CS388L Introduction to Mathematical Logic

Contents

1 Part 1

1.1 Propositional Formulas: Syntax

The alphabet of propositional logic includes the propositional connectives

$$\top \quad \bot \quad \neg \quad \wedge \quad \vee \quad \to$$

parentheses

$$(\quad)$$

and other symbols, called *atoms*. We assume that atoms are different from the propositional connectives and parentheses, and that there are finitely many of them. In example, we will assume that p, q, r are atoms.

By a *string* we understand a finite string of symbols in this alphabet. We define when a string is a *(propositional) formula* recursively, as follows:

- every atom is a formula
- $\top$ and $\bot$ are formulas
- if F is a formula, then $\neg F$ is a formula
- if F and G are formulas and $\odot$ is one of the binary connectives $\wedge, \vee, \to$ then $(F \odot G)$ is a formula.

Properties of formulas can often be proved by structural induction. In such a proof, we check that all atoms and the 0-place connectives $\top, \bot$ have the property P that we would like to establish, and that this property is preserved when a new formula is formed using a unary or binary connective. More precisely, we show that

- every atom has property P.
- $\top$ and $\bot$ have property P.
- if a formula F has property P, then so does $\neg F$.
- for any binary connective $\odot$, if formulas F and G have property P then so does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

For instance, we can use structural induction to prove that a binary connective never occurs at the very end of a formula, as follows. Atoms, $\top$ and $\bot$ don't contain binary connectives at all. If the last character of F is not a binary connective then the last character of $\neg F$ is not a binary connective. The last character of $(F \odot G)$ is not a binary connective.

Problem 1 A formula cannot contain two binary connectives next to each other. True or false? Prove by *structural induction*. Define property P as above.

- There is no connectives in atoms. Thus every atom has property P.
- There is no connectives in $\top$ and $\bot$. Thus $\top$ and $\bot$ has property P.
- $\neg$ is an unitary connective. Thus if a formula F has property P. Then so does $\neg F$.
- A formula neither stars nor ends with a binary connective. Thus if formulas F and G don't contain two binary connective next to each other. So does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

Problem 2 If a formula contains more than one character then its last character is not an atom. True or false? False. Consider $\neg p$.
How about the first character? True. Prove by *structural induction*.

Proof. Define property P as above.

- An atom has only one character.
- $\top$ and $\bot$ has only one character as well.
- $\neg F$ does not start with an atom.
- $(F \odot G)$ does not start with an atom as well.

Then we can conclude that property P holds for all formulas. □

We will abbreviate formulas of the form $(F \odot G)$ by dropping the outermost parentheses in them. For any formulas $F_1, F_2, \ldots, F_n (n > 2)$

$$F_1 \wedge F_2 \wedge \cdots \wedge F_n$$

will stand for

$$(\cdots(F_1 \wedge F_2) \wedge \cdots \wedge F_n).$$

The abbreviation $F_1 \vee F_2 \vee \cdots \vee F_n$ will be understood in a similar way. The expression $F \leftrightarrow G$ will be used as shorthand for

$$(F \to G) \wedge (G \to F).$$

1.2 Propositional Formulas: Semantics

The symbols f and t are called *truth values*. A *(propositional) interpretation*, or a *truth assignment*, is a function that maps atoms to truth values.

For any formula F and any interpretation I, the truth value F^I that is *assigned* to F by I is defined recursively, as follows:

- for any atom F, $F^I = I(F)$,
- $\top^I = t$, $\bot^I = f$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$.

If $F^I = t$ then we say that the interpretation I *satisfies* F and write $I \models F$.

Problem 3 For any formulas $F_1, \ldots, F_n$ $(n \geq 1)$ and any interpretation I,

$$\begin{aligned} (F_1 \wedge \cdots \wedge F_n)^I &= t \text{ iff } F_1^I = \cdots = F_n^I &= t \\ (F_1 \vee \cdots \vee F_n)^I &= f \text{ iff } F_1^I = \cdots = F_n^I &= f \end{aligned}$$

Here we only prove the first rule and the second one will be similar.
Define: $G_n = F_1 \wedge \cdots \wedge F_n$.
• When $n = 1$, $G_1^I = F_1^I$, thus $G_1^I = t \iff F_1^I = t$
• Let $k \in N$ and suppose $G_k^I = t \iff F_1^I = \cdots = F_k^I = t$, then

$$\begin{aligned} G_{k+1}^I = (G_k \wedge F_{k+1})^I &= \wedge\left(G_k^I, F_{k+1}^I\right) \\ G_{k+1}^I = t \quad &\iff \quad G_k^I = t \;\; \& \;\; F_{k+1}^I = t \\ G_{k+1}^I = t \quad &\iff \quad F_1^I = \cdots = F_k^I = F_{k+1}^I = t \end{aligned}$$

Thus, the induction hypothesis holds true for $n = k + 1$.
• By the principle of induction, $G_n^I = t \iff F_1^I = \cdots = F_n^I = t$, which is

$$(F_1 \wedge \cdots \wedge F_n)^I = t \text{ iff } F_1^I = \cdots = F_n^I = t$$

Problem 4 For any interpretation I, there exists a formula F such that I is the only interpretation satisfying F. [Existence + Uniqueness]

Proof. For atom $p_i : 1 \leq i \leq n$, define $L_i = \begin{cases} p_i & \text{if } p_i^I = t \\ \neg p_i & \text{if } p_i^I = f \end{cases}$.
Thus, by **problem 3**, I satisfies F where $F = L_1 \wedge \cdots \wedge L_n$,
For any other interpretation I'. There must $\exists k \in N$ that $p_k^I = \neg p_k^{I'}$. Thus, $L_k^{I'} = f$, $F^{I'} = f$.
So I' doesn't satisfy F. □

Problem 5 For any set S of interpretations there exists a formula F such that for all interpretation I, $I \models F$ iff $I \in S$.
Student Solution:
If set S is empty, $F = \bot$. Else, let $S = \{I_i : 1 \leq i \leq n\}$ be a set of interpretation. For each I_i, let F_i, s.t. I_i is the only interpretation satisfying F_i (by Problem 4).
If define $F = (F_1 \vee \cdots \vee F_n)$, since $F_i^{I_i} = t, F^{I_i} = t$. Thus, $I \in S \Rightarrow I \vdash F$.
Now, suppose $I' \notin S$, this is to say, $\forall i, F_i^{I'} = f$. Thus, $I \notin S \Rightarrow I \not\models F$, this is the same as $I \models F \Rightarrow I \in S$. Thus, $I \in S \iff I \vdash F$.

Proof. $I \models F$
$\iff$ {Construction of F }
$I \models F_1 \vee \cdots \vee F_n$
$\iff$ {by Problem 3}
for some i, $I \models F_i$
$\iff$ {by Problem 4, choice of F_i}
for some i, $I = I_i$
$\iff$ {set notation }
$I \in \{I_i : 1 \leq i \leq n\}$

□

1.3 Tautologies and Equivalence

A propositional formula F is a *tautology* if every interpretation satisfies F.

Problem 6 Determine which of the formulas

$$(p \to q) \vee (q \to p)$$
$$((p \to q) \to p) \to p$$
$$((p \to q) \to r) \to ((p \to q) \to (p \to r))$$

are tautologies.

p	q	1	2
f	f	t	t
f	t	t	t
t	f	t	t
t	t	t	t

A formula F is **equivalent** to a formula G (symbolically, $F \sim G$) if, for every interpretation I, $F^I = G^I$.

Problem 7 (a) We know that conjunction and disjunction are associative:

$$(F \wedge G) \wedge H \sim F \wedge (G \wedge H)$$
$$(F \vee G) \vee H \sim F \vee (G \vee H)$$

Determine whether equivalence has a similar property:

$$(F \leftrightarrow G) \leftrightarrow H \sim F \leftrightarrow (G \leftrightarrow H)$$

Yes, check the corresponding truth table.

(b) We know that implication distributes over conjunction:

$$F \rightarrow (G \wedge H) \sim (F \rightarrow G) \wedge (F \rightarrow H)$$

Find a similar transformation for $(F \vee G) \rightarrow H$.

$$\begin{aligned}(F \vee G) \rightarrow H &\sim \neg H \rightarrow \neg(F \vee G)\\ &\sim \neg H \rightarrow (\neg F \wedge \neg G)\\ &\sim (\neg H \rightarrow \neg F) \wedge (\neg H \rightarrow \neg G)\\ &\sim (F \rightarrow H) \wedge (G \rightarrow H)\end{aligned}$$

Problem 8 We know that conjunction distributes over disjunction and that disjunction distributes over conjunction:

$$F \wedge (G \vee H) \sim (F \wedge G) \vee (F \wedge H)$$
$$F \vee (G \wedge H) \sim (F \vee G) \wedge (F \vee H)$$

Do these connectives distribute over equivalence?

$$F \wedge (G \leftrightarrow H) \not\sim (F \wedge G) \leftrightarrow (F \wedge H)$$
$$F \vee (G \leftrightarrow H) \sim (F \vee G) \leftrightarrow (F \vee H)$$

Problem 9 De Morgan's laws

$$\neg(F \wedge G) \sim \neg F \vee \neg G$$
$$\neg(F \vee G) \sim \neg F \wedge \neg G$$

show how to transform a formula of the form $\neg(F \odot G)$ when $\odot$ is $\wedge$ or $\vee$. Find similar transformations for the cases when $\odot$ is $\rightarrow$ or $\leftrightarrow$. (use truth table)

$$\neg(F \rightarrow G) \sim F \wedge \neg G$$
$$\neg(F \leftrightarrow G) \sim \begin{cases} F \leftrightarrow \neg G \\ \neg F \leftrightarrow G \end{cases}$$

Problem 10 To simplify a formula means to find an equivalent formula that is shorter. Simplify the formulas:

$$F \vee (F \wedge G) \sim F$$
$$F \wedge (F \vee G) \sim F$$
$$F \vee (\neg F \wedge G) \sim F \vee G$$

1.4 Quiz 1

A propositional formula contains 100 occurrences of atoms and zero-place connectives. How many occurrence of binary connectives does it have?

