle\\ &+ 2\delta^{ij}\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{ij}(\boldsymbol{x}_2)}T(\boldsymbol{x}_3)\rangle + 2\delta^{ij}\langle\frac{\delta T(\boldsymbol{x}_1)}{\delta g^{ij}(\boldsymbol{x}_3)}T(\boldsymbol{x}_2)\rangle + 2\delta^{ij}\langle\frac{\delta T(\boldsymbol{x}_2)}{\delta g^{ij}(\boldsymbol{x}_3)}T(\boldsymbol{x}_1)\rangle\\ &- \lambda^2\Delta^2\left[\phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_2)\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\langle\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle\right.\\ &\qquad+ \phi_0(\boldsymbol{x}_2)\phi_0(\boldsymbol{x}_3)\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_3)\rangle\\ &\qquad\left.+ \phi_0(\boldsymbol{x}_1)\phi_0(\boldsymbol{x}_3)\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle\right] \end{aligned} \tag{4.1.16}$$

4.1.2. Renormalisation group flow

Correlation functions in the perturbed theory can be evaluated by means of the perturbative expansion in the bare coupling constant g_0,

$$\begin{aligned}\langle\ldots\rangle = \langle\ldots\rangle_0 - g_0\int \mathrm{d}^d\boldsymbol{u}_1\langle\ldots\mathcal{O}(\boldsymbol{u}_1)\rangle_0\\ + \frac{g_0^2}{2}\int \mathrm{d}^d\boldsymbol{u}_1\mathrm{d}^d\boldsymbol{u}_2\langle\ldots\mathcal{O}(\boldsymbol{u}_1)\mathcal{O}(\boldsymbol{u}_2)\rangle_0 + O(g_0^3). \end{aligned} \tag{4.1.17}$$

Generically the integrals in (4.1.17) diverge. In order to remedy the problem, first we will introduce IR and UV cutoffs Λ_{IR} and Λ_{UV}. Then, in order to remove the divergence, one must introduce some counterterms S_{ct}, which depend on the cutoffs in such a way that the full theory is divergence-free when the cutoffs are removed. We can extract the divergence by looking at the insertion of $e^{-g_0\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathcal{O}}$ in (4.1.17), which gives

$$\begin{aligned}e^{-g_0\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathcal{O}} &= 1 - g_0\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathrm{d}^d\boldsymbol{u}_1\mathcal{O}(\boldsymbol{u}_1) + \frac{g_0^2}{2}\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathrm{d}^d\boldsymbol{u}_1\mathrm{d}^d\boldsymbol{u}_2\mathcal{O}(\boldsymbol{u}_1)\mathcal{O}(\boldsymbol{u}_2) + \ldots\\ &= 1 - g_0\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathrm{d}^d\boldsymbol{u}_1\mathcal{O}(\boldsymbol{u}_1) + \frac{g_0^2 C S_{d-1}}{2\lambda}(\Lambda_{IR}^\lambda - \Lambda_{UV}^\lambda)\int_{\Lambda_{UV}}^{\Lambda_{IR}}\mathrm{d}^d\boldsymbol{u}_1\mathcal{O}(\boldsymbol{u}_1) + \ldots \end{aligned} \tag{4.1.18}$$

where we used the OPE (1.4.9) and C denotes the OPE constant. We omit the descendants, since they are less singular and we are interested in the extraction of the singularity only. S_{d-1} denotes the volume of the $(d-1)$-dimensional sphere,

$$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{4.1.19}$$

As we can see, whether the divergence occurs in the UV or IR depends on whether $\lambda < 0$ or $\lambda > 0$ or equivalently whether $\Delta > d$ or $\Delta < d$. In the following chapters of this thesis we will be interested in the case where the deforming operator is *relevant in the IR*, *i.e*, $\lambda > 0$. In such a case the $\Lambda_{UV} \to 0$ limit can be taken in (4.1.18) and we are left with the IR cutoff $\Lambda = \Lambda_{IR}$ only. On the other hand operators with $\lambda < 0$ are called *irrelevant* in the IR as they mostly modify the UV physics. Finally if $\Delta = d$ the operator is marginal and its impact on the physics depends on the specifics of the theory. It is possible that a conformal theory deformed by a marginal operator remains conformal.

From expression (4.1.18) we can interpret the renormalisation procedure in a several equivalent ways.

1. The action requires a counterterm at order g_0^2 to keep the perturbative expansion finite.

2. Operators mix. The counterterm changes the form of the operator $\mathcal{O}$ in the perturbed theory. To define $\mathcal{O}_g$ in the perturbed theory one must take $\mathcal{O}$ and mix with some other operators as well, in this case the identity operator. This is the same procedure as for obtaining normal ordered operators.

3. Couplings run. Instead of mixing the operators, the contribution from the counterterm can be absorbed into the coupling constant. As we can see in (4.1.18) the counterterm has a form of the source term for $\mathcal{O}$ in the generating functional. Therefore we can write

$$g(\Lambda) = g_0 - \frac{g_0^2 C}{\lambda} \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \Lambda^{\lambda} + O(g_0^3). \tag{4.1.20}$$

Having the expression (4.1.20), one can finalise the renormalisation procedure by fixing the value $g = g(\Lambda_0)$ at some arbitrary renormalisation scale. This is an additional information included into the quantum theory. At this point the value of the original perturbing parameter g_0 in (4.1.17) is fixed, but its meaning as a size of the perturbation is valid only at the energy scale Λ_0. Therefore it is useful to remove g_0 from all formulae and use the scale-dependent coupling instead. Its changes due to the change of the renormalisation scale can be analysed by means of the *renormalisation group (RG) equations*. To do it define dimensionless coupling ϕ by

$$g(\Lambda) = \phi(\Lambda)\Lambda^{-\lambda}. \tag{4.1.21}$$

Then one can check that (4.1.20) is a solution to the following differential equation

$$\beta(\phi) \equiv -\frac{\mathrm{d}\phi}{\mathrm{d}\log\Lambda} = -\lambda\phi + \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}C\phi^2 + O(\phi^3), \tag{4.1.22}$$

known as the *renormalisation group (RG) equation*, where the function $\beta(\phi)$ is known as the *beta-function*. One can regard the right hand side of this equation as a vector field in the space of couplings. The solution to the renormalisation group equation is therefore known as the *renormalisation group (RG) flow*. Points where $\beta(\phi) = 0$ are called *fixed points*.

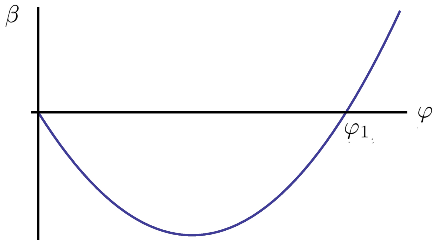


Figure 4.1: *The plot of the β-function (4.1.22) for $C > 0$. Note the 2-fixed points: one at $\phi = 0$ is the UV fixed point while one at $\phi = \frac{\lambda}{2\pi C}$ is the IR fixed point.*

Assuming C to be a positive constant of order unity, the β-function is as illustrated in figure 4.1 and we obtain an RG flow to a nearby IR fixed point at

$$\phi = \phi_1 + O(\lambda^2), \qquad \phi_1 \equiv \frac{\lambda}{2\pi C} \ll 1. \tag{4.1.23}$$

If instead C vanishes or is negative, then the nature of the IR theory will depend on the higher order coefficients in the β-function. We will not consider these cases here. For positive C then, since ϕ is small throughout the flow, we may remove higher order terms in the β-function by a suitable field redefinition $\phi = \bar{\phi} + O(\bar{\phi}^3)$. In the following, we will assume this has been accomplished and work with the purely quadratic β-function. Results for the general case may then be found by undoing the field redefinition, generating corrections at subleading orders in the expansion parameter λ.

Expanding the quadratic β-function about the IR fixed point, we find

$$\beta = \lambda(\phi - \phi_1) + 2\pi C(\phi - \phi_1)^2. \tag{4.1.24}$$

In the IR CFT, $\mathcal{O}$ thus has dimension $\Delta_{IR} = 3 + \lambda$ while the OPE coefficient is unchanged.

The entire RG flow may be obtained by integrating the β function directly, yielding

$$\left(\frac{\Lambda_0}{\Lambda}\right)^{\lambda} = \frac{\phi_0}{\phi}\frac{(\phi_1 - \phi)}{(\phi_1 - \phi_0)}, \tag{4.1.25}$$

where we imposed the boundary condition $\phi(\Lambda_0) = \phi_0$ for $0 < \phi_0 < \phi_1$. Inverting, we find

$$\phi(\Lambda) = \phi_1 \left[1 + \frac{\phi_1}{g}\Lambda^{-\lambda}\right]^{-1}, \tag{4.1.26}$$

where

$$g = \phi_1 \Lambda_0^{-\lambda} \left[\frac{\phi_1}{\phi_0} - 1\right]^{-1}. \tag{4.1.27}$$

Consequently, as we remove the cutoff,

$$\phi(\Lambda) \to g\Lambda^{\lambda} \qquad \text{as} \qquad \Lambda \to 0, \tag{4.1.28}$$

allowing us to identify g as the dimensionful renormalised coupling in the UV CFT. Note that the renormalised coupling depends on the renormalisation scale Λ_0. From now on, we will assume that the correlation functions in the deformed theory are renormalised and we are allowed to use the renormalised constant g.

The conformal perturbation theory is a starting point for an analysis of a vast class of quantum field theories. In the following sections we will compute 2- and 3-point functions in the perturbed CFT introduced in this section in the leading order in λ. Eventually, we would like to present the results in momentum space, as this will be useful for our cosmological analysis. For that, we should discuss the form of correlation functions in momentum space. Then, we will return to the computations of the correlation functions in the deformed theory.

4.1.3. Nearly-marginal deformations

In this section we return to the discussion of the conformal field theories deformed by an IR relevant operator,

$$S = S_{CFT} + g\int \mathrm{d}^3\boldsymbol{x}\mathcal{O}(\boldsymbol{x}). \tag{4.1.29}$$

From now on we will work in $d = 3$ spacetime dimensions as this case will be relevant for the cosmological analysis. We assume $\mathcal{O}$ is a scalar conformal primary operator of dimension $\Delta = d - \lambda$, where $0 < \lambda < 1$. In this section we will work in first order expansion in λ, so we assume $\lambda \ll 1$.

Finally, let us define the normalisations of 2- and 3-point functions via its OPE expansion in position space (1.4.9) denoting $C_{12} = \alpha$. This leads to

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 = \frac{\alpha}{x_{12}^{6-2\lambda}}, \tag{4.1.30}$$

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 = \frac{\alpha C}{x_{12}^{3-\lambda}x_{23}^{3-\lambda}x_{31}^{3-\lambda}}. \tag{4.1.31}$$

The Fourier transform of the 2-point function can be carried out by means of equation (2.1.3) and reads

$$\begin{aligned}\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle_0 &= \frac{2^{-3+2\lambda}\pi^{\frac{3}{2}}\alpha\Gamma\left(\lambda-\frac{3}{2}\right)}{\Gamma(3-\lambda)}p^{3-2\lambda} \\ &= \frac{\pi^2\alpha}{12}p^{3-2\lambda} + O(\lambda),\end{aligned} \tag{4.1.32}$$

where we expanded the prefactor in λ but kept the dilatation dimension exactly.

The 3-point function can be expressed in terms of the triple-K integral by equation (2.2.3),

$$\begin{aligned}\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle_0 &= \frac{\alpha C\pi^3 2^{3\lambda-\frac{1}{2}}}{\Gamma\left(3-\frac{3\lambda}{2}\right)\Gamma^3\left(\frac{3-\lambda}{2}\right)}I_{\frac{1}{2}\{\frac{3}{2}-\lambda,\frac{3}{2}-\lambda,\frac{3}{2}-\lambda\}} \\ &= \frac{\alpha C\pi^3}{3\lambda}\left[p_1^{3-3\lambda}+p_2^{3-3\lambda}+p_3^{3-3\lambda}\right] + O(\lambda^0).\end{aligned} \tag{4.1.33}$$

In the last line we expanded the result in the leading order in λ, while keeping the exact dilatation dimension $3-3\lambda$ and maintaining the symmetry of the correlation function. Note that the result is singular for $\lambda \to 0$. Therefore, it can be obtained by the method presented in section 2.6.4. One expands two out of three Bessel functions up to order $3/2$, so that all remaining terms do not lead to the singularity at $x = 0$. Then, one uses formula (2.A.51) and the result can be expanded to the leading order in λ.

4.1.4. OPE constants and expansion in λ

The result (4.1.33) shows that the leading λ term of the 3-point function in a pure CFT is singular. In particular there is no smooth $\lambda \to 0$ limit. We would like to investigate the λ behaviour of various n-point functions before we start calculataing them.

Let us introduce the following notation. Let $F(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ be any function in position space that depends on λ. Let $\hat{F}$ be its Fourier transform. If the most singular term in the power series expansion of $\hat{F}$ is $1/\lambda^\alpha$ for some $\alpha \in \bar{\mathbb{R}}$, then we define the symbol $\propto$ as

$$F \propto \frac{1}{\lambda^\alpha}, \qquad \hat{F} \propto \frac{1}{\lambda^\alpha} \tag{4.1.34}$$

In other words the only information we care about is the most singular behaviour in λ in momentum space. Assume $F \propto \frac{1}{\lambda^\alpha}$ and $G \propto \frac{1}{\lambda^\beta}$. Clearly, we have the following properties:

1. $(F+G) \propto \frac{1}{\lambda^{\max(\alpha,\beta)}}$,
2. $(\hat{F} \cdot \hat{G}) \propto \frac{1}{\lambda^{\alpha+\beta}}$,
3. if F and G depend on different variables, i.e. $F = F(\boldsymbol{x}_1, \dots, \boldsymbol{x}_p)$ and $G = G(\boldsymbol{x}_{p+1}, \dots, \boldsymbol{x}_n)$, then $(F \cdot G) \propto \frac{1}{\lambda^{\alpha+\beta}}$,
4. $\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 \propto 1$,
5. $\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \propto \frac{1}{\lambda}$,
6. $\frac{1}{x^{3+\alpha\lambda}} \propto \frac{1}{\lambda}$ for any non-zero $\alpha \propto 1$,
7. $\frac{1}{x^3} \propto 1$ due to the equation eqrefe:2ptFourier.

The last two statements follow from (2.1.3).

Now we want to argue that the leading order singularity of the n-point function of $\mathcal{O}$ is $1/\lambda^{n-2}$. To show it, assume that the only relevant interaction is between $\mathcal{O}$ themselves, *i.e.*, we can consider the OPE,

$$\mathcal{O}(\boldsymbol{x})\mathcal{O}(0) \sim \frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{x^{3-\lambda}}\mathcal{O}(0), \tag{4.1.35}$$

where we denote $c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}} = C$ with C defined in (1.4.9). We will return to the validity of this assumption in section 4.1.7. Taking the $\boldsymbol{x}_1 \to \boldsymbol{x}_2$ limit in the three-point function (4.1.33) we have

$$\frac{1}{\lambda} \propto \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{x_1 \to x_2}{\longrightarrow} \frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{x_1^{3-\lambda}}\langle \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0. \tag{4.1.36}$$

Note that the Fourier transform of the right-hand side is particularly simple, as different variables are separated. Since the two-point function is of order $O(1)$ in λ and the Fourier transform of $1/x_1^{3-\lambda}$ is of order $1/\lambda$, comparison with the three-point function gives $c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}} \propto 1$.

Now we can take the n-point function and use the OPE $(n-2)$ times,

$$\begin{aligned}\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\dots\mathcal{O}(\boldsymbol{x}_n)\rangle_0 &\longrightarrow \frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{\boldsymbol{x}_1^{3-\lambda}}\frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{\boldsymbol{x}_2^{3-\lambda}}\cdots\frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{\boldsymbol{x}_{n-2}^{3-\lambda}}\langle \mathcal{O}(\boldsymbol{x}_{n-1})\mathcal{O}(\boldsymbol{x}_n)\rangle_0 \\ &\propto \frac{1}{\lambda^{n-2}}.\end{aligned} \tag{4.1.37}$$

4.1.5. Scalar 2-point function

Let us now compute the 2-point function of $\mathcal{O}$ in the perturbed theory,

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(-\phi\Lambda^{-\lambda})^n \mathcal{I}_n, \tag{4.1.38}$$

where $\mathcal{I}_n$ is an unperturbed CFT correlator with n integrated scalar insertions,

$$\mathcal{I}_n = \int \mathrm{d}^3\boldsymbol{z}_1 \ldots \mathrm{d}^3\boldsymbol{z}_n \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{z}_1)\ldots\mathcal{O}(\boldsymbol{z}_n)\rangle_0. \tag{4.1.39}$$

To regulate the integral the range of integration is restricted so that no two insertion points approach closer than the cutoff distance Λ. As our intention is to work to leading order in λ, it is sufficient to compute only the leading singular behaviour of $\mathcal{I}_n$ as $\lambda \to 0$. We will see by the argument to follow that $\mathcal{I}_n \sim \lambda^{-n}$ in this limit; combined with the prefactor $\phi^n \sim \lambda^n$, each term in the sum (4.1.38) then makes an order one contribution.

Beginning with the integral $\mathcal{I}_1$, we may formally impose the cutoff by inserting two Heaviside step functions,

$$\mathcal{I}_1 = \int \mathrm{d}^3\boldsymbol{z}_1 \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{z}_1)\rangle_0 \Theta(|\boldsymbol{z}_1-\boldsymbol{x}_1|-\Lambda)\Theta(|\boldsymbol{z}_1-\boldsymbol{x}_2|-\Lambda). \tag{4.1.40}$$

As we are not interested in contact terms in the 2-point function (4.1.38) we will assume that $x_{12} = |\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg \Lambda$. If we now vary with respect to the cutoff Λ, we pick up contributions from the two spherical shells surrounding $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{I}_1}{\mathrm{d}\Lambda} &= \int \mathrm{d}^3\boldsymbol{z}_1 \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{z}_1)\rangle_0 [-\delta(|\boldsymbol{z}_1-\boldsymbol{x}_1|-\Lambda)-\delta(|\boldsymbol{z}_1-\boldsymbol{x}_2|-\Lambda)] \\ &= -2(4\pi\Lambda^2)\frac{C}{\Lambda^{3-\lambda}}\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 + \ldots,\end{aligned} \tag{4.1.41}$$

where in the second line we used the OPE (1.4.9). As $\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0$ is independent of Λ, upon integrating we obtain

$$\mathcal{I}_1 = -\frac{8\pi C}{\lambda}(\Lambda^\lambda - f x_{12}^\lambda)\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 + \ldots, \tag{4.1.42}$$

where f is an arbitrary constant, the x_{12}^λ dependence being fixed on dimensional grounds as no other scales are present. The ellipsis indicates omitted contributions from the remaining terms in the OPE (1.4.9). Crucially, these contributions cannot take the form of $1/\lambda$ poles unless there are other terms in the OPE scaling as $x_{12}^{-3+m\lambda}$ for some nonzero constant m. As a simplifying assumption, we will therefore assume that such terms, if present at all, are of subleading order in λ, *i.e.*, the associated OPE coefficient is of order λ or greater.[1] Physically, this means

[1]An exception to this will be the stress tensor, although as we will see in Section 4.1.7 its inclusion does not affect our present results.

that at leading order $\mathcal{O}$ is the only operator becoming marginal in the limit $\lambda \to 0$. The equation (4.1.42) then captures the leading behaviour in this limit.

To determine the constant of integration f we require this limit to be non-singular, fixing $f = 1$. As $\lambda \to 0$ we then obtain a logarithmic dependence on the cutoff Λ signalling a Weyl anomaly,

$$\lim_{\lambda \to 0} \mathcal{I}_1 = 8\pi C \log\left(\frac{x_{12}}{\Lambda}\right) \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 + \dots . \tag{4.1.43}$$

For $\lambda > 0$, on the other hand, there is no Weyl anomaly and we may safely remove the cutoff. Sending $\Lambda \to 0$, we find

$$\mathcal{I}_1 = \frac{8\pi C}{\lambda} \frac{\alpha}{x_{12}^{6-3\lambda}} + \dots \tag{4.1.44}$$

The apparently singular behaviour of this result as $\lambda \to 0$ is simply an artefact of removing the cutoff.

Proceeding now to the general integral $\mathcal{I}_n$, we first introduce step functions to regulate the separation between all possible pairs of insertion points enforcing $z_{ij} > \Lambda$, $|\boldsymbol{z}_i - \boldsymbol{x}_1| > \Lambda$ and $|\boldsymbol{z}_i - \boldsymbol{x}_2| > \Lambda$ for all $i, j = 1 \dots n$ such that $i < j$. Differentiating with respect to the cutoff, in place of (4.1.41) we now obtain

$$\frac{\mathrm{d}\mathcal{I}_n}{\mathrm{d}\Lambda} = -4\pi\Lambda^2 B_n \frac{C}{\Lambda^{3-\lambda}} \mathcal{I}_{n-1} + \dots, \tag{4.1.45}$$

where the combinatorial factor B_n counts the number of step functions we had initially, *i.e.*, the number of pairs we can form by bringing together the n insertion points z_i, either amongst themselves or with either $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$, namely

$$B_n = \binom{n}{2} + 2n = \frac{1}{2} n(n+3). \tag{4.1.46}$$

Note that in writing (4.1.45) we have effected a dilute gas approximation in which contributions to the integral $\mathcal{I}_n$ from configurations in which more than two insertion points coincide are neglected. This approximation is justified since the phase space associated with these configurations is comparatively small while the value of the integrand is comparable.

Prior to integrating (4.1.45), it is useful to trade Λ for

$$y = 1 - \left(\frac{\Lambda}{x_{12}}\right)^{\lambda}, \tag{4.1.47}$$

so that

$$\frac{\mathrm{d}\mathcal{I}_n}{\mathrm{d}y} = n(n+3)\frac{2\pi C}{\lambda} x_{12}^{\lambda} \mathcal{I}_{n-1} + \dots \tag{4.1.48}$$

To fix the arbitrary constant of integration, we then require that $\lambda^n \mathcal{I}_n \to 0$ as $\lambda \to 0$, i.e., the constant is chosen so as to cancel the leading $1/\lambda^n$ pole as $\lambda \to 0$.

Divergences due to subleading poles will be cancelled by the subleading terms we omitted in (4.1.45). Given that $y \to 0$ as $\lambda \to 0$, we find that at each order n the constants arising from integration with respect to y vanish, yielding

$$\mathcal{I}_n = \frac{n!(n+3)!}{3!}\Big(\frac{2\pi C}{\lambda}x_{12}^{\lambda}\Big)^n\frac{y^n}{n!}\mathcal{I}_0 + \ldots = \frac{\alpha}{6}(n+3)!\,\phi_1^{-n}y^n x_{12}^{(n+2)\lambda-6} + \ldots, \tag{4.1.49}$$

with ϕ_1 as given in (4.1.23).

The 2-point function in the perturbed theory may now be evaluated at leading order in λ courtesy of (4.1.38),

$$\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = \frac{\alpha}{6}\,x_{12}^{2\lambda-6}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)\Big[-\frac{\phi}{\phi_1}\Big(\frac{x_{12}^{\lambda}}{\Lambda^{\lambda}}-1\Big)\Big]^n. \tag{4.1.50}$$

Sending $\Lambda \to 0$ and taking note of the Λ-dependence of ϕ in (4.1.26), we may re-express this result as a sum of exact CFT 2-point functions with shifted dimensions:

$$\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = \frac{1}{6}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)\Big(-\frac{g}{\phi_1}\Big)^n\langle\mathcal{O}_{\Delta_n}(\boldsymbol{x}_1)\mathcal{O}_{\Delta_n}(\boldsymbol{x}_2)\rangle_0, \tag{4.1.51}$$

where g is the renormalised coupling defined in (4.1.27) and

$$\Delta_n = \Delta - \frac{n\lambda}{2} = 3 - \frac{\lambda}{2}(n+2). \tag{4.1.52}$$

Summing up the binomial series, we find

$$\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = \alpha x_{12}^{2\lambda-6}\Big[1+\frac{g}{\phi_1}x_{12}^{\lambda}\Big]^{-4}. \tag{4.1.53}$$

Finally, to transform to momentum space using (2.1.3),

$$\langle\!\langle\mathcal{O}_{\Delta_n}(\boldsymbol{p})\mathcal{O}_{\Delta_n}(-\boldsymbol{p})\rangle\!\rangle_0 = \frac{\pi^2}{12}\,p^{3-(n+2)\lambda}(1+O(\lambda)), \tag{4.1.54}$$

and then resum to find

$$\langle\!\langle\mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = \frac{\pi^2}{12}\,\alpha p^{3-2\lambda}\Big[1+\frac{g}{\phi_1}p^{-\lambda}\Big]^{-4}. \tag{4.1.55}$$

The result (4.1.55) is correct to leading order in λ after expansion in the renormalised coupling g.

4.1.6. Scalar 3-point function

Since we aim at finding the leading term of the expansion of the 3-point functions in λ, we must first recognize all possible places where λ may appear. Apart from the explicit dependence of the CFT correlation functions on the conformal dimension $\Delta = d - \lambda$, the form of our solutions will depend on the relations between λ and ratios of momenta. We define two cases, particularly important from the point of view of cosmology, which we will discuss in section 5.6.

- *Equilateral case*, where all momenta and distances are of order 1 with respect to λ, *i.e.*,

$$x_{12}^{\lambda}(1 + O(\lambda)) = x_{23}^{\lambda}(1 + O(\lambda)) = x_{31}^{\lambda}(1 + O(\lambda)), \tag{4.1.56}$$

  or equivalently

$$p_1^{-\lambda}(1 + O(\lambda)) = p_2^{-\lambda}(1 + O(\lambda)) = p_3^{-\lambda}(1 + O(\lambda)). \tag{4.1.57}$$

- *Local* or *squashed case*, where one of the momentum becomes small in comparison to the remaining two, say

$$\frac{p_3}{p_2} \sim \frac{p_1}{p_2} \sim \lambda, \qquad\qquad p_1^{-\lambda}(1 + O(\lambda)) = p_3^{-\lambda}(1 + O(\lambda)). \tag{4.1.58}$$

This means that after the expansion in λ is performed, we can impose two additional conditions on the form of the correlation functions by adjusting the subleading terms. First, we require that the dilatation dimension of all correlation functions is exact, which means that the total power of magnitudes of momenta is fixed and contains the parameter λ. Secondly, we want to impose the consistency relation between equilateral and local limits. Since

$$\begin{aligned} -\frac{\partial}{\partial g} \langle \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2) \rangle &= -\frac{\partial}{\partial g} \langle \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2) e^{-g \int \mathcal{O}} \rangle_0 \\ &= \int \mathrm{d}^3 \boldsymbol{z} \, \langle \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{z}) e^{-g \int \mathcal{O}} \rangle_0 \\ &= \int \mathrm{d}^3 \boldsymbol{z} \, \langle \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{z}) \rangle, \end{aligned} \tag{4.1.59}$$

in the zero-momentum limit

$$\langle\!\langle \mathcal{O}(0) \mathcal{O}(\boldsymbol{p}) \mathcal{O}(-\boldsymbol{p}) \rangle\!\rangle = -\frac{\partial}{\partial g} \langle\!\langle \mathcal{O}(\boldsymbol{p}) \mathcal{O}(-\boldsymbol{p}) \rangle\!\rangle. \tag{4.1.60}$$

We require that this relation holds in the renormalised theory.

Let us start our analysis with the equlateral case, where there is effectively a single scale

$$L^{\lambda} = x_{12}^{\lambda}(1 + O(\lambda)) = x_{23}^{\lambda}(1 + O(\lambda)) = x_{31}^{\lambda}(1 + O(\lambda)). \tag{4.1.61}$$

By repeating the arguments of the previous section one finds the leading order 3-point function for separated insertions,

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle = \frac{\alpha\, C}{x_{12}^{\Delta}x_{23}^{\Delta}x_{31}^{\Delta}}\Big[1+\frac{g}{\phi_1}L^{\lambda}\Big]^{-6}. \tag{4.1.62}$$

The power of minus six appearing in this result arises because here we are summing a binomial series derived from the combinatorial factor $B_n = \binom{n}{2} + 3n$ in place of (4.1.46), encoding the presence of an additional fixed scalar insertion. Note also that, since the dimension of $\mathcal{O}$ differs from the spatial dimension only by λ, the general CFT correlator with an arbitrary number of integrated scalar insertions is invariant under special conformal transformations at leading order in λ, constraining the arbitrary functions arising from integrating with respect to the cutoff to be of the form L^{λ}.

If instead the x_{ij}^{λ} are no longer all of comparable magnitude, we find ourselves in a limit where the OPE is applicable, and utilising our previous result (4.1.53) we find, e.g., when $x_{12}^{\lambda} \ll x_{23}^{\lambda} \approx x_{31}^{\lambda}$,

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle = \frac{\alpha\, C}{x_{12}^{\Delta}x_{23}^{2\Delta}}\Big[1+\frac{g}{\phi_1}x_{23}^{\lambda}\Big]^{-4}. \tag{4.1.63}$$

We may then combine (4.1.62) and the three limiting cases of the form (4.1.63) into a single result applicable at leading order for all configurations,

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle = \alpha\, C\prod_{i<j} x_{ij}^{-\Delta}\Big[1+\frac{g}{\phi_1}x_{ij}^{\lambda}\Big]^{-2}. \tag{4.1.64}$$

Expanding out the binomial series, the leading order 3-point function in the perturbed theory may alternatively be expressed as a sum of exact CFT 3-point functions with shifted dimensions,

$$\begin{aligned}\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle = \sum_{\ell_1,\ell_2,\ell_3=0}^{\infty} (\ell_1+1)(\ell_2+1)(\ell_3+1)\Big(-\frac{g}{\phi_1}\Big)^{\ell_t}\times \\ \times\ \langle \mathcal{O}_{\Delta_1}(\boldsymbol{x}_1)\mathcal{O}_{\Delta_2}(\boldsymbol{x}_2)\mathcal{O}_{\Delta_3}(\boldsymbol{x}_3)\rangle_0,\end{aligned} \tag{4.1.65}$$

where

$$\Delta_i = \Delta - \frac{\lambda}{2}(\ell_t-\ell_i), \qquad \ell_t = \ell_1+\ell_2+\ell_3. \tag{4.1.66}$$

This expanded form of the 3-point function is useful for performing the Fourier transform to momentum space, as we must do in order to ultimately connect with standard inflationary results. At leading order in λ, the result is very similar to (4.1.33). The sum can be easily done and the result, written in accordance with the consistency relation (4.1.60), is

$$\begin{aligned}\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle &= \frac{\pi^3\alpha\, C}{3\lambda}p_3^{-\lambda}\big[1+\frac{g}{\phi_1}p_3^{-\lambda}\big]^{-1}\Big(\sum_{j=1}^{3}p_j^{3-2\lambda}\Big[1+\frac{g}{\phi_1}p_j^{-\lambda}\Big]^{-2}\Big)\\ &= \frac{4\pi C}{\lambda}p_3^{-\lambda}\Big[1+\frac{g}{\phi_1}p_3^{-\lambda}\Big]^{-1}\sum_{j=1}^{3}\langle\!\langle \mathcal{O}(\boldsymbol{p}_j)\mathcal{O}(-\boldsymbol{p}_j)\rangle\!\rangle, \qquad (4.1.67)\end{aligned}$$

where the reference momentum p_3 is chosen as the largest of the three momenta and hence is implicitly nonzero.

In fact, on dimensional grounds $\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = p^{3-2\lambda}F(gp^{-\lambda})$ for some function F, since when g vanishes the 2-point correlator in the perturbed theory must reduce to that of the exact CFT, for which $\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle_0 \sim p^{3-2\lambda}$. This yields the Callan-Symanzik equation

$$0 = \Big(\frac{\partial}{\partial \log p} + \lambda g\frac{\partial}{\partial g} - 3 + 2\lambda\Big)\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle, \qquad (4.1.68)$$

which, when combined with (4.1.60), gives

$$\lambda g\langle\!\langle \mathcal{O}(0)\mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = \Big(-3+2\lambda+\frac{\partial}{\partial \log p}\Big)\langle\!\langle \mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle. \qquad (4.1.69)$$

This result will be useful in section 7.1.3 when we discuss the inflationary consistency relation for the scalar bispectrum [72].

4.1.7. Introducing the stress tensor

In this subsection, we now generalise our discussion of conformal perturbation theory to include the stress tensor. After first establishing the definition of this operator in the perturbed theory, we return to the unperturbed CFT to consider the form of 3-point correlators with mixed scalar and stress tensor insertions. From these correlators we may read off the corresponding OPEs, and hence understand the behaviour of stress tensor insertions in correlators of the perturbed theory.

Working henceforth in renormalised perturbation theory, the action takes the form

$$S = S_{CFT} + \int \mathrm{d}^3\boldsymbol{x}\sqrt{g}\,\phi\Lambda^{\lambda}\mathcal{O}, \qquad (4.1.70)$$

and we will assume the two sources ϕ and g_{ij} are functionally independent of one another. Now, in a completely general theory, the renormalised scalar operator $\mathcal{O}$ may depend on either of the sources ϕ and g_{ij}. Such a dependence, however, generally introduces additional nearly marginal scalar operators into the spectrum,[2]

[2]Unless it is possible to re-express these operators in terms of $\mathcal{O}$ and T_{ij} at higher order in λ.

e.g., $\delta\mathcal{O}/\delta\phi$ with dimension $3-2\lambda$ or $g^{ij}\delta\mathcal{O}/\delta g^{ij}$ with dimension $3-\lambda$. From a bulk perspective, this would then correspond to the introduction of additional light scalar fields besides the inflaton resulting in a multi-scalar model. Since our present aim is to concentrate on the single-field case, we will assume that $\mathcal{O}$ is independent of the sources ϕ and g_{ij}, at least to the leading order in λ at which we work.

In this case, the stress tensor T_{ij} in the perturbed theory is related to the stress tensor $\mathcal{T}_{ij}$ in the unperturbed CFT according to

$$T_{ij} = \mathcal{T}_{ij} - \phi\Lambda^{\lambda}\mathcal{O}g_{ij}. \tag{4.1.71}$$

It follows that the transverse traceless piece of these stress tensors is then identical.

We can use the results of the embeding formalism to determine the most general form of the 3-point functions of the stress-energy tensor $\mathcal{T}_{ij}$ and the scalar operator $\mathcal{O}$ in the pure CFT. The 2-point function is given by (1.4.31) with $\Delta = d = 3$,

$$\langle \mathcal{T}_{ij}(\boldsymbol{x}_1)\mathcal{T}_{kl}(\boldsymbol{x}_2)\rangle_0 = \frac{\alpha_T}{x_{12}^6}\left[\frac{1}{2}\big(I_{ik}(x)I_{jl}(x) + I_{il}(x)I_{jk}(x)\big) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] \tag{4.1.72}$$

where α_T is an undetermined normalisation constant and

$$\mathcal{I}_{ij,kl}(x) = \frac{1}{2}\big(I_{ik}(x)I_{jl}(x) + I_{il}(x)I_{jk}(x)\big) - \frac{1}{3}\delta_{ij}\delta_{kl}, \tag{4.1.73}$$

where I_{ij} is the representation of inversions (1.4.27). The 3-point correlators we will need are [22],

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{T}_{ij}(\boldsymbol{x}_3)\rangle_0 = \frac{\tilde{C}}{x_{12}^{3-2\lambda}x_{23}^3x_{31}^3}\, t_{ij}(X), \tag{4.1.74}$$

$$\langle \mathcal{T}_{ij}(\boldsymbol{x}_1)\mathcal{T}_{kl}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 = \frac{\alpha_{TT}}{x_{12}^{3+2\lambda}x_{23}^{3-2\lambda}x_{31}^{3-2\lambda}}\,\mathcal{I}_{ij,mn}(x_{31})\mathcal{I}_{kl,rs}(x_{23})t_{mn,rs}(X). \tag{4.1.75}$$

In these formulae,[3]

$$X_i = -\frac{x_{31i}}{x_{31}^2} - \frac{x_{23i}}{x_{23}^2}, \qquad t_{ij}(X) = \frac{X_iX_j}{X^2} - \frac{1}{3}\delta_{ij}, \qquad \tilde{C} = -\frac{9\alpha}{8\pi} + O(\lambda),$$
$$t_{ij,kl}(X) = -5t_{ij}(X)t_{kl}(X) - t_{ik}(X)\delta_{jl} - t_{jl}(X)\delta_{ik} + \frac{4}{3}t_{ij}(X)\delta_{kl} + \frac{4}{3}t_{kl}(X)\delta_{ij}$$
$$+\frac{1}{3}\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{9}\delta_{ij}\delta_{kl} + O(\lambda), \tag{4.1.76}$$

and we note in particular that the overall normalisation $\tilde{C}$ of the 3-point correlator with a single stress tensor insertion is fixed by the trace Ward identity (3.6.9),

[3] The specific coefficients appearing here derive from solving (3.6) and (6.20) in [22] at leading order in λ, where c in (3.4) of [22] equals α_{TT} here.

see also [73]. The second 3-point function (4.1.75) was obtained in section 1.4.4, equation (1.4.36). The first one can be obtained by the embedding formalism in the same way.

Expanding out the 3-point correlators in the limit when two insertion points coincide, we obtain the following OPE contributions with scaling dimensions close to three,

$$\mathcal{T}_{ij}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2) = A_{ij}(x_{12})\mathcal{O}(\boldsymbol{x}_2) + B_{ijkl}(x_{12})\mathcal{T}_{kl}(\boldsymbol{x}_2) + \ldots, \tag{4.1.77}$$

$$\mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2) = \frac{C}{x_{12}^{3-\lambda}}\,\mathcal{O}(\boldsymbol{x}_2) + C_{ij}(x_{12})\mathcal{T}_{ij}(\boldsymbol{x}_2) + \ldots, \tag{4.1.78}$$

where[4]

$$A_{ij}(x) = -\frac{9\alpha}{8\pi x^3}\Big(\frac{x_i x_j}{x^2} - \frac{1}{3}\delta_{ij}\Big) + O(\lambda), \tag{4.1.79}$$

$$\begin{aligned} B_{ijkl}(x) = \frac{\alpha_{TT}}{\alpha_T x^{3-\lambda}}\Bigg(&-\frac{5}{x^4}x_i x_j x_k x_l + \frac{7}{3x^2}x_k x_l \delta_{ij} - \frac{1}{x^2}x_k x_{(i}\delta_{j)l} \\ &- \frac{1}{x^2}x_l x_{(i}\delta_{j)k} + \delta_{k(i}\delta_{j)l}\Bigg) + O(\lambda), \end{aligned} \tag{4.1.80}$$

$$C_{ij}(x) = \frac{9\alpha}{8\pi\alpha_T}\frac{1}{x^{3-2\lambda}}\Big(\frac{x_i x_j}{x^2} - \frac{1}{3}\delta_{ij}\Big) + O(\lambda). \tag{4.1.81}$$

Our first task is now to verify that the presence of the stress tensor in the $\mathcal{O}\mathcal{O}$ OPE does not modify our earlier computations of the scalar 2- and 3-point functions. Fortunately, this is indeed the case. To illustrate this, let us consider the regulated integral $\mathcal{I}_1$ given in (4.1.40). Varying with respect to the cutoff Λ and using the OPE (4.1.78), the r.h.s. of our earlier result (4.1.41) acquires a new contribution

$$\begin{aligned} -\int \mathrm{d}^3\boldsymbol{z}_1\big[&C_{ij}(\boldsymbol{z}_1-\boldsymbol{x}_1)\langle\mathcal{T}_{ij}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0\delta(|\boldsymbol{z}_1-\boldsymbol{x}_1|-\Lambda) \\ &+ C_{ij}(\boldsymbol{z}_1-\boldsymbol{x}_2)\langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{T}_{ij}(\boldsymbol{x}_2)\rangle_0\delta(|\boldsymbol{z}_1-\boldsymbol{x}_2|-\Lambda)\big] \\ = -[\langle\mathcal{T}_{ij}(\boldsymbol{x}_1)&\mathcal{O}(\boldsymbol{x}_2)\rangle_0 + \langle\mathcal{O}(\boldsymbol{x}_1)\mathcal{T}_{ij}(\boldsymbol{x}_2)\rangle_0]\int \mathrm{d}^3\boldsymbol{y}\,C_{ij}(\boldsymbol{y})\delta(y-\Lambda). \end{aligned} \tag{4.1.82}$$

Now, in this particular example, the correlators out the front happen to vanish, but in general this will not be the case when we start from the integral $\mathcal{I}_n$ containing n integrated scalar insertions. Rather, the point is that the residual correlators factor out leaving the integral over a spherical shell of the OPE coefficient C_{ij}. From (4.1.81), this OPE coefficient is isotropic and traceless, and so its integral

[4]To rewrite A_{ij} as a well-defined distribution over $\mathbb{R}^3$ one may use differential regularisation as discussed in section 2.1.2 and 2.1.3. The remaining OPE coefficients B_{ij} and C_{ij} already have well-defined Fourier transforms.

over a spherical shell simply vanishes. We thus obtain no new corrections to our earlier results for the scalar 2- and 3-point functions.[5]

Let us next consider how to evaluate correlators in the perturbed theory involving one or more fixed insertions of the stress tensor. Repeating our above argument, we cannot contract an integrated scalar insertion with a fixed stress tensor to generate a fixed scalar insertion, since the OPE coefficient A_{ij} in (4.1.79) is likewise isotropic and traceless. Even if A_{ij} were not isotropic and traceless, its scaling as x^{-3} means that any resulting correlator would be suppressed by a factor of λ. Specifically, recall that each integrated scalar insertion obtained by expanding $\langle\exp(-\int\phi\Lambda^{-\lambda}\mathcal{O})\rangle_0$ carries a factor of ϕ which contributes one power of λ. In the case we considered in Section 4.1.5 (namely, contracting a fixed scalar insertion with an integrated scalar insertion to generate a fixed scalar insertion), this factor of λ was offset by a factor of $1/\lambda$ arising from integrating the OPE coefficient $C/x^{3-\lambda}$ over a spherical shell, as we saw in (4.1.41) and (4.1.42). In the present case, however, the scaling of A_{ij} as x^{-3} means that we do not acquire this compensating factor, hence any fixed scalar insertion obtained from the contraction of an integrated scalar insertion with a fixed stress tensor will be suppressed by a factor of λ relative to leading order. For this same reason we may also ignore operators in the OPE with scaling dimensions not close to three.

In summary then, at leading order in λ we find that insertions of the stress tensor are essentially inert. Contractions of integrated scalars with fixed stress tensors produce no contribution, and contractions of integrated scalars cannot generate stress tensor insertions. Thus, only scalar insertions participate in the leading order resummation process.

4.1.8. 2-point functions

Having ascertained the rules of the resummation process when stress tensor insertions are included, it remains to evaluate the specific 2- and 3-point correlators that will appear in our holographic formulae. No new methods are required for this analysis, only a straightforward application of those developed above for correlators with scalar insertions.

Beginning with the 2-point functions, at leading order in λ we find that

$$\langle T^{\perp}_{ij}(\boldsymbol{x}_1)T^{\perp}_{kl}(\boldsymbol{x}_2)\rangle = \langle \mathcal{T}^{\perp}_{ij}(\boldsymbol{x}_1)\mathcal{T}^{\perp}_{kl}(\boldsymbol{x}_2)\rangle_0, \tag{4.1.83}$$

$$\langle T^{\perp}_{ij}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle = \Big[1+\frac{g}{\phi_1}x_{12}^{\lambda}\Big]^{-2}\langle \mathcal{T}^{\perp}_{ij}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\rangle_0 = 0. \tag{4.1.84}$$

[5] In fact, the isotropy of C_{ij} alone would be sufficient to establish this, since we would obtain residual correlators involving the trace of the stress tensor which vanish by the trace Ward identities, noting that in the regulated correlators none of the insertion points are coincident.

By $T^{\perp}$ and $\mathcal{T}^{\perp}$ we denote here the projection onto the transverse-traceless part of the operator. The first of these results is a straightforward reflection of the fact that integrated scalar insertions yield no contribution when contracted with fixed stress tensor insertions. As for the second result, although we may contract each of the integrated scalar insertions against the single fixed scalar insertion, each contraction simply returns a fixed scalar insertion and so, after resumming the binomial series to obtain the middle expression, we arrive at a pure CFT 2-point function of two operators with mismatched dimensions which vanishes.

Using the definition of the helicity projected stress-energy tensor (2.9.7) and equation (2.1.3) applied to (4.1.72), in momentum space we have

$$\langle\!\langle T^{(s)}(\boldsymbol{p})T^{(s')}(-\boldsymbol{p})\rangle\!\rangle = \langle\!\langle \mathcal{T}^{(s)}(\boldsymbol{p})\mathcal{T}^{(s')}(-\boldsymbol{p})\rangle\!\rangle_0 = \frac{\pi^2}{48}\alpha_T p^3 \delta^{ss'}, \tag{4.1.85}$$

$$\langle\!\langle T^{(s)}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = 0. \tag{4.1.86}$$

4.1.9. $\langle \mathcal{O}\mathcal{O}T_{ij}\rangle$

Focusing next on the 3-point functions, in the case where all the x_{ij}^{λ} are comparable, at leading order in λ we find

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)T_{ij}^{\perp}(\boldsymbol{x}_3)\rangle = \Big[1+\frac{g}{\phi_1}L^{\lambda}\Big]^{-4}\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{T}_{ij}^{\perp}(\boldsymbol{x}_3)\rangle_0. \tag{4.1.87}$$

The power featuring in the prefactor is the same as that for the scalar 2-point function since in both cases we are resumming the binomial series resulting from contracting against two fixed scalar insertions. Through consideration of the various limiting cases using the OPEs (4.1.77) and (4.1.78) as well as the 2-point results (4.1.53), (4.1.83) and (4.1.84), we may further refine this to

$$\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)T_{ij}^{\perp}(\boldsymbol{x}_3)\rangle = \Big[1+\frac{g}{\phi_1}x_{12}^{\lambda}\Big]^{-4}\langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{T}_{ij}^{\perp}(\boldsymbol{x}_3)\rangle_0. \tag{4.1.88}$$

To convert this result to momentum space, it is useful to first re-express it as a sum of exact CFT 3-point functions with shifted dimensions. From the explicit form of the CFT correlator in (4.1.74), it follows that to leading order

$$x_{12}^{2n\lambda}\langle \mathcal{O}_{\Delta}(\boldsymbol{x}_1)\mathcal{O}_{\Delta}(\boldsymbol{x}_2)\mathcal{T}_{ij}(\boldsymbol{x}_3)\rangle_0 = \langle \mathcal{O}_{\Delta-n\lambda}(\boldsymbol{x}_1)\mathcal{O}_{\Delta-n\lambda}(\boldsymbol{x}_2)\mathcal{T}_{ij}(\boldsymbol{x}_3)\rangle_0, \tag{4.1.89}$$

where $\Delta = 3-\lambda$. The result (4.1.88) may then be cast in the desired form,

$$\langle \mathcal{O}_{\Delta}(\boldsymbol{x}_1)\mathcal{O}_{\Delta}(\boldsymbol{x}_2)\mathcal{T}_{ij}^{\perp}(\boldsymbol{x}_3)\rangle = \frac{1}{6}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)\Big(-\frac{g}{\phi_1}\Big)^n\times \\ \times\ \langle \mathcal{O}_{\Delta_n}(\boldsymbol{x}_1)\mathcal{O}_{\Delta_n}(\boldsymbol{x}_2)\mathcal{T}_{ij}^{\perp}(\boldsymbol{x}_3)\rangle_0, \tag{4.1.90}$$

where $\Delta_n = \Delta - n\lambda/2$. In the momentum space the 3-point function on the right hand side follows from (3.6.8),

$$\langle\!\langle \mathcal{O}_{\Delta_n}(\boldsymbol{p}_1)\mathcal{O}_{\Delta_n}(\boldsymbol{p}_2)\mathcal{T}^{\perp}_{ij}(\boldsymbol{p}_3)\rangle\!\rangle_0 = \Pi^{kl}_{ij}(\boldsymbol{p}_3)p_1^k p_1^l\, c(\lambda) I_{\frac{5}{2}\{\frac{3}{2}-\lambda(1+\frac{n}{2}),\frac{3}{2}-\lambda(1+\frac{n}{2}),\frac{3}{2}\}}, \tag{4.1.91}$$

where $c(\lambda)$ is expressed by (3.6.10) in terms of the 2-point function normalisation constant α.

Firstly, for the quasi-equilateral case in which all three momenta are comparable, we can expand the triple-K integral in λ and take the leading order contribution. Notice that for the integral $I_{\frac{5}{2}\{\frac{3}{2}\frac{3}{2}\frac{3}{2}\}}$ the condition (2.5.8) is never satisfied. This means that this integral is finite and therefore

$$\lim_{\lambda\to 0} I_{\frac{5}{2}\{\frac{3}{2}-\lambda(1+\frac{n}{2}),\frac{3}{2}-\lambda(1+\frac{n}{2}),\frac{3}{2}\}} = I_{\frac{5}{2}\{\frac{3}{2}\frac{3}{2}\frac{3}{2}\}}. \tag{4.1.92}$$

In other words

$$\lim_{\lambda\to 0}\langle\!\langle \mathcal{O}_{\Delta_n}(\boldsymbol{p}_1)\mathcal{O}_{\Delta_n}(\boldsymbol{p}_2)\mathcal{T}^{\perp}_{ij}(\boldsymbol{p}_3)\rangle\!\rangle_0 = \langle\!\langle \mathcal{O}_3(\boldsymbol{p}_1)\mathcal{O}_3(\boldsymbol{p}_2)\mathcal{T}^{\perp}_{ij}(\boldsymbol{p}_3)\rangle\!\rangle_0, \tag{4.1.93}$$

since the value of the constant $c(\lambda)$ is also continuous in λ and from (3.6.10) we read

$$c(0) = \frac{\pi^2}{12}\alpha. \tag{4.1.94}$$

In total, in the limit we just obtain (3.6.13).

The last element is the total dilatation dimension of (4.1.91), which we want to keep fixed. This is equal to $3-\lambda(2+n)$. We obtain the final result in the equilateral case,

$$\langle\!\langle \mathcal{O}_{\Delta}(\boldsymbol{p}_1)\mathcal{O}_{\Delta}(\boldsymbol{p}_2)\mathcal{T}^{\perp}_{ij}(\boldsymbol{p}_3)\rangle\!\rangle = \mathcal{A}_{eq}(p_1,p_2,p_3)\Pi_{ijkl}(\boldsymbol{p}_3)p_1^k p_1^l, \tag{4.1.95}$$

where

$$\mathcal{A}_{eq}(p_1,p_2,p_3) = \frac{\alpha\pi^2}{6}p_3^{-2\lambda}\left[1+\frac{g}{\phi_1}p_3^{-\lambda}\right]^{-4}\frac{(-a_{123}^3 + a_{123}b_{123} + c_{123})}{a_{123}^2} \tag{4.1.96}$$

and the symmetric polynomial in amplitudes of momenta are

$$a_{123} = p_1 + p_2 + p_3, \qquad a_{123} = p_1p_2 + p_1p_3 + p_2p_3, \qquad c_{123} = p_1p_2p_3. \tag{4.1.97}$$

In the helicity basis this may be re-expressed as

$$\langle\!\langle \mathcal{O}_{\Delta}(\boldsymbol{p}_1)\mathcal{O}_{\Delta}(\boldsymbol{p}_2)T^{(s_3)}(p_3)\rangle\!\rangle = \frac{J^2}{8\sqrt{2}p_3^2}\mathcal{A}_{eq}(p_1,p_2,p_3), \tag{4.1.98}$$

where J^2 is defined in (2.6.18).

The remaining cases are then the two squeezed limits $p_1 \to 0$ and $p_3 \to 0$. In this case, however

$$\langle\!\langle \mathcal{O}(0)\mathcal{O}(\boldsymbol{p})T^{(s)}(-\boldsymbol{p})\rangle\!\rangle = -\frac{\partial}{\partial g}\langle\!\langle \mathcal{O}(\boldsymbol{p})T^{(s)}(-\boldsymbol{p})\rangle\!\rangle = 0, \tag{4.1.99}$$

so the 3-point function vanishes in the local limit.

4.1.10. λ dependence of 3-point functions

Two-point functions for the stress-energy tensor $\mathcal{T}_{ij}$ in a pure CFT are fixed by (2.1.16)

$$\langle\!\langle \mathcal{T}^{(s)}(\boldsymbol{x})\mathcal{O}(-\boldsymbol{x})\rangle\!\rangle_0 = 0, \qquad \langle\!\langle \mathcal{T}^{(s_1)}(\boldsymbol{x})\mathcal{T}^{(s_2)}(-\boldsymbol{x})\rangle\!\rangle_0 = c_T p^3 \delta^{s_1 s_2} \propto 1. \tag{4.1.100}$$

The three-point function $\langle T\mathcal{O}\mathcal{O}\rangle$ is completely fixed in terms of two-point functions and we have found in the previous section

$$\langle\!\langle \mathcal{T}^{(s)}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle_0 \propto 1. \tag{4.1.101}$$

Moreover we have

$$\langle\!\langle \mathcal{T}^{(s)}(\boldsymbol{p}_1)\mathcal{T}^{(s_2)}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle_0 \propto \lambda^\gamma, \tag{4.1.102}$$
$$\langle\!\langle \mathcal{T}^{(s_1)}(\boldsymbol{p}_1)\mathcal{T}^{(s_2)}(\boldsymbol{p}_2)\mathcal{T}^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle_0 \propto 1, \tag{4.1.103}$$

for some parameter γ. We will show in this section that the crossing symmetry of the CFT requires that $\gamma \geqslant 1$ (or $\gamma = -1$).

Write schematically the OPEs as follows

$$\mathcal{O}(\boldsymbol{x})\mathcal{O}(0) \sim \frac{c^{\mathcal{O}}_{\mathcal{O}\mathcal{O}}}{x^{3-\lambda}}\mathcal{O}(0) + \frac{(c^{T}_{\mathcal{O}\mathcal{O}})^{ij}}{x^{3-2\lambda}}\mathcal{T}_{ij}(0), \tag{4.1.104}$$
$$\mathcal{O}(\boldsymbol{x})\mathcal{T}_{ij}(0) \sim \frac{(c^{\mathcal{O}}_{\mathcal{O}T})_{ij}}{x^{3}}\mathcal{O}(0) + \frac{(c^{T}_{\mathcal{O}T})^{kl}_{ij}}{x^{3-\lambda}}\mathcal{T}_{kl}(0), \tag{4.1.105}$$
$$\mathcal{T}_{ij}(\boldsymbol{x})\mathcal{T}_{kl}(0) \sim \frac{(c^{\mathcal{O}}_{TT})_{ijkl}}{x^{3+\lambda}}\mathcal{O}(0) + \frac{(c^{T}_{TT})^{mn}_{ijkl}}{x^{3}}\mathcal{T}_{mn}(0). \tag{4.1.106}$$

Since we are interested only in the singularities in λ, we omit the tensor structure on OPE constants as it does not matter in our considerations. Moreover, we omit tensor indices on $\mathcal{T}_{ij}$ writing just $\mathcal{T}$. Similarly as in the previous chapter, we can analyse various limits in the three-point functions in order to extract the behaviour of the OPE coefficients.

$$\begin{aligned}
\frac{1}{\lambda} &\propto \langle \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_1\to\boldsymbol{x}_2}{\longrightarrow} \frac{c^{\mathcal{O}}_{\mathcal{O}\mathcal{O}}}{x_1^{3-\lambda}}\langle \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0, \\
1 &\propto \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_1\to\boldsymbol{x}_2}{\longrightarrow} \frac{c^{\mathcal{O}}_{\mathcal{O}T}}{\boldsymbol{x}_1^{3}}\langle_0 \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0, \\
1 &\propto \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_2\to\boldsymbol{x}_3}{\longrightarrow} \frac{c^{T}_{\mathcal{O}\mathcal{O}}}{x_2^{3-2\lambda}}\langle \mathcal{T}(\boldsymbol{x}_2)\mathcal{T}(\boldsymbol{x}_3)\rangle_0, \\
\lambda^\gamma &\propto \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{T}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_1\to\boldsymbol{x}_2}{\longrightarrow} \frac{c^{\mathcal{O}}_{TT}}{x_1^{3+\lambda}}\langle \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0, \\
\lambda^\gamma &\propto \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{T}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_2\to\boldsymbol{x}_3}{\longrightarrow} \frac{c^{T}_{\mathcal{O}T}}{x_2^{3-\lambda}}\langle \mathcal{T}(\boldsymbol{x}_2)\mathcal{T}(\boldsymbol{x}_3)\rangle_0, \\
1 &\propto \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{T}(\boldsymbol{x}_2)\mathcal{T}(\boldsymbol{x}_3)\rangle_0 \overset{\boldsymbol{x}_1\to\boldsymbol{x}_2}{\longrightarrow} \frac{c^{T}_{TT}}{x_1^{3}}\langle \mathcal{T}(\boldsymbol{x}_2)\mathcal{T}(\boldsymbol{x}_3)\rangle_0
\end{aligned} \tag{4.1.107}$$

for some $\gamma \in \mathbb{R}$. In particular by definition (4.1.75) α_{TT} is proportional to $c_{TT}^{\mathcal{O}}$. From these we read

$$\begin{aligned} c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}} &\propto 1, & c_{\mathcal{O}\mathcal{O}}^{T} &\propto \lambda, \\ c_{\mathcal{O}T}^{\mathcal{O}} &\propto 1, & c_{\mathcal{O}T}^{T} &\propto \lambda^{\gamma+1}, \\ \alpha_{TT} \sim c_{TT}^{\mathcal{O}} &\propto \lambda^{\gamma+1}, & c_{TT}^{T} &\propto 1. \end{aligned}$$

In order to find a condition on γ consider the four-point function $\langle \mathcal{T}\mathcal{T}\mathcal{O}\mathcal{O}\rangle_0$. Consider two possible OPE limits:

$$\begin{aligned} \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{T}(\boldsymbol{x}_2)\, \mathcal{O}(\boldsymbol{y}_1)\mathcal{O}(\boldsymbol{y}_2)\rangle_0 &\sim \frac{c_{TT}^{\mathcal{O}} c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{x_1^{3+\lambda} y_1^{3-\lambda}} \langle \mathcal{O}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{y}_2)\rangle_0 + \frac{c_{TT}^{T} c_{\mathcal{O}\mathcal{O}}^{T}}{x_1^3 y_1^{3-2\lambda}} \langle \mathcal{T}(\boldsymbol{x}_2)\mathcal{T}(\boldsymbol{y}_2)\rangle_0 \\ &\propto \lambda^{\gamma-1} + 1, \end{aligned} \tag{4.1.108}$$

$$\begin{aligned} \langle \mathcal{T}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{y}_1)\, \mathcal{T}(\boldsymbol{x}_2)\mathcal{O}(\boldsymbol{y}_2)\rangle_0 &\sim \frac{(c_{\mathcal{O}T}^{\mathcal{O}})^2}{x_1^3 x_2^3} \langle \mathcal{O}(\boldsymbol{y}_1)\mathcal{O}(\boldsymbol{y}_2)\rangle_0 + \frac{(c_{\mathcal{O}T}^{T})^2}{x_1^{3-\lambda} x_2^{3-\lambda}} \langle \mathcal{T}(\boldsymbol{y}_1)\mathcal{T}(\boldsymbol{y}_2)\rangle_0 \\ &\propto 1 + \lambda^{2\gamma}. \end{aligned} \tag{4.1.109}$$

This means that the conformal bootstrap requires $\gamma \geqslant 1$ or $\gamma = -1$. We will not consider the case $\gamma = -1$ here as it is not physical from the point of view of the AdS/CFT correspondence as we will see in section 7.1.5.

