Relativistic Hydrodynamics

Let us start with the simplest system without external magnetic fields and quantum anomalies

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu J^\mu = 0 . \end{cases}$$

By Eckart PR. 58.919, given a dimensionless velocity vector u^μ s.t. $u^\mu u_\mu \equiv -1$, one can uniquely construct a scalar and a transverse vector from an arbitrary vector O^μ by

$$\begin{cases} O \equiv -O^\mu u_\mu \\ \tilde{O}^\mu \equiv P^{\mu\nu} O_\nu \end{cases}$$ with the projection op. $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$.

[$O^\mu \equiv g^{\mu\nu} O_\nu \equiv (g^{\mu\nu} + u^\mu u^\nu) O_\nu - u^\mu u^\nu O_\nu$]

so that $O^\mu \equiv O u^\mu + \tilde{O}^\mu$ $\quad (P^\mu{}_\nu \equiv \delta^\mu_\nu + u^\mu u^\nu ,\ P^{\mu\nu} \equiv g^{\mu\lambda} P^\mu{}_\lambda)$

Thus generally the charge current $J^\mu \equiv \underbrace{N u^\mu}_{\text{nondissipative}} + \underbrace{j^\mu}_{\text{dissipative}}$, with $N \equiv J^\nu u_\nu$.

Similarly, given a tensor of type (2,0) $T^{\mu\nu}$, one can construct

scalar $\mathcal{E} \equiv T^{\mu\nu} u_\mu u_\nu$, $P \equiv P^{\mu\nu} u_\mu u_\nu$ $\quad$ [$T^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\rho} T_{\alpha\rho} = (P^{\mu\alpha} - u^\mu u^\alpha)(P^{\nu\rho} - u^\nu u^\rho) T_{\alpha\rho}$]

transverse vector $q^\mu \equiv -P^{\mu\nu} u^\rho T_{\nu\rho}$

transverse-traceless tensor $t_{\mu\nu} \equiv \frac{1}{2}\left(P_{\mu\alpha} P_{\nu\beta} + P_{\alpha\nu} P_{\mu\beta} - \frac{2}{d} P_{\mu\nu} P_{\alpha\beta}\right) T^{\alpha\beta}$

s.t. $T^{\mu\nu} = \underbrace{\mathcal{E} u^\mu u^\nu + P P^{\mu\nu}}_{\text{nondissipative}} + \underbrace{(q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}}_{\text{dissipative}}$

Our hydro-variables will be $\{u^\mu, T, \mu\}$

Here $\mathcal{E}$, N, and P are scalars so will be function of scalars $T, \mu, \partial_\mu u^\mu, u^\mu \partial_\mu T, u^\mu \partial_\mu \mu$

In contrast, j^μ, q^μ, and $t^{\mu\nu}$ will be functions of transverse vectors like $P_{\mu\nu}\partial^\nu T$, $P_{\mu\nu}\partial^\nu \mu$ etc...

(Clearly the non-vanishing terms in j^μ, q^μ and $t^{\mu\nu}$ start with first derivatives of hydrovariables, since $P^{\mu\nu} u_\nu \equiv 0$)

Zeroth Order

There is no derivatives on hydro-variables, so the is NO dependence on j^μ, p^μ, and $t^{\mu\nu}$.

On the other hand, the splitting coefficient has explicit physical meaning: a static fluid with particle number $N = n(T,\mu)$, pressure $P = p(T,\mu)$, and energy $\mathcal{E} = \varepsilon(T,\mu)$

Thus constitutive relation $\begin{cases} J^\mu = n u^\mu \\ T^{\mu\nu} = \varepsilon u^\mu u^\nu + p P_{\mu\nu} \equiv (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} \end{cases}$

(Even with the existence of external EM!) With the assumption that $F^{\mu\nu}$ is of order $O(p)$, to the zeroth order we only need to focus on the zeroth order of $J^{(0)}_\mu = \mathcal{N} u_\mu$. But $F^{\mu\nu}$ is anti-symmetric, so (to the first order)

$$u_\mu \partial_\nu T^{\mu\nu} = u_\mu F^{\mu\nu} J_\nu = \mathcal{N} u_\mu u_\nu F^{\mu\nu} \equiv 0.$$

At zeroth order, we will find an "accidental" conservation law of entropy $\partial_\mu S^\mu = 0$.

$\triangleright$ $\partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \Rightarrow 0 = u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu \left[(\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} \right]$

$$= u_\nu \left[(\partial_\mu \varepsilon + \partial_\mu p) u^\mu u^\nu + (\varepsilon + p)\left((\partial_\mu u^\mu) u^\nu + u^\mu \partial_\mu u^\nu \right) + g^{\mu\nu} \partial_\mu p \right]$$

$u_\mu u^\mu \equiv -1$
$\Downarrow$
$u_\mu \partial_\nu u^\mu \equiv 0$

$$= -(\partial_\mu \varepsilon + \partial_\mu p) u^\mu - (\varepsilon + p) \partial_\mu u^\mu + (\varepsilon + p) u_\nu u^\mu \partial_\mu u^\nu + u^\mu \partial_\mu p$$

$$= -u^\mu \partial_\mu \varepsilon - u^\mu \partial_\mu p - (\varepsilon + p) \partial_\mu u^\mu + u^\mu \partial_\mu p = -u^\mu \partial_\mu \varepsilon - (\varepsilon + p) \partial_\mu u^\mu$$

Using the identity $\varepsilon + P = \mu n + Ts$ and the first law of thermodynamics $dE = TdS - pdV + \mu dN \Rightarrow dP = n d\mu + s dT \Rightarrow \partial_\mu P = n \partial_\mu \mu + s \partial_\mu T$, then the above result can be simplified as

$$0 = u^\mu \partial_\mu (P - \mu n - Ts) - (\mu n + Ts) \partial_\mu u^\mu$$

$$= u^\mu \partial_\mu P - u^\mu n \partial_\mu \mu - u^\mu \mu \partial_\mu n - u^\mu s \partial_\mu T - u^\mu T \partial_\mu s - (\mu n + Ts) \partial_\mu u^\mu$$

$$= -u^\mu \mu \partial_\mu n - u^\mu T \partial_\mu s - \mu n \partial_\mu u^\mu - Ts \partial_\mu u^\mu$$

$$= -\mu \partial_\mu (n u^\mu) - T \partial_\mu (s u^\mu) \equiv -\mu \underbrace{\partial_\mu \langle J^\mu \rangle}_{\text{zero by conservation}} - T \partial_\mu s^\mu = -T \partial_\mu s^\mu$$

$\triangleleft$

Frame Choice

When we go to higher order of fluctuations, a subtlety that is no evident in zeroth order emerges: the notion of local hydrovariables $\{T, \mu, u^\mu\}$ seem to be NOT uniquely defined. In fact, all hydro-variables are uniformly and uniquely defined in equilibrium, but out of equilibrium there should be NO canonical microscopic definition because they cannot be measured.

This can be easily seen from the "conservation law allowed redefinition"

$$\begin{cases} T(x) \mapsto T'(x) \equiv T(x) + \delta T \\ \mu(x) \mapsto \mu'(x) \equiv \mu(x) + \delta\mu \\ u^\mu(x) \mapsto u^{\mu\prime}(x) \equiv u^\mu(x) + \delta u \end{cases}$$

Keeping the observable $\langle T^{\mu\nu} \rangle$ and $\langle J^\mu \rangle$ unchanged

where δT, $\delta\mu$, δu^μ are functionals of at least first-order spatial fluctuation of thermodynamic (well-defined) hydro-variables.

† Note that δu^μ must satisfy the constraint $u_\mu u^\mu \equiv -1 \Rightarrow u_\mu \delta u^\mu = 0$, so it must be transverse. (so is of at least first order derivatives)

Q: Why we say such redefinition problem does not exist to the zeroth order?

A: Using the fact that $T^{\mu\nu}$ and J^μ are invariant under frame choice,

$$\delta \mathcal{E} \equiv \delta(u_\mu u_\nu T^{\mu\nu}) = 2 u_\mu \delta u_\nu T^{\mu\nu} = 2 u_\mu \delta u_\nu \left(\mathcal{E} u^\mu u^\nu + P P^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu} \right)$$

Since both q^μ and $t^{\mu\nu}$ are transverse, $u_\mu q^\mu = u_\mu t^{\mu\nu} = 0$.

$$\Rightarrow \delta \mathcal{E} = 2 u_\mu \delta u_\nu \left((\mathcal{E} + P) u^\mu u^\nu + P g^{\mu\nu} \right).$$

Again since δu_μ is transverse, we conclude that $\delta \mathcal{E} \equiv 0$.

