

Kondo Effect and Kondo Problem

Schrieffer-Wolff Transformation, Spin Scattering, and Asymptotic Freedom

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Anderson Model

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Resistance Minimum

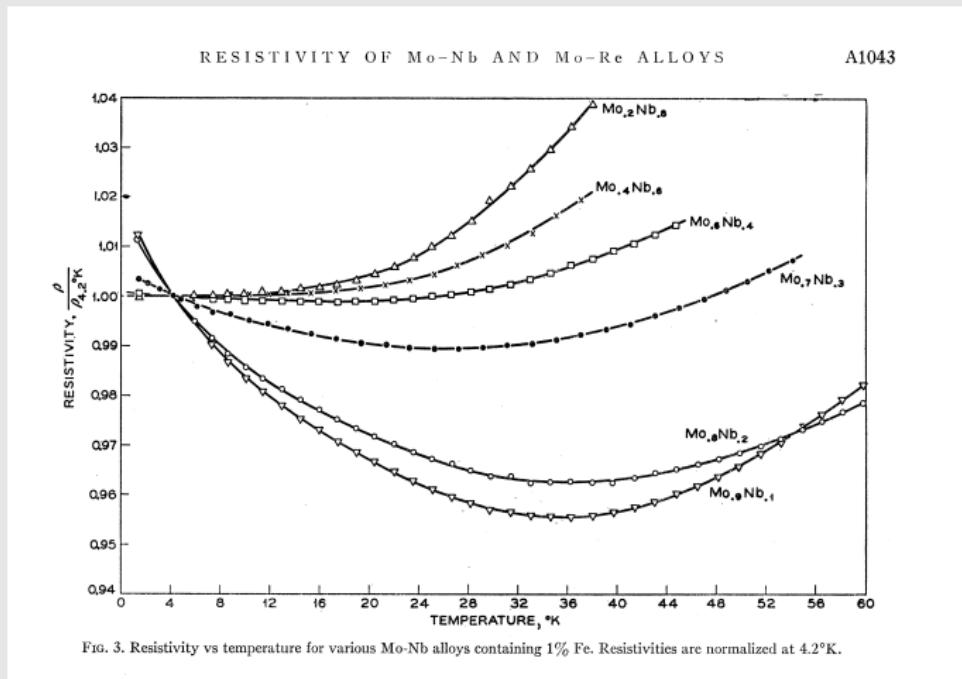


Figure: Resistance minimum for 1% Fe in a series of Mo-Nb alloys (Sarachik et al. Phys. Rev. 135, 1041(1964))

How to Understand This?

- ✳ We know that for usual impurities like defects, conductivity (relaxation time) is not temperature-dependent.
- ✳ Even if we include the contribution of electron-phonon scattering, Boltzmann equation still tells us that resistivity *drops* in the manner of $\rho \sim \frac{1}{T} \propto T^5$ at low temperature region.
- ✳ It happens only for **magnetic impurities** (Here is Fe) \implies There must be some microscopic mechanism that magnetism start to play a role at low temperature.

Anderson Model (Anderson *Phys. Rev.* **124**, 41(1961))

$$H = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{\sigma} \varepsilon_d d_{\sigma}^\dagger d_{\sigma} + \textcolor{red}{Un_{d\uparrow}n_{d\downarrow}} + \sum_{k\sigma} V_k (c_{k\sigma}^\dagger d_{\sigma} + h.c.)$$

Strong Correlation of impurities is necessary!

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Strong Correlation of impurities is necessary!

Mean-Field Solution (Hartree-Fock)

MF treatment of Hubbard term

$$H_d = \sum_{\sigma} \varepsilon_d d_{\sigma}^{\dagger} d_{\sigma} + U(\langle n_{d\downarrow} \rangle n_{d\uparrow} + \langle n_{d\uparrow} \rangle n_{d\downarrow} - \langle n_{d\uparrow} \rangle \langle n_{d\downarrow} \rangle)$$

Treating both $V_k \equiv V$ and the host DOS $\rho(\varepsilon) = \rho_0$ as constants, and denoting $\Delta \equiv \pi V^2 \rho(\varepsilon)$, then after integrating out the itinerate electrons we obtain the Green function for impurities

$$G_{d\sigma}(\omega) = \frac{1}{\omega - E_{d\sigma} + i\Delta}, \quad \text{where } E_{d\sigma} = \varepsilon_d + \rho_0 |V|^2 \ln \left| \frac{\varepsilon_d - D}{\varepsilon_d + D} \right| + U \langle n_{d\bar{\sigma}} \rangle$$

So DOS and the occupation number are

$$\rho_{d\sigma}(\varepsilon) \equiv -\frac{1}{\pi} \text{Im} G_{d\sigma} = \frac{1}{\pi} \frac{\Delta}{(\varepsilon - E_{d\sigma})^2 + \Delta^2},$$

$$\langle n_{d\sigma} \rangle \equiv \int_{-\infty}^{\mu} d\varepsilon n_F(\varepsilon) \rho_{d\sigma}(\varepsilon) = \frac{1}{2} - \arctan \frac{E_{d\sigma} - \mu}{\Delta},$$

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Success: Magnetic Impurities

Anderson obtain a $T = 0$ phase diagram for proof of local moments for impurities

$$\langle S_z \rangle \sim \langle n_{d\uparrow} - n_{d\downarrow} \rangle \simeq \frac{1}{\pi} \arctan \frac{2\langle S_z \rangle U}{\Delta}$$

Only when $\frac{U}{\pi\Delta} > 1$ will the moment exist.

