Theory of Hydrodynamic Transport: Past and Future

Research Proposal Examination

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Outline

Complexity of Condensed Matter

The fundamental Hamiltonian of condensed matter incorperates many parts

$$H = H_{\text{el}} + H_{\text{ph}} + H_{\text{imp}} + H_{\text{el-el}} + H_{\text{el-ph}} + H_{\text{el-imp}} + \cdots .$$

We already have full understanding of the system with **weakly-interacting quasiparticles**, like Fermi liquid.

But for **strongly-correlated** systems, or systems **without** well-defined quasiparticle description,

- Theoretically, all perturbative techniques become invalid;
- Numerically,
  - Exact diagnalization is limited by the exponential grow of the dimension of many-body Hilbert space;
  - QMC is known to fail due to fermion sign problem;
  - DMRG or Tensor Network are still under development for $d > 2$;
  - ...

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"The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles.
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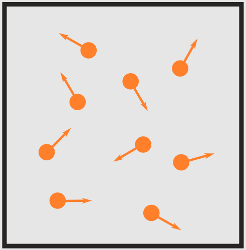
Question

How to grasp the correct physics without perturbative RG?

Traditional Hydrodynamics



$\Longrightarrow$

Traditional Hydrodynamics

All dynamic and thermodynamic properties of liquids/gas are described by (Landau&Lifshitz (1959))

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{v}) = 0$$

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla(\nabla \cdot \boldsymbol{v})$$

$$\rho \left(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \right) = \nabla(\kappa \nabla T) + \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial v_\ell}{\partial x_\ell} \right)^2 + \zeta(\nabla \boldsymbol{v})^2.$$

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What hydrodynamic theory does here is to reduce the 6^N microscopic degree of freedom $\{\boldsymbol{x}_1, \boldsymbol{p}_1; \boldsymbol{x}_2, \boldsymbol{p}_2; \cdots\}$ to just five phenomelogical variable — velocity field $\boldsymbol{v}(\boldsymbol{r})$, in-equilibrium density ρ and entropy s.

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Question

- Are transport measurements performed in the hydrodynamic regime?
- What is the correct low-energy effect degree of freedom?
- What is the EOM of such hydro-variables?
- How to compute transport coefficients and compare with experiments?

Contents

Assumption of Hydrodynamics

What is Hydrodynamics?

"Hydrodynamics is the effective theory describing the relaxation of an interacting classical or quantum system towards thermal equilibrium."

— Hartnoll, Lucas, and Sachdev (2018)

We call the relevant fields of such effective field theory **hydro-variables**.

Assumption

- Hydro-variables consist of conserved quantities and symmetry-breaking fields;
- The system reaches thermodynamic equilibrium at each instant of time;
- Physical observables are completely determined by hydro-variables.

To validate the second assumption, we demand the driving frequency of external fields to satisfy $\omega\tau_{\text{th}} \ll 1$, where the natural thermalization time scale (Damle&Sachdev, PRB, **56**, 8714 (1997)) $\tau_{\text{th}} \sim \hbar/k_B T$. This is the criterion of "hydrodynamic regime".

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Realization of Hydrodynamic Regime

All conductivities are measured at (low) frequency ω under a low, but **non-zero temperature** T. So we can easily tuned ω to meet the criterion $\hbar\omega \ll k_B T$.

$$\begin{array}{c}\text{Transport Coefficients}\\ \text{from Hydrodynamics}\end{array} \Longleftrightarrow \begin{array}{c}\text{Transport Coefficients}\\ \text{from Experiments}\end{array}.$$

As a contrast, in finding the universal conductivity around superfluid-insulator transition, for example,

- Many theorerical works (Fisher *et al.*, PRL, **64**, 587 (1990); Cha *et al.*, PRB, **44**, 6883 (1991)), as well as exact diagnolization (Runge, PRB, **45**, 13136 (1992)), are done at $T = 0$ so violate the criterion;
- Even the finite-temperature Numerical Monte Carlo (Sorensen *et al.*, PRL, **69**, 828 (1992)), involving analytic continuation that is insentitive to the temperature range, is questioned to fall out of the correct hydrodynamic regime.

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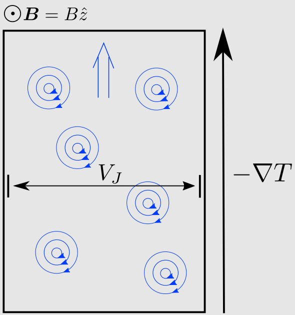
Advantages of Hydrodynamic Theory

It provides simple, direct, and novel predictions/explanations to experiments that microscopic calculation hard to cover.