Similarly

$$\delta P = \frac{1}{d} \delta P_{\mu\nu} T^{\mu\nu} \equiv \frac{1}{d} \delta (u_\mu u_\nu + g_{\mu\nu}) \cdot T^{\mu\nu} = \frac{2}{d} u_\mu \delta u_\nu \cdot T^{\mu\nu} = \frac{\delta \mathcal{E}}{d} = 0.$$

$$\delta \mathcal{N} = -\delta(u_\mu J^\mu) = -\delta u_\mu \cdot J^\mu = -\delta u_\mu (\mathcal{N} u^\mu + j^\mu) \xrightarrow[\text{are transverse}]{\text{both } j^\mu \text{ and } \delta u_\mu} 0.$$

Q: Why does the first-order terms suffer frame choice?

A:

$$\delta q^\mu \equiv -\delta \left(P^{\mu\nu} u^\sigma T_{\nu\sigma} \right) = -\delta P^{\mu\nu} \cdot u^\sigma T_{\nu\sigma} - P^{\mu\nu} \delta u^\sigma \cdot T_{\nu\sigma}$$

$$= -(u^\mu \delta u^\nu + u^\nu \delta u^\mu) u^\sigma T_{\nu\sigma} - u^\mu u^\nu \delta u_\sigma T_{\nu\sigma} - \delta u_\sigma T^{\mu\sigma}$$

$$= \underbrace{-2 u^\mu u^\sigma \delta u^\nu T_{\nu\sigma}}_{\text{already shown in } \delta\mathcal{E}} - u^\nu u^\sigma \delta u^\mu T_{\nu\sigma} - \delta u_\sigma T^{\mu\sigma}$$

$$= -u^\nu u^\sigma \delta u^\mu \left(\mathcal{E} u_\nu u_\sigma + P_{\nu\sigma} P + (q_\nu u_\sigma + q_\sigma u_\nu) + t_{\nu\sigma} \right) - \delta u_\sigma \left(\mathcal{E} u^\mu u^\sigma + P^{\mu\sigma} p + (q^\mu u^\sigma + q^\sigma u^\mu) + t^{\mu\sigma} \right)$$

$$= -\delta u^\mu \cdot \mathcal{E} - \delta u^\mu p - \underbrace{\delta u_\sigma \cdot q^\sigma u^\mu}_{\text{of second-order}} - \delta u_\sigma \cdot t^{\mu\sigma}$$

$$\simeq -(\mathcal{E} + p) \delta u^\mu$$

$$\delta j^\mu = \delta(P^{\mu\nu} J_\nu) \equiv \delta P^{\mu\nu} \cdot J_\nu = (u^\mu \delta u^\nu + u^\nu \delta u^\mu)(\mathcal{N} u_\nu + j_\nu) = -\mathcal{N} \delta u^\mu + \underbrace{u^\nu \delta u^\mu j_\mu}_{\text{of second order}}$$

$$\simeq -\mathcal{N} \delta u^\mu.$$

$$\delta t^{\mu\nu} \equiv \frac{1}{2} \delta \left(P^{\mu\alpha} P^{\nu\beta} + P^{\nu\alpha} P^{\mu\beta} - \frac{2}{d} P^{\mu\nu} P^{\alpha\beta} \right) T^{\alpha\beta}$$

$$= \cdots = 0.$$

The above arguments, particularly to the zeroth order, is independence of the concrete form of $T^{\mu\nu}$ and J^μ. But the relation to the zeroth order $\mathcal{E}\to\varepsilon$, $\mathcal{P}\to p$, and $\mathcal{N}\to n$, does change with fluctuations

$$\begin{cases} \mathcal{E} = \underbrace{\varepsilon(T,\mu)}_{\text{in equilibrium}} + \underbrace{f_{\mathcal{E}}(\partial T, \partial\mu, \partial u^\mu)}_{\text{out of equilibrium}} \\ \mathcal{P} = p(T,\mu) + f_{\mathcal{P}}(\partial T, \partial\mu, \partial u^\mu) \\ \mathcal{N} = n(T,\mu) + f_{\mathcal{N}}(\partial T, \partial\mu, \partial u^\mu) \end{cases}$$

Under the redefinition of hydrovariables, the above relation will also be modified as

$$\begin{cases} \mathcal{E} = \varepsilon(T',\mu') + f'_{\mathcal{E}}(\partial T', \partial\mu', \partial u'^\mu) \\ \mathcal{P} = p(T',\mu') + f'_{\mathcal{P}}(\partial T', \partial\mu', \partial u'^\mu) \\ \mathcal{N} = n(T',\mu') + f'_{\mathcal{N}}(\partial T', \partial\mu', \partial u'^\mu) \end{cases}$$

$\{T, \mu, u^\mu\}$

but as is proved above, $\delta\mathcal{E} = \delta\mathcal{P} = \delta\mathcal{N} = 0$, or

$$\begin{cases} f'_{\mathcal{E}} = f_{\mathcal{E}} + \frac{\partial\varepsilon}{\partial T}\delta T + \frac{\partial\varepsilon}{\partial\mu}\delta\mu \\ f'_{\mathcal{P}} = f_{\mathcal{P}} + \frac{\partial p}{\partial T}\delta T + \frac{\partial\varepsilon}{\partial\mu}\delta\mu \\ f'_{\mathcal{N}} = f_{\mathcal{N}} + \frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial\mu}\delta n \end{cases}$$

We prefer to choose the independent δT and $\delta\mu$ s.t. $f'_{\mathcal{E}}$ and $f'_{\mathcal{N}}$ are always zero (so that $\mathcal{E}$ stays to be ε and $\mathcal{N}$ stays to be n)

As for δu^μ, One can choose δu_μ s.t. $j^\mu = 0$. This is referred to "Eckart frame", implying no charge flow in the local rest frame of the fluid.

One can also choose δu_μ s.t. $q^\mu = 0$. This is referred to "Landau frame", meaning no energy flow in the local rest frame of the fluid.

We prefer to work in Landau frame.

First Order

So the left work is to express the coefficients P, $t^{\mu\nu}$, and j^μ in terms of first-order fluctuated hydro-variables. The most important thing is that whatever "gauge" we choose, constitutive relations must only involve in hydro-variable redefinition invariant terms! (Bhattacharya et al. JHEP 05. 147 (2014))

For P, we need frame-choice invariant scalars. The most general forms we have are

$\{u^\mu\partial_\mu T,\ u^\mu\partial_\mu\mu,\ \partial_\mu u^\mu,\ P^{\mu\nu}\partial_\mu u_\nu\}$.

$$u^\nu u^\mu\partial_\mu u_\nu + \partial^\mu u_\mu = \partial^\mu u_\mu$$

$$\Rightarrow \{u^\mu\partial_\mu T,\ u^\mu\partial_\mu\mu,\ \partial_\mu u^\mu\}$$

For transvers vectors j^μ satisfying $u_\mu j^\mu = 0$, it's natural to demand a projection operator. The non-vanishing term we have are $P^{\mu\nu}\partial_\nu T$, $P^{\mu\nu}\partial_\nu\mu$, and $P^{\mu\nu}u_\lambda\partial^\lambda u_\nu$, and $F^{\mu\nu}u_\nu$ if external EM exists. But besides this, Levi-Civita tensor can also achieve this by $\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$. Although the existence of such term requires breaking of parity invariance (or equivalently existence of chiral anomaly).

For transverse traceless symmetric tensor $t^{\mu\nu}$ satisfying $u_\mu t^{\mu\nu} = t^{\mu\nu}u_\nu = 0$, similarly we expect two transverse projection outside the expression $\sigma_{\mu\nu}$, also the symmetric and traceless property require

$$\sigma_{\mu\nu} = \partial_\mu u_\nu + \partial_\nu u_\mu - \frac{g^{\mu\nu}}{2}\partial_\lambda u_\lambda$$ Landau & Lifshitz ch 15, cl 136

Thus we can summarize all possible one-derivate terms of hydro variables (we need) as following:

	All Data	EOM	Independent Chosen Data
Scalars	$u^\mu\partial_\mu T$, $u^\mu\partial_\mu\mu$, $\partial_\mu u^\mu$	$u_\mu\nabla_\nu T^{\mu\nu}=0$ $\nabla_\mu J^\mu = 0$	$\partial_\mu u^\mu$
Transverse Vectors	$P^{\mu\nu}\partial_\nu T$, $P^{\mu\nu}\partial_\nu\mu$, $P^{\mu\nu}u^\lambda\partial_\lambda u_\nu$ $F^{\mu\nu}u_\nu$ (if external EM exists)	$P^{\mu\nu}\partial^\lambda T_{\lambda\nu}=0$	$P^{\mu\nu}\partial_\nu T$, $P^{\mu\nu}\partial_\nu\mu$ $F^{\mu\nu}u_\nu$
Pseudovectors* (If no parity inversion)	$\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$ $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$ (if EM exists)		$\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$ $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$
Transverse-traceless Tensor	$\sigma^{\mu\nu} \equiv P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}g_{\alpha\beta}\partial_\lambda u^\lambda\right)$		$\sigma^{\mu\nu}$

Since all possible terms must satisfy the zeroth-order EOM, we can simplify our world by choosing only the independent one listed on the last column above.