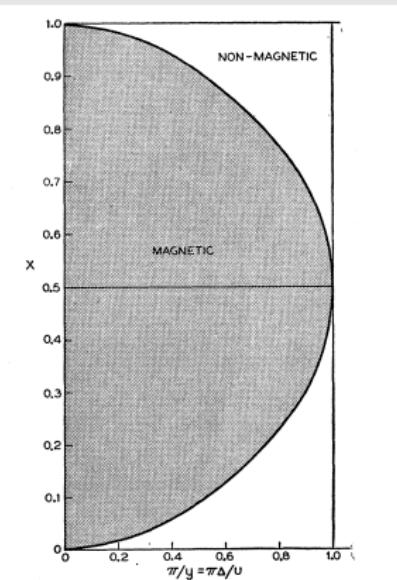


FIG. 4. Regions of magnetic and nonmagnetic behavior.
Curve gives x_c vs $\pi/\gamma_c = \pi\Delta/U$.

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Magnetic Phase

Impurities are magnetic only at large **U limit** with small DOS of itinerant electrons.

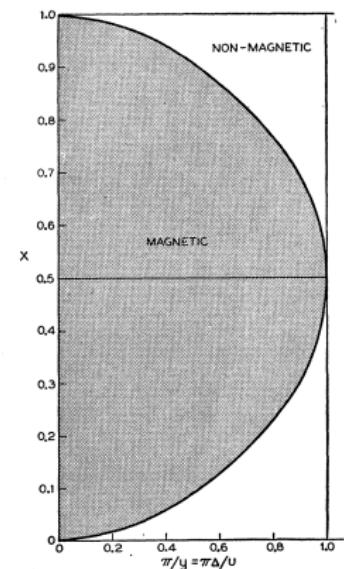


FIG. 4. Regions of magnetic and nonmagnetic behavior.
Curve gives x_e vs $\pi/\gamma_e = \pi\Delta/U$.

Figure: $x \equiv \varepsilon_d/U$

Flaws of Mean-Field Treatment

- ① We obtain a phase transition even at a finite system.
 - Since $\langle S_z \rangle \neq 0$, this is a usual symmetry-breaking phase transition (rather than quantum phase transition characterized by the long-range behavior of Green functions). However, by definition, in Ginzberg-Landau theory the order parameter becomes nonzero only when we go to the thermodynamics limit.
- ② The correlation time scale is much shorter than the lifetime of Impurity State
 - Correlation time scale is proportional to $1/U$ while from the imaginary part of Green function the lifetime of this state is proportional to $1/\Delta$. Existence of magnetic moment requires $U \gg \Delta$, so $1/U \ll 1/\Delta$ is again unacceptable.
- ③ The state that minimizes the Hartree-Fock energy is NOT the real ground state.
 - In strongly-correlated systems like Heisenberg model for antiferromagnetic case, the quantum fluctuation is extremely important that single particle picture will totally breakdown!

Low Energy Effective Hamiltonian

Let us focus on single impurity.

- ✳ Large U limit and half-filling condition highly suppressed the mobility of impurity electrons.
- ✳ So the charge degree of freedom is quenched while **superexchange mechanism** (Anderson *Phys. Rev.* **79**, 350 (1950)) still works for spin degree of freedom. Therefore, the low-energy effective Hamiltonian must be spin Hamiltonian for impurity electrons.

To realize the above physical confinement, we should separate the entire many-body Hilbert space as orthogonal subspaces with doubly occupied $|\psi_2\rangle$, singly occupied $|\psi_1\rangle$, or empty states $|\psi_0\rangle$ and project out the unphysical first and last one.

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Schrieffer-Wolff Transformation

(Schrieffer&Wolff *Phys. Rev.* **149**, 491 (1966))

With projection operator $P_2 = n_{d\uparrow}n_{d\downarrow}$, $P_1 = n_{d\uparrow}(1 - n_{d\downarrow}) + n_{d\downarrow}(1 - n_{d\uparrow})$, and $P_0 = \mathbb{1} - P_1 - P_2$ we have explicitly

$$H_{00} \equiv P_0 H P_0$$

$$= \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma},$$

$$H_{11} \equiv P_1 H P_1$$

$$= \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \varepsilon_d,$$

$$H_{22} \equiv P_2 H P_2$$

$$= \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + 2\varepsilon_d + U,$$

$$H_{01} \equiv P_0 H P_1$$

$$= \sum_{k\sigma} V_k c_{k\sigma}^\dagger d_\sigma n_{d\sigma} (1 - n_{d\bar{\sigma}})$$

$$H_{12} \equiv P_1 H P_2$$

$$= \sum_{k\sigma} V_k c_{k\sigma}^\dagger d_\sigma n_{d\sigma} n_{d\bar{\sigma}}$$

$$H_{02} \equiv P_0 H P_2 = 0.$$

Schrieffer-Wolff Transformation (Schrieffer&Wolff *Phys. Rev.* **149**, 491 (1966))

Schrieffer-Wolff Transformation

Schrieffer-Wolff transformation is a canonical transformation diagonalizing the *many-body Hamiltonian* with distinct number of occupations

$$\begin{aligned}\tilde{H} &\equiv e^S H e^{-S} = H + [S, H] + \frac{1}{2!} [S, [S, H]] + \dots \\ &= H_0 + V + [S, H_0] + [S, V] + \frac{1}{2!} [S, [S, H_0 + V]] + \dots\end{aligned}$$

such that $V + [S, H_0] \equiv 0$. So up to the second order we are left with

$$\tilde{H}^{(2)} = H_0 + [S, V] + \frac{1}{2} [S, [S, H_0]] = H_0 + \frac{1}{2} [S, H_0].$$

This is also some renormalization procedure for gapped system!