Contents

Nernst Signal of $La_{2-x}Sr_xCuO_4$



Nernst Signal
$e_N := -E_y/\partial_x T$

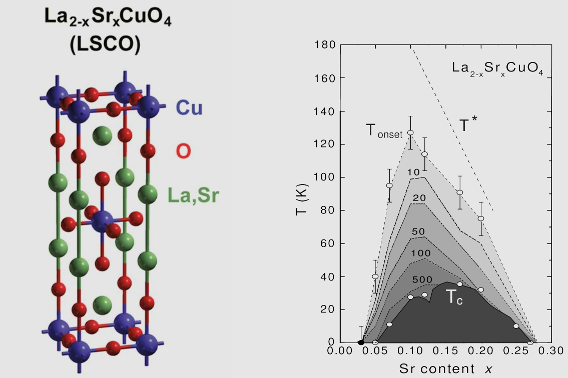
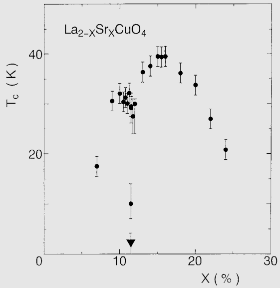


Figure: **Lattice Structure and Phase diagram of LSCO**. The Nernst coefficient on Contour $\nu \equiv e_N/B$. Extracted from Wang *et al.* PRB, **73**, 024510 (2006).

Quantum Criticality



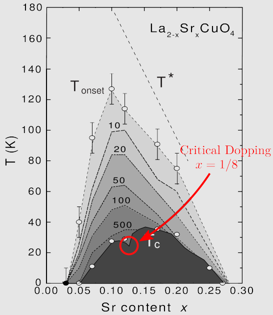
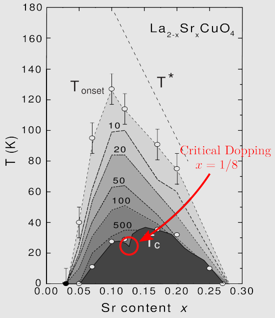


Figure: Sharp drop of the superconducting transition temperature. Extracted from Kumagai *et al.*, J. Superconductivity, **7**, 1 (1993)

Quantum Criticality

- Zero field μSR measurements provide evidence for AFM order of Cu moments (Kumagai *et al.*, J. Superconductivity, **7**, 1 (1993))
- Elastic/Inelastic Neutron Scattering reveals magnetically ordered SDW states (Yamada *et al.*, PRB, **57**, 6165 (1998))
- 14keV (Hard) X-ray diffraction measurements reveals CDW states (Croft *et al.*, PRB, **89**, 224513 (2014))



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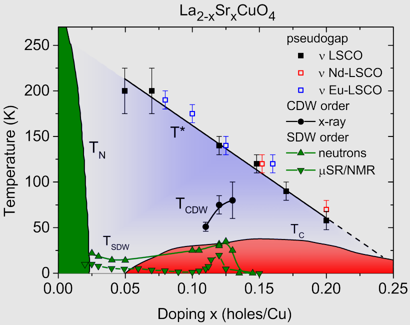

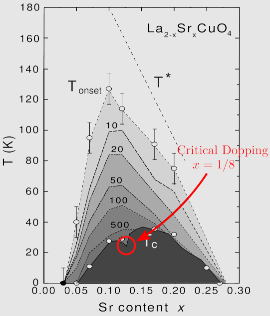

Figure: Onset temperature of each type of experiment in LSCO phase diagram. Extracted from Croft *et al.*, PRB, **89**, 224513 (2014).

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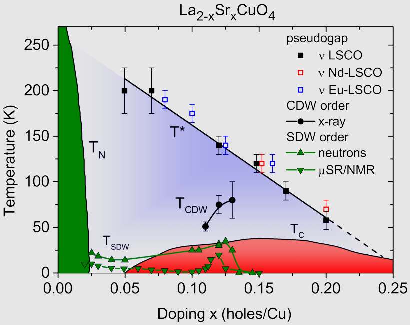

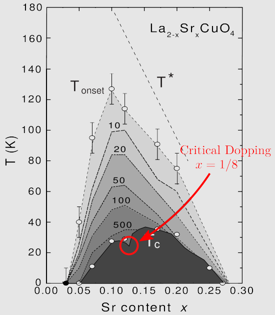

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Physics close to the sharp dip around doping $x = 1/8$ demands a quantum critical theory, or CFT.

Bose-Hubbard Model

The simplest model for superfluid-insulator phase transition of vortice liquid is

Bose-Hubbard Model

$$H = -t\sum_{\langle ij\rangle} b_i^\dagger b_j - \mu\sum_i n_i + \frac{U}{2}\sum_i n_i(n_i-1)$$

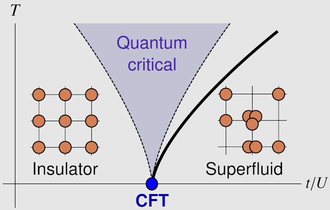

Figure: Extracted from Witczak-Krempa *et al.*, Nat. Phys., **10**, 5 (2014).