So to the first order of fluctuation, we have (we prefer to work with μ/T rather than single μ)

$$\begin{cases} P = p - \zeta\,\partial_\lambda u^\lambda \\ j^\mu = -\sigma T P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) - \chi_T P^{\mu\nu}\partial_\nu T \\ t^{\mu\nu} = -\eta\,\sigma^{\mu\nu} \end{cases}$$

ζ: bulk viscosity
σ: charge conductivity
η: shear viscosity

or in all, Lucas & Fong J. Phys.: Condens. Matt. 30. 053001 (2018)

$$\begin{cases} T^{\mu\nu} = \varepsilon u^\mu u^\nu + p P^{\mu\nu} - \zeta P^{\mu\nu}\partial_\lambda u^\lambda - \eta P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2g_{\alpha\beta}}{d}\partial_\lambda u^\lambda\right) \\ J^\mu = n u^\mu - \sigma T P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) - \chi_T P^{\mu\nu}\partial_\nu T \end{cases}$$

This result only use relativity and parity symmetry. To prove that $\zeta, \eta, \sigma \geq 0$ and $\chi_T \equiv 0$, entropy argument is demanded.

Restriction from Entropy Production Rate

Similar to what we have done in the zeroth order, we need to find some unknown expression for the entropy current. Using the same trick:

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad u_\mu \nabla_\nu T^{\mu\nu} = u_\mu \nabla_\nu T^{(0)\mu\nu} + u_\mu \nabla_\nu T^{(1)\mu\nu} = 0$$

Inserting $T^{\mu\nu} \equiv T^{(0)\mu\nu} + T^{(1)\mu\nu} \equiv \left((\varepsilon+p)u^\mu u^\nu + p g^{\mu\nu}\right) + T^{(1)\mu\nu}$, and using the result for $T^{(0)\mu\nu}$ in the zeroth order, we have

$$0 \equiv -\mu \nabla_\mu (\underbrace{n u^\mu}_{J^{(0)\mu}}) - T \nabla_\mu (s u^\mu) + u_\mu \nabla_\nu T^{(1)\mu\nu}$$

$$= -\mu \underbrace{\nabla_\mu J^\mu}_{\text{zero by charge conservation}} - \mu \nabla_\mu J^{(1)\mu} - T \nabla_\mu (s u^\mu) + u_\mu \nabla_\nu T^{(1)\mu\nu}$$

$$\Rightarrow \nabla_\mu (s u^\mu) \equiv -\frac{\mu}{T} \nabla_\mu J^{(1)\mu} + \frac{u_\mu}{T} \nabla_\nu T^{(1)\mu\nu}$$

$$= -\nabla_\mu \left(\frac{\mu}{T} J^{(1)\mu}\right) - J^{(1)\mu} \nabla_\mu \left(\frac{\mu}{T}\right) + \frac{1}{T} \underbrace{\nabla_\nu \left(u_\mu T^{(1)\mu\nu}\right)}_{\text{zero because } T^{(1)\mu\nu} \text{ is transverse}} - \frac{1}{T} T^{(1)\mu\nu} \nabla_\nu u_\mu$$

$$\Rightarrow \nabla_\mu \underbrace{\left(s u^\mu + \frac{\mu}{T} J^{(1)\mu}\right)}_{\text{New entropy current } S^\mu} = -J^{(1)\mu} \nabla_\mu \left(\frac{\mu}{T}\right) - \frac{1}{T} T^{(1)\mu\nu} \nabla_\nu u_\mu$$

Since $\frac{\mu}{T}$, T, and u_μ are independent, the non-negativity of entropy production rate requires,

$$-J^{(1)\mu} \nabla_\mu \left(\frac{\mu}{T}\right) \geq 0 \quad \text{and} \quad T^{(1)\mu\nu} \nabla_\nu u_\mu \geq 0, \quad \text{respectively}$$

Inserting the general form of

$$J^{(1)\mu} = -\sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) - \chi_T P^{\mu\nu} \partial_\nu T$$

$$T^{(1)\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta} \partial_\lambda u^\lambda\right) - \zeta P^{\mu\nu} \partial_\lambda u^\lambda$$

The first inequality requires

$$-\sigma T \left[\left(u_\mu \partial_\mu \left(\frac{\mu}{T}\right)\right)^2 + \left(\partial_\mu \left(\frac{\mu}{T}\right)\right)^2\right] - \chi_T P^{\mu\nu} \partial_\mu \left(\frac{\mu}{T}\right) \partial_\nu T \geq 0$$

$$\Rightarrow \quad \sigma \geq 0, \quad \chi_T \equiv 0$$

The second inequality requires

$$-\eta P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta} \partial_\lambda u^\lambda\right) \partial_\nu u_\mu - \zeta \underbrace{P^{\mu\nu} (\partial_\nu u_\mu)}_{\Downarrow} \cdot \partial_\lambda u^\lambda \geq 0$$

$$\zeta g^{\mu\nu} (\partial_\nu u_\mu) \cdot \partial_\lambda u^\lambda \equiv \zeta (\partial_\lambda u^\lambda)^2 \Rightarrow \zeta \geq 0$$

For the first part, because $\sigma^{\mu\nu}$ is symmetric, when contracting with an arbitrary tensor of type (0,2), it's equivalent to calculate $\sigma^{\mu\nu} A_{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}(A_{\mu\nu}+A_{\nu\mu})$. Also, since $\sigma^{\mu\nu}$ is traceless $\sigma^{\mu\mu}=0$, we can subtract the trace of $(A_{\mu\nu}+A_{\nu\mu})$ without changing the result. So finally,

$$-\eta\,\sigma^{\mu\nu}\partial_\nu u_\mu = -\frac{\eta}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} \quad\Rightarrow\quad \eta \geq 0.$$

Relativistic Hydrodynamics with External E.M.

$$\text{EoM}\quad \begin{cases} \partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu \\ \partial_\mu J^\mu = 0. \end{cases}$$

Zeroth order constitutive relation won't change and still $\partial_\mu s^\mu = \partial_\mu(su^\mu)=0$. But when we go to the first order, Lorentz symmetry allows

$$\begin{cases} P = p - \zeta\,\partial_\lambda u^\lambda \\ t^{\mu\nu} = -\eta\,\sigma^{\mu\nu} \\ j^\mu = -c_1 P^{\mu\nu}\partial_\nu\mu - c_2 P^{\mu\nu}F_{\nu\lambda}u^\lambda - c_3 P^{\mu\nu}\partial_\nu T \end{cases}$$

$$\equiv -\sigma T P^{\mu\nu}\left[\partial_\nu\left(\frac{\mu}{T}\right) - \frac{1}{T}F_{\nu\lambda}u^\lambda\right] - \chi_T P^{\mu\nu}\partial_\nu T - c\,P^{\mu\nu}F_{\nu\lambda}u^\lambda.$$

To prove that $\zeta, \eta, \sigma \geq 0$ and $\chi_T = c = 0$, still entropy argument is demanded.

One must be clear that the above trick finding the entropy current should be modified!

$$u_\mu\partial_\nu T^{\mu\nu} = u_\mu F^{\mu\nu}J_\nu \Rightarrow u_\mu\partial_\nu\left(T^{(0)\mu\nu}+T^{(1)\mu\nu}\right) = u_\mu F^{\mu\nu}\left(J^{(0)}_\nu - J^{(1)}_\nu\right)$$

$$\underbrace{u_\mu\partial_\nu T^{(0)\mu\nu}}_{-\mu\,\partial_\mu(nu^\mu) - T\,\partial_\mu(su^\mu)} + u_\mu\partial_\nu T^{(1)\mu\nu} = u_\mu F^{\mu\nu}\left(N u_\nu + J_\nu\right) = 0 + u_\mu F^{\mu\nu}J^{(1)}_\nu$$

$$J^{(0)\mu} \equiv J^\mu - J^{(1)\mu}$$

$$\Rightarrow\quad \underbrace{\frac{\mu}{T}\partial_\mu J^{(1)\mu}}_{\partial_\mu\left(\frac{\mu}{T}J^{(1)\mu}\right) - J^{(1)\mu}\partial_\mu\left(\frac{\mu}{T}\right)} - \partial_\mu s^\mu + \underbrace{\frac{u_\mu}{T}\partial_\nu T^{(1)\mu\nu}}_{\frac{1}{T}\partial_\nu\left(u_\mu T^{(1)\mu\nu}\right) - \frac{1}{T}T^{(1)\mu\nu}\partial_\nu u_\mu} = \frac{u_\mu}{T}F^{\mu\nu}J^{(1)}_\nu$$

($u_\mu T^{(1)\mu\nu}$: transverse so vanishes)

$$\Rightarrow\quad \partial_\mu\underbrace{\left(su^\mu - \frac{\mu}{T}J^{(1)\mu}\right)}_{\text{entropy current } S^\mu} = -J^{(1)\mu}\left[\partial_\mu\left(\frac{\mu}{T}\right) + F_{\mu\nu}\frac{u^\nu}{T}\right] - \frac{1}{T}T^{(1)\mu\nu}\partial_\nu u_\mu$$

Clearly the independent non-negativity for the first part tells $\sigma \geq 0$, $\chi_T = 0$, and $c = 0$, while that for the second part tells the same $\zeta \geq 0$ and $\eta \geq 0$

Relativistic Hydrodynamics with External E.M. and Chiral Anomalies

$$\text{EoM} \begin{cases} \partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \\ \partial_\mu J^\mu = C E^\mu B_\mu \end{cases}, \text{ where } E^\mu = F^{\mu\nu} u_\nu, \; B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}$$

Note that the anomalous term is of order $\mathcal{O}(p^2)$, so the zeroth-order equilibrium with $s^\mu = s u^\mu$ is still valid here.