Now we can prove a general statement assuming concerning the divergence of any correlation functions in the CFT, provided $\gamma \geqslant 1$ in (4.1.102). We expect the divergence of the form

$$\langle \mathcal{T}^{(s_1)}(\boldsymbol{x}_1)\dots\mathcal{T}^{(s_m)}(\boldsymbol{x}_m)\mathcal{O}(\boldsymbol{y}_1)\dots\mathcal{O}(\boldsymbol{y}_n)\rangle_0 \propto \begin{cases} \frac{1}{\lambda^{n-2}} & \text{for } n \geqslant 2, \\ 1 & \text{otherwise} \end{cases} \tag{4.1.110}$$

depend on the number of insertions of the scalar operator only.

The statement is true for two and three-point functions. Assume that the statement is true for any m, n such that $m+n \leqslant k-1$. We will show that the statement is true for $m+n=k$. For simplicity denote

$$G_{m,n} = \langle \mathcal{T}(\boldsymbol{x}_1)\dots\mathcal{T}(\boldsymbol{x}_m)\mathcal{O}(\boldsymbol{y}_1)\dots\mathcal{O}(\boldsymbol{y}_n)\rangle_0. \tag{4.1.111}$$

Consider various possible expansions

- Assume $m \geqslant 2$ and expand $\mathcal{T}\mathcal{T}$. We have

$$\begin{aligned} G_{m,n} &\sim \langle \mathcal{T}\dots\mathcal{T}(\boldsymbol{x}_{m-2})\mathcal{O}\dots\mathcal{O}(\boldsymbol{y}_n)\left(\frac{c_{TT}^{\mathcal{O}}}{x_m^{3+\lambda}}\mathcal{O}(\boldsymbol{x}_{m-1}) + \frac{c_{TT}^{T}}{x_m^3}\mathcal{T}(\boldsymbol{x}_{m-1})\right)\rangle_0 \\ &= \frac{c_{TT}^{\mathcal{O}}}{x_m^{3+\lambda}} G_{m-2,n+1} + \frac{c_{TT}^{T}}{x_m^3} G_{m-1,n} \\ &\propto \lambda^{\gamma}\cdot\frac{1}{\lambda^{\max(n-1,0)}} + \frac{1}{\lambda^{\max(n-2,0)}} \propto \frac{1}{\lambda^{\max(n-2,0)}}, \end{aligned} \tag{4.1.112}$$

where we used the fact that $\gamma \geqslant 1$.

- Similarly, assume $n \geqslant 2$ and expand $\mathcal{O}\mathcal{O}$. We have

$$G_{m,n} \sim \langle \mathcal{T} \ldots \mathcal{T}(\boldsymbol{x}_m)\mathcal{O} \ldots \mathcal{O}(\boldsymbol{y}_{n-2}) \left(\frac{c_{\mathcal{O}\mathcal{O}}^{\mathcal{O}}}{y_n^{3-\lambda}} \mathcal{O}(\boldsymbol{y}_{n-1}) + \frac{c_{\mathcal{O}\mathcal{O}}^{T}}{y_n^{3-2\lambda}} \mathcal{T}(\boldsymbol{y}_{n-1}) \right) \rangle_0$$
$$\propto \frac{1}{\lambda} \cdot \frac{1}{\lambda^{\max(n-3,0)}} + \frac{\lambda}{\lambda} \cdot \frac{1}{\lambda^{\max(n-4,0)}} \propto \frac{1}{\lambda^{n-2}}. \tag{4.1.113}$$

- Take $m \geqslant 1$ and $n \geqslant 1$ and expand $\mathcal{T}\mathcal{O}$. We have

$$G_{m,n} \sim \langle \mathcal{T} \ldots \mathcal{T}(\boldsymbol{x}_{m-1})\mathcal{O} \ldots \mathcal{O}(\boldsymbol{y}_{n-1}) \left(\frac{c_{\mathcal{O}T}^{\mathcal{O}}}{x_m^{3}} \mathcal{O}(\boldsymbol{y}_{n}) + \frac{c_{\mathcal{O}T}^{T}}{x_m^{3-\lambda}} \mathcal{T}(\boldsymbol{y}_{n}) \right) \rangle_0$$
$$\propto \frac{1}{\lambda^{\max(n-2,0)}} + \lambda^{\gamma} \cdot \frac{1}{\lambda^{\max(n-3,0)}} \propto \frac{1}{\lambda^{\max(n-2,0)}}. \tag{4.1.114}$$

4.1.11. $\langle \mathcal{O} T_{ij} T_{kl} \rangle$ and $\langle T_{ij} T_{kl} T_{mn} \rangle$

Next consider the 3-point function with two stress tensor insertions and one scalar in the perturbed theory. Following arguments analogous to those we have used above, we obtain

$$\langle\!\langle T_{ij}^{\perp}(\boldsymbol{x}_1) T_{kl}^{\perp}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{x}_3) \rangle\!\rangle = \left[1 + \frac{g}{\phi_1} x_{12}^{\lambda} \right]^{-2} \langle\!\langle \mathcal{T}_{ij}^{\perp}(\boldsymbol{x}_1) \mathcal{T}_{kl}^{\perp}(\boldsymbol{x}_2) \mathcal{O}(\boldsymbol{x}_3) \rangle\!\rangle_0. \tag{4.1.115}$$

In momentum space the correlation function appearing on the right hand side is determined uniquely up to one multiplicative constant α_{TT} defined in (4.1.75) and in previous section we found that $\alpha_{TT} = O(\lambda)$. Now we will show that the triple-K integral defining this 3-point function does not have a singularity when $\lambda \to 0$, hence in the leading order in λ,

$$\langle\!\langle T^{(s_1)}(\boldsymbol{p}_1) T^{(s_2)}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle = 0 \tag{4.1.116}$$

and this correlation function will not contribute to our analysis of the holographic cosmology. However, it will be useful to find its helicity structure anyway.

The 3-point function is given by equations (3.9.8) . One can inspect all triple-K integrals building the solution and find that the only integral diverging in the limit $\lambda \to 0$ is the $J_{0\{000\}}$ integral. This divergence is linear, as can be found by explicit calculations. However, the coefficient α_3 that multiplies this integral vanishes linearly in the limit and hence the whole 3-point function is finite. This means that, in order to find the leading contribution to the 3-point function, we may consider the $\lambda \to 0$ limit. For $\Delta = 3$ the result is given by (3.9.23) - (3.9.25). Projecting onto the helicity basis one finds

$$\langle\!\langle \mathcal{T}^{(s_1)}(\boldsymbol{p}_1) \mathcal{T}^{(s_2)}(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle\!\rangle_0 = \alpha_{TT}' \frac{p_1 p_2 (p_1 + p_2 + 4p_3)(p_1 + p_2 - p_3)^2}{8 a_{123}^2} \delta^{s_1 s_2}, \tag{4.1.117}$$

where α'_{TT} is an undetermined constant proportional to α_{TT}. In particular,

$$\langle\!\langle \mathcal{T}^{(+)}(\boldsymbol{p}_1)\mathcal{T}^{(-)}(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle\!\rangle_0 = 0. \tag{4.1.118}$$

The last correlation function is

$$\langle T^{(s_1)}T^{(s_2)}T^{(s_3)}\rangle = \langle \mathcal{T}^{(s_1)}\mathcal{T}^{(s_2)}\mathcal{T}^{(s_3)}\rangle_0 \tag{4.1.119}$$

in the leading order in λ, since the perturbing operator does not alter the transverse-traceless part of the stress-energy tensor. The exact expression is given by (2.9.19) and (2.9.20).

4.2. One-loop Feynman integrals

In this section we would like to present a novel method for the evaluation of the 1-loop Feynman diagrams with 3 vertices in odd dimensions. The method is based on the triple-K integrals and does not require *any* reduction scheme.

4.2.1. Momentum space integrals

Consider the momentum space integral

$$\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{k^{\mu_1}\ldots k^{\mu_r}}{|\boldsymbol{k}|^{2\delta_3}|\boldsymbol{k}-\boldsymbol{p}_1|^{2\delta_2}|\boldsymbol{k}+\boldsymbol{p}_2|^{2\delta_1}}. \tag{4.2.1}$$

Such an integral appears in calculations of the 1-loop Feynman diagrams. Denote by $S^{\nu_1\cdots\nu_{2m}}$ a completely symmetric tensor built from metrics only, with each coefficient equal to 1, *e.g.*,

$$S^{\nu_1\nu_2\nu_3\nu_4} = \delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4} + \delta^{\nu_1\nu_3}\delta^{\nu_2\nu_4} + \delta^{\nu_1\nu_4}\delta^{\nu_2\nu_3}. \tag{4.2.2}$$

We will also denote $S^{\nu_1\cdots\nu_{2m+1}} = 0$ for an odd number of indices.

Using Schwinger parameters

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}s\, s^{\alpha-1}e^{-sA}\mathrm{d}s, \qquad \alpha > 0. \tag{4.2.3}$$

we can rewrite (4.2.1) as

$$\begin{aligned}
&\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{k^{\mu_1}\ldots k^{\mu_r}}{\boldsymbol{k}^{2\delta_3}|\boldsymbol{k}-\boldsymbol{p}_1|^{2\delta_2}|\boldsymbol{k}+\boldsymbol{p}_2|^{2\delta_1}} \\
&= \Gamma^{-3}\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} k^{\mu_1}\ldots k^{\mu_r}\int_{\mathbb{R}^3_+}\mathrm{d}\vec{s}\, s_1^{\delta_1-1}s_2^{\delta_2-1}s_3^{\delta_3-1}\times \\
&\qquad \times \exp\left[-(s_3\boldsymbol{k}^2 + s_2|\boldsymbol{k}-\boldsymbol{p}_1|^2 + s_1|\boldsymbol{k}+\boldsymbol{p}_2|^2)\right],
\end{aligned} \tag{4.2.4}$$

where we use the following abbreviations

$$\Gamma^3 = \Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3), \qquad \mathrm{d}\vec{s} = \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3. \tag{4.2.5}$$

Denoting $s_t = s_1 + s_2 + s_3$, we rewrite the expression in the exponent as

$$s_3\boldsymbol{k}^2 + s_2|\boldsymbol{k} - \boldsymbol{p}_1|^2 + s_1|\boldsymbol{k} + \boldsymbol{p}_2|^2 = s_t l^2 + \Delta, \tag{4.2.6}$$

where

$$\boldsymbol{l} = \boldsymbol{k} + \frac{s_1\boldsymbol{p}_2 - s_2\boldsymbol{p}_1}{s_t}, \qquad \Delta = \frac{s_1 s_2 p_3^2 + s_1 s_3 p_2^2 + s_2 s_3 p_1^2}{s_t}. \tag{4.2.7}$$

We can now re-express the integral (4.2.1) as

$$\Gamma^{-3}\int_{\mathbb{R}^3_+} \mathrm{d}\vec{s}\, s_1^{\delta_1-1}s_2^{\delta_2-1}s_3^{\delta_3-1}e^{-\Delta}\int\frac{\mathrm{d}^d\boldsymbol{l}}{(2\pi)^d}e^{-s_t l^2}\prod_{j=1}^{r}\left(l^{\mu_j} + \frac{s_2 p_1^{\mu_j} - s_1 p_2^{\mu_j}}{s_t}\right). \tag{4.2.8}$$

This expression can be expanded and split up into a sum of integrals. The integral over $\boldsymbol{l}$ gives some moment of a Gaussian random variable. For any a such that $\operatorname{Re} a > 0$ we can find

$$\int\frac{\mathrm{d}^d\boldsymbol{l}}{(2\pi)^d}l^{2m}e^{-al^2} = \frac{\Gamma\left(\frac{d}{2}+m\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\cdot\frac{1}{a^{\frac{d}{2}+m}}, \tag{4.2.9}$$

$$\int\frac{\mathrm{d}^d\boldsymbol{l}}{(2\pi)^d}l^{\mu_1}\dots l^{\mu_{2m}}e^{-al^2} = \frac{S^{\mu_1\cdots\mu_{2m}}}{(4\pi)^{\frac{d}{2}}2^m a^{\frac{d}{2}+m}}, \tag{4.2.10}$$

and integrals with an odd number of l vanish. The calculations of the integral (4.2.1) therefore boil down to the calculation of several integrals of the form

$$i_{d,m,\{\delta_j\}} = \frac{1}{(4\pi)^{\frac{d}{2}}2^m\Gamma^3}\int_{\mathbb{R}^3_+}\mathrm{d}\vec{s}\, s_t^{-\frac{d}{2}-m}s_1^{\delta_1-1}s_2^{\delta_2-1}s_3^{\delta_3-1}e^{-\Delta}. \tag{4.2.11}$$

This expression gives the coefficient of the completely symmetric tensor $S^{\mu_1\cdots\mu_{2m}}$ when we evaluate (4.2.8). In fact, the coefficient of all the tensors in (4.2.8) can similarly be expressed in terms of $i_{d,m,\{\delta_j\}}$ for some values of m and δ_j. (In this case, however, the δ_j parameters are no longer equal to those in (4.2.1), since each momentum in (4.2.8) is accompanied by a Schwinger parameter.)

Let us now express the integral (4.2.11) in terms of the triple-K integral (2.5.1). Defining $\delta_t = \delta_1 + \delta_2 + \delta_3$, we make the following substitution in (4.2.11)

$$s_j = \frac{v_1v_2 + v_1v_3 + v_2v_3}{2v_j} = \frac{V}{2v_j}, \qquad j = 1,2,3, \tag{4.2.12}$$

giving

$$i_{d,m,\{\delta_j\}} = \frac{2^{\frac{d}{2}-\delta_t}}{(4\pi)^{\frac{d}{2}}\Gamma^3}\int_{\mathbb{R}^3_+}\mathrm{d}\vec{v}\, V^{\delta_t-d-2m}\prod_{j=1}^{3}v_j^{\frac{d}{2}+m-\delta_j-1}e^{-\frac{v_j p_j^2}{2}}. \tag{4.2.13}$$

Observing that

$$V = v_1 v_2 v_3 \left(v_1^{-1} + v_2^{-1} + v_3^{-1}\right) \tag{4.2.14}$$

and introducing a new Schwinger parameter t to exponentiate the term in brackets, we find

$$\begin{aligned}
i_{d,m,\{\delta_j\}} &= \frac{2^{\frac{d}{2}-\delta_t}}{(4\pi)^{\frac{d}{2}} \Gamma^3 \Gamma(d+2m-\delta_t)} \int_0^\infty \mathrm{d}t\, t^{d+2m-\delta_t-1} \times \\
&\quad \times \int_{\mathbb{R}^3_+} \mathrm{d}\vec{v} \prod_{j=1}^{3} v_j^{-\frac{d}{2}-m+\delta_t-\delta_j-1} e^{-\frac{v_j p_j^2}{2} - \frac{t}{v_j}} \\
&= \frac{2^{-d-3m+\delta_t}}{(4\pi)^{\frac{d}{2}} \Gamma^3 \Gamma(d+2m-\delta_t)} \int_0^\infty \mathrm{d}t\, t^{d+2m-\delta_t-1} \times \\
&\quad \times \int_{\mathbb{R}^3_+} \mathrm{d}\vec{u} \prod_{j=1}^{3} p_j^{d+2m-2\delta_t+2\delta_j} u_j^{-\frac{d}{2}-m+\delta_t-\delta_j-1} e^{-u_j - \frac{t p_j^2}{2u_j}}.
\end{aligned} \tag{4.2.15}$$

Using the standard formula [59]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-u-\frac{z^2}{4u}} u^{-\nu-1} \mathrm{d}u, \qquad |\arg z| < \frac{\pi}{4} \tag{4.2.16}$$

we now obtain our final result

$$\begin{aligned}
i_{d,m,\{\delta_j\}} &= \frac{2^{-\frac{d}{2}-2m+4}}{(4\pi)^{\frac{d}{2}} \Gamma^3 \Gamma(d+2m-\delta_t)} \times \\
&\qquad \times \int_0^\infty \mathrm{d}t (\sqrt{2t})^{\frac{d}{2}+m-2} \prod_{j=1}^{3} p_j^{\frac{d}{2}+m-\delta_t+\delta_j} K_{\frac{d}{2}+m-\delta_t+\delta_j}(\sqrt{2t} p_j) \\
&= \frac{2^{-\frac{d}{2}-2m+4}}{(4\pi)^{\frac{d}{2}} \Gamma^3 \Gamma(d+2m-\delta_t)} I_{\frac{d}{2}+m-1\{\frac{d}{2}+m-\delta_t+\delta_j\}},
\end{aligned} \tag{4.2.17}$$

where $\Gamma^3 = \Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)$ and I stands for the triple-K integral (2.5.1).

4.2.2. Triple-K integrals via Feynman parametrisation

Instead of Schwinger parameters, one can use the more familiar Feynman parametrisation in order to evaluate (4.2.1). Using standard results [74], we can write (4.2.1) as

$$\frac{\Gamma(\delta_t)}{\Gamma^3} \int_{[0,1]^3} \mathrm{d}X \prod_{j=1}^{3} x_j^{\delta_j-1} \int \frac{\mathrm{d}^d \boldsymbol{l}}{(2\pi)^d} \frac{1}{(l^2+D)^{\delta_t}} \prod_{j=1}^{r} \left(l^{\mu_j} + x_2 p_1^{\mu_j} - x_1 p_2^{\mu_j}\right), \tag{4.2.18}$$

where

$$\mathrm{d}X = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \delta(x_1 + x_2 + x_3 - 1), \tag{4.2.19}$$

$$\boldsymbol{l} = \boldsymbol{k} - x_2 \boldsymbol{p}_1 + x_1 \boldsymbol{p}_2, \tag{4.2.20}$$

$$D = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2. \tag{4.2.21}$$

Looking at the coefficient of the tensor $S^{\mu_1 \cdots \mu_{2m}}$ defined in the previous section, we obtain

$$i_{d,m,\{\delta_j\}} = \frac{\Gamma\left(\delta_t - m - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} 2^m \Gamma^3} \int_{[0,1]^3} \mathrm{d}X x_1^{\delta_1 - 1} x_2^{\delta_2 - 1} x_3^{\delta_3 - 1} D^{\frac{d}{2} + m - \delta_t}. \tag{4.2.22}$$

Comparing with (4.2.17), we then find

$$\begin{aligned} I_{\alpha\{\beta_1\beta_2\beta_3\}} &= \int_0^\infty \mathrm{d}x \, x^\alpha \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x) \\ &= 2^{\alpha-3} \Gamma\left(\frac{\alpha - \beta_t + 1}{2}\right) \Gamma\left(\frac{\alpha + \beta_t + 1}{2}\right) \times \\ &\quad \times \int_{[0,1]^3} \mathrm{d}X \, D^{\frac{1}{2}(\beta_t - \alpha - 1)} \prod_{j=1}^3 x_j^{\frac{1}{2}(\alpha - 1 - \beta_t) + \beta_j}, \end{aligned} \tag{4.2.23}$$

where $\beta_t = \beta_1 + \beta_2 + \beta_3$ and I is the triple-K integral (2.5.1). The inverse is

$$\int_{[0,1]^3} \mathrm{d}X \, D^a x_1^{b_1} x_2^{b_2} x_3^{b_3} = \frac{1}{2^{a+b_t-1}\Gamma(-a)\Gamma(2a + b_t + 3)} I_{b_t + a + 2\{b_j + a + 1\}}, \tag{4.2.24}$$

where $b_t = b_1 + b_2 + b_3$.

4.2.3. Feynman integrals

Evaluation of the 1-loop Feynman integrals with 3 vertices in odd dimensions is now absolutely straightforward. As an example, consider the integral we will encounter in following sections,

$$I^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{k^{\mu_1}(\boldsymbol{k} - \boldsymbol{p}_1)^{\nu_1}(\boldsymbol{k} + \boldsymbol{p}_2)^{\mu_2} k^{\nu_2} (\boldsymbol{k} - \boldsymbol{p}_1)^{\mu_3} (\boldsymbol{k} + \boldsymbol{p}_2)^{\nu_3}}{k^2 |\boldsymbol{k} - \boldsymbol{p}_1|^2 |\boldsymbol{k} + \boldsymbol{p}_2|^2}. \tag{4.2.25}$$

This expressions depends on the magnitudes of momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ only. Let us extract from this integral the coefficients of the tensors

$$t_1 = p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2} p_1^{\mu_3} p_1^{\nu_3}, \qquad t_2 = \delta^{\mu_1\nu_2}\delta^{\mu_2\nu_3}\delta^{\mu_3\nu_1}, \tag{4.2.26}$$

where the convention (2.3.4) is assumed.

Let us start with the tensor t_1. By comparing the integral (4.2.25) to (4.2.1) we see that the structure $p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_3^{\nu_2}$ appears due to four $\boldsymbol{l}$'s only, when the substitution (4.2.7) is applied and

$$\boldsymbol{k} = \boldsymbol{l} + \frac{s_2 \boldsymbol{p}_1 - s_1 \boldsymbol{p}_2}{s_t} = \boldsymbol{l} + \frac{-(s_1 + s_2)\boldsymbol{p}_2 - s_2 \boldsymbol{p}_3}{s_t}. \tag{4.2.27}$$

For the last part in (4.2.25) we have, for example,

$$\boldsymbol{k} - \boldsymbol{p}_1 = \boldsymbol{l} + \frac{-(s_1 + s_3)\boldsymbol{p}_1 - s_1 \boldsymbol{p}_2}{s_t} = \boldsymbol{l} + \frac{-s_3 \boldsymbol{p}_1 + s_1 \boldsymbol{p}_3}{s_t}. \tag{4.2.28}$$

Combining these results we find that the coefficient of the tensor t_1 can be written in terms of the following integral,

$$C_{d,m,\{\delta_j\},\{n_j\}} = \frac{(-1)^{n_t}}{(4\pi)^{\frac{d}{2}} 2^m \Gamma^3} \int_{\mathbb{R}^3_+} \mathrm{d}\vec{s}\, s_t^{-\frac{d}{2}-m-n_t} s_1^{\delta_1+n_1-1} s_2^{\delta_2+n_2-1} s_3^{\delta_3+n_3-1} e^{-\Delta} \tag{4.2.29}$$

where

$$d = 3, \qquad m = 0, \qquad \delta_j = 1, \qquad n_j = 2 \tag{4.2.30}$$

for $j = 1, 2, 3$. The value of (4.2.29) can be expressed in terms of the integrals (4.2.17),

$$C_{d,m,\{\delta_j\},\{n_j\}} = \frac{(-1)^{n_t} 2^{-\frac{d}{2}-2m-n_t+4}}{(4\pi)^{\frac{d}{2}} \Gamma^3 \Gamma(d + 2m - \delta_t + n_t)} I_{\frac{d}{2}+m+n_t-1\{\frac{d}{2}+m-\delta_t+\delta_j+n_j\}} \tag{4.2.31}$$

where $\Gamma^3 = \prod_{j=1}^3 \Gamma(\delta_j)$.

Using the fact that

$$K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{3}{2}}}(1 + x) \tag{4.2.32}$$

we find straightforwardly the coefficient of the tensor t_1 in (4.2.25) to be

$$\begin{aligned} C_{3,0,\{1,1,1\},\{2,2,2\}} &= \frac{1}{30\,720} \int_0^\infty \mathrm{d}x e^{-(p_1+p_2+p_3)x} x^2 (1 + p_1 x)(1 + p_2 x)(1 + p_3 x) \\ &= \frac{1}{3\,840\, a_{123}^6} \left[a_{123}^3 + 3 a_{123} b_{123} + 15 c_{123} \right], \end{aligned} \tag{4.2.33}$$

where

$$a_{123} = p_1 + p_2 + p_3, \qquad b_{123} = p_1 p_2 + p_1 p_3 + p_2 p_3, \qquad c_{123} = p_1 p_2 p_3 \tag{4.2.34}$$

are the standard symmetric polynomials in amplitudes of momenta.

The same method can be applied for the tensor t_2. Notice that the only source of such a structure is a completely symmetric tensor $S^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}$. Therefore, we must find the integral following from six $\boldsymbol{l}$'s when the substitution (4.2.7) is applied. The result is given by the integral $i_{3,3,\{111\}} = C_{3,3,\{111\},\{000\}}$. Using the fact that

$$K_{\frac{5}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^{\frac{5}{2}}}(x^2 + 3x + 3) \tag{4.2.35}$$

the integral reads

$$\begin{aligned} C_{3,3,\{1,1,1\},\{0,0,0\}} &= \frac{1}{30\,720} \left(\frac{2}{\pi}\right)^{\frac{3}{2}} I_{\frac{7}{2}\{\frac{5}{2}\frac{5}{2}\frac{5}{2}\}} \\ &= \frac{1}{30\,720} \int_0^\infty \mathrm{d}x e^{-(p_1+p_2+p_3)x} x^{-4} \prod_{j=1}^{3} (3 + 3p_j x + p_j^2 x^2). \end{aligned} \tag{4.2.36}$$

This integral diverges and requires a regularisation. We use our standard regularisation (2.5.7), which corresponds to the shift in the α-parameter of the triple-K integral by ϵ. This leads to the shift in the power of the integration variable: $x^{-4+\epsilon}$. The integral can be calculated with the non-vanishing ϵ and expressed in terms of several gamma functions. Then the $\epsilon \to 0$ limit can be taken. In this way one finds

$$\begin{aligned} C_{3,3,\{1,1,1\},\{0,0,0\}} = \frac{1}{30\,720\, a_{123}^3} \big[& 3a_{123}^6 - 9a_{123}^4 b_{123} + 3a_{123}^2 b_{123}^2 \\ & + 3a_{123}^3 c_{123} + 3a_{123} b_{123} c_{123} + 2c_{123}^2 \big]. \end{aligned} \tag{4.2.37}$$

4.3. Free theories

In this section we will calculate 2- and 3-point functions of the stress-energy tensor in free theories in dimension $d = 3$. We will find these results useful in section 7.1. All results for 3-point functions can be given in terms of the ten form factors discussed in section 2.A.1. However, since we will be interested in applications to cosmology, we will present the results in the helicity formalism defined in section 2.9.

4.3.1. Definitions

We consider a collection of the following fields,

1. $\mathcal{N}_\phi$ minimal scalars ϕ,
2. $\mathcal{N}_\chi$ conformal scalars χ,
3. $\mathcal{N}_\psi$ fermions ψ,
4. $\mathcal{N}_A$ gauge fields A_i.

All fields transform in the adjoint representation of the $SU(N)$ group. The dimension of the representation is

$$d_A = N^2 - 1. \tag{4.3.1}$$

The action is

$$S = \frac{1}{g_{\rm YM}^2}\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{tr}\left[\frac{1}{4}F^I_{ij}F^I_{ij} + \frac{1}{2}\partial_i\phi^J\partial^i\phi^J + \frac{1}{2}\partial_\chi\phi^K\partial^i\chi^K + \bar{\psi}^L\gamma^i\partial_i\psi^L \right.$$
$$\left. + \text{ neglected interactions}\right]. \tag{4.3.2}$$

where for all fields, $\Phi = \Phi^a T^a$, and

$$\mathrm{tr}(T^aT^b) = \mathrm{tr}(t^a_A t^b_A) = \delta^{ab}, \tag{4.3.3}$$

where t^a_A are the representation matrices for the adjoint representation. The gamma matrices satisfy $\{\gamma_i,\gamma_j\} = -2\delta_{ij}$. Conformal scalars and fermions are conformal, while minimal scalars and gauge fields in $d=3$ are not.

The propagators following from the action (4.3.2) are

$$\langle\phi^I(\boldsymbol{p})\phi^J(\boldsymbol{p}')\rangle = (2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')g^2_{\rm YM}\frac{\delta^{IJ}}{p^2}, \tag{4.3.4}$$

$$\langle\psi^I(\boldsymbol{p})\bar{\psi}^J(\boldsymbol{p}')\rangle = (2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')g^2_{\rm YM}\frac{\delta^{IJ}\mathrm{i}\gamma^i p_i}{p^2}, \tag{4.3.5}$$

$$\langle A^I_i(\boldsymbol{p})A^J_j(\boldsymbol{p}')\rangle = (2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')g^2_{\rm YM}\frac{\delta^{IJ}}{p^2}\left(\delta_{ij} - (1-\xi)\frac{p_ip_j}{p^2}\right), \tag{4.3.6}$$

where for the gauge field we use the R_ξ gauge. The propagator for the conformal scalars is identical to the propagator for the minimal scalars. In general, one needs to consider the ghost fields as well, but since we work in the zeroth order free theory, these can be neglected. We present the analysis for the ghost fields in section 4.3.7.

The stress-energy tensor (1.1.55) is

$$T_{ij} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{ij}}\bigg|_{g_{ij}=\delta_{ij}} = T^A_{ij} + T^\phi_{ij} + T^\psi_{ij} + T^\chi_{ij}, \tag{4.3.7}$$

where the contributions from the various fields (suppressing the interactions, as well as the ghost and gauge-fixing terms which we discuss in section 4.3.7) in (4.3.2) are

$$T^A_{ij} = \frac{1}{g^2_{\rm YM}}\mathrm{tr}\big[F^I_{ik}F^{\ \ k}{}_{I j} - \delta_{ij}\frac{1}{4}F^I_{kl}F^{klI}\big], \tag{4.3.8}$$

$$T^\phi_{ij} = \frac{1}{g^2_{\rm YM}}\mathrm{tr}\big[\partial_i\phi^J\partial_j\phi^J - \delta_{ij}\frac{1}{2}\partial_k\phi^J\partial^k\phi_J\big], \tag{4.3.9}$$

$$T^\chi_{ij} = \frac{1}{g^2_{\rm YM}}\mathrm{tr}\big[\partial_i\chi^K\partial_j\chi^K - \delta_{ij}\frac{1}{2}\partial_k\chi^K\partial^k\chi^K + \frac{1}{8}\left(\delta_{ij}\partial^2 - \partial_i\partial_j\right)(\chi^K)^2\big], \tag{4.3.10}$$

$$T^{\psi}_{ij} = \frac{1}{g^2_{\mathrm{YM}}}\mathrm{tr}\big[\frac{1}{2}\bar{\psi}^L\gamma_{(i}\overleftrightarrow{\partial}_{j)}\psi^L - \delta_{ij}\frac{1}{2}\bar{\psi}^L\,\overleftrightarrow{\not{\partial}}\,\psi^L\big], \tag{4.3.11}$$

where $\overleftrightarrow{\partial}_i = \overrightarrow{\partial}_i - \overleftarrow{\partial}_i$. Note that the trace of the stress-energy tensors for both conformally coupled scalars and for massless fermions vanishes on shell. This is a consequence of the Weyl invariance discussed in section 1.1.5.

In momentum space we have

$$T^{\phi}_{ij}(\boldsymbol{p}) = \frac{1}{2g^2_{\mathrm{YM}}}P^{ab}_{ij}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}k_a(\boldsymbol{k}-\boldsymbol{p})_b\,\mathrm{tr}:\phi^J(\boldsymbol{k})\phi^J(\boldsymbol{p}-\boldsymbol{k}):, \tag{4.3.12}$$

$$T^{\chi}_{ij}(\boldsymbol{p}) = \tilde{T}^{\chi}_{ij}(\boldsymbol{p}) - \frac{1}{8g^2_{\mathrm{YM}}}p^2\pi_{ij}(\boldsymbol{p})\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{tr}:\chi^J(\boldsymbol{k})\chi^J(\boldsymbol{p}-\boldsymbol{k}):, \tag{4.3.13}$$

$$T^{\psi}_{ij}(\boldsymbol{p}) = \frac{\mathrm{i}}{g^2_{\mathrm{YM}}}\hat{P}^{ab}_{ij}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{k}\right)_b\mathrm{tr}:\bar{\psi}^L(\boldsymbol{k})\gamma_a\psi^L(\boldsymbol{p}-\boldsymbol{k}):, \tag{4.3.14}$$

where

$$P^{ab}_{ij} = 2\delta^{(a}_i\delta^{b)}_j - \delta_{ij}\delta^{ab}, \qquad \hat{P}^{ab}_{ij} = \delta^{(a}_i\delta^{b)}_j - \delta_{ij}\delta^{ab} \tag{4.3.15}$$

and the projector π_{ij} is defined in (2.1.13). By $\tilde{T}^{\chi}_{ij}$ we denote the stress-energy tensor for free scalars with substitution $\phi \mapsto \chi$.

In order to simplify the stress-energy tensor for the gauge fields notice that in 3 spacetime dimensions the space of 2-forms is isomorphic to the space of vectors by means of the Hodge isomorphism. Introducing the Hodge-dual field strength G^I_i by

$$F^I_{ij} = \epsilon_{ijk}G^{kI}, \qquad G^{iI} = -\frac{1}{2}\epsilon^{ijk}F^I_{jk} \tag{4.3.16}$$

we find the propagator

$$\langle G^a_i(\boldsymbol{p})G^b_j(\boldsymbol{p}')\rangle = -(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')g^2_{\mathrm{YM}}\pi_{ij}(\boldsymbol{p}). \tag{4.3.17}$$

The stress-energy tensor (4.3.8) simplifies to

$$T^A_{ij} = \frac{1}{g^2_{\mathrm{YM}}}\,P^{ab}_{ij}\,\mathrm{tr}\,G^I_aG^I_b. \tag{4.3.18}$$

We define the correlation functions of the stress-energy tensor in the natural way as

$$\langle T_{i_1j_1}(\boldsymbol{p}_1)\cdots T_{i_nj_n}(\boldsymbol{p}_n)\rangle = \int\mathcal{D}\Phi\,T_{i_1j_1}(\boldsymbol{p}_1)\cdots T_{i_nj_n}(\boldsymbol{p}_n)e^{-S_{QFT}[\Phi]} \tag{4.3.19}$$

where Φ denotes collectively all fields in the QFT. One can obtain the correlation functions by taking functional derivatives of (1.3.17) with respect to the metric.

This procedure leads to a new insertion of the stress tensor when functional derivative acts on $S_{QFT}[\Phi, g_{ij}]$, but also leads to additional semi- and ulta-local terms. One source of such terms are the factors of $1/\sqrt{g(x)}$ in (1.3.17) and for this reason some authors, see for example [22] define the correlators without such factors, *i.e.*,

$$< T_{i_1 j_1}(\boldsymbol{x}_1) \cdots T_{i_n j_n}(\boldsymbol{x}_n) > \equiv (-2)^n \frac{\delta}{\delta g^{i_1 j_1}(\boldsymbol{x}_1)} \cdots \frac{\delta}{\delta g^{i_n j_n}(\boldsymbol{x}_n)} \int \mathcal{D}\Phi \, e^{-S_{QFT}} \bigg|_{g_{ij}=\delta_{ij}} \tag{4.3.20}$$

Note however that these correlators differ from (4.3.19) (hence the different notation: $< T ... >$ instead of $\langle T ... \rangle$) because the stress tensor of the theory in a curved background also depends on g_{ij} and the functional differentiation leads to additional insertions of $\delta T_{ij}(\boldsymbol{x}_1)/\delta g^{kl}(\boldsymbol{x}_2)\big|_0$. A careful evaluation of all such semi-local terms is given in section 4.1 of [75].

All results will be presented in terms of the symmetric polynomials in amplitudes of momenta,

$$\begin{gathered} a_{123} = p_1 + p_2 + p_3, \qquad b_{123} = p_1 p_2 + p_1 p_3 + p_2 p_3, \qquad c_{123} = p_1 p_2 p_3, \\ a_{ij} = p_i + p_j, \qquad\qquad b_{ij} = p_i p_j, \end{gathered} \tag{4.3.21}$$

where $i, j = 1, 2, 3$. Furthermore by J^2 we denote the object defined in (2.6.18),

$$J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3). \tag{4.3.22}$$

4.3.2. 2-point functions

The decomposition of the 2-point function of the stress-energy tensor is given by (2.1.16) and reads

$$\langle\!\langle T_{ij}(\boldsymbol{p}) T_{kl}(-\boldsymbol{p}) \rangle\!\rangle = \Pi_{ijkl}(\boldsymbol{p}) A(p) + \pi_{ij}(\boldsymbol{p}) \pi_{kl}(\boldsymbol{p}) B(p), \tag{4.3.23}$$

where the projectors π_{ij} and Π_{ijkl} are defined in (2.1.13) and (2.1.14).

The contribution to the 2-point function from each of the individual fields follow from simple Feynman rules and read

$$\begin{aligned} A_\phi = B_\phi &= \frac{1}{256} \mathcal{N}_\phi d_A p^3, \qquad A_\psi = \frac{1}{128} \mathcal{N}_\psi d_A p^3, \qquad B_\psi = 0, \\ A_A = B_A &= \frac{1}{256} \mathcal{N}_A d_A p^3, \qquad A_\chi = \frac{1}{256} \mathcal{N}_\chi d_A p^3, \qquad B_\chi = 0, \end{aligned} \tag{4.3.24}$$

and thus in total we have

$$A = \frac{1}{256} \mathcal{N}_{(A)} d_A p^3, \qquad B = \frac{1}{256} \mathcal{N}_{(B)} d_A p^3, \tag{4.3.25}$$

where

$$\mathcal{N}_{(A)} = \mathcal{N}_A + \mathcal{N}_\phi + \mathcal{N}_\chi + 2\mathcal{N}_\psi, \qquad \mathcal{N}_{(B)} = \mathcal{N}_A + \mathcal{N}_\phi. \tag{4.3.26}$$

For conformal theories these results lead to the following normalisation constants c_T defined in (2.5.56),

$$c_T^{\chi} = \frac{1}{256}, \qquad c_T^{\psi} = \frac{1}{128}. \tag{4.3.27}$$

4.3.3. Minimal scalars

The 3-point function for minimal scalars is given by

$$\langle\!\langle T^{\phi}_{ij}(\boldsymbol{p}_1) T^{\phi}_{kl}(\boldsymbol{p}_2) T^{\phi}_{mn}(\boldsymbol{p}_3)\rangle\!\rangle = \mathcal{N}_{\phi} d_A P^{ab}_{ij} P^{cd}_{kl} P^{ef}_{mn} I_{abcdef}(p_1,p_2,p_3), \tag{4.3.28}$$

where $I_{abcdef}(p_1,p_2,p_3)$ is the integral (4.2.25). By extracting the required coefficients and applying the helicity projectors (2.9.2) one arrives at the following expressions

$$\langle\!\langle T_{\phi}(\boldsymbol{p}_1) T_{\phi}(\boldsymbol{p}_2) T_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle = \frac{\mathcal{N}_{\phi} d_A}{128} \left[a_{123}^3 - 2a_{123}b_{123} + 2c_{123}\right], \tag{4.3.29}$$

$$\begin{aligned} \langle\!\langle T_{\phi}(\boldsymbol{p}_1) T_{\phi}(\boldsymbol{p}_2) T^{(+)}_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle &= -\frac{\mathcal{N}_{\phi} d_A J^2}{1\,024\sqrt{2}\, p_3^2 a_{123}^2} \times \\ &\times \left[3a_{12}p_3^2 + 2(3a_{12}^2 - 4b_{12})p_3 + a_{12}(3a_{12}^2 - 4b_{12})\right], \end{aligned} \tag{4.3.30}$$

$$\begin{aligned} \langle\!\langle T_{\phi}(\boldsymbol{p}_1) T^{(+)}_{\phi}(\boldsymbol{p}_2) T^{(+)}_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle &= -\frac{\mathcal{N}_{\phi} d_A}{8\,192\, a_{123}^2 b_{23}^2} (p_1 - a_{23})^2 \times \\ &\left[5p_1^7 + 20a_{23}p_1^6 + (29a_{23}^2 + 6b_{23})p_1^5\right. \\ &\quad + a_{23}(17a_{23}^2 + 21b_{23})p_1^4 + 3a_{23}^2(a_{23}^2 + 8b_{23})p_1^3 + 2a_{23}^3(a_{23}^2 + 3b_{23})p_1^2 \\ &\quad \left. + (3a_{23}^6 - 6a_{23}^4 b_{23} - 32b_{23}^3)p_1 + a_{23}^5(a_{23}^2 - 3b_{23})\right], \end{aligned} \tag{4.3.31}$$

$$\begin{aligned} \langle\!\langle T_{\phi}(\boldsymbol{p}_1) T^{(+)}_{\phi}(\boldsymbol{p}_2) T^{(-)}_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle &= -\frac{\mathcal{N}_{\phi} d_A}{8192\, b_{23}^2} (p_1^2 - a_{23}^2 + 4b_{23})^2 \times \\ &\times \left[5p_1^3 - (a_{23}^2 + 2b_{23})p_1 + a_{23}(a_{23}^2 - 3b_{23})\right], \end{aligned} \tag{4.3.32}$$

$$\begin{aligned} \langle\!\langle T^{(+)}_{\phi}(\boldsymbol{p}_1) T^{(+)}_{\phi}(\boldsymbol{p}_2) T^{(+)}_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle &= -\frac{\mathcal{N}_{\phi} d_A J^2}{32\,768\sqrt{2}\, a_{123}^4 c_{123}^2} \times \\ &\times \left[3a_{123}^9 - 7a_{123}^7 b_{123} + 5a_{123}^6 c_{123} - 64c_{123}^3\right], \end{aligned} \tag{4.3.33}$$

$$\begin{aligned} \langle\!\langle T^{(+)}_{\phi}(\boldsymbol{p}_1) T^{(+)}_{\phi}(\boldsymbol{p}_2) T^{(-)}_{\phi}(\boldsymbol{p}_3)\rangle\!\rangle &= -\frac{\mathcal{N}_{\phi} d_A J^2}{32\,768\sqrt{2}\, a_{123}^2 c_{123}^2} (p_3 - a_{12})^2 \times \\ &\times \left[3p_3^5 + 4a_{12}p_3^4 + (a_{12}^2 - 2b_{12})p_3^3\right. \\ &\quad \left. + a_{12}(a_{12}^2 - 3b_{12})p_3^2 + 4a_{12}^2(a_{12}^2 - 3b_{12})p_3 + a_{12}^3(3a_{12}^2 - 7b_{12})\right]. \end{aligned} \tag{4.3.34}$$

All remaining 3-point functions for minimal scalars may be found from these via permutations and/or a parity transformation.

Turning now to evaluate the semi-local terms in the holographic formulae, for minimal scalars

$$\Upsilon^{\phi}_{ijkl}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{1}{2}(\delta_{ij} T^{\phi}_{kl} + P_{ijkl} T^{\phi})\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{4.3.35}$$

We thus have

$$\langle\!\langle T^{\phi}_{ij}(\boldsymbol{p}_1)\Upsilon^{\phi}_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{2}\delta_{kl}\langle\!\langle T^{\phi}_{ij}(\boldsymbol{p}_1)T^{\phi}_{mn}(-\boldsymbol{p}_1)\rangle\!\rangle - \frac{1}{2}P_{klmn}\langle\!\langle T^{\phi}_{ij}(\boldsymbol{p}_1)T^{\phi}(-\boldsymbol{p}_1)\rangle\!\rangle, \tag{4.3.36}$$

from which we may extract the helicity-projected components

$$\langle\!\langle T_{\phi}(\boldsymbol{p}_1)\Upsilon^{(s_3)}_{\phi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -\frac{3}{2}B_{\phi}(p_1)\Theta^{(s_3)}_1(p_i), \tag{4.3.37}$$

$$\langle\!\langle T_{\phi}(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}_{\phi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -B_{\phi}(p_1)\theta^{(s_2 s_3)}(p_i), \tag{4.3.38}$$

$$\langle\!\langle T^{(s_1)}_{\phi}(\boldsymbol{p}_1)\Upsilon_{\phi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.39}$$

$$\langle\!\langle T^{(s_1)}_{\phi}(\boldsymbol{p}_1)\Upsilon^{(s_3)}_{\phi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -\frac{3}{8}A_{\phi}(p_1)\theta^{(s_1 s_3)}(p_i), \tag{4.3.40}$$

$$\langle\!\langle T^{(s_1)}_{\phi}(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}_{\phi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = 0. \tag{4.3.41}$$

4.3.4. Fermions

The 3-point function for fermions is

$$\langle\!\langle T^{\psi}_{ij}(\boldsymbol{p}_1)T^{\psi}_{kl}(\boldsymbol{p}_2)T^{\psi}_{mn}(\boldsymbol{p}_3)\rangle\!\rangle = \frac{1}{4}\mathcal{N}_{\psi} d_A \hat{P}^{ab}_{ij}\hat{P}^{cd}_{kl}\hat{P}^{ef}_{mn}\Gamma_{ubvfwd}\times \\ \times \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{p^u(\boldsymbol{k}-\boldsymbol{p}_1)^v(\boldsymbol{k}+\boldsymbol{p}_2)^w(2\boldsymbol{k}-\boldsymbol{p}_1)_a(2\boldsymbol{k}+\boldsymbol{p}_2)_c(2\boldsymbol{k}-\boldsymbol{p}_1+\boldsymbol{p}_2)_e}{k^2|\boldsymbol{k}-\boldsymbol{p}_1|^2|\boldsymbol{k}+\boldsymbol{p}_2|^2}, \tag{4.3.42}$$

where

$$\begin{aligned}\Gamma_{ubvfwd} &= \mathrm{tr}(\gamma_u\gamma_b\gamma_v\gamma_f\gamma_w\gamma_d) \\ &= -2\delta_{ub}P_{vwdf} + 2\delta_{uv}P_{bwdf} - 2\delta_{uf}P_{bwvd} + 2\delta_{uw}P_{bfvd} - 2\delta_{ud}P_{bfvw},\end{aligned} \tag{4.3.43}$$

recalling that $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$.

Evaluating the integral explicitly, we find

$$\langle\!\langle T_{\psi}(\boldsymbol{p}_1)T_{\psi}(\boldsymbol{p}_2)T_{\psi}(\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.44}$$

$$\langle\!\langle T_{\psi}(\boldsymbol{p}_1)T_{\psi}(\boldsymbol{p}_2)T^{(+)}_{\psi}(\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.45}$$

$$\langle\!\langle T_{\psi}(\boldsymbol{p}_1)T^{(+)}_{\psi}(\boldsymbol{p}_2)T^{(+)}_{\psi}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{\mathcal{N}_{\psi}d_A}{2048\,b_{23}^2}\,a_{23}(a_{23}^2 - 3b_{23})(p_1^2 - a_{23}^2)^2, \tag{4.3.46}$$

$$\langle\!\langle T_\psi(\boldsymbol{p}_1)T_\psi^{(+)}(\boldsymbol{p}_2)T_\psi^{(-)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{\mathcal{N}_\psi d_A}{2048\, b_{23}^2}\times$$
$$\times\, a_{23}(a_{23}^2-3b_{23})(p_1^2-a_{23}^2+4b_{23})^2, \tag{4.3.47}$$

$$\langle\!\langle T_\psi^{(+)}(\boldsymbol{p}_1)T_\psi^{(+)}(\boldsymbol{p}_2)T_\psi^{(+)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{\mathcal{N}_\psi d_A J^2}{8192\sqrt{2}\, a_{123}^4 c_{123}^2}\times$$
$$\times\left[a_{123}^9-2a_{123}^7 b_{123}+a_{123}^6 c_{123}+32c_{123}^3\right], \tag{4.3.48}$$

$$\langle\!\langle T_\psi^{(+)}(\boldsymbol{p}_1)T_\psi^{(+)}(\boldsymbol{p}_2)T_\psi^{(-)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{\mathcal{N}_\psi d_A J^2}{8192\sqrt{2}\, a_{123}^2 c_{123}^2}(p_3-a_{12})^2\times$$
$$\left[p_3^5+a_{12}p_3^4-b_{12}p_3^3+a_{12}^2(a_{12}^2-3b_{12})p_3+a_{12}^3(a_{12}^2-2b_{12})\right]. \tag{4.3.49}$$

The correlators with only one trace may be written in the condensed form

$$\langle\!\langle T_\psi(\boldsymbol{p}_1)T_\psi^{(s_2)}(\boldsymbol{p}_2)T_\psi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{2}\big(A_\psi(p_2)+A_\psi(p_3)\big)\theta^{(s_2 s_3)}(p_i). \tag{4.3.50}$$

This result is in fact fully determined by the trace Ward identity (1.3.42).

All these results are in perfect agreement with the predictions of section (3.10). The values of the two undetermined constants α_1 and α_2 can be read of from the full solution before the projection onto helicity basis is performed,

$$\alpha_1^\psi = -\frac{1}{1\,920}, \qquad \alpha_2^\psi = \frac{1}{96}. \tag{4.3.51}$$

The value of the 2-point normalisation constant c_T agrees with (4.3.27) and the value of the constant c_g defined in (3.11.20) is $c_g^\psi = -\frac{1}{8}$. This can be checked by the following calculations of the semi-local terms. By explicit calculations we find the operator

$$\Upsilon^\psi_{ijkl}(\boldsymbol{x}_1,\boldsymbol{x}_2) = C^{(\mathcal{M})}_{ijklmn}\mathcal{M}^{mn}(\boldsymbol{x}_1)\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)+C^{(\mathcal{J})}_{ijklm}\mathcal{J}^m(\boldsymbol{x}_1)\partial^n\delta(\boldsymbol{x}_1-\boldsymbol{x}_2), \tag{4.3.52}$$

where partial derivatives are taken with respect to $\boldsymbol{x}_1$, and the local operators

$$\mathcal{M}_{mn} = \frac{1}{g^2_{\text{YM}}}\operatorname{tr}\left[\frac{1}{2}\bar{\psi}^L\gamma_m\overleftrightarrow{\partial}_n\psi^L\right], \qquad \mathcal{J}_m = \frac{1}{g^2_{\text{YM}}}\operatorname{tr}\left[\frac{1}{4}\bar{\psi}^L\gamma_m\psi^L\right], \tag{4.3.53}$$

are associated with the coefficients

$$C^{(\mathcal{M})}_{ijklmn} = \delta_{i(k}\delta_{l)j}\delta_{mn}-\frac{1}{2}\delta_{ij}\delta_{m(k}\delta_{l)n}-\frac{1}{2}\delta_{m(k}\delta_{l)(i}\delta_{j)n}, \tag{4.3.54}$$
$$C^{(\mathcal{J})}_{ijklm} = \delta_{i(k}\delta_{l)j}\delta_{mn}+\delta_{ij}\delta_{m(k}\delta_{l)n}-\delta_{ij}\delta_{kl}\delta_{mn}-\delta_{m(k}\delta_{l)(i}\delta_{j)n}. \tag{4.3.55}$$

Operator $\mathcal{J}_m$ is the current operator (2.7.37), so it has a definite conformal dimension $\Delta = d-1$. Therefore,

$$\langle T^\psi_{kl}\mathcal{J}_m\rangle = 0, \qquad \langle T^\psi_{kl}\mathcal{M}_{mn}\rangle = \langle T^\psi_{kl}T^\psi_{mn}\rangle, \tag{4.3.56}$$

from which it follows that

$$\langle\!\langle T^{\psi}_{ij}(\boldsymbol{p}_1)\Upsilon^{\psi}_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = C^{(\mathcal{M})}_{klmnab}\langle\!\langle T^{\psi}_{ij}(\boldsymbol{p}_1)T^{\psi}_{ab}(-\boldsymbol{p}_1)\rangle\!\rangle. \tag{4.3.57}$$

Projecting into the helicity basis, the components appearing in the holographic formulae are

$$\langle\!\langle T_{\psi}(\boldsymbol{p}_1)\Upsilon^{(s_3)}_{\psi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.58}$$

$$\langle\!\langle T_{\psi}(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}_{\psi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.59}$$

$$\langle\!\langle T^{(s_1)}_{\psi}(\boldsymbol{p}_1)\Upsilon_{\psi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.60}$$

$$\langle\!\langle T^{(s_1)}_{\psi}(\boldsymbol{p}_1)\Upsilon^{(s_3)}_{\psi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{2}A_{\psi}(p_1)\theta^{(s_1 s_3)}(p_i), \tag{4.3.61}$$

$$\langle\!\langle T^{(s_1)}_{\psi}(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}_{\psi}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{16}A_{\psi}(p_1)\Theta^{(s_1 s_2 s_3)}(p_i). \tag{4.3.62}$$

4.3.5. Conformal scalars

As discussed in section 4.3.1, the stress tensor T^{χ}_{ij} for conformal scalars may be decomposed as $T^{\chi}_{ij} = \tilde{T}^{\chi}_{ij} - \mathcal{C}_{ij}$, where $\tilde{T}^{\chi}_{ij}$ is the stress tensor for minimal scalars and $\mathcal{C}_{ij}$ is the improvement term. Here we prefer to invert this relation by writing $T^{\phi}_{ij} = \tilde{T}^{\phi}_{ij} + \mathcal{C}_{ij}$, where $\tilde{T}^{\phi}_{ij}$ is the stress-energy tensor for conformal scalars evaluated with $\chi \mapsto \phi$. In this sense we can regard the minimal scalar as a deformation of the conformal scalar. The improvement term takes the form

$$\mathcal{C}_{ij}(\boldsymbol{p}_1) = \frac{1}{8g^2_{\mathrm{YM}}}p^2\pi_{ij}(\boldsymbol{p})\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{tr}\,{:}\chi^J(\boldsymbol{k})\chi^J(\boldsymbol{p}-\boldsymbol{k}){:} \tag{4.3.63}$$

Due to the presence of the projection operator π_{ij}, it follows that $T^{(s)}_{\phi}(\boldsymbol{p}) = \widetilde{T}^{(s)}_{\phi}(\boldsymbol{p})$ and hence the conformal scalar 3-point function involving three helicities is equal to that for minimal scalars. Similarly, the correlator

$$\langle\!\langle \widetilde{T}_{\phi}\widetilde{T}^{(s_2)}_{\phi}\widetilde{T}^{(s_3)}_{\phi}\rangle\!\rangle = \langle\!\langle T_{\phi}T^{(s_2)}_{\phi}T^{(s_3)}_{\phi}\rangle\!\rangle - \langle\!\langle \mathcal{C}T^{(s_2)}_{\phi}T^{(s_3)}_{\phi}\rangle\!\rangle, \tag{4.3.64}$$

where the latter term may be evaluated from the integral

$$\langle\!\langle \mathcal{C}T^{\phi}_{kl}T^{\phi}_{mn}\rangle\!\rangle = \frac{1}{2}\mathcal{N}_{\phi}d_A p_1^2 P^{ab}_{kl}P^{cd}_{mn}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\boldsymbol{k}_a(\boldsymbol{k}+\boldsymbol{p}_2)_b(\boldsymbol{k}-\boldsymbol{p}_1)_c(\boldsymbol{k}+\boldsymbol{p}_2)_d}{k^2|\boldsymbol{k}-\boldsymbol{p}_1|^2|\boldsymbol{k}+\boldsymbol{p}_2|^2}. \tag{4.3.65}$$

Thus, to evaluate the conformal scalar 3-point function involving two helicities, only this integral needs to be computed since we already have the result for minimal scalars. Finally, evaluating the trace $\widetilde{T}_{\phi}(\boldsymbol{p})$ directly, it is straightforward to show that the conformal scalar 3-point function involving only one helicity vanishes.

In light of these considerations, the 3-point functions for the conformal scalar field χ are

$$\langle\!\langle T_\chi(\boldsymbol{p}_1) T_\chi(\boldsymbol{p}_2) T_\chi(\boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{4.3.66}$$

$$\langle\!\langle T_\chi(\boldsymbol{p}_1) T_\chi(\boldsymbol{p}_2) T_\chi^{(s_3)}(\boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{4.3.67}$$

$$\langle\!\langle T_\chi(\boldsymbol{p}_1) T_\chi^{(s_2)}(\boldsymbol{p}_2) T_\chi^{(s_3)}(\boldsymbol{p}_3) \rangle\!\rangle = \frac{1}{4} \frac{\mathcal{N}_\chi}{\mathcal{N}_\psi} \langle\!\langle T_\psi(\boldsymbol{p}_1) T_\psi^{(s_2)}(\boldsymbol{p}_2) T_\psi^{(s_3)}(\boldsymbol{p}_3) \rangle\!\rangle, \tag{4.3.68}$$

$$\langle\!\langle T_\chi^{(s_1)}(\boldsymbol{p}_1) T_\chi^{(s_2)}(\boldsymbol{p}_2) T_\chi^{(s_3)}(\boldsymbol{p}_3) \rangle\!\rangle = \frac{\mathcal{N}_\chi}{\mathcal{N}_\phi} \langle\!\langle T_\phi^{(s_1)}(\boldsymbol{p}_1) T_\phi^{(s_2)}(\boldsymbol{p}_2) T_\phi^{(s_3)}(\boldsymbol{p}_3) \rangle\!\rangle. \tag{4.3.69}$$

The form of the functional derivative of the stress-energy tensor with respect to the metric for conformal scalar in $d = 3$ was found in section 2.4.4, equations (2.4.37) - (2.4.38). The precise form of these prefactors is not important, however, since

$$\langle T_{ij}^\chi \mathcal{O}^\chi \rangle = 0 \tag{4.3.70}$$

due to the differing conformal dimension of the two operators, hence

$$\langle\!\langle T_{ij}^\chi(\boldsymbol{p}_1) \Upsilon_{klmn}^\chi(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = -\frac{1}{2} \delta_{kl} \langle\!\langle T_{ij}^\chi(\boldsymbol{p}_1) T_{mn}^\chi(-\boldsymbol{p}_1) \rangle\!\rangle. \tag{4.3.71}$$

From this it follows that the c_g constant defined in (3.11.20) vanishes. The helicity-projected components appearing in the holographic formulae are then

$$\langle\!\langle T_\chi(\boldsymbol{p}_1) \Upsilon_\chi^{(s_3)}(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{4.3.72}$$

$$\langle\!\langle T_\chi(\boldsymbol{p}_1) \Upsilon_\chi^{(s_2 s_3)}(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{4.3.73}$$

$$\langle\!\langle T_\chi^{(s_1)}(\boldsymbol{p}_1) \Upsilon_\chi(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = 0, \tag{4.3.74}$$

$$\langle\!\langle T_\chi^{(s_1)}(\boldsymbol{p}_1) \Upsilon_\chi^{(s_3)}(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = -\frac{3}{8} A_\chi(p_1) \theta^{(s_1 s_3)}(p_i), \tag{4.3.75}$$

$$\langle\!\langle T_\chi^{(s_1)}(\boldsymbol{p}_1) \Upsilon_\chi^{(s_2 s_3)}(\boldsymbol{p}_2, \boldsymbol{p}_3) \rangle\!\rangle = 0. \tag{4.3.76}$$

As in case of the fermions, one can confront our result with the most general form of the 3-point function (3.11.21) - (3.11.25). We find the perfect agreement with the following values of the parameters

$$\alpha_1^\chi = \frac{1}{3\,840}, \qquad \alpha_2^\chi = 0, \qquad c_T^\chi = \frac{1}{256}, \qquad c_g^\chi = 0. \tag{4.3.77}$$

4.3.6. Gauge fields

Using the dual field strength (4.3.16) the contribution from gauge fields to the 3-point function is given by

$$\begin{aligned}\langle\!\langle T^A_{ij}(\boldsymbol{p}_1)T^A_{kl}(\boldsymbol{p}_2)T^A_{mn}(\boldsymbol{p}_3)\rangle\!\rangle &= -\mathcal{N}_A d_A P^{ab}_{ij}P^{cd}_{kl}P^{ef}_{mn}\times\\ &\times\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\pi_{ac}(\boldsymbol{k})\pi_{be}(\boldsymbol{k}-\boldsymbol{p}_1)\pi_{df}(\boldsymbol{k}+\boldsymbol{p}_2).\end{aligned} \tag{4.3.78}$$

Upon closer examination, this integral may equivalently be expressed in terms of the 2- and 3-point functions for minimal scalars,

$$\begin{aligned}\frac{\mathcal{N}_\phi}{\mathcal{N}_A}\langle\!\langle T^A_{ij}(\boldsymbol{p}_1)T^A_{kl}(\boldsymbol{p}_2)T^A_{mn}(\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)T^\phi_{kl}(\boldsymbol{p}_2)T^\phi_{mn}(\boldsymbol{p}_3)\rangle\!\rangle\\ &- Q_{klmnab}\langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)T^{\phi ab}(-\boldsymbol{p}_1)\rangle\!\rangle - Q_{mnijab}\langle\!\langle T^\phi_{kl}(\boldsymbol{p}_2)T^{\phi ab}(-\boldsymbol{p}_2)\rangle\!\rangle\\ &- Q_{ijklab}\langle\!\langle T^\phi_{mn}(\boldsymbol{p}_3)T^{\phi ab}(-\boldsymbol{p}_3)\rangle\!\rangle,\end{aligned} \tag{4.3.79}$$

where

$$Q_{ijklmn} = P_{ijac}P_{klbd}\delta^{cd}\hat{P}^{ab}_{mn}. \tag{4.3.80}$$

This result is a consequence of the fact that G_i may be identified with the operator $\partial_i\phi$, where ϕ is a massless scalar field. The appearance of the various semi-local terms in (4.3.79) then reflects the fact that $\partial_i G^i$ vanishes identically, while $\partial^2\phi$ vanishes on-shell only.