And the critical theory is given by (Fisher *et al.*, PRB, **40**, 1 (1989))

(2+1)-D Conformal Field Theory

$$S = \int \mathrm{d}^2 r\, \mathrm{d}\tau \left[|\partial_\tau \psi|^2 + v^2 |\nabla\psi|^2 - g|\psi|^2 + \frac{u}{2}|\psi|^4 \right].$$

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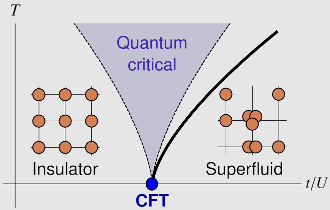

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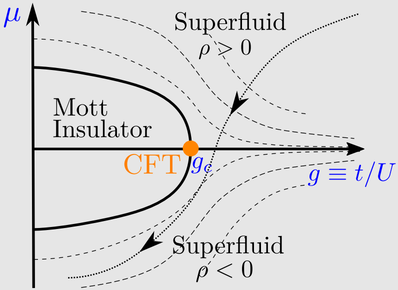
Linear Response Around Critical Point

To study the Nernt effect of LSCO (dotted line on the RHS), we are going to perturb such CFT with

- a chemical potential fluctuation $-\nabla\mu$ (induced by external electric fields)
- an external magnetic field B
- a thermal fluctuation $-\nabla T$
- a small density of impurities

and calculate the corresponding linear reponses.



Perturbative CFT

$$S = \int \mathrm{d}^2r\,\mathrm{d}\tau \left[|\partial_\tau - \mu\psi|^2 + v^2|(\nabla - i2e\boldsymbol{A})\psi|^2 - g|\psi|^2 + \frac{u}{2}|\psi|^4 + \mathcal{L}_{\text{imp}} \right].$$

Conservation Law

The Lorentz-invariant microscopic action gives rise to two *macroscopic* **Ward identities** (Herzog, J. Phys. A, **42**, 34 (2009)) (as **conservation laws**)

Conservation Laws

$$\nabla_\mu \langle T^{\mu\nu} \rangle = F^{\nu\lambda} \langle J_\lambda \rangle, \quad \partial_\mu \langle J^\mu \rangle = 0,$$

where

- (macroscopic) current operator $\langle J^\mu \rangle \equiv -\frac{\delta}{\delta A_\mu} W[g, A]$,
- (macroscopic) stress-energy tensor $\langle T^{\mu\nu} \rangle \equiv = \frac{-2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} W[g, A]$,
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Hydro-variables

By assumption, we thus have three (dual) hydro-variables: chemical potential μ, temperature T, and (normalized) velocity u^μ.

Constitutive Relation — Zeroth Order

By Eckart, Phys. Rev., **58**, 10 (1940), given a vector u^μ, one can always decompose the two tensors with the projection operator $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ that

$$J^\mu = \mathcal{N} u^\mu + j^\mu,$$
$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} P^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu},$$

where scalar $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{P}$, vector j^μ and q^ν, and tensor $t^{\mu\nu}$ are formally contractions with whether velocity vector or projection operator

scalar	$\mathcal{N} \equiv J^\mu u_\mu, \quad \mathcal{E} \equiv T^{\mu\nu} u_\mu u_\nu, \quad \mathcal{P} \equiv P^{\mu\nu} T_{\mu\nu},$
transverse vector	$j^\mu \equiv P^{\mu\nu} J_\nu, \quad q^\mu \equiv -P^{\mu\nu} u^\rho T_{\nu\rho},$
transverse-traceless tensor	$t_{\mu\nu} \equiv \frac{1}{2}\left(P_{\mu\alpha}P_{\nu\beta} + P_{\alpha\nu}P_{\mu\beta} - \frac{2}{d} P_{\mu\nu}P_{\alpha\beta}\right) T^{\alpha\beta}$

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where scalar $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{P}$, vector j^μ and q^ν, and tensor $t^{\mu\nu}$ are formally contractions with whether velocity vector or projection operator

Choosing the proper frame of reference, we get the familiar constitutive relation of ideal fluids

Zeroth Order

$$\mathcal{N} = \rho(T,\mu), \quad \mathcal{E} = \varepsilon(T,\mu) + P(T,\mu), \quad \mathcal{P} = P(T,\mu).$$

so that accidentally **the entropy current is conserved**

$$\partial_\mu s^\mu \equiv \partial_\mu (s u^\mu) = 0.$$

Constitutive Relation — First Order

Subtleties on "Frame Choice" (Kovtun, J. Phys. A, 45, 47 (2012))

The notion of local hydro-variables $\{T, \mu, u^\mu\}$ are NOT uniquely defined up to the conservation laws.

Generally,

$$\begin{cases} T(\boldsymbol{r}) \mapsto T'(\boldsymbol{r}) \equiv T + \delta T, \\ \mu(\boldsymbol{r}) \mapsto \mu'(\boldsymbol{r}) \equiv \mu + \delta\mu, \\ u^\mu(\boldsymbol{r}) \mapsto u^{\mu\prime}(\boldsymbol{r}) \equiv u^\mu + \delta u^\mu, \end{cases} \Longrightarrow \begin{cases} \delta\mathcal{E} = \delta\mathcal{P} = \delta\mathcal{N} = 0, \\ \delta j^\mu \simeq -\mathcal{N}\delta u^\mu, \quad \delta q^\mu \simeq (\varepsilon + P)\delta u^\mu, \\ \delta t^{\mu\nu} \simeq 0. \end{cases}$$

Even the zeroth-order constitutive relation will suffer from such redefinition!

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Gauge fix of the first-order spatial gradient expansion

- We choose δT and $\delta\mu$ such that $\mathcal{E} \equiv \varepsilon$ and $\mathcal{N} \equiv \rho$;
- We choose δu^μ such that the first-order $q^\mu \equiv 0$ (**Landau Frame**).