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Schrieffer-Wolff Transformation

After lengthy calculation of commutators, up to some constant, we obtain the second order effective Hamiltonian

$$H_{\text{eff}}^{(2)} = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'\sigma\sigma'} V_k^* V_{k'} \left(\frac{d_\sigma^\dagger d_{\sigma'}}{\varepsilon_d - \varepsilon_{k'}} c_{k\sigma} c_{k'\sigma'}^\dagger + \frac{d_\sigma d_{\sigma'}^\dagger}{U + \varepsilon_d - \varepsilon_{k'}} c_{k\sigma}^\dagger c_{k'\sigma'} \right).$$

Introducing the second quantized spin operator $S_{kk'} \equiv \sum_{\alpha\beta} c_{k\alpha}^\dagger \frac{\boldsymbol{\sigma}_{\alpha\beta}}{2} c_{k\beta}$ and $S_d \equiv \sum_{\mu\nu} d_\mu^\dagger \frac{\boldsymbol{\sigma}_{\mu\nu}}{2} d_\nu$, we finally get the Low-energy Effective Spin Hamiltonian

s-d Hamiltonian

$$H_{sd} = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'} JS_{kk'} \cdot S_d,$$

Warmup: Interactive Green Function

Functional derivative of partition function with auxilliary fields tells us

$$\begin{aligned} \text{---} \rightarrow = & \text{---} \rightarrow + \left(\text{---} \overset{*}{\underset{|}{\text{---}}} + \text{---} \overset{*}{\underset{|}{\text{---}}} + \dots \right) \\ & + \left(\text{---} \overset{*}{\underset{\diagup \diagdown}{\text{---}}} + \text{---} \overset{*}{\underset{\diagup \diagdown}{\text{---}}} + \text{---} \overset{*}{\underset{\diagup \diagdown}{\text{---}}} \right) \\ & + \left(\text{---} \overset{*}{\underset{\diagup \diagdown}{\text{---}}} + \text{---} \overset{*}{\underset{\diagup \diagdown}{\text{---}}} + \dots \right) + \dots, \end{aligned}$$

Warmup: Relaxation Time from Impurities Scattering

Interactive Green function for itinerate electrons can be perturbatively re-arranged to form Dyson equation

$$G(i\omega_n, \mathbf{k}) = \frac{1}{(G_0)_k^{-1} - \Sigma_k} \equiv \frac{1}{i\omega_n - \xi_k - \Sigma_k},$$

where up to single-loop level

$$\Sigma_{k,k'} \equiv \langle \mathbf{k} | T_n | \mathbf{k}' \rangle \equiv \left(\begin{array}{c} \text{---} \\ \vdots \\ \text{---} + \text{---} \end{array} \right).$$

So from the analytic properties of retarded T -matrix we have

Relaxation Time

$$\frac{1}{2\tau} \equiv \text{Im } \Sigma^R(\varepsilon_k, \mathbf{k}) \equiv \text{Im} \langle \mathbf{k} | T^R | \mathbf{k} \rangle = \pi \left\langle \sum_{\mathbf{k}'} |\langle \mathbf{k}' | T^R | \mathbf{k} \rangle|^2 \delta(\varepsilon_k - \varepsilon_{k'}) \right\rangle_{\text{imp}},$$

First Order Scattering

If we just truncate at the lowest order

$$\begin{aligned}
 \frac{1}{2\tau} &= \frac{1}{2\tau} = \frac{1}{2\tau} \left[\frac{1}{2\pi} \sum_{m_s} (S_d^z)^2 + \frac{1}{2\pi} \sum_{m_s} (S_d^\pm)^2 \right] \\
 &= \frac{\pi J^2 n}{2S+1} \sum_{m_s} (S(S+1) + 2m_s) = \pi J^2 n S(S+1)
 \end{aligned}$$

where the DOS n is a constant for itinerant electrons.

- ⊗ Clearly this result has no temperature dependence so it CANNOT contribute to the behavior of resistance minimum.
- ⊗ In fact, one can easily see that the only possible involvement of temperature is from the fermionic distribution of intermediate states. That's why we have to take the free Green function into account, i.e., go to the second order of the perturbative series of T^R .