It has 99 occurrence of binary connectives.
Claim: A propositional formula with n occurrences of atoms and zero-place connectives has $n-1$ occurrences of binary connectives.

Proof.

- For every atom, it contains 1 occurrence of atom and 0 occurrences of binary connective.
- For $\top$ and $\bot$, it contains 1 occurrence of atom and 0 occurrences of binary connective.
- If a formula F with n occurrences of atoms and zero-place connectives has $n-1$ occurrences of binary connectives, then $\neg F$ has n occurrences of atoms and zero-place connectives has $n-1$ occurrences of binary connectives. This is because that there is no additional binary connectives comparing $\neg F$ with F.
- If formula F and G has the above property, this is to say, F with n_F occurrences of atoms and zero-place connectives has $n_F - 1$ occurrences of binary connectives. G with n_G occurrences of atoms and zero-place connectives has $n_G - 1$ occurrences of binary connectives. Then for $(F \odot G)$, it contains $(n_F + n_G$ occurrences of atoms and zero-place connectives, and $(n_F - 1) + (n_G - 1) + 1 = n_F + n_G - 1$ occurrences of binary connectives. Then the above property also holds for $(F \odot G)$.

Thus, by structural induction, the property holds for any propositional formula. $\square$

2 Part 2

2.1 Adequate Sets of Connectives

Problem 11 For any formula, there exists an equivalent formula that contains no connectives other than (i) $\wedge$ and $\neg$; (ii) $\vee$ and $\neg$. (structural induction)

(i) $\wedge$ and $\neg$

- There is no connective in atoms. Thus every atom has property P.
- $\top \sim p \vee \neg p \sim \neg(p \wedge \neg p)$; $\perp \sim p \wedge \neg p$. Thus $\top$ and $\perp$ has property P.
- By induction hypothesis, there exist $F' \sim F$, that contains no connective other than $\wedge$ and $\neg$. Clearly, $\neg F \sim \neg F'$. So $\neg F$ has property P.
- $F \wedge G \sim F' \wedge G'$;
  $F \vee G \sim F' \vee G' \sim \neg(\neg F' \wedge \neg G')$; (*De Morgan's Law*)
  $F \to G \sim F' \to G' \sim \neg(F' \wedge \neg G')$;
  Thus, for any binary connective $\odot$, if formulas F and G has property P, then so does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

(ii) $\vee$ and $\neg$

- There is no connective in atoms. Thus every atom has property P.
- $\top \sim p \vee \neg p$; $\perp \sim \neg(p \vee \neg p)$. Thus $\top$ and $\perp$ has property P.
- By induction hypothesis, there exist $F' \sim F$, that contains no connective other than $\wedge$ and $\neg$. Clearly, $\neg F \sim \neg F'$. So $\neg F$ has property P.
- $F \wedge G \sim F' \wedge G' \sim \neg(\neg F' \vee \neg G')$; (*De Morgan's Law*)
  $F \vee G \sim F' \vee G'$;
  $F \to G \sim F' \to G' \sim \neg(F' \wedge \neg G') \sim \neg F' \vee G'$
  Thus, for any binary connective $\odot$, if formulas F and G has property P, then so does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

Problem 12 For any formula, there exists an equivalent formula that contains no connectives other than (i) $\to$ and $\neg$; (ii) $\to$ and $\perp$. (structural induction)

(i) $\to$ and $\neg$

- There is no connective in atoms. Thus every atom has property P.
- $\top \sim p \to p$; $\perp \sim \neg(p \to p)$. Thus $\top$ and $\perp$ has property P.
- By induction hypothesis, there exist $F' \sim F$, that contains no connective other than $\to$ and $\neg$. Clearly, $\neg F \sim \neg F'$. So $\neg F$ has property P.

- $F \wedge G \sim F' \wedge G' \sim F' \wedge \neg(\neg G') \sim \neg(F' \rightarrow \neg G')$;
  $F \vee G \sim F' \vee G' \sim \neg(\neg F' \wedge \neg G') \sim \neg F' \rightarrow G'$;
  $F \rightarrow G \sim F' \rightarrow G'$;
  Thus, for any binary connective $\odot$, if formulas F and G has property P, then so does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

(ii) $\rightarrow$ and $\perp$

- There is no connective in atoms. Thus every atom has property P.
- $\top \sim p \rightarrow p$ $(\perp \rightarrow p)$; $\perp$ is trivial. Thus $\top$ and $\perp$ has property P.
- By induction hypothesis, there exist $F' \sim F$, that contains no connective other than $\rightarrow$ and $\perp$. Clearly, $\neg F \sim \neg F' \sim F' \rightarrow \perp$. So $\neg F$ has property P.
- $F \wedge G \sim F' \wedge G' \sim \neg(F' \rightarrow \neg G') \sim [F' \rightarrow (G' \rightarrow \perp)] \rightarrow \perp$;
  $F \vee G \sim F' \vee G' \sim \neg F' \rightarrow G' \sim (F' \rightarrow \perp) \rightarrow G'$;
  $F \rightarrow G \sim F' \rightarrow G'$
  Thus, for any binary connective $\odot$, if formulas F and G has property P, then so does $(F \odot G)$.

Then we can conclude that property P holds for all formulas.

Problem 13 Any propositional formula equivalent to $\neg p$ contains $\neg$ or $\perp$.
Lemma: Any propositional formula containing neither $\neg$ nor $\perp$ under the interpretation that all the atoms are evaluated true will be interpreted as true.

Proof. Proof by structural induction.

i Atom. Trivially.

ii $\top$. $\top^I = t$.

iii $F \odot G$. Suppose $F^I = t$, $G^I = t$,

- $(F \wedge G)^I = (F^I \wedge G^I) = t$;
- $(F \vee G)^I = (F^I \vee G^I) = t$;
- $(F \rightarrow G)^I = (F^I \rightarrow G^I) = t$.

By the structural induction, the lemma follows.
However, any propositional formula equivalent to $\neg p$ can not be interpreted as true under the interpretation that all the atoms are evaluated true.
Thus, Any propositional formula equivalent to $\neg p$ contains $\neg$ or $\perp$. □

2.2 Normal Forms

A *literal* is an atom or the negation of an atom. A propositional formula is said to be *negation normal form* if

- it contains no connectives other than conjunction, disjunction, and negation, and
- every negation in it is part of a literal.

Problem 14 Any formula is equivalent to a formula in negation normal form.

Proof. Use structure induction.

- F is atom a, $\neg F = \neg a$, both F and $\neg F$ are in NNF.
- if F is $\perp$, $F \sim p \wedge \neg p$, $\neg F \sim p \vee \neg p$. Similar for the case where $F = \top$. Both F and $\neg F$ are in NNF.
- Let F and $\neg F$ have NNF. The negation of them are still NNF.
- Suppose $F, \neg F, G, \neg G$ have NNF, this is to say, we have $F \sim F', \neg F \sim F'', G \sim G', \neg G \sim G''$, where F', F'', G', G'' are in NNF.
  - $F \wedge G \sim F' \wedge G', \quad \neg(F \wedge G) \sim \neg F \vee \neg G \sim F'' \vee G''$;
  - $F \vee G \sim F' \vee G', \quad \neg(F \vee G) \sim \neg F \wedge \neg G \sim F'' \wedge G''$;
  - $F \to G \sim \neg F \vee G \sim F'' \vee G'$; $\quad \neg F \to \neg G \sim F \vee \neg G \sim F' \vee G''$.

□

A $simple\ conjunction$ is a formula of the form $L_1 \wedge \cdots \wedge L_n (n \geq 1)$, where $L_1, \ldots, L_n$ are literals. A formula is in *disjunctive normal form (DNF)* if it has the form $C_1 \vee \cdots \vee C_m (m \geq 1)$, where $C_1, \ldots C_m$ are simple conjunctions.

Problem 15 Any formula is equivalent to a formula in disjunctive normal form.

Proof. For any formula F, let all the interpretations that satisfy F are $I_1, I_2, \ldots, I_n$, and define set $S = \{I_1, I_2, \ldots, I_n\}$.
By Problem 5, there $\exists$ a formula F' such that for all interpretations I, $I \models F'$ iff $I \in S$, where $F' = F'_1 \vee \ldots \vee F'_n$, where F'_i is the formula that I_i is the only interpretation satisfying it. From problem 4, we know that F'_i is a simple conjunction of literals.

- if $I \in S$ (i.e. $I \models F$), then $I \models F'$;
- if $I \notin S$ (i.e. $I \not\models F$), then $I \not\models F'$.

For all interpretations I, $F^I = F'^I$, so F is equivalent to F'. Since F' is in the form of DNF, F is equivalent to a formula in DNF. □

A $simple\ disjunction$ is a formula of the form $L_1 \vee \cdots \vee L_n (n \geq 1)$, where $L_1, \ldots, L_n$ are literals. Simple disjunctions are also called *clauses*. A formula is in *conjunctive normal form (CNF)* if it has the form $D_1 \wedge \cdots \wedge D_m (m \geq 1)$, where $D_1, \ldots D_m$ are simple disjunctions.

Problem 16 Let F be a formula in disjunctive normal form. Show that $\neg F$ is equivalent to a formula in conjunctive normal form.

Proof. Since F is in DNF, $F = C_1 \vee \cdots \vee C_m$ $(m \geq 1)$, where $C_i = L_1 \wedge \cdots \wedge L_n$.
We have $\neg F \sim \neg C_1 \wedge \cdots \wedge \neg C_m$ $(m \geq 1)$, where $\neg C_i \sim \overline{L_1} \vee \cdots \vee \overline{L_n}$.
Define $D_i = \neg C_i$, thus, $\neg F$ has the form $D_1 \wedge \cdots \wedge D_m (m \geq 1)$, where $D_1, \ldots, D_m$ are simple disjunctions.
Thus, $\neg F$ is equivalent to a formula in conjunctive normal form. □

Problem 17 Any formula is equivalent to a formula in conjunctive normal form.