Constitutive Relation — First Order

List all possible terms allowed by symmetry as following (Bhattacharya *et al.*, JHEP, **05**, 147 (2014))
(here we assume **time-reversal symmetry** and **parity-inversion symmetry**)

	All Data	EOM	Independent Data
Scalars	$u^\mu\partial_\mu T, u_\mu\partial_\mu\mu, \partial_\mu u^\mu$	$u_\mu\nabla_\nu T^{\mu\nu}=0$ $\partial_\mu J^\mu=0$	$\partial_\mu u^\mu$
Transerves Vectors	$P^{\mu\nu}\partial_\nu T, P^{\mu\nu}\partial_\nu\mu$ $P^{\mu\nu}u^\lambda\partial_\lambda u_\nu, F^{\mu\nu}u_\nu$	$P^{\mu\nu}\nabla^\lambda T_{\mu\nu}=0$	$P^{\mu\nu}\partial_\nu T, P^{\mu\nu}\partial_{\nu\mu}$ $F^{\mu\nu}u_\nu$
Transverse-traceless Tensor	$\sigma^{\mu\nu}\equiv P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta+\partial_\beta u_\alpha-\frac{2}{d}g_{\alpha\beta}\partial_\lambda u^\lambda\right)$		$\sigma^{\mu\nu}$

Constitutive Relation — First Order

Combine all possible terms linearly, we get

First Order

$$j^\mu = -\sigma_Q T P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) - \chi_T P^{\mu\nu}\partial_\nu T, \qquad \sigma_Q : \text{Universal Conductivity}$$

$$\mathcal{P} = P - \zeta\partial_\lambda u^\lambda, \qquad \zeta : \text{Bulk Viscosity}$$

$$t^{\mu\nu} = -\eta P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{2}g_{\alpha\beta}\partial_\lambda u^\lambda\right). \qquad \eta : \text{Shear Viscosity}$$

The constraint of non-negative entropy production

$$\partial_\mu S^\mu \equiv \partial_\mu\left(su^\mu + \frac{\mu}{T}J^{(1)\mu}\right) = -J^{(1)\mu}\partial_\mu\left(\frac{\mu}{T}\right) - \frac{1}{T}T^{(1)\mu\nu}\partial_\nu u_\mu \geq 0$$

requires

$$\chi_T \equiv 0, \quad \zeta > 0, \quad \sigma_Q > 0, \quad \eta > 0.$$

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$$\chi_T \equiv 0, \quad \zeta > 0, \quad \sigma_Q > 0, \quad \eta > 0.$$

Hydrodynamic EOM

Following Kandanoff&Martin, Ann. Phys., **24**, 419 (1969), the linear response over equilibrium state can be obtained by introducing a purtabative Hamiltonian

$$\mathcal{H} \to \mathcal{H} - \int \mathrm{d}^2 x \left(\frac{\delta T}{T} (\varepsilon - \mu \rho) + \delta \mu \rho + \delta u^{\mu} T_{\mu 0} \right),$$

where we explicitly split out the perturbative source of energy density, charge density, and momentum densities.

Linearized Hydrodynamic equations of motion are read out from conservation laws

Conservation Laws

$$\partial_t \rho + \nabla \cdot \left\{ \rho \boldsymbol{v} + \sigma_Q \left[-\nabla \mu + \frac{\mu}{T} \nabla T + \boldsymbol{v} \times \boldsymbol{B} \right] \right\} = 0$$

$$\partial_t \varepsilon + \nabla \cdot \Big((\varepsilon + P) \boldsymbol{v} \Big) = 0$$

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Thermoelectric Response

Thermoelectric response coefficients are defined as

$$\begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{Q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma} & \boldsymbol{\alpha} \\ T\boldsymbol{\alpha} & \bar{\boldsymbol{\kappa}} \end{pmatrix} \begin{pmatrix} -\nabla\mu \\ -\nabla T \end{pmatrix},$$

where electric field $\boldsymbol{E} = -\nabla\mu$. We are also interested in

- **Thermal Conductivity**: heat current induced by thermal gradient but in the absence of charge current $\boldsymbol{Q} = -\boldsymbol{\kappa}\nabla T$. So $\boldsymbol{\kappa} \equiv \bar{\boldsymbol{\kappa}} - T\boldsymbol{\alpha}\boldsymbol{\sigma}^{-1}\boldsymbol{\alpha}$,
- **Nernst Response**: electric field induced by thermal gradient but in the absense of charge current $\boldsymbol{E} = -\boldsymbol{\theta}\nabla T$. So $\boldsymbol{\theta} \equiv -\boldsymbol{\sigma}^{-1}\boldsymbol{\alpha}$.