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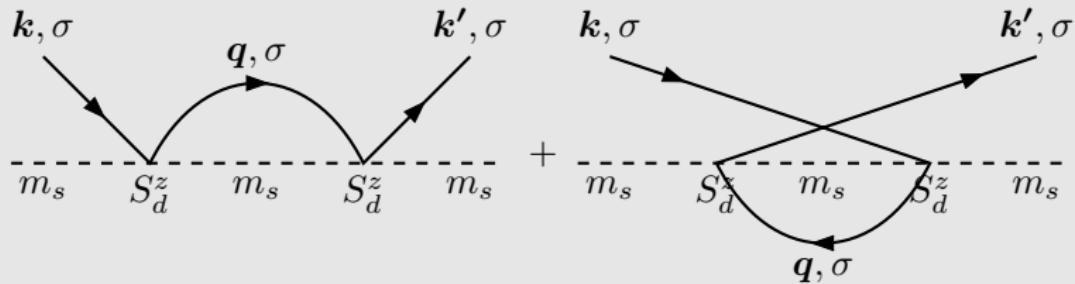
$$\begin{aligned}
 \frac{1}{2\tau} &= \frac{1}{2\tau} = \frac{1}{2\pi} \sum_{m_s} \left(S_d^z (m_s) + S_d^\pm (m_s) \right) \\
 &= \frac{\pi J^2 n}{2S+1} \sum_{m_s} (S(S+1) + 2m_s) = \pi J^2 n S(S+1)
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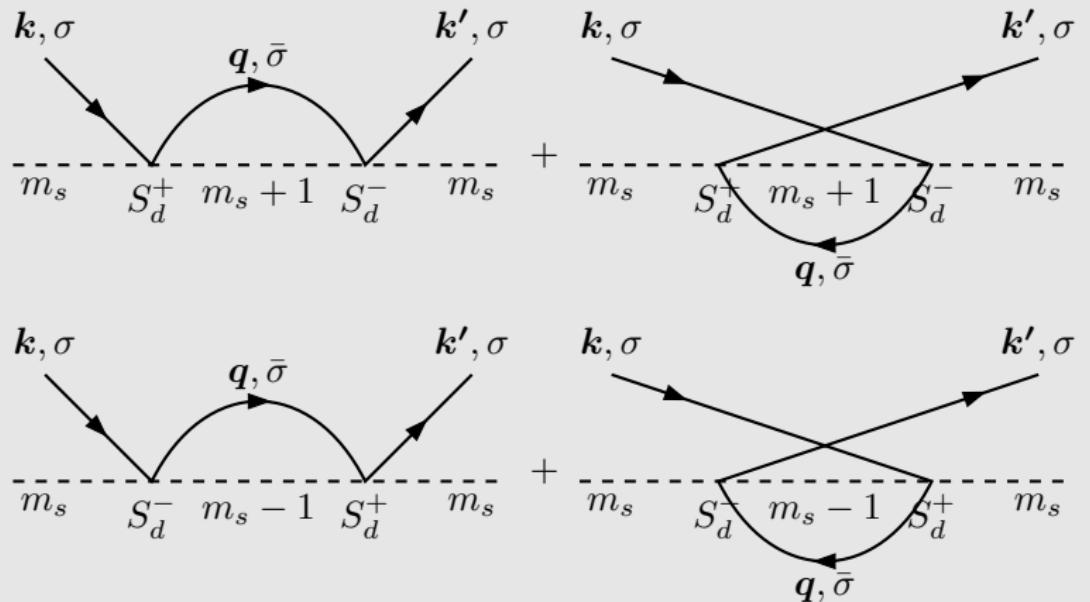
Second Order Scattering (Non Spin-flip)

Things becomes dramatically complicated when we go to the second order. As is seen below, there will be six diagrams contributing to relaxation time¹



¹One can easily check that Single spin-flip terms like $\langle |\hat{S}_d^z \hat{G}_0 \hat{S}_d^\pm| \rangle$ vanish because single creation or annihilation operator will survive after contraction, whose expectation value is always zero.

Second Order Scattering (Spin-flip)



Example of Calculation of Non Spin-flip Process

The contraction rules are similar to each other. Let us take one typical process of from state $|\mathbf{k}, \uparrow\rangle$ to $|\mathbf{k}', \uparrow\rangle$ as an example²

$$= J^2 \left\langle \Omega \left| \color{red}{c_{\mathbf{k}\uparrow}} \sum_{p_1 p_2 p_3 p_4} \left(c_{p_1\uparrow}^\dagger c_{p_2\uparrow} - c_{p_1\downarrow}^\dagger c_{p_2\downarrow} \right) S_d^z \hat{G}_0 \left(c_{p_3\uparrow}^\dagger c_{p_4\uparrow} - c_{p_3\downarrow}^\dagger c_{p_4\downarrow} \right) S_d^z \color{blue}{c_{\mathbf{k}'\uparrow}^\dagger} \right| \Omega \right\rangle.$$

²The QFT language we use here is similar to the work of Abrikosov, *Physics Physique Fizika*, 2, 5 (1965)

Example of Calculation of Non Spin-flip Process

- All the factorized feynman diagrams (bubble diagrams) coming from contraction within the same physical operators should be absorbed in the normalization of path integral.

So the only non-vanishing term is

$$\begin{aligned}
 & J^2 \sum_{p_1 p_2 p_3 p_4} \langle \Omega | c_{k\uparrow} c_{p_1\uparrow}^\dagger c_{p_2\uparrow} S_d^z \hat{G} c_{p_3\uparrow}^\dagger c_{p_4\uparrow} S_d^z c_{k'\uparrow}^\dagger | \Omega \rangle \\
 &= J^2 \sum_{p_1 p_2 p_3 p_4} \langle c_{k\uparrow} c_{p_1\uparrow}^\dagger \rangle S_d^z \frac{\langle c_{p_2\uparrow} c_{p_3\uparrow}^\dagger \rangle}{\varepsilon_k - \varepsilon_{p_2} + i\delta} S_d^z \langle c_{p_4\uparrow} c_{k'\uparrow}^\dagger \rangle \\
 &+ J^2 \sum_{p_1 p_2 p_3 p_4} \langle c_{k\uparrow} c_{p_3\uparrow}^\dagger \rangle S_d^z \frac{\langle c_{p_1\uparrow}^\dagger c_{p_4\uparrow} \rangle}{\varepsilon_k - \varepsilon_{p_4} + i\delta} S_d^z \langle c_{p_2\uparrow} c_{k'\uparrow}^\dagger \rangle \\
 &= J^2 \sum_p \frac{1 - f(\varepsilon_p)}{\varepsilon_k - \varepsilon_p + i\delta} S_d^z + J^2 \sum_p \frac{f(\varepsilon_p)}{\varepsilon_k - \varepsilon_p + i\delta} S_d^z,
 \end{aligned}$$

which is NOT temperature-dependent again.

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 \end{aligned}$$

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Logarithmical Term

All the contribution from other diagrams can be obtained by simply replacing S_d^z by S^\pm . After dropping all temperature-independent terms, we arrive at the final simple expression³

$$\begin{aligned} & \langle \mathbf{k}, \sigma | T^{R(2)} | \mathbf{k}', \sigma' \rangle \\ & \xrightarrow{\text{temp.-depend}} -J^2 \sum_p [S^+, S^-] \frac{f(\varepsilon_p)}{\varepsilon_k - \varepsilon_p + i\delta} = -J^2 \sum_p \frac{f(\varepsilon_p)}{\varepsilon_k - \varepsilon_p + i\delta} 2S_d^z \\ & = -2J^2 S^z \left(n \mathcal{P.V.} \int_{-D}^D d\varepsilon_p \frac{f(\varepsilon_p)}{\varepsilon_k - \varepsilon_p + i\delta} - i\pi n \right) = 2J^2 n S_d^z \ln \left| \frac{D}{k_B T} \right| \end{aligned}$$

³Imaginary parts are dropped because they are temperature-independent.