$\forall F$

$\sim \quad \{\forall I, F^I = \neg(\neg(F^I)) = (\neg\neg F)^I\}$

$\neg\neg F$

$\sim \quad$ {by Problem 15, $\neg F \sim G$ in DNF}

$\neg G$

$\sim$ {by Problem 16, $\neg G \sim H$ in CNF}

H

2.3 Satisfiability and Entailment

A *set* Γ is *satisfiable* if there exists an interpretation that satisfies all formulas in Γ, and *unsatisfiable* otherwise.

Problem 18 A set Γ of literals is satisfiable iff there is no atom A for which both A and $\neg A$ belong to Γ.

- $\rightarrow$ If a set Γ of literals is satisfiable, then there $\exists$ an interpretation I that satisfies all formulas in Γ. Clearly, for any atom A, A^I and $(\neg A)^I$ can not be satisfied at the same time. Thus, there is no atom A for which both A and $\neg A$ belong to Γ.
- $\leftarrow$ If there is no atom A for which both A and $\neg A$ belong to Γ. Thus, we can find interpretation I for any atom A based on the following rule:

$$A^I = \begin{cases} t & \text{if } A \in \Gamma \\ f & \text{if } \neg A \in \Gamma \\ f & \text{otherwise} \end{cases}$$

Thus for any literal $L \in \Gamma$, we have $I(L) = t$. Clearly, this set Γ of literal is satisfiable.

A set Γ of formulas *entails* a formula F (symbolically, $\Gamma \models F$), if every interpretation that satisfies all formulas in Γ satisfies F also.

Problem 19 For any formulas $F_1, \ldots, F_n, G$, the following conditions are equivalent:

(1) $F_1, \ldots, F_n \models G$,
(2) $(F_1 \wedge \cdots \wedge F_n) \to G$ is tautology,
(3) the set $\{F_1, \ldots, F_n, \neg G\}$ is unsatisfiable.

(1) By definition, $\forall I s.t. F_1^I = t, \ldots, F_n^I = t$, we have $G^I = t$.
(2) $(F_1 \wedge \cdots \wedge F_n) \to G$ is tautology means that $\forall I s.t. (F_1 \wedge \cdots \wedge F_n)^I = t$, we have $G^I = t$. From problem 3, we know that $(F_1 \wedge \cdots \wedge F_n)^I = t \sim F_1^I = t, \ldots, F_n^I = t$. This is (1). Thus we have $(1) \iff (2)$.
(3) the set $\{F_1, \ldots, F_n, \neg G\}$ is unsatisfiable means that there $\not\exists I s.t. F_1^I = t, \ldots, F_n^I = t, \neg G^I = t$. This is to say, $\forall I s.t. F_1^I = t, \ldots, F_n^I = t$, we have $\neg G^I = f$. Thus, $G^I = t$. This turn out to be (1). Thus we have $(1) \iff (3)$.

Thus, we have $(1) \iff (2) \iff (3)$.

2.4 Clausification

To *clausify* a formula F means to find a formula F' that may contain some new atoms, not occurring in F, such that

- F' is in conjunctive normal form,
- any interpretation satisfying F' satisfies F, and
- any interpretation satisfying F can be extended to the new atoms so that it will satisfy F'.

Here is an algorithm for clausifying a propositional formula:

```
begin
  Γ ← ∅
  while F is not CNF do
    A ← a new atom
    G ← a minimal non-literal subformula of F
    F ← the result of replacing G in F by A
    Δ ← the set of clauses of the CNF of A ↔ G
    Γ ← Γ ∪ Δ
  end
  return the conjunction of F with clauses Γ
end
```

Problem 20 (i) Apply this algorithm to the formula $p \vee \neg(q \to r)$. (ii) Determine whether this formula is equivalent to the result of its clausification.

(ii). The formula is not equivalent to the result of its clausification for the latter has new atom whose interpretation is not defined in the interpretation for the original formula.

2.5 Quiz 3

The set of *canonical* propositional formulas is defined recursively:

- every literal is a canonical formula;
- if formulas F and G are canonical then the formulas $F \vee G$ and $F \to G$ are canonical also.

Every formula is equivalent to a canonical formula. True or false?

False, $\perp$ is not equivalent to a canonical formula.

Proof. Use structural induction over the definition of canonical formula.

- every literal can not be equivalent to $\perp$;
- suppose formulas F and G are not equivalent to $\perp$. There $\exists$ an interpretation I, that $G^I = t$. Thus,

$$\begin{aligned} (F \vee G)^I = & \quad \vee(F^I, G^I) = \quad t \\ (F \to G)^I = & \quad \to (F^I, G^I) = \quad t \end{aligned}$$

  Clearly, neither $(F \vee G)$ nor $(F \to G)$ is equivalent to $\perp$.

Then we can conclude that no canonical formula is equivalent to $\perp$. $\square$

3 Part 3

3.1 Positive Programs

A *positive rule* is a propositional formula of the form $F \to G$, where F and G contain no connectives other than $\top$, $\bot$, $\wedge$, and $\vee$. The antecedent F is called the *body* of the rule, and the consequent G is called its *head*. Rules are often written with the head on the left and body on the right: $G \leftarrow F$. A rule of the form $G \leftarrow \top$ is often identified with its head G. Rules of the form $\bot \leftarrow F$ are called *constraints* and are usually written as $\leftarrow F$.

A *positive (logic) program* is a set of positive rules.

A positive rule is *flat* if (i) its head is $\bot$, or an atom, or a disjunction of several atoms, and (ii) its body is $\top$, or an atom, or a conjunction of several atoms. Any propositional formula can be transformed into an equivalent set of flat positive rules by converting it to conjunctive normal form and then rewriting each of its simple disjunctions

$$A_1 \vee \cdots A_m \vee \neg A_{m+1} \vee \cdots \vee \neg A_n$$

as the rule

$$A_1 \vee \cdots A_m \vee \leftarrow A_{m+1} \wedge \cdots \wedge A_n$$

*The expression in the head is understood as $\bot$ if $m = 0$; the expression in the body is understood as $\top$ if $m = n$.

3.2 Minimal Models

In the theory of logic programs it is customary to identify an interpretation with the set of atoms to which it assigns the value t. For instance, the interpretation that assigns t to the atom p and f to all other atoms can be viewed as the singleton $\{p\}$.

A *model* of a set Γ of formulas is an interpretation that satisfies all formulas in Γ. A model M of Γ is *minimal* if no proper subset of M is a model of Γ. For example, the program

$$\begin{gathered} p, \\ q \leftarrow r \end{gathered} \tag{1}$$

has 3 models:

$$\{p\}, \{p, q\}, \{p, q, r\};$$

only the first of them is minimal.

Problem 21 Find all models of the program

$$\begin{gathered} p \vee q, \\ q \vee r, \\ q \leftarrow p \wedge r. \end{gathered}$$

Determine which of these models are minimal.

$$\{p, q, r\}$$
$$\{p, q\}, \{q, r\}$$
$$\{q\}$$

Problem 22 Consider the positive program consisting of the rules $A \vee B$ for all pairs of distinct atoms A, B. If the total number of atoms is n, then how many models does this program have? How many of them are minimal?
Claim: A model of the above positive program must have at least $n-1$ atoms.

Proof. Suppose we have a model M with less than $n-1$ atoms. $\exists$ two distinct atoms A and B, where $A \vee B$ is one of the rules while $A \notin M, B \notin M$. Thus, $(A \vee B)^M = \vee(A^M, B^M) = f \vee f = f$. Thus, M failed to satisfy all rules. Contradiction. $\square$

From the above claim, we can see that the minimal model should have $n-1$ atoms. $C_n^{n-1} = n$. There is one more model which contains all the atoms. Thus, the total number of models is $n+1$.

Problem 23 Let Γ be a positive program such that the head of each of its rules is an atom. Show that the intersection of all models of Γ is the only minimal model of Γ.
Claim: Given a positive program Γ such that the head of each of its rules is an atom, for two model M_1, M_2, $M_1 \cap M_2$ is model as well.

Proof.

- For the head p of each rule in the form $p \leftarrow F$:
  - If $p \in M_1 \cap M_2$, then the rule is satisfied by $M_1 \cap M_2$.
  - Otherwise, suppose $p \notin M_i$, then $F^{M_i} = f$. As F contains no connectives other than $\top, \bot, \wedge$ and $\vee$, a subset of M_i doesn't satisfy F as well (Contrapositive of Problem 24). Thus, $F^{M_1 \cap M_2} = f$, the rule is also satisfied by $M_1 \cap M_2$.
- For the head p of each rule in the form $p \leftarrow \top$:
  $p \in M_1, p \in M_2$, thus $p \in M_1 \cap M_2$, the rule is satisfied under $M_1 \cap M_2$

$\square$

By induction, $\cap M_i$ is model, which is also the subset of all models. Thus, the intersection of all models of Γ is the only minimal model.

If Γ_1, Γ_2 are sets of formulas such that $\Gamma_1 \subseteq \Gamma_2$ then every model of Γ_2 is a model of Γ_1. But we cannot assert, in general, that every minimal model of Γ_2 is a minimal model of Γ_1. For instance, if we add the rule $r \leftarrow p$ to program (1) then it will get a new minimal model, $\{p, q, r\}$. In this sense, the concept of a minimal model is "nonmonotonic."

Problem 24 Let F be a formula containing no connectives other than $\top, \bot, \wedge, \vee$, and let M_1, M_2 be sets of atoms such that $M_1 \subseteq M_2$. Show that if M_1 satisfies F then so does M_2.

Proof. Use structural induction:

- If F is an atom, M_1 satisfies F then so does M_2;
- $\top, \bot$ is trivial;
- There is no $\neg$ in F, we don't need to consider this case;
- Suppose given M_1 satisfy both F and G, and $M_1 \subseteq M_2$, we have M_2 satisfy both F and G as well.
  - For $F \wedge G$, if $(F \wedge G)^{M_1} = \wedge(F^{M_1}, G^{M_1}) = t$, then $F^{M_1} = t$ and $G^{M_1} = t$. From induction hypothesis, we have $F^{M_2} = t$ and $G^{M_2} = t$. Thus, $(F \wedge G)^{M_2} = \wedge(F^{M_2}, G^{M_2}) = t$, induction hypothesis holds true.
  - For $F \vee G$, $(F \vee G)^{M_1} = \vee(F^{M_1}, G^{M_1}) = t$, then $F^{M_1} = t$ or $G^{M_1} = t$. From induction hypothesis, we have $F^{M_2} = t$ or $G^{M_2} = t$. Thus, $(F \vee G)^{M_2} = \vee(F^{M_2}, G^{M_2}) = t$,induction hypothesis holds true;
  - there is no $\rightarrow$ in F, we don't need to consider this case.