All of them can be obtained after a lengthy re-arrangement of hydrodynamic EOM, with the poles

Cyclotron Resonance

$$\omega = \pm\omega_c + i\gamma, \quad \omega_c \equiv \frac{v^2}{c^2}\frac{2B}{(\varepsilon + P)/\rho c}, \quad \gamma \equiv \sigma_Q \frac{v^2}{c^2}\frac{B^2}{\varepsilon + P}.$$

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Thermoelectric Response

On the aspect of "particle",

$$\sigma_{xx} = \sigma_Q \frac{\omega + i\gamma + i\omega_c^2/\gamma}{(\omega + i\gamma)^2 - \omega_c^2}, \qquad \sigma_{xy} = -\frac{\rho}{B}\frac{\gamma^2 + \omega_c^2 - 2i\gamma\omega}{(\omega + i\gamma)^2 - \omega_c^2},$$

$$\alpha_{xx} = \frac{\rho}{T}\frac{i\omega}{(\omega + i\gamma)^2 - \omega_c^2}, \qquad \alpha_{xy} = -\frac{\gamma}{B}\frac{\gamma^2 + \omega_c^2 - i\gamma\omega}{(\omega + i\gamma)^2 - \omega_c},$$

$$\bar{\kappa}_{xx} = s\frac{i\omega - \gamma}{(\omega + i\gamma)^2 - \omega_c^2}, \qquad \bar{\kappa}_{xy} = -s\frac{\omega_c}{(\omega + i\gamma) - \omega_c^2},$$

while on the aspect of "vortex",

$$\rho_{xx} = \frac{1}{\sigma_Q}\frac{\omega(\omega + i\omega_c^2/\gamma + i\gamma)}{(\omega + i\omega_c^2/\gamma) - \omega_c^2}, \qquad \rho_{xy} = \frac{B}{\rho}\frac{(\omega_c^2/\gamma)^2 + \omega_c^2 - 2i(\omega_c^2/\gamma)\gamma\omega}{(\omega + i\omega_c^2/\gamma)^2 - \omega_c^2},$$

$$\theta_{xx} = \frac{s}{\rho}\frac{(\omega_c^2/\gamma)^2 + \omega_c^2 - i(\omega_c^2/\gamma)\omega}{(\omega + i\omega_c^2/\gamma)^2 - \omega_c^2}, \qquad \theta_{xy} = -\frac{B}{T}\frac{i\omega}{(\omega + i\omega_c^2/\gamma) - \omega_c^2},$$

$$\kappa_{xx} = s\frac{i\omega - \omega_c^2/\gamma}{(\omega + i\omega_c^2/\gamma) - \omega_c^2}, \qquad \kappa_{xy} = s\frac{\omega_c}{(\omega + i\omega_c^2/\gamma) - \omega_c^2}.$$

Byproduct: Nontrivial Self Duality

Under the interchange of densities of electrons and vortices $\rho \leftrightarrow B$, and the reverse of the universal conductivity $\sigma_Q \leftrightarrow 1/\sigma_Q$, the cyclotron resonance ω_c remains unchanged whlie $\gamma \leftrightarrow \gamma_v \equiv \omega_c^2/\gamma$.

A surprising self-duality was encoded in the thermoelectric response coefficients

Particle-Vortex Duality

$$\sigma_{xx}, \sigma_{xy}, \alpha_{xx}, \alpha_{xy}, \bar{\kappa}_{xx}, \bar{\kappa}_{xy}$$
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$$\rho_{xx}, -\rho_{xy}, -\theta_{xy}, -\theta_{xx}, \kappa_{xx}, -\kappa_{xy},$$

which can be interpreted as a bulk EM duality in its AdS correspondence (Herzog *et al.*, PRD, **75**, 8 (2009)).

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Nernst Signal

In the presence of momentum relaxation (characterized by τ), the measured Nernst signal corresponds to the transverse component

$$e_N \equiv \theta_{yx} = \frac{k_B}{2e}\frac{\varepsilon + P}{k_B T \rho}\left[\frac{\omega_c/\tau}{(\omega_c^2/\gamma + 1/\tau) + \omega_c^2}\right].$$

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Using the previous study of temperature dependence of thermodynamics quantities near the critical point (Chubukov *et al.*, PRB, **49**, 17 (1994) and Sachdev (1994)) that

$$\varepsilon = k_B T\left(\frac{k_B T}{\hbar v}\right)^2 \Phi_\varepsilon, \quad P = k_B T\left(\frac{k_B T}{\hbar v}\right)^2 \Phi_P,$$

where Φ_ε and Φ_P are dimensionless numbers, and sum of them can be estimated with ε-expansion (Damle&Sachdev, PRB, **56**, 8714 (1997))

$$\Phi_s + \Phi_P = 4\pi^2/45 + \mathcal{O}(3-d).$$

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- Taking a typical scatter time $\tau \sim 10^{-12}\mathrm{s}$ (from simply Drude formula), the velocity is estimated to be $\hbar v = 47\mathrm{meV}\cdot\text{Å}$. This value is in accordance with previous experimental ranges (for example, Balasky *et al.*, Science, **284**, 5417 (1999)).
- Working in full relativistic regime where $s \sim T^2$ and $\varepsilon + P \sim T^3$, and inserting $\rho \equiv (x - x_c)/(2a^2) = -0.025/a^2$, we have the temperature-dependent cyclotron frequency

$$\omega_c = 6.2\,\mathrm{GHz}\cdot\frac{B}{1\mathrm{T}}\cdot\left(\frac{35\,\mathrm{K}}{T}\right)^3.$$

Comparison

Finally, we can plot the phase diagram of e_N as a function of temperature T and magnetic field B and compare that with experiments

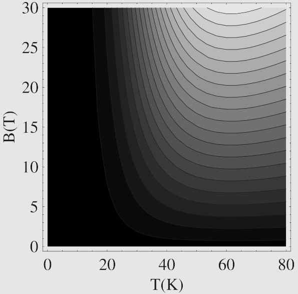

Figure: Estimated $e_N(B,T)$ for $x = 0.12$ and $T > T_c \approx 30K$. The strengh ranges up to $10\mu\mathrm{V}/\mathrm{K}$. Extracted from Hartnoll *et al.*, PRB, **76**, 144502 (2007).