But since parity inversion is broken by chiral anomaly, in general we can write

$$\begin{cases} P = p - \zeta \partial_\lambda u^\lambda \\ t^{\mu\nu} = -\eta \sigma^{\mu\nu} \\ j^\mu = -c_1 P^{\mu\nu}\partial_\nu \mu - c_2 P^{\mu\nu} F_{\nu\lambda} u^\lambda - c_3 P^{\mu\nu}\partial_\nu T - c_4 \cdot \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \\ \qquad - c_5 \cdot \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} \end{cases}$$

$$\equiv \underbrace{-\sigma T P^{\mu\nu}\left[\partial_\nu \mu - \frac{1}{T}F^{\mu\nu}u_\nu\right] - \chi_T P^{\mu\nu}\partial_\nu T}_{j^\mu_{EM}} - \underset{\text{"vorticity"}}{\xi\, \omega^\mu} - \xi_B B^\mu$$

Following the same procedure above constructing entropy current, we're left with

$$-\frac{\mu}{T} C E^\mu B_\mu + \frac{\mu}{T}\partial_\mu\left(\frac{\mu}{T}\right) - \partial_\mu (s u^\mu) + \frac{u_\mu}{T}\partial_\nu T^{(1)\mu\nu} = \frac{u_\mu}{T} F^{\mu\nu} J^{(1)}_\nu$$

$$\Downarrow$$

$$\partial_\mu\left(s u^\mu - \frac{\mu}{T}J^{(1)\mu}\right) = -J^{(1)\mu}\left[\partial_\mu\left(\frac{\mu}{T}\right) + F_{\mu\nu}\frac{u_\nu}{T}\right] - \frac{1}{T}T^{(1)\mu\nu}\partial_\nu u_\mu - \frac{\mu}{T} C E^\mu B_\mu$$

This time we can no longer take the entropy current $S^\mu = s u^\mu - \frac{\mu}{T}J^{(1)\mu}$ as before, because the anomalous term can overwhelm the other ones and flip the sign of entropy production!

To solve this problem, we have to assign another form of entropy current.

In fact, as an vector, generally we can write

$$S^\mu = s u^\mu - \frac{\mu}{T}J^{(1)\mu} + D_B B^\mu + D\omega^\mu.$$

So the non-negativity of entropy production gives

$$\partial S^\mu = \partial_\mu\left(s u^\mu - \frac{\mu}{T}J^{(1)\mu}\right) + \partial_\mu\left(D_B B^\mu + D\omega^\mu\right)$$

$$= \underbrace{-J^{(1)\mu}_{EM}\left[\partial_\mu\left(\frac{\mu}{T}\right) + F_{\mu\nu}\frac{u^\nu}{T}\right]} - J^{(1)\mu}_{Anomaly}\left[\partial_\mu\left(\frac{\mu}{T}\right) + F_{\mu\nu}\frac{u^\nu}{T}\right] \underbrace{- \frac{1}{T}T^{(1)\mu\nu}\partial_\nu u_\mu}$$

$$-\frac{\mu}{T} C E^\mu B_\mu + \partial_\mu\left(D_B B^\mu + D\omega^\mu\right) \geq 0$$

The blue terms are nothing but those we encountered previously. We require them to be non-negative respectively (s.t. still $\zeta, \sigma, \eta \geq 0$ and $\chi_T \equiv 0$). As for the other terms

$$-J^{(1)\mu}_{Anomaly}\left[\partial_\mu\left(\frac{\mu}{T}\right)+\frac{1}{T}E_\mu\right]-\frac{\mu}{T}CE^\mu B_\mu+\partial_\mu\left(D_B B^\mu+D\omega^\mu\right)\geq 0.$$

$$\Rightarrow \left(\xi\omega^\mu+\xi_B B^\mu\right)\left[\partial_\mu\left(\frac{\mu}{T}\right)+\frac{1}{T}E_\mu\right]-\frac{\mu}{T}CE^\mu B_\mu+\partial_\mu D_B\cdot B^\mu+D_B\partial_\mu B^\mu+\partial_\mu D\cdot\omega^\mu+D\partial_\mu\omega^\mu\geq 0$$

But ideal fluid eq. tells (???)

$$\begin{cases}\partial_\mu\omega^\mu=-\dfrac{2}{\varepsilon+p}\omega^\mu\left(\partial_\mu P-nE_\mu\right)\\ \partial_\mu B^\mu=-2\omega_\mu E^\mu+\dfrac{2}{\varepsilon+p}\left(-B_\mu\partial^\mu P+nE^\mu B_\mu\right)\end{cases}$$

And $\partial_\mu D\equiv\dfrac{\partial D}{\partial p}\partial_\mu P+\dfrac{\partial D}{\partial\left(\frac{\mu}{T}\right)}\partial_\mu\left(\dfrac{\mu}{T}\right)$, (Ditto for D_B)

Then we can rearrange them into

$$\omega^\mu\partial_\mu\left(\frac{\mu}{T}\right)\cdot\left[\xi+\frac{\partial D}{\partial\left(\frac{\mu}{T}\right)}\right]+\omega^\mu E_\mu\left[\frac{\xi}{T}-2D_B+2\frac{nD}{\varepsilon+p}\right]+\omega^\mu\partial_\mu p\left[\frac{\partial D}{\partial p}-2\frac{D}{\varepsilon+p}\right]$$

$$+B^\mu\partial_\mu\left(\frac{\mu}{T}\right)\left[\xi_B+\frac{\partial D_B}{\partial\left(\frac{\mu}{T}\right)}\right]+B^\mu E_\mu\left[\frac{\xi_B}{T}-\frac{\mu}{T}C+2\frac{nD_B}{\varepsilon+p}\right]+B^\mu\partial_\mu P\left[\frac{\partial D_B}{\partial p}-2\frac{D_B}{\varepsilon+p}\right]\geq 0$$

All terms in the square braket must be zero!

This gives, for example,

$$\begin{cases}\dfrac{\partial D}{\partial p}-2\dfrac{D}{\varepsilon+p}=0\\ \xi+\dfrac{\partial D}{\partial\left(\frac{\mu}{T}\right)}=0\end{cases}$$

To solve this, we can use thermodynamic identity $dp=sdT+nd\mu\Rightarrow\left(\dfrac{\partial T}{\partial p}\right)_{\tilde\mu}=\dfrac{T}{\varepsilon+p}$

$$\left(\frac{\partial T}{\partial\tilde\mu}\right)_p=-\frac{nT^2}{\varepsilon+p}$$

arbitrary function

So $D=T^2f(\tilde\mu)$

Similarly $D_B=T^2g(\tilde\mu)$.

But other relation restrict $\begin{cases}g(\tilde\mu)=\dfrac{1}{2}\dfrac{df}{d\tilde\mu}\\ C\tilde\mu=g(\tilde\mu)\end{cases}\Rightarrow\begin{cases}f(\tilde\mu)=\dfrac{C}{3}\tilde\mu^3\\ g(\tilde\mu)=\dfrac{C}{2}\tilde\mu^2\end{cases}$

Finally we find

$$\begin{cases}\xi=C\left(\mu^2-\dfrac{2}{3}\dfrac{n\mu^3}{\varepsilon+p}\right)\\ \xi_B=C\left(\mu^2-\dfrac{1}{2}\dfrac{n\mu^2}{\varepsilon+p}\right)\end{cases}$$

Restriction on the Local Entropy Production.

The second law of thermodynamics requires $\partial_\mu S^\mu \geq 0$ for some entropy current. But to the zeroth order $S^\mu = s u^\mu$, so from the fundamental thermodynamic relation $Ts = \varepsilon + p - \mu n$, we can construct a covariant version of entropy current by multiplying u^μ on both sides,

$$T s u^\mu = \varepsilon u^\mu + p u^\mu - \mu n u^\mu$$

$$\Downarrow$$

$$TS^\mu = -T^{\mu\nu} u_\nu + p u^\mu - \mu J^\mu \quad \text{(Israel Physica 106A (1981) 204)}$$

Using $T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \hat{T}^{\mu\nu}$, $J^\mu = n u^\mu + \hat{J}^\mu$, and $Ts = (\varepsilon + p - \mu n)$, the RHS can also be written equivalently as

$$TS^\mu = -\hat{T}^{\mu\nu} u_\nu - \mu \hat{J}^\mu + T s u^\mu$$

$$= Ts^\mu - \hat{T}^{\mu\nu} u_\nu - \mu \hat{J}^\mu \quad \text{(Bhattacharya et al. JHEP 04 (2011) 125)}$$

Linear Response of Quantum Critical Point

(Hartnoll, Kovtun, Müller, and Sachdev PRB 76, 144502 (2007))

We're considering (2+1)-D quantum critical point of superfluid-insulator phase transition.