□

Problem 25 For any positive program Γ and any constraint $\leftarrow F$, an interpretation M is a **minimal** model of $\Gamma \cup \{\leftarrow F\}$ iff M is a **minimal** model of Γ and does not satisfy F.

$\leftarrow$ If M is a **minimal** model of Γ and does not satisfy F, prove that M is a **minimal** model of $\Gamma \cup \{\leftarrow F\}$.

- M is a model of $\Gamma \cup \{\leftarrow F\}$:
  Clearly, M satisfies Γ. M doesn't satisfy F, thus M satisfy $\{\leftarrow F\}$. Thus, M satisfies $\Gamma \cup \{\leftarrow F\}$.
- M is a minimal model of $\Gamma \cup \{\leftarrow F\}$:
  Suppose there is a subset M' of M which is a model for $\Gamma \cup \{\leftarrow F\}$ as well. Thus, M' must also be a model of Γ. However, we already know that M is a minimal model of Γ. Contradiction. Thus, M is a minimal model of $\Gamma \cup \{\leftarrow F\}$.

$\rightarrow$ If M is a **minimal** model of $\Gamma \cup \{\leftarrow F\}$, prove that M is a **minimal** model of Γ and does not satisfy F.

- Sicne $\{\leftarrow F\}^M = t$, $F^M = f$, thus, M does not satisfy F.
- Clearly, M must be a model of Γ.
- Suppose there is a subset M' of M which is a model for Γ. From the contrapositive of Problem 24, M' doesn't satisfy F as well. Thus, from $\leftarrow$, we know that M' is a model of $\Gamma \cup \{\leftarrow F\}$. However, we already know that M is a **minimal** model of $\Gamma \cup \{\leftarrow F\}$. Contradiction. Thus, M is a **minimal** model of Γ.

Problem 26 If M is a minimal model of a positive program Γ then every atom from M occurs in the head of one of the rules of Γ.

Proof. Suppose there is an atom p in minimal model M that doesn't occur in the head of any of the rules. Define $M' = M \backslash \{p\}$. For any rule $H \leftarrow B$ in Γ, if

- if M satisfies H, M' satisfies H as well, so M' satisfies $H \leftarrow B$.
- if M doesn't satisfy H, M' doesn't neither. Since M satisfies $H \leftarrow B$, M must don't satisfy B. From Problem 24, M' doesn't satisfy B as well. Thus, M' satisfy rule $H \leftarrow B$. Since we already know M is a minimal model. Contradiction.

□

3.3 Quiz 4

Let Γ be a positive program containing the rule $p \leftarrow q$. Show that if p doesn't occur in the heads of the other rules of Γ then every minimal model of Γ satisfies the formula $p \leftrightarrow q$.

Proof. For any minimal model M of Γ,

- If $q \in M$, since the rule $p \leftarrow q$ must be satisfied by M, then $p \in M$ as well. Clearly, M satisfies $p \leftrightarrow q$.
- If $q \notin M$, the rule $p \leftarrow q$ is satisfied. Suppose $p \in M$, define $M' = M \backslash \{p\}$. Clearly, the rule $p \leftarrow q$ is satisfied by M' as well. For all other rules $H \leftarrow B$ in Γ:
  - If M satisfies H, since p doesn't occur in H, M' satisfies H as well. Then M' satisfies this rule.
  - If M doesn't satisfy H, since M satisfies $H \leftarrow B$, M must not satisfy B. By problem 24, M', a subset of M, doesn't satisfy B as well. Thus, M' also satisfies this rule.

This is to say, M' satisfies Γ, which contradicts that M is a minimal model of Γ. Thus, our assumption that $p \in M$ must be false. This is to say, if $q \notin M$, $p \notin M$ as well. Clearly, M satisfies $p \leftrightarrow q$.

□

3.4 Application: Graph Coloring

We would like to assign one of three colors to each vertex of a graph G so that adjacent vertices will have different colors. This problem can be reduced to generating a minimal model of a positive program. The atoms in this program are expressions of the form $color(v, c)$, where v is a vertex of G and c is one of the colors $blue, red, yellow$. The program consists of the rules

$$color(v, blue) \vee color(v, red) \vee color(v, yellow) \tag{2}$$

for all vertices v, and the constrains

$$\leftarrow color(v, c) \wedge color(v', c) \tag{3}$$

for all pairs v, v' of adjacent vertices of G and all colors c.

Problem 27 A set X of atoms is a minimal model of program (13), (3) iff

- for every vertex v there is exactly one color c such that $color(v, c) \in X$, and
- for every pair of atoms $color(v, c), color(v', c')$ from X, if v is adjacent to v', then $c \neq c'$.

Proof. X of atoms is a minimal model of program (2), (3)
$\iff$ { by problem 25}

- X is a minimal model of (2) for all vertex v, and
- X does not satisfy $color(v, c) \wedge color(v', c)$ for all pairs v, v' of adjacent vertices of G and all colors c.

$\iff$ { by definition}

- for every vertex v there is exactly one color c such that $color(v, c) \in X$, and
- for every pair of atoms $color(v, c), color(v', c')$ from X, if v is adjacent to v' then $c \neq c'$.

□

4 Part 4

4.1 Logic Programs and Stable Models

In this part of the course, a *rule* is a propositional formula of the form $F \to G$ where F and G contain no connectives other than $\top, \bot, \wedge, \vee$, and $\neg$ (that is to say, $F \to G$ has no implications other than the one explicitly shown). The notational conventions and terminology related to positive rules that are introduced in the first paragraph of Part 3 of these lecture notes apply to these more general rules as well. A *(logic) program* is a set of rules. A rule is *flat* if (i) its head is $\bot$, or a literal, or a disjunction of several literals, and (ii) it body is $\top$, or a literal, or a conjunction of several literals.

For any rule R and any set M of atoms, the *reduct* of R with respect to M is the positive rule obtained from R by replacing each maximal subformula of the form $\neg F$ with $\top$ if M satisfies that subformula, and with $\bot$ otherwise. For instance, the reduct of the rule

$$p \leftarrow q \wedge \neg(p \wedge \neg r)$$

with respect to {p} is

$$p \leftarrow q \wedge \bot$$

Indeed, $\{p\}$ satisfies the formula $p \wedge \neg r$, and consequently doesn't satisfy its negation; in the reduct, its negation is replaced with $\bot$. The reduct of the same rule with respect to $\{q\}$ is

$$p \leftarrow q \wedge \top$$

because $\{q\}$ doesn't satisfy the formula $p \wedge \neg r$, and consequently satisfies its negation; in the reduct, its negation is replaced with $\top$.

The *reduct* of a program Γ with respect to M is the positive program consisting of the reducts of the rules of Γ with respect to M.

A set M of atoms is a *stable model* of a logic program Γ if M is a minimal model of the reduct of Γ with respect to M. The use of the term "stable model of Γ" is justified by the fact that every stable model of Γ in the sense of this definition is a model of Γ. Indeed, the reduct Γ' of Γ with respect to M is obtained from Γ by replacing some subformulas not satisfied by M with $\bot$; consequently M is a model of the reduct iff M is a model of Γ.

For instance, $\{p, q\}$ is a stable model of the program

$$\begin{aligned} p &\leftarrow q, \\ q &\leftarrow \neg r \end{aligned} \tag{4}$$

because the reduct of this program with respect to $\{p, q\}$ is the positive program

$$\begin{aligned} p &\leftarrow q, \\ &q \end{aligned}$$

and $\{p, q\}$ is a minimal model of this positive program.

Problem 28 Does program (4) have stable models other than $\{p, q\}$?
Let's check all possible combinations:

- $\emptyset$, this is not a model of the original program Γ.
- $\{p\}$, this is not a model of the original program Γ.
- $\{q\}$, this is not a model of the original program Γ.
- $\{r\}$, the reduct of Γ with respect to $\{r\}$ is:

$$p \leftarrow q,$$
$$q \leftarrow \bot$$

  the minimal model of the reduct is $\emptyset$. Thus, $\{r\}$ is not a stable model of program Γ.

- $\{p, q\}$, the reduct of Γ with respect to $\{p, q\}$ is

$$p \leftarrow q,$$
$$q$$

  the minimal model of the reduct is $\{p, q\}$. Thus, $\{p, q\}$ is a stable model of program Γ.

- $\{p, r\}$, the reduct of Γ with respect to $\{p, r\}$ is

$$p \leftarrow q,$$
$$q \leftarrow \bot$$

  the minimal model of the reduct is $\emptyset$. Thus, $\{p, r\}$ is not a stable model of program Γ.

- $\{q, r\}$, this is not a model of the original program Γ.
- $\{p, q, r\}$, the reduct of Γ with respect to $\{p, q, r\}$ is

$$p \leftarrow q,$$
$$q \leftarrow \bot$$

  the minimal model of the reduct is $\emptyset$. Thus, $\{p, q, r\}$ is not a stable model of program Γ.

Thus, program (4) don't have any stable model other than $\{p, q\}$.

Problem 29 For each of the one-rule programs

$$p \leftarrow \neg q$$

and

$$p \leftarrow \neg\neg p$$

find its minimal models and its stable models.

$p \leftarrow \neg q$	$p \leftarrow \neg\neg p$
Minimal model: $\{q\}$ and $\{p\}$	Minimal model: $\emptyset$
Stable model: $\{p\}$	Stable model: $\emptyset$ and $\{p\}$.

The answer to the last program shows that minimal models of a program are not necessarily stable, and stable models are not necessarily minimal. **But if a program is positive then its stable models are identical to its minimal models.**

The comparison of the one-rule program $p \vee q$ and $p \leftarrow \neg q$ shows that two equivalent programs may have different stable models.