Turning now to evaluate the semi-local terms appearing in the holographic formulae, a short calculation reveals the operator

$$\Upsilon^A_{ijkl}(\boldsymbol{x}_1,\boldsymbol{x}_2) = -\frac{1}{2}\Big[\delta_{ij}T^A_{kl} + P_{ijkl}T^A + Q_{ijklmn}T^{Amn}\Big]\delta(\boldsymbol{x}_1-\boldsymbol{x}_2). \tag{4.3.81}$$

Making use of the fact that the 2-point functions for gauge fields and for minimal scalars coincide, see (4.3.24), it then follows that

$$\begin{aligned}\frac{\mathcal{N}_\phi}{\mathcal{N}_A}\langle\!\langle T^A_{ij}(\boldsymbol{p}_1)\Upsilon^A_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)\Upsilon^\phi_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle\\ &- \frac{1}{2}Q_{klmnab}\langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)T^{\phi ab}(-\boldsymbol{p}_1)\rangle\!\rangle.\end{aligned} \tag{4.3.82}$$

4.3.7. Ghosts and gauge-fixing terms

To evaluate the gauge field contribution to 3-point functions we must gauge-fix and introduce ghost fields. This procedure generates a new contribution to the stress tensor that depends on the gauge-fixing part of the Lagrangian. Here we show that this part does not contribute to the 3-point functions. The general argument is based on the fact that the full Lagrangian for the gauge field is

$$S_{\mathrm{YM}} = \frac{1}{g^2_{\mathrm{YM}}}\int\mathrm{d}^3\boldsymbol{x}\ \mathrm{tr}\left[\frac{1}{4}F^I_{ij}F^{ijI} + \delta_B\mathcal{O}\right], \tag{4.3.83}$$

where $\mathcal{O}$ is a gauge-fixing part containing ghosts and δ_B is an infinitesimal BRST transformation. The full stress tensor is therefore

$$T_{ij}^{\mathrm{YM}} = T_{ij}^{A} + T_{ij}^{\mathrm{gf}}, \tag{4.3.84}$$

where T_{ij}^{gf} is a BRST-exact operator. Since physical states correspond to the cohomology of the BRST transformation, T_{ij}^{gf} vanishes when acting on such states. Therefore, inside any vacuum correlation function, T_{ij}^{YM} can be replaced by T_{ij}^{A}.

As this is a formal argument, we will also present now an explicit perturbative proof that the gauge-fixing part does not contribute to any correlation functions. We work in the R_ξ gauge and to first order in g_{YM}^2. The ghost part and gauge-fixing part of the action may be written as

$$S_\xi = -\frac{1}{g_{\mathrm{YM}}^2}\int \mathrm{d}^3\boldsymbol{x}\, \mathrm{tr}\left[\frac{\xi}{2}(B^I)^2 + A_i^I \partial^i B^I\right], \tag{4.3.85}$$

$$S_{\mathrm{gh}} = \frac{1}{g_{\mathrm{YM}}^2}\int \mathrm{d}^3\boldsymbol{x}\ \mathrm{tr}\left[\partial_i \bar{c}^I \partial^i c^I\right]. \tag{4.3.86}$$

where $\bar{c}^I$ and c^I are the antighost and the ghost fields, and B^I is the BRST auxiliary field. All fields are in the adjoint representation and are regarded as traceless hermitian matrices. The full Yang-Mills theory is given by the action

$$S_{\mathrm{YM}} = \frac{1}{g_{\mathrm{YM}}^2}\int \mathrm{d}^3\boldsymbol{x}\ \mathrm{tr}\left[\frac{1}{4}F_{ij}^I F^{ijI}\right] + S_\xi + S_{\mathrm{gh}}. \tag{4.3.87}$$

This leads to the following propagators

$$\langle\!\langle B^I(\boldsymbol{p}) B^J(-\boldsymbol{p})\rangle\!\rangle = 0, \qquad \langle\!\langle B^I(\boldsymbol{p}) F_{ij}^J(-\boldsymbol{p})\rangle\!\rangle = 0, \tag{4.3.88}$$

and

$$-\langle\!\langle A_i^I(\boldsymbol{p}) B^J(-\boldsymbol{p})\rangle\!\rangle = \langle\!\langle (\partial_i \bar{c}^I)(\boldsymbol{p})\, c^J(-\boldsymbol{p})\rangle\!\rangle = \delta^{IJ}\frac{\mathrm{i} g_{\mathrm{YM}}^2 p_i}{p^2}. \tag{4.3.89}$$

Here, by $(\partial_i \bar{c}^{Ia})(\boldsymbol{p})$, we denote the Fourier transform of $\partial_i \bar{c}^{Ia}(\boldsymbol{x})$.

The stress tensor and the Υ tensor defined in (2.9.8) corresponding to each component of the action is given by

$$T_{ij}^A = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{tr}[F_{ik}^I F_j{}^{kI} - \delta_{ij}\frac{1}{4}F_{kl}^I F^{klI}], \tag{4.3.90}$$

$$T_{ij}^\xi = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{tr}[-P_{ij}^{kl} A_k^I \partial_l B^I + \delta_{ij}\frac{\xi}{2}(B^I)^2], \tag{4.3.91}$$

$$T_{ij}^{\mathrm{gh}} = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{tr}[P_{ij}^{kl}\partial_k \bar{c}\partial_l c], \tag{4.3.92}$$

$$\Upsilon_{ijkl}^A = -\frac{1}{2}\left[\delta_{ij} T_{kl}^A + P_{ijkl} T^A + Q_{ijklmn} T^{Amn}\right]\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2), \tag{4.3.93}$$

$$\Upsilon_{ijkl}^\xi = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{tr}[-\delta_{i(k}\delta_{l)j} A_m^I \partial^m B^I + \delta_{ij} A_{(k}^I \partial_{l)} B^I - \delta_{i(k}\delta_{l)j}\frac{\xi}{2}(B^I)^2]\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2), \tag{4.3.94}$$

$$\Upsilon^{\text{gh}}_{ijkl} = \frac{1}{g^2_{\text{YM}}}\,\text{tr}[\delta_{i(k}\delta_{l)j}\partial_m\bar{c}^I\partial^m c^I - \delta_{ij}\partial_{(k}\bar{c}^I\partial_{l)}c^I]\delta(\boldsymbol{x}_1-\boldsymbol{x}_2). \tag{4.3.95}$$

where Q_{ijklmn} is defined in (4.3.80). The full stress tensor and Υ tensor is a sum

$$T^{\text{YM}}_{ij} = T^A_{ij} + T^\xi_{ij} + T^{\text{gh}}_{ij}, \qquad \Upsilon^{\text{YM}}_{ijkl} = \Upsilon^A_{ijkl} + \Upsilon^\xi_{ijkl} + \Upsilon^{\text{gh}}_{ijkl}. \tag{4.3.96}$$

The mechanism for cancellation of ghost and gauge-fixing terms is very general. Let us consider a set of general gauge-invariant operators $\mathcal{F}^{(\alpha)}$ of arbitrary tensor structure, indexed by α, quadratic in field strengths F^I. Consider moreover gauge dependent terms $\mathcal{B}^{(\alpha)}$ and ghost terms $\mathcal{C}^{(\alpha)}$ of the schematic form

$$\mathcal{B}^{(\alpha)} = \frac{1}{g^2_{\text{YM}}}\,\text{tr}\left[A^{iI}\hat{O}^{A,(\alpha)}_i[B^I] + \hat{O}^{B,(\alpha)}[(B^I)^2]\right], \tag{4.3.97}$$

$$\mathcal{C}^{(\alpha)} = \frac{1}{g^2_{\text{YM}}}\,\text{tr}\left[\partial^i\bar{c}^I\hat{O}^{C,(\alpha)}_i[c^I]\right], \tag{4.3.98}$$

where $\hat{O}^{A,(\alpha)}_i$ is linear in B^I, $\hat{O}^{B,(\alpha)}$ is quadratic in B^I and $\hat{O}^{C,(\alpha)}_i$ is linear in c^I, but are otherwise operators of arbitrary tensor structure which *may* contain derivatives but no other fields. We consider operators $\mathcal{O}^{(\alpha)} = \mathcal{F}^{(\alpha)} + \mathcal{B}^{(\alpha)} + \mathcal{C}^{(\alpha)}$ and their n-point function in the Yang-Mills theory with the action (4.3.87). The stress tensor and the Υ tensor are of this form. We find

$$\begin{aligned}\langle \mathcal{O}^{(1)}\mathcal{O}^{(2)}\dots\mathcal{O}^{(n)}\rangle = {} & \langle \mathcal{F}^{(1)}\mathcal{F}^{(2)}\dots\mathcal{F}^{(n)}\rangle + \\ & + \langle \mathcal{B}^{(1)}\mathcal{F}^{(2)}\dots\mathcal{F}^{(n)}\rangle + \langle \mathcal{F}^{(1)}\mathcal{B}^{(2)}\dots\mathcal{F}^{(n)}\rangle + \dots \\ & + \langle \mathcal{B}^{(1)}\mathcal{B}^{(2)}\mathcal{F}^{(3)}\dots\mathcal{F}^{(n)}\rangle + \text{ permutations} \\ & + \dots \\ & + \langle \mathcal{B}^{(1)}\mathcal{B}^{(2)}\dots\mathcal{B}^{(n)}\rangle + \langle \mathcal{C}^{(1)}\mathcal{C}^{(2)}\dots\mathcal{C}^{(n)}\rangle\end{aligned} \tag{4.3.99}$$

since there is no interaction between ghosts and any other fields at leading order in g^2_{YM}. We will now show that all terms but the first one cancel.

To begin, we observe that all terms containing at least one $\mathcal{F}$ and at least one $\mathcal{B}$ vanish. Indeed, when Wick's theorem is applied, there must be at least one contraction between F and B fields, or between B and another B field, which gives zero by (4.3.88).

Now consider the term with $\mathcal{B}$ operators only. When expanded, it has 2^n terms, but every term containing $(B^I)^2$ must evaluate to zero as there must be at least one B-B contraction. Therefore, only one term survives, namely

$$\langle \mathcal{B}^{(1)}\dots\mathcal{B}^{(n)}\rangle = \frac{1}{g^{2n}_{\text{YM}}}\langle \text{tr}\left(A^{j_1 I_1}\hat{O}^{A,(1)}_{j_1}[B^{I_1}]\right)\cdot\ldots\cdot\text{tr}\left(A^{j_n I_n}\hat{O}^{A,(n)}_{j_n}[B^{I_n}]\right)\rangle. \tag{4.3.100}$$

The only non-vanishing way of contracting fields is to contract auxiliary fields with gauge fields. This gives precisely the same possible contractions as in the ghost part, which is

$$\langle \mathcal{C}^{(1)} \dots \mathcal{C}^{(n)} \rangle = \frac{1}{g_{\text{YM}}^{2n}} \langle \text{tr}\left(\partial^{j_1} \bar{c}^{I_1} \hat{O}_{j_1}^{C,(1)}[c^{I_1}] \right) \cdot \ldots \cdot \text{tr}\left(\partial^{j_n} \bar{c}^{I_n} \hat{O}_{j_n}^{C,(n)}[c^{I_n}] \right) \rangle. \quad (4.3.101)$$

It follows that if $-\hat{O}_i^{A,(\alpha)} = \hat{O}_i^{C,(\alpha)}$ for all α, then (4.3.100) and (4.3.101) cancel each other out, due to (4.3.89) and the anti-commuting nature of ghost fields.

In our case, and to this order in g_{YM}^2, there are no gauge boson interactions and so the gauge group G is effectively $U(1)^{\dim G}$. For $\langle T_{i_1 j_1}^{\text{YM}} \, T_{i_2 j_2}^{\text{YM}} \, T_{i_3 j_3}^{\text{YM}} \rangle$ we therefore find

$$\mathcal{F}^\alpha = T^A_{i_\alpha j_\alpha}, \qquad \mathcal{B}^\alpha = T^\xi_{i_\alpha j_\alpha}, \qquad \mathcal{C}^\alpha = T^{\text{gh}}_{i_\alpha j_\alpha}, \quad (4.3.102)$$

and

$$-\hat{O}^{A,(\alpha)}_{i_\alpha j_\alpha, k} = \hat{O}^{C,(\alpha)}_{i_\alpha j_\alpha, k} = P_{i_\alpha j_\alpha k l} \partial^l \quad (4.3.103)$$

for $\alpha = 1, 2, 3$. For $\langle \Upsilon^{\text{YM}}_{ijkl} \, T^{\text{YM}}_{mn} \rangle$, we have

$$\begin{aligned} \mathcal{F}^{(1)} &= \Upsilon^A_{ijkl}, & \mathcal{B}^{(1)} &= \Upsilon^\xi_{ijkl}, & \mathcal{C}^{(1)} &= \Upsilon^{\text{gh}}_{ijkl}, \\ \mathcal{F}^{(2)} &= T^A_{mn}, & \mathcal{B}^{(2)} &= T^\xi_{mn}, & \mathcal{C}^{(2)} &= T^{\text{gh}}_{mn} \end{aligned} \quad (4.3.104)$$

and

$$\begin{aligned} -\hat{O}^{A,(1)}_{ijkl,m} &= \hat{O}^{C,(1)}_{ijkl,m} = \delta_{i(k} \delta_{l)j} \partial_m - \delta_{ij} \delta_{m(k} \partial_{l)}, \\ -\hat{O}^{A,(2)}_{mn,i} &= \hat{O}^{C,(2)}_{mn,i} = P_{mnik} \partial^k. \end{aligned} \quad (4.3.105)$$

It follows that the contribution due to the gauge-fixing part of the action indeed cancels out.

4.3.8. Minimal scalars from conformal scalars

In the previous sections we computed all relevant 3-point functions and semi-local terms by direct computation of 1-loop Feynman diagrams. In this section we elucidate the structure of these correlators by ascertaining the extent to which they are determined by Ward identities.

As noted previously, the stress tensor for minimal scalars may be decomposed as

$$T^\phi_{ij} = \widetilde{T}^\phi_{ij} - \frac{1}{8} (\delta_{ij} \partial^2 - \partial_i \partial_j) \mathcal{O}_1, \qquad \mathcal{O}_1 = \frac{1}{g^2_{\text{YM}}} \text{tr}[(\phi^J)^2], \quad (4.3.106)$$

where $\widetilde{T}^\phi_{ij}$ is the stress tensor for a conformal scalar field and $\mathcal{O}_1$ is a dimension one scalar operator. The 3-point functions of T^ϕ_{ij} may thus be expressed in terms of 3-point functions of the conformal fields $\widetilde{T}^\phi_{ij}$ and $\mathcal{O}_1$. Specifically, we find

$$\langle\!\langle T^{(s_1)}_\phi(\boldsymbol{p}_1) T^{(s_2)}_\phi(\boldsymbol{p}_2) T^{(s_3)}_\phi(\boldsymbol{p}_3) \rangle\!\rangle = \langle\!\langle \widetilde{T}^{(s_1)}_\phi(\boldsymbol{p}_1) \widetilde{T}^{(s_2)}_\phi(\boldsymbol{p}_2) \widetilde{T}^{(s_3)}_\phi(\boldsymbol{p}_3) \rangle\!\rangle\,, \quad (4.3.107)$$

$$
\begin{aligned}
\langle\!\langle T_\phi(\boldsymbol{p}_1)T_\phi^{(s_2)}(\boldsymbol{p}_2)T_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1)\widetilde{T}_\phi^{(s_2)}(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle \\
&+ \frac{p_1^2}{4}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\widetilde{T}_\phi^{(s_2)}(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle\,, \qquad (4.3.108)
\end{aligned}
$$

$$
\begin{aligned}
\langle\!\langle T_\phi(\boldsymbol{p}_1)T_\phi(\boldsymbol{p}_2)T_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1)\widetilde{T}_\phi(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle \\
&+ \frac{p_1^2}{4}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\widetilde{T}_\phi(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3) + \frac{p_2^2}{4}\langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1)\mathcal{O}_1(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle \\
&+ \frac{p_1^2p_2^2}{16}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_1(\boldsymbol{p}_2)\widetilde{T}_\phi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle, \qquad (4.3.109)
\end{aligned}
$$

$$
\begin{aligned}
\langle\!\langle T_\phi(\boldsymbol{p}_1)T_\phi(\boldsymbol{p}_2)T_\phi(\boldsymbol{p}_3)\rangle\!\rangle &= \langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1)\widetilde{T}_\phi(\boldsymbol{p}_2)\widetilde{T}_\phi(\boldsymbol{p}_3)\rangle\!\rangle \\
&+ \left[\frac{p_1^2}{4}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\widetilde{T}_\phi(\boldsymbol{p}_2)\widetilde{T}_\phi(\boldsymbol{p}_3)\rangle\!\rangle + 2\text{ perm.}\right] \\
&+ \left[\frac{p_1^2p_2^2}{16}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_1(\boldsymbol{p}_2)\widetilde{T}_\phi(\boldsymbol{p}_3)\rangle\!\rangle + 2\text{ perm.}\right] \\
&+ \frac{p_1^2p_2^2p_3^2}{64}\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_1(\boldsymbol{p}_2)\mathcal{O}_1(\boldsymbol{p}_3)\rangle\!\rangle. \qquad (4.3.110)
\end{aligned}
$$

Recalling that gauge fields contribute the same as minimal scalars, the computation of general 3-point functions thus reduces to computing a set of 2- and 3-point functions in a CFT. These correlators are in turn (almost) uniquely determined by Ward identities, as we know from chapter 2. In the following section we will apply the conformal field theory formalism to see it explicitly.

4.3.9. Trace Ward identity

In light of the above, we are interested in correlation functions of the stress tensor and of a scalar operator $\mathcal{O}_\Delta$ of dimension $\Delta = 1$ in a three-dimensional CFT. The trace Ward identities were introduced in section (1.3.4) and presented in (4.1.8) - (4.1.16). Here we work in a pure CFT, hence by setting $\phi_0 = 0$ and projecting into the helicity basis, we obtain the complete set of trace Ward identities

$$
\langle\!\langle T(\boldsymbol{p}_1)\mathcal{O}_\Delta(\boldsymbol{p}_2)\mathcal{O}_\Delta(\boldsymbol{p}_3)\rangle\!\rangle = -\Delta\big[\langle\!\langle \mathcal{O}_\Delta(\boldsymbol{p}_2)\mathcal{O}_\Delta(-\boldsymbol{p}_2)\rangle\!\rangle + \langle\!\langle \mathcal{O}_\Delta(\boldsymbol{p}_3)\mathcal{O}_\Delta(-\boldsymbol{p}_3)\rangle\!\rangle\big], \qquad (4.3.111)
$$

$$
\langle\!\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)\mathcal{O}_\Delta(\boldsymbol{p}_3)\rangle\!\rangle = 2\langle\!\langle \Upsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)\mathcal{O}_\Delta(\boldsymbol{p}_3)\rangle\!\rangle, \qquad (4.3.112)
$$

$$
\langle\!\langle T(\boldsymbol{p}_1)T^{(s_2)}(\boldsymbol{p}_2)\mathcal{O}_\Delta(\boldsymbol{p}_3)\rangle\!\rangle = 2\langle\!\langle \Upsilon^{(s_2)}(\boldsymbol{p}_1,\boldsymbol{p}_2)\mathcal{O}_\Delta(\boldsymbol{p}_3)\rangle\!\rangle, \qquad (4.3.113)
$$

$$
\begin{aligned}
\langle\!\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)T(\boldsymbol{p}_3)\rangle\!\rangle &= 2\,[\langle\!\langle T(\boldsymbol{p}_1)\Upsilon(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle + \langle\!\langle T(\boldsymbol{p}_2)\Upsilon(\boldsymbol{p}_3,\boldsymbol{p}_1)\rangle\!\rangle + \\
&+ \langle\!\langle T(\boldsymbol{p}_3)\Upsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)\rangle\!\rangle]\,, \qquad (4.3.114)
\end{aligned}
$$

$$\langle\!\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = 2\big[\langle\!\langle T(\boldsymbol{p}_1)\Upsilon^{(s_3)}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle + \langle\!\langle T(\boldsymbol{p}_2)\Upsilon^{(s_3)}(\boldsymbol{p}_1,\boldsymbol{p}_3)\rangle\!\rangle \\ + \langle\!\langle \Upsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle\big], \tag{4.3.115}$$

$$\langle\!\langle T(\boldsymbol{p}_1)T^{(s_2)}(\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = \frac{1}{2}\big(A(p_2)+A(p_3)\big)\theta^{(s_2s_3)}(\boldsymbol{p}_i) \\ + 2\left[\langle\!\langle T^{(s_2)}(\boldsymbol{p}_2)\Upsilon^{(s_3)}(\boldsymbol{p}_1,\boldsymbol{p}_3)\rangle\!\rangle + \langle\!\langle T^{(s_3)}(\boldsymbol{p}_3)\Upsilon^{(s_2)}(\boldsymbol{p}_1,\boldsymbol{p}_2)\rangle\!\rangle \right. \\ \left. + \langle\!\langle T(\boldsymbol{p}_1)\Upsilon^{(s_2s_3)}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle\right], \tag{4.3.116}$$

where $A(p)$ is the transverse traceless piece of the stress tensor 2-point function defined in (2.1.16) and the Υ symbols are defined in (2.A.117) - (2.A.119).

Further insight may be distilled from the trace Ward identities (4.3.111) - (4.3.116) by replacing the semi-local contact terms on the right hand side with 2-point functions of T_{ij} and $\mathcal{O}_\Delta$. On general grounds, the Υ tensor has an expansion in terms of local operators of dimension less than or equal to d, and for fermions and conformal scalars we computed this explicitly in (4.3.52) and (2.4.37) - (2.4.38). Then, as we found in the analysis leading to (4.3.57) and (4.3.71), only operators of dimension d contribute to the correlator $\langle T_{ij}\Upsilon_{klmn}\rangle$, permitting it to be expressed in terms of $\langle T_{ij}T_{kl}\rangle$. Substituting into (4.3.111) - (4.3.116) our previous results (4.3.58) and (4.3.72) for the semi-local terms $\langle T_{ij}\Upsilon_{klmn}\rangle$, we obtain

$$\langle\!\langle T_\chi(\boldsymbol{p}_1)T_\chi(\boldsymbol{p}_2)T_\chi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = \langle\!\langle T_\psi(\boldsymbol{p}_1)T_\psi(\boldsymbol{p}_2)T_\psi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = 0, \tag{4.3.117}$$

$$\langle\!\langle T_\chi(\boldsymbol{p}_1)T_\chi^{(s_2)}(\boldsymbol{p}_2)T_\chi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{4}\big(A_\chi(p_2)+A_\chi(p_3)\big)\theta^{(s_2s_3)}(p_i), \tag{4.3.118}$$

$$\langle\!\langle T_\psi(\boldsymbol{p}_1)T_\psi^{(s_2)}(\boldsymbol{p}_2)T_\psi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{1}{2}\big(A_\psi(p_2)+A_\psi(p_3)\big)\theta^{(s_2s_3)}(p_i). \tag{4.3.119}$$

Thus, all our earlier results in (4.3.44) - (4.3.49) involving the trace T_ψ are in fact a consequence of the Ward identities (noting also (4.3.50)), and similarly for all our results in (4.3.66) - (4.3.69) involving T_χ. For the latter, note that $A_\chi(p) = (\mathcal{N}_\chi/2\mathcal{N}_\psi)A_\psi(p)$ from (4.3.24), hence from (4.3.119) we have

$$\langle\!\langle T_\chi(\boldsymbol{p}_1)T_\chi^{(s_2)}(\boldsymbol{p}_2)T_\chi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle = \frac{\mathcal{N}_\chi}{4\mathcal{N}_\psi}\langle\!\langle T_\psi(\boldsymbol{p}_1)T_\psi^{(s_2)}(\boldsymbol{p}_2)T_\psi^{(s_3)}(\boldsymbol{p}_3)\rangle\!\rangle. \tag{4.3.120}$$

As well as confirming earlier calculations, these formulae additionally serve as a check of the overall sign in our 3-point function integrals.

To check the results of our 3-point function calculations for minimal scalars using (4.3.107), we must also evaluate the semi-local terms on the right hand sides of (4.3.111) - (4.3.116) involving the correlator $\langle\mathcal{O}_1\widetilde{\Upsilon}^\phi_{ijkl}\rangle$, where $\widetilde{\Upsilon}^\phi_{ijkl}$ denotes the Υ tensor for conformal scalars. The expansion for this latter quantity may be read off from (2.4.37). The correlator $\langle\mathcal{O}_1\widetilde{\Upsilon}^\phi_{ijkl}\rangle$ receives contributions only from terms of dimension one in this expansion, and so we find

$$\langle\!\langle \widetilde{\Upsilon}_\phi(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = \frac{1}{16} (a_{12}^2 - 2b_{12} - p_3^2) \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_3) \mathcal{O}_1(-\boldsymbol{p}_3) \rangle\!\rangle, \qquad (4.3.121)$$

$$\langle\!\langle \widetilde{\Upsilon}_\phi^{(s_2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = \frac{3}{32} p_3^2 \Theta_3^{(s_2)}(p_i) \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_3) \mathcal{O}_1(-\boldsymbol{p}_3) \rangle\!\rangle. \qquad (4.3.122)$$

Substituting these expressions into (4.3.111) - (4.3.116), we obtain

$$\langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1) \mathcal{O}_1(\boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = - \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_2) \mathcal{O}_1(-\boldsymbol{p}_2) \rangle\!\rangle - \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_3) \mathcal{O}_1(-\boldsymbol{p}_3) \rangle\!\rangle, \qquad (4.3.123)$$

$$\langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1) \widetilde{T}_\phi(\boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = \frac{1}{8} (a_{12}^2 - 2b_{12} - p_3^2) \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_3) \mathcal{O}_1(-\boldsymbol{p}_3) \rangle\!\rangle, \qquad (4.3.124)$$

$$\langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1) \widetilde{T}_\phi^{(s_2)}(\boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = \frac{3}{16} p_3^2 \Theta_3^{(s_2)}(p_i) \langle\!\langle \mathcal{O}_1(\boldsymbol{p}_3) \mathcal{O}_1(-\boldsymbol{p}_3) \rangle\!\rangle, \qquad (4.3.125)$$

$$\langle\!\langle \widetilde{T}_\phi(\boldsymbol{p}_1) \widetilde{T}_\phi(\boldsymbol{p}_2) \widetilde{T}_\phi(\boldsymbol{p}_3) \rangle\!\rangle = 0, \qquad (4.3.126)$$

where for the latter equation we used (4.3.71). The trace Ward identities thus supply all terms appearing on the right hand sides of (4.3.107) that involve the trace $\widetilde{T}_\phi$.

4.3.10. Conformal Ward identities

In the previous section we showed how the trace Ward identities determine the 3-point functions involving the trace of the stress tensor in terms of 2-point functions. Thus to determine all correlation functions, it remains to find

$$\langle T^{(s_1)} T^{(s_2)} T^{(s_3)} \rangle, \quad \langle \mathcal{O}_1 \widetilde{T}_\phi^{(s_2)} \widetilde{T}_\phi^{(s_3)} \rangle, \quad \langle \mathcal{O}_1 \mathcal{O}_1 \widetilde{T}_\phi^{(s_3)} \rangle, \quad \langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle, \quad \langle \mathcal{O}_1 \mathcal{O}_1 \rangle. \qquad (4.3.127)$$

In fact we have already computed all these correlation functions. Due to the secondary Ward identity (3.6.9) the correlation function $\langle T_{ij}^\chi \mathcal{O}_1 \mathcal{O}_1 \rangle$ is determined uniquely in terms of the 2-point function normalisation constant $c_{\mathcal{O}}$, which was evaluated in (2.4.39). The case of $\langle T_{ij}^\chi T_{kl}^\chi \mathcal{O}_1 \rangle$ was studied extensively throughout the entire chapter 2 of the thesis. The most general helicity projected result is given by (2.9.18) and the constants appearing there are given by (2.5.73). All these results can be checked by the exact calculations based on the Feynman rules. We find the exact agreement, which is an excellent check on our results of the chapter 2. For completeness, the helicity projected correlation functions (we suppress a common overall factor of $d_A \mathcal{N}_\phi$) are

$$\langle\!\langle \mathcal{O}_1(\boldsymbol{p}) \mathcal{O}_1(-\boldsymbol{p}) \rangle\!\rangle = \frac{1}{4p}, \qquad (4.3.128)$$

$$\langle\!\langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_1(\boldsymbol{p}_2) \mathcal{O}_1(\boldsymbol{p}_3) \rangle\!\rangle = \frac{1}{c_{123}}, \qquad (4.3.129)$$

$$\langle\!\langle \widetilde{T}_\phi^{(s_1)}(\boldsymbol{p}_1)\mathcal{O}_1(\boldsymbol{p}_2)\mathcal{O}_1(\boldsymbol{p}_3)\rangle\!\rangle = \frac{J^2}{16\sqrt{2}}\frac{(2p_1+a_{23})}{a_{123}^2 b_{23}\boldsymbol{p}_1^2}, \tag{4.3.130}$$

$$\langle\!\langle \widetilde{T}_\phi^{(+)}(\boldsymbol{p}_1)\widetilde{T}_\phi^{(-)}(\boldsymbol{p}_2)\mathcal{O}_1(\boldsymbol{p}_3)\rangle\!\rangle = \frac{1}{2048\, b_{12}^2 p_3}\,(p_3^2 - a_{12}^2 + 4b_{12})^2(a_{12}^2 + 2b_{12} - 5p_3^2), \tag{4.3.131}$$

$$\begin{aligned}\langle\!\langle \widetilde{T}_\phi^{(+)}(\boldsymbol{p}_1)\widetilde{T}_\phi^{(+)}(\boldsymbol{p}_2)\mathcal{O}_1(\boldsymbol{p}_3)\rangle\!\rangle &= \frac{(a_{12}-p_3)^2}{2048\, a_{123}^2 b_{12}^2 p_3}\,\big[-5p_3^6 - 20a_{12}p_3^5 \\ &- \ (29a_{12}^2 + 6b_{12})p_3^4 - 8a_{12}(2a_{12}^2 + 3b_{12})p_3^3 + a_{12}^2(a_{12}^2 - 36b_{12})p_3^2 \\ &+ 4a_{12}^3(a_{12}^2 - 6b_{12})p_3 + a_{12}^6 - 6a_{12}^4 b_{12} + 32b_{12}^3\big]\,.\end{aligned} \tag{4.3.132}$$

Substituting these expressions into (4.3.107), along with those in (4.3.123), we recover all the results for minimal scalars listed in (4.3.29) - (4.3.34) that involve the trace T_ϕ.

Part II

Holographic cosmology

Chapter 5

General relativity and cosmology

5.1. Basic tools

In this section we would like to discuss some basic tools which we will use in our analysis. These tools cover well-known, standard results in general relativity, see [70, 76, 77]. Our conventions follow [70], we will work both in Euclidean and mostly plus $(- + + \ldots +)$ Lorentzian signatures and

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\alpha\mu}, \tag{5.1.1}$$

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}. \tag{5.1.2}$$

5.1.1. ADM decomposition

The ADM formalism (Arnowitt, Deser, Misner) is a Hamiltonian approach to gravity. Consider a D dimensional manifold with Riemannian or Lorentzian structure g and a foliation Σ_z parametrised by z. In the Lorentzian case the foliation is assumed to be spacelike, *i.e.*, the metric induced on Σ_z is Riemannian. In such case z is interpreted as time. For that reason, both in the Lorentzian and Euclidean cases, we will denote the derivative with respect to the z coordinate by a dot.

The normal unit vector to Σ_z and the induced metric $\gamma_{\mu\nu}$ on Σ_z can be written as

$$n^\mu = \frac{\partial^\mu z}{|\partial^\mu z|_g}, \qquad \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \tag{5.1.3}$$

The tensor $\gamma^\mu_\nu(z)$ is a projector onto Σ_z and therefore $\gamma^\mu_\nu|_{\Sigma_z} = \delta^\mu_\nu$. Finally, we can introduce a vector field z^μ by $z^\mu \partial_\mu z = 1$. The lapse and shift functions N and N^μ

are defined as the normal and tangent components of z^μ to Σ_z,

$$Nn^\mu = (g_{\alpha\beta}z^\alpha n^\beta)n^\mu, \qquad N^\mu = \gamma^\mu_\alpha z^\alpha. \tag{5.1.4}$$

With all these definitions one can check that the following 1-forms

$$\mathrm{d}\hat{x}^\mu = \mathrm{d}x^\mu - (N^\mu + Nn^\mu)\mathrm{d}z \tag{5.1.5}$$

constitute a basis for the cotangent space $T^*\Sigma_z$. The full metric $g_{\mu\nu}$ can be then decomposed as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \sigma N^2\mathrm{d}z^2 + \gamma_{\mu\nu}(\mathrm{d}\hat{x}^\mu + N^\mu\mathrm{d}z)(\mathrm{d}\hat{x}^\nu + N^\nu\mathrm{d}z). \tag{5.1.6}$$

The parameter σ is a sign, $\sigma = \pm 1$, depending on the signature:

- $\sigma = +1$ for the Euclidean signature,
- $\sigma = -1$ for the mostly plus Lorentzian signature $(- + + \ldots +)$.

Each slice of the foliation Σ_z can be characterised by its intrinsic and extrinsic curvatures, respectively,

$$\hat{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}[\gamma_{\alpha\beta}], \tag{5.1.7}$$

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu} = \gamma^\alpha_\mu \nabla_\alpha n_\nu \\ &= \frac{1}{2N}\left(\dot{\gamma}_{\mu\nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu\right), \end{aligned} \tag{5.1.8}$$

where $\mathcal{L}_{\boldsymbol{n}}$ is the Lie derivative in the direction of the vector $\boldsymbol{n}$ and $\hat{\nabla}_\mu$ is a covariant derivative of the metric γ. One can show that $\hat{\nabla}_\mu$ is the projection of the full covariant derivative ∇_μ onto Σ_z. Now various projections of the full Riemann tensor $R_{\mu\nu\rho\sigma}$ on directions parallel and transverse to the foliation can be expressed in terms of the following *Gauss-Codazzi* equations,

$$\gamma^\alpha_\mu\gamma^\beta_\nu\gamma^\gamma_\rho\gamma^\delta_\sigma R_{\alpha\beta\gamma\delta} = \hat{R}_{\mu\nu\rho\sigma} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}, \tag{5.1.9}$$

$$\gamma^\alpha_\mu n^\beta R_{\alpha\beta} = \hat{\nabla}_\alpha K^\alpha_\mu - \hat{\nabla}_\mu K, \tag{5.1.10}$$

$$n^\alpha n^\beta R_{\mu\alpha\nu\beta} = -n^\alpha\nabla_\alpha K_{\mu\nu} - K_{\mu\alpha}K^\alpha{}_\nu, \tag{5.1.11}$$

where $K = K^\alpha_\alpha$. Finally, one can manipulate the Gauss-Codazzi equations in order to obtain the following form

$$K^2 - K_{\mu\nu}K^{\mu\nu} - \hat{R} = 2G_{\mu\nu}n^\mu n^\nu, \tag{5.1.12}$$

$$\hat{\nabla}_\alpha K^\alpha_\mu - \hat{\nabla}_\mu K = G_{\alpha\beta}\gamma^\alpha_\mu n^\beta, \tag{5.1.13}$$

$$\mathcal{L}_{\boldsymbol{n}}K_{\mu\nu} + KK_{\mu\nu} - 2K_\mu{}^\alpha K_{\alpha\nu} - \hat{R}_{\mu\nu} = -\gamma^\alpha_\mu\gamma^\beta_\nu R_{\alpha\beta}, \tag{5.1.14}$$

where $G_{\mu\nu}$ denotes Einstein tensor for the full metric $g_{\mu\nu}$. The right hand sides of these equations can be connected to the stress-energy tensor of matter via the Einstein equations

5.1.2. Action principle

In this thesis we will consider Einstein equations both in Euclidean and Lorentzian signatures. We will consider almost exclusively a single scalar coupled to matter, for which the action is

$$S = \frac{\sigma}{2\kappa^2} \int \mathrm{d}^D \boldsymbol{x} \sqrt{|g|} \left[-R + \partial_\mu \Phi \partial^\mu \Phi + 2V(\Phi) \right] \tag{5.1.15}$$

where $\kappa^2 = 8\pi G_D$, G_D is the D-dimensional Newton constant and $V(\Phi)$ is a potential for Φ. This action can be supplemented by the Gibbons-Hawking term if the boundary of the manifold is non-empty.

In this normalisation there is an overall κ^{-2} factor in front of the entire action. This is a natural normalisation from the string theory point of view as we will discuss in section 6.1.1. Another convention commonly used is a cosmological convention, where $1/(2\kappa^2)$ multiplies the Ricci scalar only. It can be obtained from (5.1.15) by a redefinition of the scalar field as $\Phi \mapsto \kappa\Phi$ together with the redefinition of the coupling constants in the potential. Finally, in section 6.6 we will use yet another convention, where we redefine the potential in (5.1.15) according to $V(\Phi) \mapsto \kappa^2 V(\Phi)$.

The equations of motion following from (5.1.15) are

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \qquad \Box\Phi - V'(\Phi) = 0, \tag{5.1.16}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \qquad \kappa^2 T_{\mu\nu} = \frac{2\sigma}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} S_{\text{matter}}. \tag{5.1.17}$$

The sign is chosen such that in both Euclidean and Lorentzian signatures we obtain the same stress-energy tensor. In case of (5.1.15),

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi + 2V(\phi) \right]. \tag{5.1.18}$$

Let us now return to the ADM formalism. Using Gauss-Codazzi equations one finds that the action (5.1.15) takes form

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int \mathrm{d}^D \boldsymbol{x} \sqrt{\gamma} N \bigg[& K_{\mu\nu} K^{\mu\nu} - K^2 + \frac{1}{N^2} \left(\dot{\Phi} - N^\mu \partial_\mu \Phi \right)^2 \\ & + \sigma \left(-\hat{R} + \gamma^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2V(\Phi) \right) \bigg]. \end{aligned} \tag{5.1.19}$$

Using (5.1.8) one can rewrite the action purely in terms of N, N^μ and $\gamma_{\mu\nu}$. Since N and N^μ have no dynamics, the Hamilton equations are *constraint equations*

$$\frac{\delta S}{\delta N} = 0, \qquad \frac{\delta S}{\delta N^\mu} = 0 \tag{5.1.20}$$

known as *Hamiltonian* and *momentum constraints* respectively. The canonical momenta for the metric and the scalar field are

$$\Pi_{\mu\nu} = \frac{-2}{\sqrt{\gamma}} \frac{\delta}{\delta\dot{\gamma}^{\mu\nu}}(\kappa^2 L) = K_{\mu\nu} - \gamma_{\mu\nu} K, \tag{5.1.21}$$

$$\Pi_\Phi = \frac{1}{\sqrt{\gamma}} \frac{\delta L}{\delta\dot{\Phi}}(\kappa^2 L) = \frac{1}{N^2}\dot{\Phi} \tag{5.1.22}$$

and the Hamilton equations are

$$\dot{\Pi}_{\mu\nu} = -\frac{-2}{\sqrt{\gamma}} \frac{\delta}{\delta\gamma^{\mu\nu}}(\kappa^2 L), \qquad \dot{\Pi}_\Phi = -\frac{1}{\sqrt{\gamma}} \frac{\delta}{\delta\Phi}(\kappa^2 L). \tag{5.1.23}$$

All these equations result in (5.1.12) - (5.1.14) with right hand sides expressed via the Einstein equations.

In section 6.6 we will use the gauge choice where $N = 1$ and $N^\mu = 0$, so the metric (5.1.6) takes form

$$\mathrm{d}s^2 = \sigma \mathrm{d}z^2 + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{5.1.24}$$

In this case direction z is perpendicular to the foliation at each point, therefore we may assume that the Latin indices take values $i = 1, 2, \ldots, D-1$ while Greek indices take values $\mu = z, 1, 2, \ldots, D-1$. In this gauge the extrinsic curvature (5.1.8) is

$$K_{ij} = \frac{1}{2}\dot{\gamma}_{ij} \tag{5.1.25}$$

and the Gauss-Codazzi equations read

$$K^2 - K_{ij}K^{ij} - R = T_{zz}, \tag{5.1.26}$$

$$\nabla_i K^i_j - \nabla_j K = \frac{1}{2} T_{jz}, \tag{5.1.27}$$

$$\dot{K}^i_j + K K^i_j - R^i_j - \frac{1}{2}\left(T^i_j - \frac{1}{d-1}\delta^i_j T\right), \tag{5.1.28}$$

where $T = T^\mu_\mu$ is the trace of the stress-energy tensor and $\dot{K}^i_j = \partial_z(\gamma^{ik}K_{kj})$.

5.1.3. Vacuum solutions

An important class of spacetimes are solutions with the cosmological constant Λ. The action (5.1.15) takes form

$$S = \frac{\sigma}{2\kappa^2} \int \mathrm{d}^D \boldsymbol{x} \sqrt{|g|}\left[-R + 2\Lambda\right], \tag{5.1.29}$$

The homogeneous and isotropic solutions to the Einstein equations can be found as follows. Let $\mathbb{R} \times S^D$ be a compactification of the usual Minkowski space $\mathbb{R}^{D+1}$ with coordinates $X_0, X_1, \dots X_D$ and the standard metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 + \sum_{j=1}^{D} \mathrm{d}X_j^2. \tag{5.1.30}$$

Euclidean *anti-de Sitter space* (EAdS, AdS), known also as the *hyperbolic space*, is defined as a Riemannian submanifold of $\mathbb{R}^{D+1}$ satisfying

$$-X_0^2 + \sum_{j=1}^{D} X_j^2 = -L_{\mathrm{AdS}}^2, \tag{5.1.31}$$

with the induced metric. The parameter $L_{\mathrm{AdS}} > 0$ is the *radius* of the AdS space. This version of AdS is called Euclidean, since the induced metric is positive definite. To see it notice that by definition $|X_0| \geqslant L_{\mathrm{AdS}}$ so one can introduce the coordinates

$$X_0 = L_{\mathrm{AdS}} \cosh u, \qquad X_j = L_{\mathrm{AdS}} \sinh u \, \boldsymbol{e}_j, \quad j = 1, \dots, D, \tag{5.1.32}$$

where $\boldsymbol{e}_j$ are the standard coordinates for the unit sphere $S^{D-1} \subseteq \mathbb{R}^D$. These coordinates cover the entire AdS space and the induced metric is

$$\mathrm{d}s^2 = L_{\mathrm{AdS}}^2 \left[\mathrm{d}u^2 + \sinh^2 u \, \mathrm{d}\Omega_{D-1}^2 \right], \tag{5.1.33}$$

which is Euclidean indeed.

Due to the fact that $|X_0| \geqslant L_{\mathrm{AdS}}$, the manifold has two identical components as in figure 5.1. By Euclidean AdS we mean a single component with $X_0 > 0$. In a similar fashion one can define a Lorentzian version of AdS by starting from space with two time-like directions. Such a Lorentzian AdS is a connected space and its relation to the Euclidean AdS is similar to the relation between Schwarzschield and Euclidean Schwartzschield geometry. In this thesis we will concentrate on the Euclidean version of AdS, but interesting phenomena can arise when the Lorentzian dynamics is considered.

Let us define two more coordinate systems that will be useful in the upcoming analysis. Consider coordinates on AdS denoted by $(z, \boldsymbol{x})$, where $\boldsymbol{x}$ is a $(D-1)$ dimensional vector and $z \in (0, \infty)$. The coordinates are defined as

$$X_0 = \frac{L_{\mathrm{AdS}}}{2z}(1 + x^2 + z^2), \tag{5.1.34}$$

$$X_j = \frac{L_{\mathrm{AdS}} x_j}{z}, \quad j = 1, \dots, D-1, \tag{5.1.35}$$

$$X_D = \frac{L_{\mathrm{AdS}}}{2z}(1 - x^2 - z^2). \tag{5.1.36}$$

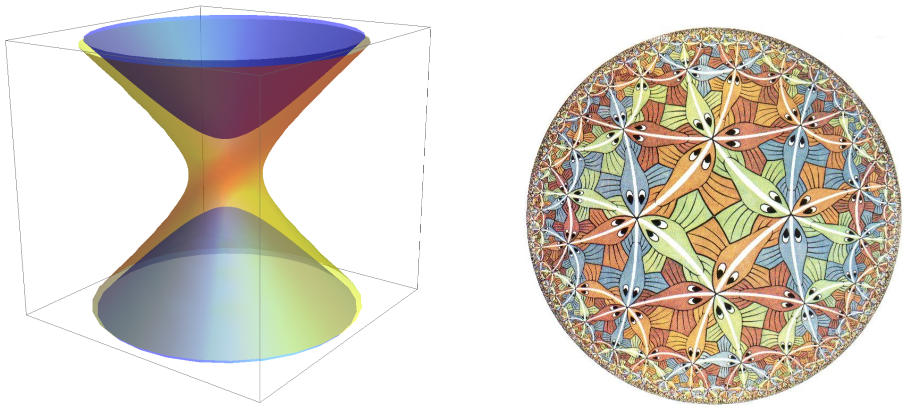


Figure 5.1: *On the left: two dimensional hyperbolic (AdS) space (blue) and de Sitter space (yellow). Note, however, that the metric on the dS space should be Lorentzian, not Euclidean as suggested by the plot. On the right: a model of the hyperbolic space as a Poincaré disc. The boundary of the disc is infinitely far away from its center. (M.C.Escher, 'Circle Limit III')*

These coordinates do not cover the entire AdS space, since $X_0+X_D = L_{\mathrm{AdS}}/z > 0$. However, the induced metric is particularly simple,

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}\boldsymbol{x}^2\right]. \tag{5.1.37}$$

Yet another useful coordinate system we are going to use in cosmology can be obtained by the substitution

$$z = e^{Hr}, \qquad \boldsymbol{y} = L_{\mathrm{AdS}}\boldsymbol{x}, \qquad H = L_{\mathrm{AdS}}^{-1}, \tag{5.1.38}$$

which leads to the metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + e^{-2Hr}\mathrm{d}\boldsymbol{y}^2. \tag{5.1.39}$$

Lorentzian *de Sitter* (dS) space can be defined as a Lorentzian submanifold of the Minkowski space $\mathbb{R}^{D+1}$ satisfying

$$-X_0^2 + \sum_{j=1}^{D} X_j^2 = L_{\mathrm{dS}}^2, \tag{5.1.40}$$

with the induced metric. In this case the substitution

$$X_0 = L_{\mathrm{dS}}\sinh u, \qquad X_j = L_{\mathrm{dS}}\cosh u\, \boldsymbol{e}_j, \quad j = 1,\ldots,D, \tag{5.1.41}$$

leads to the Lorentzian metric

$$\mathrm{d}s^2 = L_{\mathrm{dS}}^2\left[-\mathrm{d}u^2 + \cosh^2 u\, \mathrm{d}\Omega_{D-1}^2\right]. \tag{5.1.42}$$

Essentially the same substitutions as for the AdS metric with some signs changed lead to the following metrics on the dS spacetime,

$$\mathrm{d}s^2 = \frac{L_{\mathrm{dS}}^2}{z^2}\left[-\mathrm{d}z^2 + \mathrm{d}\boldsymbol{x}^2\right], \tag{5.1.43}$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\mathrm{d}\boldsymbol{y}^2, \qquad H = L_{\mathrm{dS}}^{-1}. \tag{5.1.44}$$

As we can see the Euclidean AdS space and Lorentzian dS space can be related by some analytic continuations of coordinates or radii. In some sense we can think about the dS space as a space with the positive square of the radius $L^2 > 0$ and the AdS space as the same space with $L^2 < 0$. Indeed, both de Sitter and anti-de Sitter space are homogeneous spaces,

$$\mathrm{AdS}_D \equiv \frac{SO(D,1)}{SO(D)}, \qquad \mathrm{dS}_D \equiv \frac{SO(D,1)}{SO(D-1,1)} \tag{5.1.45}$$

and hence they admit metrics with a constant curvature. By direct calculations one shows that all metrics discussed have the constant Ricci scalar,

$$R = \frac{D(D-1)}{L^2}, \tag{5.1.46}$$

where $L^2 = L_{\mathrm{dS}}^2$ for de Sitter space and $L^2 = -L_{\mathrm{AdS}}^2$ for anti-de Sitter space. This means that the AdS space solves vacuum Einstein equations with a negative cosmological constant while the dS space solves them with a positive one,

$$\Lambda = \frac{(D-1)(D-2)}{2L^2}. \tag{5.1.47}$$

In particular the full Riemann and Ricci tensors can be rewritten in terms of the metrics as

$$R_{\mu\nu\rho\sigma} = \frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{5.1.48}$$

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha} = \frac{D-1}{L^2}g_{\mu\nu}. \tag{5.1.49}$$

Both de Sitter and anti-de Sitter spaces have a boundary. In the original coordinates the boundary occurs for $|X_0| \to \infty$, which corresponds to $z \to 0$ in (5.1.43, 5.1.37). The boundary of dS space is spacelike, but the boundary of (Lorentizan) AdS is timelike. Although the boundary is infinitely far away from the interior of AdS, in has a peculiar property that – in the Lorentzian version – light reaches the boundary in a finite global time. Therefore the Lorentzian AdS space is not hyperbolic. This means that, in order to have an unambiguous evolution, initial conditions must be supplied by a specification of boundary conditions.

Since both de Sitter and anti-de Sitter space are quocient spaces of the Lie group $SO(D,1)$, their symmetry group is precisely $SO(D,1)$. For the AdS space in coordinates (5.1.37), the subgroup isomorphic to the Poincaré group $P_D = SO(D) \times \mathbb{R}^D \subseteq SO(1,D)$ acts as rotations and translations

$$x^\mu \mapsto \Lambda^\mu_\alpha x^\alpha + a^\mu, \tag{5.1.50}$$

for a matrix of rotation Λ in $\mathbb{R}^D$ and a vector $\boldsymbol{a} \in \mathbb{R}^D$, leaving the $z = 0$ plane invariant. The scalings by λ act as

$$z \mapsto \lambda z, \qquad x^\mu \mapsto \lambda x^\mu \tag{5.1.51}$$

and the inversions,

$$z \mapsto \frac{z}{z^2 + \vec{x}^2}, \qquad x^\mu \mapsto \frac{x^\mu}{z^2 + \vec{x}^2}. \tag{5.1.52}$$

These transformations generate the conformal group $SO(D,1)$ and they map the boundary at $z = 0$ to itself. Furthermore at the boundary these transformations reduce to conformal transformations (1.1.12).

For more details on the geometry of dS and AdS spaces see [78, 79]. As a side comment notice that if one started with the de Sitter space construction (5.1.40) in Euclidean rather than Lorentzian space with the positive sign in front of X_0^2, one would obtain the usual sphere S^D of radius L_{dS}.

5.2. From cosmology to inflation

5.2.1. The shape of the Universe

Einstein equations of relativity provide a unique opportunity to investigate the history and the dynamics of our entire Universe. In the year 1927 Georges Lemaître theoretised the possibility that our Universe can expand or contract with time [80]. The prediction of the expanding Universe was confirmed by Edwin Hubble in 1929 [81], based on observations of distant galaxies. Hubble found that there exists a correlation between the distance R to a galaxy and its velocity v,

$$v = H_0 R, \tag{5.2.1}$$

where H_0 is a constant known as the *Hubble constant*. The Hubble constant changes with time and its value today is still quite uncertain. The direct observations of distant galaxies yield values,

$$H_0^{\mathrm{HST}} = 73.8 \pm 2.4 \ (\mathrm{km/s})/\mathrm{Mpc}, \tag{5.2.2}$$

$$H_0^{\mathrm{6dF}} = 67.0 \pm 3.2 \ (\mathrm{km/s})/\mathrm{Mpc}, \tag{5.2.3}$$

where pc stands for a parsec, $1\mathrm{pc} \approx 30.86 \cdot 10^{15}\,\mathrm{m}$. The H_0^{HST} value was obtained by observation via the Hubble Space Telescope [82], while H_0^{6dF} follows from the 6dF

Galaxy Survey [83]. On the other hand, the values of Hubble constant obtained by the satellites WMAP [84] and Planck [85] are respectively,

$$H_0^{\mathrm{WMAP9}} = 69.32 \pm 0.80\ (\mathrm{km/s})/\mathrm{Mpc}, \tag{5.2.4}$$

$$H_0^{\mathrm{Planck}} = 67.80 \pm 0.77\ (\mathrm{km/s})/\mathrm{Mpc}. \tag{5.2.5}$$

These measurements have much smaller uncertainty, but they are not direct. The value of the Hubble constant is obtained from the measurements of the Cosmic Microwave Background by the fit to the ΛCDM model, which we will discuss in section 5.6. The direct measurements do not depend on the underlying model, but have much bigger uncertainty. For a comparison of various measurements of the Hubble constant, see figure 5.2.

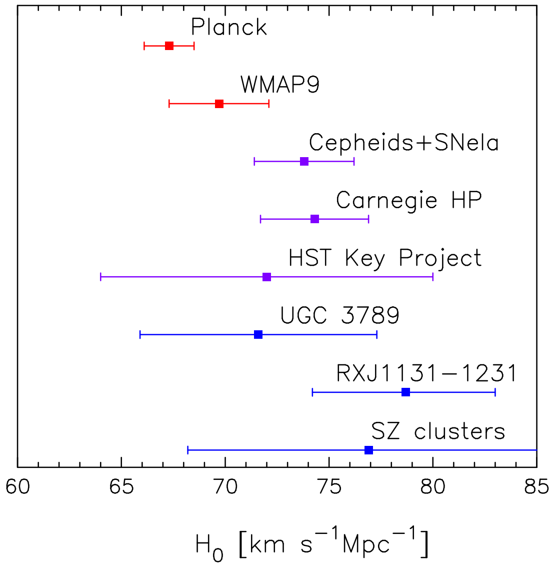


Figure 5.2: *The values of the Hubble constant today measured by various experiments, from top to bottom: [85, 84, 82, 86, 87, 88, 89, 90]. The first two measurements are indirect results based on the measurements of the Cosmic Microwave Background. The next three values are obtained by direct observations of velocities of distant galaxies. The remaining three measurements are based on geometrical methods such as gravitational lensing and redshift. Source: [91].*

One could be concerned that the value of the Hubble constant depends both on position and direction of the observation in spacetime. However, all observations mentioned above show that this is not the case and our Universe is homogeneous and isotropic on cosmological scales. This means that on average the density of matter is constant at any point and at any direction. Obviously, there exist regions with large matter densities due to the gravitational interactions, such as galaxies, planets or black holes, but on scales much larger than the size of a galaxy the Universe looks extremely homogeneous and isotropic.

With the assumption of homogeneity and isotropy we can build a basic but – as it turns out – very realistic model of our Universe. As references, consult standard textbooks, *e.g.*, [70, 77, 92, 93]. The isotropy of the Universe requires that the metric locally takes form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[e^{2B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\right], \tag{5.2.6}$$

where $\mathrm{d}\Omega_2^2$ is a volume element of the unit sphere and $a(t)$ and $B(r)$ are arbitrary functions. The homogeneity requires that the Ricci scalar is constant in space, which leads to the *Friedman-Robertson-Walker (FRW) metric*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega_2^2\right]. \tag{5.2.7}$$

The constant k is related to the spatial curvature of a single time slice at $t = 0$. It can be assumed that $k \in \{-1, 0, +1\}$, as one can always rescale r appropriately.

The two parameters in the FRW metric: $a(t)$ and k depend on the content of the Universe. The *scale factor* $a(t)$ determines the physical distances in space and its dependence on time describes the velocity of the expansion or contraction. If we choose two points with coordinates $(t, \boldsymbol{x}_j)$, $j = 1, 2$, then their physical distance on the slice of constant time at t is $R = a(t)d(\boldsymbol{x}_1, \boldsymbol{x}_2)$, where d is a geodesic distance in the slice. Therefore the rate of change of the distance R between the two points is

$$v = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{R}{a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{5.2.8}$$

By comparing with empirical law (5.2.1) we have

$$H = \frac{\dot{a}}{a}, \tag{5.2.9}$$

where $\dot{a}$ denotes the derivative of $a(t)$ with respect to t. Note that H changes with time. The values of various parameters in our Universe today will be denoted by superscript 0, *e.g.*, H_0.

The constant k in the metric (5.2.7) determines spatial geometry of the Universe,

1. If $k = 0$ then the FRW metric reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\right]. \tag{5.2.10}$$

   The term in brackets is a flat space metric, so spatial slices are flat. Such a case is known as the *flat universe* and all observations point to the conclusion, that our Universe is flat.

2. If $k = 1$ then by the substitution $r = \sin\phi$ we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\phi^2 + \sin^2\phi\,\mathrm{d}\Omega_2^2\right]. \tag{5.2.11}$$

The term in brackets is a metric on the unit sphere S^3. Such a universe, called *closed*, has spheres as spatial slices with radius changing in time according to $a(t)$.

3. If $k = -1$ then by the substitution $r = \sinh u$ we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}u^2 + \sinh^2 u\,\mathrm{d}\Omega_2^2\right]. \tag{5.2.12}$$

The term in brackets is the metric on the 3-dimensional hyperbolic (EAdS) space we found in (5.1.33). Topologically, the constant time sections are diffeomorphic with $\mathbb{R}^3$ and such a model of a universe is called *open*.

5.2.2. Dynamics

With the most general homogeneous and isotropic Universe described by the FRW metric (5.2.7), we can now analyse the dynamics of the scale factor $a(t)$ described by the Einstein equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{5.2.13}$$

where $G_{\mu\nu}$ is the Einstein tensor for the FRW metric (5.2.7). We use supergravity conventions where the Newton constant multiplies the entire action (5.1.15).

In this section we will consider a simple model where the stress-energy tensor is given *a priori*. In particular we give no action principle for matter fields and we simply assume that the stress-energy tensor is that of the perfect isotropic fluid,

$$\kappa^2 T^\mu_{\ \nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{5.2.14}$$

where ρ and p are density and pressure of matter. We may assume in this section that the contribution from the cosmological constant Λ is incorporated into the stress-energy tensor.

Each form of matter or energy satisfies some equation of state, $F(\rho, p) = 0$. We will assume a simple form of this equation,

$$p = w\rho \tag{5.2.15}$$

where w is a number. The most typical values of w are listed in the following table,

Type of matter	w parameter
Relativistic particles: radiation	$w = \frac{1}{3}$
Non-relativistic particles: matter	$w = 0$
Cosmological constant: dark energy	$w = -1$

Furthermore, the stress-energy tensor must satisfy the conservation equation $\nabla_\mu T^\mu_\nu = 0$. By evaluating this on the FRW background the equation following from $\nu = 0$ component reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{5.2.16}$$

where dot denotes a derivative with respect to t. By solving this equation we find the dependence between the energy density and the scale factor for all types of matter described by the equation of state (5.2.15). We find

$$\rho(t) = \rho_0\, a(t)^{-3(1+w)} \tag{5.2.17}$$

for some integration constant ρ_0.