Subtracting magnetization (both charge & energy) from the total current, we're left with EOM of ~~transport~~ current still of the same form

$$\begin{cases} \partial_\mu J^\mu = 0 \\ \partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu}. \end{cases} \quad \text{where} \quad F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix} \quad \text{(Magnetohydrodynamics)}$$

We've already shown the constitutive relations for J^μ and $T^{\mu\nu}$,

$$\begin{cases} J^\mu = \rho u^\mu - \sigma T P^{\mu\nu} \left[\partial_\nu \left(\frac{\mu}{T}\right) - \frac{1}{T} F_{\nu\lambda} u^\lambda \right] \\ T^{\mu\nu} = \varepsilon u^\mu u^\nu + p P^{\mu\nu} - \zeta \partial_\lambda u^\lambda - \eta P^{\mu\alpha} P^{\nu\beta} \left[\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta} \partial_\lambda u^\lambda \right] \end{cases}$$

Following Kandanoff & Martin Ann. Phys 24, 419 (1963), let us introduce a perturbative Hamiltonian through

$$\bigstar \quad H \Rightarrow H - \int d^{d+1}x \left(\underbrace{\frac{\delta T}{T}}_{\text{sources}} \overbrace{(\varepsilon - \mu\rho)}^{\text{energy density}} + \underbrace{\delta\mu}_{\text{sources}} \cdot \overbrace{\rho}^{\text{charge density}} + \underbrace{\delta u^\mu}_{\text{sources}} \cdot \overbrace{T_{\mu 0}}^{\text{momentum density}} \right)$$

in the same order of δT and $\delta\mu$

Since $u^\mu u_\mu \equiv -1 \Rightarrow u^\mu \delta u_\mu = 0$, the variation on 3-velocity δu^μ only has TWO independent component $\delta u^\mu = (0, v_x, v_y)$

In static equilibrium (in the proper reference frame), we have

$$J^\mu = \begin{pmatrix} \bar{\rho} \\ 0 \\ 0 \end{pmatrix}, \quad T^{\mu\nu} = \begin{pmatrix} \bar{\varepsilon} & & \\ & \bar{p} & \\ & & \bar{p} \end{pmatrix}, \text{ and } u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \left(\text{and } P^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right).$$

When we switch on perturbations, the thermodynamic variables will also change with δT and $\delta\mu$ in the way of

$$\begin{cases} \delta\rho = \left.\frac{\partial\rho}{\partial\mu}\right|_T \delta\mu + \left.\frac{\partial\rho}{\partial T}\right|_\mu \delta T \\ \delta\varepsilon = \left.\frac{\partial\varepsilon}{\partial\mu}\right|_T \delta\mu + \left.\frac{\partial\varepsilon}{\partial T}\right|_\mu \delta T \\ \delta p = \left.\frac{\partial p}{\partial\mu}\right|_T \delta\mu + \left.\frac{\partial p}{\partial T}\right|_\mu \delta T = \bar{\rho}\,\delta\mu + \bar{s}\,\delta T \quad (dP \equiv s\,dT + \rho\,d\mu) \end{cases}$$

The Linearized EOM for perturbative hydro-variables can be read from charge / stress-energy tensor conservation (at first order gradient expansion and variations)

$$\partial_\mu J^\mu = 0 \Rightarrow \partial_0 \delta J^0 + \partial_i \delta J^i = 0$$

$$\partial_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu \Rightarrow \begin{cases} \partial_0 T^{00} + \partial_i T^{i0} = J_0 F^{00} + J_i F^{i0} \\ \partial_0 T^{0i} + \partial_j T^{ji} = J_0 F^{0i} + J_j F^{ji} \end{cases}$$

where

$$\delta J^0 = \delta\rho\cdot u^0 + \rho \underbrace{\delta u^0}_{0} + \overset{\text{out of equilibrium}}{\delta j^0} = \delta\rho - \sigma T (\delta P^{0\nu}) \left[\partial_\nu\left(\frac{\mu}{T}\right) - \frac{1}{T} F_{\nu\lambda} u^\lambda \right] + \overbrace{\sigma T P^{0\nu} \delta[\quad]_\nu}^{\text{second order}}$$

But to the first order $P^{\mu\nu} \equiv u^\mu u^\nu + g^{\mu\nu} = \begin{pmatrix} 1 & v_x & v_y \\ v_x & v_x^2 & v_x v_y \\ v_y & v_y v_x & v_y^2 \end{pmatrix} + \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & v_x & v_y \\ v_x & 1 & 0 \\ v_y & 0 & 1 \end{pmatrix}$, so $\delta P^{\mu\nu} = \begin{pmatrix} 0 & v_x & v_y \\ v_x & 0 & 0 \\ v_y & 0 & 0 \end{pmatrix}$.

leaving the simple relation $\delta J^0 = \delta\rho$.

For the spatial components,

$$\delta J^i = \delta\rho\cdot \underbrace{u^i}_{0} + \rho\,\delta u^i + \delta j^i = \rho v^i + \delta j^i$$

Writing $j^\mu \equiv -\sigma T P^{\mu\nu} \left[\partial_\nu\left(\frac{\mu}{T}\right) + \frac{1}{T} F_{\nu\lambda} u^\lambda \right]$

$$= -\sigma P^{\mu\nu} \left[\partial_\nu \mu - \frac{\mu}{T} \partial_\nu T + F_{\nu\lambda} u^\lambda \right]$$

Then the non-vanishing component of δj^i is

$$\delta j^i = -\sigma (\delta P^{i\nu}) \underbrace{[\cdots]_\nu}_{\substack{\text{zero because of uniform} \\ T,\ \mu \text{ and } u^\mu}} - \sigma P^{i\nu} \delta[\cdots]_\nu = -\sigma P^{ii} \delta[\cdots]_i$$

$$= -\sigma \left[\delta(\partial_i \mu) - \underbrace{\delta\left(\frac{\mu}{T} \partial_i T\right)}_{} + F_{i\lambda} \delta(u^\lambda) \right]$$

$$\frac{\mu+\delta\mu}{T+\delta T} \partial_i (T+\delta T) = \frac{\mu+\delta\mu}{T}\left(1 - \frac{\delta T}{T}\right) \partial_i \delta T \simeq \frac{\mu}{T} \partial_i \delta T + \text{higher orders}$$

$$\Rightarrow \delta j^i = -\sigma \left[\partial_i(\delta\mu) - \frac{\mu}{T} \partial_i(\delta T) + \vec{B} \times \vec{v} \right]$$

~~Nothing but~~ charge current J (Because we're working in Landau frame, where equilibrium charge current is always ~~zero~~)

So charge conservation $\partial_t(\delta\rho) + \partial_i(\delta J^i) = 0$ gives

$$\partial_t(\delta\rho) + \vec{\nabla}\cdot\left\{ \rho\vec{v} + \sigma\left[-\vec{\nabla}(\delta\mu) + \frac{\mu}{T}\vec{\nabla}(\delta T) + \vec{v}\times\vec{B} \right] \right\} = 0 \quad \text{----} \ 1$$

Similarly,

$$\delta T^{00} = \delta(\varepsilon u^0 u^0) + \underbrace{\delta(p P^{00}) - \zeta\,\delta(P^{00} \partial_\lambda u^\lambda) - \eta\,\delta(\sigma^{\mu\nu})}_{\text{second order}}$$

$$= \delta\varepsilon \cdot u^0 u^0 = \delta\varepsilon.$$

Recall that

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P^{\mu\nu} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \text{ and } \delta P^{\mu\nu} = \begin{pmatrix} 0 & v_x & v_y \\ v_x & 0 & 0 \\ v_y & 0 & 0 \end{pmatrix}$$

$$\delta T^{0i} = \delta(\varepsilon u^0 u^i) + \delta(p P^{0i}) - \zeta\,\delta(P^{0i}\partial_\lambda u^\lambda) - \eta\,\delta(\sigma^{0i})$$

$$= \varepsilon u^0 \delta u^i + p\,\delta(P^{0i}) + \text{higher orders}$$

$$= (\varepsilon + p) v^i \leftarrow \text{Energy Current } J^E$$

(So Heat Current $\vec{Q} \equiv \vec{J}^E - \mu\vec{J} = (\varepsilon+p)\vec{v} - \rho\mu\vec{v} - \rho\vec{j} = sT\vec{v} - \rho\vec{j}$)

Thus $\nabla_\mu T^{\mu 0} = F^{\mu 0} J_\mu \Rightarrow \partial_t(\delta T^{00}) + \partial_i(\delta T^{i0}) = E^i \delta J_i = 0$

$$\partial_t(\delta\varepsilon) + \vec{\nabla}\cdot\left((\varepsilon+p)\vec{v}\right) = 0 \quad \text{----} \; (2)$$