A *choice rule* is a rule with the head of the form $A \vee \neg A$, where A is an atom. In the next problem we consider programs consisting of simple choice rules.

Problem 30 Find all stable models of the programs

(a) $p \vee \neg p$,

(b) $p_i \vee \neg p_i, \qquad (i = 1, \ldots, n)$

(a)
- $\emptyset$ The corresponding reduct is $p \vee \top$, thus the stable model is $\emptyset$.
- $\{p\}$ The corresponding reduct is $p \vee \bot$, thus the stable model is $\{p\}$.

(b) According to (a), either $\emptyset$ or $\{p_i\}$ is a stable model of each rule. And each rule is independent. Thus, any subset of models will be a stable model. There is 2^n of them.

Problem 31 (a) Find a stable model of the programs

$$\begin{aligned} p &\leftarrow \neg q \\ q &\leftarrow \neg r \end{aligned} \tag{5}$$

(b) Find a stable model of the program

$$p_{i+1} \leftarrow \neg p_i \qquad (i = 1, \ldots, 100).$$

(a) Check each combinations, stable model: $\{q\}$

(b) Follow the same pattern as step (a), we only need to find such a model.

$$\{p_2, p_4, \ldots, p_{98}, p_{100}\}$$

Problem 32 (a) Find all stable model of the programs

$$p \leftarrow \neg p$$

and

$$\begin{aligned} p &\leftarrow \neg q, \\ q &\leftarrow \neg p. \end{aligned}$$

For $\quad p \leftarrow \neg p$

- $\emptyset$, it is not a model of the above program.
- $\{p\}$, the corresponding reduct is $p \leftarrow \bot$, whose minimal model is $\emptyset$.

For $\begin{aligned} p &\leftarrow \neg q, \\ q &\leftarrow \neg p. \end{aligned}$

- $\emptyset$, it is not a model of the above program.
- $\{p\}$, the corresponding reduct is $\begin{aligned} p &\leftarrow \top, \\ q &\leftarrow \bot. \end{aligned}$, whose minimal model is $\{p\}$, which is the stable model of the program.
- $\{q\}$, the corresponding reduct is $\begin{aligned} p &\leftarrow \bot, \\ q &\leftarrow \top. \end{aligned}$, whose minimal model is $\{q\}$, which is the stable model of the program.
- $\{p, q\}$, the corresponding reduct is $\begin{aligned} p &\leftarrow \bot, \\ q &\leftarrow \bot. \end{aligned}$, whose minimal model is $\emptyset$.

(b) Determine how the stable models of the last program change if we add to it the rules

$$\begin{aligned} r &\leftarrow p, \\ r &\leftarrow q. \end{aligned}$$

$\{p, r\}$ and $\{q, r\}$

Problem 33 For any program Γ and any constraint $\leftarrow F$, a set M of atoms is a stable model of $\Gamma \cup \{\leftarrow F\}$ iff M is a stable model of Γ and does not satisfy F.

Proof. Prove by chain of equivalence.
M is a stable model of Γ
$\Longleftrightarrow$ (By definition of stable model)
M is a minimal model of reduct$_M(\Gamma \cup \{\leftarrow F\})$, which is a positive program.
$\Longleftrightarrow$ (By problem 25)
M is a minimal model of reduct$_M(\Gamma)$, and doesn't satisfy reduct$_M(F)$.
$\Longleftrightarrow$ (By definition of stable model)
M is a stable model of Γ, and doesn't satisfy F. $\square$

Problem 34 If M is a stable model of a program Γ then every atom from M occurs in the head of one of the rules of Γ.

Proof. For a stable model M of Γ, suppose Γ' is the reduct of Γ with respect to M.
We have M is a minimal model of Γ' where Γ' is a positive program.
According to problem 26, every atom from M occurs in the head of one of the rules of Γ'.
Since Γ' is the reduct of Γ, then every atom which occurs in the head of one of the rules of Γ' must occurs in the head of one of the rules of Γ.
Thus, every atom from M must also occur in the head of one of the rules of Γ. $\square$

For any set Γ of clauses, by Γ^* we denote the flat program consisting of

- the choice rules $A \vee \neg A$ for all atoms A occurring in Γ, and
- the constraints $\leftarrow \overline{L_1} \wedge \cdots \wedge \overline{L_n}$ for all clauses $L_1 \vee \cdots \vee L_n$ from Γ.

Problem 35 The stable models of Γ^* are identical to the models of Γ.

Proof. Let $\Gamma^* = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the set of choice rules $A \vee \neg A$, and Γ_2 is the set of constraints $\leftarrow \overline{L_1} \wedge \cdots \wedge \overline{L_n}$.
By construct, Γ and Γ^* use the same set of atoms. For $\forall$ stable model M of Γ, by the statement of problem 30, M is a stable model of Γ_1. Also, M satisfies all the clauses in the form $L_1 \vee \cdots \vee L_n$ from Γ. Clearly, M doesn't satisfy any of the constraints $\overline{L_1} \wedge \cdots \wedge \overline{L_n}$. Thus, M does not satisfy Γ_2. By the statement of problem 33, M is a stable model of Γ^*.
For $\forall$ stable model M' of Γ^*, M' doesn't satisfy any of the constraints $\overline{L_1} \wedge \cdots \wedge \overline{L_n}$. Clearly, M satisfies $L_1 \vee \cdots \vee L_n$ for all clauses in the form $L_1 \vee \cdots \vee L_n$ in Γ. Hence, M' is a stable model of Γ.
Thus, the stable models of Γ^* are identical to the models of Γ.

$\square$

5 Part 5

5.1 Natural Deduction

In this part of the course we consider, for simplicity, propositional formulas that do not contain the connective $\top$.
A *sequent* is an expression of the form

$$\Gamma \Rightarrow F \tag{6}$$

("F under assumption Γ"), where Γ is a finite set of formulas. If Γ is written as $\{G_1, \ldots, G_n\}$, we will drop the braces and write (6) as

$$G_1, \ldots, G_n \Rightarrow F \tag{7}$$

Intuitively, a sequent (7) has the same meanning as the formula

$$(G_1 \wedge \ldots \wedge G_n) \rightarrow F \tag{8}$$

(as the formula F if $n = 0$).

We define below which sequents are considered *axioms* and provide a list of *inference rules*. A *proof* is a list of sequents $S_1, \ldots, S_n$ such that each S_i is either an axiom or can be derived from some of the sequents $S_1, \ldots, S_{i-1}$ by one of the inference rules.
Axioms are sequents of the forms

$$F \Rightarrow F$$

and

$$\Rightarrow F \vee \neg F$$

("the law of the excluded middle")
Inference Rules In the list below, F, G, H are formulas, and Γ, Δ, Σ are finite sets of formulas. Most inference rules are classifed into *introduction rules* (the left column) and *elimination rules* (the right column); two exceptions are the *contradiction rule (C)* and the *weakening rule (W)*.

$$(\wedge I)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G} \qquad (\wedge E)\frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow F} \quad \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow G}$$

$$(\vee I)\frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow F \vee G} \quad \frac{\Gamma \Rightarrow G}{\Gamma \Rightarrow F \vee G} \qquad (\vee E)\frac{\Gamma \Rightarrow F \vee G \quad \Delta, F \Rightarrow H \quad \Sigma, G \Rightarrow H}{\Gamma, \Delta, \Sigma \Rightarrow H}$$

$$(\rightarrow I)\frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad (\rightarrow E)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \rightarrow G}{\Gamma, \Delta \Rightarrow G}$$

$$(\neg I)\frac{\Gamma, F \Rightarrow \bot}{\Gamma \Rightarrow \neg F} \qquad (\neg E)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow \neg F}{\Gamma, \Delta \Rightarrow \bot}$$

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F}$$

$$(W)\ \frac{\Gamma \Rightarrow H}{\Gamma, \Delta \Rightarrow H}$$

To prove a sequent S means to find a proof with the last sequent S. To prove a formula F means to prove the sequent $\Rightarrow F$. For instance, here is a proof of the formula $(p \wedge q) \to (p \vee q)$. along with its "translation into English".

$p \wedge q \Rightarrow p \wedge q$	("Assume $p \wedge q$.")
$p \wedge q \Rightarrow p$	("Then p ")
$p \wedge q \Rightarrow p \vee q$	("and consequently $p \vee q$.")
$\Rightarrow p \vee q$	("and consequently $p \vee q$.")

To clarify why a given list of sequents is a proof we will explain, next to every sequent, how its presence in the proof is justified by the axioms and inference rules. It is also convenient to introduce abbreviations (A1, A2, A3, ...) for the assumptions used in the proof:

A1.	$p \wedge q.$	
1.	$A1 \Rightarrow p \wedge q$	– axiom.
2.	$A1 \Rightarrow p$	– $(\wedge E), 1.$
3.	$A1 \Rightarrow p \vee q$	– $(\vee I), 2.$
4.	$\Rightarrow (p \wedge q) \to (p \vee q)$	– $(\to I), 3.$

Problem 36 $(p \wedge q \wedge r) \to (p \wedge r)$.

A1.	$(p \wedge q) \wedge r.$	
1.	$A1 \Rightarrow (p \wedge q) \wedge r$	– axiom.
2.	$A1 \Rightarrow p \wedge q$	– $(\wedge E), 1.$
3.	$A1 \Rightarrow r$	– $(\wedge E), 1.$
4.	$A1 \Rightarrow p$	– $(\wedge E), 2.$
5.	$A1 \Rightarrow p \wedge r$	– $(\wedge I), 3, 4.$
6.	$\Rightarrow (p \wedge q \wedge r) \to (p \wedge r)$	– $(\to I), 4.$

Problem 37 $((p \wedge q) \to r)) \to (p \to (q \to r))$.