Finally, we can insert the FRW metric (5.2.7) to the Einstein equations (5.2.13) and derive the equations for the evolution of the scale factor, known as *Friedman equations*,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}, \tag{5.2.18}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p). \tag{5.2.19}$$

In fact these two equations are not independent, as they are related by Bianchi identities. Before we substitute the information on the matter content delivered by the equations (5.2.15) and (5.2.17) let us describe some general features of the Friedman equations. Since the left hand side of (5.2.18) is positive, if the universe is dominated by the negative cosmological constant giving $\rho < 0$ as follows from (5.1.18), one necessarily needs $k = -1$. On the other hand if $\rho > 0$ and $p > 0$ equation (5.2.19) leads to the conclusion that $\ddot{a} < 0$, which means the expansion of the Universe slows down. Thus not earlier than time

$$\tau_H \leqslant \frac{a}{\dot{a}} = H^{-1} \tag{5.2.20}$$

ago we would have $a = 0$. This means that the isotropic and homogeneous universe with positive energy density has an *initial singularity* or the *Big Bang.* A finite time ago all distances between all points were zero. We can assume that this initial singularity happens at $t = 0$. A distance from the initial singularity that a null particle can travel in time t is then

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a'}{a'}\frac{1}{a'H(a')}. \tag{5.2.21}$$

This is a maximal distance that can be seen at time t after the initial singularity and is called the *Hubble horizon*.

The Friedman equations can be solved exactly for the matter types (5.2.15). Exact solutions can be found in [70], and for the universe filled with matter or radiation the following conclusions apply:

1. If $k = 1$ then at some finite time $\dot{a} = 0$ and after that $\dot{a} < 0$. This means the the universe collapses to $a = 0$ at some finite time in the future.

2. If $k = 0$ then the universe expands forever, but $\lim_{t\to\infty} \dot{a}(t) = 0$.

3. If $k = -1$ then the universe expands forever and $\lim_{t\to\infty} \dot{a}(t) > 0$.

The solutions for $k = 0$ are particularly simple and read

$$a(t) = a_0 t^{\frac{2}{3(1+w)}}, \tag{5.2.22}$$

where a_0 is some constant. This turns out to be the physical case, since all observations as well as the theory of inflation suggest that our Universe is extremely close to be flat and we can set $k = 0$ henceforth.

A determination of the value of k in our Universe can be achieved as follows. Define

$$\Omega = \frac{\rho}{3H^2}, \qquad \rho_{\mathrm{crit}} = 3H_0^2 \tag{5.2.23}$$

and rewrite (5.2.18) as

$$\Omega - 1 = \frac{k}{a^2H^2}, \qquad \Omega = \frac{\rho}{\rho_{crit}}. \tag{5.2.24}$$

Therefore $k = 0$ ($k < 0$, $k > 0$) only if $\Omega = 1$ ($\Omega < 1$, $\Omega > 1$), which corresponds to $\rho = \rho_{\mathrm{crit}}$ ($\rho < \rho_{\mathrm{crit}}$, $\rho > \rho_{\mathrm{crit}}$). The measurements of the density of the Universe [84, 91, 94, 95, 96] lead to the conclusion that today $\rho \cong \rho_{\mathrm{crit}}$ and therefore we will consider models with $k = 0$ only. Note, however, that the value of Ω depends on time and if it is not exactly equal to one, the geometry of the Universe will be more and more curved in time, see figure 5.3

5.2.3. Need for inflation

The standard Big Bang cosmology is a very successful model of the cosmological evolution, confirmed by hundreds of measurements and observations. However, it presents a few problems, which require an explanation:

1. **Horizon problem.** Due to the expansion of the Universe, two distant regions separated by an angle of about $1°$ in the sky should have never been in a causal contact with one another. This means that one should expect to see

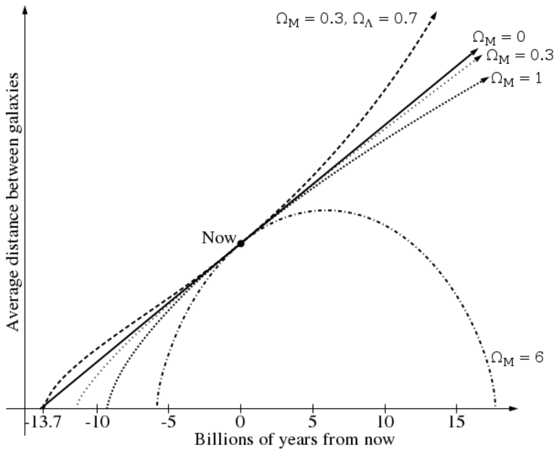


Figure 5.3: *The graph shows the evolution of the scale factor $a(t)$ depending on the content of the universe. Since our Universe is dominated by dark energy and matter, one can write $\Omega = \Omega_m + \Omega_\Lambda$ where Ω_m is a density of matter with $w = 0$ in the equation of state* (5.2.15) *and Ω_Λ is a density of dark energy with $w = -1$. In our Universe $\Omega_m \sim 0.3$, $\Omega_\Lambda \sim 0.7$, as we will discuss in section 5.6.*

about one million causally disconnected regions in the sky. If so, it is a very disturbing fact that all the Universe looks so homogeneous. Without interactions, the causally disconnected domains should not equilibrate, leading to variable formations in each region. While it is theoretically possible that the initial conditions were extremely homogeneous, it does not seem very probable.

2. **Flatness problem.** The radius of the constant time slices at time t of the FRW metric is

$$R = \frac{H^{-1}}{\sqrt{|\Omega - 1|}}. \tag{5.2.25}$$

For $k = 0$ one finds $R = \infty$, while for $k = \pm 1$ we find the appropriate radius of the sphere or the hyperbolic space $R = a^{-1}$. By looking at (5.2.24), one finds that the radius of curvature always grows in time. Since the Universe we observe today is very close to be flat, this implies that the initial conditions were extremely fine-tuned. Again, since this seems very improbable, either we should find a reason why $R = \infty$ exactly or explain why the curvature was extremely small at the very early Universe.

3. **Entropy problem.** The flatness problem can be reiterated in terms of the entropy problem. Since for relativistic particles the entropy $S \sim a^3 T^3$, one

can show that the small initial curvature implies small entropy of the Universe. Since the evolution of the Universe is approximately adiabatic, and the observed entropy now is huge, one needs a mechanism that generates a large amounts of entropy at the very early Universe.

4. **Magnetic monopoles.** Many fundamental theories such as string theory requires an existence of magnetic monopoles. Since these were not observed [97, 98], there should exist a mechanism that dilutes them to the densities small enough for them not to be detected.

5. **Spectrum of the Cosmic Microwave Background.** The whole Universe is filled with an electromagnetic radiation known as the *Cosmic Microwave Background* (CMB). Interestingly, the CMB was detected by accident [99], and since then it became the most important prediction of the Big Bang theory, since it delivers quantitative data. The CMB radiation is almost thermal with temperature today $T = 2.725\,\mathrm{K}$, but the small fluctuations of order $\Delta T/T \approx 10^{-5}$ carry an important imprint of the inflationary era. We will discuss the properties of the CMB in section 5.6.

All the problems listed above would be solved, if there was a phase in the very early Universe when

$$\dot{a} > 0, \qquad \ddot{a} > 0. \tag{5.2.26}$$

Such a phase is called *inflation*, since the Universe is 'inflated': all distances in spacetime grow very rapidly. For the inflation to take place, equation (5.2.19) implies that

$$p < -\frac{\rho}{3}. \tag{5.2.27}$$

This condition then can be satisfied by an addition of a positive cosmological constant that has $p = -\rho$. In next section we will find a particular example of matter that satisfies this peculiar condition.

It turns out that the inflation in principle solves all the five problems listed above. Particular quantitative analysis requires a specific model, but the generic features are as follows

1. **Horizon problem.** The horizon problem would be solved, if the Hubble horizon had shrunk during the inflation, so that two points that were in a causal contact before the inflation would become causally disconnected afterwards, see figure 5.4. For this to happen it is enough that in equation (5.2.21),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{aH} < 0 \quad \Longleftrightarrow \quad \ddot{a} > 0. \tag{5.2.28}$$

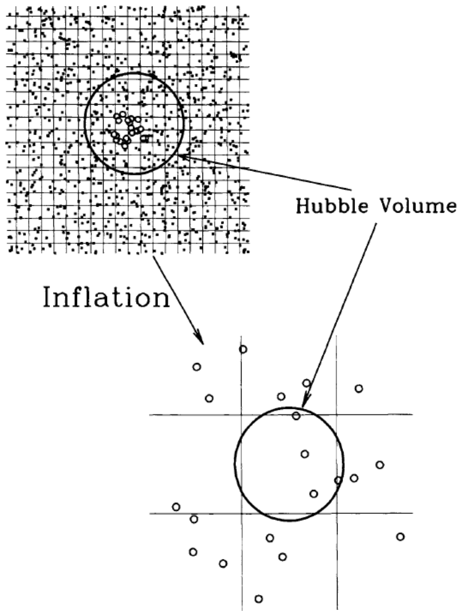


Figure 5.4: *Idea of inflation: the spacetime geometry grows exponentially fast, so that the objects in a causal constant before the inflation are separated by huge distances afterwards. Source: [92].*

2. **Flatness problem.** Looking at the equation (5.2.24) we can write

$$\frac{|\Omega - 1|_{\text{end}}}{|\Omega - 1|_{\text{start}}} = \left(\frac{\dot{a}_{\text{start}}}{\dot{a}_{\text{end}}}\right)^2 . \tag{5.2.29}$$

   During the inflationary stage $\dot{a}$ grows very rapidly, usually exponentially fast. Therefore the size of the spatial curvature at the end of the inflation is many orders of magnitudes smaller than at the beginning. Therefore, even if $k \neq 0$ in our Universe, after the inflation $|\Omega - 1|$ is extremely small.

3. **Entropy problem.** The inflation itself is an adiabatic process, but at some point the Universe should exhibit a phase transition to the radiation dominated phase. During such a transition the energy stored previously in highly energetic degrees of freedom that driven the inflation must be transferred into low energy degrees of freedom and the production of a huge number of particles takes place. This period of the history of our Universe is known as *reheating*. We will not discuss the entropy problem in detail, but one can estimate that the entropy produced in this way would be sufficient to explain the entropy of the Universe today.

4. **Magnetic monopoles.** As we are not focusing on this problem, let us only mention that the magnetic monopoles problem would be solved if the energy required for their creation was sufficiently large. Then, the magnetic monopoles would have not been created during or after the inflation, which would have necessarily diluted their number present at the pre-inflationary universe.

5. **Spectrum of the Cosmic Microwave Background.** Today, this is the most robust argument for the inflation, since the Cosmic Microwave Background (CMB) delivers a quantitative checks on inflationary models. We will discuss the features of the CMB in the following sections.

5.3. Inflation

In the previous section we discussed why it is strongly believed that the very early Universe underwent a phase known as the inflation. In this section we will build some basic models and analyse their phenomenology. The simplest model is based on the observation that a universe dominated by the positive dark energy expands exponentially. This simple fact will resolve the horizon problem, the flatness problem and the magnetic monopoles problem. Since it is the Cosmic Microwave Background (CMB) that delivers the most interesting information about inflation, various models will differ by the characteristics of the predicted spectrum of the CMB. Therefore most of the remaining part of this chapter will be devoted to quite non-trivial calculations of the features of the CMB. The fundamentals of the inflationary theory is covered in standard textbooks, *e.g.*, [93, 92, 100, 101, 102, 103].

5.3.1. Pure dark energy

The most primitive model of inflation is an expansion of the universe due to a positive cosmological constant. In such case the Friedman equations (5.2.18) and (5.2.19) with $k = 0$ lead to the de Sitter solution discussed in section 5.1.3 with the scale parameter

$$a(t) = a_e e^{H(t-t_e)}, \tag{5.3.1}$$

where $H = \dot{a}/a$ is a constant Hubble parameter and a_e and t_e are arbitrary constants. t_e is interpreted as the time when the inflation ends and a_e is the scale factor at t_e. In such a model the scale factor grows exponentially fast,

$$\frac{a_e}{a_b} = e^{H(t_e-t_b)}, \tag{5.3.2}$$

where a_b denotes the scale factor at the beginning of the inflation. One can show that this model solves the horizon and flatness problems if the inflation took place sometime between 10^{-36} and 10^{-32} seconds after the Big Bang. During the inflation the Universe grew by the factor of at least 10^{78}.

The main drawback of this simple model is the absence of any other degrees of freedom. Therefore the inflation never ends and the Standard Model particles are not produced. In particular there is no Cosmic Microwave Background. On the other hand, one can expect that due to the extreme dilution, the magnetic monopoles problem could be solved, if the monopoles were heavy enough.

5.3.2. Introducing inflaton

A more realistic model than the dark energy model requires a single scalar field Φ called *inflaton* with the potential energy dominating over the kinetic energy. We will consider the Lorentzian action in the supergravity normalisation,

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\sqrt{|g|}\,\left[R - \partial_\mu\Phi\partial^\mu\Phi - 2V(\Phi)\right]. \tag{5.3.3}$$

Let us now find a *background solution*, *i.e.*, the solution consistent with the flat FRW cosmology,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2, \qquad\qquad \Phi(t,\boldsymbol{x}) = \phi(t), \tag{5.3.4}$$

depending on time t only. The derivatives with respect to t will be denoted by a dot. The *Friedman equations* following from (5.1.16) are

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{2}{(D-1)(D-2)}\left[\frac{1}{2}\dot\phi^2 + V(\phi)\right], \tag{5.3.5}$$

$$\frac{\ddot a}{a} = -\frac{2}{D-1}\left[\frac{1}{2}\dot\phi^2 - \frac{1}{D-2}V(\phi)\right], \tag{5.3.6}$$

$$0 = \ddot\phi + (D-1)\frac{\dot a}{a}\dot\phi + V'(\phi). \tag{5.3.7}$$

As previously, the second equation follows from the first one when the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ are used. Moreover, by comparing both equations we find

$$\dot H = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot a}{a} = -\frac{1}{D-2}\dot\phi^2. \tag{5.3.8}$$

The right hand side of this equality is non-positive, which leads to the conclusion that H is a non-increasing function of time. This means that $\dot\phi$ can change sign only at times t_0 such that $\dot H(t_0) = 0$, therefore ϕ is a piecewise monotonic function on intervals where $\dot H < 0$. Thus we can invert $\phi(t)$ and express H in terms of the field through the function W defined as

$$H(t) = -\frac{1}{D-2}W(\phi(t)). \tag{5.3.9}$$

Equation (5.3.8) leads to

$$W'(\phi) = \dot{\phi}. \tag{5.3.10}$$

Finally, one can substitute these results to (5.3.7) to find that the potential V can be expressed in terms of W as

$$V(\phi) = -\frac{1}{2}W'^2(\phi) + \frac{D-1}{2(D-2)}W^2(\phi). \tag{5.3.11}$$

All together, we have shown that the equations of motion (5.3.5) - (5.3.7) for the scalar coupled to the gravity (5.3.3) can be reduced to the set of first order equations,

$$H = \frac{\dot{a}}{a} = -\frac{1}{D-2}W(\phi), \tag{5.3.12}$$

$$\dot{\phi} = W'(\phi), \tag{5.3.13}$$

$$V(\phi) = -\frac{1}{2}W'^2(\phi) + \frac{D-1}{2(D-2)}W^2(\phi), \tag{5.3.14}$$

on the domain of monotonicity of ϕ. For its relations to supergravity, the function W is called a *fake superpotential.* The applications of the fake superpotential to cosmology date back to [104], where it was called as *Hubble function.*

5.3.3. Slow-roll inflation

Let us finally see how the model (5.3.3) leads to inflation. The stress-energy tensor (5.1.18) has the form (5.2.14) with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{5.3.15}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5.3.16}$$

Therefore, if the potential energy dominates the kinetic energy, *i.e.*, $\dot{\phi}^2 \ll V(\phi)$ we have $\rho \approx -p$ approximately. We recognize the equation of state for dark energy and effectively we obtain the model discussed in section 5.3.1 with approximately exponential expansion (5.3.1) with

$$H \approx \sqrt{\frac{2V(\phi)}{(D-1)(D-2)}}. \tag{5.3.17}$$

The conditions we imposed means that the field rolls slowly down the potential and hence the name: *slow-roll inflation.* This model of inflation was developed by Andrei Linde in [105]. It is also reasonable to assume that $|\ddot{\phi}| \ll 1$, so that the slow-rolling phase lasts long enough.

In the further analysis it is convenient to define the following parameters,

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{5.3.18}$$

Note that the conventions for the definition of η vary among the textbooks. These variables are well-defined in any model based on the action (5.3.3). If, however, one sticks to the slow-roll inflation, then the above assumptions lead to the conclusion that the background solution is essentially given by the exponential model with the cosmological constant only, and ϵ and η measure small departures from such a geometry. In the discussion here we assumed $\dot{\phi}^2 \ll V(\phi)$ and $|\ddot{\phi}| \ll 1$ that can be expressed as $|\epsilon_{SR}| < 1$ and $|\eta_{SR}| < 1$. We added a subsript SR to indicate that these assumptions hold only in the slow-roll inflationary models. In such cases, equations (5.3.5) - (5.3.7) lead to the following set of slow-roll conditions,

$$\begin{aligned}
&\dot{\phi}^2 \ll V(\phi), && \frac{V'^2}{V} \ll H^2, \\
&\ddot{\phi} \ll (D-1)H\dot{\phi}, && V'' \ll H^2, \\
&\epsilon_{SR} \approx \frac{1}{D-2}\left(\frac{V'}{V}\right)^2 && \eta_{SR} \approx \frac{V''}{V} - \epsilon_{SR}.
\end{aligned} \tag{5.3.19}$$

In the slow-roll inflation the universe expands due to the large positive cosmological constant, as in the model discussed in section 5.3.1. The power of the inflaton is that now one can consider small fluctuations around the background solution, both for the inflaton and for the metric. These fluctuations, quantum in nature at the beginning of the inflation, are stretched to gigantic sizes in the process. Therefore, small inhomogeneities in the inflaton field lead to the emergence of the structure in the Universe such as galaxies and stars. Since the initial quantum fluctuations were essentially random, the visible structure today reflects this randomness and their various statistical characteristics can be measured in the CMB. The exact features of the CMB depend on a particular model under considerations.

Also the entropy problem can be solved by coupling the inflaton and metric to the Standard Model particles. However, we will not pursue this direction and the discussion of the problem can be found in the standard textbooks.

The slow-roll inflation is only one of a vast number of inflationary models. The most popular models are:

1. *Old inflation*: the original Guth's proposal [106]. The model is based on the quantum tunnelling of the inflaton from an unstable vacuum to a stable one.

2. *Eternal inflation* [107, 108]: taking the old inflation to a new level. The Universe is filled up with a slowly-changing dilaton field that has a complicated

landscape of vacuua. Therefore, the inflation lasts forever, but its speed differs in time and space.

3. *Hiltop inflation* [109, 110]. This model is closely related to the slow-roll inflation. The difference is that the inflation starts as the inflaton starts rolling down the hill from an unstable critical point of the potential. We will find this kind of the inflationary model in section 7.2. The inflaton profile resembles an instantonic solution.

4. *Large-field models* such as *chaotic inflation* [111], or *natural inflation* [112]. In these models one obtains larger tensor amplitudes than in other inflationary models.

5. *Multi-field inflation*, *e.g.*, [113, 114, 115]: there is more than one inflating field.

6. *String gas cosmology* [116, 117]: the inflation is embedded into string theory.

7. *Holographic inflation*: the main point of this thesis

and many more.

All the presented models share two additional features. Firstly, the inflating fields are spin-0 scalars. If this assumption is not met, some particular direction in the Universe would be preffered, but this is not observed. Secondly, all inflationary models predict at least one more unknown particle: the inflaton. One can wonder whether the only scalar particle in the Standard Model, the Higgs boson, could be the inflaton [118]. However, due to the small mass of the Higgs and unitarity issues it is very unlikely that this is the case [119, 120].

5.4. Inflaton on de Sitter background

5.4.1. Perturbation of inflaton

As a starter we will calculate the spectrum of perturbations of the inflaton, assuming a fixed gravitational background given by the de Sitter metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2, \qquad a(t) = e^{Ht}. \tag{5.4.1}$$

This is the usual setup of the quantum field theory on the curved background [71, 121]. We will keep $a(t)$ parameter explicitly for now, since we will use these results later on. In general we will denote the Hubble parameter

$$H = \frac{\dot{a}}{a}. \tag{5.4.2}$$

On the de Sitter solution H is a true constant and equal to the parameter in (5.4.1) also denoted by H. We want to solve the equation of motion $\Box\Phi - V'(\Phi) = 0$ on this background, which reads

$$\ddot{\Phi} + (D-1)\frac{\dot{a}}{a}\dot{\Phi} - \frac{1}{a^2}\Box_0\Phi + V'(\Phi) = 0, \tag{5.4.3}$$

where $\Box_0 = \delta^{ij}\partial_i\partial_j$ is the d'Alambertian in the $\boldsymbol{x}$ directions. We will solve (5.4.3) perturbatively by assuming that the potential has the following form

$$V(\Phi) = V_0 + \frac{1}{2}m^2\Phi^2 + \frac{a_3}{3}\Phi^3 + \frac{a_4}{4}\Phi^4 + \dots \tag{5.4.4}$$

and we will work with the perturbative expansion in all couplings a_j, $j \geqslant 3$. The field Φ then has a perturbative expansion in all couplings as well. We will consider only first order perturbation by writing

$$\Phi(t,\boldsymbol{x}) = \phi(t) + \delta\phi(t,\boldsymbol{x}). \tag{5.4.5}$$

The equation for the background solution $\phi(t)$ following from (5.4.3) is

$$\ddot{\phi} + dH\dot{\phi} + m^2\phi = 0, \tag{5.4.6}$$

where $d = D-1$. For the de Sitter background the equation becomes linear with two independent solutions

$$\phi(t) = C_+ t^{\Delta_+} + C_- t^{\Delta_-} \tag{5.4.7}$$

with two undetermined constants $C_\pm$ and

$$\Delta_\pm = \frac{dH}{2}\left[-1 \pm \sqrt{1 - \left(\frac{2m}{dH}\right)^2}\,\right]. \tag{5.4.8}$$

Let us now turn to the perturbation $\delta\phi$. First assume $a_n = 0$ for all n. Since we consider the flat FRW universe, we can Fourier transform $\delta\phi$ in $\boldsymbol{x}$ directions by writing

$$\begin{aligned}
\delta\phi(t,\boldsymbol{x}) &= \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\delta\phi(t,\boldsymbol{k}) \\
&= \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\left[a_{\boldsymbol{k}}u_k(t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + a^*_{\boldsymbol{k}}u^*_k(t)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right], \\
&= \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[a_{\boldsymbol{k}}u_k(t) + a^*_{-\boldsymbol{k}}u^*_k(t)\right],
\end{aligned} \tag{5.4.9}$$

where $d = D - 1$, $u_k(t)$ and $u_k^*(t)$ are the conjugate Fourier modes satisfying (5.4.3) and $a_{\boldsymbol{k}}$ and $a_{-\boldsymbol{k}}^*$ are conjugate coefficients. The $u_k(t)$ modes depend on the magnitude $k = |\boldsymbol{k}|$, since (5.4.3) reads

$$\ddot{u}_k + dH\dot{u}_k + \left(\frac{k^2}{a^2} + m^2\right) u_k = 0. \tag{5.4.10}$$

Notice the following:

- For wavelengths $\lambda \ll H^{-1}$ or equivalently $k \gg aH$ we can neglect the $dH\dot{u}_k$ term in (5.4.10) and the equation becomes the equation for the harmonic oscillator with the frequency dependent on time. Such a regime is called *subhorizon*.
- For *superhorizon* modes with $k \ll aH$ or equivalently $\lambda \gg H^{-1}$ we can neglect the k^2/a^2 term in (5.4.10) and the resulting equation has constant coefficients. If $m = 0$, then the mode is constant outside the horizon and if $m \neq 0$ but small, the solution depends slightly on k.

The equation of motion (5.4.10) can be simplified by means of the substitutions

$$u_k(t) = \frac{v_k(t)}{a(t)}, \qquad \mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)}. \tag{5.4.11}$$

The new time variable τ is called the *conformal time*. The equation (5.4.10) reads now

$$v_k'' + v_k \left[-\tfrac{1}{2}(d-1)\frac{a''}{a} - \tfrac{1}{4}(d-1)(d-3)\frac{a'^2}{a^2} + \left(k^2 + m^2 a^2\right)\right] = 0, \tag{5.4.12}$$

where prime denotes the derivative with respect to τ.

From now on let us work on the de Sitter background,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\mathrm{d}\boldsymbol{x}^2. \tag{5.4.13}$$

We set $a(t) = e^{Ht}$ and by choosing the integration constants we have

$$\tau = -\frac{1}{H}e^{-Ht}, \qquad a(\tau) = -\frac{1}{H\tau}. \tag{5.4.14}$$

With this choice the far past in the original time variable t corresponds to $\tau \to -\infty$, but the far future corresponds to $\tau \to 0$. In general $\tau < 0$. The equation (5.4.12) simplifies to

$$v_k'' + v_k \left[k^2 - \frac{1}{\tau^2}\left(\frac{1}{4}(d^2 - 1) - \frac{m^2}{H^2}\right)\right] = 0. \tag{5.4.15}$$

The general solution is

$$v_k(\tau) = \sqrt{-\tau}\left[c_1 H_\nu^{(1)}(-k\tau) + c_2 H_\nu^{(2)}(-k\tau)\right], \tag{5.4.16}$$

where

$$\nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}} \tag{5.4.17}$$

and $H_\nu^{(j)}$, $j = 1, 2$ denote the Hankel functions of the first and second kind, see [59] and c_j are two undetermined integration constants. We should fix the integration constants by requiring that in the far past, *i.e.*, for $\tau \to -\infty$ we have only incoming planar waves,

$$v_k(-\infty) = \frac{e^{-\mathrm{i}k\tau}}{\sqrt{2k}}, \tag{5.4.18}$$

where the normalisation is fixed by the Wronskian condition $v^* v' - v v'^* = -\mathrm{i}$. Such a choice of vacuum is called the *Bunch-Davies vacuum.* This leads to the solution

$$\begin{aligned} v_k(\tau) &= \frac{\sqrt{\pi}}{2} e^{\mathrm{i}(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau) \\ &= \sqrt{\frac{\mathrm{i}\tau}{\pi}} K_\nu(\mathrm{i}k\tau), \end{aligned} \tag{5.4.19}$$

where K_ν is the Bessel function K. This is the first appearance of the Bessel K functions in cosmology. Note that in the massless limit $m = 0$ the solution simplifies to elementary functions,

$$v_k(\tau) = \frac{e^{-\mathrm{i}k\tau}}{\sqrt{2k}} \left(1 + \frac{\mathrm{i}}{k\tau}\right). \tag{5.4.20}$$

5.4.2. 2-point function

At this point we can quantise the system by the imposition of the canonical commutation relations,

$$[a_{\boldsymbol{k}}, a^\dagger_{-\boldsymbol{k}'}] = (2\pi)^d \delta(\boldsymbol{k} + \boldsymbol{k}'), \tag{5.4.21}$$

and the 2-point function in momentum space is then

$$\langle \delta\phi(\tau, \boldsymbol{k}) \delta\phi(\tau, \boldsymbol{k}') \rangle = (2\pi)^d \delta(\boldsymbol{k} + \boldsymbol{k}') \frac{|\tau|}{\pi} |K_\nu(\mathrm{i}k\tau)|^2. \tag{5.4.22}$$

If no state is specified, the expectation value is always taken in the Bunch-Davies vacuum. We can return to the original time variable and use (5.4.14). Then we can expand the result around $t = \infty$ to find its late time behaviour to be

$$\langle \delta\phi(\boldsymbol{k}) \delta\phi(\boldsymbol{k}') \rangle \approx (2\pi)^d \delta(\boldsymbol{k} + \boldsymbol{k}') \frac{\Gamma^2(\nu) 4^{\nu-1}}{a^2 k \pi} \left(\frac{k}{aH}\right)^{1-2\nu}. \tag{5.4.23}$$

When the late time behaviour is considered, we will omit the indication of the time dependence in the cosmological correlation functions. For a massless field in $D = 4$ we have $\nu = 3/2$ and the 2-point function simplifies at late times to

$$\langle \delta\phi(\boldsymbol{k}) \delta\phi(\boldsymbol{k}') \rangle \approx (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') \frac{H^2}{2k^3}. \tag{5.4.24}$$

We call a scalar 2-point function *scale invariant* if its dependence on momentum is k^{-d}, where $d = D-1$. The reason is that such a 2-point function does not depend on any dimensionful parameters other than H.

The *spectral index* or *tilt* n_ϕ is defined as

$$n_\phi - d = \frac{\mathrm{d}}{\mathrm{d}\log k} \log\langle\langle \delta\phi(\boldsymbol{k})\delta\phi(-\boldsymbol{k})\rangle\rangle. \tag{5.4.25}$$

For a scale-invariant 2-point function we find $n_\phi = 0$. As in the first part of the thesis by a double bracket we denote the 2-point function without the Dirac delta,

$$\langle \delta\phi(\boldsymbol{k})\delta\phi(\boldsymbol{k}')\rangle = (2\pi)^d \delta(\boldsymbol{k}+\boldsymbol{k}')\langle\langle \delta\phi(\boldsymbol{k})\delta\phi(-\boldsymbol{k})\rangle\rangle \tag{5.4.26}$$

with a similar notation for the higher-point correlation functions, see (2.3.1).

5.4.3. 3-point function

Now let us turn to higher-point functions. We will discuss an example of calculations of the 3-point function of the inflaton perturbation in case of $m = 0$. The 3-point function for a free field vanishes and therefore we should expect that the interactions play the most important role here. Let us consider a simple potential of the form

$$V(\Phi) = \frac{g}{3}\Phi^3. \tag{5.4.27}$$

The 3-point function is by definition

$$\begin{aligned} &\langle \Omega(\tau)|:\delta\phi(\tau,\boldsymbol{k}_1)\delta\phi(\tau,\boldsymbol{k}_2)\delta\phi(\tau,\boldsymbol{k}_3):|\Omega(\tau)\rangle = \\ &\qquad = \langle 0|U_\tau^{-1}:\delta\phi(\tau,\boldsymbol{k}_1)\delta\phi(\tau,\boldsymbol{k}_2)\delta\phi(\tau,\boldsymbol{k}_3):U_\tau|0\rangle, \end{aligned} \tag{5.4.28}$$

where $|\Omega(\tau)\rangle = U_\tau|0\rangle$ denotes the state obtained by the evolution of the Minkowski vacuum $|0\rangle$ in the far past to the time τ. This is by definition the Bunch-Davies vacuum. Here we denote the evolution operator

$$U_\tau = \exp\left(-\mathrm{i}\int_{-\infty}^{\tau} H_{\mathrm{int}}(\tau')\mathrm{d}\tau'\right), \tag{5.4.29}$$

and in our case

$$H_{\mathrm{int}}(\tau) = \frac{g}{3}\int \mathrm{d}^d\boldsymbol{x}\, a^d(\tau)\Phi^3(\tau,\boldsymbol{x}), \tag{5.4.30}$$

where $d = D-1$. By expanding in the coupling g and keeping the leading term only we have

$$\begin{aligned} &\langle :\delta\phi(\tau,\boldsymbol{k}_1)\delta\phi(\tau,\boldsymbol{k}_2)\delta\phi(\tau,\boldsymbol{k}_3):\rangle \approx \\ &\qquad \approx -\mathrm{i}\int_{-\infty}^{\tau} \mathrm{d}\tau' a^d(\tau')\langle 0|[:\delta\phi(\tau,\boldsymbol{k}_1)\delta\phi(\tau,\boldsymbol{k}_2)\delta\phi(\tau,\boldsymbol{k}_3):, H_{\mathrm{int}}(\tau')]|0\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{-\mathrm{i}g}{3} \int_{-\infty}^{\tau} \mathrm{d}\tau' a^d(\tau') \int \frac{\mathrm{d}^D \boldsymbol{q}_1}{(2\pi)^D} \int \frac{\mathrm{d}^D \boldsymbol{q}_2}{(2\pi)^D} \times \\
&\qquad \times \langle 0 | [: \delta\phi(\tau, \boldsymbol{k}_1) \delta\phi(\tau, \boldsymbol{k}_2) \delta\phi(\tau, \boldsymbol{k}) :, : \delta\phi(\tau', \boldsymbol{q}_1) \delta\phi(\tau', \boldsymbol{q}_2 - \boldsymbol{q}_1) \delta\phi(\tau', -\boldsymbol{q}_2) :] | 0 \rangle \\
&= \frac{-\mathrm{i}g}{3} \int_{-\infty}^{\tau} \mathrm{d}\tau' a^d(\tau') \int \frac{\mathrm{d}^D \boldsymbol{q}_1}{(2\pi)^D} \int \frac{\mathrm{d}^D \boldsymbol{q}_2}{(2\pi)^D} \times \\
&\qquad \times \Big[v_{k_1}(\tau) v_{k_2}(\tau) v_{k_3}(\tau) v^*_{q_1}(\tau') v^*_{|\boldsymbol{q}_2 - \boldsymbol{q}_1|}(\tau') v^*_{q_2}(\tau') \langle 0 | a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a^\dagger_{-\boldsymbol{q}_1} a^\dagger_{\boldsymbol{q}_1 - \boldsymbol{q}_2} a^\dagger_{\boldsymbol{q}_2} | 0 \rangle \\
&\qquad - v_{q_1}(\tau) v_{|\boldsymbol{q}_2 - \boldsymbol{q}_1|}(\tau) v_{q_2}(\tau) v^*_{k_1}(\tau') v^*_{k_2}(\tau') v^*_{k_3}(\tau') \langle 0 | a_{\boldsymbol{q}_1} a_{\boldsymbol{q}_2 - \boldsymbol{q}_1} a_{-\boldsymbol{q}_2} a^\dagger_{-\boldsymbol{k}_1} a^\dagger_{-\boldsymbol{k}_2} a^\dagger_{-\boldsymbol{k}_3} | 0 \rangle \Big] \\
&= 4g \, \mathrm{Im} \left[v_{k_1}(\tau) v_{k_2}(\tau) v_{k_3}(\tau) \int_{-\infty}^{\tau} \mathrm{d}\tau' a^d(\tau') v^*_{k_1}(\tau') v^*_{k_2}(\tau') v^*_{k_3}(\tau') \right]. \qquad (5.4.31)
\end{aligned}
$$

We should comment on the choice of the vacuum. In case of a free theory the vacuum $|0\rangle$ is defined as the usual Minkowski vacuum at far past with the canonical normalisation coming from the Wronskian condition. In case of interactions one should take the Minkowski vacuum at far past of the full interacting theory. One can obtain such a vacuum state by the standard procedure of the projection of the free theory vacuum state on the subspace of minimal energy by deforming the integration contour into a slightly imaginary direction, see [21, 72]. In the case of the integral in (5.4.31) the deformation is $\tau' \mapsto \tau' + \mathrm{i}\epsilon|\tau'|$, where ϵ is a small parameter. This corresponds to the choice of the Bunch-Davies vacuum. Moreover, it regulises the integral, which is usually severely divergent.

Let us finalise the calculations by considering de Sitter background (5.4.14). Using (5.4.19) the integral to be done is

$$
(-1)^d \int_{-\infty}^{\tau} \mathrm{d}\tau' \left(\frac{\mathrm{i}}{\pi H^2 \tau'} \right)^{\frac{d}{2}} K_\nu(\mathrm{i}k_1\tau') K_\nu(\mathrm{i}k_2\tau') K_\nu(\mathrm{i}k_3\tau'). \qquad (5.4.32)
$$

This is the first instance of the triple-K integral in our cosmological analysis. We will not pursue its evaluation, since the example of the inflaton on the fixed background is not physical. In the next section we will consider its backreaction on the gravity and then we will arrive at the triple-K integrals with $\tau \to 0$. In the late time limits we will usually omit the implicit time dependence in the correlation function.

5.5. Cosmological perturbations

In the previous sections we considered perturbations of inflaton field on the fixed gravitational background. However, we should also consider the perturbations of gravity, since the inflaton is coupled to gravity in the action (5.3.3). In total we expect three propagating degrees of freedom: one scalar and two transverse-traceless spin-2. To see it, we must either fix the gauge freedom in the choice of the metric or to find such a combination of the variables that is gauge independent. It will be convenient to work in the ADM formalism.

5.5.1. ADM formulation

Consider the ADM decomposition (5.1.6) of the metric,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) \tag{5.5.1}$$

where

$$\begin{aligned} &N = 1 + \delta N(t,\boldsymbol{x}), \qquad N_i = g_{ij}N^j = \delta N_i(t,\boldsymbol{x}),\\ &\gamma_{ij} = a^2(t)(\delta_{ij} + h_{ij}(t,\boldsymbol{x})), \end{aligned} \tag{5.5.2}$$

and $i,j = 1,2,3$. For example, the perturbation of g_{tt} in (5.5.1) is

$$\delta g_{tt} = -(2\delta N + \delta N^2) + \frac{\delta N_i \delta N_i}{a^2}, \tag{5.5.3}$$

where the repeated Latin indices are summed with the Kronecker delta as a metric. The perturbations of shift vector and h_{ij} can be decomposed as

$$\delta N_i = a^2(\partial_i \nu + \nu_i), \qquad h_{ij} = -2\psi\delta_{ij} + 2\partial_i\partial_j\chi + 2\partial_{(i}\omega_{j)} + \gamma_{ij}, \tag{5.5.4}$$

where ν, χ and ψ are scalars, ν_i and ω_i are transverse vectors satisfying $\partial_i\nu_i = \partial_i\omega_i = 0$ and γ_{ij} is symmetric transverse and traceless, $\sum_i \gamma_{ii} = 0$.

We will work in the *comoving gauge* where the perturbations of inflaton vanish, *i.e.*,

$$\Phi(t,\boldsymbol{x}) = \phi(t), \qquad \delta\phi(t,\boldsymbol{x}) = 0. \tag{5.5.5}$$

This means that all degrees of freedom are encoded in two functions that we will be denoted as $\zeta(t,\boldsymbol{x})$ and $\hat{\gamma}_{ij}(t,\boldsymbol{x})$ and call *primordial perturbations.* The first function is a *scalar perturbation* of the curvature and the second one is a symmetric transverse-traceless *tensor perturbation*, *i.e.*, gravity waves. They are defined as perturbations of the spatial part of the metric,

$$\gamma_{ij} = a^2 e^{2\zeta}[e^{\hat{\gamma}}]_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + \hat{\gamma}_{ij} + \frac{1}{2}\hat{\gamma}_{ik}\hat{\gamma}_{kj} + \dots\right). \tag{5.5.6}$$

Now one can use the formulae listed above to find perturbations of the entire metric (5.5.1) in the ADM formalism. It can be shown [75] that the ζ and $\hat{\gamma}$ variables can be defined – up to second order in perturbations – in an gauge independent way as follows,

$$\zeta = -\psi - \frac{H}{\dot{\phi}}\delta\phi \tag{5.5.7}$$

$$- \psi^2 + \left(\dot{H} - \frac{H\ddot{\phi}}{\dot{\phi}}\right)\frac{\delta\phi^2}{2\dot{\phi}^2} + \frac{H}{\dot{\phi}^2}\delta\phi\delta\dot{\phi} + \frac{H}{\dot{\phi}}(\partial_k\chi + \omega_k)\partial_k\delta\phi + \frac{1}{4}\hat{\pi}_{ij}X_{ij}, \tag{5.5.8}$$

$$\hat{\gamma} = \gamma_{ij} + \hat{\Pi}_{ijkl} X_{kl}, \tag{5.5.9}$$

where

$$X_{ij} = -\frac{\partial_i \delta\phi \partial_j \delta\phi}{a^2 \dot{\phi}^2} - \frac{2\delta N_{(i}\partial_{j)}\delta\phi}{a^2\dot{\phi}} - \frac{\delta\phi \dot{h}_{ij}}{\dot{\phi}} - 2\partial_i(\partial_k \chi + \omega_k)h_{jk} \tag{5.5.10}$$
$$- \ (\partial_k\chi + \omega_k)\partial_k h_{ij} + \partial_i(\partial_k\chi + \omega_k)\partial_j(\partial_k\chi + \omega_k) + 2\phi\gamma_{ij} - \frac{1}{2}\gamma_{ik}\gamma_{kj}$$

and $\hat{\pi}_{ij}$ and $\hat{\Pi}_{ijkl}$ are defined in (2.1.20) and (2.1.21). If one is interested in 2-point functions only, then only the leading, first order terms in (5.5.7) and (5.5.9) can be considered. These can be found in the standard textbooks.

5.5.2. Equations of motion

Now the action (5.3.3) can be rewritten in the ADM formalism as in (5.1.19),

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D \boldsymbol{x}\sqrt{|g|}N\left[K_{\mu\nu}K^{\mu\nu} - K^2 + \frac{1}{N^2}\left(\dot{\Phi} - N^\mu\partial_\mu\Phi\right)^2 \right.$$
$$\left. + \left(\hat{R} - \gamma^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2V(\Phi)\right)\right], \tag{5.5.11}$$

where

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu} = \frac{1}{2N}\left(\dot{\gamma}_{\mu\nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu\right). \tag{5.5.12}$$

Equations of motion for the first order perturbations can be worked out from this action, see *e.g.*, [122] and they read

$$0 = \ddot{\zeta}_k + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{\zeta}_k + \frac{k^2}{a^2}\zeta_k, \tag{5.5.13}$$

$$0 = \ddot{\hat{\gamma}}_k + 3H\dot{\hat{\gamma}}_k + \frac{k^2}{a^2}\hat{\gamma}_k, \tag{5.5.14}$$

where the Fourier modes are defined as

$$\zeta(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[a_{\boldsymbol{k}}\zeta_k(t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + a^*_{\boldsymbol{k}}\zeta^*_k(t)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{5.5.15}$$

$$\hat{\gamma}^{(s)}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[b^{(s)}_{\boldsymbol{k}}\hat{\gamma}_k(t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + b^{(s)*}_{\boldsymbol{k}}\hat{\gamma}^*_k(t)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{5.5.16}$$

Here H and ϵ stand for the Hubble parameter and the slow-roll parameter,

$$H = \frac{\dot{a}}{a}, \qquad\qquad \epsilon = -\frac{\dot{H}}{H^2}, \tag{5.5.17}$$

which are fixed by the background solution, but are generally time-dependent. In slow-roll inflation one assumes $|\epsilon| < 1$, but the equations (5.5.13) and (5.5.14) are exact as long as one considers first order perturbations on top of the FRW solution.

As we can see, equation (5.5.14) is exactly the same as equation (5.4.10) for a free massless scalar on the de Sitter background. Its solution in conformal time τ is therefore given by (5.4.19). The first equation (5.5.13) for the scalar perturbation is similar to (5.4.10) up to the $\dot{\epsilon}/\epsilon$ term. In the leading order in slow-roll we can account for this correction at late times by solving the free massless equation and evaluating the cosmological parameters at the *horizon crossing time*, *i.e.*, at time $t_*(\boldsymbol{k})$ such that $\boldsymbol{k} = a(t_*)H(t_*)$. We will also denote $a_* = a(t_*)$ and $H_* = H(t_*)$. The reason is that at late times ζ is approximately constant as we have seen in (5.4.19), while at early times the field is in the vacuum and its wavefunction is accurately given by the WKB approximation. For more details, see [72].

5.5.3. 2-point functions

Using canonical quantisation we promote the coefficients $a_{\boldsymbol{k}}$ and $b^{(s)}_{\boldsymbol{k}}$ to operators satisfying

$$[a_{\boldsymbol{k}}, a^{\dagger}_{-\boldsymbol{k}'}] = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}'), \qquad [b^{(s)}_{\boldsymbol{k}}, b^{(s')\dagger}_{-\boldsymbol{k}'}] = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\delta^{ss'}. \tag{5.5.18}$$

The 2-point functions of the primordial perturbations are

$$\langle\zeta(t,\boldsymbol{k})\zeta(t,\boldsymbol{k}')\rangle = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')|\zeta_{\boldsymbol{k}}(t)|^2, \tag{5.5.19}$$

$$\langle\hat{\gamma}^{(s)}(t,\boldsymbol{k})\hat{\gamma}^{(s')}(t,\boldsymbol{k}')\rangle = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')\delta^{ss'}|\hat{\gamma}_{\boldsymbol{k}}(t)|^2. \tag{5.5.20}$$

Notice that these expressions depend on the amplitude of the momentum $\boldsymbol{k}$ only. If one can solve equations (5.5.13) and (5.5.14) exactly, this will give an exact 2-point function in the absence of interactions.

Similarly as in case of the inflaton, we can define the *spectral indices* as

$$n_S(k) - 4 = \frac{\mathrm{d}}{\mathrm{d}\log k}\log\langle\!\langle\zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle, \tag{5.5.21}$$

$$n_T(k) - 3 = \frac{\mathrm{d}}{\mathrm{d}\log k}\log\langle\!\langle\hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s)}(-\boldsymbol{k})\rangle\!\rangle, \tag{5.5.22}$$

which measure a departure of the 2-point functions from scale invariance. For historical reasons, $n_S = 1$ and $n_T = 0$ correspond to the scale-invariant scalar and tensor 2-point functions respectively.

In case of the slow-roll inflation, one can solve equations (5.5.13) and (5.5.14) in the leading order in ϵ, which leads to the late time correlation functions,

$$\langle\!\langle\zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle_{SR} = \frac{\kappa^2 H_*^2}{4\epsilon_* k^3}\left[1 + O(\epsilon_*, \eta_*)\right], \tag{5.5.23}$$

$$\langle\!\langle\hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s')}(-\boldsymbol{k})\rangle\!\rangle_{SR} = \frac{2\kappa^2 H_*^2\delta^{ss'}}{k^3}\left[1 + O(\epsilon_*, \eta_*)\right], \tag{5.5.24}$$

where the asterisk denotes the evaluation at the horizon crossing time t_* satisfying $k = a_* H_* = a(t_*)H(t_*)$. Their dependence on the momentum in hidden in the fact that the Hubble parameter and ϵ_{SR} are evaluated at the horizon crossing time. Using the conditions (5.3.19) one finds

$$n_S - 1 = \frac{\mathrm{d}}{\mathrm{d}\log k} \log \frac{H_*^2}{\epsilon_*} \approx \frac{1}{H_*} \frac{\mathrm{d}}{\mathrm{d}t_*} \log \frac{H_*^2}{\epsilon_*} \approx 2\eta_* - 4\epsilon_*, \tag{5.5.25}$$

$$n_T = \frac{\mathrm{d}}{\mathrm{d}\log k} \log H_*^2 = \frac{1}{H_*} \frac{\mathrm{d}}{\mathrm{d}t_*} \log H_*^2 \approx -2\epsilon_*. \tag{5.5.26}$$

Note that these results are of order one in slow-roll parameters. Therefore, the slow-roll inflation predicts only minor deviations from the scale-invariance of the 2-point functions.

It is customary to define *the power spectra* Δ_S^2 and Δ_T^2 as

$$\Delta_S^2(k) = \frac{k^3}{2\pi^2} \langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k}) \rangle\!\rangle_{SR} = \frac{\kappa^2 H_*^2}{8\pi^2 \epsilon_*} \left[1 + O(\epsilon_*, \eta_*)\right], \tag{5.5.27}$$

$$\Delta_T^2(k) = \frac{2k^3}{\pi^2} \langle\!\langle \hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s)}(-\boldsymbol{k}) \rangle\!\rangle_{SR} = \frac{4\kappa^2 H_*^2}{\pi^2} \left[1 + O(\epsilon_*, \eta_*)\right], . \tag{5.5.28}$$

The expressions for the 2-point functions expanded to the higher order in slow-roll are known [123, 124]. For the comparison with our holographic cosmology, we will need the first order results for the scalar perturbations,

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k}) \rangle\!\rangle_{SR} = \frac{\kappa^2 H_*^2}{4\epsilon_* k^3} \left[1 + (2 - \log 2 - \gamma_E)(2\epsilon_* + \eta_*) - \epsilon_*\right]. \tag{5.5.29}$$

In general one can show [125] that for any non-negative integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}\log k^n}(n_S - 1) \sim \text{slow-roll}^{n+1}, \tag{5.5.30}$$

where by 'slow-roll' we denote small parameters appearing in the higher terms of the expansion (5.5.29). These comprise ϵ and η but also other slow-roll parameters following from the estimates of higher derivatives of V and ϕ in the background solution.

5.5.4. Hamiltonian formalism

For the analysis of 3-point functions of the primordial perturbations it is convenient to work in the Hamiltonian formalism. The reason is that, in order to calculate the 3-point functions as in section 5.4.3, we must evaluate the interaction Hamiltonians for the primordial perturbations ζ and $\hat{\gamma}$. To do it define the canonical momenta

$$\Pi = \frac{\delta}{\delta\dot{\zeta}}(\kappa^2 L), \qquad\qquad \Pi_{ij} = \frac{\delta}{\delta\dot{\hat{\gamma}}}(\kappa^2 L), \tag{5.5.31}$$

where L is the Lagrangian in (5.5.11). Next, since the perturbation $\hat{\gamma}_{ij}$ is transverse-traceless, we can project it onto the helicity basis as explained in section 2.9. We define

$$\hat{\gamma}^{(s)}(\boldsymbol{k}) = \frac{1}{2}\hat{\gamma}_{ij}(\boldsymbol{k})\epsilon^{(s)}_{ij}(-\boldsymbol{k}), \qquad \Pi^{(s)}(\boldsymbol{k}) = \frac{1}{2}\Pi_{ij}(\boldsymbol{k})\epsilon^{(s)}_{ij}(-\boldsymbol{k}), \tag{5.5.32}$$

where the projectors $\epsilon^{(s)}_{ij}$ are defined in (2.9.2).

The dynamics is given by the Hamilton equations which in momentum space read,

$$\begin{aligned}
\dot{\zeta}(\boldsymbol{k}) &= (2\pi)^3 \frac{\delta}{\delta\Pi(-\boldsymbol{k})}(\kappa^2\mathcal{H}), & \dot{\hat{\gamma}}^{(s)}(\boldsymbol{k}) &= \frac{1}{2}(2\pi)^3\frac{\delta}{\delta\Pi^{(s)}(-\boldsymbol{k})}(\kappa^2\mathcal{H}),\\
\dot{\Pi}(\boldsymbol{k}) &= -(2\pi)^3 \frac{\delta}{\delta\zeta(-\boldsymbol{k})}(\kappa^2\mathcal{H}), & \dot{\Pi}^{(s)}(\boldsymbol{k}) &= -\frac{1}{2}(2\pi)^3\frac{\delta}{\delta\hat{\gamma}^{(s)}(-\boldsymbol{k})}(\kappa^2\mathcal{H}),
\end{aligned} \tag{5.5.33}$$

where $\mathcal{H}$ is the Hamiltonian following from (5.5.11). In order to extract it, one needs to find the interactions between the primordial perturbations in (5.5.11). We can divide the Hamiltonian into a free and interacting part,

$$\mathcal{H} = H_{\text{free}} + H_{\text{int}}. \tag{5.5.34}$$

The free part can be written as $H_{\text{free}} = H_{\zeta\zeta} + H_{\hat{\gamma}\hat{\gamma}}$, where the subscript denotes types of interations. Starting from (5.5.11) one can show [122, 72, 103] that

$$\kappa^2 H_{\zeta\zeta} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[\frac{1}{4a^3\epsilon}\Pi(\boldsymbol{k})\Pi(-\boldsymbol{k}) + a\epsilon k^2\zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\right], \tag{5.5.35}$$

$$\kappa^2 H_{\hat{\gamma}\hat{\gamma}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[\frac{4}{a^3}\Pi^{(s)}(\boldsymbol{k})\Pi^{(s)}(-\boldsymbol{k}) + \frac{a}{4}k^2\hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s)}(-\boldsymbol{k})\right]. \tag{5.5.36}$$

One can also check that these Hamiltonians lead to equations (5.5.13) and (5.5.14). These Hamiltonians are valid in general, as the equations (5.5.13) and (5.5.14) hold in general. Note that the lack of interactions between ζ and $\hat{\gamma}$ explains why

$$\langle\!\langle \zeta(\boldsymbol{k})\hat{\gamma}(-\boldsymbol{k})\rangle\!\rangle = 0. \tag{5.5.37}$$

For the interaction part of the Hamiltonian we use the perturbative expansion. We look for the triple interactions in (5.5.11) since we are interested in 3-point functions only and without loop corrections. On general grounds we can write

$$H_{\text{int}} = H_{\zeta\zeta\zeta} + H_{\zeta\zeta\hat{\gamma}} + H_{\zeta\hat{\gamma}\hat{\gamma}} + H_{\hat{\gamma}\hat{\gamma}\hat{\gamma}}, \tag{5.5.38}$$

where the particular components are complicated, theory dependent expressions. Before we analyse their form closer, let us discuss the case of the slow-roll inflation, where the Hamiltonian (5.5.38) is known.

5.5.5. 3-point functions in slow-roll inflation

Expressions for the four terms in the Hamiltonian (5.5.38) have been found in the slow-roll approximation, see [75]. In this case, however, various tricks introduced in [72] allow to simplify the calculations considerably, see also [100, 103, 40]. For example, it turns out that by using the following field redefinition,

$$\zeta = \zeta_c + \left(\frac{\ddot{\phi}}{2\dot{\phi}H} + \frac{\epsilon}{4} \right) \zeta_c^2 + \frac{\epsilon}{2}\partial^{-2}(\zeta_c \partial^2 \zeta_c) + \dots, \tag{5.5.39}$$

one can rewrite the essential part of $H_{\zeta\zeta\zeta}$ as

$$H_{\zeta\zeta\zeta} = -\frac{1}{\kappa^2} \int \mathrm{d}^4\boldsymbol{x}\, 4\epsilon^2 a^5 H \dot{\zeta}_c^2 \partial^{-2}\dot{\zeta}_c + \dots \tag{5.5.40}$$

where the omitted terms are higher order in slow-roll parameters or vanish outside the horizon. Using the interaction Hamiltonian following from this interaction one finds in the same way as described in section 5.4.3

$$\langle\!\langle \zeta_c(\boldsymbol{k}_1)\zeta_c(\boldsymbol{k}_2)\zeta_c(\boldsymbol{k}_3) \rangle\!\rangle_{SR} = \frac{\kappa^4 H_*^4}{\epsilon_* c_{123}^3} \frac{\sum_{i<j} k_i^2 k_j^2}{4a_{123}} + \dots \tag{5.5.41}$$

in the leading order in slow-roll parameters. The full 3-point function in the original variable in the leading order in slow-roll parameters ϵ_* and η_* defined in (5.3.18) reads

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3) \rangle\!\rangle_{SR} = \frac{\kappa^4 H_*^4}{32\epsilon_*^2} \frac{1}{c_{123}^3} \Bigg[2\eta_* \sum_j k_j^3 + \\ + \epsilon_* \left(a_{123}^3 - 2a_{123}b_{123} - 16c_{123} + 8\frac{b_{123}^2}{a_{123}} \right) \Bigg], \tag{5.5.42}$$

where a_{123}, b_{123} and c_{123} are symmetric polynomials in amplitudes of momenta,

$$a_{123} = k_1 + k_2 + k_3, \qquad b_{123} = k_1k_2 + k_1k_3 + k_2k_3, \qquad c_{123} = k_1k_2k_3. \tag{5.5.43}$$

The index SR reminds us that this form of the 3-point fucntion holds for the slow-roll inflation in the first order in ϵ_* and η_* in late time approximation.

In a similar fashion one can obtain the remaining correlation functions for the primordial perturbations,

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3) \rangle\!\rangle_{SR} = \frac{\kappa^4 H_*^4}{16\sqrt{2}\epsilon_*} \frac{J^2}{a_{123}^2 c_{123}^3 k_3^2} \left[a_{123}^3 - a_{123}b_{123} - c_{123} \right], \tag{5.5.44}$$

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3) \rangle\!\rangle_{SR} = -\frac{\kappa^4 H_*^4}{128 b_{23}^5 k_1^2} (k_1^2 - a_{23}^2)^2 \left[(k_1^2 - a_{23}^2 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}} \right], \tag{5.5.45}$$

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(-)}(\boldsymbol{k}_3)\rangle\!\rangle_{SR} = -\frac{\kappa^4 H_*^4}{128\, b_{23}^5 k_1^2}(k_1^2 - a_{23}^2 + 4b_{23})^2 \times \\ \times \left[(k_1^2 - a_{23}^2 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}}\right]. \tag{5.5.46}$$

$$\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle_{SR} = \frac{\kappa^4 H_*^4}{64\sqrt{2}}\frac{J^2 a_{123}^2}{c_{123}^5}(a_{123}^3 - a_{123}b_{123} - c_{123}), \tag{5.5.47}$$

$$\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(-)}(\boldsymbol{k}_3)\rangle\!\rangle_{SR} = \frac{\kappa^4 H_*^4}{64\sqrt{2}}\frac{J^2}{a_{123}^2 c_{123}^5}(k_3 - a_{12})^4 \times \\ \times\ (a_{123}^3 - a_{123}b_{123} - c_{123}), \tag{5.5.48}$$

where

$$a_{ij} = k_i + k_j, \qquad\qquad b_{ij} = k_i k_j \tag{5.5.49}$$

for $i, j = 1, 2, 3$ and J^2 is defined in (2.6.18) as

$$J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3). \tag{5.5.50}$$

As long as the Hamiltonians in do not contain parity violating terms, the correlation functions are symmetric in the sense that

$$\langle \zeta\zeta\hat{\gamma}^{(+)}\rangle = \langle \zeta\zeta\hat{\gamma}^{(-)}\rangle, \qquad \langle \zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\rangle = \langle \zeta\hat{\gamma}^{(-)}\hat{\gamma}^{(-)}\rangle, \\ \langle \hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\rangle = \langle \hat{\gamma}^{(-)}\hat{\gamma}^{(-)}\hat{\gamma}^{(-)}\rangle, \qquad \langle \hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}\rangle = \langle \hat{\gamma}^{(-)}\hat{\gamma}^{(-)}\hat{\gamma}^{(+)}\rangle, \tag{5.5.51}$$

so only the relative number of helicities matters. All the above results can be found in [72, 75]. More on the structure of 3-point correlation functions in the slow-roll inflation, including parity violating terms can be found in [126, 127, 29, 128].

5.5.6. Response functions

In the previous sections we calculated 2- and 3-point functions of scalar and tensor perturbations produced during the slow-roll inflation and propagated to late times. We were starting from the inflationary action (5.5.11) and we worked mostly in the leading order in the slow-roll parameters ϵ and η, (5.3.18).

In this section we want to return to the Hamiltonian formalism of section 5.5.4 and introduce a more general analysis *valid for any inflationary scenario* with the scalar and spin-2 propagating degrees of freedom. For the details, consult [39, 40, 75].

First notice that the free Hamiltonian (5.5.35) and (5.5.36) is valid in general, since the equations (5.5.13) and (5.5.14) are derived without any approximations.