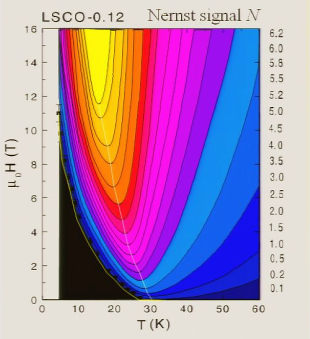

Figure: Measured $e_N(B,T)$. Converted from Fig (7) of Wang *et al.*, PRB, **73**, 024510 (2006). The unit is also $\mu\mathrm{V}/\mathrm{K}$.

The End?

Figure: When Landau teach his Jackass students about QM

Contents

Chiral Anomaly

Anomaly

Anomaly is the phenomenon of the non-conservation of a classical symmetry in the quantum limit.

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Anomaly is the phenomenon of the non-conservation of a classical symmetry in the quantum limit.

Consider applying $\boldsymbol{B} = B\hat{z}$ to the single RH (3+1)-D Weyl fermion $H = \hbar v_F \boldsymbol{\sigma} \cdot \boldsymbol{p}$

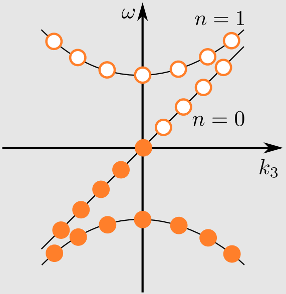

Figure: Spectrum of Landau level when $\boldsymbol{E} \parallel \boldsymbol{B}$ is applied — breakdown of charge conservation (Nielsen&Ninomiya, Phys. Lett. B, **130**, 6 (1983))

Effects on Classical Hydrodynamics

When chiral anomaly is present, the equation of macroscopic charge conservation should be replaced by

$$\partial_\mu \langle J^\mu \rangle = -\frac{C}{8}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = C\boldsymbol{E}\cdot\boldsymbol{B}, \quad C = \frac{\pm 1}{4\pi^2}.$$

If we follow the previous steps working out the constitutive relation for J^μ and $T_{\mu\nu}$, a new row of data start to emerge due to the **breakdown of parity-inversion symmetry** (Bhattacharya *et al.*, JHEP, **05**, 147 (2014)).

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	All Data	EOM	Independent Data
Pseudo-	$\omega^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$		ω^μ
vectors	$B^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$		B^μ

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So after entropy production argument (Son&Surówka, PRL, **103**, 191601 (2009))

$$J^\mu = \rho u^\mu - \sigma_Q T P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \sigma_Q E^\mu + \xi\omega^\mu + \xi_B B^\mu$$

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{\mu^3\rho}{\varepsilon + P}\right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{\rho\mu^2}{\varepsilon + P}\right).$$

Gravitational Anomaly

Motivated by Luttinger's pioneer work on thermal transport (Luttinger PR, **135**, A1505 (1964)), thermal gradient can be brought in through the gravitation fields. So a curved space with Riemann curvature $R_{\alpha\beta\mu\nu}$ should also be taken into account.

Using Fujikawa's (1979) technique, the full conservation laws are (Alvarez-Gaumé&Witten, Nuclear Phys. B, **234**, 2 (1984))

Conservation Law

$$\nabla_\mu J^\mu = -\frac{C}{8}\varepsilon^{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{G}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}R^\alpha_{\beta\mu\nu}R^\beta_{\alpha\rho\sigma},$$

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu}J_\mu - \frac{G}{16\pi^2}\nabla_\mu\left(\varepsilon^{\rho\sigma\alpha\beta}F_{\rho\sigma}R^{\nu\mu}_{\alpha\beta}\right), \quad G = \frac{\pm 1}{24}$$

Coefficent G is intrinsic that even in flat space (as is done in experiments), the charge current will displays some dependence on that!

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A naive guess is that we can obtain the transport properties for one chiral fluid, and then add them together to get the net conductivities.But **this is NOT TRUE** for periodic system since

$$0 = \int \mathrm{d}^3\boldsymbol{x}\,\partial_\mu J^\mu \neq \int \mathrm{d}^3\boldsymbol{x}\,C\boldsymbol{E}\cdot\boldsymbol{B}.$$

Failure on Single Weyl Node

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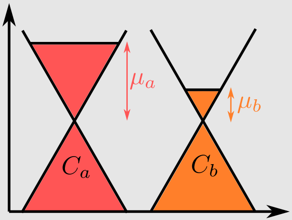
On the lattice where Nielsen-Ninomiya theorem applies

$$0 = \int \mathrm{d}^3\boldsymbol{x}\,\partial_\mu J^\mu_{\text{total}} = \int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{E}\cdot\boldsymbol{B}\sum_a C_a = 0$$

- Expect balance for a finite DC conductivity: UV Physics of intervalley scattering (Son&Spivak, PRB, **88**, 104412 (2013))
- Similar argument for the heat current $\partial_i Q^i_a = 2G_a T\nabla T\cdot\boldsymbol{B}$.