Kondo Effect (Kondo, *Progress of Theoretical Physics*, vol. 1, 32 (1964))

By Summing up all diagrams up to single-loop level, performing the integral over k' and taking the trace of impurity spins, the relaxation time is corrected as

$$\frac{1}{2\tau} = \pi n J^2 S(S+1) \left(1 + 2nJ \ln \left| \frac{D}{k_B T} \right| \right)^2 \simeq \pi n J^2 S(S+1) \left(1 + 4nJ \ln \left| \frac{D}{k_B T} \right| \right)$$

Including the electron-photon contribution, we finally have

Logarithmical Correction of Resistance

$$R(T) \sim AT^5 + R(0) \left(1 - 4nJ \ln \left| \frac{k_B T}{D} \right| + \dots \right),$$

where local minimum exists for *antiferromagnetic couplings*.

Kondo Effect (Kondo, *Progress of Theoretical Physics*, vol. 1, 32 (1964))

In order to test the logarithmic dependence on T we have made a comparison of (23) with an experiment¹⁰⁾ made on the alloys of iron with gold, in which the resistivity has been measured to very low temperatures. In Fig. 1 the three curves drawn represent the three functions, $0.20 - 0.0078 \log T$, $0.077 - 0.004 \times \log T$ and $0.034 - 0.0016 \log T$, respectively from the above, in units of $\mu\Omega\text{cm}$. The agreement is quite good, particularly the steep rise at the lowest temperature is well represented by a logarithmic function. From (22) both the constant terms and the coefficients of the logarithmic terms must be proportional to concentration. Nominal solute concentration is indicated in the figure. The indicated values

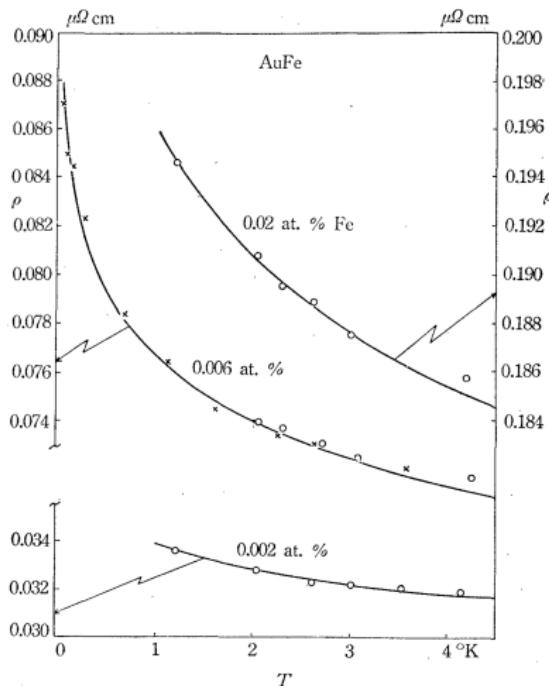


Fig. 1. Comparison of experimental and theoretical ρ - T curves for dilute AuFe alloys.

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Is This the End of the Story? Kondo Problem!

Serious Contradiction (Kondo Problem)

- ✳ Logarithmical-divergent term is necessary to explain the resistance minimum and works perfectly to match the experimental results
- ✳ If we continue decreasing the temperature, the logarithmical term tells us that the resistance of alloy will goes to infinity at zero temperature, which, is thoroughly unacceptable.

Contradiction is always the prelude of deep physics:

- ✳ **There will be one critical temperature or critical energy scale under which our perturbative treatment of scattering processes break down**
- ✳ Since perturbation theory is built on the assumption of small coupling constant, this fact tells us that **the coupling constant will run with the decrease of energy scale** \Rightarrow RG flow!

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Decomposition of Many-body Hilbert Space

Scaling the conduction band into two parts $(-D/b, D/b)$ and $(-D, -D/b] \cup [D/b, D)$ for $b > 1$ and decomposing the many-body Hilbert space into three kinds⁴ (with the same notation as before)

- ✳ ψ_1 has no conduction electron or hole in the upper and lower edge
- ✳ ψ_0 has at least one hole in the lower edge
- ✳ ψ_2 has at least one conduction electron in the upper edge.

Anderson consider the **anisotropic** s-d Hamiltonian in his original paper for future discussion (Anderson, *Journal of Physics C: Solid State Physics* **3**, 2436 (1970))

Anisotropic sd Hamiltonian

$$H_{\text{sd}} = \sum_{kk'} \left(J_z S^z (c_{k\uparrow}^\dagger c_{k'\uparrow} - c_{k\downarrow}^\dagger c_{k'\downarrow}) + J_+ S^+ c_{k\downarrow}^\dagger c_{k'\uparrow} + J_- S^- c_{k\uparrow}^\dagger c_{k'\downarrow} \right).$$

⁴Because doubly excited intermediate states are of high order

Decomposition of Many-body Hilbert Space

Scaling the conduction band into two parts $(-D/b, D/b)$ and $(-D, -D/b] \cup [D/b, D)$ for $b > 1$ and decomposing the many-body Hilbert space into three kinds⁴ (with the same notation as before)

- ✳ ψ_1 has no conduction electron or hole in the upper and lower edge
- ✳ ψ_0 has at least one hole in the lower edge
- ✳ ψ_2 has at least one conduction electron in the upper edge.