For the interaction part, one can consider the most general forms of various interactions. For example, the most general form $H_{\zeta\zeta\zeta}$ is

$$H_{\zeta\zeta\zeta} = \frac{1}{\kappa^2}\int [[\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3]]\,[\mathcal{A}(k_i)\zeta(-\boldsymbol{k}_1)\zeta(-\boldsymbol{k}_2)\zeta(-\boldsymbol{k}_3) + \mathcal{B}(k_i)\Pi(-\boldsymbol{k}_1)\zeta(-\boldsymbol{k}_2)\zeta(-\boldsymbol{k}_3)$$
$$+ \mathcal{C}(k_i)\zeta(-\boldsymbol{k}_1)\Pi(-\boldsymbol{k}_2)\Pi(-\boldsymbol{k}_3) + \mathcal{D}(k_i)\Pi(-\boldsymbol{k}_1)\Pi(-\boldsymbol{k}_2)\Pi(-\boldsymbol{k}_3)]\,, \tag{5.5.52}$$

where Π is the canonical momentum associated with ζ defined in (5.5.31), $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are some functions of momenta depending on the specifics of the theory and

$$[[\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3]] = (2\pi)^3\delta(\sum_j \boldsymbol{k}_j)\frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}_3}{(2\pi)^3}. \tag{5.5.53}$$

Exact expressions for the functions $\mathcal{A}$ - $\mathcal{D}$ depend on a particular model. For slow-roll inflation these were calculated in [40].

Now we can define the *response functions* $\Omega_{[2]}$ and $\Omega_{[3]}$ as in the usual response theory,

$$\Pi(\boldsymbol{k}_1) = \Omega_{[2]}(-\boldsymbol{k}_1)\zeta(\boldsymbol{k}_1) + \int [[\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3]]\,\Omega_{[3]}(\boldsymbol{k}_2,\boldsymbol{k}_3)\zeta(-\boldsymbol{k}_2)\zeta(-\boldsymbol{k}_3) + \ldots \tag{5.5.54}$$

where

$$[[\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3]] = (2\pi)^3\delta(\sum_j \boldsymbol{k}_j)\frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}_3}{(2\pi)^3}. \tag{5.5.55}$$

One can show that the linear response function $\Omega_{[2]}(\boldsymbol{k})$ depends on the magnitude of the momentum $k = |\boldsymbol{k}|$ only. Similarly, the response function $\Omega_{[3]}(\boldsymbol{k}_1,\boldsymbol{k}_2)$ can be written entirely in terms of the momenta amplitudes

$$k_1 = |\boldsymbol{k}_1|, \qquad k_2 = |\boldsymbol{k}_2|, \qquad k_3 = |\boldsymbol{k}_3| = |-\boldsymbol{k}_1 - \boldsymbol{k}_2| \tag{5.5.56}$$

as discussed in section 2.2.2. We will usually omit the arguments of the response functions, assuming $\Omega_{[3]} = \Omega_{[3]}(k_1,k_2,k_3)$.

The Hamilton equations (5.5.33) lead to the equations satisfied by the response functions,

$$0 = \dot{\Omega}_{[2]}(k) + \frac{1}{2a^3\epsilon}\Omega_{[2]}^2(k) + 2a\epsilon k^2, \tag{5.5.57}$$

$$0 = \dot{\Omega}_{[3]}(k_1,k_2,k_3) + \frac{1}{2a^3\epsilon}\big(\Omega_{[2]}(k_1) + \Omega_{[2]}(k_2) + \Omega_{[2]}(k_3)\big)\Omega_{[3]}(k_1,k_2,k_3)$$
$$+\ \mathcal{X}(k_1,k_2,k_3), \tag{5.5.58}$$

where

$$\mathcal{X}(k_1,k_2,k_3) = 3\mathcal{A}_{123} + \mathcal{B}_{123}\Omega_{[2]}(k_1) + \mathcal{B}_{213}\Omega_{[2]}(k_2) + \mathcal{B}_{312}\Omega_{[2]}(k_3)$$
$$+\ \mathcal{C}_{123}\Omega_{[2]}(k_2)\Omega_{[2]}(k_3) + \mathcal{C}_{213}\Omega_{[2]}(k_1)\Omega_{[2]}(k_3) + \mathcal{C}_{312}\Omega_{[2]}(k_1)\Omega_{[2]}(k_2)$$
$$+\ 3\mathcal{D}_{123}\Omega_{[2]}(k_1)\Omega_{[2]}(k_2)\Omega_{[2]}(k_3). \tag{5.5.59}$$

Here we write $\mathcal{A}_{ijk} = \mathcal{A}(k_i, k_j, k_k)$, so for example $\mathcal{A}_{213} = \mathcal{A}(k_2, k_1, k_3)$ and similarly for other functions.

The solution to the linear order response function can be deduced from (5.5.13). Indeed, if

$$\Omega_{[2]}(k) = 2a^3\epsilon\frac{\dot{\zeta}_k}{\zeta_k} \tag{5.5.60}$$

then (5.5.13) implies (5.5.57). In the conformal time τ defined in (5.4.11) the solution for $\Omega_{[3]}$ is

$$\Omega_{[3]}(\tau, k_j) = -\left(\prod_i \frac{1}{\zeta_{k_i}(z)}\right)\int_{-\infty}^{\tau} \mathrm{d}\tau' \mathcal{X}(\tau', k_j)\prod_i \zeta_{k_i}(\tau'). \tag{5.5.61}$$

The 3-point function in the leading order of interaction can be written as

$$\begin{aligned}\langle\zeta(\tau,\boldsymbol{k}_1)\zeta(\tau,\boldsymbol{k}_2)\zeta(\tau,\boldsymbol{k}_3)\rangle &= -4\kappa^{-2}(2\pi)^3\delta(\sum_i \boldsymbol{k}_j)\prod_i |\zeta_{k_i}(z)|^2\times\\ &\times\ \mathrm{Im}\Big[\Big(\prod_i \frac{1}{\zeta_{k_i}(z)}\Big)\int_{-\infty}^{\tau} \mathrm{d}\tau' \mathcal{X}(\tau', k_j)\prod_i \zeta_{k_i}(\tau')\Big]\\ &= (2\pi)^3\delta(\sum_i \boldsymbol{k}_i)\ 4\kappa^{-2}\mathrm{Im}\left[\Omega_{[3]}(\tau, k_j)\right]\prod_i |\zeta_{k_i}(z)|^2.\end{aligned} \tag{5.5.62}$$

Notice that the form of this expression agrees with (5.4.31). In case of (5.4.31) the interaction Hamiltonian was very simple, $H_{\mathrm{int}} \sim \Phi^3$, hence we had $\mathcal{X}(\tau', k_i) \sim a^d(\tau')$ there. Here the Hamiltonian is given by (5.5.35), so we can interpret it as a vertex for Feynman diagrams in the quasi-de Sitter background.

In a similar fashion one can introduce the response functions for all 2- and 3-point functions and express the correlation functions by them. For further use define the linear response function E for the transeverse-traceless perturbation,

$$\Pi^{(s)}(\boldsymbol{k}) = E(-\boldsymbol{k})\hat{\gamma}^{(s)}(\boldsymbol{k}), \tag{5.5.63}$$

where $\Pi^{(s)}$ is defined in (5.5.32). The expressions for all response functions can be found in [40] and [75]

5.6. Experimental evidence for inflation

As we mentioned at the beginning of this chapter, the observation of the inhomogeneities in the Cosmic Microwave Background (CMB) is a strong evidence for the inflation. In this section we will shortly discuss how the calculations carried out in the previous sections fit into the actual measurements.

The most recent measurements of the Cosmic Microwave Background (CMB) were carried out by the Planck satellite [85], see figure 5.5, and are in the good

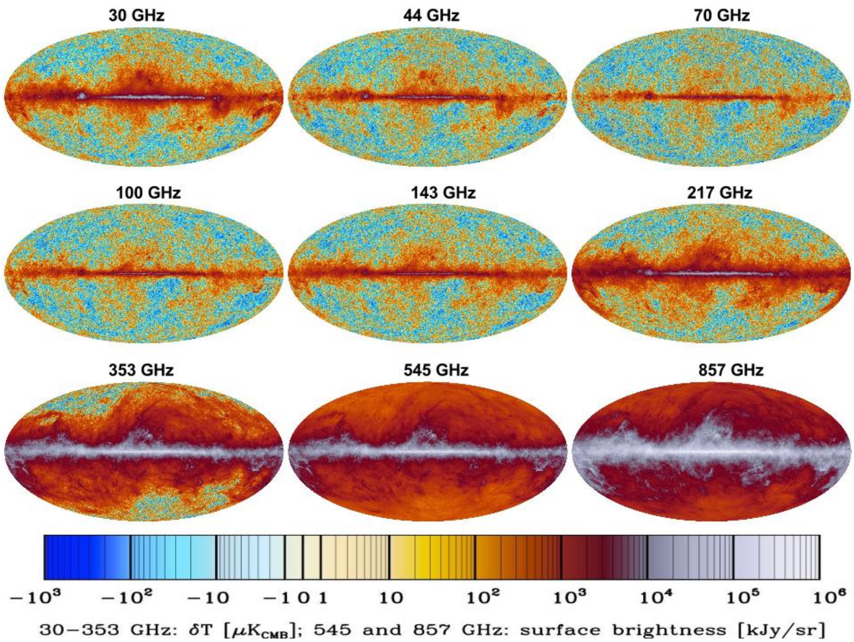


Figure 5.5: *The maps of the Cosmic Microwave background obtained by the Planck satellite [85] in all its channels. The bright area in the middle is the Milky Way galaxy. Before the analysis of the Cosmic Microwave Background can be carried out, such bright objects must be cut off, so that only the residual CMB is analysed.*

agreement with the previous measurements, *e.g.*, [84, 129, 99]. The CMB is an electromagnetic, almost thermal radiation filling up the Universe. Its temperature is $T = 2.7260 \pm 0.0013$ K [130] and the small fluctuations on top of the thermal distribution are of the order $\Delta T/T \approx 10^{-5}$.

Due to the evolution of the Universe, the shape of the spectrum of the CMB in the figure 5.6 does not resemble a simple shape given by (5.5.23). There is a few well-understood reasons for that. Firstly, as we live at a fixed position within the Universe, we cannot measure the 3-dimensional distribution of the radiation, but rather a 2-dimensional one extending on the sphere of the sky. Therefore, in order to compare with the experiment, all theoretical computations must be recalculated in terms of spherical observables. For example, by using the spherical harmonics we can write

$$\zeta(\boldsymbol{x}) = \int_0^\infty \mathrm{d}k \sum_{l,m} \zeta_{lm}(k) Z_{klm}(x, \theta, \phi), \tag{5.6.1}$$

where

$$Z_{klm}(z, \theta, \phi) = \sqrt{\frac{2}{\pi}} k j_l(kx) Y_{lm}(\theta, \phi), \tag{5.6.2}$$

where j_l denotes the spherical Bessel function. By orthonormality of Z_{klm} we have

$$\langle \zeta_{lm}(k)\zeta_{l'm'}(k')\rangle = \langle \zeta(\boldsymbol{k})\zeta(\boldsymbol{k}')\rangle \delta_{ll'}\delta_{mm'}. \tag{5.6.3}$$

Similar, but more involved calculations are required for 3-point functions, see [131, 132].

Another problem is that while the inflationary models predict an average spectrum, we observe its particular realisation in our part of the Universe. For example, the monopole moment $l = 1$ is unknown, since it is impossible to measure the move of the Earth with respect to the background geometry. This means that we must account for this *cosmic variance* by adding additional error bars on all our observations. It turns out that on average the cosmic variance behaves like $l^{-1/2}$, therefore it is negligible for large multiple numbers l.

The final problem is the evolution of the perturbations ζ and $\hat{\gamma}$ from the inflationary period to our time. This is a non-trivial problem that requires solving equations of motion for the evolution of matter and electromagnetic field coupled to gravity, see [133, 134, 92, 100]. These equations are usually solved numerically and the resulting spectrum is fitted to the data. In this way one can predict the spectrum of the CMB based on the spectrum of ζ and $\hat{\gamma}$. The actual measurements and the fit to the ΛCDM model is presented in figure 5.6.

Any inflationary model delivers predictions for correlation functions of the primordial perturbations. Up to date only the scalar power spectrum $\langle\zeta\zeta\rangle$ was measured precisely. It is convenient to parametrise the spectrum as

$$\Delta_S^2(k) = \frac{k^3}{2\pi^2}\langle\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\rangle = \Delta^2(k_0)\left(\frac{k}{k_0}\right)^{n_S(k)-1}, \tag{5.6.4}$$

where k_0 is some reference scale and n_S is a spectral index defined in (5.5.21). For the Planck satellite the reference scale is $k_0 = 0.05$ Mpc^{-1}. The spectral index was determined by the measurements of the CMB at various wavelengths. The Planck satellite measured nine different frequencies between 30 and 857 GHz. While it is enough to determine the average value of the spectral index, its dependence on the momentum could not be determined precisely. By expanding in $\log k$ one can parametrise

$$\langle\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\rangle = \frac{2\pi^2}{k^3}\Delta^2(k_0)\left(\frac{k}{k_0}\right)^{n_S - 1 + \frac{1}{2}\frac{\mathrm{d}n_S}{\mathrm{d}\log k}\log\frac{k}{k_0} + \frac{1}{6}\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}\log^2\frac{k}{k_0} + \ldots}, \tag{5.6.5}$$

where n_S, $\frac{\mathrm{d}n_S}{\mathrm{d}\log k}$ and $\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}$ are evaluated at k_0. The first derivative of n_S is known as the *running* of the spectral index, the second derivative is the running of the running and so on. It turns out that the spectrum is almost scale-invariant with a very small amplitude,

$$\Delta^2(k_0) = (2.23 \pm 0.16)\cdot 10^{-9} \tag{5.6.6}$$

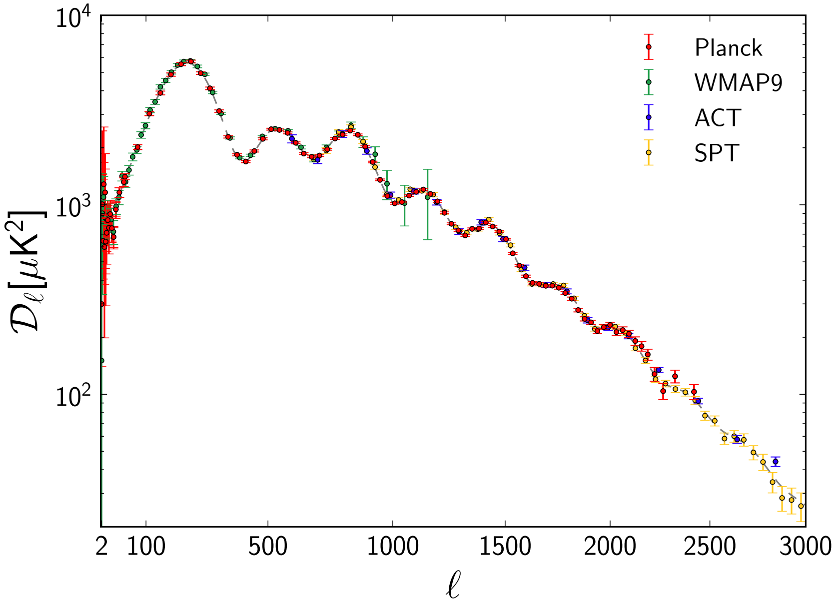


Figure 5.6: *The power spectrum (2-point function) of the Cosmic Microwave Background and the fit of the ΛCDM model. The data denoted by Planck and WMAP9 come from the two satellites. ACT stands for the Atacama Cosmology Telescope [135], while SPT for the South Pole Telescope [136, 137]. Source: [85].*

so that indeed $\Delta(k_0) \sim 10^{-5}$.[1] The values of the remaining parameters depend slightly on whether one assumes that $\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2} = 0$ or not, see table 5.1. When

	$\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2} = 0$	$\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}$ fitted
$n_S(k_0) - 1$	-0.0404 ± 0.0063	-0.0432 ± 0.068
$\frac{\mathrm{d}n_S}{\mathrm{d}\log k}(k_0)$	-0.013 ± 0.009	0.000 ± 0.016
$\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}(k_0)$	0	0.017 ± 0.016

Table 5.1: *Values of the spectral tilt and its running measured by the Plack satellite, [138]. The fitted values depend on the assumption whether the second running $\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}$ is assumed to vanish or not.*

combined with previous measurements, the value of the spectral index is $n_S(k_0) - 1 = -0.0392 \pm 0.0054$, which excludes $n_S = 1$ at 6σ level. On the other hand

[1] All results in this section are given with the uncertainty at 68% level.

the inflationary models predict that the subsequent runnings of the spectral index should be suppressed by higher and higher powers of slow-roll parameters, (5.5.30).

The amplitude of the tensor modes in our Universe is very small. So far, no detection of the tensor modes has been reported [91], but hopefully new data from the Planck satellite to be released in 2014 may shed some new light on the perturbations of $\hat{\gamma}$.

The 3-point function of the scalar mode ζ has not been detected yet. Since all 3-point functions are functions of three amplitudes of momenta, one usually considers some typical configurations parametrised by a single constant f_{NL}. The *equilateral* and *local non-Gaussianities* are [139, 132],

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle_{\text{local}} = 2\Delta^4(k_0) f_{NL}^{\text{local}} \left[\frac{1}{(k_1k_2)^{4-n_S}} + \frac{1}{(k_2k_3)^{4-n_S}} + \frac{1}{(k_3k_1)^{4-n_S}}\right], \tag{5.6.7}$$

$$\begin{aligned} \langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle_{\text{equil}} = 6\Delta^4(k_0) f_{NL}^{\text{equiv}} \Bigg[& -\frac{2}{(k_1k_2k_3)^{2(4-n_S)/3}} \\ & -\frac{1}{(k_1k_2)^{4-n_S}} - \frac{1}{(k_2k_3)^{4-n_S}} - \frac{1}{(k_3k_1)^{4-n_S}} \\ & + \left(\frac{1}{k_1^{(4-n_S)/3} k_2^{2(4-n_S)/3} k_3^{4-n_S}} + 5 \text{ permutations}\right)\Bigg]. \end{aligned} \tag{5.6.8}$$

The Planck satellite results are [140],

$$f_{NL}^{\text{local}} = 2.7 \pm 5.8, \qquad f_{NL}^{\text{equil}} = -42 \pm 75. \tag{5.6.9}$$

We can compare the results listed so far with the general form of the slow-roll correlation functions (5.5.23) and (5.5.24). As we can see, the *tensor-to-scalar ratio*,

$$r_{SR}^2(k) = \frac{\Delta_T^2}{\Delta_S^2} = 32\epsilon_*. \tag{5.6.10}$$

With the sensitivity of the Planck satellite, one obtains an estimate $\epsilon_* \lesssim 0.01$. Similarly, comparing the forms of non-Gaussianities (5.6.7) and (5.6.8) with the 3-point function (5.5.42) we find $f_{NL} = O(\epsilon_*)$ in slow-roll models, which is consistent with the measurements.

Let finish this section with a comment on how the remaining cosmological parameters such as the Hubble constant or the curvature Ω can be extracted. The basic idea is that the evolution of the CMB depends heavily on the content of the Universe and its expansion rate. For example, the position of the first peak in the spectrum in figure 5.6 is determined mostly by the abundance of the dark energy in the Universe.

The accepted model that fits all the features of the spectrum in figure 5.6 is called Λ*CDM model* (Cosmological Constant & Cold Dark Matter). The basic

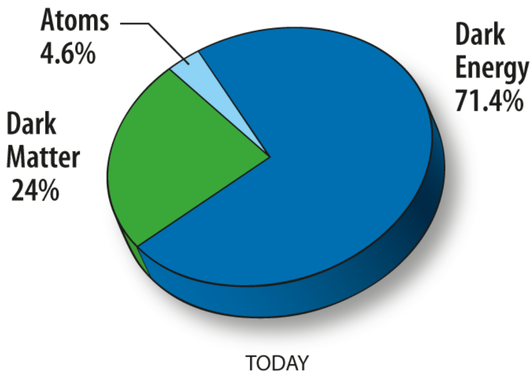


Figure 5.7: *Approximate content of our Universe today, based on the Planck and WMAP results.*

version contains six parameters: the amplitude $\Delta^2(k_0)$ and the spectral index $n_S(k_0)$ of the primordial scalar fluctuations, the Hubble constant H_0, densities of matter and dark energy respectively,

$$\Omega_m = \frac{\rho_m}{\rho_{\rm crit}}, \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\rm crit}}, \tag{5.6.11}$$

where $\rho_{\rm crit}$ is the critical density defined in (5.2.23) and the optical depth τ. This last parameter specifies a probability $P = e^{-\tau}$ that the photon emitted after the decoupling of the CMB from other particles was scattered. In principle this parameter is computable, but known astrophysical models deliver only some crude estimates. The measured value of the optical depth is $\tau = 0.089 \pm 0.014$. The values of the remaining parameters determined by the Planck satellite are

$$\Omega_m = 0.314 \pm 0.020, \qquad \Omega_\Lambda = 0.686 \pm 0.020, \tag{5.6.12}$$

see figure 5.7. Note that $\Omega_m + \Omega_\Lambda = 1.00 \pm 0.03$. This is consistent with the flat Universe and the inflation as discussed in section 5.2.2.

With the current resolution in the measurements of the CMB, the ΛCDM model can be extended to contain more parameters. In particular, one can measure the abundance of our usual baryonic matter to be about 4.5%. These measurements are in great agreement with other observations, *e.g.*, [94, 95]. Also, the age of our Universe can be calculated to be $t_0 = 13.81 \pm 0.06$ Gyr.

Chapter 6

Gauge/gravity duality

6.1. Fundamentals of AdS/CFT

6.1.1. Strings and branes

Gauge/gravity duality is an exact correspondence between theories containing (quantum) gravity and quantum field theories without gravity in one spectime dimension less. The existence of such correspondence was a standing conjuncture for some time, called a *holographic principle* [141, 142], motivated by various 'co-incidences' such as:

1. The Bekenstein-Hawking entropy of a black hole shows that the entropy is proportional to the area of the black hole and not its volume. This suggests that the degrees of freedom of quantum gravity scale with one less dimension than the dimension of spacetime.

2. In string theory, there is a relation between couplings of open and closed strings, $g_{\text{closed}} = g_{\text{open}}^2$. Since closed string carry gravitions, while open strings do not, it suggests a relation between gravitational and non-gravitational description of the low-energy limit of the string theory. The same type of correspondence was observed in a relation between $\mathcal{N} = 4$ super-Yang-Mills theory and $\mathcal{N} = 8$ supergravity in $d = 4$.

3. The effective description of flux tubes in a strongly interacting quantum gauge theory resembles the description of propagating strings. An interaction of flux tubes can be represented as a diagram of interacting strings. The genus of the Riemann surface on which such a diagram can be drawn is related to the combination $\lambda = g_{\text{string}} N$, where N is a rank of the gauge group, usually $SU(N)$. This suggests that the large N limit of a strongly interacting gauge

theory may be describable in terms of weakly interacting string theory or even classical gravity.

In 1997 Juan Maldacena [143] presented a quantitative version of the holographic principle, known as the *AdS/CFT correspondence* and refined in [144, 145]. For the pedagogical reviews see [146, 147, 148, 149]. The correspondence states that type IIB string theory on the spacetime $\mathrm{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory in $d = 4$.

Let us start with 10-dimensional super-string theory with the bosonic part of the action,

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\boldsymbol{x}\, \sqrt{\gamma}\left[\left(\gamma^{mn} g_{\mu\nu}(\boldsymbol{x}) + \epsilon^{mn} B_{\mu\nu}(\boldsymbol{x})\right) \partial_m x^\mu \partial_n x^\nu + \alpha' R_\gamma \Phi(\boldsymbol{x})\right], \tag{6.1.1}$$

where γ^{mn} is a metric on the world-sheet Σ, $g_{\mu\nu}$ and $B_{\mu\nu}$ are the background metric and B-field, R_γ is a Ricci scalar for γ and Φ is a dilaton. The low energy effective action can be obtained by calculating the β-functions for the coupling fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ and by requiring that the string theory is conformally invariant, $\beta = 0$. These equations become equations of motion for the low energy fields and one can find an appropriate action

$$S_0 = \frac{1}{(2\pi)^7 \alpha'^4} \int \mathrm{d}^{10}\boldsymbol{x} \sqrt{-g}\, e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right], \tag{6.1.2}$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{6.1.3}$$

What the Polyakov action (6.1.1) does not contain are the non-perturbative contributions from branes. The action (6.1.2) is the NS-NS sector of $\mathcal{N} = 2$ supergravity in 10 dimensions, which contains other dynamics fields, such as forms: C_0, C_2 and C_4. We will not need here the details of the supergravity theory, therefore let us consider the effective 10-dimensional action for a single Dp-brane,

$$S_p = -\frac{1}{2(2\pi)^7 (p+2)! \alpha'^4} \int \mathrm{d}^{10}\boldsymbol{x} \sqrt{-g} F_{p+2}^2, \tag{6.1.4}$$

where $F_{p+2} = \mathrm{d}A_{p+1}$ is a field strength of the appropriate gauge field. In components,

$$A_{p+1} = \frac{1}{(p+1)!} A_{\mu_1 \ldots \mu_{p+1}} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge x^{\mu_{p+1}}. \tag{6.1.5}$$

Let us look for the simplest solution to the equations of motion following from $S = S_0 + S_p$, where the brane fills a hyperplane $\mathbb{R}^{p+1} \subseteq \mathbb{R}^{10}$. To be precise, write $\boldsymbol{x} = (\vec{x}, \vec{y})$ where by x^μ for $\mu = 0, 1, \ldots, p$ we denote the coordinates parallel to the brane and by $y^u = x^{p+u}$, $u = 1, 2, \ldots, 9 - p$ coordinates perpendicular to the brane. Moreover, denote $y = |\vec{y}|$ a distance of a given point $\boldsymbol{x}$ to the brane.

The 1/2-BPS solution to the equations of motion in the Einstein frame is

$$\mathrm{d}s_E^2 = H^{-\frac{1}{2}}\mathrm{d}\vec{x}^2 + H^{\frac{1}{2}}\mathrm{d}\vec{y}^2, \tag{6.1.6}$$

$$e^{\Phi} = g_s H^{\frac{3-p}{4}}, \tag{6.1.7}$$

$$F_{8-p} = \frac{1}{g_s} *_{9-p} \mathrm{d}H, \tag{6.1.8}$$

where $H = H(y)$ is a harmonic function,

$$H(y) = 1 + \frac{Q_p}{y^{7-p}}, \qquad Q_p = (4\pi)^{\frac{5-d}{2}}\Gamma\left(\frac{7-p}{2}\right)\alpha'^{\frac{7-p}{2}}N, \tag{6.1.9}$$

where N is an integral charge of the brane.

6.1.2. AdS/CFT correspondence

As we can see in (6.1.7) for $p = 3$ the dilaton is constant, which suggests a scale-invariant low energy physics. Let us concentrate on $p = 3$ case and by the rule of superposition, we may consider a stack of N coincident $D3$-brances. The metric is then

$$\mathrm{d}s_E^2 = \left(1 + \frac{L^4}{y^4}\right)^{-\frac{1}{2}}\mathrm{d}\vec{x}^2 + \left(1 + \frac{L^4}{y^4}\right)^{\frac{1}{2}}(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_5^2) \tag{6.1.10}$$

where

$$L^4 = 4\pi g_s N \alpha'^2 \tag{6.1.11}$$

and we wrote the $\mathrm{d}\vec{y}^2$ metric in spherical coordinates. We would like to consider the *decoupling limit*,

$$\begin{aligned} &g_s \to 0, && \alpha' \to 0, \\ &u = \frac{y}{\alpha'} = \text{fixed}, && \lambda = 4\pi g_s N = \frac{L^4}{\alpha'^2} = \text{fixed}. \end{aligned} \tag{6.1.12}$$

The parameter λ is known as the *'t Hooft coupling*. The metric (6.1.10) in this limit takes form

$$\mathrm{d}s_E^2 = \alpha'\left[\frac{u^2}{\sqrt{4\pi g_s N}}\mathrm{d}\vec{x}^2 + \frac{\sqrt{4\pi g_s N}}{u^2}(\mathrm{d}u^2 + u^2\mathrm{d}\Omega_5^2)\right]. \tag{6.1.13}$$

This is a metric on $\mathrm{AdS}_5 \times S^5$ space as one can find out by substituting $z = \sqrt{4\pi g_s N}u^{-1}$, which leads to the metric in the form (5.1.37),

$$\mathrm{d}s_E^2 = \frac{L^2}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}\vec{x}^2\right] + L^2\mathrm{d}\Omega_5^2. \tag{6.1.14}$$

As we can see the radius L_{AdS} of the AdS is equal to the radius L_{S^5} of the sphere, $L_{\mathrm{AdS}} = L_{S^5} = L$.

In order to retrieve the dynamical degrees of freedom in the decoupling limit, we substitute the metric (6.1.13) back to the Polyakov action (6.1.1). The factors of α' cancel and we are left with the supergravity theory on the $\mathrm{AdS}_5 \times S^5$ background.

On the other hand the low energy limit of the quantum field theory living on the branes is known and unique. Heuristically, we can argue that such a theory consists of all open strings ending on two out of N branes. If the branes are slightly separated, then the strings between any two different branes are massive and in the low-energy limit only the strings ending on the same brane survive. The symmetry of such a theory is $U(1)^N$. In the limit where the separation goes to zero, this symmetry is enhanced to the full $U(N)$. At this point one can remove one $U(1)$ factor corresponding to the overall position of the branes, hence leaving the $SU(N)$ theory. The configuration (6.1.6) - (6.1.8) is 1/2-BPS supersymmetric with $\mathcal{N} = 4$ supersymmetry. Theory with such symmetries is the unique $\mathcal{N} = 4$ super-Yang-Mills. The bosonic part of the action is

$$S = \int \mathrm{d}^4\boldsymbol{x}\ \mathrm{Tr}\left[-\frac{1}{4g_{\mathrm{YM}}^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu X D^\mu X + \frac{g_{\mathrm{YM}}^2}{4}[X,X]^2\right] + \frac{\theta}{8\pi^2}\int \mathrm{d}^4\boldsymbol{x}\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{6.1.15}$$

The only coupling in the theory g_{YM} is given by the open string coupling as $g_{\mathrm{YM}}^2 = 4\pi g_s$.

We will not analyse the properties of the $\mathcal{N} = 4$ super-Yang-Mills theory here, since we will not make much use of them in the following parts of the thesis. The most important property is that this theory is fully conformal even on the quantum level.

Despite the fact that the AdS/CFT correspondence was checked only for a certain class of limits, it is strongly believed that it holds in general. The full correspondence states that the type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ both having the same radius L is equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory in its superconformal phase with the gauge group $SU(N)$ and the gauge coupling g_{YM} related via

$$g_{\mathrm{YM}}^2 = 4\pi g_s = 4\pi\langle e^\Phi\rangle, \qquad L^4 = 4\pi g_s N \alpha'^2. \tag{6.1.16}$$

The θ-vacuum state in $\mathcal{N} = 4$ super-Yang-Mills corresponds to the $\mathrm{AdS}_5 \times S^5$ solution with constant dilaton and the value of the axion C_0, $\theta = 2\pi C_0$. Various limits of the correspondence are listed in table 6.1.

The final task is to reduce the theory on $\mathrm{AdS}_5 \times S^5$ to the theory on AdS_5 alone. One can start with the Fourier analysis on S^5 to perform a Kaluza-Klein reduction along the sphere. For the dilaton it reads

$$\Phi = \sum_k \Phi_k Y_k(S^5), \tag{6.1.17}$$

Type IIB super-string theory on $\mathbf{AdS_5 \times S^5}$	**$\mathcal{N} = 4$ super-Yang-Mills**
Classical string theory $g_s \to 0$ Expansion in g_s	't Hooft limit of $\mathcal{N} = 4$ SYM $N \to \infty$ Expansion in N^{-2}
IIB (quantum) supergravity $\alpha' \to 0$ Expansion in α'	Large λ limit of $\mathcal{N} = 4$ SYM $\lambda \to \infty$ Expansion in $\lambda^{-1/2}$
Classical IIB supergravity $\alpha', g_s \to 0$	Large λ and N limit of $\mathcal{N} = 4$ SYM $\lambda, N \to \infty$

Table 6.1: *The AdS/CFT correspondence and its various limits.*

where $Y_k(S^5)$ is a set of spherical harmonics. However, since the radii of the AdS and the sphere in (6.1.14) are equal, one needs to keep the entire tower of fields. However, it can be shown that the 5-dimensional action

$$S = \frac{1}{2\kappa_5^2} \int \mathrm{d}^5\boldsymbol{x} \sqrt{|g|} \left[R + \frac{12}{L_{\mathrm{AdS}}^2} - \partial_\mu \Phi_k \partial^\mu \Phi_k - m_k^2 \Phi_k^2 \right], \tag{6.1.18}$$

where

$$m_k^2 L_{\mathrm{AdS}}^2 = k(k-4) \tag{6.1.19}$$

has AdS_5 as its background solution and this solution extends to the solution of the 10-dimensional action. The 5-dimensional theory is called a *consistent truncation* of the 10-dimensional one. The effective 5-dimensional Newton constant is

$$2\kappa_5^2 = 16\pi G_5 = \frac{4\pi^2}{N^2} \tag{6.1.20}$$

as we will show in the next section.

In general, the action (6.1.18) has additional interaction terms between various fields. On the CFT side each field Φ_k corresponds to a 1/2-BPS conformal primary operator of the conformal dimension $\Delta_k = k$. In general, there is a one-to-one correspondence between the conformal primary operators in the boundary QFT and the fields in the bulk.

One of the most remarkable features of the AdS/CFT correspondence that can be read from table 6.1 is the fact, that the strongly coupled regime of the QFT corresponds to a weakly coupled, classical gravity and *vice versa*. Therefore, the AdS/CFT correspondence gives a unique opportunity to investigate quantitative features of a strongly coupled QFT in terms of supergravity or – the other way around – strongly coupled, quantum gravity in terms of a weakly coupled, perturbative QFT.

6.1.3. Non-conformal branes

For the applications to cosmology we will be interested in a correspondence between a 4-dimensional gravitational theory and a 3-dimensional quantum field theory. Such a form of gauge/gravity duality was worked out in [150, 151, 152], based on [153, 154, 155]. In our case the duality follows from the considerations of a D2-brane geometry in the string theory of type IIA. In general, the duality follows from two descriptions of the same system associated with a stack of a large number of Dp-branes. The supergravity solutions (6.1.6) - (6.1.8) are given in the previous section. By a suitable substitution one can show that for general p they lead to $\mathrm{AdS}_{p+2} \times S^{8-p}$ geometry in the decoupling limit,

$$\mathrm{d}s^2 = L_{AdS}^2 \left[\left(\frac{2}{5-p} \right)^2 \left(\frac{\mathrm{d}z^2}{z^2} + z^2 \mathrm{d}\vec{x}^2 \right) + \mathrm{d}\Omega_{8-p}^2 \right], \tag{6.1.21}$$

$$e^{\Phi} = (g_s^2 N)^{\frac{7-p}{2(5-p)}} \frac{1}{N} c_1 z^{\frac{(p-3)(p-7)}{2(p-5)}}, \tag{6.1.22}$$

$$F_{8-p} = N c_2 \mathrm{d}\Omega_{8-p}, \tag{6.1.23}$$

where c_1, c_2 and L_{AdS} are constants that depend on p and α'.

The field equations can be reduced along the sphere as in the case of $\mathrm{AdS}_5 \times S^5$. The lower dimensional action reads

$$S = c_3 (N g_d^2)^{\frac{p-3}{5-p}} N^2 \int \mathrm{d}^{p+2}\boldsymbol{x} \sqrt{G} e^{\gamma\Phi} \left[R + \beta \partial_\mu \Phi \partial^\mu \Phi + C \right], \tag{6.1.24}$$

where

$$\gamma = \frac{2(p-3)}{7-p}, \tag{6.1.25}$$

$$\beta = -4 \frac{(p-1)(p-4)}{(p-7)^2}, \tag{6.1.26}$$

$$C = -2 \frac{(p-9)(p-7)}{(p-5)^2}, \tag{6.1.27}$$

and c_3 is a constant that depends on p only. From this form we see that the inverse of the $(p+2)$-dimensional Newton constant is

$$\frac{1}{2\kappa_{p+2}^2} = \frac{1}{16\pi G_{p+2}} = c_3 \lambda_p^{\frac{p-3}{5-p}} N^2 \tag{6.1.28}$$

and has a quadratic dependence on N for any p. Here we defined $\lambda_p = N g_d^2$, which is a counterpart of the 't Hooft coupling (6.1.12).

On the quantum field theory side one expects to find a non-conformal field theory if $p \neq 3$. It turns out that the dual QFT is similar to $\mathcal{N} = 4$ super-Yang-

Mills (6.1.15), the bosonic part of the action is

$$S = \int \mathrm{d}^{p+1}\boldsymbol{x}\ \mathrm{Tr}\left[-\frac{1}{4g_p^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu X D^\mu X + \frac{g_p^2}{4}[X,X]^2\right]. \tag{6.1.29}$$

The crucial difference is, however, that for $p \neq 3$ the coupling constant g_p is dimensionful. This is the only source of the scale in the theory and we can define an effective, dimensionless coupling

$$g_{\mathrm{eff}}^2(\Lambda) = g_d^2 N \Lambda^{p-3}, \tag{6.1.30}$$

which depends on the energy scale Λ.

A field theory with such a dependence is called a QFT with *generalised conformal structure*. The reason is that one can promote the coupling constant to the background field ϕ_0 and associate the Weyl transformation

$$\phi_0 = \frac{1}{g_d^2}, \qquad\qquad \delta_\sigma\phi_0 = -(p-3)\sigma\phi_0. \tag{6.1.31}$$

Such a theory is Weyl invariant. Note, however, that we are interested in correlation functions with non-vanishing value of the coupling ϕ_0, which breaks the conformal invariance of the correlation functions. However, the only scale that can appear is through the effective coupling constant, which leads to the following form of the 2-point function,

$$\langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = f(g_{\mathrm{eff}}^2(\boldsymbol{x}), N, \ldots)\frac{1}{x^{2\Delta}}, \tag{6.1.32}$$

where f is a function of dimensionless parameters in the theory. In this case, x is the only allowed dimensionful quantity, hence here

$$g_{\mathrm{eff}}^2(\boldsymbol{x}) = g_d^2 N x^{3-p}. \tag{6.1.33}$$

6.1.4. Realisations of AdS/CFT

Having discussed the foundations of the AdS/CFT correspondence in previous sections we can state the correspondence in a way, that will be useful for further uses. The equivalence of the *bulk theory*, *i.e.*, the gravitational theory living in the AdS space and the *boundary theory* is simply the statement that the partition functions of the two are equal,

$$Z_{\mathrm{bulk}}[\phi_0^{(j)}] = Z_{\mathrm{boundary}}[\phi_0^{(j)}], \tag{6.1.34}$$

where $\phi_0^{(j)}$ is a set of sources supported on the boundary of AdS. The partition function of the bulk theory is a complicated object, but in the supergravity approximation $\alpha' \to 0$ we can write

$$Z_{\mathrm{sugra}}[\phi_0^{(j)}] = \int_{\Phi^{(j)}\sim\phi_0^{(j)}} \mathcal{D}\Phi^{(j)} e^{-S_{\mathrm{sugra}}[\Phi^{(j)}]}, \tag{6.1.35}$$

where S_{sugra} is the supergravity action and the path integral extends over all bulk fields with the asymptotics at the boundary given by the QFT sources $\phi_0^{(j)}$. In a classical supergravity approximation, one can use the saddle-point approximation which leads to

$$S_{\text{sugra, on-shell}}[\phi_0^{(j)}] = -W_{\text{QFT}}[\phi_0^{(j)}]. \tag{6.1.36}$$

On the left hand side we have the action of the supergravity evaluated on the solution to the equations of motion with the boundary conditions implemented by $\phi_0^{(j)}$. On the right hand side we have an effective action of the dual QFT with sources $\phi_0^{(j)}$ evaluated in the large N, large λ limit.

The full duality states that the excitations of $\mathcal{N} = 4$ super-Yang-Mills represented as various operators acting on the θ-vacuum state correspond to the excitations of the gravity fields. In particular, there exist states in the QFT that correspond to the non-geometric, quantum description of gravity in the bulk.

We should explain what we mean by the boundary values of the fields. For example the AdS metric (5.1.37) diverges at the boundary at $z = 0$. The correct procedure for extracting the boundary sources from the bulk fields is the *conformal compactification*. Consider a metric $g_{\mu\nu}$ and a scalar field Φ in the bulk. The value of these at $z = 0$ usually either vanishes or is infinite. The boundary values are defined as the limits

$$g_{(0)ij}(\boldsymbol{x}) = \lim_{z\to 0} G^2(z,\boldsymbol{x}) g_{\mu\nu}(z,\boldsymbol{x}), \tag{6.1.37}$$

$$\phi_{(0)}(\boldsymbol{x}) = \lim_{z\to 0} F^{d-\Delta}(z,\boldsymbol{x}) \Phi(z,\boldsymbol{x}), \tag{6.1.38}$$

where G and F are smooth non-negative *defining functions*, having a single zero at the boundary, *i.e.*, $G|_{z=0} = F|_{z=0} = 0$ and $\mathrm{d}G|_{z=0}, \mathrm{d}F|_{z=0} \neq 0$. Δ here is some numerical parameter which later on will be identified as the conformal dimension of the dual operator in the CFT. For example, one can take $G(z,\boldsymbol{x}) = F(z,\boldsymbol{x}) = z$. However, the defining functions are not unique and the conformal compactification determines the boundary sources $g_{(0)\mu\nu}$ and $\phi_{(0)}$ up to an arbitrary rescaling, the Weyl transformation in other words. Thus, the boundary theory possess the conformal structure uniquely induced by the bulk fields.

It was shown that the expansion of the bulk metric near the boundary always takes the following form, known as the *Fefferman-Graham* expansion,

$$\mathrm{d}s^2 = L_{\text{AdS}}^2 \left[\frac{\mathrm{d}z^2}{z^2} + \frac{1}{z^2} \gamma_{ij}(z,\boldsymbol{x}) \mathrm{d}\boldsymbol{x}^2 \right], \tag{6.1.39}$$

where

$$\gamma_{ij}(z,\boldsymbol{x}) = g_{(0)ij}(\boldsymbol{x}) + o(z), \tag{6.1.40}$$

where $o(z)$ denote terms that vanish in $z \to 0$ limit. In the leading order (6.1.39) is the AdS metric and in general the spacetime satisfying (6.1.39, 6.1.40) asymptotically is called *asymptotically locally AdS* or AlAdS for short.

From now on we will work in the Euclidean signature rather than the Lorentzian one. We consider both the gravitational and the boundary theory to be Euclidean. This will not alter our predictions, since we will mostly work in the near-boundary region. For the issues of the Lorentzian version of the AdS/CFT correspondence, see [156, 157].

6.2. Holographic renormalisation

The discussion of the AdS/CFT correspondence was general enough to avoid a few pitfalls. The main issue is that expressions on both sides of the equality (6.1.36) suffer from severe divergences when the Lagrangian formalism is used. The supergravity action is usually infinite, since the cosmological constant term leads to the integral of the form $\Lambda \int \mathrm{d}^d \boldsymbol{x} \sqrt{g}$ which is proportional to the volume of AdS, which is infinite. One can try reviving the equality by imposing a cutoff $\epsilon > 0$ on the z variable in the supergravity action. Such a prescription leads to a regularised action, but it is not clear what type of the regularisation should be used on the QFT side.

Therefore, the equality (6.1.36) is a statement about full, renormalised theories. On the QFT side it is well-understood what it means. On the gravity side it means that the supergravity action should be accompanied by *local counterterms*. Such counterterms should be added to the regularised action and supported on the cutoff surface only, so they do not interfere with the bulk dynamics. Moreover they should be fully covariant, which means they should be built up with bulk fields and their covariant derivatives on the regulating surface. After the counterterms are added to the regularised action, the finite $\epsilon \to 0$ limit exists. The described procedure is known as the *holographic renormalisation* and was developed in [158, 159, 160, 161, 162, 163, 164].

By the functional differentiation of the equality (6.1.36) we find the correlation functions in the dual QFT,

$$\langle \mathcal{O}(\boldsymbol{x}_1) \rangle = \left. \frac{\delta S_{\text{on shell}}}{\delta \phi_0(\boldsymbol{x}_1)} \right|_{\text{All } \phi_0^{(j)}=0}, \tag{6.2.1}$$

$$\langle \mathcal{O}(\boldsymbol{x}_1) \mathcal{O}(\boldsymbol{x}_2) \rangle = - \left. \frac{\delta^2 S_{\text{on shell}}}{\delta \phi_0(\boldsymbol{x}_1) \delta \phi_0(\boldsymbol{x}_2)} \right|_{\text{All } \phi_0^{(j)}=0}, \tag{6.2.2}$$

$$\langle \mathcal{O}(\boldsymbol{x}_1) \ldots \mathcal{O}(\boldsymbol{x}_n) \rangle = (-1)^{n+1} \left. \frac{\delta^n S_{\text{on shell}}}{\delta \phi_0(\boldsymbol{x}_1) \ldots \delta \phi_0(\boldsymbol{x}_n)} \right|_{\text{All } \phi_0^{(j)}=0}. \tag{6.2.3}$$

In what follows, we will work in the Euclidean signature both on the AdS and CFT sides of the correspondence. By d we denote the spacetime dimension of the boundary field theory, so that the bulk theory lives in $D = d + 1$ dimensions. The *radial direction* in the bulk will be denoted by z coordinate and the directions parallel to the boundary will be denoted by bold symbols $\boldsymbol{x}$. Latin indices will be used for the $\boldsymbol{x}$ coordinates only, so that $\mu = z, 1, 2, \ldots, d$ and $i = 1, 2, \ldots, d$. The bulk metric will be denoted by $g_{\mu\nu}$ and the boundary metric acting as a source for the stress-energy tensor by $g_{(0)ij}$.

6.2.1. Free scalar field

Let us consider first a free scalar field Φ of mass m on a fixed AdS background

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}\boldsymbol{x}^2\right]. \tag{6.2.4}$$

We set $L^2_{\mathrm{AdS}} = 1$, so all distances are dimensionless, measured in units of L_{AdS}. The action is

$$S = \frac{1}{2}\int \mathrm{d}z\mathrm{d}^d\boldsymbol{x}\sqrt{g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right). \tag{6.2.5}$$

The equation of motion is $(\Box_g - m^2)\Phi = 0$, which leads to

$$-z^2\partial_i\partial_i\Phi - z^2\partial_z\partial_z\Phi + (d-1)z\partial_z\Phi + m^2\Phi = 0. \tag{6.2.6}$$

By $\partial_i\partial_i = \Box_0$ we denote the Laplacian in the directions $\boldsymbol{x}$ parallel to the boundary of AdS at $z = 0$. Since these directions are flat, we can Fourier transform Φ,

$$\Phi(z, \boldsymbol{p}) = \int \mathrm{d}^d\boldsymbol{x} e^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\Phi(z, \boldsymbol{x}). \tag{6.2.7}$$

In particular $\Box_0 \mapsto -p^2$. The equation of motion becomes

$$-z^2\partial_z\partial_z\Phi + (d-1)z\partial_z\Phi + (z^2p^2 + m^2)\Phi = 0. \tag{6.2.8}$$

This equation can be solved exactly and the most general solution is

$$\Phi(z, \boldsymbol{p}) = C_1 z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(pz) + C_2 z^{\frac{d}{2}} I_{\Delta-\frac{d}{2}}(pz), \tag{6.2.9}$$

where C_1 and C_2 are undetermined constants, K_ν and I_ν are Bessel functions defined in (2.A.22) and (2.A.21) and Δ is the larger solution to the equation

$$m^2 = \Delta(\Delta - d), \qquad\qquad \Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right). \tag{6.2.10}$$

Notice that the equation of motion (6.2.8) as well as the solution (6.2.9) resemble equation (5.4.3) and expression (5.4.19) known from cosmology. This is expected, since the AdS space is closely related to dS space and small differences are mostly due to our choice of the metric (6.2.4) rather than its FRW form (5.4.1).

If $\Delta - \frac{d}{2}$ is non-integral, the expansion of (6.2.9) in z is

$$\begin{aligned}\Phi(z, \boldsymbol{p}) = z^{d-\Delta} \big[&\left(\phi_{(0)}(\boldsymbol{p}) + z^2\phi_{(2)}(\boldsymbol{p}) + \ldots\right) \\ &+ \left(z^{2\Delta-d}\phi_{(2\Delta-d)}(\boldsymbol{p}) + z^{2\Delta-d+2}\phi_{(2\Delta-d+2)}(\boldsymbol{p}) + \ldots\right)\big].\end{aligned} \tag{6.2.11}$$

Since this expansion solves the second order differential equation (6.2.8), only two constants can be independent. Indeed, either by looking at the expansion of (6.2.9) or by substituting (6.2.11) back to the equation of motion (6.2.8), one finds that the constants $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$ remain undetermined, while the remaining constants can be expressed in terms of them. For example, all terms in the first bracket read

$$\phi_{(2n)} = \frac{1}{2n(2n-d-2\Delta)} \Box_0 \phi_{(2n-2)}, \quad n = 1, 2, 3, \ldots \tag{6.2.12}$$

On general grounds, if one adds some interaction terms to the action (6.2.5), the solution will still contain two undetermined integration constants. For the evolution problem in the bulk to be well-defined, one must specify the boundary conditions, which is a relation between the two constants. Furthermore, in the general setting the two modes with z^{Δ} and $z^{d-\Delta}$ dependence must always appear, since these are solutions to free field equation. In this thesis we will always use the *Dirichlet boundary conditions*, *i.e.*,

$$\phi_0 = \phi_{(0)}, \tag{6.2.13}$$

where ϕ_0 is a source in the boundary QFT. As we will see formally in the following sections, $\phi_{(2\Delta-d)}$ is related to the expectation value of the operator dual to Φ.

6.2.2. 2-point function

Looking at the expansion (6.2.11) we see that for the correct definition of the boundary source in (6.1.38), we should take Δ there to be equal to the parameter Δ defined in (6.2.10). Note that if we take as the defining function in (6.1.38) $F(z) = \lambda z$ with $\lambda > 0$, then the value of the boundary source changes accordingly. By a comparison with (1.3.37) we see that $\phi_0 = \phi_{(0)}$ transforms as a source for the scalar conformal primary operator of dimension Δ. Therefore we have found that the bulk field Φ of mass (6.2.10) is dual to the scalar conformal primary of dimension Δ.

We would like to find the 2-point function of the operator $\mathcal{O}$ dual to the field Φ. From the first part of the thesis we know that

$$\langle\!\langle \mathcal{O}(\boldsymbol{p}) \mathcal{O}(-\boldsymbol{p}) \rangle\!\rangle \sim p^{2\Delta-d}. \tag{6.2.14}$$

The only coefficient with this dependence on momentum in (6.2.11) is $\phi_{(2\Delta-d)}$ coefficient. Indeed, since this coefficient is not determined by the near-boundary expansion, its value depends non-locally on the source $\phi_{(0)}$ as we should expect.

To find the dependence of $\phi_{(2\Delta-d)}$ on $\phi_{(0)}$ define the *bulk-to-boundary propagator*

$$\mathcal{K}_{\Delta}(z, p) = \frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma\left(\Delta - \frac{d}{2}\right)} p^{\Delta-\frac{d}{2}} z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(pz), \tag{6.2.15}$$

where K_ν is the Bessel function K defined in (2.A.22). For simplicity it will be useful to substitute $\Delta = d - \lambda$ and define

$$K_\lambda(z,p) = \mathcal{K}_{d-\lambda}(z,p) = \frac{2^{-\frac{d}{2}+\lambda+1}}{\Gamma\left(\frac{d}{2}-\lambda\right)} p^{\frac{d}{2}-\lambda} z^{\frac{d}{2}} K_{\frac{d}{2}-\lambda}(pz). \tag{6.2.16}$$

The bulk-to-boundary propagator is normalised in such a way, that the linear solution to (6.2.8), regular for $z \to \infty$, with the boundary condition

$$\lim_{z\to 0} z^{-(d-\Delta)} \Phi(z,\boldsymbol{p}) = \phi_0(\boldsymbol{p}) \tag{6.2.17}$$

is

$$\Phi(z,\boldsymbol{p}) = \mathcal{K}_\Delta(z,p)\phi_0(\boldsymbol{p}) = K_\lambda(z,p)\phi_0(\boldsymbol{p}) \tag{6.2.18}$$

The regularity at $z = \infty$ follows from the fact that the metric (6.2.4) covers only a part of the entire AdS space.

By (6.2.11) the 2-point function of the dual operator is proportional to the z^Δ coefficient in the bulk-to-boundary propagator, which leads exactly to (6.2.14). This is an *ad hoc* procedure and the correct treatment will be presented in the following section. The coefficient $\phi_{(2\Delta-d)}$ is sometimes called a *response*, since it measures how the field Φ responds to the first order perturbation of its boundary value $\phi_{(0)}$.

As we have seen, the unitarity bound (1.2.23) for scalar operators required that $\Delta \geqslant \frac{d}{2} - 1$. Using (6.2.10) it implies the following *Breitenlohner-Friedman bound* on the mass of the scalar field in AdS,

$$m^2 L_{\text{AdS}}^2 \geqslant -\frac{d^2}{4} + 1. \tag{6.2.19}$$

6.2.3. Adding interactions

In this section we will finish our preparations for the procedure of holographic renormalisation by introducing interactions for the scalar field dual to a conformal primary of dimension Δ. We will work up to the second order in the source ϕ_0 and – to simplify the formulae – we will work in $D = 4$ dimensional bulk, which corresponds to a $d = 3$ dimensional CFT. The action for the scalar field is

$$S = \int \mathrm{d}z \mathrm{d}^3\boldsymbol{x} \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} m^2 \Phi^2 - \frac{1}{3} a_3 \Phi^3 \right), \tag{6.2.20}$$

where $g_{\mu\nu}$ is the fixed AdS background metric (6.2.4) and the dimension of the dual operator Δ and the mass m are

$$\Delta = 3 - \lambda, \qquad m^2 = \Delta(\Delta - d) = \lambda(\lambda - 3), \tag{6.2.21}$$

where $0 \leqslant \lambda < 1$ but not necessarily small.

The equation of motion is $-\Box_g \Phi + m^2\Phi = a_3\Phi^2$ which in momentum space gives

$$L_\lambda(z,p)\Phi(z,\boldsymbol{p}) = a_3 \int \frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d}\Phi(z,\boldsymbol{q})\Phi(z,\boldsymbol{p}-\boldsymbol{q}), \tag{6.2.22}$$

where the differential operator $L_\lambda(z,p)$ is

$$L_\lambda(z,p) = z^2p^2 - z^2\partial_z^2 + 2z\partial_z + \lambda(\lambda-3). \tag{6.2.23}$$

Besides the bulk-to-boundary propagator (6.2.16) which solves the equations

$$\left\{\begin{array}{l} L_\lambda(z,p)K_\lambda(z,p) = 0 \\ \lim_{z\to 0} z^{-\lambda}K_\lambda(z,p) = 1 \\ K_\lambda(\infty,p) = 0 \end{array}\right. \tag{6.2.24}$$

we can define the *bulk-to-bulk* propagator $G(z,p;\zeta)$ that solves the linearised equations of motion with the initial data in the bulk,

$$\left\{\begin{array}{l} L_\lambda(z,p)G_\lambda(z,p;\zeta) = \zeta^4\delta(z-\zeta) \\ \lim_{z\to 0} z^{-\lambda}G_\lambda(z,p;\zeta) = 0 \\ G_\lambda(\infty,p;\zeta) = 0 \end{array}\right. \tag{6.2.25}$$

This solves uniquely to

$$G_\lambda(z,p;\zeta) = \left\{\begin{array}{ll} (z\zeta)^{3/2}I_{\frac{3}{2}-\lambda}(pz)K_{\frac{3}{2}-\lambda}(p\zeta) & \text{for } z \leqslant \zeta, \\ (z\zeta)^{3/2}I_{\frac{3}{2}-\lambda}(p\zeta)K_{\frac{3}{2}-\lambda}(pz) & \text{for } z > \zeta, \end{array}\right. \tag{6.2.26}$$

where on the right hand side K and I are Bessel functions. The expansion in z of the bulk-to-bulk propagator is

$$G_\lambda(z,p;\zeta) = \left\{\begin{array}{ll} \frac{z^{3-\lambda}}{3-2\lambda}K_\lambda(\zeta,p) + O(z^{5-\lambda}) & \text{for } z \leqslant \zeta, \\ O(z^\lambda) & \text{for } z > \zeta. \end{array}\right. \tag{6.2.27}$$

Using these results a general solution to the problem

$$\left\{\begin{array}{l} L_\lambda(z,p)\Phi(z,\boldsymbol{p}) = f(z,\boldsymbol{p}), \\ \Phi(0,\boldsymbol{p}) = \Phi(\infty,\boldsymbol{p}) = 0 \end{array}\right. \tag{6.2.28}$$

for some function f is

$$\Phi(z,\boldsymbol{p}) = \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^4}G_\lambda(z,p;\zeta)f(\zeta,\boldsymbol{p}). \tag{6.2.29}$$

6.3. Renormalisation for $\lambda = 0$

We will start with the holographic renormalisation for the scalar field of the conformal dimension $\Delta = d$, *i.e.*, $\lambda = 0$. The first step in the renormalisation procedure is to find out what is the most general near-boundary expansion of the solution Φ. We will work perturbatievly in a_3, which means that the solution is assumed to be a power series in a_3,

$$\Phi = \Phi_0 + a_3\Phi_1 + a_3^2\Phi_2 + \dots \tag{6.3.1}$$

The equations of motion split into the infinite set,

$$(-\Box_g + m^2)\Phi_0 = 0, \tag{6.3.2}$$

$$(-\Box_g + m^2)\Phi_1 = \Phi_0^2, \tag{6.3.3}$$

$$(-\Box_g + m^2)\Phi_2 = 2\Phi_0\Phi_1, \tag{6.3.4}$$

and so on. The first equation is the free field equation (6.2.8) we have already solved and its near boundary expansion is

$$\Phi_0 = z^\lambda \left(\phi_{(0)} + z^2\phi_{(2)} + z^{3-2\lambda}\phi_{(3-2\lambda)} + \dots\right), \tag{6.3.5}$$

with $\phi_{(0)}$ and $\phi_{(3-2\lambda)}$ being undetermined coefficients. At this point the procedure depends on whether $\lambda = 0$ or $\lambda > 0$. In the first case, when the solution to Φ_0 is substituted into the right hand side of the second equation of motion (6.3.3), one finds that Φ_1 should contain logarithmic terms in z. If $\lambda > 0$, however, we can see that Φ_1 contains a sequence of terms by z^λ 'higher' than Φ_0.

Looking at the perturbative term at order n, one can see that if $\lambda > 0$, *i.e.*, if $\Delta < d$, then higher order solutions Φ_n start with the source terms of order $z^{n\lambda}$. Therefore the boundary behaviour of Φ is not altered by the interactions and $z^{d-\Delta}$ remains the leading term in Φ. This is interpreted as turning on a source for the IR relevant operator $\mathcal{O}$ in the dual CFT. Such an operator modifies the low energy physics fn the QFT, which corresponds to the interior of the bulk. On the other hand if $\lambda > 0$ *i.e.*, if $\Delta > d$ and the dual operator $\mathcal{O}$ is irrelevant, then $\Phi_n \sim z^{-n\lambda}$. In such case the near-boundary behaviour of the field is modified, which corresponds to the UV physics in the dual QFT. In case $\lambda = 0$ the dual operator is marginal, hence only a logarithmic correction appears.

Let us first analyse case $\lambda = 0$. For this case the calculations are shorter and the final solution exhibits some interesting features.