Hydrodynamic EOM with Intervalley Scattering

In the proper reference frame, the charge current has the form of (Lucas *et al.*, PNAS, **113**, 34 (2016))

$$\boldsymbol{J}_a = (\boldsymbol{J}_a)_{\text{non-chiral}} + \mathcal{D}_1 \nabla \times \boldsymbol{v} + \frac{D_2}{2}\boldsymbol{B}, \quad T_a^{\mu\nu} = (T_a^{\mu\nu})_{\text{non-chiral}}.$$

$$\mathcal{D}_1 = \frac{C\mu^2}{2}\left(1 - \frac{2}{3}\frac{\rho\mu}{\varepsilon + P}\right) - \frac{4G\mu\rho T^2}{\varepsilon + P}, \quad \mathcal{D}_2 = C\mu\left(1 - \frac{1}{2}\frac{\rho\mu}{\varepsilon + P}\right) - \frac{GT^2\rho}{\varepsilon + P}.$$

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Phenomenological Hydrodynanic EOM

$$\partial_\mu J_a^\mu = -\sum_b \left(\mathcal{R}_{ab}\nu_b + \mathcal{S}_{ab}\beta_b\right) \qquad \text{(charge)}$$

$$\partial_\mu T_a^{\mu 0} = \sum_b \left(\mathcal{U}_{ab}\nu_b + \mathcal{V}_{ab}\beta_b\right) \qquad \text{(energy)}$$

where $\nu_a \equiv \mu_a/T_a$, $\beta_a \equiv 1/T_a$, and

$$\sum_b \mathcal{R}_{ab} = \sum_b \mathcal{S}_{ab} = \sum_b \mathcal{U}_{ab} = \sum_b \mathcal{V}_{ab} \equiv 0.$$

Conductivity Matrix

After solving the linearized hydrodynamic EOM and reading out the conductivity (where $\boldsymbol{B} = B\hat{z}$), one gets (Lucas *et al.*, PNAS, **113**, 34 (2016))

$$\sigma_{ij} = \begin{pmatrix} \sum_a \dfrac{\rho_a^2 \Gamma_a}{\Gamma_a^2 + B^2 \rho_a^2} & \sum_a \dfrac{\rho_a^3 B}{\Gamma_a^2 + B^2 \rho_a^2} & \\ \sum_a \dfrac{-\rho_a^3 B}{\Gamma_a^2 + B^2 \rho_a^2} & \sum_a \dfrac{\rho_a^2 \Gamma_a}{\Gamma_a^2 + B^2 \rho_a^2} & \\ & & \sum_a \dfrac{\rho_a^2}{\Gamma_a} + \mathfrak{s} B^2 \end{pmatrix},$$

where Γ_a **represents the momentum relaxation rate** and

$$\mathfrak{s} \equiv T \begin{pmatrix} C_a & \mu C_a \end{pmatrix} \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} C_b \\ \mu C_b \end{pmatrix}.$$

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Drude Physics

$$\sigma = i\mathcal{D}/\pi(\omega + i\Gamma)$$

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Electrical Negative Magnetoresistance

$$\sigma_{ij} \equiv \sigma_{ij}^{\text{Drude}}(B) + \mathfrak{s}B_iB_j.$$

Thermoelectric Response

Thermalelectric Negative Magnetoresistance

$$\alpha_{ij} \equiv \alpha_{ij}^{\text{Drude}} + \mathfrak{a} B_i B_j, \quad \bar{\kappa} \equiv \bar{\kappa}_{ij}^{\text{Drude}} + \mathfrak{b} B_i B_j$$

where

$$\mathfrak{a} = 2T^2 \begin{pmatrix} 0 \\ G_a \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} C_b \\ \mu C_b \end{pmatrix},$$

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All anomalous terms for α_{ij} and $\bar{\kappa}_{ij}$ vanish if $G_a = 0$. So **observation of thermolelectric negative magneotoresistance is a manifestation of gravitational anomaly!**

Experimental Results

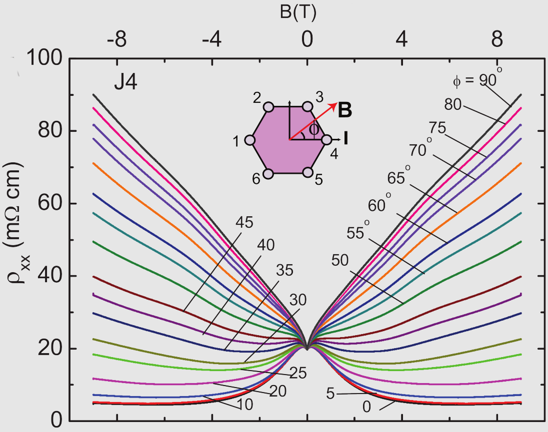


Figure: Evidence for the axial current in $\mathrm{Na_3Bi}$. Extraced from Xiong *et al.*, Science, **350**, 6259 (2015).