A1.	$(p \wedge q) \to r.$	
1.	$A1 \Rightarrow (p \wedge q) \to r$	– axiom.
A2.	$p.$	
2.	$A2 \Rightarrow p$	– axiom.
A3.	$q.$	
3.	$A3 \Rightarrow q$	– axiom.
4.	$A2, A3 \Rightarrow p \wedge q$	– $(\wedge I), 2, 3.$
5.	$A1, A2, A3 \Rightarrow r$	– $(\to E), 1, 4.$
6.	$A1, A2 \Rightarrow q \to r$	– $(\to I), 5.$
7.	$A1 \Rightarrow p \to (q \to r)$	– $(\to I), 6.$
8.	$\Rightarrow ((p \wedge q) \to r)) \to (p \wedge r)$	– $(\to I), 7.$

Problem 38 $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

A1.	$p \to (q \to r).$	
A2.	$p \to q.$	
A3.	$p.$	
1.	$A1 \Rightarrow p \to (q \to r)$	– axiom.
2.	$A2 \Rightarrow p \to q$	– axiom.
3.	$A3 \Rightarrow p$	– axiom.
4.	$A1, A3 \Rightarrow q \to r$	– $(\to E), 1, 3.$
5.	$A2, A3 \Rightarrow q$	– $(\to E), 2, 3.$
6.	$A1, A2, A3 \Rightarrow r$	– $(\to E), 4, 5.$
7.	$A1, A2 \Rightarrow (p \to r)$	– $(\to I), 6.$
8.	$A1 \Rightarrow (p \to q) \to (p \to r)$	– $(\to I), 7.$
9.	$\Rightarrow (p \to (q \to r)) \to ((p \to q) \to (p \to r))$	– $(\to I), 8.$

Problem 39 $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$.

A1.	$\neg(p \vee q)$.	
A2.	p.	
1.	$A1 \Rightarrow \neg(p \vee q)$	– axiom.
2.	$A2 \Rightarrow p$	– axiom.
3.	$A2 \Rightarrow p \vee q$	– $(\vee I), 2$.
4.	$A1, A2 \Rightarrow \perp$	– $(\neg E), 1, 3$.
5.	$A1 \Rightarrow \neg q$	– $(\neg I), 4$.
A3.	q.	
6.	$A3 \Rightarrow q$	– axiom.
7.	$A3 \Rightarrow p \vee q$	– $(\vee I), 6$.
8.	$A1, A3 \Rightarrow \perp$	– $(\neg E), 1, 7$.
9.	$A1 \Rightarrow \neg q$	– $(\neg I), 8$.
10.	$A1 \Rightarrow (\neg p \wedge \neg q)$	– $(\wedge I), 5, 9$.
11.	$\Rightarrow \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	– $(\rightarrow I), 10$.
A4.	$\neg p \wedge \neg q$.	
A5.	$p \vee q$.	
A6.	p.	
12.	$A4 \Rightarrow \neg p \wedge \neg q$	– axiom.
13.	$A5 \Rightarrow p \vee q$	– axiom.
14.	$A6 \Rightarrow p$	– axiom.
15.	$A4 \Rightarrow \neg p$	– $(\wedge E), 12$.
16.	$A4, A6 \Rightarrow \perp$	– $(\neg E), 14, 15$.
A7.	q.	
17.	$A7 \Rightarrow q$	– axiom.
18.	$A4 \Rightarrow \neg q$	– $(\wedge E), 12$.
19.	$A4, A7 \Rightarrow \perp$	– $(\neg E), 17, 18$.
20.	$A4, A5 \Rightarrow \perp$	– $(\vee E), 13, 16, 19$.
21.	$A4 \Rightarrow \neg(p \vee q)$	– $(\vee E), 13, 16, 20$.
22.	$\Rightarrow (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$	– $(\rightarrow I), 21$.
23.	$\Rightarrow \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$	– $(\wedge I), 11, 22$.

Problem 40 $(p \to q) \to (\neg q \to \neg p)$.

A1.	$p \to q$.	
A2.	$\neg q$.	
1.	$A1 \Rightarrow p \to q$	– axiom.
2.	$A2 \Rightarrow \neg q$	– axiom.
3.	$p \Rightarrow p$	– axiom.
4.	$A1, p \Rightarrow q$	– $(\to E), 1, 3$.
5.	$A1, A2, p \Rightarrow \bot$	– $(\neg E), 2, 4$.
6.	$A1, A2 \Rightarrow \neg p$	– $(\neg I), 5$.
7.	$A1 \Rightarrow \neg q \to \neg p$	– $(\to I), 6$.
8.	$\Rightarrow (p \to q) \to (\neg q \to \neg p)$	– $(\to I), 7$.

Problem 41 $(p \wedge \neg p) \to q$.

A1.	$p \wedge \neg p$.	
1.	$A1 \Rightarrow p \wedge \neg p$	– axiom.
2.	$A1 \Rightarrow p$	– $(\wedge E), 1$.
3.	$A1 \Rightarrow \neg p$	– $(\wedge E), 1$.
4.	$A1 \Rightarrow \bot$	– $(\neg E), 2, 3$.
5.	$A1 \Rightarrow q$	– $(C), 4$.
6.	$\Rightarrow (p \wedge \neg p) \to q$	– $(\to I), 5$.

Problem 42 $((p \wedge q) \vee r) \to (p \vee r)$.

A1.	$(p \wedge q) \vee r$.	
1.	$A1 \Rightarrow (p \wedge q) \vee r$	– axiom.
A2.	$p \wedge q$	
2.	$A2 \Rightarrow p \wedge q$	– axiom.
3.	$A2 \Rightarrow p$	– $(\wedge E), 2$.
4.	$A2 \Rightarrow p \vee r$	– $(\vee I), 3$.
A3.	r	
5.	$A3 \Rightarrow r$	– axiom.
6.	$A3 \Rightarrow p \vee r$	– $(\vee I), 5$.
7.	$A1 \Rightarrow p \vee r$	– $(\vee E), 1, 4, 6$.
8.	$\Rightarrow ((p \wedge q) \vee r) \to (p \vee r)$	– $(\to I), 7$.

Problem 43 $p \to (q \to p)$.

1.	$p \Rightarrow p$	– axiom.
2.	$p, q \Rightarrow p$	– $(W), 1.$
3.	$p \Rightarrow q \to p$	– $(\to I), 2.$
4.	$\Rightarrow p \to (q \to p)$	– $(\to I), 3.$

Problem 44 $p \leftrightarrow \neg\neg p$.

A1.	$\neg\neg p$	
1.	$A1 \Rightarrow \neg\neg p$	– axiom.
2.	$\Rightarrow p \vee \neg p$	– axiom.
A2.	p	
3.	$A2 \Rightarrow p$	– axiom.
A3.	$\neg p$	
4.	$A3 \Rightarrow \neg p$	– axiom.
5.	$A1, A3 \Rightarrow \perp$	– $(\neg E), 4, 1.$
6.	$A1, A3 \Rightarrow p$	– $(C), 5.$
7.	$A1 \Rightarrow p$	– $(\vee E), 2, 3, 6.$
8.	$\Rightarrow \neg\neg p \to p$	– $(\to I), 7.$
9.	$A2, A3 \Rightarrow \perp$	– $(\neg E), 3, 4.$
10.	$A2 \Rightarrow \neg\neg p$	– $(\neg I), 9.$
11.	$\Rightarrow p \to \neg\neg p$	– $(\to I), 10.$
12.	$\Rightarrow p \leftrightarrow \neg\neg p$	– $(\wedge I), 8, 11.$

Problem 45 $(p \to q) \vee (q \to p)$.

1.	$\Rightarrow p \vee \neg p$	– axiom.
2.	$p \Rightarrow p$	– axiom.
3.	$p, q \Rightarrow p$	– $(W), 2.$
4.	$p \Rightarrow q \to p$	– $(\to I), 3.$
5.	$p \Rightarrow (p \to q) \vee (q \to p)$	– $(\vee I), 4.$
6.	$\neg p \Rightarrow \neg p$	– axiom.
7.	$p, \neg p \Rightarrow \perp$	– $(\neg E), 2, 6.$
8.	$p, \neg p \Rightarrow q$	– $(C), 7.$
9.	$\neg p \Rightarrow p \to q$	– $(\to I), 8.$
10.	$\neg p \Rightarrow (p \to q) \vee (q \to p)$	– $(\vee I), 9.$
11.	$\Rightarrow (p \to q) \vee (q \to p)$	– $(\vee E), 1, 5, 10.$

Problem 46 $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$.

A1.	$\neg(p \wedge q)$	
1.	$A1 \Rightarrow \neg(p \wedge q)$	– axiom.
2.	$\Rightarrow p \vee \neg p$	– axiom.
A2.	p	
3.	$A2 \Rightarrow p$	– axiom.
4.	$q \Rightarrow q$	– axiom.
5.	$A2, q \Rightarrow p \wedge q$	– $(\wedge I), 3, 4.$
6.	$A1, A2, q \Rightarrow \perp$	– $(\neg E), 5, 1.$
7.	$A1, A2 \Rightarrow \neg q$	– $(\neg I), 6.$
8.	$A1, A2 \Rightarrow \neg p \vee \neg q$	– $(\vee I), 7.$
A3.	$\neg p$	
9.	$A3 \Rightarrow \neg p$	– axiom.
10.	$A3 \Rightarrow \neg p \vee \neg q$	– $(\vee I), 9.$
11.	$A1 \Rightarrow \neg p \vee \neg q$	– $(\vee E), 2, 8, 10.$
12.	$\Rightarrow \neg(p \wedge q) \rightarrow \neg p \vee \neg q$	– $(\rightarrow I), 11.$
A4.	$\neg p \vee \neg q$	
13.	$A4 \Rightarrow \neg p \vee \neg q$	– axiom.
A5.	$\neg p$	
14.	$A5 \Rightarrow \neg p$	– axiom.
15.	$p \Rightarrow p$	– axiom.
16.	$A5, p \Rightarrow \perp$	– axiom.
17.	$A5, p, q \Rightarrow \perp$	– $(W), 16.$
18.	$A5 \Rightarrow \neg(p \wedge q)$	– $(\neg I), 17.$
A6.	$\neg q$	
19.	$A6 \Rightarrow \neg q$	– axiom.
20.	$A6, q \Rightarrow \perp$	– axiom.
21.	$A6, p, q \Rightarrow \perp$	– $(W), 20.$
22.	$A6 \Rightarrow \neg(p \wedge q)$	– $(\neg I), 21.$
23.	$A4 \Rightarrow \neg(p \wedge q)$	– $(\vee E), 13, 18, 22.$
24.	$\Rightarrow \neg p \vee \neg q \rightarrow \neg(p \wedge q)$	– $(\rightarrow I), 23.$
25.	$\Rightarrow \neg p \vee \neg q \leftrightarrow \neg(p \wedge q)$	– $(\wedge I), 12, 24.$

Problem 47 $(p \vee q) \leftrightarrow (\neg p \rightarrow q)$.