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$$H_{\text{sd}-\ell\ell} = \sum_{k_\ell k'_\ell} \left(J_z S^z (c_{k_\ell\uparrow}^\dagger c_{k'_\ell\uparrow} - c_{k_\ell\downarrow}^\dagger c_{k'_\ell\downarrow}) + J_+ S^+ c_{k_\ell\downarrow}^\dagger c_{k'_\ell\uparrow} + J_- S^- c_{k_\ell\uparrow}^\dagger c_{k'_\ell\downarrow} \right)$$

$$H_{\text{sd}-\ell h} = \sum_{k_\ell k'_h} \left(J_z S^z (c_{k_\ell\uparrow}^\dagger c_{k'_h\uparrow} - c_{k_\ell\downarrow}^\dagger c_{k'_h\downarrow}) + J_+ S^+ c_{k_\ell\downarrow}^\dagger c_{k'_h\uparrow} + J_- S^- c_{k_\ell\uparrow}^\dagger c_{k'_h\downarrow} \right)$$

$$H_{\text{sd}-h\ell} = \sum_{k_h k'_\ell} \left(J_z S^z (c_{k_h\uparrow}^\dagger c_{k'_\ell\uparrow} - c_{k_h\downarrow}^\dagger c_{k'_\ell\downarrow}) + J_+ S^+ c_{k_h\downarrow}^\dagger c_{k'_\ell\uparrow} + J_- S^- c_{k_h\uparrow}^\dagger c_{k'_\ell\downarrow} \right)$$

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where

$$\sum_k \equiv \sum_{|k| \in (0, D/b)} + \sum_{|k| \in (D/b, D)} \equiv \sum_{k_\ell} + \sum_{k_h}$$

Effective Hamiltonian

Our Hamiltonian do not involve processes in H_{02} and H_{20} subspaces. Thus, the effective Hamiltonian on subspace H_{11} should take entirely the same form as we have developed before for Anderson model with Schrieffer-Wolff transformation

$$H_{\text{eff}} = H_{11} + H_{10} \frac{1}{E - H_{00}} H_{01} + H_{12} \frac{1}{E - H_{22}} H_{21},$$

where (to the lowest order)

$$H_{11} = H_0 + H_{\text{sd-}\ell\ell},$$

$$H_{00} = H_0 + H_{\text{sd-}hh} \text{ for } h \in (-D, -D/b) \simeq H_0,$$

$$H_{22} = H_0 + H_{\text{sd-}hh} \text{ for } h \in (D/b, D) \simeq H_0$$

and

$$H_{01} = H_{\text{sd-}h\ell} \text{ for } h \in (-D, -D/b), \quad H_{10} = H_{\text{sd-}\ell h} \text{ for } h \in (-D, -D/b)$$

$$H_{21} = H_{\text{sd-}h\ell} \text{ for } h \in (D/b, D), \quad H_{12} = H_{\text{sd-}\ell h} \text{ for } h \in (D/b, D),$$

Non Impurity Spin-flip Scattering: S^z - S^z Part

Rearranging the operators, we have (remember we are working in empty-edge subspace)

$$H_{12} \frac{1}{E - H_{22}} H_{21} = \frac{3J_z^2}{4} \sum_{k_{\ell 1} k_{\ell 2}} \sum_{\sigma} \frac{c_{k_{\ell 1} \sigma}^\dagger c_{k_{\ell 2} \sigma}}{E - D + \varepsilon_{k_{\ell 2}}} \times n_0 D \left(1 - \frac{1}{b}\right),$$

$$H_{10} \frac{1}{E - H_{00}} H_{01} = \frac{3J_z^2}{4} \sum_{k_{\ell 1} k_{\ell 2}} \sum_{\sigma} \frac{c_{k_{\ell 1} \sigma}^\dagger c_{k_{\ell 2} \sigma}}{E - D - \varepsilon_{k_{\ell 2}}} \times n_0 D \left(1 - \frac{1}{b}\right),$$

where again DOS is approximated to be flat $n \equiv n_0$.

Non Impurity Spin-flip Scattering: S^+ - S^- Part

$$H_{12} \frac{1}{E - H_{22}} H_{21} = J_- J_+ \left\{ \left(\frac{1}{2} - S^z \right) \sum_{k_{\ell 1} k_{\ell 2}} \frac{c_{k_{\ell 1} \uparrow}^\dagger c_{k_{\ell 2} \uparrow}}{E - D + \varepsilon_{k_{\ell 2}}} \right. \\ \left. + \left(\frac{1}{2} + S^z \right) \sum_{k_{\ell 1} k_{\ell 2}} \frac{c_{k_{\ell 1} \downarrow}^\dagger c_{k_{\ell 2} \downarrow}}{E - D + \varepsilon_{k_{\ell 2}}} \right\} \times n_0 D \left(1 - \frac{1}{b} \right),$$

$$H_{10} \frac{1}{E - H_{00}} H_{01} = J_- J_+ \left\{ \left(\frac{1}{2} - S^z \right) \sum_{k_{\ell 1} k_{\ell 2}} \frac{c_{k_{\ell 1} \uparrow}^\dagger c_{k_{\ell 2} \uparrow}}{E - D - \varepsilon_{k_{\ell 2}}} \right. \\ \left. + \left(\frac{1}{2} + S^z \right) \sum_{k_{\ell 1} k_{\ell 2}} \frac{c_{k_{\ell 1} \downarrow}^\dagger c_{k_{\ell 2} \downarrow}}{E - D - \varepsilon_{k_{\ell 2}}} \right\} \times n_0 D \left(1 - \frac{1}{b} \right),$$

where identity $S^- S^+ = \frac{1}{2} - S^z$ is applied.