Experimental Results

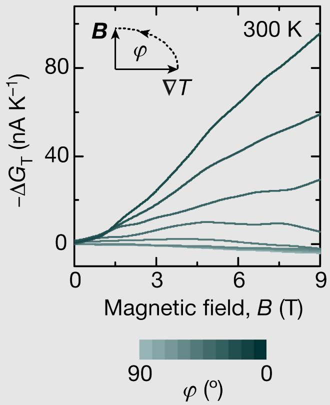
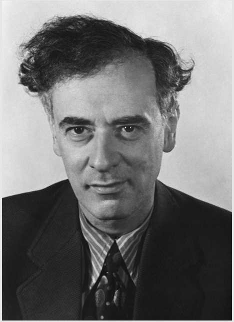


Figure: Evidence for the axial-gravitational anomaly in NbP. Extracted from Gooth *et al.*, Nature, **547**, 7663 (2017).

Contents

Failure of Landau's Paradigm



- **Landau's Fermi Liquid Theory**
  - Quasiparticle with mass etc. dressed by interactions;
  - One-to-one correspondence between Fermi liquid and Fermi gas;
- **Ginzberg-Landau Theory**
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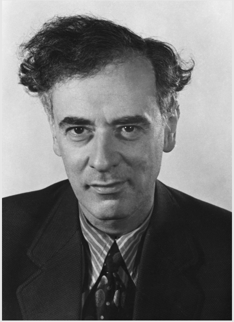

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SPT: Haldane Chain, IQHE, TI, etc.

- Protected by symmetry
- Anomalous edges
- No topological order
- Short-range entanglement

SET: Spin Liquids, FQHE, etc

- Topological ordered
- Bulk anyonic excitation
- Long-range entanglement

Intrinsic Anomlous Hall Effect

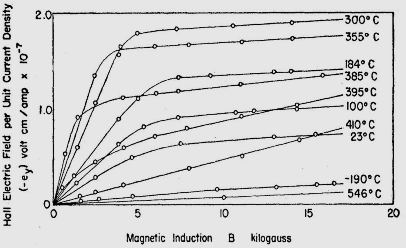

Figure: Saturate dependence of Hall Resistivity on the perpendicular magnetic field. Extracted from Paugh&Rostoker, RMP, **25**, 151 (1953)

Following KL theory (Karplus&Luttinger, PR, **95**, 5 (1954)) where anomalous group velocity is introduced due to SOC, the topological nature of AHE was revealed (Haldane, PRL, **93**, 206602 (2004)) that

$$\sigma_0^{ab}(\mu) = \frac{e^2}{\hbar}\frac{1}{\Omega N}\sum_{\boldsymbol{k}n}\mathcal{F}_n^{ab} f_n^0(\boldsymbol{k},\mu).$$

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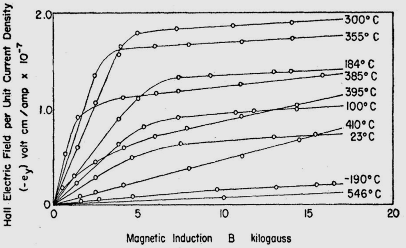

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AHE in Weyl Semimetal

- All transverse transport coefficient of Hydrodynamic theory with anomalies vanish when $B \to 0$

$$\sigma_{xy} = \sum_a \frac{B\rho_a^3}{\Gamma_a^2 + B^2\rho_a^2}, \quad \alpha_{xy} = \sum_a \frac{B\rho_a^2 s_a}{\Gamma_a^2 + B^2\rho_a^2}, \quad \bar{\kappa}_{xy} = \sum_a \frac{BT\rho_a s_a^2}{\Gamma_a^2 + B^2\rho_a^2}.$$

- But in principle, ther would be non-vanishing AHE in Weyl semimetal if the symmetry of Weyl nodes are low (Yang, Lu, and Ran, PRB, **84**, 075129 (2011))

$$\sigma_{ij} = \frac{e^2}{2\pi\hbar}\varepsilon_{ijk}\nu_k, \quad \boldsymbol{\nu} = \sum_i (-1)^{\pm 1}\boldsymbol{p}_i.$$

So there must be some intrinsic topological terms we missed in the hydrodynamic EOM (even with anomalies)!

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Further Steps

- Guessing the missing term in general hydrodynamic EOM (without relativistic structure).
- Obtaining the topological correction to all thermoelectric coefficients.
- Going to nonlinear level to derive more transport properties, and comparing with the non-trivial reciprocal relation founded by Xu and Ran (to be published)

$$\frac{\partial \sigma_{ab}}{\partial A_c} + \frac{\partial \sigma_{bc}}{\partial A_a} + \frac{\partial \sigma_{ca}}{\partial A_b} \equiv 0, \quad a, b, c \text{ stands for eletrical or thermal.}$$

- Studying more realistic experiments with lower lattice symmetries (more terms will emerge).
- Finding out the origin of such topological term.
- $\cdots$

Acknowlegement

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Collaborator: Xu Yang
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Thanks!