And $\delta T^{ij} = \underbrace{\delta(\varepsilon u^i u^j)}_{0} + \delta(p \underbrace{P^{ij}}_{\delta_{ij}}) - \zeta\,\delta(P^{ij}\partial_\lambda u^\lambda) - \eta\,\delta(\sigma^{ij})$

$$= \delta p \cdot \delta^{ij} - \zeta\,\delta^{ij}\partial_\lambda(\delta u^\lambda) - \eta\,\delta\left[P^{i\alpha}P^{j\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta}\partial_\lambda u^\lambda\right)\right]$$

$$= \delta p\cdot\delta^{ij} - \zeta\,\delta^{ij}\overbrace{\partial_k v^k}^{\text{Einstein Summation}} - \eta\cdot 2\,\delta(P^{i0})P^{jj}\underbrace{(\cdots)_{0j}}_{\text{always zero}} - \eta P^{ii}P^{jj}\delta(\cdots)_{ij}$$

the order matters!!!

$$= \delta p\cdot\delta^{ij} - \zeta\,\delta^{ij}\partial_k v^k - \eta\left(\partial^i v^j + \partial^j v^i - \delta^{ij}\partial_k v^k\right) \qquad (d=2)$$

Thus $\nabla_\mu T^{\mu i} = F^{i\mu}J_\mu \Rightarrow \partial_t(\delta T^{0i}) + \partial_j(\delta T^{ji}) = \hat{y}F^{21}J_x + \hat{x}F^{12}J_y = \hat{x}BJ_y - \hat{y}BJ_x = \delta\vec{J}\times\vec{B}$

$$\partial_t\left((\varepsilon+p)\vec{v}\right) + \vec{\nabla}p - \zeta\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta\left[\cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})} + \nabla^2\vec{v} - \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}\right] = \delta\vec{J}\times\vec{B}$$

$$\partial_t\left((\varepsilon+p)\vec{v}\right) + \vec{\nabla}p - \zeta\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta\nabla^2\vec{v} - \delta\vec{J}\times\vec{B} = 0 \quad \text{----} \; (3)$$

For simplicity let us switch on only magnetic field $F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix}$ (Magnetohydrodynamics).

The prescription we switch on the perturbation tells us it is $\varepsilon(r,t) - \mu n(r,t)$ that plays the role of conjugate field of $\frac{\delta T}{T}$. Thus we can re-written the energy conservation EoM as

$$\partial_t\varepsilon + \vec{\nabla}\cdot\vec{J}^E = 0 \Rightarrow \partial_t(\varepsilon - \mu\rho) + \vec{\nabla}\cdot\underbrace{(\vec{J}^E - \mu\vec{J})}_{\vec{Q}} = 0$$

Given the perturbation sources as $\{\delta\mu, \frac{\delta T}{T}, \delta u^\mu\}$, standard linear response theory tells, an operator (of hydrovariables) $A[n, \rho, T^{0i}]$ can be calculated from

$$A(\vec{k},\omega) = \frac{G_{A,\varepsilon-\mu\rho}(\vec{k},\omega) - G_{A,\varepsilon-\mu\rho}(\vec{k},0)}{i\omega}\,\frac{\delta T(\vec{k},0)}{T}$$

$$+ \frac{G_{A,\rho}(\vec{k},\omega) - G_{A,\rho}(\vec{k},0)}{i\omega}\,\delta\mu(\vec{k},0)$$

$$+ \frac{G_{A,T^{0i}}(\vec{k},\omega) - G_{A,T^{0i}}(\vec{k},0)}{i\omega}\,v_i$$

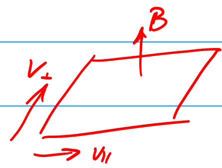
Retarded Green function for conserved density

In fourier space, the hydrodynamic equation reads

$$\begin{cases} \partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0 \\ \partial_t \varepsilon + \vec{\nabla}\cdot\vec{J}^E = 0 \\ \partial_t v_i + \partial_i \pi^{ji} = 0 \end{cases} \Rightarrow \begin{cases} \partial_t \delta\rho + \vec{\nabla}\cdot\left[\rho\vec{v} + \sigma\left(-\vec{\nabla}\mu + \frac{\mu}{T}\vec{\nabla}T + \vec{v}\times\vec{B}\right)\right] = 0 \\ \partial_t \delta\varepsilon + \vec{\nabla}\cdot\left((\varepsilon+p)\vec{v}\right) = 0 \\ \partial_t\left((\varepsilon+p)\vec{v}\right) + \vec{\nabla}p - \zeta\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta^2\vec{\nabla}^2\vec{v} + \left[\rho\vec{v} + \sigma_Q\left(-\vec{\nabla}\mu + \frac{\mu}{T}\vec{\nabla}T + \vec{v}\times\vec{B}\right)\right]\times\vec{B} = 0 \end{cases}$$

Switching to variable $\{\delta\mu, \delta T, v\}$, we have

$$\begin{cases} \left(\frac{\partial\bar{\rho}}{\partial\bar{\mu}}\Big|_T \partial_t\delta\mu + \frac{\partial\bar{\rho}}{\partial\bar{T}}\Big|_\mu \partial_t\delta T\right) + \vec{\nabla}\cdot\left[\rho\vec{v} + \sigma_Q\left(-\vec{\nabla}\mu + \frac{\mu}{T}\vec{\nabla}T + \vec{v}\times\vec{B}\right)\right] = 0 \\ \left(\frac{\partial\bar{\varepsilon}}{\partial\bar{\mu}}\Big|_T \partial_t\delta\mu + \frac{\partial\bar{\varepsilon}}{\partial\bar{T}}\Big|_\mu \partial_t\delta T\right) + \vec{\nabla}\cdot\left((\varepsilon+p)\vec{v}\right) = 0. \\ \partial_t\left((\varepsilon+p)\vec{v}\right) + \underline{\vec{\nabla}p} - \underbrace{\zeta\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta\vec{\nabla}^2\vec{v}}_{\text{higher orders}} - \left[\rho\vec{v} + \sigma_Q\left(-\vec{\nabla}\mu + \frac{\mu}{T}\vec{\nabla}T + \vec{v}\times\vec{B}\right)\right]\times\vec{B} = 0 \end{cases}$$



In momentum space $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, splitting $\vec{v} \equiv \underbrace{\frac{k_i k_j}{k^2} v^j}_{v_\perp} + \underbrace{\left(\delta_{ij} - \frac{k_i k_j}{k^2} v^j\right)}_{v_\parallel}$, we have four independent EOM

$$\begin{cases} -i\omega\left(\frac{\partial\bar{\rho}}{\partial\bar{\mu}}\Big|_T \delta\mu + \frac{\partial\bar{\rho}}{\partial\bar{T}}\Big|_\mu \delta T\right) + ik\left(\rho v_\parallel + \sigma_Q B v_\perp\right) + k^2\sigma_Q\left(\delta\mu - \frac{\mu}{T}\delta T\right) = 0 \\ -i\omega\left(\frac{\partial\bar{\varepsilon}}{\partial\bar{\mu}}\Big|_T \delta\mu + \frac{\partial\bar{\varepsilon}}{\partial\bar{T}}\Big|_\mu \delta T\right) + ik(\varepsilon+p)v_\parallel = 0 \\ -i\omega(\varepsilon+p)v_\parallel + ik(\rho\delta\mu + s\delta T) + k^2(\zeta+\eta)v_\parallel - \rho B v_\perp + \sigma_Q B^2 v_\parallel = 0 \\ -i\omega(\varepsilon+p)v_\perp + \eta k^2 v_\perp + \rho B v_\parallel - ik\sigma_Q B\left(\delta\mu - \frac{\mu}{T}\delta T\right) + \sigma_Q B^2 v_\perp = 0 \end{cases}$$

To ensure the existence of non-zero solution of $\{\delta\mu, \delta T, v_\parallel, v_\perp\}$, we demand that

$$\det \mathcal{M} \equiv \det\begin{pmatrix} -i\omega\frac{\partial\bar{\rho}}{\partial\bar{\mu}}\Big|_T + k^2\sigma_Q & -i\omega\frac{\partial\bar{\rho}}{\partial\bar{T}}\Big|_\mu - k^2\sigma_Q\frac{\mu}{T} & ik\rho & ik\sigma_Q B \\ -i\omega\frac{\partial\bar{\varepsilon}}{\partial\bar{\mu}}\Big|_T & -i\omega\frac{\partial\bar{\varepsilon}}{\partial\bar{T}}\Big|_\mu & ik(\varepsilon+p) & 0 \\ ik\rho & iks & -i\omega(\varepsilon+p) + \sigma_Q B^2 + k^2(\zeta+\eta) & -\rho B \\ -ik\sigma_Q B & ik\sigma_Q B\frac{\mu}{T} & \rho B & -i\omega(\varepsilon+p) + \sigma_Q B^2 + \eta k^2 \end{pmatrix} = 0$$