Single Impurity Spin-flip Scattering

$$H_{12} \frac{1}{E - H_{22}} H_{21} = -\frac{J_z J_+}{2} \sum_{k_{\ell 1} k_{\ell 2}} \sum_{\sigma} \frac{c_{k_{\ell 1} \sigma}^{\dagger} c_{k_{\ell 2} \sigma}}{E - D - \varepsilon_{k_{\ell 2}}} \times n_0 D \left(1 - \frac{1}{b}\right),$$
$$H_{10} \frac{1}{E - H_{00}} H_{01} = -\frac{3 J_z J_-}{2} \sum_{k_{\ell 1} k_{\ell 2}} \sum_{\sigma} \frac{c_{k_{\ell 1} \sigma}^{\dagger} c_{k_{\ell 2} \sigma}}{E - D - \varepsilon_{k_{\ell 2}}} \times n_0 D \left(1 - \frac{1}{b}\right).$$

Single-loop β -function

Dropping all the trivial shift on effective Hamiltonian (which has no effect on partition function) and recognizing the corresponding terms, we obtain

$$J_{\pm}(b) = J_{\pm} + J_z J_{\pm} n D \left(1 - \frac{1}{b}\right) \left(\frac{1}{E - D + \varepsilon_k} + \frac{1}{E - D - \varepsilon_k}\right),$$

$$J_z(b) = J_z + J_+ J_- n D \left(1 - \frac{1}{b}\right) \left(\frac{1}{E - D + \varepsilon_k} + \frac{1}{E - D - \varepsilon_k}\right).$$

Since we are interested in low-energy behavior of effective Hamiltonian, both the kinetic energy of itinerate electron E and the internal excitation energy ε_k are negligible comparing with the band width. Therefore

Single-loop β -function

$$\beta(J_{\pm}) \equiv \frac{dJ_{\pm}}{d \ln b} = 2n J_z J_{\pm}, \quad \beta(J_z) \equiv \frac{dJ_z}{d \ln b} = 2n J_+ J_-.$$

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RG Flow

The integral curve of beta-function is $J_z^2 - J_{\pm}^2 = \text{const.}$



Figure: RG Flow of Coupling Constants.

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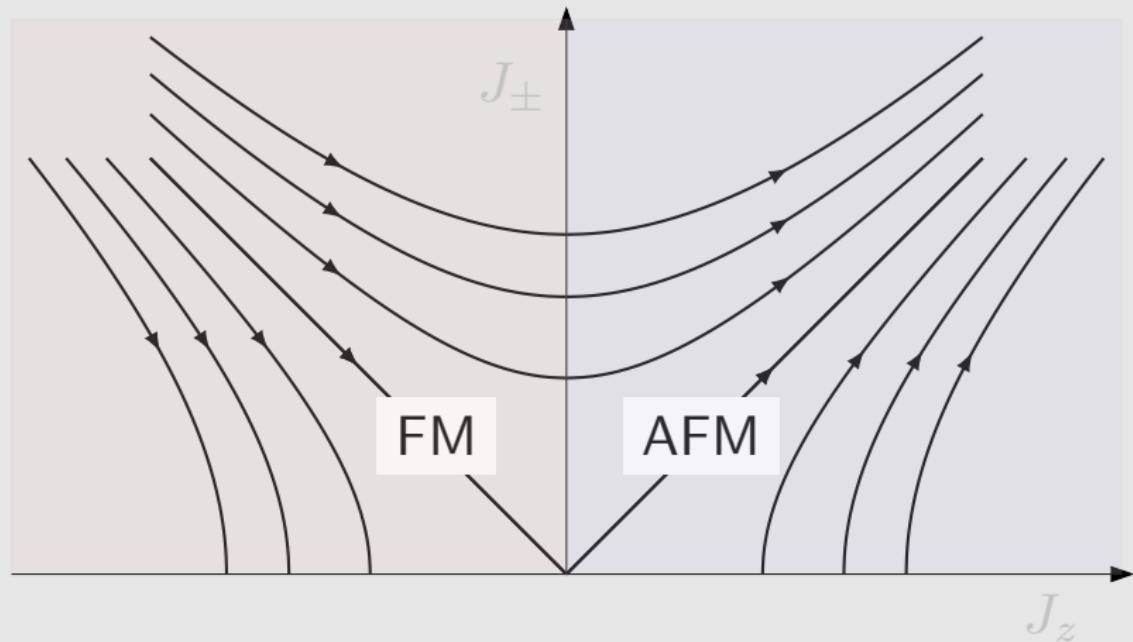