A1.	$p \vee q$	
1.	$A1 \Rightarrow p \vee q$	– axiom.
2.	$p \Rightarrow p$	– axiom.
3.	$\neg p \Rightarrow \neg p$	– axiom.
4.	$p, \neg p \Rightarrow \perp$	– $(\neg E), 2, 3.$
5.	$p, \neg p \Rightarrow q$	– $(C), 4.$
6.	$q \Rightarrow q$	– axiom.
7.	$A1, \neg p \Rightarrow q$	– $(\vee E), 1, 5, 6.$
8.	$A1 \Rightarrow \neg p \rightarrow q$	– $(\rightarrow I), 7.$
9.	$\Rightarrow (p \vee q) \rightarrow (\neg p \rightarrow q)$	– $(\rightarrow I), 8.$
A2.	$\neg p \rightarrow q$	
10.	$A2 \Rightarrow \neg p \rightarrow q$	– axiom.
11.	$A2, \neg p \Rightarrow q$	– $(\rightarrow E), 3, 10.$
12.	$A2, \neg p \Rightarrow p \vee q$	– $(\vee I), 11.$
13.	$p \Rightarrow p$	– axiom.
14.	$p \Rightarrow p \vee q$	– axiom.
15.	$\Rightarrow p \vee \neg p$	– axiom.
16.	$A2 \Rightarrow p \vee q$	– $(\vee E), 15, 12, 14.$
17.	$\Rightarrow (\neg p \rightarrow q) \rightarrow (p \vee q)$	– $(\rightarrow I), 16.$
18.	$\Rightarrow (p \vee q) \leftrightarrow (\neg p \rightarrow q)$	– $(\wedge I), 9, 17.$

5.2 Soundness and Completeness of Natural Deduction

A sequent (6) is called *tautological* if the corresponding formula (8) is a tautology (in other words, if F is entailed by the assumptions $G_1, \ldots, G_n$). An inference rule is said to be *sound* if, for any instance

$$\frac{S_1 \ \ldots \ S_k}{S}$$

of this rule such that the premises $S_1, \ldots, S_k$ are tautological, the conclusion S is tautological also.

Problem 48 Rules $(\rightarrow I)$ and $(\vee E)$ are sound.

$$\boxed{(\rightarrow I) \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G}}$$

Goal: If $(\Gamma^\wedge \wedge F) \rightarrow G$ is tautology, then $\Gamma^\wedge \rightarrow (F \rightarrow G)$ is tautology.

Proof. If $(\Gamma^\wedge \wedge F) \rightarrow G$ is tautology, then $\forall$ interpretation I, $((\Gamma^\wedge \wedge F) \rightarrow G)^I = t$.

- If $(\Gamma^\wedge)^I = f$, then $(\Gamma^\wedge \rightarrow (F \rightarrow G))^I = t$.
- If $F^I = f$, then $(F \rightarrow G)^I = t$, thus $(\Gamma^\wedge \rightarrow (F \rightarrow G))^I = t$.
- If $G^I = t$, then $(F \rightarrow G)^I = t$, thus $(\Gamma^\wedge \rightarrow (F \rightarrow G))^I = t$.

Otherwise, $(\Gamma^\wedge \wedge F) \rightarrow G$ can't be tautology.
Thus, $\forall$ interpretation I, $(\Gamma^\wedge \rightarrow (F \rightarrow G))^I = t$. □

$$\boxed{(\vee E) \qquad \frac{\Gamma \Rightarrow F \vee G \quad \Delta, F \Rightarrow H \quad \Sigma, G \Rightarrow H}{\Gamma, \Delta, \Sigma \Rightarrow H}}$$

Goal: If $\left.\begin{array}{c} \Gamma^\wedge \rightarrow (F \vee G) \\ (\Delta^\wedge \wedge F) \rightarrow H \\ (\Sigma^\wedge \wedge G) \rightarrow H \end{array}\right\}$ are tautology, then $(\Gamma^\wedge \wedge \Delta^\wedge \wedge \Sigma^\wedge) \rightarrow H$ is tautology.

Proof. From problem 19, we only need to prove that the set $\{\Gamma^\wedge, \Delta^\wedge, \Sigma^\wedge, \neg H\}$ is unsatisfiable.
Suppose $\exists I$ that the satisfies the set $\{\Gamma^\wedge, \Delta^\wedge, \Sigma^\wedge, \neg H\}$. Then $(\Gamma^\wedge)^I = (\Delta^\wedge)^I = (\Sigma^\wedge)^I = (\neg H)^I = \neg(H^I) = t$, thus, $(H^I) = f$.
Since $\Gamma^\wedge \rightarrow (F \vee G)$ is tautology, then $(\Gamma^\wedge \rightarrow (F \vee G))^I = t$. Thus, $(F \vee G)^I = \vee(F^I, G^I) = t$. This is to say, either F^I or G^I is true.
If $F^I = t$, then $((\Delta^\wedge \wedge F) \rightarrow H)^I = f$. If $G^I = t$, then $((\Sigma^\wedge \wedge G) \rightarrow H)^I = f$.
This contradicts that both rules are tautology. Thus, the set $\{\Gamma^\wedge, \Delta^\wedge, \Sigma^\wedge, \neg H\}$ is unsatisfiable. □

Since all rules of natural deduction are sound, and all axioms are tautological, we can conclude that every sequent that can be proved by natural deduction is tautological. In this sense, the system of natural deduction is sound.

Problem 49 Let $A_1, \dots, A_n$ be the list of all atoms, and let I be an interpretation. Define the literals $L_1, \dots, L_n$ as follows:

$$L_i = \begin{cases} A_i & \text{if } I \models A_i, \\ \neg A_i & \text{otherwise.} \end{cases}$$

Show that for every formula F,

(a) if $I \models F$ then the sequent $L_i, \dots, L_n \Rightarrow F$ can be proved by natural deduction;

(b) if $I \not\models F$ then the sequent $L_i, \dots, L_n \Rightarrow \neg F$ can be proved by natural deduction;

Proof. Prove by structural induction.

Case 1. When F is an atom, say $F = A$.

(a) if $I \models F \Rightarrow I \models A \Rightarrow \exists L_i = A$, thus

1.	$L_i \Rightarrow F$	– axiom.
2.	$L_1 \cdots L_n \Rightarrow F$	– $(W), 1.$

(b) if $I \not\models F \Rightarrow I \not\models F \Rightarrow \exists L_i = \neg A$, thus

1.	$L_i \Rightarrow \neg F$	– axiom.
2.	$L_1 \cdots L_n \Rightarrow \neg F$	– $(W), 1.$

Case 2. We don't need to consider $\top$ (not part of formula). When $F = \bot$.

(a) Nothing satisfies $\bot$, no need to consider this case.

(b) if $I \not\models F$,

1.	$\bot \Rightarrow \bot$	– axiom.
2.	$\Rightarrow \neg\bot$	– $(\neg I), 1.$
3.	$L_1 \cdots L_n \Rightarrow \neg\bot$	– $(W), 2.$

Case 3. If F has the above property, then so does $\neg F$.
Induction hypothesis: $I \models F$ then the sequent $L_i, \ldots, L_n \Rightarrow F$ is provable. $I \not\models F$ then the sequent $L_i, \ldots, L_n \Rightarrow \neg F$ is provable.

(a) if $I \models \neg F$, $I \not\models F$, thus the sequent $L_i, \ldots, L_n \Rightarrow \neg F$ is provable.

(b) if $I \not\models \neg F$ then $I \models F$, thus

1.	$L_1 \cdots L_n \Rightarrow F$	– hypothesis.
2.	$\neg F \Rightarrow \neg F$	– axiom.
3.	$L_1 \cdots L_n, \neg F \Rightarrow \perp$	– $(\neg E), 1, 2.$
4.	$L_1 \cdots L_n \Rightarrow \neg\neg F$	– $(\neg I), 3.$

Case 4.1 If F_1, F_2 has the above property, then so does $F = F_1 \wedge F_2$.

(a) if $I \models F \Rightarrow I \models F_1, F_2$, then

1.	$L_1 \cdots L_n \Rightarrow F_1$	– hypothesis.
2.	$L_1 \cdots L_n \Rightarrow F_2$	– hypothesis.
3.	$L_1 \cdots L_n \Rightarrow F_1 \wedge F_2$	– $(\wedge I), 1, 2.$

(b) if $I \not\models F \Rightarrow I \not\models F_1$ or $I \not\models F_2$.

- if $I \not\models F_1$, then $L_i, \ldots, L_n \Rightarrow \neg F_1$, thus

1.	$F_1 \wedge F_2 \Rightarrow F_1 \wedge F_2$	– axiom.
2.	$F_1 \wedge F_2 \Rightarrow F_1$	– $(\wedge E), 1.$
3.	$L_1 \cdots L_n \Rightarrow \neg F_1$	– hypothesis.
4.	$L_1 \cdots L_n, F_1 \wedge F_2 \Rightarrow \perp$	– $(\neg E), 2, 3.$
5.	$L_1 \cdots L_n \Rightarrow \neg(F_1 \wedge F_2)$	– $(\neg I), 4.$

- if $I \not\models F_2$, then $L_i, \ldots, L_n \Rightarrow \neg F_2$, thus

1.	$F_1 \wedge F_2 \Rightarrow F_1 \wedge F_2$	– axiom.
2.	$F_1 \wedge F_2 \Rightarrow F_2$	– $(\wedge E), 1.$
3.	$L_1 \cdots L_n \Rightarrow \neg F_2$	– hypothesis.
4.	$L_1 \cdots L_n, F_1 \wedge F_2 \Rightarrow \perp$	– $(\neg E), 2, 3.$
5.	$L_1 \cdots L_n \Rightarrow \neg(F_1 \wedge F_2)$	– $(\neg I), 4.$

Case 4.2 If F_1, F_2 has the above property, then so does $F = F_1 \vee F_2$.