In the long wavelength limit $k \to 0$, the above eq. simplifies drastically as

$$\det\begin{pmatrix} -i\omega \left.\frac{\partial\rho}{\partial\mu}\right|_T & -i\omega\left.\frac{\partial\rho}{\partial T}\right|_\mu & 0 & 0 \\ -i\omega\left.\frac{\partial\varepsilon}{\partial\mu}\right|_T & -i\omega\left.\frac{\partial\varepsilon}{\partial T}\right|_\mu & 0 & 0 \\ 0 & 0 & -i\omega(\varepsilon+p)+\sigma_Q B^2 & -\rho B \\ 0 & 0 & \rho B & -i\omega(\varepsilon+p)+\sigma_Q B^2 \end{pmatrix}$$

$$= -\omega^2 \cdot \det\begin{pmatrix} \left.\frac{\partial\rho}{\partial\mu}\right|_T & \left.\frac{\partial\rho}{\partial T}\right|_\mu \\ \left.\frac{\partial\varepsilon}{\partial\mu}\right|_T & \left.\frac{\partial\varepsilon}{\partial T}\right|_\mu \end{pmatrix} \cdot \det\begin{pmatrix} -i\omega(\varepsilon+p)+\sigma_Q B^2 & -\rho B \\ \rho B & -i\omega(\varepsilon+p)+\sigma_Q B^2 \end{pmatrix} = 0$$

$$\Leftarrow \quad i\omega(\varepsilon+p) - \sigma_Q B^2 = \pm i\rho B, \quad \text{or}$$

$$\omega = \pm\omega_c - i\gamma, \text{ with } \quad \omega_c \equiv \frac{\rho B}{\varepsilon+p}, \quad \gamma = \frac{\sigma_Q B^2}{\varepsilon+p}$$

The initial value $\{\delta\mu^0, \delta T^0, v_\parallel^0, v_\perp^0\}$ problem can thus be solved by Laplacian T.F.

Clearly in the long wavelength limit, the energy/charge conservation tells

$$\delta\mu = \frac{i}{\omega}\delta\mu^0, \quad \delta T = \frac{i}{\omega}\delta T^0.$$

while momentum conservation tells (keeping the first order fluctuations, and recall that $v^0 \equiv 0$)

$$\begin{cases} -i\omega(\varepsilon+p)v_\parallel + (\rho\nabla\mu + s\nabla T) - B\rho v_\perp + B^2\sigma_Q v_\parallel = 0 \\ -i\omega(\varepsilon+p)v_\perp + B\rho v_\parallel - B\sigma_Q\left(\nabla\mu - \frac{\mu}{T}\nabla T\right) + B^2\sigma_Q v_\perp = 0 \end{cases}$$

$$\Rightarrow \quad v_\parallel = \frac{-B^2\sigma_Q(sT+\mu\rho)\frac{\nabla T}{T} + is(\varepsilon+p)\omega\nabla T + i(\varepsilon+p)\rho\omega\nabla\mu}{B^2\rho^2 + \left(B^2\sigma_Q - i(p+\varepsilon)\omega\right)^2}$$

(Recall that $\varepsilon+p = sT+\mu\rho$)

$$= \frac{-\frac{B^2\sigma_Q}{\varepsilon+p}\frac{\nabla T}{T} + i\frac{\omega s}{\varepsilon+p}\nabla T - i\frac{\omega\rho}{\varepsilon+p}\nabla\mu}{\left(\frac{B\rho}{\varepsilon+p}\right)^2 - \left(i\frac{B^2\sigma_Q}{\varepsilon+p}+\omega\right)^2} = \frac{\frac{B^2\sigma_Q}{\varepsilon+p}\frac{\nabla T}{T} - i\frac{\omega sT}{\varepsilon+p}\frac{\nabla T}{T} + i\frac{\omega\rho}{\varepsilon+p}\nabla\mu}{(\omega+i\gamma)^2 - \omega_c^2}$$

$$= \frac{\gamma\frac{\nabla T}{T} - i\omega\frac{Ts}{\varepsilon+p}\frac{\nabla T}{T} + i\frac{\omega\omega_c}{B}\nabla\mu}{(\omega+i\gamma)^2 - \omega_c^2}$$

$$v_\perp = \frac{BsT\rho\nabla T + B\mu\sigma_Q\left(-B^2\sigma_Q + i(\varepsilon+p)\omega\right)\nabla T + T\rho^2\nabla\mu + T\sigma_Q\left(B^2\sigma_Q - i(\varepsilon+p)\omega\right)\nabla\mu}{T\left(B^2\sigma_Q - i(\varepsilon+p)\omega\right)^2 + TB^2\rho^2}$$

$$= \frac{\frac{Bs\rho}{(\varepsilon+p)^2}\nabla T + \frac{\mu}{T}\frac{B\sigma_Q}{\varepsilon+p}\left(-\frac{B^2\sigma_Q}{\varepsilon+p} + i\omega\right)\nabla T + \frac{\rho^2}{(\varepsilon+p)^2}\nabla\mu + \frac{\sigma_Q}{\varepsilon+p}\left(\frac{\sigma_Q B^2}{\varepsilon+p} - i\omega\right)\nabla\mu}{-\left(i\frac{B^2\sigma_Q}{\varepsilon+p} + \omega\right)^2 + \left(\frac{B\rho}{\varepsilon+p}\right)^2}$$

$$= \frac{-\frac{Bs\rho}{(\varepsilon+p)^2}\nabla T - \frac{\mu}{T}\frac{B\sigma_Q}{\varepsilon+p}\left(-\frac{B^2\sigma_Q}{\varepsilon+p}+i\omega\right)\nabla T - \frac{B\rho^2}{(\varepsilon+p)^2}\nabla\mu + \frac{B\sigma_Q}{\varepsilon+p}\left(\frac{\sigma_Q B^2}{\varepsilon+p}-i\omega\right)\nabla\mu}{(\omega+i\gamma)^2-\omega_c^2}$$

$$= \frac{-\frac{\omega_c^2}{B\rho}sT\frac{\nabla T}{T} - \frac{\mu}{B}\gamma(i\omega-\gamma)\frac{\nabla T}{T} - \frac{\omega_c^2}{B}\nabla\mu + \frac{1}{B}\gamma(i\omega-\gamma)\nabla\mu}{(\omega+i\gamma)^2-\omega_c^2}$$

Thus for the charge current,

$$\vec{J} = \rho\vec{v} + \sigma_Q\left(-\vec{\nabla}\mu + \vec{v}\times\vec{B} + \mu\frac{\vec{\nabla}T}{T}\right)$$

2° The longitudinal parts of charge current are

$$J_x = \rho v_\parallel - \sigma_Q\partial_x\mu + \sigma_Q B v_\perp + \sigma_Q\mu\frac{\partial_x T}{T}$$

$$= -\partial_x\mu\left\{\frac{-i\rho\frac{\omega\omega_c}{B}}{(\omega+i\gamma)^2-\omega_c^2} + \sigma_Q - \sigma_Q B\cdot\frac{-\frac{\omega_c^2}{B} + \frac{B\sigma_Q}{\varepsilon+p}\left(\frac{\sigma_Q B^2}{\varepsilon+p}-i\omega\right)}{(\omega+i\gamma)^2-\omega_c^2}\right\}$$

$$-\frac{\partial_x T}{T}\left\{\frac{-\rho\gamma + i\omega\frac{Ts\rho}{\varepsilon+p} + \sigma_Q B\left[\frac{\omega_c^2}{B\rho}sT + \frac{\mu}{B}\gamma(i\omega-\gamma)\right]}{(\omega+i\gamma)^2-\omega_c^2} - \sigma_Q\mu\right\}$$