6.3.1. Renormalised action

For $\lambda = 0$ the expansion of the zeroth order solution Φ_0 is simply

$$\Phi_0 = \phi_{(0)} + z^2\phi_{(2)} + z^3\phi_{(3)} + \dots, \tag{6.3.6}$$

with undetermined $\phi_{(0)}$ and $\phi_{(3)}$. The first order solution requires a logarithmic terms and by the substitution to the equation (6.2.22) we find the expansion,

$$\begin{aligned}\Phi = \phi_{(0)} + a_3\xi_{(0)} + 2a_3\psi_{(0)}\log z + z^2\left(\phi_{(2)} + a_3\xi_{(2)} + 2a_3\psi_{(2)}\log z\right) \\ + \; z^3\left(\phi_{(3)} + a_3\xi_{(3)} + 2a_3\psi_{(3)}\log z\right) + \ldots \end{aligned} \tag{6.3.7}$$

in the first order in a_3. The boundary data is $\phi_{(0)} + a_3\xi_{(0)}$ and the undetermined coefficients are $\phi_{(3)}$ and $\xi_{(3)}$. In the notation of [165], $\phi_{(n)} = \phi_{\{0\}(n)}$ and $\xi_{(n)} = \phi_{\{1\}(n)}$. By substituting this expression to (6.2.22) we find that:

- There are no terms of order z^1,
- $\phi_{(3)}$ and $\xi_{(3)}$ are not determined by a near boundary analysis,
- The remaining coefficients are given as

$$\begin{aligned}\psi_{(0)} &= \frac{1}{6}\phi_{(0)}^2, \\ \phi_{(2)} &= \frac{1}{2}\Box_0\phi_{(0)}, \\ \psi_{(2)} &= \frac{1}{2}\Box_0\psi_{(0)}, \\ \xi_{(2)} &= \frac{1}{2}\Box_0\xi_{(0)} + \frac{1}{2}\Box_0\psi_{(0)} + \frac{1}{2}\phi_{(0)}\Box_0\phi_{(0)}, \\ \psi_{(3)} &= -\frac{1}{3}\phi_{(0)}\phi_{(3)}, \end{aligned} \tag{6.3.8}$$

where $\Box_0 = \partial_i\partial_i$.

Our main goal is to regulate the action (6.2.5) by imposing a cutoff $z \geqslant \epsilon$ and then show that by adding suitable counterterms the finite $\epsilon \to 0$ limit exists. The counterterms must be supported on the cutoff surface $z = \epsilon$ and be covariant in Φ as discussed at the beginning of this section. The regulated action reads

$$\begin{aligned} S_{\text{reg}} &= \int_{z\geqslant\epsilon} \mathrm{d}z\mathrm{d}^3\boldsymbol{x}\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}m^2\Phi^2 - \frac{1}{3}a_3\Phi^3\right) \\ &= \int_{z\geqslant\epsilon} \mathrm{d}z\mathrm{d}^3\boldsymbol{x}\sqrt{g}\left[\frac{1}{2}\Phi\left(-\Box_g + m^2 - a_3\Phi^2\right)\Phi + \frac{a_3}{6}\Phi^3\right] \\ &= \frac{a_3}{6}\int_{z\geqslant\epsilon}\mathrm{d}z\mathrm{d}^3\boldsymbol{x}\sqrt{g}\Phi^3 - \frac{1}{2}\int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{g}g^{zz}\Phi\partial_z\Phi, \end{aligned} \tag{6.3.9}$$

where the use of the equations of motion (6.2.22) leads to the boundary term. The metric induced from (6.2.4) on the slices of constant z is

$$(\gamma_z)_{ij} = \frac{1}{z^2}\delta_{ij} = \frac{1}{z^2}g_{(0)ij} \tag{6.3.10}$$

The bulk part of the regulated action can be rewritten as

$$\frac{a_3}{6}\int_{z\geqslant\epsilon}\mathrm{d}z\mathrm{d}^3\boldsymbol{x}\sqrt{g}\Phi^3 = a_3\int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\left(\frac{1}{18}\phi_{(0)}^3+\frac{1}{2}z^2\phi_{(0)}^2\phi_{(2)}-\frac{1}{2}z^3\log z\phi_{(0)}^2\phi_{(3)}\right). \tag{6.3.11}$$

The divergent part of the regularized action, after some integrations by parts, can be written down entirely in terms of the undetermined coefficients on the $z=\epsilon$ slice.

$$\begin{aligned} S_{\text{reg, div}} = -\int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\bigg[z^2\bigg(&\frac{1}{2}(\phi_{(0)}+a_3\xi_{(0)})\Box_0\phi_{(0)}+\frac{1}{2}a_3\phi_{(0)}\Box_0\xi_{(0)} \\ &+\frac{1}{2}a_3\phi_{(0)}^2\Box_0\phi_{(0)}+\frac{1}{3}a_3\log z\phi_{(0)}^2\Box_0\phi_{(0)}\bigg)+\frac{1}{9}\phi_{(0)}^3\bigg]. \end{aligned} \tag{6.3.12}$$

It is easy to check that these divergent terms can be packaged into a local functional of the bulk field, so that we can write the following counterterm,

$$S_{\text{ct}} = \int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\left(\frac{1}{2}\Phi\Box_z\Phi+\frac{1}{9}a_3\Phi^3+\frac{1}{3}a_3\Phi^2\Box_z\Phi\right). \tag{6.3.13}$$

where $\Box_z$ is the Laplacian for the metric $(\gamma_z)_{ij}$ on the slice of constant z. When these counterterms are added to the regulated action, the variation of $S_{\text{sub}} = S_{\text{reg}}+S_{\text{ct}}$ gives

$$\delta S_{\text{sub}} = \int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\delta\Phi\left[-3\phi_{(3)}+a_3\left(-3\xi_{(3)}+\frac{4}{3}\phi_{(0)}\phi_{(3)}+2\phi_{(0)}\phi_{(3)}\ln z\right)\right]. \tag{6.3.14}$$

The 1-point function of $\mathcal{O}$ can be obtained from (6.2.1) by the differentiation with respect to the sources,

$$\begin{aligned} \langle\mathcal{O}_\Phi\rangle &= -\frac{1}{\sqrt{g_{(0)ij}}}\frac{\delta W_{\text{CFT}}}{\delta\phi_0} = \frac{1}{\sqrt{g_{(0)ij}}}\frac{\delta S_{\text{ren}}}{\delta\phi_{(0)}} \\ &= \lim_{z\to0}\frac{1}{z^3}\frac{1}{\sqrt{\gamma_z}}\frac{\delta\Phi}{\delta\phi_{(0)}}\frac{\delta S_{\text{sub}}}{\delta\Phi} \\ &= -3\left(\phi_{(3)}+a_3\xi_{(3)}\right)+\frac{4}{3}a_3\phi_{(0)}\phi_{(3)}. \end{aligned} \tag{6.3.15}$$

6.3.2. 2-point function

Since the analysis of the previous section was carried out to second order in the source $\phi_{(0)}$, we can now compute 2- and 3-point functions in the dual CFT by taking derivatives of the 1-point function (6.3.15) with respect to the source and then turning the source off. As we would like to have the results in momentum space, we can use the following trick. The Fourier transform of the source in general dimension d is

$$\phi_{(0)}(\boldsymbol{p}) = \int \mathrm{d}^d\boldsymbol{x} e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\phi_{(0)}(\boldsymbol{x}), \tag{6.3.16}$$

hence

$$\frac{\delta}{\delta\phi(\boldsymbol{x})} = \int \mathrm{d}^d\boldsymbol{p}\, \frac{\delta\phi(-\boldsymbol{p})}{\delta\phi(\boldsymbol{x})}\frac{\delta}{\delta\phi(-\boldsymbol{p})} = \int \mathrm{d}^d\boldsymbol{p}\, e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\frac{\delta}{\delta\phi(-\boldsymbol{p})}. \tag{6.3.17}$$

From this one finds

$$\langle\mathcal{O}(\boldsymbol{p}_1)\ldots\mathcal{O}(\boldsymbol{p}_n)\rangle = (2\pi)^{(n-1)d}(-1)^{n-1}\frac{\delta^{n-1}\langle\mathcal{O}_\Phi(\boldsymbol{p}_1)\rangle}{\delta\phi_{(0)}(-\boldsymbol{p}_2)\ldots\delta\phi_{(0)}(-\boldsymbol{p}_n)}. \tag{6.3.18}$$

In our case the 2-point function is

$$\begin{aligned}\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\rangle &= (2\pi)^3 3\frac{\delta\phi_{(3)}(\boldsymbol{p}_1)}{\delta\phi_{(0)}(-\boldsymbol{p}_2)}\\ &= (2\pi)^3\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)p_1^3,\end{aligned} \tag{6.3.19}$$

where we used the fact that the linear solution (6.2.16) for $\lambda = 0$ reads

$$\begin{aligned}\Phi_0(z,\boldsymbol{p}) &= K(z,p)\phi_{(0)}(\boldsymbol{p})\\ &= e^{-pz}(1+zp)\phi_{(0)}(\boldsymbol{p})\\ &= \left[1-\frac{1}{2}p^2z^2+\frac{1}{3}p^3z^3+O(z^4)\right]\phi_{(0)}(\boldsymbol{p}).\end{aligned} \tag{6.3.20}$$

6.3.3. From AdS/CFT to triple-K integrals

For the 3-point function we need to solve equations of motion to first order in a_3. Since we do not want to modify the sources, we assume that the only non-vanishing part of the source is $\phi_{(0)}$ and the subleading terms vanish, *i.e.*, $\xi_{(0)} = 0$.

Let us start by rewriting the equation of motion in momentum space. Here we will reintroduce the parameter λ, defined in (6.2.21), so that we can use these results in the $\lambda > 0$ case as well. Using the convolution we can write the equation of motion (6.3.3) in momentum space as

$$\begin{aligned}L_\lambda(z,p)\Phi_1(z,\boldsymbol{p}) &= \int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\Phi_0(z,\boldsymbol{q})\Phi_0(z,\boldsymbol{p}-\boldsymbol{q})\\ &= \int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}K_\lambda(z,\boldsymbol{p})K_\lambda(z,\boldsymbol{p}-\boldsymbol{q})\phi_{(0)}(\boldsymbol{q})\phi_{(0)}(\boldsymbol{p}-\boldsymbol{q}).\end{aligned} \tag{6.3.21}$$

The solution, using the bulk-to-bulk propagator $G_\lambda(z, p; \zeta)$, can be written as

$$\begin{aligned}\Phi_1(z, \boldsymbol{p}) &= \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^4} G_\lambda(z, \boldsymbol{p}; \zeta) \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} K_\lambda(\zeta, \boldsymbol{q}) K_\lambda(\zeta, \boldsymbol{p}-\boldsymbol{q}) \phi_{(0)}(\boldsymbol{q}) \phi_{(0)}(\boldsymbol{p}-\boldsymbol{q}) \\ &= \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \phi_{(0)}(\boldsymbol{q}) \phi_{(0)}(\boldsymbol{p}-\boldsymbol{q}) \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^4} G_\lambda(z, \boldsymbol{p}; \zeta) K_\lambda(\zeta, \boldsymbol{q}) K_\lambda(\zeta, \boldsymbol{p}-\boldsymbol{q}).\end{aligned} \tag{6.3.22}$$

Since the whole dependence on z is in the second integral, we must compute

$$I_\lambda(z, \boldsymbol{p}, \boldsymbol{q}) = \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^4} G_\lambda(z, \boldsymbol{p}; \zeta) K_\lambda(\zeta, \boldsymbol{q}) K_\lambda(\zeta, \boldsymbol{p}-\boldsymbol{q}) \tag{6.3.23}$$

However, as it stands, this integral is divergent. Holographic renormalisation tells us that if we regulate the integral by imposing the cutoff $z \geqslant \epsilon$, then the divergent terms will be subtracted by the contribution following from the counterterm action (6.3.13) when the limit $\epsilon \to 0$ is taken. One can see it explicitly, by solving the quations of motion following from the counterterm action. On the other hand, we have shown how the fully renormalised 1-point function with sources (6.3.15) is expressed in terms of various coefficients of z in the regulated integral (6.3.23),

$$I_\lambda(z, \boldsymbol{p}, \boldsymbol{q}) = \int_\epsilon^\infty \frac{\mathrm{d}\zeta}{\zeta^4} G_\lambda(z, \boldsymbol{p}; \zeta) K_\lambda(\zeta, \boldsymbol{q}) K_\lambda(\zeta, \boldsymbol{p}-\boldsymbol{q}). \tag{6.3.24}$$

Since this integral is directly related to the 3-point function, Poincaré invariance implies that it can be expressed in terms of magnitudes of three momenta,

$$p_1 = |\boldsymbol{p}|, \qquad p_2 = |\boldsymbol{q}|, \qquad p_3 = |-\boldsymbol{p}-\boldsymbol{q}|. \tag{6.3.25}$$

Notice also that the integral (6.3.24) resembles the triple-K integral (2.5.1). There are however subtle differences. Firstly, the bulk-to-bulk propagator is not equal to the Bessel function K. Secondly, the integral has an explicit cutoff. Finally, we are not interested in the value of the entire integral, but only at some particular terms. Nevertheless, as we will see in section 6.4.3, if $\lambda > 0$, then (6.3.24) can be reduced to a genuine triple-K integral.

In order to write down the 3-point function, denote by $I_\lambda^{(\alpha)}(\boldsymbol{k}, \boldsymbol{q})$ the coefficient of the z^α term in $I_\lambda(z, \boldsymbol{k}, \boldsymbol{q})$ in (6.3.24) and define the reduced correlation function as in (2.3.1),

$$\langle \mathcal{O}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle = (2\pi)^d \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3) \langle\!\langle \mathcal{O}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle. \tag{6.3.26}$$

6.3.4. 3-point function

According to (6.3.15), the 3-point function for $\lambda = 0$ case is

$$\langle\!\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle\!\rangle = a_3 \left[-3 \left(I_0^{(3)}(\boldsymbol{p}_1, \boldsymbol{p}_2) + I_0^{(3)}(\boldsymbol{p}_1, \boldsymbol{p}_3) \right) + \frac{4}{9}(p_2^3 + p_3^3) \right]. \tag{6.3.27}$$

Since for $\lambda = 0$ all Bessel functions in the propagators reduce to elementary functions 2.A.24, the integral (6.3.24) can be computed exactly and in the $\epsilon \to 0$ limit we find

$$\begin{aligned} I_0(z, p_1, p_2, p_3) &= \frac{1}{9} + \frac{1}{3} \log\left(\frac{z}{\epsilon}\right) + z^2 \left[-\frac{5}{36} p_1^2 - \frac{1}{4} p_2^2 - \frac{1}{4} p_3^2 - \frac{1}{6} p_1^2 \log\left(\frac{z}{\epsilon}\right) \right] \\ &+ z^3 \left[\frac{1}{9} (p_1^2 p_2 + 5 \text{ permutations}) - \frac{1}{9} p_1 p_2 p_3 - \frac{1}{9} \sum_i p_i^3 \log(p_1 + p_2 + p_3) \right. \\ &\left. + \left(\frac{5}{27} - \frac{\gamma_E}{9} \right) \sum_i p_i^3 - \frac{1}{27} p_1^3 - \frac{1}{9} (p_2^3 + p_3^3) \log z - \frac{1}{9} p_1^3 \log \epsilon \right] + O(z^4). \end{aligned} \tag{6.3.28}$$

Notice that the solution (6.3.28) contains a homogeneous and divergent part with $\xi_{(0)} \neq 0$. This is indicated by the leading $\frac{1}{9} - \frac{1}{3} \log \epsilon$ term. It can be removed by subtracting the homogeneous solution. After the subtraction it can be checked, that (6.3.28) satisfies all relations (6.3.8) obtained by a near boundary analysis. For example, we see that the coefficient of $z^3 \log z$ is $-(p_2^3 + p_3^3)/9$. On the other hand the near-boundary analysis predicts that this should be equal to

$$2\psi_{(3)} = -\frac{2}{3} \phi_{(0)} \phi_{(3)} \quad \overset{\mathcal{F}}{\longmapsto} \quad -\frac{1}{9} \int \frac{\mathrm{d}^3 \boldsymbol{q}}{(2\pi)^3} \phi_{(0)}(\boldsymbol{q}) \phi_{(0)}(\boldsymbol{p} - \boldsymbol{q}) \cdot \left[q^3 + (\boldsymbol{p} - \boldsymbol{q})^3 \right]. \tag{6.3.29}$$

Comparing with (6.3.22) and using (6.3.25) we see that both results match.

The formula (6.3.27) gives now

$$\begin{aligned} \langle\!\langle \mathcal{O}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle &= \frac{2a_3}{3} \left[p_1 p_2 p_3 - (p_1^2 p_2 + 5 \text{ permutations}) \right. \\ &\left. + (\gamma_E - 1)(p_1^3 + p_2^3 + p_3^3) + (p_1^3 + p_2^3 + p_3^3) \log(p_1 + p_2 + p_3) \right]. \end{aligned} \tag{6.3.30}$$

Notice that the local term $\frac{4}{3} a_3 \phi_{(0)} \phi_{(3)}$ in (6.3.15) was essential for the solution being symmetric in all momenta.

At this point we can check if the AdS/CFT result agrees with the developments of the first part of the thesis. The most general form of the 3-point function of three primary scalar operators of dimension $\Delta = 3$ in $d = 3$ dimensional CFT is given by the triple-K integral (2.5.3),

$$\begin{aligned} \langle\!\langle \mathcal{O}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle\!\rangle &= C_{123} J_{0+\epsilon\{000\}} = C_{123} I_{\frac{3}{2}+\epsilon\{\frac{3}{2}\frac{3}{2}\frac{3}{2}\}} = \\ &= C_{123} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} \left[\frac{1}{3\epsilon} (p_1^3 + p_2^3 + p_3^3) \right. \\ &- \frac{1}{3} \left(p_1 p_2 p_3 - (p_1^2 p_2 + 5 \text{ permutations}) + (\gamma_E - \tfrac{4}{3})(p_1^3 + p_2^3 + p_3^3) \right. \\ &\left. \left. + (p_1^3 + p_2^3 + p_3^3) \log(p_1 + p_2 + p_3) \right) \right]. \end{aligned} \tag{6.3.31}$$

The leading singularity should be removed by appropriate counterterms and then both expressions (6.3.30) and (6.3.31) agree for

$$C_{123} = -2a_3 \left(\frac{2}{\pi}\right)^{\frac{3}{2}} + O(\epsilon) \tag{6.3.32}$$

up to local terms of the form $p_1^3 + p_2^3 + p_3^3$, which can be matched by a choice of the subleading terms in C_{123}. We find the perfect agreement.

6.4. Renormalisation for $\lambda > 0$

6.4.1. Renormalised action

Holographic renormalisation of the 3-point functions in the case $\lambda > 0$ follows the same lines as the $\lambda = 0$ case. The equation of motion (6.2.22) leads to following expansion of Φ up to first order in a_3,

$$\begin{aligned}\Phi = z^\lambda \big[&\left(\phi_{(0)} + a_3\xi_{(0)} + z^2\phi_{(2)} + a_3 z^2\xi_{(2)} + z^{3-2\lambda}\phi_{(3-2\lambda)} + a_3 z^{3-2\lambda}\xi_{(3-2\lambda)} + \ldots\right)\\ &+ a_3 z^\lambda \left(\psi_{(0)} + z^2\psi_{(2)} + z^{3-2\lambda}\psi_{(3-2\lambda)} + \ldots\right)\big]\end{aligned} \tag{6.4.1}$$

By starting with $\phi_{(0)} + a_3\xi_{(0)}$ equations of motion can be recursively solved as in the case $\lambda = 0$. We find that:

- There are no terms of order z^1 or any other order different than written above,
- $\phi_{(3-2\lambda)}$ and $\xi_{(3-2\lambda)}$ are not determined by a near boundary analysis,
- The remaining coefficients written explicitly in the expansion above are the only divergent terms apart from $\psi_{(3-2\lambda)}$ which gives a finite contribution,
- these coefficients are

$$\begin{aligned}\psi_{(0)} &= \frac{1}{3\lambda(1-\lambda)}\phi_{(0)}^2,\\ \phi_{(2)} &= \frac{1}{2(1-2\lambda)}\Box_0\phi_{(0)},\\ \psi_{(2)} &= \frac{(1+\lambda)(2-3\lambda)}{3\lambda C_\lambda(2+\lambda)}\phi_{(0)}\Box_0\phi_{(0)} + \frac{2(\partial_i\phi_{(0)})^2}{3\lambda(1-\lambda)(1-3\lambda)(2+\lambda)},\\ \xi_{(2)} &= \frac{1}{2\lambda(1-2\lambda)}\Box_0\xi_{(0)},\end{aligned} \tag{6.4.2}$$

where $\Box_0 = \partial_i\partial_i$ and $C_\lambda = (1-\lambda)(1-2\lambda)(1-3\lambda)$.

The regularized action following from (6.2.5) is given by (6.3.12). The integration of the bulk part gives in this case

$$\frac{a_3}{6}\int_{z\geqslant\epsilon} \mathrm{d}z\mathrm{d}^3\boldsymbol{x}\sqrt{g}\Phi^3 = \tag{6.4.3}$$
$$= a_3\int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\left(\frac{1}{18(1-\lambda)}z^{3\lambda}\phi_{(0)}^3 + \frac{1}{2(1-3\lambda)}z^{2+3\lambda}\phi_{(0)}^2\phi_{(2)} + \ldots\right).$$

The counterterm action that kills all divergences is

$$S_{\text{ct}} = \int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\left(\frac{1}{2(1-2\lambda)}\Phi\Box_z\Phi + \frac{\lambda}{2}\Phi^2 + \frac{a_3}{9(1-\lambda)}\Phi^3 + \frac{a_3}{3C_\lambda}\Phi^2\Box_z\Phi\right). \tag{6.4.4}$$

The variation of the regularised action plus the counterterm action leads to the 1-point function in presence of the sources,

$$\langle\mathcal{O}_\Phi\rangle = (2\lambda-3)(\phi_{(3-2\lambda)} + a_3\xi_{(3-2\lambda)}) \tag{6.4.5}$$

6.4.2. 2-point function

The 2-point function follows from the expansion of the bulk-to-boundary propagator (6.2.16)

$$K_\lambda(z,p) = z^\lambda + z^{2+\lambda}\frac{p^2}{2(2\lambda-1)} + z^{3-\lambda}\frac{\Gamma\left(\lambda-\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}\left(\frac{p}{2}\right)^{3-2\lambda} + O(z^{4+\lambda}). \tag{6.4.6}$$

Combining with (6.4.5) and (6.2.18) we find

$$\langle\!\langle\mathcal{O}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\!\rangle = \frac{4^{\lambda-1}\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}p^{3-2\lambda}. \tag{6.4.7}$$

6.4.3. 3-point function

In the same way as in section 6.3.4 we find the 3-point function to be

$$\langle\!\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = (2\lambda-3)\left(I_\lambda^{(3-\lambda)}(\boldsymbol{p}_1,\boldsymbol{p}_2) + I_\lambda^{(3-\lambda)}(\boldsymbol{p}_1,\boldsymbol{p}_3)\right), \tag{6.4.8}$$

where $I_\lambda^{(3-\lambda)}(\boldsymbol{p},\boldsymbol{q})$ is the coefficient of $z^{3-\lambda}$ in $I_\lambda(r,\boldsymbol{p},\boldsymbol{q})$ defined in (6.3.23).

We will analyse the integral (6.3.23) for $\lambda > 0$ and we will show that it simplifies to a genuine triple-K integral. Let us first consider the integral (6.3.23) for $\zeta < r$,

$$I_\lambda^{(1)}(z,p_1,p_2,p_3) = r^{3/2}K_{\frac{3}{2}-\lambda}(p_1 z)\int_\epsilon^z\frac{\mathrm{d}\zeta}{\zeta^4}\zeta^{3/2}I_{\frac{3}{2}-\lambda}(p_1\zeta)K_\lambda(\zeta,p_2)K_\lambda(\zeta,p_3). \tag{6.4.9}$$

In this region the function is analytic, so we may integrate it term-by-term in its series expansion. Our primary aim is to find a term standing at front of $z^{3-\lambda}$, so let us omit explicit coefficients at the moment. Using expansions

$$\begin{aligned} x^{3/2}K_{\frac{3}{2}-\lambda}(x) \sim K_\lambda(x,p) &= \bullet x^\lambda + \bullet x^{2+\lambda} + \bullet x^{4+\lambda} + \ldots + \bullet x^{3-\lambda} + \bullet x^{5-\lambda} + \ldots, \\ x^{3/2}I_{\frac{3}{2}-\lambda}(x) &= \bullet x^{3-\lambda} + \bullet x^{5-\lambda} + \ldots. \end{aligned} \tag{6.4.10}$$

we find

$$\begin{aligned} I_\lambda^{(1)} = & \left(\bullet z^\lambda + \bullet z^{2+\lambda} + \ldots + \bullet z^{3-\lambda} + \ldots\right) \times \\ & \times \int_\epsilon^z \mathrm{d}\zeta \left(\bullet \zeta^{-1+\lambda} + \bullet \zeta^{1+\lambda} + \bullet \zeta^{2-\lambda} + \ldots\right). \end{aligned} \tag{6.4.11}$$

In particular we see the following features:

1. The term with the lowest power of ζ is $\zeta^{-1+\lambda}$. This integrates to ζ^λ which is non-singular when $\epsilon \to 0$. Therefore the limit exists and vanishes.

2. Each term under the integral is of the form $\zeta^{n\pm\lambda}$ with $n \in \mathbb{Z}$. In the upper limit it integrates to $z^{n+1\pm\lambda}$. Similarly the expression in front of the integral has the same form. There are four expressions in total, so the final power of z can be $n \pm 2\lambda$ or an integer. Therefore $I_\lambda^{(1)}$ does not contribute to the $z^{3-\lambda}$ term.

3. The form of the expansion agrees with the near-boundary analysis.

Exact calculations give

$$I_\lambda^{(1)}(z,p_1,p_2,p_3) = \frac{z^{2\lambda}}{\lambda(3-2\lambda)} + \frac{10p_1^2 + \lambda(5-2\lambda)(p_2^2+p_3^2)}{2\lambda(2+\lambda)(2\lambda-5)(2\lambda-3)(2\lambda-1)} z^{2+2\lambda} + \ldots. \tag{6.4.12}$$

Now consider $\zeta > z$ and the integral

$$\begin{aligned} I_\lambda^{(2)}(z,p_1,p_2,p_3) = & \frac{(p_2p_3)^{3/2-\lambda}}{2^{1-2\lambda}\Gamma^2\left(\frac{3}{2}-\lambda\right)} z^{3/2} I_{\frac{3}{2}-\lambda}(p_1 z) \times \\ & \times \int_z^\infty \mathrm{d}\zeta \sqrt{\zeta} K_{\frac{3}{2}-\lambda}(p_1\zeta) K_{\frac{3}{2}-\lambda}(p_2\zeta) K_{\frac{3}{2}-\lambda}(p_3\zeta). \end{aligned} \tag{6.4.13}$$

Since the integral is analytic, we can series expand the integrand and integrate term-by-term all terms that vanish in the $z \to \infty$ limit. In this way we will obtain all coefficients of z^α for $\alpha < 0$. This leads to

$$\begin{aligned} I_\lambda^{(2)} = & \left(\bullet z^{3-\lambda} + \bullet z^{5-\lambda} \ldots\right) \times \\ & \times \int_z^\infty \mathrm{d}\zeta \left(\bullet \zeta^{-4+3\lambda} + \bullet \zeta^{-2+3\lambda} + \bullet \zeta^{-1+\lambda} + \bullet \zeta^{3\lambda} + \ldots\right). \end{aligned} \tag{6.4.14}$$

We can check our argument by computing all integrals explicitly. The exact calculations give

$$I^{(2)}_\lambda(z, p_1, p_2, p_3) = \frac{z^{2\lambda}}{3(\lambda-1)(2\lambda-3)} + \frac{2(7-8\lambda)p_1^2 + 3(\lambda-1)(2\lambda-5)(p_2^2+p_3^2)}{6(2\lambda-5)(2\lambda-3)(3\lambda-1)(2\lambda-1)(\lambda-1)} z^{2+2\lambda} + \ldots . \tag{6.4.15}$$

Adding $I^{(1)}$ and $I^{(2)}$ we find

$$I_\lambda(z, p_1, p_2, p_3) = \frac{z^{2\lambda}}{3\lambda(1-\lambda)} + \left(-\frac{(1+\lambda)(2-3\lambda)(p_2^2+p_3^2)}{6\lambda C_\lambda (2+\lambda)} + \frac{p_2^2+p_3^2-p_1^2}{3\lambda(1-\lambda)(1-3\lambda)(2+\lambda)} \right) z^{2+2\lambda} + \ldots \tag{6.4.16}$$

which is in perfect agreement with the predictions of the near-boundary analysis.

The remaining – and the most important task – is to find the coefficient of $z^{3-\lambda}$ in the expression (6.4.13). The only term which escaped our considerations is the value of the indefinite integral at infinity. This term, which is of order z^0 is multiplied by $z^{3/2} I_{3/2-\lambda}(pz)$. Due to the expansion (6.4.10) we finally find

$$\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{a_3 (p_1 p_2 p_3)^{\frac{3}{2}-\lambda}}{2^{\frac{1}{2}-3\lambda}\Gamma^3\left(\frac{3}{2}-\lambda\right)} \times \\ \times \int_z^\infty \mathrm{d}\zeta \sqrt{\zeta} K_{\frac{3}{2}-\lambda}(p_1\zeta) K_{\frac{3}{2}-\lambda}(p_2\zeta) K_{\frac{3}{2}-\lambda}(p_3\zeta) \Big|_{z^0} \tag{6.4.17}$$

where $|_{z^0}$ denotes the extraction of the coefficient of z^0. We would like to substitute $z = 0$ as the integration limit, but in such a case the integral diverge. The reason is that there are terms in (6.4.13) with negative powers of z, which are at the end multiplied by $z^{3/2} I_{3/2-\lambda}(pz)$. We know all such terms by the near-boundary analysis and we already checked that they organize themselves correctly. However, if we insisted on taking $z = 0$ as the integration limit, we would find

$$\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{a_3}{2^{\frac{1}{2}-3\lambda}\Gamma^3\left(\frac{3}{2}-\lambda\right)} I_{\frac{1}{2}\{\frac{3}{2}-\lambda, \frac{3}{2}-\lambda, \frac{3}{2}-\lambda\}}, \tag{6.4.18}$$

where $I_{\alpha\{\beta_1\beta_2\beta_3\}}$ is a genuine triple-K integral (2.5.1). This expression, however, exists by the analytic continuation as discussed in section 2.5.1, since the condition (2.5.8) is not satisfied. Therefore one can indeed take $z = 0$ limit in (6.4.17) when the resulting expression is understood as the analytic continuation of the triple-K integral. Summarising, we found the renormalised 3-point function equals to the genuine triple-K integral without any additional local terms. From the mathematical point of view this is a consequence of the uniqueness of the analytic continuation. From the physical point of view it follows from the fact that no local terms of the appropriate dimension can be added to the action of the dual CFT.

There is no simple analytic expression for the triple-K integral in (6.4.18), but one can carry out the calculations as power series in λ. By the method described in section 2.6.4 it is very easy to find the first term of order λ^{-1}. By expanding two out of three Bessel K functions one can isolate the terms that lead to singularity in λ. The result is

$$\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle = -\frac{2}{3\lambda}(p_1^3+p_2^3+p_3^3)+O(\lambda^0). \tag{6.4.19}$$

As we can see there is no finite limit $\lim_{\lambda\to 0}\langle\!\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle\!\rangle$ and in particular the solution of the case $\lambda=0$ is not a limit of the $\lambda>0$ case.

6.5. General holographic renormalisation

6.5.1. Renormalisation of the metric

The renormalisation of other fields, most notably the metric, follows the same lines as the renormalisation of the scalar field. Since the field dual to the metric is the stress-energy tensor of the dual CFT, it has a definite dimension $\Delta=d$. The Fefferman-Graham expansion reads

$$\mathrm{d}s^2=\frac{1}{z^2}\left[\mathrm{d}z^2+\gamma_{ij}(z,\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j\right], \tag{6.5.1}$$

where

$$\begin{aligned}
&\gamma_{ij}(z,\boldsymbol{x})=g_{(0)ij}+z^2g_{(2)ij}+\ldots+z^dg_{(d)ij}+\ldots && \text{if } d \text{ is odd,}\\
&\gamma_{ij}(z,\boldsymbol{x})=g_{(0)ij}+z^2g_{(2)ij}+\ldots+z^d(g_{(d)ij}+2\tilde{g}_{(d)ij}\log z)+\ldots && \text{if } d \text{ is even.}
\end{aligned} \tag{6.5.2}$$

In this expansion $g_{(0)ij}$ is a source for the stress-energy tensor of the dual theory and $g_{(d)ij}$ is the response. All other coefficients can be expressed in terms of these by means of the Einstein equations.

The procedure of the holographic renormalisation follows the same lines as for the scalar fields. The equations of motion are given by (5.1.12) - (5.1.14). For a pure AdS space we use (5.1.49) in order to find the Einstein and Ricci tensors for the metric (6.5.1). One can find that for $d>2$,

$$g_{(2)ij}=\frac{1}{d-2}\left(R_{(0)ij}-\frac{1}{2(d-1)}R_{(0)}g_{(0)ij}\right), \tag{6.5.3}$$

where $R_{(0)ij}$ is the Ricci tensor for the boundary metric $g_{(0)ij}$. For $d=3$ the counterterm action reads

$$S_{ct}=\frac{1}{2\kappa^2}\int_{z=\epsilon}\mathrm{d}^3\boldsymbol{x}\sqrt{\gamma_z}\left[d(d-1)+\frac{1}{d-2}R\right]. \tag{6.5.4}$$

In general dimension d the renormalised 1-point function of the stress-energy tensor is

$$\langle T_{ij}\rangle = \frac{d}{2\kappa^2} g_{(d)ij} + X[g_{(0)ij}, \phi_{(0)}], \tag{6.5.5}$$

where $X[g_{(0)ij}, \phi_{(0)}]$ is a computable functional of the boundary data similar to the $\frac{4}{3}a_3\phi_{(0)}\phi_{(3)}$ term in (6.3.15). Such a term contributes to coincident points of the correlation functions only. In case $d = 3$ for a pure gravity or the gravity coupled to the scalar field dual to the operator of dimension $d - \lambda$ with $0 < \lambda \ll 1$, the functional X vanishes.

6.5.2. Non-conformal branes

The results for the case of asymptotically power-law spacetimes can be obtained from the results for asymptotically AdS_{2s+1} spacetimes via a dimensional reduction on the $(2s-3)$-dimensional torus according to the results of [152]. Then, one can analytically continue the results in s, hence obtaining a more general form of the gauge/gravity duality. The most general asymptotic solution for the metric and the scalar field is [162, 150],

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[\mathrm{d}z^2 + \gamma_{ij}(z, \boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j\right], \tag{6.5.6}$$

$$\Phi = -\frac{\gamma^2}{\gamma^2 - \beta}\log z + \kappa(z, \boldsymbol{x}), \tag{6.5.7}$$

where

$$\gamma_{ij} = g_{(0)ij} + z^2 g_{(2)ij} + \ldots + z^{2s}(g_{(2s)ij} + \tilde{g}_{(2s)ij}) + \ldots, \tag{6.5.8}$$

$$\kappa = \kappa_{(0)} + z^2\kappa_{(2)} + \ldots + z^{2s}(\kappa_{(2s)} + \tilde{\kappa}_{(2s)}) + \ldots \tag{6.5.9}$$

and the value of s is

$$s = \frac{d}{2} + \frac{\gamma^2}{\gamma^2 - \beta}, \tag{6.5.10}$$

where γ and β parameters are defined in (6.1.25) and (6.1.26). Notice that $s = d/2$ for AdS asymptiotics. The coefficients $g_{(0)ij}$ and $\kappa_{(0)}$ are arbitrary sources, from which all $g_{(2n)ij}$ and $\kappa_{(2n)}$ with $n < s$ are locally determined. Similarly as in case of the of AdS asymptiotics, one can show that these coefficients are directly related to the expectation value of the boundary stress-energy tensor and the scalar operator [162, 150],

$$\langle T_{ij}\rangle = \frac{1}{2\kappa^2}(2s g_{(2s)ij}), \tag{6.5.11}$$

$$\langle \mathcal{O}\rangle = -\frac{4s}{2s-d}e^{\kappa_{(0)}}\kappa_{(2s)}. \tag{6.5.12}$$

Here the dimension of the dual operator is $\Delta = 4$.

6.5.3. Hamiltonian formalism

In general only the transverse-traceless part of the response functions $g_{(d)ij}$ in case of AdS asymptotics and $g_{(2s)ij}$ in case of power-law asymptotics is not determined locally in terms of the source. The Ward identities of the dual CFT imply that

$$0 = \langle T^i_i \rangle + (d - \Delta)\phi_{(0)}\langle \mathcal{O} \rangle, \tag{6.5.13}$$

$$0 = \nabla^i \langle T_{ij} \rangle + \langle \mathcal{O} \rangle \nabla_i \phi_{(0)}, \tag{6.5.14}$$

so that the longitudinal and trace parts of the response functions are fixed. This conclusion is valid for both AdS and power-law asymptotics. In general the first equation (6.5.13) can receive anomalous contribution from the logarithmic terms in the Fefferman-Graham expansion. In $d = 3$, however, the anomaly associated with the renormalisation of the stress-energy tensor does not occur. The anomaly is present if one considers scalar operators of a specific dimensions, *e.g.*, $\Delta = d$.

To analyse the Ward identities from the point of view of gravity it will be useful to discuss the Hamiltonian formalism. The analysis is simpler to perform in coordinates leading to the metric (5.1.39). In this case the counterpart of the Fefferman-Graham expansion (6.5.2) is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}(r, \boldsymbol{x})\mathrm{d}x^i \mathrm{d}x^j, \tag{6.5.15}$$

where

$$\gamma_{ij}(z, \boldsymbol{x}) = e^{2r} g_{(0)ij} + e^{0r} g_{(2)ij} + \ldots + (e^{-(d-2)r} g_{(d)ij} + r e^{-(d-2)r} \tilde{g}_{(d)ij}) + \ldots \tag{6.5.16}$$

when d is even and

$$\gamma_{ij}(z, \boldsymbol{x}) = e^{2r} g_{(0)ij} + e^{0r} g_{(2)ij} + \ldots + e^{-(d-2)r} g_{(d)ij} + \ldots \tag{6.5.17}$$

otherwise. The expansion of the scalar field dual to the operator of conformal dimension Δ reads

$$\Phi = e^{(\Delta - d)r}\left[\phi_{(0)} + e^{-2r}\phi_{(2)} + \ldots + e^{-(2\Delta - d)r}\phi_{(2\Delta - d)} + \ldots\right]. \tag{6.5.18}$$

The $re^{-(2\Delta-d)r}$ term, which corresponds to the logarithmic term in (6.3.7), can appear if $2\Delta - d$ is a non-negative integer.

Consider the trace Ward identity (6.5.13), which follows from the application of the operator (1.3.41) to the effective action. We would like to rewrite (1.3.41) covariantly in terms of the bulk fields. Using the expansions written above notice that

$$\dot{\gamma}_{ij} \sim 2\gamma_{ij}, \qquad \dot{\Phi} \sim -(d - \Delta)\Phi, \tag{6.5.19}$$

in leading order, where the dot denotes the derivative with respect to r. In coordinates (6.5.15) the extrinsic curvature (5.1.8) is given by (5.1.25) and therefore

$$\delta_D = \int \mathrm{d}^d\boldsymbol{x} \left[-2K^{ij}\frac{\delta}{\delta\gamma^{ij}} + (d-\Delta)\dot{\phi}_0\frac{\delta}{\delta\phi_0} \right] + O(e^{-r}) = \partial_r + O(e^{-r}). \quad (6.5.20)$$

In the Hamiltonian approach to the holographic renormalisation [149, 166], see also [161, 167, 168, 169, 170, 171, 172], one organises the expansion of the fields in eigenfunctions of the dilatation operator (6.5.20) rather than the radial coordinate r. The both expansions agree in the leading term, but differ by some subleading and recursively computable expressions. We will not pursue the details of the Hamiltonian approach here, but we will comment on two important points.

Firstly, it is easy to evaluate the trace anomaly in the Hamiltonian approach. One can expect that organisation in the eigenfunctions of the dilatation operator will be disturbed by the terms linear in r rather than exponential. These terms can be directly mapped to the anomalies of the dual CFT as explained in [159, 158].

Secondly, in the Hamiltonian formalism the response functions should be related to the canonical momentum. Indeed, the Hamiltonian approach shows that the equations (6.3.15, 6.4.5, 6.5.5, 6.5.11) can be rewritten as

$$\langle T_{ij} \rangle = -\frac{1}{\kappa^2}(\Pi_{ij})_{(d)}, \qquad \langle \mathcal{O} \rangle = \frac{1}{\kappa^2}(\Pi_\Phi)_{(\Delta)}. \quad (6.5.21)$$

where Π_{ij} and Π_Φ are the canonical momenta (5.1.21) and (5.1.22). The subscript denotes the projection onto the subspace of the eigenfunctions $A_{(n)}$ of the dilatation operator (6.5.20) satisfying

$$\delta_D A_{(n)}(\boldsymbol{x}) = -nA_{(n)}. \quad (6.5.22)$$

The advantage of the Hamiltonian formulation is that the equations (6.5.21) hold in general, in any order of perturbation. One can check that they reduce themselves to the expressions (6.3.15) and (6.4.5) for the second order perturbations.

The expressions for the canonical momenta in (6.5.21) are given by (5.1.21) and (5.1.22). In this chapter we used the gauge $N = 1$ and $N_i = 0$ for the metric (6.5.15), and so the canonical momenta read

$$\Pi_{ij} = K_{ij} - \gamma_{ij}K, \qquad \Pi_\Phi = \dot{\Phi}. \quad (6.5.23)$$

In case of asymptotically power-law spacetimes one can follow the same lines and obtain

$$\Pi_{ij} = e^{\gamma\Phi}\left[K_{ij} - \gamma_{ij}(K + \gamma\partial_r\Phi)\right], \quad (6.5.24)$$

see [150, 152] for more details.

6.6. Holographic cosmology

6.6.1. dS/CFT correspondence

Soon after the AdS/CFT correspondence was established [143, 144, 145], people turned their attention to a possible duality between CFTs and gravitational theories on spaces with positive cosmological constant. The first proposals of the *dS/CFT correspondence* [173, 174] were based on the observation that by switching the radial coordinate in AdS to the time coordinate in dS one should be able to apply the AdS/CFT correspondence to cosmology. The main reason to use a holographic approach to cosmology is a possibility to investigate the effects of quantum gravity in the very early Universe. In fact it is the only tool that allows us to study the strongly coupled gravitational systems. In principle one would like to resolve the classical singularity by allowing the Universe to undergo some phase transition from its non-geometric description to the the geometric one.

Let us now present a short history of the holographic cosmology. Despite the great success of the regular AdS/CFT correspondence, the details of the dS/CFT duality turned out to be difficult in realisation. First approaches were to try to extract the dS/CFT correspondence from the AdS/CFT duality. However, the analytic continuation of the AdS/CFT correspondence requires the existence of conformal operators in a CFT with complex conformal dimensions [174]. This leads to the conclusion that the dual CFT is non-unitary. Furthermore, such a duality is problematic from the string theory point of view, as it requires supersymmetric de Sitter solutions, see [175, 176, 177, 178].

Another possibility is to consider the de Sitter spacetime as a subspace of a brane moving in some asymptotically locally AdS background geometry [179]. If the tension of the brane is large enough, then the geometry of the universe on the brane resembles the FRW geometry with a positive cosmological constant. Then one can use the regular AdS/CFT correspondence for the entire configuration. Using this kind of reasoning one can propose the dS/dS dualities [180, 181, 182], where the cosmological theory on the de Sitter space corresponds to the quantum field theory on another de Sitter background.

Another approach is to assume that our Universe is in some metastable de Sitter vacuum state, while the true, possibly supersymmetric vacuum state of the string theory is the anti-de Sitter state [183, 175, 184, 185, 186]. Such a model is viable if the tunnelling time is long enough. The problem with these types of models is that the reinterpretation of the radial direction in AdS in terms of the time direction in our Universe leads to the conclusion that the potential for the scalar field is unbounded from below. A possible resolution is to reduce the space of solutions to the equations of motion to such combinations that have finite energy [187, 188]. Such models are characterised by a cyclic evolution from big bangs to big crunches.

The idea of a complicated vacuum structure for the quantum gravity with various vacua having positive, negative or vanishing cosmological constants lead to the idea of multiverse and eternal inflation. There have been attempts to understand these types of scenarios based on the original formulation of the dS/CFT correspondence [189, 190].

Finally, let us discuss the form of the dS/CFT correspondence known as the wavefunction of the Universe approach [191]. This idea is based on the observation that the generating functional of the gravitational theory on the Euclidean AdS $Z[\phi_{(0)}] = e^{-\Psi[\phi_{(0)}]}$ leads to the effective action Ψ which, after the Wick rotation, can be interpreted as a wave functional of the Universe [72, 192, 193]. In this approach one can apply the appropriate analytic continuations on the field theory side leading to some $Sp(N)$ non-unitary field theories [194, 195].

With any workable model of the dS/CFT correspondence it is natural to ask questions about its prediction for the cosmological observables such as the spectrum of the Cosmic Microwave Background. These issues were addressed by several papers [196, 197, 198, 199, 200, 201]. Using the philosophy of the wavefunction of the Universe approach one can schematically write

$$\Psi[\gamma_{ij}] = 1 + \mathrm{i} \int \delta g^{ij} \langle T_{ij} \rangle - \frac{1}{2} \iint \delta g^{ij} \delta g^{kl} \langle T_{ij} T_{kl} \rangle + \ldots \tag{6.6.1}$$

and by inverting this relation at late times one finds

$$\langle \delta g^{ij} \delta g^{kl} \rangle \sim \frac{1}{\langle T_{ij} T_{kl} \rangle}. \tag{6.6.2}$$

This is a very imprecise statement and we will derive the correct holographic formulae in section 6.6.3. From this, one can expect that the scalar perturbations ζ are connected to the non-vanishing trace of the stress-energy tensor of the dual field theory, while $\hat{\gamma}_{ij}$ is related to its transverse-traceless part. The violation of the scale-invariance of the CMB spectrum is therefore related to the deformation of the dual CFT by some relevant operator. Furthermore,

1. The size of the deformation is related to the amplitude of the power spectrum. Smallness of the amplitude is translated into large N of the dual field theory.

2. The conformal dimension Δ of the deforming operator is related to the spectral index. Smallness of the spectral index is reflected in the fact that the dimension $\Delta = d - \lambda$ is close to marginality, *i.e.*, $0 < \lambda \ll 1$.

3. The dependence of the spectral index on the momentum is translated into the dependence of the beta function for the coupling of the deforming operator on the renormalisation scale.

While it is straightforward to obtain the expected properties of the cosmological 2-point functions, considerations of the 3-point functions of the scalar perturbations ζ [202, 203] led to complicated results.

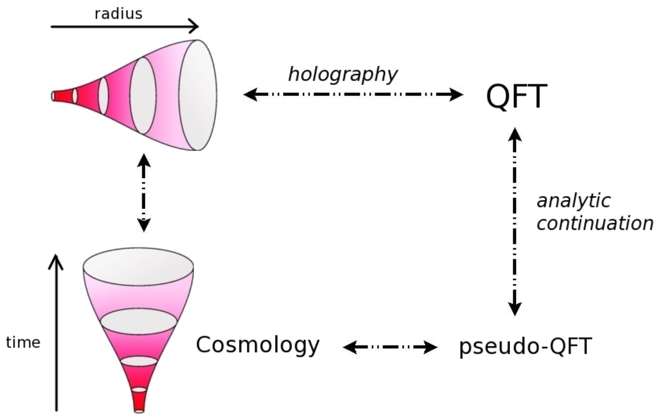


Figure 6.1: *The proposed 'dS/CFT' correspondence. The cosmological observables are mapped to the analytically continued correlation function of a QFT. See the main text for the details.*

In this thesis we will present a version of the duality that leads to a well-defined, AdS/CFT-based procedure which allows to compute all correlation functions in cosmology using the information of the dual QFT. In our procedure, see figure 6.1, we start with a cosmological solution and use the cosmology/domain wall correspondence to map it to the domain wall-like solution. This is a well-established procedure, which we will discuss in the next section and – up to the second order in perturbations – it leads to the substitutions

$$\kappa^2 \mapsto \bar{\kappa}^2 = -\kappa^2, \qquad\qquad k \mapsto p = -\mathrm{i}k, \tag{6.6.3}$$

where k denotes a magnitude of any momentum. Then, the genuine AdS/CFT can be used to map the cosmological observables to the correlation functions in a QFT. The 'dS/CFT' correspondence we propose is a correspondence between cosmologies and psuedo-QFTs that arises via translating the continuation (6.6.3) to the QFT side. It leads to,

$$N^2 \mapsto \bar{N}^2 = -N^2, \qquad\qquad k \mapsto p = -\mathrm{i}k. \tag{6.6.4}$$

We do not claim that the pseudo-QFT is a quantum field theory in any sense. Its 'correlation functions' do not satisfy Wightman/Osterwalder-Schrader axioms and we do not seek any interpretation of the pseudo-QFT in terms of non-unitary QFTs. The 'correlation functions' of the pseudo-QFT are analytically continued genuine correlation functions of a QFT realised in the AdS/CFT correspondence. Since our procedure is based on two well-developed dualities, it is a very robust realisation of the dS/CFT philosophy. Furthermore, as we will see, it is in perfect agreement with the inflationary paradigm if one considers strongly coupled QFTs.

The details of the cosmology/pseudo-QFT correspondence were developed by Paul McFadden and Kostas Skenderis in a series of papers [38, 204, 39, 40, 75]. I had a great opportunity to join their work at its final stages when the physical models of the holographic cosmology could be built [1, 2] and tested [205].

In the following sections we will present the cosmology/domain wall correspondence and then we will apply the AdS/CFT correspondence to develop the holographic formulae for cosmological observables in section 6.6.3. In the next chapter we will analyse two models of holographic cosmology. The first one will show that the inflation can be consistently reinterpreted in the language of holography. Then we will present a class of models with strongly coupled gravity at early times. Such models are inaccessible by the standard gravitational methods of cosmology.

6.6.2. Cosmology/domain wall correspondence

The cosmology/domain wall correspondence is a strict realisation of the observation that the de Sitter space and anti-de Sitter space metrics (5.1.39) and (5.1.44) differ by some signs. This suggests that there should exist a correspondence between Lorentzian solutions to Einstein equations with positive cosmological constant and Euclidean solutions with a negative one. Such a correspondence is known as the *cosmology/domain wall correspondence* and was analysed in [206, 207, 208, 209]. The name *domain wall* comes from the interpretation of the Euclidean solutions as a form of instantonic solutions interpolating between two asymptotic regimes. We will see this behaviour in section 7.2.1.

To give a precise statement of the cosmology/domain wall correspondence consider the action (5.3.3) and write the flat FRW metric as

$$\mathrm{d}s^2 = \sigma \mathrm{d}z^2 + a^2(z)\mathrm{d}\boldsymbol{x}^2, \qquad\qquad \Phi(z, \boldsymbol{x}) = \phi(z) \tag{6.6.5}$$

There is a one-to-one correspondence between the Lorentzian FRW solutions with $\sigma = -1$ and the Euclidean domain wall solutions with $\sigma = +1$. For simplicity we will consider the flat FRW geometries only, the general case can be found in [206, 207]. Furthermore there is a one-to-one correspondence between perturbations on top of the background solutions on both sides of the correspondence.

In order to write the correspondence in the simplest way, consider the action (5.3.3) with a redefined potential $V \mapsto \kappa^2 V$,

$$S = \frac{\sigma}{2\kappa^2} \int \mathrm{d}^D\boldsymbol{x} \sqrt{|g|} \left[-R + \partial_\mu \Phi \partial^\mu \Phi + 2\kappa^2 V(\Phi) \right]. \tag{6.6.6}$$

The correspondence states that for every cosmological solution with $\sigma = -1$ there is a Euclidean domain wall solution with $\sigma = +1$ that can be obtained by the following substitutions,

$$\kappa^2 \mapsto \bar{\kappa}^2 = -\kappa^2, \qquad\qquad k \mapsto p = -\mathrm{i}k, \tag{6.6.7}$$

where p and k denote magnitudes of any momenta. By $\boldsymbol{k}$ we will denote momentum in Lorentzian cosmologies as in chapter 5, while by $\boldsymbol{p}$ we will denote the continued momentum in Euclidean domain walls. Via the AdS/CFT correspondence the momentum $\boldsymbol{p}$ appears as the momentum in the dual CFT. Furthermore for ambiguous symbols such as the Newton constant, we will place the bar over the symbol to denote its Euclidean origin.

In order to prove the cosmology/domain wall correspondence, one can work out equations of motion for the background solutions and perturbations around them for both cosmologies and domain walls. For cosmologies we have already worked out all necessary expressions. One can follow the same lines for the domain walls. Keeping σ explicitly, one arrives at the following equations of motion:

1. The background equations corresponding to (5.3.12) - (5.3.14) are

$$H = \frac{\dot{a}}{a} = -\frac{1}{D-2} W(\phi), \tag{6.6.8}$$

$$\dot{\phi} = W'(\phi), \tag{6.6.9}$$

$$2\sigma\kappa^2 V(\phi) = \frac{1}{2} W'^2(\phi) - \frac{D-1}{2(D-2)} W^2(\phi), \tag{6.6.10}$$

   Therefore on the level of the background solution $\sigma \mapsto -\sigma$ is equivalent to $\kappa^2 \mapsto -\kappa^2$.

2. Equations of motion for the first order perturbations in cosmology are (5.5.13) and (5.5.14). Following the same procedure for the domain walls, see [39], one obtains,

$$0 = \ddot{\zeta}_k + \left(dH + \frac{\dot{\epsilon}}{\epsilon} \right) \dot{\zeta}_k - \sigma \frac{k^2}{a^2} \zeta_k, \tag{6.6.11}$$

$$0 = \ddot{\hat{\gamma}}_k + dH \dot{\hat{\gamma}}_k - \sigma \frac{k^2}{a^2} \hat{\gamma}_k, \tag{6.6.12}$$

   therefore indeed $\sigma \mapsto -\sigma$ corresponds to $k^2 \mapsto p^2 = -k^2$. Here we denote

$$\epsilon = -\frac{\dot{H}}{H^2} = (D-2) \frac{W'^2}{W^2}. \tag{6.6.13}$$

   Note that these equations lead to $k = \pm \mathrm{i}p$ and they do not determine the sign. The sign is fixed by the choice of vacuum. In cosmology we demand that in the far past the vacuum agrees with the Minkowski vacuum, $\zeta_k, \hat{\gamma}_k \sim \exp(-\mathrm{i}k\tau)$, see equation (5.4.18). Similar considerations for the Euclidean domain walls require the asymptotic behaviour $\zeta_p, \hat{\gamma}_p \sim \exp(-p|\tau|)$, hence $p = -\mathrm{i}k$.

3. The same form of the correspondence holds for the equations of motions for the second order perturbations. For details, see [40, 75]. The idea is to work out the Hamiltonians for the perturbations in (5.5.38) while keeping the sign explicitly. At the end it turns out that the substitutions

$$\kappa^2 \mapsto \bar{\kappa}^2 = -\kappa^2, \qquad k_j \mapsto p_j = -\mathrm{i}k_j, \quad j = 1, 2, 3 \tag{6.6.14}$$

are equivalent to the substitution $\sigma \mapsto -\sigma$. One can heuristically see it for the scalar field on the fixed AdS/dS background: the bulk-to-boundary propagator in AdS (6.2.16) resembles the v_k mode (5.4.19) with $k \mapsto p = -\mathrm{i}k$ and $\tau \mapsto -\tau$.

Using the AdS/CFT correspondence we can now translate the substitutions (6.6.7) into the language of the dual field theory. The required formula is (6.1.20) which leads to the following continuations

$$N^2 \mapsto \bar{N}^2 = -N^2, \qquad k \mapsto p = -\mathrm{i}k. \tag{6.6.15}$$

In context of [194, 195] the continuation $N^2 \mapsto -N^2$ for the group $SU(N)$ corresponds to the continuation $N \mapsto -N$ for the $SO(N)$ group. The commutation relations for the $SO(-N)$ show that this group should be in fact interpreted as the $Sp(N)$ group. The $Sp(N)$-theory contains matter violating the spin-statistics theorem. In our approach, however, the pseudo-QFT in figure 6.1 is not claimed to be a quantum field theory.

6.6.3. Holographic correlation functions

The analysis presented in this section follows from [39]. The idea is to consider variations of the 1-point function of the stress-energy tensor both from the point of view of cosmology and the dual field theory. We want to expand $\delta\langle T_{ij}(\boldsymbol{p}_1)\rangle$ in terms of the ζ and $\hat{\gamma}_{ij}$ variables defined in (5.5.7) and (5.5.9) up to the second order and compare with the definition of the response functions (5.5.54). Note, however, that we will work now in the Euclidean set-up. In principle we need to repeat all steps of section 5.5.1 for the Euclidean metric. The only difference, however, is the sign in (5.5.1), which reads now

$$\mathrm{d}s^2 = N^2\mathrm{d}z^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}z)(\mathrm{d}x^j + N^j\mathrm{d}z). \tag{6.6.16}$$

The parametrisation of the perturbations is exactly the same as in section 5.5.1.

From the QFT point of view, to linear order in the sources, the variation of the 1-point function for the stress-energy tensor is

$$\delta\langle T_{ij}(\boldsymbol{x})\rangle = -\int \mathrm{d}^3\boldsymbol{y}\sqrt{g_{(0)}}\left(\frac{1}{2}\langle T_{ij}(\boldsymbol{x})T_{kl}(\boldsymbol{y})\rangle\delta g^{kl}_{(0)}(\boldsymbol{y}) + \langle T_{ij}(\boldsymbol{x})\mathcal{O}(\boldsymbol{y})\rangle\delta\phi_{(0)}(\boldsymbol{y})\right). \tag{6.6.17}$$

In momentum space the 2-point function has the decomposition given by (2.1.16). Decomposing the boundary metric in terms of the variables defined in (5.5.2) we have

$$\delta\langle T_{ij}\rangle = \frac{1}{2}A(\boldsymbol{p})\hat{\gamma}_{(0)ij} - 2B(\boldsymbol{p})\psi_{(0)}\pi_{ij} - \langle T_{ij}(\boldsymbol{p})\mathcal{O}(-\boldsymbol{p})\rangle\delta\phi_{(0)}. \tag{6.6.18}$$

Now we can vary the 1-point function of the stress-energy tensor (6.5.21) from the AdS point of view. We have

$$\delta\langle T_{ij}\rangle = \frac{1}{\bar{\kappa}^2}[\delta K\hat{\gamma}_{ij} - \delta K_{ij}]_{(3)} = -\frac{1}{\bar{\kappa}^2}\left[2\dot{\psi}\hat{\gamma}_{ij} + p^2\dot{\chi}\pi_{ij} + \frac{1}{2}\dot{\gamma}_{ij}\right]_{(3)}. \tag{6.6.19}$$

In the present gauge we have $N = 1$ and $N_i = 0$, which translates to $\phi = \nu = \nu_i = 0$ in the variables defined in section 5.5.1. Furthermore, the constraint equations (5.1.20) in this gauge lead to

$$2\dot{\psi} = \dot{\phi}\delta\phi, \tag{6.6.20}$$

$$\begin{aligned} p^2\dot{\chi} &= \frac{p^2\psi}{a^2H} - \epsilon\dot{\zeta} = \frac{p^2\psi}{a^2H} + \frac{\bar{\kappa}^2\bar{\Omega}_{[2]}\zeta}{2a^3} \\ &= \left(\frac{p^2}{a^2H} + \frac{\bar{\kappa}^2\bar{\Omega}_{[2]}}{2a^3}\right)\psi + \frac{\bar{\kappa}^2\bar{\Omega}_{[2]}H}{2a^3\dot{\phi}}\delta\phi, \end{aligned} \tag{6.6.21}$$

$$\omega_i = 0. \tag{6.6.22}$$

Here we have used the definition of the domain wall response function $\bar{\Omega}_{[2]}$ as in (5.5.54) and expanded ζ according to its definition (5.5.7). Now, using the definition of the response function $\bar{E}$ in (5.5.63), we find

$$\delta\langle T_{ij}\rangle = \left[\frac{2\bar{E}}{a^3}\hat{\gamma}_{ij} - \left(\frac{p^2}{\bar{\kappa}^2a^2H} + \frac{\bar{\Omega}_{[2]}}{2a^3}\right)\psi\pi_{ij} - \left(\frac{H\bar{\Omega}_{[2]}}{2a^3\dot{\phi}}\pi_{ij} + \frac{\dot{\phi}}{\bar{\kappa}^2}\hat{\gamma}_{ij}\right)\delta\phi\right]_{(3)}. \tag{6.6.23}$$

Since the scale factor a has dilatation weight -1, comparison with (6.6.18) gives

$$A(\boldsymbol{p}) = 4\bar{E}_{(0)}(\boldsymbol{p}), \qquad B(\boldsymbol{p}) = \frac{1}{4}\bar{\Omega}_{[2](0)}(\boldsymbol{p}). \tag{6.6.24}$$

In these formulae, the subscript zero indicates taking the piece of the response functions with zero weight under dilatations. In accord with the Hamiltonian approach to the holographic renormalisation, to extract this piece correctly, one first expands the response functions in eigenfunctions of the dilatation operator, then determines the terms with eigenvalues less than zero through an asymptotic analysis of the response function equations of motion (5.5.57) and (5.5.58). The weight zero pieces of the response functions are then obtained by subtracting these terms with negative dilatation weight from the full response functions and taking the limit $z \to \infty$. The relevant issue here is that the subtraction of terms with negative weight (which diverge as $z \to \infty$) may induce a change in the zero weight (finite) part as well.