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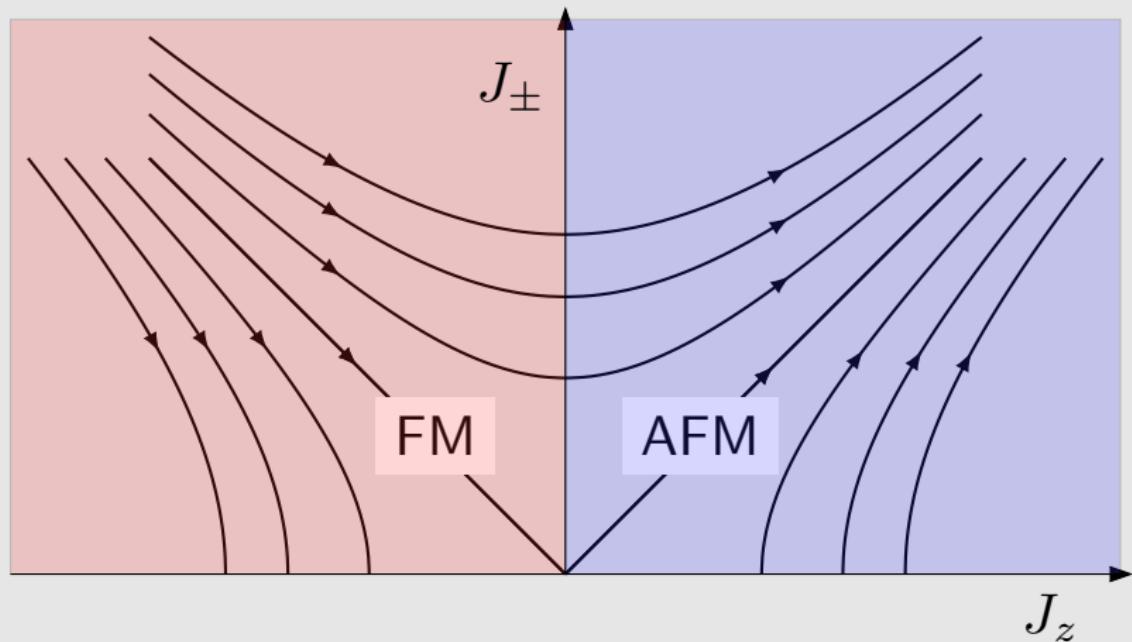


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Focusing on the case where $J_z = J_{\pm} \equiv J$, and denote $J(1) \equiv J_0$, then

$$\frac{dJ}{d \ln b} = 2nJ^2 \implies J(b) = \frac{J_0}{1 - 2nJ_0 \ln b} \equiv \frac{J_0}{1 + 2nJ(0) \ln \frac{D(b)}{D}}.$$

Since $D(b)$ has the unit of energy, or $k_B T$, the above RG flow exhibits an apparent **IR divergence** at

$$T_K = D e^{-1/2nJ_0} = D(b) e^{-1/2nJ(b)}$$

and **UV freedom** at $T \rightarrow \infty \implies$ **Asymptotic Freedom!**

Direct Result from RG Flow

- ⊗ Existence of T_K explains the breakdown of perturbative methods.
- ⊗ Divergence of J require the formation of impurity-conduction-electron *singlet*

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Universality of Asymptotic Freedom

Asymptotic freedom exists in many fields of physics

- * In QCD Gross and Wilczek found the vanishing of strong interaction between quarks in SU(N) Yang-Mills theory (Gross&Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973)), which is crucial in explanation of the quarks-hadron confinement.

must be gauge independent in lowest order, could have any sign at $g=0$. We have calculated Z_1 and Z_3 for the above Lagrangian, and we find that¹²

$$\beta_V = -(g^3/16\pi^2)\frac{11}{3}C_2(G) + O(g^5), \quad (8)$$

where $C_2(G)$ is the quadratic Casimir operator of the adjoint representation of the group G : $\sum_{b,c} c_{abc} \times c_{abc} = C_2(G) \delta_{aa}$ [e.g., $C_2(\text{SU}(N)) = N$]. The solution of (3) is then $\bar{g}^2(t) = g^2/(1 - 2\beta_V g^{-1}t)$, and $\bar{g} \rightarrow 0$ as $t \rightarrow \infty$ as long as the physical coupling constant g is in the domain of attraction of the origin.¹³

Universality of Asymptotic Freedom

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- * In **CMP** Polyakov also found IR divergence and UV freedom of the antiferromagnetic coupling in (1+1)D spacetime (NL σ model) (Polyakov, *Physics Lett. B*, **59**, 1 (1975))

After repeating the partial integration sufficient number of times we get the coupling constant for the momentum q :

$$f(q) = \frac{f}{1 - (N - 2)(1/2\pi) \log \Lambda/q}. \quad (11)$$

It is evident, that (11) is correct only for such q for which $f(q) \ll 1$.

Now let us calculate the correlation function:

$$C(R) = \langle \mathbf{n}(0) \cdot \mathbf{n}(R) \rangle \quad (12)$$

Universality of Asymptotic Freedom

Asymptotic freedom exists in many fields of physics

- ⊗ In **TQFT** Witten propose a non-abelian bosonization to map (1+1)D Dirac fermion to Wess-Zumino-Witten model and found a similar RG flow (Witten *Commun. Math. Phys.*, **92** (1984))

$$\int \frac{i(N-2)}{16\pi} \text{Tr} \partial_\mu g_0 \partial_\mu g_0^{-1} \ln \left(\frac{\Lambda^2}{\mu^2} \right) d^2x, \quad (18)$$

where Λ is a momentum space cut-off and μ is a renormalization mass. From this we read off the one loop beta function

$$\beta(\lambda, n) = -\frac{\lambda^2(N-2)}{4\pi} \left[1 - \left(\frac{\lambda^2 n}{4\pi} \right)^2 \right] \quad (19)$$

which, as claimed, vanishes for $\lambda^2 = \left| \frac{4\pi}{n} \right|$.

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③ Summary and Outlook

Summary and Outlook

What have we covered?

- ✳ Resistance Minimum \Rightarrow Logarithmical Correction
- ✳ Logarithmical Divergence \Rightarrow Contradiction on Perturbative Results
- ✳ Poor Man's Scaling \Rightarrow Asymptotic Freedom
- ✳ Divergence of Antiferromagnetic Coupling \Rightarrow Kondo Singlet

What has been developed since then?

- ✳ Single Impurity
 - Local Fermi Liquid (Nozieres, *J. Low Temp.* **17**, 31 (1974))
 - Numerical RG (Wilson, *Rev. Mod. Phys.* **47**, 773 (1975))
- ✳ Multi-channel Impurities
 - Slave-fermion and Large-N (Cox et al., *Phys. Rev. Lett.*, **71**, 1613 (1993))
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Thanks!