For the part proportional to $-\nabla\mu$, we have

$$J_x = -\partial_x\mu\left\{\sigma_Q\omega\left[i\frac{\rho\omega_c}{B\sigma_Q} + \frac{1}{\omega}(\omega^2+2i\gamma\omega-\gamma^2-\omega_c^2) + \frac{\omega_c^2}{\omega} + \frac{\gamma^2}{\omega} - i\gamma\right]\Big/\left((\omega+i\gamma)^2-\omega_c^2\right)\right\}$$

$$= -\partial_x\mu\left\{\sigma_Q\omega\left[i\frac{\frac{B\rho}{(\varepsilon+p)}}{\frac{B^2\sigma_Q}{\varepsilon+p}}\omega_c + \left(\omega+2i\gamma-\cancel{\frac{\gamma^2}{\omega}}-\cancel{\frac{\omega_c^2}{\omega}}\right) + \cancel{\frac{\omega_c^2}{\omega}} + \cancel{\frac{\gamma^2}{\omega}} - i\gamma\right]\Big/\left((\omega+i\gamma)^2-\omega_c^2\right)\right\}$$

$$= -\partial_x\mu\left\{\sigma_Q\cdot\frac{\omega(\omega+i\gamma)}{(\omega+i\gamma)^2-\omega_c^2}\right\}$$

$$\Rightarrow \sigma_{xx} = \sigma_Q\frac{\omega(\omega+i\gamma)}{(\omega+i\gamma)^2-\omega_c^2}$$

while for the part proportional to $-\frac{\nabla T}{T}$, we have

$$J_x = -\frac{\partial_x T}{T}\left\{-\rho\gamma + i\omega\frac{Ts}{B}\frac{\rho B}{\varepsilon+p} + \underbrace{\sigma_Q\cdot\frac{1}{\rho}\frac{B^2\rho^2}{(\varepsilon+p)^2}}_{\frac{B^2\sigma_Q}{\varepsilon+p}\frac{B\rho}{\varepsilon+p}\frac{sT}{B}}sT + \sigma_Q\mu\gamma(i\omega-\gamma) - \sigma_Q\mu\left((\omega+i\gamma)^2-\omega_c^2\right)\right\}\Big/\left[(\omega+i\gamma)^2-\omega_c^2\right]$$

$$= -\frac{\partial_x T}{T}\left\{-\rho\gamma + i\omega\frac{sT}{B}\omega_c + \gamma\omega_c\frac{sT}{B} + \sigma_Q\mu\left(i\omega\gamma-\cancel{\gamma^2}-\omega^2-2i\omega\gamma+\cancel{\gamma^2}-\omega_c^2\right)\right\}\Big/\left[(\omega+i\gamma)^2-\omega_c^2\right]$$

$$= -\frac{\partial_x T}{T}\left\{ -\rho\gamma + \frac{sT}{B}\left(i\omega\omega_c + \gamma\omega_c\right) - \sigma_Q\mu\left(\omega^2 - i\omega\gamma - \omega_c^2\right)\right\} \Big/ \left[(\omega+i\gamma)^2 - \omega_c^2\right]$$

$$= -\frac{\partial_x T}{T}\left\{ \frac{\left(-\rho + \frac{sT}{\cancel{B}}\cdot\frac{\cancel{B}\rho}{\varepsilon+p}\right)\gamma + \frac{sT}{B}i\omega\omega_c - \sigma_Q\mu\,\omega(\omega-i\gamma) + \sigma_Q\mu\cdot\frac{B^2\rho^2}{(\varepsilon+p)^2}}{(\omega+i\gamma)^2-\omega_c^2}\right\}$$

$$= -\frac{\partial_x T}{T}\left\{ \frac{\left(-(\varepsilon+p)+sT\right)\rho}{\varepsilon+p}\gamma + \frac{sT}{B}i\omega\left(\omega_c + \frac{\sigma_Q B\mu}{sT}i(\omega-i\gamma)\right) + \frac{\sigma_Q B^2}{\varepsilon+p}\cdot\frac{\mu\rho^2}{\varepsilon+p}\right\} \Big/ \left[(\omega+i\gamma)^2-\omega_c^2\right]$$

$$= -\frac{\partial_x T}{T}\left\{ \cancel{\frac{-\mu\rho\cdot\rho}{\varepsilon+p}\gamma} + \frac{sT}{B}i\omega\omega_c\left[1 + \frac{i}{\omega_c}\cdot\frac{\frac{\sigma_Q B^2}{\varepsilon+p}}{\frac{\rho B}{\varepsilon+p}}\cdot\frac{\rho\mu}{sT}(\omega-i\gamma)\right] + \cancel{\gamma\frac{\mu\rho^2}{\varepsilon+p}}\right\} \Big/ \left[(\omega+i\gamma)^2-\omega_c^2\right]$$

$$= -\partial_x T\cdot \alpha_{xx}(\omega),$$

where

$$\alpha_{xx}(\omega) = \frac{s}{B}\,\frac{i\omega\cdot\omega_c\left[1 + i\frac{\gamma\rho\mu}{\omega_c^2 sT}(\omega-i\gamma)\right]}{(\omega+i\gamma)^2-\omega_c^2}.$$

2° The transverse part of charge current are

$$J_x = \rho v_\perp - \sigma_Q B v_\parallel$$

$$= -\partial_y\mu\left\{\left(\frac{\rho}{B}\omega_c^2 - \frac{\rho}{B}\gamma(i\omega-\gamma) + \sigma_Q\, i\,\omega\omega_c\right)\Big/\left[(\omega+i\gamma)^2-\omega_c^2\right]\right\}$$

$$\quad -\frac{\partial_y T}{T}\left\{\left[\frac{\omega_c^2}{B}sT + \frac{\rho\mu}{B}\gamma(i\omega-\gamma) + \sigma_Q B\cdot\left(\gamma - i\omega\frac{sT}{\varepsilon+p}\right)\right]\Big/\left[(\omega+i\gamma)^2-\omega_c^2\right]\right\}$$

$$= -\partial_y\mu\left\{\frac{\rho}{B}\left[\omega_c^2 - i\omega\gamma + \gamma^2 + i\sigma_Q\omega\cdot\frac{B}{\cancel{\rho}}\,\frac{\cancel{\rho}B}{\varepsilon+p}\right]\Big/\left[(\omega+i\gamma)^2-\omega_c^2\right]\right\}$$

$$\quad -\frac{\partial_y T}{T}\left\{\frac{\frac{1}{B}\left[\omega_c^2 sT + \rho\mu i\omega\gamma - \rho\mu\gamma^2 + (\varepsilon+p)\cdot\frac{\sigma_Q B^2}{\varepsilon+p}\gamma - i\omega\cdot\frac{\sigma_Q B^2}{\varepsilon+p}sT\right]}{(\omega+i\gamma)^2-\omega_c^2}\right\}$$

$$= -\partial_y\mu\cdot\left\{\frac{\rho}{B}\left(\gamma^2+\omega_c^2 - 2i\omega\gamma\right)\Big/\left((\omega+i\gamma)^2-\omega_c^2\right)\right\}$$

$$\quad -\frac{\partial_y T}{T}\left\{\frac{1}{B}\left[\omega_c^2 sT + \rho\mu i\omega\gamma + sT\gamma^2 - i\omega\gamma sT\right]\Big/\left((\omega+i\gamma)^2-\omega_c^2\right)\right\}$$

$$= \sigma_{xy} \cdot (-\partial_y \mu) + \alpha_{xy}(-\partial_y T)$$

where

$$\sigma_{xy}(\omega) = \frac{\rho}{B} \cdot \frac{\gamma^2 + \omega_c^2 - 2i\omega\gamma}{(\omega+i\gamma)^2 - \omega_c^2}, \quad \alpha_{xy}(\omega) = \frac{s}{B} \cdot \frac{\gamma^2+\omega_c^2 - i\omega\gamma(1-\rho\mu/sT)}{(\omega+i\gamma)^2-\omega_c^2}$$

In the same sense, substituting $\vec{v}$ into heat current $\vec{J}^Q = \vec{J} - \mu \vec{E}$ we can also obtain $\bar{\alpha}_{xx}$, $\bar{\alpha}_{xy}$, $\bar{\kappa}_{xx}$, and $\bar{\kappa}_{xy}$ s.t.

$$\bar{\alpha}_{ij} \equiv T\alpha_{ij}$$

and

$$\bar{\kappa}_{xx} = -\frac{(\varepsilon+p)^2}{TB\rho} \cdot \frac{\omega_c\gamma}{(\omega+i\gamma)^2-\omega_c^2} \cdot \left\{ 1 - i\omega\frac{s^2T^2\omega_c^2 + \gamma^2\mu^2\rho^2}{\gamma\omega_c^2(\varepsilon+p)^2} - \omega^2\frac{\mu^2\rho^2}{\omega_c^2(\varepsilon+p)^2} \right\}$$

$$\bar{\kappa}_{xy} = -\left(\frac{Ts^2}{B\rho}\right) \frac{\omega_c^2}{(\omega+i\gamma)^2-\omega_c^2} \cdot \left\{ 1 + \frac{2\sigma_Q\mu B(\omega-i\gamma)}{Ts\,\omega_c} - \left(\frac{\mu\sigma_Q B}{Ts}\right)^2 \right\}$$

Self-duality

In the region where both B and ρ are small (precisely, $B \ll T^2$, $\rho \ll T^2$), then fundamental thermodynamic relation can be written as $Ts \simeq \varepsilon + p$, and we will observe a self-duality under the exchange $\rho \Leftrightarrow B$ and $\sigma_Q \Leftrightarrow 1/\sigma_Q$ so that

$$\sigma_{xx},\ \sigma_{xy},\ \alpha_{xx},\ \alpha_{xy},\ \bar{\kappa}_{xx},\ \bar{\kappa}_{xy}$$

$$\updownarrow$$

$$\rho_{xx},\ -\rho_{xy},\ -\vartheta_{xy},\ -\vartheta_{xx},\ \kappa_{xx},\ -\kappa_{xy}$$

This amazing relation is explained as particle-vortex duality in Herzog, Kovtun, Sachdev & Son PRD 75, 085020 (2007).