Fortunately, however, we are saved from having to carry out any of this analysis in detail by virtue of the fact that the cosmological formulae such as (5.5.61) and (5.5.62) for the cosmological 3-point functions involve taking the imaginary part of the cosmological response function at late times. The counterterms one subtracts to obtain the weight-zero piece of the domain-wall response functions through the procedure described above are all analytic functions of k^2, and hence under the continuation $k^2 = -p^2$, these terms remain real and so do not contribute to the imaginary part of the cosmological response functions.

By the cosmology/domain wall correspondence (6.6.7) we arrive at the final expressions

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{-1}{2\,\mathrm{Im}\langle\!\langle T(\boldsymbol{p})T(-\boldsymbol{p})\rangle\!\rangle} = \frac{-1}{8\ \mathrm{Im}\,B(p)}, \tag{6.6.25}$$

$$\langle\!\langle \hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s')}(-\boldsymbol{k})\rangle\!\rangle = \frac{-\delta^{ss'}}{\mathrm{Im}\,A(p)}, \tag{6.6.26}$$

where $p = -ik$ and the functions A and B are defined via the decomposition (2.1.16).

For the 3-point function we must expand the canonical momentum in (6.6.19) up to second order in perturbations. The calculations are rather involved and presented in [40] and [75]. For the trace of stress-energy tensor, after many cancellations, we are left with the simple result

$$\begin{aligned}(\dot{h} - h^{ij}\dot{h}_{ij})(\boldsymbol{p}_1) = \frac{1}{a^3}\bar{\Omega}_{[2]}(p_1)\psi(\boldsymbol{p}_1) + \int [[\mathrm{d}\boldsymbol{p}_2\mathrm{d}\boldsymbol{p}_3]]\frac{1}{a^3}\big[-\bar{\Omega}_{[3]}(p_i) + \bar{\Omega}_{[2]}(p_1) \\ + \frac{3}{2}\big(\bar{\Omega}_{[2]}(p_2) + \bar{\Omega}_{[2]}(p_3)\big)\big]\psi(-\boldsymbol{p}_2)\psi(-\boldsymbol{p}_3) + \ldots\end{aligned} \tag{6.6.27}$$

where $[[\mathrm{d}\boldsymbol{p}_2\mathrm{d}\boldsymbol{p}_3]]$ is defined in (5.5.55). By taking the part of the dilatation weight three and comparing to (5.5.54) we arrive at the final expression

$$\begin{aligned}\frac{\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle}{\prod_i \langle\!\langle \zeta(\boldsymbol{k}_i)\zeta(-\boldsymbol{k}_i)\rangle\!\rangle} = 2\,\mathrm{Im}\,\Big[\langle\!\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)T(\boldsymbol{p}_3)\rangle\!\rangle \\ + \sum_i \langle\!\langle T(\boldsymbol{p}_i)T(-\boldsymbol{p}_i)\rangle\!\rangle - 2\left(\langle\!\langle T(\boldsymbol{p}_1)\Upsilon(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle + 2 \text{ cyclic perms.}\right)\Big]\end{aligned} \tag{6.6.28}$$

where

$$\Upsilon(\boldsymbol{x}_1, \boldsymbol{x}_2) = \delta^{ij} \delta^{kl} \frac{\delta T_{ij}(\boldsymbol{x}_1)}{\delta g^{kl}(\boldsymbol{x}_2)} \tag{6.6.29}$$

and $\Upsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is the Fourier transform of $\Upsilon(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

The formulae (6.6.25, 6.6.26, 6.6.28) were derived for asymptotically AdS domain walls but it turns out that they also hold in the case of asymptotically power-law domain walls [39, 40, 75].

The holographic formulae for all remaining 3-point functions, valid both for AdS as well as power-law asymptotics, were developed in [75] and read

$$\begin{aligned}
&\langle\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\rangle \\
&\quad = -\frac{1}{256}\Big(\prod_i \mathrm{Im}[B(p_i)]\Big)^{-1} \times \mathrm{Im}\Big[\langle\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)T(\boldsymbol{p}_3)\rangle\rangle + 4\sum_i B(p_i) \\
&\qquad\qquad - 2\Big(\langle\langle T(\boldsymbol{p}_1)\Upsilon(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\rangle + \text{ cyclic perms.}\Big)\Big],
\end{aligned} \tag{6.6.30}$$

$$\begin{aligned}
&\langle\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\rangle \\
&\quad = -\frac{1}{32}\Big(\mathrm{Im}[B(p_1)]\mathrm{Im}[B(p_2)]\mathrm{Im}[A(p_3)]\Big)^{-1} \times \qquad (6.6.31)\\
&\qquad \times\ \mathrm{Im}\Big[\langle\langle T(\boldsymbol{p}_1)T(\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\rangle - 2\big(\Theta_1^{(s_3)}(p_i)B(p_1) + \Theta_2^{(s_3)}(p_i)B(p_2)\big) \\
&\qquad\qquad - 2\ \Big(\langle\langle \Upsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\rangle + \langle\langle T(\boldsymbol{p}_1)\Upsilon^{(s_3)}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\rangle \\
&\qquad\qquad\qquad + \langle\langle T(\boldsymbol{p}_2)\Upsilon^{(s_3)}(\boldsymbol{p}_1,\boldsymbol{p}_3)\rangle\rangle\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
&\langle\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(s_2)}(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\rangle \\
&\quad = -\frac{1}{4}\Big(\mathrm{Im}[B(p_1)]\mathrm{Im}[A(p_2)]\mathrm{Im}[A(p_3)]\Big)^{-1} \times \qquad (6.6.32)\\
&\quad \times\ \mathrm{Im}\Big[\langle\langle T(\boldsymbol{p}_1)T^{(s_2)}(\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\rangle - \frac{1}{2}\big(A(p_2) + A(p_3)\big)\theta^{(s_2 s_3)}(p_i) \\
&\qquad\qquad -\ B(p_1)\Theta^{(s_2 s_3)}(p_i) - 2\langle\langle T(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\rangle \\
&\qquad\qquad - 2\langle\langle T^{(s_3)}(\boldsymbol{p}_3)\Upsilon^{(s_2)}(\boldsymbol{p}_1,\boldsymbol{p}_2)\rangle\rangle - 2\langle\langle T^{(s_2)}(\boldsymbol{p}_2)\Upsilon^{(s_3)}(\boldsymbol{p}_1,\boldsymbol{p}_3)\rangle\rangle\Big],
\end{aligned}$$

$$\begin{aligned}
&\langle\langle \hat{\gamma}^{(s_1)}(\boldsymbol{k}_1)\hat{\gamma}^{(s_2)}(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\rangle \\
&\quad = -\Big(\prod_i \mathrm{Im}[A(p_i)]\Big)^{-1} \times \mathrm{Im}\Big[2\langle\langle T^{(s_1)}(\boldsymbol{p}_1)T^{(s_2)}(\boldsymbol{p}_2)T^{(s_3)}(\boldsymbol{p}_3)\rangle\rangle \\
&\qquad\qquad -\ \frac{1}{2}\Theta^{(s_1 s_2 s_3)}(p_i)\sum_i A(p_i) \\
&\qquad\qquad - 4\Big(\langle\langle T^{(s_1)}(\boldsymbol{p}_1)\Upsilon^{(s_2 s_3)}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\rangle + \text{ cyclic perms.}\Big)\Big].
\end{aligned} \tag{6.6.33}$$

The imaginary part in these formulae is taken after the analytic continuation

(6.6.7) or (6.6.14) is made. Our notation for the various contractions of helicity tensors is given in appendix 2.A.9. The operator Υ and it variations is defined in (2.9.8) - (2.9.10).

Note that all quantities appearing on the right hand sides of these formulae relate to the dual QFT. Each right hand side consists of an overall prefactor constructed from the 2-point function multiplying a sum of the appropriate 3-point function along with various semi-local terms. The semi-local terms vanish when all operators are at separate points, but they may be non-zero if two of the operators are coincident. In the case of (6.6.28), it was shown in [40] that these semi-local terms contribute to 'local'-type non-Gaussianity (5.6.7) and therefore cannot be neglected.

Chapter 7

Holographic models of cosmology

7.1. Holographic inflation

In this section we will present a simple model of the holographic cosmology. We will consider a dual QFT given by a small slightly relevent perturbation of a CFT. We analysed such a model from the QFT point of view in section 4.1.3. We will use the holographic formulae (6.6.25, 6.6.26) and (6.6.30) - (6.6.33) in order to find the cosmological predictions. In the next section we will show that the recovered cosmology is a hilltop inflation.

7.1.1. Holographic formulae

Exact holographic formulae expressing tree-level cosmological 2- and 3-point functions in terms of correlation functions of the dual QFT were derived in section 6.6.3. Specifically, these holographic formulae relate cosmological correlators to correlators of the stress tensor in the dual QFT. In the present case, the dual QFT is a small deformation of a CFT by a slightly relevant operator. Therefore, starting from the action (4.1.1) we may use the Ward identities (4.1.8) - (4.1.16) to replace insertions of the trace of the stress tensor with insertions of the deforming operator $\mathcal{O}$. Moreover, when working at the leading order in λ, the analytic continuations appearing in the formulae (5.5.23, 5.5.24) and (6.6.30) - (6.6.33) simply amount to a few changes of sign and may be trivially implemented. Explicitly, for adjoint fields $d_A = N^2 - 1$, where N is the rank of the gauge group, so in the leading N order all correlators are proportional to N^2. The continuation $N^2 \to -N^2$ according to (6.6.15) then contributes an overall sign. The continuation $p = -\mathrm{i}k$

on the magnitude of the momenta sends, e.g., $p^{3-2\lambda} \to \mathrm{i}p^{3-2\lambda}(1+O(\lambda))$, since $(-\mathrm{i})^{\lambda} = 1+O(\lambda)$. As the underlying CFT correlators are homogeneous functions of p whose overall scaling is $p^{3+n\lambda}$ for some integer n, this second continuation merely contributes an overall factor of i to correlators, which is eliminated upon taking the imaginary part as directed by the holographic formulae. We summarise the simplified holographic formulae obtained by these elementary manipulations below.

The leading order cosmological 2-point functions are given by

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{1}{2\lambda^2 g^2 \langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle}, \tag{7.1.1}$$

$$\langle\!\langle \hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s')}(-\boldsymbol{k})\rangle\!\rangle = \frac{\delta^{ss'}}{A(k)}, \tag{7.1.2}$$

where ζ is the scalar perturbation and $\hat{\gamma}^{(s)}$ the graviton and

$$\langle\!\langle T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle = \frac{1}{2}A(p)\delta^{ss'}. \tag{7.1.3}$$

is defined by the decomposition (2.1.16). Alternatively, the power spectra (5.5.27) and (5.5.28) are

$$\Delta^2_S(k) = \frac{k^3}{4\pi^2\lambda^2 g^2 \langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle}, \qquad \Delta^2_T(k) = \frac{2k^3}{\pi^2 A(k)}. \tag{7.1.4}$$

The cosmological 3-point functions at leading order in λ are

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle = \frac{g\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(\boldsymbol{k}_2)\mathcal{O}(\boldsymbol{k}_3)\rangle\!\rangle - \sum_{j=1}^{3}\langle\!\langle \mathcal{O}(\boldsymbol{k}_j)\mathcal{O}(-\boldsymbol{k}_j)\rangle\!\rangle}{4\lambda^3 g^4 \prod_{j=1}^{3}\langle\!\langle \mathcal{O}(\boldsymbol{k}_j)\mathcal{O}(-\boldsymbol{k}_j)\rangle\!\rangle}, \tag{7.1.5}$$

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\gamma^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle = \frac{-\lambda g\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(\boldsymbol{k}_2)T^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle - \langle\!\langle \mathcal{O}(\boldsymbol{k}_3)T^{(s)}(-\boldsymbol{k}_3)\rangle\!\rangle}{2\lambda^3 g^3 \langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle\langle\!\langle \mathcal{O}(\boldsymbol{k}_2)\mathcal{O}(-\boldsymbol{k}_2)\rangle\!\rangle A(\boldsymbol{k}_3)}, \tag{7.1.6}$$

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\gamma^{(s_2)}(\boldsymbol{k}_2)\gamma^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{1}{\lambda g\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle A(k_2)A(k_3)}\times \\ \times \Bigg[\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)T^{(s_2)}(\boldsymbol{k}_2)T^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle &+ \frac{1}{4}\lambda g\Theta^{(s_2s_3)}\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle \\ - 2\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\Upsilon_T^{(s_2s_3)}(-\boldsymbol{k}_1)\rangle\!\rangle\Bigg]&.\end{aligned} \tag{7.1.7}$$

The 2-point terms in the numerators of the holographic formulae above are important, despite the fact they represent semi-local contact terms, *i.e.*, terms that are non-analytic in only one of the three momentum magnitudes, or equivalently in position space, terms that contribute only when two of the three insertion points are brought together. Due to the product of three 2-point functions in the denominator of the holographic formulae, the 2-point terms in the numerator produce a net contribution to the cosmological correlator that is non-analytic in *two* momenta, and hence may contribute to so-called local-type non-Gaussianity. It is therefore essential to correctly compute and retain these semi-local contributions in the numerator of the holographic formulae.

7.1.2. Power spectra

Inserting (4.1.55) into the holographic formula (6.6.25), we find the scalar power spectrum

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{6}{\pi^2\alpha\lambda^2 g^2 k^3}\, k^{2\lambda}\Big[1+\frac{g}{\phi_1}k^{-\lambda}\Big]^4, \tag{7.1.8}$$

where g is a renormalised, dimensionful coupling constant (4.1.21), ϕ_1 is a position of the IR fixed point (4.1.23) and α is a normalisation constant of the 2-point function (4.1.30). Referring back to (4.1.27), we see the renormalised dimensionless coupling ϕ is essentially arbitrary as it depends on the initial condition $g(\Lambda_0) = g$ when we integrated the β-function. This arbitrariness corresponds to our freedom to rescale the operator $\mathcal{O}$ in the perturbed action. It is therefore useful to repackage this arbitrariness into a momentum scale k_0 defined by

$$k_0^{-\lambda} \equiv \frac{\phi_1}{g} = \Big(\frac{\phi_1}{\phi_0}-1\Big)\Lambda_0^{\lambda}. \tag{7.1.9}$$

The 2-point function reads now

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{24\, C^2}{\alpha\lambda^4 k^3}\Big(\frac{k}{k_0}\Big)^{-2\lambda}\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^4. \tag{7.1.10}$$

In terms of the parametrisation (5.5.27) we find

$$\Delta_S^2(k) = \frac{1}{16}\Delta_S^2(k_0)\Big(\frac{k}{k_0}\Big)^{-2\lambda}\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^4, \tag{7.1.11}$$

$$\Delta_S^2(k_0) = \frac{192\, C^2}{\pi^2\alpha\lambda^4}. \tag{7.1.12}$$

The spectral tilt (5.5.25) is

$$n_S - 1 \equiv \frac{\mathrm{d}\log\Delta_S^2(k)}{\mathrm{d}\log k} = 2\lambda - 4\lambda\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^{-1}, \tag{7.1.13}$$

and so for small momenta $k/k_0 \ll 1$ we obtain a red-tilted spectrum with $n_S - 1 \approx -2\lambda$, while for large momenta $k/k_0 \gg 1$ we obtain a blue tilt $n_S - 1 \approx 2\lambda$. This behaviour reflects the fact that $\mathcal{O}$ is marginally relevant in the UV but marginally irrelevant in the IR, i.e., $\Delta_{UV} = 3-\lambda$ while $\Delta_{IR} = 3+\lambda$ as we will see in section 7.2.1.

The tensor power spectrum at leading order in λ may be found by inserting (4.1.85) into (6.6.26), giving

$$\langle\!\langle \hat{\gamma}^{(s)}(\boldsymbol{k})\hat{\gamma}^{(s')}(-\boldsymbol{k})\rangle\!\rangle = \frac{24}{\pi^2 \alpha_T k^3}\,\delta^{ss'}. \tag{7.1.14}$$

or equivalently

$$\Delta_T^2(k) = \frac{48}{\pi^4 \alpha_T} \tag{7.1.15}$$

The exact scale invariance of the tensor power spectrum at leading order in λ is due to the fact that α_{TT} defined in (4.1.10) vanishes in the leading order in λ and thus the correlator $\langle \mathcal{T}^{(s)}\mathcal{T}^{(s')}\mathcal{O}\rangle_0$ vanishes, as we showed in section 4.1.10.

One can see this behaviour from the AdS/CFT point of view as well. In fact, α_{TT} (and hence B_{ijkl} in (4.1.80)) vanishes for the CFT dual to Einstein gravity. To see this, note first that α_{TT} is a property of the UV CFT alone (as opposed to the full perturbed theory), and so may be extracted from an AdS/CFT calculation on an exact AdS background. For the CFT correlator $\langle \mathcal{T}_{ij}\mathcal{T}_{kl}\mathcal{O}\rangle_0$ to be nonzero we would require a nonvanishing graviton-graviton-scalar coupling in the expansion of the bulk action about this background. Given a bulk action of the form

$$\frac{1}{2\kappa^2}\int \mathrm{d}^4\boldsymbol{x}\sqrt{-g}[R - \partial_\mu\Phi\partial^\mu\Phi - 2\kappa^2 V(\Phi)], \tag{7.1.16}$$

perturbing about an AdS background involves setting $\Phi = \phi_0 + \delta\phi$, where ϕ_0 is a constant. A graviton-graviton-scalar vertex may then only come from the expansion of $\sqrt{-g}V(\Phi)$, yet since the background is a solution of the scalar field equation of motion this term is a tadpole and vanishes.[1]

For the CFT dual to Einstein gravity then, the correlator $\langle \mathcal{T}_{ij}\mathcal{T}_{kl}\mathcal{O}\rangle_0$ and hence α_{TT} must vanish. In consequence in the QFT one cannot generate a fixed stress tensor insertion from contracting an integrated scalar insertion with a fixed stress tensor insertion, at least at leading order in λ. (At higher order, however, this should still be possible in order to generate a nontrivial momentum dependence in the tensor 2-point function, and hence a nonvanishing tilt in the inflationary tensor power spectrum through the holographic formula (7.1.4).)

Yet another reason for the vanishing $\langle \mathcal{T}_{ij}\mathcal{T}_{kl}\mathcal{O}\rangle_0$ correlator in the leading order in λ is the observation of [75] that the leading order cosmological correlator $\langle\!\langle \zeta\hat{\gamma}^{(s_2)}\hat{\gamma}^{(s_3)}\rangle\!\rangle$ has the helicity structure of $\theta^{(s_2 s_3)}$ defined in (2.A.108). As we saw in (4.1.117), the helicity structure of $\langle \mathcal{O}\mathcal{T}^{(s_2)}\mathcal{T}^{(s_3)}\mathcal{O}\rangle_0$ is $\delta^{s_2 s_3}$ and these two facts turn out to be inconsistent when the holographic formula (6.6.32) is applied.

[1]This may also be seen from the explicit calculation of the cubic interaction terms in [72]. When the time derivative of the background scalar field vanishes we must use the gauge (3.2); the graviton-graviton-scalar vertex is then given by the first line of (3.17), which vanishes after taking into account (3.11).

We stress also that this conclusion is specific to Einstein gravity; for more general bulk actions it may be possible to obtain a nonvanishing α_{TT} which would allow integrated scalars to contract with fixed stress tensors. It would be interesting to explore this further in specific models.

7.1.3. Three scalars

Inserting our earlier results (4.1.55) and (4.1.67) into (7.1.5) and using (7.1.9), we obtain

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{576\, C^4}{\alpha^2\lambda^7}\Big(-1+2\Big[1+\Big(\frac{k_3}{k_0}\Big)^{\lambda}\Big]^{-1}\Big) \\ &\times \sum_{i<j} k_i^{-3}k_j^{-3}\Big(\frac{k_i}{k_0}\Big)^{-2\lambda}\Big(\frac{k_j}{k_0}\Big)^{-2\lambda}\Big[1+\Big(\frac{k_i}{k_0}\Big)^{\lambda}\Big]^4\Big[1+\Big(\frac{k_j}{k_0}\Big)^{\lambda}\Big]^4, \qquad (7.1.17)\end{aligned}$$

where C is the OPE constant defined in (4.1.31). Equivalently,

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle = -\frac{1}{2}(n_S(k_3)-1)\sum_{i<j}\langle\!\langle \zeta(\boldsymbol{k}_i)\zeta(-\boldsymbol{k}_i)\rangle\!\rangle\langle\!\langle \zeta(\boldsymbol{k}_j)\zeta(-\boldsymbol{k}_j)\rangle\!\rangle, \quad (7.1.18)$$

where k_3 is the nonzero reference momentum chosen as the largest of the three momenta. Thus, at leading order in λ, we obtain a scalar 3-point function of purely local form.

To set up a comparison with the standard inflationary results in section 7.2.4 it is useful to record here that in the quasi-equilateral case (4.1.57) where all the $k_i^{-\lambda}$ are comparable, (7.1.18) reduces to

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{576\, C^4}{\alpha^2\lambda^7}\Big(-1+2\Big[1+\Big(\frac{k_3}{k_0}\Big)^{\lambda}\Big]^{-1}\Big)\Big(\frac{k_3}{k_0}\Big)^{-4\lambda}\times \\ &\times\ \Big[1+\Big(\frac{k_3}{k_0}\Big)^{\lambda}\Big]^8\sum_{i<j}k_i^{-3}k_j^{-3}, \qquad (7.1.19)\end{aligned}$$

where the interpretation of the terms in the prefactor will become transparent in section 7.2.4.

In the squeezed limit where, e.g., k_1 becomes small, the leading behaviour of (7.1.18) is

$$\lim_{q_1\to 0}\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle \approx -(n_S(k_3)-1)\langle\!\langle \zeta(\boldsymbol{k}_3)\zeta(-\boldsymbol{k}_3)\rangle\!\rangle\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(-\boldsymbol{k}_1)\rangle\!\rangle \quad (7.1.20)$$

in accordance with the consistency relation of [72]. Indeed, a simple yet independent way to understand this behaviour is to observe that

$$\begin{aligned}
-(n_S(k)-1)\langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle &= -\langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle \frac{\partial}{\partial \log k}\log \Delta_S^2(k) \\
&= \Big(-3+\frac{\partial}{\partial \log k}\Big)\langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle \qquad (7.1.21)\\
&= \lambda g \langle\!\langle \mathcal{O}(0)\mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle - 2\lambda \langle\!\langle \mathcal{O}(\boldsymbol{k})\mathcal{O}(-\boldsymbol{k})\rangle\!\rangle,
\end{aligned}$$

where in the first line we used the holographic formula for the power spectrum (6.6.25) and in the second line we used our earlier relation (4.1.69). Thus, when $k_1 \ll k_2 \approx k_3$, the leading behaviour of (λ times) the numerator of the holographic formula (7.1.5) is

$$\begin{aligned}
\lambda g \langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(\boldsymbol{k}_2)\mathcal{O}(\boldsymbol{k}_3)\rangle\!\rangle - \sum_{i=1}^{3} \lambda \langle\!\langle \mathcal{O}(\boldsymbol{k}_i)\mathcal{O}(-\boldsymbol{k}_i)\rangle\!\rangle \approx \\
\approx\ (n_S(k_3)-1)\langle\!\langle \mathcal{O}(\boldsymbol{k}_3)\mathcal{O}(-\boldsymbol{k}_3)\rangle\!\rangle, \qquad (7.1.22)
\end{aligned}$$

since $\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle \ll \langle\!\langle \mathcal{O}(\boldsymbol{k}_3)\mathcal{O}(-\boldsymbol{k}_3)\rangle\!\rangle$. Meanwhile, the leading behaviour of (λ times) the denominator of the holographic formula (7.1.5) is

$$\begin{aligned}
&\frac{1}{4\lambda^4 g^4 \langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle \langle\!\langle \mathcal{O}(\boldsymbol{k}_3)\mathcal{O}(-\boldsymbol{k}_3)\rangle\!\rangle^2} \\
&\qquad = \frac{1}{\langle\!\langle \mathcal{O}(\boldsymbol{k}_3)\mathcal{O}(-\boldsymbol{k}_3)\rangle\!\rangle} \langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(-\boldsymbol{k}_1)\rangle\!\rangle \langle\!\langle \zeta(\boldsymbol{k}_3)\zeta(-\boldsymbol{k}_3)\rangle\!\rangle, \qquad (7.1.23)
\end{aligned}$$

and so overall we recover the leading behaviour (7.1.20). These considerations also illustrate the importance of the semi-local contact terms in the numerator of the holographic formula (7.1.5).

7.1.4. Two scalars and a graviton

For two scalars and a graviton, in the quasi-equilateral case where all three momenta are of comparable magnitude, inserting our previous results (4.1.55), (4.1.85), (4.1.86), (4.1.96) and (4.1.98) into the holographic formula (7.1.6), we find

$$\begin{aligned}
\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(s)}(\boldsymbol{k}_3)\rangle\!\rangle = \frac{144\, C^2}{\sqrt{2}\pi^2\lambda^4\alpha\alpha_T}\frac{[1+(k_3/k_0)^{\lambda}]^4}{(k_3/k_0)^{2\lambda}} \times \\
\times\ \frac{J^2}{a_{123}^2 c_{123}^3 k_3^2}\big[a_{123}^3 - a_{123}b_{123} - c_{123}\big], \qquad (7.1.24)
\end{aligned}$$

where J^2, a_{123}, b_{123} and c_{123} were defined previously in (4.1.97) and (2.6.18).

7.1.5. Two gravitons and one scalar

Turning now to the case of two gravitons and one scalar, from our results in Section 4.1.9 we see that the first term in the numerator of the holographic formula (7.1.7) vanishes at leading order in λ. Any nonzero contribution must then come from the remaining semi-local terms in the numerator.

The operator $\Upsilon_T^{(s_2 s_3)}$ in (7.1.7) is defined as in (2.9.10),

$$\Upsilon_T^{(s_2 s_3)}(\boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{1}{4}\epsilon^{(s_2)ij}(-\boldsymbol{k}_2)\epsilon^{(s_3)kl}(-\boldsymbol{k}_3)\Upsilon^T_{ijkl}(\boldsymbol{k}_2, \boldsymbol{k}_3) \tag{7.1.25}$$

of the local operator Υ^T_{ijkl} constructed by taking the variation of the stress tensor with respect to the metric,

$$\frac{\delta T_{ij}(\boldsymbol{x}_1)}{\delta g^{kl}(\boldsymbol{x}_2)}\Big|_{g=\delta} \equiv \Upsilon^T_{ijkl}(\boldsymbol{x}_1)\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{7.1.26}$$

From the expression (4.1.71) for the stress tensor, we have

$$\Upsilon^T_{ijkl} = \Upsilon^{\mathcal{T}}_{ijkl} + g\mathcal{O}\delta_{i(k}\delta_{l)j}, \tag{7.1.27}$$

and hence

$$\Upsilon_T^{(s_2 s_3)}(\boldsymbol{k}_2, \boldsymbol{k}_3) = \Upsilon_{\mathcal{T}}^{(s_2 s_3)}(\boldsymbol{k}_2, \boldsymbol{k}_3) + \frac{1}{4}g\mathcal{O}(\boldsymbol{k}_2)\theta^{(s_2 s_3)} \tag{7.1.28}$$

where $\theta^{(s_2 s_3)}$ is defined in (2.A.108) and $\Upsilon^{\mathcal{T}}_{ijkl}$ is defined as in (7.1.26) replacing T_{ij} with its pure CFT part $\mathcal{T}_{ij}$. The contact term appearing in the numerator of the holographic formulae (7.1.7) is then

$$\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\Upsilon_T^{(s_2 s_3)}(\boldsymbol{k}_2, \boldsymbol{k}_3)\rangle\!\rangle = \langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\Upsilon_{\mathcal{T}}^{(s_2 s_3)}(\boldsymbol{k}_2, \boldsymbol{k}_3)\rangle\!\rangle + \frac{1}{4}g\langle\!\langle \mathcal{O}(\boldsymbol{k}_1)\mathcal{O}(-\boldsymbol{k}_1)\rangle\!\rangle\theta^{(s_2 s_3)}. \tag{7.1.29}$$

Since the dilatation dimension of $\Upsilon^{\mathcal{T}}_{ijkl}$ is three, the only way the first correlator on the right hand side could be nonzero would be if an integrated insertion of $\mathcal{O}$ could be contracted with the fixed $\Upsilon^{\mathcal{T}}_{ijkl}$ insertion to generate a fixed insertion of $\mathcal{O}$; otherwise we would be left with a CFT 2-point function of two operators with different dimensions which vanishes identically. Contractions of this form are governed by the OPE

$$\mathcal{O}(\boldsymbol{x}_1)\Upsilon^{\mathcal{T}}_{ijkl}(\boldsymbol{x}_2, \boldsymbol{x}_3) = D_{ijkl}(\boldsymbol{x}_{12})\mathcal{O}(\boldsymbol{x}_2) + \ldots, \tag{7.1.30}$$

where D_{ijkl} scales as $1/x_{12}^3$. Due to this scaling, there can be no contribution to (7.1.29) at order λ^0 (*cf.* our earlier arguments in Section 4.1.7 concerning A_{ij}); rather, the lowest possible order is λ.

In fact, we expect one further cancellation to occur for the CFT dual to Einstein gravity: since in slow-roll inflation the correlator $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ is of order λ^0 (see

(7.2.40) and (7.2.41)), recalling that $g = \phi_1 k_0^\lambda$ is of order λ we see the numerator of the holographic formula (7.1.7) must be of order λ^2. To achieve this, the OPE coefficient D_{ijkl} should have a transverse traceless piece of the right form to produce a cancellation at order λ between the correlators on the right hand side of (7.1.29). Since D_{ijkl} is a property of the UV CFT it should be computable from AdS/CFT, although we will not pursue this calculation here.

To obtain the first nonvanishing contribution to $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ therefore requires an extension of our present analysis to higher order in λ. We leave this for future work, but in the meantime note that the inflationary consistency condition for $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ may be straightforwardly verified through steps parallel to those in section 7.1.3.

Specifically, on dimensional grounds, the function $A(k)$ appearing in the tensor 2-point function (7.1.3) must in general be of the form $A(k) = k^3 F(gk^{-\lambda})$ for some function F, since when g vanishes we recover the exact CFT correlator with $A_0(k) \sim k^3$. On the one hand, then, we have the Callan-Symanzik equation

$$0 = \Big(\frac{\partial}{\partial \log k} + \lambda g \frac{\partial}{\partial g} - 3\Big)\langle\!\langle T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle, \tag{7.1.31}$$

while on the other hand, from the holographic formula for the tensor power spectrum (7.1.4), we have

$$n_T(k) \equiv \frac{\mathrm{d}\log \Delta_T^2(k)}{\mathrm{d}\log k} = 3 - \frac{\mathrm{d}\log A(k)}{\mathrm{d}\log k}. \tag{7.1.32}$$

Combining these relations we have

$$\begin{aligned}
\lambda g \langle\!\langle \mathcal{O}(0) T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle &= -\lambda g \frac{\partial}{\partial g}\langle\!\langle T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle \\
&= \Big(\frac{\partial}{\partial \log k} - 3\Big)\langle\!\langle T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle \\
&= -n_T(k)\langle\!\langle T^{(s)}(\boldsymbol{k})T^{(s')}(-\boldsymbol{k})\rangle\!\rangle.
\end{aligned} \tag{7.1.33}$$

Of course in our present leading order analysis these quantities vanish, but in general this will not be so at higher orders.

Considering now the holographic formula (7.1.7) for $\langle\zeta\hat{\gamma}\gamma\rangle$, in the squeezed limit $k_1 \ll k_2 \approx k_3$ the leading behaviour of the numerator will be governed by the first term, as the remaining semi-local contact terms, which are functions of k_1 only, become small. Approximating this first term using (7.1.33) above, and using the power spectra formulae (6.6.25)and (6.6.26), we then recover the expected consistency relation [72]

$$\begin{aligned}
&\lim_{k_1\to 0}\langle\!\langle \zeta(\boldsymbol{k}_1)\gamma^{(s_2)}(\boldsymbol{k}_2)\gamma^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle \approx \\
&\qquad \approx -n_T(k_3)\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(-\boldsymbol{k}_1)\rangle\!\rangle\langle\!\langle \hat{\gamma}^{(s_2)}(\boldsymbol{k}_3)\hat{\gamma}^{(s_3)}(-\boldsymbol{k}_3)\rangle\!\rangle.
\end{aligned} \tag{7.1.34}$$

7.1.6. Three gravitons

The result for the leading order 3-graviton correlation function is universal, since it does not depend on the deforming operator in the dual QFT. Before substituting (2.9.19) and (2.9.20) to the holographic formula (6.6.33), observe that the semi-local terms proportional to c_g in (2.9.19) and (2.9.20) match the semi-local terms of the form $\langle\!\langle T_{ij}\Upsilon_{klmn}\rangle\!\rangle$, so by definition the dependence on c_g cancels. The 3-graviton correlation function reads

$$\begin{aligned}\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{103\,680\sqrt{2}\alpha_1 J^2}{\pi^6\alpha_T^3 c_{123}^2 a_{123}^4} \\ &+ \frac{9J^2 a_{123}^2}{2\sqrt{2}\pi^4\alpha_T^2 2\sqrt{2}c_{123}^5}\left[-11a_{123}^3+25a_{123}b_{123}-17c_{123}\right], \end{aligned} \tag{7.1.35}$$

$$\begin{aligned}\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(-)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{9J^2(p_1+p_2-p_3)^2}{2\sqrt{2}\pi^4\alpha_T^2 2\sqrt{2}c_{123}^5 a_{123}^2}\left[-11p_3^5-14p_3^4 a_{12}\right. \\ &+ p_3^3(8b_{12}-3a_{12}^2)+p_3 a_{12}(3p_3+14a_{12})(3b_{12}-a12^2) \\ &\left.+ a_{12}^3(25a_{12}-11a_{12}^2)\right], \end{aligned} \tag{7.1.36}$$

where J^2, a_{123}, b_{123} and c_{123} are defined in (4.1.97) and (2.6.18). Here α_1 is an undetermined constant depending on the underlying CFT and proportional to the OPE constant c_{TT}^T defined as,

$$\mathcal{T}^{(+)}(\boldsymbol{x})\mathcal{T}^{(+)}(0) \sim c_{TT}^T \frac{1}{x^3}\mathcal{T}^{(+)}(0). \tag{7.1.37}$$

7.2. Hilltop inflation

In the previous section we calculated cosomological observables for a simple model of the holographic cosmology. The model was based on a slightly relevant deformation of a CFT. In this section we will carry out the cosmological computations showing that the dual cosmology is a particular example of the hilltop inflation.

7.2.1. Identifying the dual cosmology

Let us consider single-field inflationary cosmologies governed by the action (5.3.3)

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4\boldsymbol{x}\sqrt{-g}[R-\partial_\mu\Phi\partial^\mu\Phi-2\kappa^2 V(\Phi)]. \tag{7.2.1}$$

Note the we redefined the potential in (5.3.3) according to $V(\Phi)\mapsto\kappa^2 V(\Phi)$. Assuming flat spatial slices, the background FRW solution takes the form (5.2.10),

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2, \qquad \Phi(t,\boldsymbol{x}) = \phi(\boldsymbol{x}). \tag{7.2.2}$$

Solutions for which the evolution of the scalar field is monotonic (or at least piecewise so), as befits an RG flow (4.1.22), may be obtained from the equations (5.3.12) - (5.3.14),

$$H = \frac{\dot{a}}{a} = -\frac{1}{2}W, \qquad \dot{\phi} = W', \qquad -2\kappa^2 V = W'^2 - \frac{3}{2}W^2. \tag{7.2.3}$$

Our interest relates to cosmologies admitting regions in which the metric is asymptotically de Sitter. Such regions correspond to critical points of the potential, $V' = 0$, for which the superpotential satisfies either

$$W'(\phi) = 0, \qquad \text{or} \qquad W''(\phi) = \frac{3}{2}W(\phi). \tag{7.2.4}$$

Here we will focus on the first of these possibilities, *i.e.*, solutions for which the critical point of $V(\phi)$ is also a critical point of $W(\phi)$. Cosmologies of this type map under the domain-wall/cosmology correspondence to holographic RG flows that possess 'fake' supersymmetry, and such holographic RG flows are known to be stable both perturbatively and non-perturbatively [210].

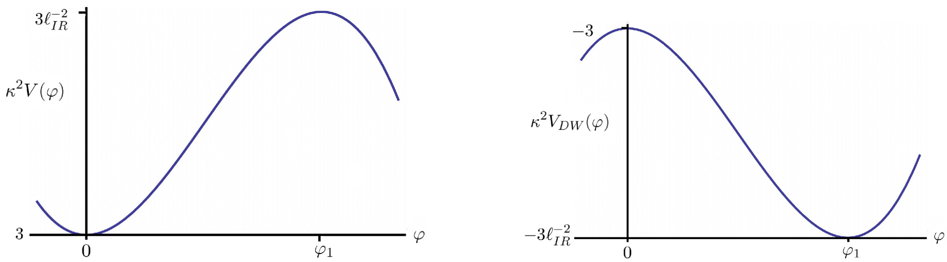


Figure 7.1: *The relevant portion of the inflationary potential (left-hand plot) is from the first hilltop at $\phi = \phi_1$ to the origin. The corresponding domain-wall potential (right-hand plot) is simply minus that of the cosmology. The factor L_{IR}^{-2} is given in* (7.2.21)*. The asymptotically AdS solution for the domain walls corresponds to the asymptotically dS cosmological solution.*

Adopting units in which the de Sitter radius is unity and shifting the scalar field so that the critical point $W'(\phi) = 0$ occurs at $\phi = 0$, the superpotential may be Taylor expanded about the critical point as

$$W(\phi) = -2 + \sum_{n=2}^{\infty} a_n \phi^n. \tag{7.2.5}$$

The corresponding potential then takes the form

$$\kappa^2 V(\phi) = 3 - a_2(2a_2 + 3)\phi^2 - 3a_3(1 + 2a_2)\phi^3 + O(\phi^4). \tag{7.2.6}$$

Since we are interested in the leading order 2- and 3-point functions, we will truncate the potential after the the ϕ^3 term. Therefore we consider superpotentials of the form

Under the domain-wall/cosmology correspondence the sign of the potential is reversed (see figure 7.1), and hence the potential of the corresponding holographic RG flow is

$$\begin{aligned}\kappa^2 V_{DW}(\phi) &= -3 + a_2(2a_2+3)\phi^2 + 3a_3(1+2a_2)\phi^3 + O(\phi^4)\\ &= -3 + \frac{1}{2}m^2\phi^2 - \frac{1}{3}g\phi^3 + O(\phi^4).\end{aligned} \tag{7.2.7}$$

The squared mass $m^2 = 2a_2(2a_3+3)$ and cubic coupling $g = -9a_3(1+2a_2)$ may be related to the operator dimension $\Delta = 3 - \lambda$ and scalar OPE coefficient C of the dual CFT via the standard AdS/CFT identifications

$$\begin{aligned} m^2 &= \lambda(\lambda-3) \qquad &\Rightarrow \qquad \lambda &= -2a_2,\\ g &= -6\pi C(1-\lambda) \qquad &\Rightarrow \qquad C &= \frac{3}{2\pi}a_3,\end{aligned} \tag{7.2.8}$$

as follows from computing holographic correlators for an interacting scalar field on an exact AdS background.

As our earlier analysis applied to CFTs with $0 < \lambda \ll 1$ and C a nonzero positive constant of order unity, we must therefore consider cosmologies with superpotential

$$W(\phi) = -2 - \frac{1}{2}\lambda\phi^2 + \frac{2}{3}\pi C\phi^3 + O(\phi^4). \tag{7.2.9}$$

From the equation of motion

$$\dot{\phi} = -\lambda\phi + 2\pi C\phi^2 + O(\phi^3), \tag{7.2.10}$$

we see there is then a second critical point of the superpotential close to the origin at

$$\phi = \phi_1 + O(\lambda^2), \qquad\qquad \phi_1 = \frac{\lambda}{2\pi C}. \tag{7.2.11}$$

as before (4.1.23). The dot denotes the derivative with respect to t. Mirroring our earlier analysis of the β-function, we now specialise to the case of the exact cubic superpotential by dropping the higher order terms in (7.2.9), although we expect that our analysis will ultimately hold more generally.[2] Integrating (7.2.10) with

[2] In particular, since ϕ is small throughout the inflationary evolution, we may perform a field redefinition of the form $\phi = \bar{\phi} + O(\bar{\phi})^3$ to re-introduce higher order terms in the Taylor expansion of the superpotential; note, however, this generates a non-canonical kinetic term in the bulk action (7.2.1). A different generalisation would be to consider superpotentials of the form $W(\phi) = -2 - (\lambda/2)\phi^2 + \tilde{C}_n\phi^n + O(\phi^{n+1})$ for which the cubic term is absent. For these models the background geometry may be solved in the same manner as we discuss here.

the initial condition $\phi(t_0) = \phi_0$ where $0 < \phi_0 < \phi_1$, we obtain

$$e^{\lambda(t-t_0)} = \frac{\phi_0}{\phi}\frac{(\phi_1 - \phi)}{(\phi_1 - \phi_0)}. \tag{7.2.12}$$

Inverting, we find

$$\phi = \phi_1 \Big[1 + \frac{\phi_1}{g} e^{\lambda t}\Big]^{-1}, \tag{7.2.13}$$

where

$$g = \phi_1 e^{\lambda t_0} \Big[\frac{\phi_1}{\phi_0} - 1\Big]^{-1} \tag{7.2.14}$$

encodes the asymptotic behaviour

$$\phi \to g e^{-\lambda t} \qquad \text{as} \quad t \to \infty. \tag{7.2.15}$$

From the holographic dictionary, g is then the renormalised coupling in the dual QFT, as indeed we would expect from comparison with (4.1.27). We may also identify the time coordinate t with the RG scale according to $t = -\log \Lambda$, since $\dot{\phi} = \beta(\phi) = -\mathrm{d}\phi/\mathrm{d}\log\Lambda$. In other words, the cosmological evolution in time is equivalent to the inverse RG flow in the dual QFT.

Integrating the equation of motion for the scale factor with boundary condition

$$a \to e^t \qquad \text{as} \qquad t \to \infty, \tag{7.2.16}$$

we find

$$a^{\lambda} = \frac{g}{\phi}\Big(1 - \frac{\phi}{\phi_1}\Big)^{1+\lambda\phi_1^2/12} \exp\Big[\frac{\lambda}{12}\phi(\phi_1 - \phi)\Big], \tag{7.2.17}$$

which evaluates to

$$a = \Big(\frac{g}{\phi_1} + e^{\lambda t}\Big)^{-\phi_1^2/12} \exp\Big[t\Big(1 + \frac{\lambda\phi_1^2}{12}\Big) + \frac{1}{12} g\phi_1 e^{\lambda t}\Big(\frac{g}{\phi_1} + e^{\lambda t}\Big)^{-2}\Big]. \tag{7.2.18}$$

In the infinite past, therefore, we obtain the asymptotic behaviour

$$a \to \Big(\frac{\phi_1}{g}\Big)^{\phi_1^2/12} \exp\Big[t\Big(1 + \frac{\lambda\phi_1^2}{12}\Big)\Big], \qquad \phi \to \phi_1 - \frac{\phi_1^2}{g} e^{\lambda t}, \tag{7.2.19}$$

as $t \to -\infty$ from which we see the difference in Hubble rate between $t \to -\infty$ and $t \to \infty$ is $\lambda\phi_1^2/12 \sim \lambda^3$. (Interestingly, this difference is proportional to the difference in free energy between UV and IR for the corresponding RG flow on an S^3 geometry [211], $\Delta H = (3/2\pi^4)\Delta F_{\text{sphere}}$.)

Notice that under an infinitesimal time shift $\delta t = -\sigma$ the background solution for a and ϕ remains invariant provided we simultaneously transform $\delta g = -\sigma\lambda g$ and $\delta a_0 = \sigma a_0$, where a_0 is the boundary scale factor introduced by a rescaling of the spatial coordinates so that $a \to a_0 e^t$ as $t \to \infty$. Thus, minus the time derivative operator in the bulk maps to the dilatation operator $\delta_D = a_0(\partial/\partial a_0) - \lambda g(\partial/\partial g)$ in the dual QFT.

To determine the parameters Δ_{IR} and C_{IR} in the IR CFT we examine the Taylor expansion of the potential about $\phi = \phi_1$. This takes the form

$$\kappa^2 V_{DW} = -3L_{IR}^{-2} + \frac{1}{2}m_{IR}^2(\phi-\phi_1)^2 - \frac{1}{3}g_{IR}(\phi-\phi_1)^3 + O\big((\phi-\phi_1)^4\big), \quad (7.2.20)$$

with

$$L_{IR}^{-2} = 1 + \frac{1}{6}\lambda\phi_1^2 + \frac{1}{144}\lambda^2\phi_1^2, \quad (7.2.21)$$

$$m_{IR}^2 = 3\lambda + \lambda^2 + \frac{1}{4}\lambda^2\phi_1^2, \quad (7.2.22)$$

$$g_{IR} = -6\pi C(1 + \lambda + \frac{1}{12}\lambda\phi_1^2), \quad (7.2.23)$$

hence from $\Delta_{IR}(\Delta_{IR}-3) = m_{IR}^2 L_{IR}^2$ and $g_{IR} = -6\pi C_{IR}(\Delta_{IR}-2)$ we obtain

$$\Delta_{IR} = 3 + \lambda - \frac{1}{12}\lambda^2\phi_1^2 + O(\lambda^7), \qquad C_{IR} = C + \frac{1}{12}\lambda\phi_1^2 + O(\lambda^4). \quad (7.2.24)$$

Referring back to our earlier discussion below (4.1.24), the fact that Δ_{IR} differs from $3+\lambda$ at order λ^4 suggests that the relation between the bulk inflaton and the dimensionless QFT coupling ϕ may be nontrivial at higher orders in λ. Such considerations may be ignored, however, in our present leading order analysis.

7.2.2. Slow-roll parameters

Let us now evaluate the inflationary power spectra and non-Gaussianities via the slow-roll approximation. Introducing the slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{2W'^2}{W^2}, \qquad \eta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{2W''}{W}, \quad (7.2.25)$$

the cosmological power spectra and non-Gaussianities may be expressed in terms of the values of these parameters at horizon crossing, defined for some mode of momentum k as the time t_* for which $k = a(t_*)H(t_*)$. These values may be computed as follows.

Expanding (7.2.17), we find

$$\phi_* = \phi_1\Big[1 + \Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^{-1} + O(\lambda^4), \quad (7.2.26)$$

where $k_0^{-\lambda} = \phi_1/g$ in accordance with (7.1.9), and hence from (7.2.25),

$$\epsilon_* = \frac{1}{2}\lambda^2\phi_1^2\Big(\frac{k}{k_0}\Big)^{2\lambda}\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^{-4} + O(\lambda^7), \tag{7.2.27}$$

$$\eta_* = -\lambda + 2\lambda\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^{-1} + O(\lambda^4), \tag{7.2.28}$$

$$H_* = 1 + O(\lambda^3). \tag{7.2.29}$$

7.2.3. Power spectra

Noting that $\epsilon_* \sim \lambda^4$ while $\eta_* \sim \lambda$, the scalar power spectrum to leading order in λ may be found from the usual first-order slow-roll result (5.5.27) and (5.5.28) while the size of the error term may be determined from the second-order slow-roll result (5.5.29), see [123, 124] for details. This gives

$$\begin{aligned}\Delta_S^2(k) &= \frac{k^3}{2\pi^2}\langle\!\langle\zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle \\ &= \frac{\kappa^2 H_*^2}{8\pi^2\epsilon_*} + O(\lambda^{-3}) \\ &= \frac{1}{16}\Delta_S^2(k_0)\Big(\frac{k}{k_0}\Big)^{-2\lambda}\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^4 + O(\lambda^{-3}),\end{aligned} \tag{7.2.30}$$

where

$$\Delta_S^2(k_0) = \frac{4\kappa^2}{\pi^2\lambda^2\phi_1^2} = \frac{16C^2\kappa^2}{\lambda^4}. \tag{7.2.31}$$

The spectral tilt (5.5.25) is

$$n_S - 1 = -2\eta_* = 2\lambda - 4\lambda\Big[1+\Big(\frac{k}{k_0}\Big)^{\lambda}\Big]^{-1}. \tag{7.2.32}$$

in striking agreement with our previous holographic result (7.1.13). To match the overall amplitude of the power spectrum to our holographic result (7.1.11) we must normalise the dual scalar operator $\mathcal{O}$ so that to leading order in λ

$$\alpha = \frac{12}{\pi^2\kappa^2}. \tag{7.2.33}$$

This is in perfect agreement with the standard AdS/CFT normalisation of this operator as follows from the comparison of (4.1.32) to our holographic 2-point function (6.4.7). Our holographic result for the scalar power spectrum (7.1.11) is then in exact agreement with the slow-roll result (7.2.30). Given the non-trivial momentum dependence of these formulae, this agreement is quite remarkable.

The tensor power spectrum may similarly be evaluated from standard slow-roll formulae, yielding

$$\Delta_T^2(q) = \frac{2H_*^2}{\pi^2} + O(\lambda^4) = \frac{2}{\pi^2} + O(\lambda^3). \tag{7.2.34}$$

Comparing with our holographic result (7.1.15), we find exact agreement upon choosing the normalisation of $\mathcal{T}^{(s)}$ to be

$$\alpha_T = \frac{24}{\pi^2\kappa^2} = 2\alpha. \tag{7.2.35}$$

This normalisation is again in agreement with the standard AdS/CFT normalisation. As discussed earlier, to recover the leading nonzero contribution to the tensor tilt $n_T \sim \epsilon_* \sim \lambda^4$ would require an extension of our holographic analysis to higher order in λ.

7.2.4. Non-Gaussianities

The cosmological 3-point functions for scalar and tensor fluctuations were evaluated for quasi-equilateral and for squeezed momentum configurations at leading order in slow-roll in [72].[3] The various contractions of helicity tensors appearing in the formulae of [72] may be re-expressed in terms of the momenta and helicities using the formulae presented in Appendix C of [75] (see also Appendix A of [1]). After this step, the slow-roll 3-point functions reduce to completely explicit functions of the momenta.

Three scalars

In the quasi-equilateral case (4.1.57) where all the $k_i^{-\lambda}$ are of comparable magnitude, from (5.5.42) we have

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle = \frac{\kappa^4 H_*^4 \eta_*}{16\epsilon_*^2}\sum_{i<j} k_i^{-3}k_j^{-3} + O(\lambda^{-4}). \tag{7.2.36}$$

Here, the leading term is of order $H_*^4\eta_*/\epsilon_*^2 \sim \lambda^{-7}$ while all remaining terms in the slow-roll result of [72] for this correlator are proportional to $H_*^4/\epsilon_* \sim \lambda^{-4}$ and thus are subleading in the λ expansion. The leading order scalar 3-point function is therefore of the local type with $f_{NL} = 5\eta_*/6$.

Inserting our results (7.2.27)–(7.2.29) for ϵ_*, η_* and H_* with q_3 as our reference momentum, and making use of the normalisation (7.2.33), we may rewrite (7.2.36) as

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle = \frac{576\,C^4}{\alpha^2\lambda^7}\Big(-1 + 2\Big[1+\Big(\frac{k_3}{k_0}\Big)^{\lambda}\Big]^{-1}\Big)\Big(\frac{k_3}{k_0}\Big)^{-4\lambda} \times \\ \times\ \Big[1+\Big(\frac{k_3}{k_0}\Big)^{\lambda}\Big]^{8}\sum_{i<j}k_i^{-3}k_j^{-3} + O(\lambda^{-7}).\end{aligned} \tag{7.2.37}$$

[3] At leading order, the slow-roll parameters ϵ_V and η_V used in [72] are related to those here by $\epsilon_V = \epsilon$, $\eta_V = \epsilon - \eta$.

Comparing with our holographic result (7.1.19) for the corresponding quasi-equilateral case, we find perfect agreement. Again, the overall normalisation is in agreement with the canonical AdS normalisation we found in (6.4.19). Moreover, as we have already verified in Section 7.1.3, the behaviour of the holographic result in the squeezed limit (cf. (7.1.20)) is in agreement with that given by the consistency relation of [72].

Two scalars and one graviton

In the quasi-equilateral case (4.1.57), the complete result for this correlator at first order in slow-roll is

$$\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(\pm)}(\boldsymbol{k}_3)\rangle\!\rangle = \frac{\kappa^4 H_*^4}{16\sqrt{2}\epsilon_*}\frac{J^2}{a_{123}^2 c_{123}^3 k_3^2}\big[a_{123}^3 - a_{123}b_{123} - c_{123}\big], \qquad (7.2.38)$$

where the elementary symmetric polynomials a_{123}, b_{123} and c_{123} were given in (4.1.97) and J^2 was defined in (2.6.18).

Inserting our results (7.2.27) and (7.2.29) for ϵ_* and H_* with k_3 as our reference momentum, and using the normalisations (7.2.33) and (7.2.35), we find

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(s)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{144\,C^2}{\sqrt{2}\pi^2\lambda^4\alpha\alpha_T}\frac{[1+(k_3/k_0)^\lambda]^4}{(k_3/k_0)^{2\lambda}}\times \\ &\qquad \times\ \frac{J^2}{a_{123}^2 c_{123}^3 k_3^2}\big[a_{123}^3 - a_{123}b_{123} - c_{123}\big]. \qquad (7.2.39)\end{aligned}$$

This result matches exactly our holographic expression (7.1.24). Once again, this agreement is striking in view of the nontrivial momentum dependence of this formula.

Two gravitons and one scalar

For completeness, the first-order slow-roll correlator for two gravitons and a scalar is given in the quasi-equilateral case (4.1.57) by

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(\pm)}(\boldsymbol{k}_2)\hat{\gamma}^{(\pm)}(\boldsymbol{k}_3)\rangle\!\rangle &= -\frac{\kappa^4 H_*^4}{128 b_{23}^5 k_1^2}(k_1^2 - a_{23}^2)^2 \times \\ &\qquad \times\ \Big[(k_1^2 - a_{23}^2 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}}\Big], \qquad (7.2.40)\end{aligned}$$

$$\begin{aligned}\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(\pm)}(\boldsymbol{k}_2)\hat{\gamma}^{(\mp)}(\boldsymbol{k}_3)\rangle\!\rangle &= -\frac{\kappa^4 H_*^4}{128\, b_{23}^5 k_1^2}(k_1^2 - a_{23}^2 + 4b_{23})^2 \times \\ &\qquad \times\ \big[(k_1^2 - a_{23}^2 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}}\big], \qquad (7.2.41)\end{aligned}$$

where we have supplemented the elementary symmetric polynomials in (4.1.97) with $a_{23} = k_2 + k_3$ and $b_{23} = k_2 k_3$. As we discussed in Section 7.1.5, however, to recover this first nonzero contribution to $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ holographically requires working to higher order in λ. Such an analysis should also be able to recover the subleading equilateral pieces of order λ^{-4} in $\langle\zeta\zeta\zeta\rangle$.

7.2.5. Discussion

In this thesis we have constructed a holographic duality between a single-field inflationary slow-roll cosmology and a three-dimensional QFT consisting of a CFT plus a single nearly marginal scalar deformation. Our methods enable explicit computations to be performed on both sides of the duality allowing our holographic framework to be directly verified. The form of inflationary correlators is seen to be controlled by the perturbative breaking of conformal symmetry in the vicinity of a fixed point of the dual QFT, completely determining the spectra and 3-point punctions (bispectra) up to a few constants. If these constants are fixed using AdS/CFT at the fixed point we obtain a model corresponding to Einstein gravity, otherwise we obtain alternative models.

It should be emphasised that our conformal perturbation expansion in the anomalous dimension $\lambda \ll 1$ leads to slow-roll parameters $\epsilon_* \sim \lambda^4$ while $\eta_* \sim \lambda$. Working at leading order in λ, we have been able to recover the slow-roll $\langle\zeta\zeta\rangle$ and $\langle\zeta\zeta\hat{\gamma}\rangle$ correlators exactly, along with the leading order pieces of all other correlators, $\langle\hat{\gamma}\hat{\gamma}\rangle$, $\langle\zeta\zeta\zeta\rangle$ and $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ (although this last correlator is zero to leading order). To obtain the remaining pieces, such as the tensor tilt or the leading non-zero part of the $\langle\zeta\hat{\gamma}\hat{\gamma}\rangle$ correlator will require further calculations at higher order in λ. Alternatively, one might look for modifications of the present set-up in which ϵ_* and η_* are both of the same order.

In this section of the thesis we were able to recover the entire momentum dependence of the cosmological correlators by resumming an infinite set of diagrams in conformal perturbation theory. Naively, in conformal perturbation theory the leading order contribution to the 2-point (3-point) functions comes from the integrated 3-point (4-point) functions. This expectation turned out to be incorrect in our case. When the dimension of the deforming scalar operator is close to three, $\Delta = 3 - \lambda$, the integrated higher point functions are singular in λ and one needs to resum an infinite number of insertions. The coefficient of the leading order singularity, however, which is the quantity entering our leading order computations, is universal. This appears to be related to a conformal anomaly present when there are marginal operators. Correlation functions of such operators contain logarithmic terms, and these are linked with the singular terms in the correlators of the $\Delta = 3 - \lambda$ operator. We hope to discuss this issue in more detail elsewhere.

The naive application of conformal perturbation theory still applies if in addition we focus on the leading short/long-wavelength behaviour of the correlators, since the higher point functions give subleading contributions in this limit. Focusing on such terms only, one may then analyse the subleading in λ contributions to the correlators. Using this approach we were also able to recover the subleading contribution in slow-roll to the scalar power spectrum (5.5.29) exactly. It would be interesting to extend this computation to the other observables as well.

Our approach may readily be extended to other bulk actions beyond standard Einstein gravity. For example, it would be interesting to consider actions allowing for a non-zero CFT 3-point function $\langle \mathcal{T}_{ij}(x_1)\mathcal{T}_{kl}(x_2)\mathcal{O}(x_3)\rangle_0$.[4] The nature of the resuming involved to calculate correlators in the perturbed QFT would be modified leading to different holographic predictions. We should also emphasise that the results from conformal perturbation theory are valid irrespectively of whether the CFT is at strong or at weak coupling. A weakly coupled CFT would correspond to a strongly coupled non-geometric bulk and we may extract from our results the phenomenology of such models.

In the thesis we found holographically that all leading order results are essentially fixed by the broken conformal invariance. It would be interesting to analyse the same question directly from the bulk perspective.

The holographic construction leads to a connection between properties of CFTs and inflationary physics. In particular, single scalar models are linked to CFTs that contain in their spectrum a single scalar operator which is nearly marginal. The presence of additional nearly marginal operators would map to the presence of bulk light scalars, and the inflationary model would then be a multi-scalar one. In general, the deforming operator would mix with other operators along the flow corresponding to the effects of entropy perturbations in the cosmology. It would be very interesting to map out all such possibilities and classify inflationary models using properties of the dual QFT. It would also be very interesting to find concrete CFT models with the required spectrum and OPE structure.

7.3. Models with strongly coupled gravity

In this section we present the predictions of the holographic cosmology based on the weakly coupled dual QFT. This means that the gravitational theory is strongly coupled at early times and possible new phenomena arise. Note that such a regime is not available to the usual cosmology.

The model was proposed and analysed in [39] and [1]. In [205] it was shown that it fits the actual cosmological data equally well as the standard ΛCDM. The analysis was based on the data acquired from the WMAP satellite and no statistical evidence in favour of any model has been found.

The model for the dual QFT is a free 3-dimensional QFT containing minimal scalars, conformal scalars, fermions and gauge fields. The necessary calculations

[4] It seems likely that the action (7.2.1) with the additional term $\Phi W_{ijkl}W^{ijkl}$, where W_{ijkl} is the Weyl tensor, would have this property.

on the QFT side were carried out in section 4.3 . The gauge/gravity duality used here is the non-conformal one described in section 6.1.3. The holographic formulae for the cosmological observables are presented in section 6.6.3.

7.3.1. 2-point functions

The 2-point functions of primordial perturbations follows from (6.6.25) and (6.6.26) applied to (4.3.24) and read

$$\langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{32}{N^2\mathcal{N}_{(B)}k^3}, \qquad \langle\!\langle \hat{\gamma}(\boldsymbol{k})\hat{\gamma}(-\boldsymbol{k})\rangle\!\rangle = \frac{256}{N^2\mathcal{N}_{(A)}k^3}, \tag{7.3.1}$$

or alternatively

$$\Delta_S^2(k) = \frac{16}{\pi^2 N^2\mathcal{N}_{(B)}}, \qquad \Delta_T^2(k) = \frac{512}{\pi^2 N^2\mathcal{N}_{(A)}}. \tag{7.3.2}$$

where $\mathcal{N}_{(A)}$ and $\mathcal{N}_{(B)}$ are defined in (4.3.26). As expected, all fields contribute to the 2-point function of the $\hat{\gamma}$ perturbations, but only non-conformal fields contribute to the scalar perturbations ζ. The tensor-to-scalar ratio (5.6.10) is

$$r^2 = \frac{\Delta_T^2}{\Delta_S^2} = \frac{32\mathcal{N}_{(B)}}{\mathcal{N}_{(A)}}, \tag{7.3.3}$$

and hence its upper bound $r^2 < 0.1$, as measured by Planck [85], translates into a constraint on the field content of the dual QFT.

The smallness of the overall amplitude $\Delta_S^2 \sim 10^{-9}$ follows from $N \sim 10^4$. This justifies the large N expansion.

The power spectra (7.3.2) are scale-invariant, $n_S - 1 = n_T = 0$. This is, however, the result based on free theories. In reality, the dual QFT is a non-Abelian gauge theory such as (6.1.29) and the free theory is the zeroth order approximation in the gauge effective coupling constant defined in (6.1.30),

$$g_{\text{eff}} = \frac{g_{\text{YM}}^2 N}{p_*}, \tag{7.3.4}$$

where p_* is the renormalisation scale. Therefore, corrections to the stress tensor 2-point function at 2-loop order[5] give rise to small deviations from scale invariance. The perturbative expansion depends on the effective dimensionless coupling

[5]Super-renormalizable theories have infrared divergences, but large N resummation leads to well-defined expressions with g_{YM}^2 effectively playing the role of an infrared regulator. The exact amplitudes are nonanalytic functions of the coupling constant [212]. Note that our analytic continuation to pseudo-QFT does not involve the coupling constant.

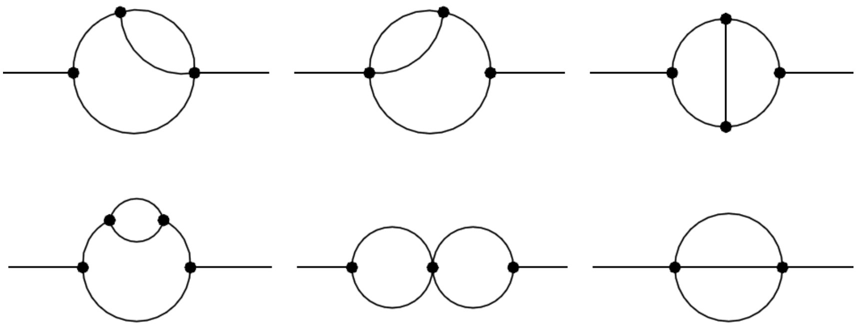


Figure 7.2: *Diagram topologies contributing at 2-loop order. Each diagram consists of an overall factor of $N^3 g^2_{\mathrm{YM}}$ multiplying an integral with superficial degree of divergence two. After dimensional regularization and renormalisation, the integrals evaluate to $\sim p^2 \log(p/p_*)$, and so overall each diagram yields a contribution to the stress tensor 2-point function of order $\sim N^3 g^2_{\mathrm{YM}} p^2 \log(p/p_*)$, or equivalently $\sim N^2 p^3 g^2_{\mathrm{eff}} \log(p/p_*)$.*

constant g^2_{eff}. Either from inspection or from direct calculation of some of the diagrams contributing at $O(g^2_{\mathrm{eff}})$ (see figure 7.2), one finds the form of the A and B factors in the decomposition of the 2-point function (2.1.16) to be

$$A(p) = C_A \bar{N}^2 p^3 \left[1 + D_A g^2_{\mathrm{eff}} \log\left(\frac{p}{p_*}\right) + O(g^4_{\mathrm{eff}})\right], \tag{7.3.5}$$

$$B(p) = C_B \bar{N}^2 p^3 \left[1 + D_B g^2_{\mathrm{eff}} \log\left(\frac{p}{p_*}\right) + O(g^4_{\mathrm{eff}})\right], \tag{7.3.6}$$

where D_A and D_B are numerical coefficients of order one whose value depends only on the field content. To compute D_A and D_B precisely requires summing all the relevant 2-loop diagrams presented in figure 7.2.

Inserting these two-loop corrected results into the holographic formulae (6.6.25) and (6.6.26), we find

$$\Delta^2_S(k) = \frac{1}{16\pi^2 N^2 C_B}\left[1 - D_B g^2_{\mathrm{eff}} \log\left(\frac{k}{k_*}\right) + O(g^4_{\mathrm{eff}})\right], \tag{7.3.7}$$

$$\Delta^2_T(k) = \frac{2}{\pi^2 N^2 C_A}\left[1 - D_A g^2_{\mathrm{eff}} \log\left(\frac{k}{k_*}\right) + O(g^4_{\mathrm{eff}})\right], \tag{7.3.8}$$

where the analytically continued effective coupling $g^2_{\mathrm{eff}} = g^2_{\mathrm{YM}} N/q$. In comparison, expanding (5.6.5) yields

$$\Delta_S^2(k) = \Delta_S^2(k_0)\left[1 + (n_S(k){-}1)\log\left(\frac{k}{k_0}\right) + O\left((n_S(k){-}1)^2\right)\right], \tag{7.3.9}$$

$$\Delta_T^2(k) = \Delta_T^2(k_0)\left[1 + n_T(k)\log\left(\frac{k}{k_0}\right) + O\left(n_T(k)^2\right)\right]. \tag{7.3.10}$$

Identifying the renormalisation scale k_* with the pivot scale k_0, we then see that the amplitudes (7.3.2) are correct to $O(g_{\rm eff}^4)$, and that the corresponding spectral tilts are

$$n_S(k){-}1 = -D_B g_{\rm eff}^2 + O(g_{\rm eff}^4), \qquad n_T(k) = -D_A g_{\rm eff}^2 + O(g_{\rm eff}^4). \tag{7.3.11}$$

Comparing with the Planck data in table 5.1 we find that $(n_s{-}1) \sim O(10^{-2})$ and hence $g_{\rm eff}^2 \sim O(10^{-2})$ also, justifying our perturbative treatment of the QFT.

To determine whether the spectral tilts are red or blue requires evaluating the signs of D_A and D_B, which will in general depend on the field content of the QFT. It is nonetheless still possible to extract predictions which are independent of the field content: for example, in these models, the scalar spectral index runs as

$$\alpha_s = \frac{\mathrm{d}n_S}{\mathrm{d}\log k} = -(n_s{-}1) + O(g_{\rm eff}^4). \tag{7.3.12}$$

This prediction is qualitatively different from slow-roll inflation, for which $\alpha_s/(n_s{-}1)$ is of first-order in slow-roll as we discussed in (5.5.30). In fact it was shown [213] that such a dependence holds for any cosmology following from a weakly coupled dual QFT. Furthermore, one can consider higher order corrections to the power spectrum (7.3.9) and find that

$$\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2} \sim g_{\rm eff} \tag{7.3.13}$$

while in slow-roll inflation $\frac{\mathrm{d}^2 n_S}{\mathrm{d}\log k^2}/(n_s{-}1)$ is of order (slow-roll)^2. If one takes the results presented in table 5.1 seriously, then the holographic models constitute a strong candidate for the inflation.

7.3.2. 3-point functions

Having computed all relevant QFT quantities in section 4.3 we can now evaluate the holographic formulae. First notice that since minimnal scalars and gauge fields are non-conformal, only these two fields can contribute to the perturbation ζ. What is more from (4.3.79) and (4.3.82) we have

$$\frac{\mathcal{N}_\phi}{\mathcal{N}_A}\Big[\langle\!\langle T^A_{ij}(\boldsymbol{p}_1)T^A_{kl}(\boldsymbol{p}_2)T^A_{mn}(\boldsymbol{p}_3)\rangle\!\rangle - 2\Big(\langle\!\langle T^A_{ij}(\boldsymbol{p}_1)\Upsilon^A_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle + \text{cyclic perms}\Big)\Big]$$
$$= \langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)T^\phi_{kl}(\boldsymbol{p}_2)T^\phi_{mn}(\boldsymbol{p}_3)\rangle\!\rangle - 2\Big(\langle\!\langle T^\phi_{ij}(\boldsymbol{p}_1)\Upsilon^\phi_{klmn}(\boldsymbol{p}_2,\boldsymbol{p}_3)\rangle\!\rangle + \text{cyclic perms}\Big). \tag{7.3.14}$$

Notice that this combination, suitably projected, appears in the numerator of all the holographic formulae for cosmological 3-point functions (6.6.31), (6.6.32) and (6.6.33). Thus, since the 2-point functions for gauge fields and minimal scalars also coincide (4.3.24), we see that gauge fields and minimal scalars necessarily make *identical* contributions to all cosmological 3-point functions.

Furthermore by comparing trace Ward identities (4.3.111) - (4.3.116) with our holographic formulae (6.6.30), (6.6.31) and (6.6.32), we immediately see that conformal fields make no contribution to the numerators of these formulae. An important consequence of this is that the $\zeta\zeta\zeta$, $\zeta\zeta\hat{\gamma}$ and $\zeta\hat{\gamma}\hat{\gamma}$ cosmological shape functions are forced to be independent of the field content of the dual QFT, as we will see below.

Therefore it is instructive to first use the trace Ward identities and the relation of minimal scalars to conformal scalars in order to express the cosmological 3-point functions in terms of CFT correlations functions. This yields

$$\langle\!\langle\zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle = -\frac{2^4}{N^4\mathcal{N}^2_{(B)}\prod_{i=1}^3\left(k_i^4\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_i)\mathcal{O}_1(-\boldsymbol{k}_i)\rangle\!\rangle\right)}$$
$$\times\left[k_1^2k_2^2k_3^2\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(\boldsymbol{k}_2)\mathcal{O}_1(\boldsymbol{k}_3)\rangle\!\rangle\right.$$
$$\left.+\,2\left(k_1^2(k_1^2-k_2^2-k_3^2)\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(-\boldsymbol{k}_1)\rangle\!\rangle + 2\text{ perm.}\right)\right], \tag{7.3.15}$$

$$\langle\!\langle\zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle = -\frac{2^{11}}{N^4\mathcal{N}_{(A)}\mathcal{N}_{(B)}k_3^3\prod_{i=1}^2\left(k_i^4\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_i)\mathcal{O}_1(-\boldsymbol{k}_i)\rangle\!\rangle\right)}$$
$$\times\Big[k_1^2k_2^2\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(\boldsymbol{k}_2)\widetilde{T}^{(s_3)}_\phi(\boldsymbol{k}_3)\rangle\!\rangle$$
$$+\,\Big(k_1^4\Theta_1^{(s_3)}(k_i)\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(-\boldsymbol{k}_1)\rangle\!\rangle + (\boldsymbol{k}_1\leftrightarrow\boldsymbol{k}_2)\Big)\Big], \tag{7.3.16}$$

$$\langle\!\langle\zeta(\boldsymbol{k}_1)\hat{\gamma}^{(s_2)}(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle = -\frac{2^{14}}{N^4\mathcal{N}^2_{(A)}k_2^3k_3^3k_1^4\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(-\boldsymbol{k}_1)\rangle\!\rangle}$$
$$\times\Big[16k_1^2\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\widetilde{T}^{(s_2)}_\phi(\boldsymbol{k}_2)\widetilde{T}^{(s_3)}_\phi(\boldsymbol{k}_3)\rangle\!\rangle$$
$$+\,k_1^4\Big(2\theta^{(s_2s_3)}(k_i)-\Theta^{(s_2s_3)}(k_i)\Big)\langle\!\langle\mathcal{O}_1(\boldsymbol{k}_1)\mathcal{O}_1(-\boldsymbol{k}_1)\rangle\!\rangle\Big], \tag{7.3.17}$$

where $\mathcal{N}_{(A)}$ and $\mathcal{N}_{(B)}$ are defined in (4.3.26) and we have made the dependence on the number of fields explicit by considering the $\mathcal{O}_1$ and $\widetilde{T}^{(s)}_\phi$ correlators to be those of a single field. The analytic continuation (6.6.7) has also been implicitly performed; the correlators appearing above are therefore those in (4.3.128) - (4.3.132) with p_i replaced by k_i. As discussed in the previous section, the trace Ward identities imply that the numerators of the holographic formulae (6.6.30)-(6.6.32) for the above correlators receive no contribution from conformal fields and are therefore proportional to $\mathcal{N}_{(B)}$, the number of non-conformal fields. The dependence of these correlators on the field content is then simply given by an overall factor, amounting to $\mathcal{N}_{(B)}$ divided by the corresponding factors in the denominators of the holographic formulae.

For the $\hat{\gamma}\hat{\gamma}\hat{\gamma}$ correlator, we find

$$\begin{aligned}
&\langle\!\langle \hat{\gamma}^{(s_1)}(\boldsymbol{k}_1)\hat{\gamma}^{(s_2)}(\boldsymbol{k}_2)\hat{\gamma}^{(s_3)}(\boldsymbol{k}_3)\rangle\!\rangle \qquad (7.3.18)\\
&\quad = -\frac{2^{24}}{N^4\mathcal{N}_{(A)}^3 k_1^3k_2^3k_3^3}\Bigg[\mathcal{N}_\psi\left(2\langle\!\langle T_\psi^{(s_1)}T_\psi^{(s_2)}T_\psi^{(s_3)}\rangle\!\rangle - \frac{\Theta^{(s_1s_2s_3)}(k_i)}{512}\sum_{i=1}^{3}k_i^3\right)\\
&\qquad + (\mathcal{N}_\phi+\mathcal{N}_\chi+\mathcal{N}_A)\left(2\langle\!\langle T_\chi^{(s_1)}T_\chi^{(s_2)}T_\chi^{(s_3)}\rangle\!\rangle - \frac{\Theta^{(s_1s_2s_3)}(k_i)}{512}\sum_{i=1}^{3}k_i^3\right)\Bigg],
\end{aligned}$$

considering again the correlators to be those of a single field so as to make the dependence on the number of fields explicit. It then turns out that

$$2\langle\!\langle T_\chi^{(+)}T_\chi^{(+)}T_\chi^{(-)}\rangle\!\rangle = \langle\!\langle T_\psi^{(+)}T_\psi^{(+)}T_\psi^{(-)}\rangle\!\rangle + \frac{\Theta^{(++-)}(q_i)}{1024}\sum_{i=1}^{3}k_i^3, \qquad (7.3.19)$$

i.e., these correlators differ only by the helicity projection of a semi-local term. This follows from (2.9.20), which depends on the normalisations of the 2-point functions only. On the other hand,

$$2\langle\!\langle T_\chi^{(+)}T_\chi^{(+)}T_\chi^{(+)}\rangle\!\rangle = \langle\!\langle T_\psi^{(+)}T_\psi^{(+)}T_\psi^{(+)}\rangle\!\rangle + \frac{\Theta^{(+++)}(k_i)}{1024}\sum_{i=1}^{3}k_i^3 + \frac{J^2}{128\sqrt{2}}\frac{c_{123}}{a_{123}^4}, \qquad (7.3.20)$$

and so these correlators differ by the helicity projections of both a semi-local and a non-local term. The non-local term reflects the fact that both solutions for the 3-point function of the stress tensor at separated points survive the $(+++)$ helicity projection, and leads in turn to a more complicated dependence on the QFT field content in the $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$ correlator.

Returning to (7.3.15) and substituting in our results for the remaining correlators, we first recover the result derived in [40],

$$\begin{aligned}
\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{512}{\mathcal{N}_{(B)}^2N^4}\Big(\prod_i k_i^{-3}\Big)\Big(-2k_1k_2k_3 - \sum_i k_i^3 + (k_1k_2^2 + 5\,\mathrm{perms})\Big)\\
&= \frac{512}{\mathcal{N}_{(B)}^2N^4}\frac{J^2}{a_{123}c_{123}^3}, \qquad (7.3.21)
\end{aligned}$$

showing that the scalar 3-point function exactly coincides with the equilateral template (5.6.8) with $n_S = 1$. In the first line, note that all the terms but the one proportional to $k_1k_2k_3$ originate from semi-local terms in the numerator of the holographic formula (6.6.30). Without their contribution we would not have been able to distinguish the equilateral shape from others involving a similar factor of $k_1k_2k_3$ in the numerator (for example, the orthogonal shape [214], for which the corresponding numerator is $-8k_1k_2k_3 - 3\sum_i k_i^3 + 3(k_1k_2^2 + 5\,\text{perms})$). In fact, due to the factor of $\prod_i k_i^{-3}$ coming from the product of 2-point functions in the denominator of the holographic formula, the semi-local term $\sum_i k_i^3$ in the numerator generates a contribution to the 3-point function of exactly the 'local' type (5.6.7). It is therefore essential to include the contribution of all semi-local terms in the holographic formulae, as we have been careful to do.

For the remaining correlators, we find

$$\begin{aligned}
\langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{2048}{\sqrt{2}N^4\mathcal{N}_{(A)}\mathcal{N}_{(B)}}\frac{J^2}{a_{123}^2c_{123}^3k_3^2}\times \\
&\times\,\Big[(a_{123}^3 - a_{123}b_{123} - c_{123}) - a_{123}k_3^2\Big], \qquad (7.3.22)\\
\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle &= -\frac{512}{N^4\mathcal{N}_{(A)}^2b_{23}^5k_1^2}(k_1^2 - a_{23}^2)^2\times \\
&\times\,\Big[(k_1^2 - a_{23}^2 + 2b_{23}) + \frac{32b_{23}^3}{a_{123}^4}\Big], \qquad (7.3.23)\\
\langle\!\langle \zeta(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(-)}(\boldsymbol{k}_3)\rangle\!\rangle &= -\frac{512}{N^4\mathcal{N}_{(A)}^2b_{23}^5k_1^2}\times \\
&\times\,(k_1^2 - a_{23}^2 + 4b_{23})^2(k_1^2 - a_{23}^2 + 2b_{23}), \qquad (7.3.24)\\
\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{1024}{\sqrt{2}N^4\mathcal{N}_{(A)}^2}\frac{J^2a_{123}^2}{c_{123}^5}\times \\
&\times\,\Big[(a_{123}^3 - a_{123}b_{123} - c_{123}) - \Big(1 - 4\frac{\mathcal{N}_\psi}{\mathcal{N}_{(A)}}\Big)\frac{64c_{123}^3}{a_{123}^6}\Big], \qquad (7.3.25)\\
\langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k}_1)\hat{\gamma}^{(+)}(\boldsymbol{k}_2)\hat{\gamma}^{(-)}(\boldsymbol{k}_3)\rangle\!\rangle &= \frac{1024}{\sqrt{2}N^4\mathcal{N}_{(A)}^2}\frac{J^2}{a_{123}^2c_{123}^5}\times \\
&\times\,(k_3 - a_{12})^4(a_{123}^3 - a_{123}b_{123} - c_{123}). \qquad (7.3.26)
\end{aligned}$$

We would now like to define corresponding shape functions, *i.e.*, we wish to write these correlators as bispectra: a product of power spectra times a shape function. To do so, we first define the dimensionless 2-point amplitudes

$$\mathcal{A}(\zeta\zeta) = k^3 \langle\!\langle \zeta(\boldsymbol{k})\zeta(-\boldsymbol{k})\rangle\!\rangle = \frac{32}{N^2\mathcal{N}_{(B)}}, \tag{7.3.27}$$

$$\mathcal{A}(\hat{\gamma}\hat{\gamma}) = k^3 \langle\!\langle \hat{\gamma}^{(+)}(\boldsymbol{k})\hat{\gamma}^{(+)}(-\boldsymbol{k})\rangle\!\rangle = \frac{256}{N^2\mathcal{N}_{(A)}}, \tag{7.3.28}$$

and similarly the dimensionless 3-point amplitudes, *e.g.*,

$$\mathcal{A}(\zeta\zeta\hat{\gamma}^{(+)}) = k_1^2 k_2^2 k_3^2 \, \langle\!\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\hat{\gamma}^{(+)}(\boldsymbol{k}_3)\rangle\!\rangle, \tag{7.3.29}$$

with analogous expressions for the other correlators. Physically, these quantitites parametrise the contribution per logarithmic interval of wavenumbers to the corresponding position-space expectation values with all insertions at the same point, *e.g.*,

$$\langle \zeta^2(\boldsymbol{x})\rangle = \frac{1}{2\pi^2}\int (\mathrm{d}\log k)\mathcal{A}(\zeta\zeta), \tag{7.3.30}$$

$$\langle \zeta^2(\boldsymbol{x})\hat{\gamma}^{(+)}(\boldsymbol{x})\rangle = \frac{1}{8\pi^4}\int \left(\prod_i \mathrm{d}\log k_i\right)\mathcal{A}(\zeta\zeta\hat{\gamma}^{(+)}), \tag{7.3.31}$$

where the latter integral ranges over all possible triangle side lengths in momentum space. (For reference, the usual logarithmic power spectrum is simply $\Delta_\zeta^2 = (1/2\pi^2)\mathcal{A}(\zeta\zeta)$.) The dimensionless 3-point amplitudes may now be naturally re-expressed as a product of dimensionless 2-point amplitudes and a purely momentum-dependent shape function:

$$\begin{aligned}
\mathcal{A}(\zeta\zeta\hat{\gamma}^{(s_3)}) &= \mathcal{A}(\zeta\zeta)\mathcal{A}(\hat{\gamma}\hat{\gamma})\mathcal{S}(\zeta\zeta\hat{\gamma}^{(s_3)}),\\
\mathcal{A}(\zeta\hat{\gamma}^{(s_2)}\hat{\gamma}^{(s_3)}) &= \mathcal{A}^2(\hat{\gamma}\hat{\gamma})\mathcal{S}(\zeta\hat{\gamma}^{(s_2)}\hat{\gamma}^{(s_3)}),\\
\mathcal{A}(\hat{\gamma}^{(s_1)}\hat{\gamma}^{(s_2)}\hat{\gamma}^{(s)}) &= \mathcal{A}^2(\hat{\gamma}\hat{\gamma})\mathcal{S}(\hat{\gamma}^{(s_1)}\hat{\gamma}^{(s_2)}\hat{\gamma}^{(s)}).
\end{aligned} \tag{7.3.32}$$

Explicitly, the shape functions are given by

$$\begin{aligned}
\mathcal{S}(\zeta\zeta\hat{\gamma}^{(+)}) &= \frac{1}{4\sqrt{2}}\frac{J^2}{a_{123}^2 c_{123} k_3^2}\Big[(a_{123}^3 - a_{123}b_{123} - c_{123}) - a_{123}k_3^2\Big],\\
\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) &= -\frac{1}{128 b_{23}^3}(k_1^2 - a_{23}^2)^2\Big[(k_1^2 - a_{23}^2 + 2b_{23}) + \frac{32 b_{23}^3}{a_{123}^4}\Big],\\
\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) &= -\frac{1}{128 b_{23}^3}(k_1^2 - a_{23}^2 + 4b_{23})^2(k_1^2 - a_{23}^2 + 2b_{23}),\\
\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) &= \frac{1}{64\sqrt{2}}\frac{J^2 a_{123}^2}{c_{123}^3}\Big[(a_{123}^3 - a_{123}b_{123} - c_{123}) - \Big(1 - 4\frac{\mathcal{N}_\psi}{\mathcal{N}_{(A)}}\Big)\frac{64 c_{123}^3}{a_{123}^6}\Big],\\
\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) &= \frac{1}{64\sqrt{2}}\frac{J^2}{a_{123}^2 c_{123}^3}(k_3 - a_{12})^4(a_{123}^3 - a_{123}b_{123} - c_{123}).
\end{aligned} \tag{7.3.33}$$

Thus, with the sole exception of $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$, all the shape functions defined in this manner are independent of the field content of the dual QFT. (Indeed, this was our motivation in selecting the factors of $\mathcal{A}(\zeta\zeta)$ and $\mathcal{A}(\hat{\gamma}\hat{\gamma})$ appearing in (7.3.32).) From our previous discussion, we see that for the shape functions involving one or more factors of ζ this property is a consequence of the trace Ward identities, which limit the field content-dependence of the corresponding 3-point functions to a single overall factor. The independence of $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$ from the QFT field content arises similarly from the fact that the corresponding 3-point function depends on the field content via an overall factor only. As we saw above, this latter property relies on both the conformal Ward identities and the precise form of the semi-local terms appearing in the holographic formula.

We have plotted the holographic shape functions in figures 7.3 and 7.4 (along with their counterparts for slow-roll inflation which we discuss in the next section). In these figures we have adopted the expedient of scaling all momenta such that $k_1 + k_2 + k_3 = 1$ (note that the shape functions are invariant under a constant rescaling of all momenta). By the usual triangle inequalities, the allowed range for any two momenta, say k_1 and k_2, is then $0 \leqslant k_1 \leqslant 1/2$ and $1/2 - k_1 \leqslant k_2 \leqslant 1/2$ as displayed. In each case, we have chosen to plot the two momenta under whose interchange the shape function is symmetric.

Note that the usual plotting convention adopted for the scalar bispectrum $\mathcal{S}(\zeta\zeta\zeta)$ (namely, ordering the momenta $k_1 \geqslant k_2 \geqslant k_3$ and then scaling k_1 to unity, with the triangle inequality then constraining $k_2 \geqslant 1 - k_3$) is not applicable to the correlators considered here, since in each case (with the sole exception of $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$) one of the three momenta is distinguished and so the required ordering of momenta cannot be accomplished without loss of generality. Without this initial ordering step, rescaling one of the momenta to unity then fails to yield an upper bound on the magnitude of the remaining momenta, resulting in a plot with unbounded area. This problem is neatly sidestepped by constraining the total perimeter of the triangle to be unity, instead of the length of one the sides.

7.3.3. Comparison with slow-roll results

Slow-roll inflation predicts the correlators of three gravitons are (5.5.47) and (5.5.48). We may recover these results exactly from our holographic model by setting

$$2\mathcal{N}_\psi = \mathcal{N}_\phi + \mathcal{N}_A + \mathcal{N}_\chi, \qquad \frac{1}{256}N^2\mathcal{N}_{(A)} = \frac{1}{\kappa^2 H_*^2}. \tag{7.3.34}$$

In particular, the latter relation is also consistent with matching the amplitude of the graviton 2-point function of slow-roll inflation and the holographic model. The first relation is consistent with that found in v2 of [29] for the special case where

$\mathcal{N}_A = \mathcal{N}_\phi = 0$. (Note however that our careful treatment of the semi-local terms in the holographic formulae enables us to correctly recover the *entire* slow-roll 3-point function (5.5.47) and (5.5.48).) For general QFT field content (for which the first relation in (7.3.34) is not satisfied), the $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$ and $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}$ holographic 3-point functions in (7.3.22) coincide precisely with the corresponding 3-point functions derived in [29] for slow-roll inflation in which one includes an additional term in the action proportional to the Weyl tensor cubed. Relative to [29], our QFT additionally contains non-conformal fields, and our treatment of the semi-local terms enables us to recover the cosmological result exactly.

The remaining slow-roll results are given by equations (5.5.44) - (5.5.48). While these differ from the predictions of the holographic model, interestingly the difference is only in the last term.

Evaluating the shape functions, for slow-roll inflation the 2-point amplitudes defined analogously to (7.3.27) are

$$\mathcal{A}_{SR}(\zeta\zeta) = \frac{\kappa^2 H_*^2}{4\epsilon_*}, \qquad \mathcal{A}_{SR}(\hat{\gamma}\hat{\gamma}) = \kappa^2 H_*^2. \tag{7.3.35}$$

The slow-roll shape functions then differ from their holographic counterparts by at most a single term:

$$\mathcal{S}_{SR}(\zeta\zeta\hat{\gamma}^{(+)}) = \mathcal{S}(\zeta\zeta\hat{\gamma}^{(+)}) + \frac{1}{4\sqrt{2}}\frac{J^2}{a_{123}c_{123}}, \tag{7.3.36}$$

$$\mathcal{S}_{SR}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) = \mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) + \frac{1}{16a_{123}c_{123}}(k_1^2 - a_{23}^2)^2\Big(1 + \frac{4c_{123}}{a_{123}^3}\Big), \tag{7.3.37}$$

$$\mathcal{S}_{SR}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) = \mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) + \frac{1}{16a_{123}c_{123}}(k_1^2 - a_{23}^2 + 4b_{23})^2, \tag{7.3.38}$$

$$\mathcal{S}_{SR}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) = \mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) + \frac{J^2}{\sqrt{2}\,a_{123}^4}\Big(1 - \frac{4\mathcal{N}_\psi}{\mathcal{N}_{(A)}}\Big), \tag{7.3.39}$$

$$\mathcal{S}_{SR}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) = \mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}). \tag{7.3.40}$$

The holographic and slow-roll shape functions, as well as the difference terms in the expressions above, are plotted in figures 7.3 and 7.4. From these figures it is apparent that the holographic and slow-roll shape functions share the same broad qualititative features in all cases except for $\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$: here, $\mathcal{S}_{SR}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$ has a simple pole as the momentum k_1 associated with ζ vanishes, whereas the corresponding holographic shape function has a zero.

At a more quantitative level, a rough indication of the distinguishability of the holographic and slow-roll shape functions may be obtained by evaluating the cosine orthogonality measure proposed in [215] (following earlier work in [216]),

$$C(\mathcal{S},\mathcal{S}') = \frac{F(\mathcal{S},\mathcal{S}')}{\sqrt{F(\mathcal{S},\mathcal{S})F(\mathcal{S}',\mathcal{S}')}}, \tag{7.3.41}$$

where the weighted inner product

$$F(\mathcal{S},\mathcal{S}') = \int \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \frac{1}{a_{123}}\mathcal{S}(k_1,k_2,k_3)\mathcal{S}'(k_1,k_2,k_3). \qquad (7.3.42)$$

Writing $k_1 = \alpha\hat{q}_1$, $k_2 = \alpha\hat{k_2}$ and $k_3 = \alpha(1-\hat{k_1}-\hat{k_2})$, the integral over α in the inner product factors out, since all shape functions we consider here are scale-invariant, *i.e.*, independent of α. This overall factor may then be discarded since its contribution to the cosine measure $C(\mathcal{S},\mathcal{S}')$ cancels between numerator and denominator. We may thus replace (7.3.42) with the two-dimensional integral

$$F(\mathcal{S},\mathcal{S}') = \int \mathrm{d}\hat{k}_1\mathrm{d}\hat{k}_2\mathcal{S}(\hat{k}_1,\hat{k}_2,1-\hat{k}_1-\hat{k}_2)\mathcal{S}'(\hat{k}_1,\hat{k}_2,1-\hat{k}_1-\hat{k}_2), \qquad (7.3.43)$$

where the shape functions here are precisely the isoperimetric shape functions plotted in figures 7.3 and 7.4.

Naively, one might expect the domain of integration would be $0<\hat{k}_1<1/2$ and $1/2-\hat{k}_1<\hat{k}_2<1/2$. Since however several of the shape functions have poles when one or more of the triangle sides are taken to zero, as we see from figures 7.3 and 7.4, one must further restrict the domain of integration in order to obtain finite inner products. The physical justification for this procedure is that any real observation is only sensitive to momenta in some range $k_{\mathrm{min}}<k_i<k_{\mathrm{max}}$. We will therefore restrict all rescaled momenta $\hat{k}_i>\epsilon$, where the cutoff $\epsilon = k_{\mathrm{min}}/2k_{\mathrm{max}} \sim 5\times 10^{-4}$. The domain of integration $0<\hat{q}_1<1/2$ and $1/2-\hat{k}_1<\hat{k}_2<1/2$ is thus further restricted by the conditions $\hat{k}_1>\epsilon$, $\hat{k}_2>\epsilon$, and $1-\hat{k}_2-\hat{k}_3>\epsilon$. For shape functions with poles at the corners, the orthogonality measure (7.3.41) will depend on the cutoff ϵ, reflecting the fact that our ability to resolve the shape functions concerned depends on how sensitive we are to the corners of the distribution.

Having thus carefully defined the orthogonality measure, one may now numerically evaluate the orthogonality measure between each holographic shape function and its slow-roll counterpart. Rounding to two decimal places,

$$\begin{aligned} &C(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) = 1.00, \qquad && C(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) = 0.33, \\ &C(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}) = 0.67, && C(\zeta\zeta\hat{\gamma}^{(+)}) = 1.00. \end{aligned} \qquad (7.3.44)$$

Values close to unity indicate nearly indistinguishable shape functions, while smaller values correspond to shape functions that are more orthogonal. (For comparison, the overlap between the standard local and equilateral shape functions evaluates to $C=0.34$ with our cutoff prescription.) Overall, these values confirm one's impression by eye from figures 7.3 and 7.4; namely, that the holographic and slow-roll shape functions are nearly indistiguishable for the cases $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$ and $\zeta\zeta\hat{\gamma}^{(+)}$, while in the case $\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$ the two shape functions may be distinguished by the presence or absence of a pole as the momentum k_1 associated with ζ vanishes.

7.3.4. Discussion

In this section we computed the complete set of 3-point functions (and defined and extracted the corresponding shapes[6]) for a class of holographic models of the very early universe based on perturbative QFT. The leading 1-loop result actually depends only on the free part of the QFT, so in particular our results are also the complete answer when the dual QFT is free. The field content of the dual theory includes gauge fields, massless fermions, massless minimal and conformal scalars and thus the parameters that can appear in the results are the number of species for each type of field. The 3-point functions could, a priori, depend on these in a complicated way, but it turns out that we get instead (nearly) universal results that are independent of all details of the dual QFT, within the class of the theories we consider. Thus, these models make clean and precise predictions.

One can trace this universality to the specific form of the holographic map, the fact that to leading order the QFT is free, symmetry considerations and properties of $d = 3$ theories. Let us explain this. Firstly, in three dimensions, vectors are dual to scalars so one may anticipate that the contribution due to gauge fields (at 1-loop order) is equal to that of the contribution due to mininal scalars, and we indeed find this to be the case. Taking this into account, the answer could then depend on three parameters, the number of conformal scalars, $\mathcal{N}_\chi$, the number of fermions, $\mathcal{N}_\psi$ and the total number of gauge fields plus minimal scalars, $\mathcal{N}_{(B)}$. The trace Ward identity of the dual QFT and the specific form of the holographic formulae then imply that, in all correlators involving at least one factor of ζ, the field content appears only as a multiplicative factor and is such that the corresponding shape functions are completely independent of the field content.

Let us now turn to correlators involving only tensors: these are effectively determined by the 3-point function of the stress tensor of a CFT. In $d > 3$ dimensions, this 3-point function is parametrised by two constants, α_1 and α_2 in (3.11.21) - (3.11.25). However, in $d = 3$ the constant α_2 is redundant and the helicity projected result (2.9.19, 2.9.20) depends on α_1 only. Furthermore, only the $\langle T^{(+)}T^{(+)}T^{(+)}\rangle$ correlation function depends on α_1, while $\langle T^{(+)}T^{(+)}T^{(-)}\rangle$ is completely determined in terms of the normalisations of 2-point function. Indeed, the shape corresponding to three positive helicity gravitons does depend on the field content, but surprisingly the shape for two positive and one negative helicity graviton is independent of the field content.

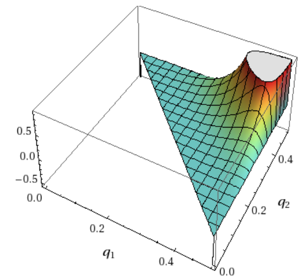
Our calculations carefully include all such semi-local contributions. In the holographic formulae for the 3-point functions, these contributions appear as terms in the numerator that are non-analytic in only one of the three momenta. Since the denominator of the holographic formulae is however non-analytic in all three mo-

[6] To our knowledge, shapes other than those relevant for purely scalar or purely tensor 3-point functions have not been discussed before.

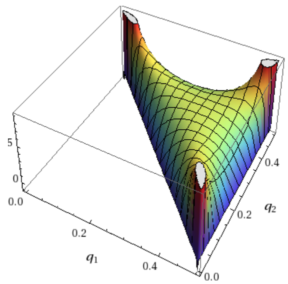
menta, the net contribution of these semi-local terms to the 3-point function is in fact non-analytic in *two* of the three momenta. Semi-local terms in the holographic formulae may thus contribute, for example, to 'local'-type non-Gaussianity behaving as $1/k_1^3k_2^3$ + perms. Contributions of this nature therefore play a crucial role in allowing different cosmological shapes to be distinguished.

To get a feeling for our results we also computed the corresponding slow-roll results and compared them with the holographic results. Firstly, comparing the power spectra one obtains a relation between the parameters N^2, $\mathcal{N}_{(A)}$ and $\mathcal{N}_{(B)}$ of the QFT and the parameters κ^2, H_*^2 and ϵ_* of the slow-roll model. Comparing the 3-point functions, we find that the $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}$ correlators agree exactly, while the $\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$ correlators can be made to agree if one imposes that that the field content satisfies the relation $2\mathcal{N}_\psi = \mathcal{N}_\phi + \mathcal{N}_A + \mathcal{N}_\chi$. As explained in [29], these slow-roll correlators are constrained by the late-time de Sitter isometries to satisfy conformal Ward identities, and thus at separated points they should be expressible in terms of the 3-point functions of conformal scalars and free fermions. Indeed, the linear combination found in v2 of [29] is the same as the one we find (setting $\mathcal{N}_\phi = \mathcal{N}_A = 0$ in our relation). By taking into account the contribution from semi-local terms, however, we are further able to correctly recover every individual term appearing in the graviton 3-point functions.

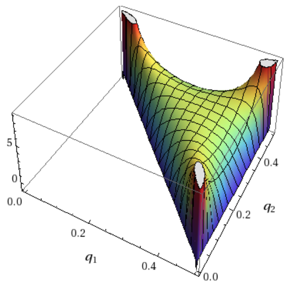
There is no apparent reason for the remaining slow-roll and holographic correlators to agree. Nevertheless we find rather similar results. To quantify the difference we used the cosine orthogonality measure of [215] to obtain a first indication of the distinguishability of the corresponding shapes. We find that the shapes for $\zeta\zeta\hat{\gamma}^{(s)}$ are nearly indistinguishable, while for $\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}$, the two shapes may be distinguished (as a consequence of differing behaviour in the squeezed limit where the momentum associated with the ζ goes to zero), with the case of $\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)}$ lying in between.



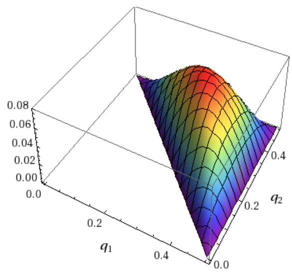
(a) $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$



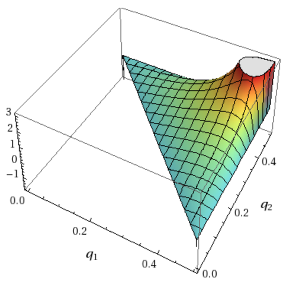
(b) $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$



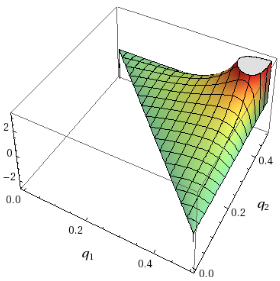
(c) $\mathcal{S}_{SR}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$



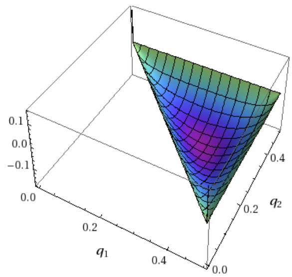
(d) $\Delta\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$



(e) $\mathcal{S}(\zeta\zeta\hat{\gamma}^{(+)})$

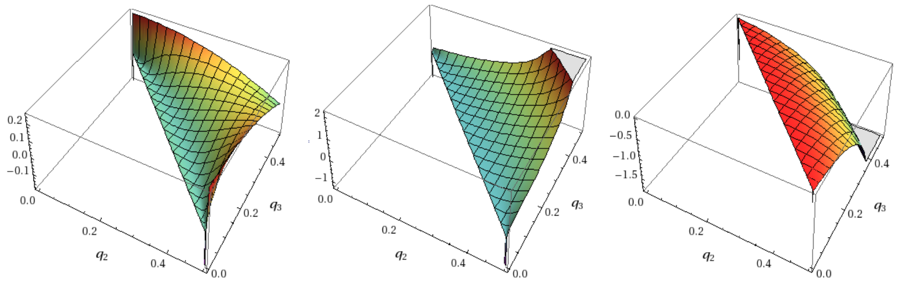


(f) $\mathcal{S}_{SR}(\zeta\zeta\hat{\gamma}^{(+)})$

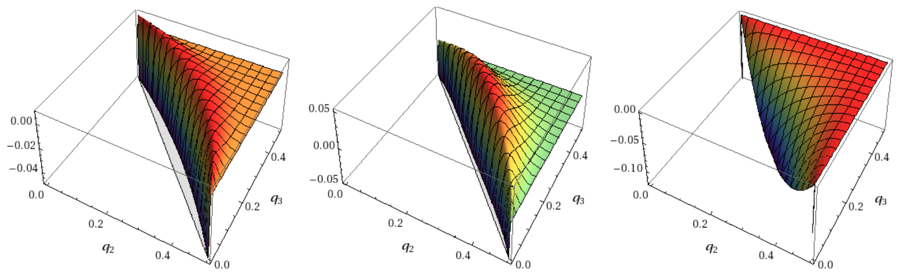


(g) $\Delta\mathcal{S}(\zeta\zeta\hat{\gamma}^{(+)})$

Figure 7.3: *Isoperimetric plots displaying the holographic and slow-roll shape functions, as well as the difference between them (e.g., $\Delta\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) = \mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}) - \mathcal{S}_{SR}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$). The invariance of the shape functions under a rescaling $k_i \to \lambda k_i$ of all momenta has been exploited to set $k_1+k_2+k_3 = 1$, constraining the allowed momentum values to those displayed. Each plot is symmetric under interchange of the appropriate momenta as expected. Note that $\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$ (shown in plot (a)) coincides for the holographic and slow-roll models. In plots (7.3b) and (7.3d) we have set $\mathcal{N}_\psi = \mathcal{N}_{(A)}$ to maximise $\Delta\mathcal{S}(\hat{\gamma}^{(+)}\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$ for illustrative purposes.*



(a) $\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$ **(b)** $\mathcal{S}_{SR}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$ **(c)** $\Delta\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(+)})$



(d) $\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$ **(e)** $\mathcal{S}_{SR}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$ **(f)** $\Delta\mathcal{S}(\zeta\hat{\gamma}^{(+)}\hat{\gamma}^{(-)})$

Figure 7.4: *Isoperimetric plots for holographic and slow-roll shape functions continued. In plots (7.4d) and (7.4e) note that both shape functions are actually finite along the line* $k_3 = 1/2 - k_2$ *(i.e.,* $k_1 = 1/2$*); we have simply restricted the plot range to exhibit the overall shape more clearly.*

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Samenvatting

> Er is meetkunde in het zoemen van de snaren, er is muziek in de plaatsing van de bollen.
> *Pythagoras, 6th c. BC.*
>
> Er is niets nieuws meer te ontdekken in de natuurkunde. Alles wat rest zijn meer en meer precieze metingen.
> *Toegeschreven aan lord Kelvin, ca. 1900.*
>
> Of je iets kan observeren of niet ligt aan welke theorie je gebruikt. Het is de theorie die bepaalt wat er te observeren valt.
> *Albert Einstein, 1926.*
>
> Niets is echt, alles is toegestaan.
> *Van „Alamut" door V. Bartol, 1938.*

Moderne natuurkunde is een directe afstammeling van de natuurfilosofie en het uiteindelijke doel van dit proefschrift is de aanvraag van de graad *philosophiae doctor*. Maar natuurkunde is niet meer een filosofie. Natuurfilosofie, in de zin van Isaac Newton en zijn voorgangers, beschrijft de wereld zoals hij is, perfect in aard, voorspelbaar en onderworpen aan de klassieke logica en objectieve observaties. Voor millennia waren mensen overtuigd dat de geïdealiseerde concepten van de wiskunde, meetkunde en andere takken van de wetenschap de wereld waar in we leven representeren en ons uiteindelijk in staat stellen de vraag „Hoe gedraagt het universum zich?" te beantwoorden. Met het grote succes van in de 19de eeuw ontwikkelde theorieën, zoals elektromagnetisme en thermodynamica, leek het antwoord op deze vraag dichterbij dan ooit.

De beruchte quote die aan heer Kelvin wordt toegeschreven kon echter onmogelijk op een slechter moment gesproken worden. Sinds 1900 werden in minder dan een halve eeuw de grondslagen van de natuurfilosofie verbrijzeld. De revolutie startte met kleine, bijna insignificante onregelmatigheden. De klassieke thermodynamica voorspelde het spectrum van zwarte lichaamstraling met eigenaardige eigenschappen bij zeer lage temperaturen. Hoewel in die tijd dit experimenteel niet te meten was, leidde het onderzoek naar dit gedrag Max Planck en zijn opvolgers naar de schokkende ontdekking van de kwantummechanica. Een andere onregelmatigheid werd geobserveerd in de Maxwell vergelijkingen van het elektromagnetisme. Hun rare, niet-Galileische transformatie-eigenschappen, samen met de metingen die wezen op de constante snelheid van het licht leidden Albert Einstein tot de ontwikkeling van de speciale en vervolgens ook de algemene relativiteitstheorie.

Kwantummechanica en de algemene relativiteitstheorie zijn de twee theorieën in de kern van de moderne natuurkunde. Hun voorspellingen, getest met een weergaloze precisie, hebben tot duizenden uitvindingen geleid van de microprocessor tot het GPS navigatiesysteem. In weerwil van hun succes, is hun filosofie in een verschrikkelijke tegenstelling met de regels van de natuurfilosofie.

1. De wereld is niet deterministisch. De regels van de klassieke logica zijn niet van toepassing. De klassieke mechanica volgt alleen uit de kwantummechanica in een toepasselijk limiet.

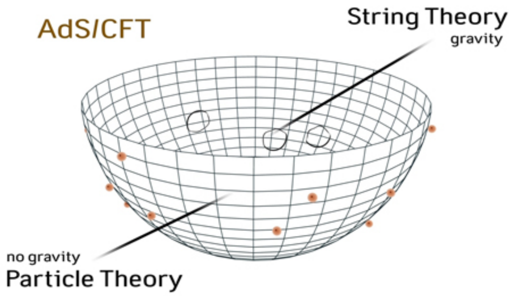
2. Tijd en ruimte zijn niet absoluut. Ze zijn beiden onderling verbonden en beiden ontwikkelen in relatie met de materiële inhoud van het universum.

3. Golf/deeltje dualiteit. Iemand kan zich niet de „aard" van kwantumobjecten voorstellen. Ze kunnen even goed beschreven worden als golven als als deeltjes hoewel het nut van een beschrijving ligt aan de situatie.

Moderne theoretische natuurkunde probeert niet de vraag Hoe gedraagt het universum zich? te beantwoorden, maar liever ontwikkelt het wiskundige modellen die het gedrag van echte fysieke objecten voorspellen. Alle succesvolle modellen zijn onderworpen aan wat beperkingen en werken alleen in hun bereik van werkelijkheid. Er kunnen meer dan n model zijn - en gewoonlijk is dat ook zo - die een gegeven natuurkundig fenomeen beschrijven. Gebruik makend van de wetenschappelijke methode, kiest iemand gewoonlijk een model dat nauwkeurigere voorspellingen geeft, simpeler is, minder parameters nodig heeft en een groot bereik van werkelijkheid heeft.

Gedurende de 20ste eeuw hebben kwantummechanica en de speciale relativiteitstheorie zich ontwikkeld tot de kwantumveldtheorie en vervolgens het standaard model van elementaire deeltjes. Het laatste missende deeltje, het Higgs boson, veertig jaar geleden getheoretiseerd, was tenslotte in de LHC gezien in 2012. Aan de andere kant zijn de gevolgen van de algemene relativiteitstheorie op een hoge precisie getest en voor de eerste keer in de geschiedenis werd er een betrouwbaar kosmologisch model voor de ontwikkeling van het universum gevormd dat in overeenstemming was met de astronomie, geologie en evolutionaire biologie. Is de natuurkunde weer compleet?

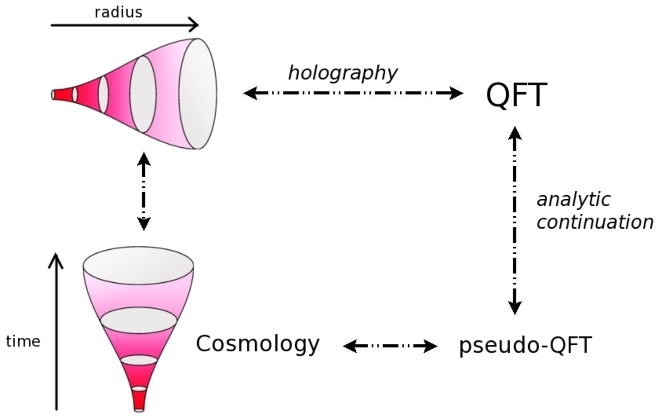
Misschien niet. Een van de meest fascinerende puzzels in de hoogenergetische natuurkunde is de kwantisatie van zwaartekracht. Het standaard model van elementaire deeltjes bevat de zwaartekracht niet en theoretische overwegingen leiden tot grote problemen wanneer iemand EinsteinŹs zwaartekracht probeert te verenigen met kwantumveldtheorieën. Voor decennia hebben generaties natuurkundigen geworsteld om zwaartekracht te kwantiseren. Terwijl de natuurkunde van alle andere krachten in de arena van ruimte en tijd speelt is de zwaartekracht ruimte en tijd. Daarom vereist de kwantisatie van zwaartekracht een compleet nieuw begrip van de structuur van het universum. Hints voor de kwantisatie van zwaartekracht volgden van veel kanten, onder andere hoogenergetische natuurkunde, zwarte gaten mechanica en kosmologie. Wederom leidden kleine onregelmatigheden en theoretische tegenstrijdigheden tot de ontwikkeling van de gauge/zwaartekracht dualiteit in 1997. Voor de eerste keer konden er daadwerkelijk berekeningen in de kwantumzwaartekracht gedaan worden.



Figuur 7.5: *Het basis idee van de gauge/zwaartekracht dualiteit. De zwaartekracht theorie zoals snaartheorie levend in de massa is equivalent met de kwantumveldtheorie zonder zwaartekracht op de grens.*

De gauge/zwaartekracht dualiteit, ook wel bekend als de AdS/CFT correspondentie of holografie, zegt dat een kwantumsysteem dat zwaartekracht, materie en andere krachten bevat compleet equivalent is met een ander systeem dat beschreven wordt door een kwantumveldtheorie zonder zwaartekracht. Een van de meest verwonderende eigenschappen van de dualiteit is dat een sterk gekoppeld niet-perturbatief stelsel van n theorie waar daadwerkelijke berekeningen praktisch onmogelijk zijn correspondeert met een zwak gekoppeld perturbatief stelsel van de duale theorie, waar natuurkundige voorspellingen gedaan kunnen worden. De natuurkunde wordt in beide theorieën even goed beschreven. Het is vanwege onze plaats en tijd in het universum dat we NewtonŹs zwaartekracht vaker ervaren dan de dynamica van de duale veldtheorie op dezelfde manier als dat we het golfkarakter van licht observeerde voor het deeltjes karakter.

De gauge/zwaartekracht dualiteit komt voort uit de analyse van de snaartheorie. Voor decennia was snaartheorie de meest veelbelovende kandidaat voor de theorie van kwantumzwaartekracht. Ondanks gebrek aan direct experimenteel bewijs deed de schoonheid van de theorie mensen geloven dat er een natuurkundige belang achter zat. Snaartheorie is een unieke theorie die zwaartekracht, materie en andere krachten combineert in een consistente kwantumtheorie. In snaartheorie komen alle deeltjes, inclusief krachtdragers zoals gravitons, voort uit verschillende manieren van trillingen van piepkleine snaren. het probleem, echter, was dat de enige bekende definitie van snaartheorie gebaseerd was op de perturbatieve groei, waar de effecten van kwantumzwaartekracht gezien worden als kleine correcties op de klassieke oplossing. De gauge/zwaartekracht dualitiet stond een nieuwe mogelijkheid toe voor de analyse van het sterk gekoppelde stelsel van snaartheorie, waar de effecten van kwantumzwaartekracht domineren.

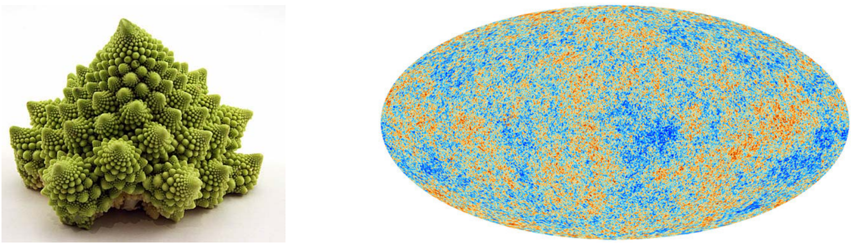


Figuur 7.6: *Stappen leidend naar de holografische beschrijving van kosmologie. Door de kosmologie/domein muur correspondentie kan iemand kosmologische observaties vertalen in schommelingen rond een of andere euclidische domein muur ruimtetijd. Dan kan de gauge/zwaartekracht dualiteit worden toegepast. Deze procedure leidt tot de holografische formules die kosmologische observaties uitdrukken in termen van de analytisch voortgezette correlatie functie van de duale kwantumveldtheorie.*

De gauge/zwaartekracht dualiteit is succesvol toegepast op verscheidene gecondenseerde materie systemen zoals de natuurkunde van supergeleiders, supergeleiders van hydrodynamica van quark-gluon plasma. Aangezien al deze fenomenen beschreven worden door wat sterk gekoppelde kwantumveldtheorieën vertaalt holografie deze dynamica in de equivalente dynamica van wat op zwaartekracht gebaseerde systemen zoals zwarte gaten. In dit proefschrift echter zullen we de gauge/zwaartekracht dualiteit in de andere richting gebruiken. Door specifieke berekeningen uit te voeren in een niet op zwaartekracht gebaseerde veldtheorie zullen we meetbare voorspellingen verkrijgen die behoren tot de heel erg vroege fases van ons universum.

Het is een huidig geaccepteerd paradigma dat gedurende de erg vroege fases het universum snel groeide terwijl de gering kwantumschommelingen daar binnen werden gestrekt tot gigantische groottes wat het begin creëerde van toekomstige sterren en sterrenstelsels. We hebben goed experimenteel bewijs, vooral door de metingen van de Cosmic Microwave Background: elektronische straling die de hele ruimte vult en schatbare informatie over het vroege universum meedraagt. Het is ook geaccepteerd dat voor de inflatie plaatsvond het universum overheerst werd door sterk gekoppelde, niet-meetkundige kwantumzwaartekracht. In zoŹn stelsel stoppen ruimte en tijd met bestaan, is de meetkundige beschrijving niet langer geldig en kan alleen holografie ons enig inzicht in deze fascinerende fase geven.

In dit proefschrift zullen we laten zien hoe iemand toegang kan krijgen tot zowel de groeiende als de pre-groeiende fases van het universum door gebruik van de gauge/zwaartekracht dualiteit. We zullen specifieke modellen tonen die de erg vroege fases van het universum beschrijven en we zullen de voorspellingen vergelijken met de huidig toegankelijke data. Zoals we zullen zien passen beide modellen goed met de data en worden ze een serieus alternatief voor de standaard theorie van inflatie. Beide modellen lossen de initiële singulariteit van het universum op door de oerknal te herinterpreteren als de uitgang van de niet-meetkundige fase van het universum. Diagram 7.6 toont hoe de holografische modellen van de kosmologie worden verkregen.



Figuur 7.7: *Een bloemkool is een voorbeeld van een bij benadering schaal invariant object. Een ander voorbeeld van een bijna schaal invariant systeem is de Cosmic Microwave Background. De grafiek rechts toont de temperatuur schommelingen van de elektromagnetische straling afhankelijk van de richting in de hemel.*

Om kwantitatieve resultaten te krijgen die uit de modellen volgen moet iemand de eigenschappen van de duale kwantumveldtheorie analyseren. In het geval van de gauge/zwaartekracht dualiteit vertoont de kwantumveldtheorie gewoonlijk extra symmetrieen zoals de invariantie van hoeken. Hoekgetrouwe symmetrie is sterk gerelateerd aan de schaal invariantie: de situatie waar de natuurkunde identiek is op verschillende schalen. Echter de meeste natuurkunde om ons heen is niet schaal invariant. Bijvoorbeeld de hydrodynamische beschrijving van water valt uit elkaar wanneer de grootte van de golven klein genoeg is om de dynamica van de enkele moleculen mee te laten spelen. Echter veel systemen zijn bij benadering schaal invariant, zie figuur 7.7.

In het proefschrift tonen we een nieuwe aanpak voor de classificatie van de correlatie functies van de hoekgetrouwe veldtheorieën direct in impuls ruimte. Onze aanpak versnelt en versimpelt de holografie analyse aanzienlijk. Bovendien zijn onze resultaten niet gelimiteerd tot de kosmologie en kunnen ze gebruikt worden in veel toepassingen van de hoekgetrouwe veldtheorieën.

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