(a) if $I \models F \Rightarrow I \models F_1$ or $I \models F_2$.

- if $I \models F_1$, then $L_i, \ldots, L_n \Rightarrow F_1$, thus

1.	$L_1 \cdots L_n \Rightarrow F_1$	– hypothesis.
2.	$L_1 \cdots L_n \Rightarrow F_1 \vee F_2$	– $(\vee I), 1.$

- if $I \models F_2$, then $L_i, \ldots, L_n \Rightarrow F_2$, thus

1.	$L_1 \cdots L_n \Rightarrow F_2$	– hypothesis.
2.	$L_1 \cdots L_n \Rightarrow F_1 \vee F_2$	– $(\vee I), 1.$

(b) if $I \not\models F \Rightarrow I \not\models F_1, F_2$, thus

1.	$L_1 \cdots L_n \Rightarrow \neg F_1$	– hypothesis.
2.	$F_1 \Rightarrow F_1$	– axiom.
3.	$L_1 \cdots L_n, F_1 \Rightarrow \bot$	– $(\neg E), 1, 2.$
4.	$L_1 \cdots L_n \Rightarrow \neg F_2$	– hypothesis.
5.	$F_2 \Rightarrow F_2$	– axiom.
6.	$L_1 \cdots L_n, F_2 \Rightarrow \bot$	– $(\neg E), 4, 5.$
7.	$F_1 \vee F_2 \Rightarrow F_1 \vee F_2$	– axiom.
8.	$L_1 \cdots L_n, F_1 \vee F_2 \Rightarrow \bot$	– $(\vee E), 3, 6, 7.$
9.	$L_1 \cdots L_n \Rightarrow \neg(F_1 \vee F_2)$	– $(\neg I), 8.$

Case 4.3 If F_1, F_2 has the above property, then so does $F = F_1 \to F_2$.

(a) if $I \models F \Rightarrow I \models F_2$ or $I \not\models F_1, F_2$.

- if $I \models F_2$, then

1.	$L_1 \cdots L_n \Rightarrow F_2$	– hypothesis.
2.	$L_1 \cdots L_n, F_1 \Rightarrow F_2$	– $(W), 1.$
3.	$L_1 \cdots L_n \Rightarrow F_1 \to F_2$	– $(\to I), 2.$

- if $I \not\models F_1, F_2$, then

1.	$L_1 \cdots L_n \Rightarrow \neg F_1$	– hypothesis.
2.	$F_1 \Rightarrow F_1$	– axiom.
3.	$L_1 \cdots L_n, F_1 \Rightarrow \bot$	– $(\neg E), 1, 2.$
4.	$L_1 \cdots L_n, F_1 \Rightarrow F_2$	– $(C), 3.$
5.	$L_1 \cdots L_n \Rightarrow F_1 \to F_2$	– $(\to I), 4.$

(b) if $I \not\models F \Rightarrow I \models F_1$ and$I \not\models F_2$, thus

1.	$L_1 \cdots L_n \Rightarrow F_1$	– hypothesis.
2.	$L_1 \cdots L_n \Rightarrow \neg F_2$	– hypothesis.
3.	$F_1 \to F_2 \Rightarrow F_1 \to F_2$	– axiom.
4.	$L_1 \cdots L_n, F_1 \to F_2 \Rightarrow F_2$	– $(\to E), 1, 3.$
5.	$L_1 \cdots L_n, F_1 \to F_2 \Rightarrow \perp$	– $(\neg E), 2, 4.$
6.	$L_1 \cdots L_n \Rightarrow \neg(F_1 \to F_2)$	– $(\neg I), 5.$

□

Problem 50 For any tautology F, the sequent $\Rightarrow F$ can be proved by natural deduction.
Let $A_1, \ldots, A_n$ be the atoms, and for $\forall k \in \mathbb{N}$, $p(k)$ the statements:

> If $k \leq n$ then $\forall L_{k+1}, \ldots, L_n$, s.t. $\forall i$, $L_i = A_i$ or $L_i = \neg A_i$, the sequent $L_{k+1}, \ldots, L_n \Rightarrow F$ is provable in natural deduction.

We will prove that $\forall k \in \mathbb{N}$, $p(k)$ holds. Then the theorem will follow as $p(n)$. Prove by mathematical induction:

- Base case: $k = 0$.
  Define $I(A_i) = \begin{cases} t & \text{if } L_i = A_i, \\ f & \text{if } L_i = \neg A_i. \end{cases}$ Then $L_i = \begin{cases} A_i & \text{if } I \models A_i, \\ \neg A_i & \text{otherwise.} \end{cases}$
  Since F is tautology, $I \models F$. Thus, we can apply Problem 49 directly to get a natural deduction proof of $L_1, L_2, \ldots, L_n \Rightarrow F$. This demonstrates $p(0)$.
- Induction step: $\forall k > 0$, let:

$$\begin{aligned} S_t = &\quad A_k, L_{k+1}, \ldots, L_n &\Rightarrow F \\ S_f = &\quad \neg A_k, L_{k+1}, \ldots, L_n &\Rightarrow F \end{aligned}$$

  Suppose we have $p(k-1)$. Applying $p(k-1)$ with $L_{(k-1)+1} = A_k$, give us a natural deduction proof of S_t. Applying $p(k-1)$ with $L_{(k-1)+1} = \neg A_k$, give us a natural deduction proof of S_f. Then we can prove $L_{k+1}, \ldots, L_n \Rightarrow F$ as follow:

1.	$A_k \vee \neg A_k$	– axiom.
2.	$A_k, L_{k+1}, \ldots, L_n \Rightarrow F$	– S_t.
3.	$\neg A_k, L_{k+1}, \ldots, L_n \Rightarrow F$	– S_f.
4.	$L_{k+1}, \ldots, L_n \Rightarrow F$	– $(\vee E), 1, 2, 3.$

By the principle of deduction, we can prove $p(n)$, the sequent $\Rightarrow F$ can be proved by natural deduction.

Problem 51 Every tautology sequent can be proved by natural deduction.

Let $\Gamma \to F$ be a tautological sequent ($\Gamma \neq \emptyset$), assuming that $\Gamma = \{\Gamma_1, ..., \Gamma_n\}$.

- By Problem 50, we get that
  $\Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_n \to F$.
- we can prove $\Gamma \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_n$ by induction on n.
  - Base case ($n = 1$):
    $\Gamma \Rightarrow \Gamma_1$, since $\Gamma = \{\Gamma_1\}$.
  - Induction:
    Assuming that $\Gamma \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_k$ if $\Gamma = \{\Gamma_1, ..., \Gamma_k\}$
    Then for $n = k + 1$,
    1. $\Gamma_1, ..., \Gamma_k \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_k$ -Induction Hypothesis
    2. $\Gamma_{k+1} \Rightarrow \Gamma_{k+1}$ -axiom
    3. $\Gamma_1, ..., \Gamma_k, \Gamma_{k+1} \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_k \wedge \Gamma_k$ -($\wedge I$),1,2

    So $\Gamma \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_k, \wedge\Gamma_{k+1}$ if $\Gamma = \{\Gamma_1, ..., \Gamma_k, \Gamma_{k+1}\}$

Thus, we have
1. $\Rightarrow (\Gamma_1 \wedge ... \wedge \Gamma_n \to F$ -Problem 50
2. $\Gamma \Rightarrow \Gamma_1 \wedge ... \wedge \Gamma_n$ -Induction Proof
3. $\Gamma \Rightarrow F$ - ($\to$ E), 1, 2

In the case that $\Gamma = \emptyset$, the formula corresponding to the sequent $\Gamma \Rightarrow F$ is actually just F, and the result follows directly from Problem 50.

In this sense, the system of natural deduction is complete.

5.3 Quiz 5

Prove by natural deduction:

$$((p \to q) \to p) \to p$$

Proof.

A1.	$(p \to q) \to p$.	
1.	$A1 \Rightarrow (p \to q) \to p$	– axiom.
2.	$\Rightarrow p \vee \neg p$	– axiom.
A2.	p.	
3.	$A2 \Rightarrow p$	– axiom.
A3.	$\neg p$.	
4.	$A3 \Rightarrow \neg p$	– axiom.
5.	$p \Rightarrow p$	– axiom.
6.	$A3, p \Rightarrow \bot$	– $(\neg E), 5, 4.$
7.	$A3, p \Rightarrow q$	– $(C), 6.$
8.	$A3 \Rightarrow p \to q$	– $(\to I), 7.$
9.	$A1, A3 \Rightarrow p$	– $(\to E), 8, 1.$
10.	$A1 \Rightarrow p$	– $(\vee E), 2, 3, 9.$
11.	$\Rightarrow ((p \to q) \to p) \to p$	– $(\to I), 10.$

□

5.4 Quiz 6

The following rule

$$\frac{\Gamma \Rightarrow F \to G \qquad \Delta \Rightarrow \neg G}{\Gamma, \Delta \Rightarrow \neg F}$$

is sound in G_3.

We need to show that if

$$\Delta \to \neg G \tag{9}$$

$$\Gamma \to (F \to G) \tag{10}$$

are tautological in G_3, then

$$\Gamma \wedge \Delta \to \neg F \tag{11}$$

is also tautological.

Proof. (9) is tautological means that $\forall I$, we have

$$(\Delta \to \neg G)^I = \to (\Delta^I, (\neg G)^I) = 1$$

From definition, we know that

$$\Delta^I \leq (\neg G)^I \tag{12}$$

(10) is tautological means that $\forall I$, we have

$$(\Gamma \to (F \to G))^I = \to (\Gamma^I, (F \to G)^I) = 1$$

From definition, we know that

$$\Gamma^I \leq (F \to G)^I \tag{13}$$

Consider the following two cases:

- if $F^I \leq G^I$, then $(F \to G)^I = \to (F^I, G^I) = 1$. Clearly, (13) holds true. We have:

  $$(\neg F)^I \geq (\neg G)^I \tag{14}$$

  This is because if both F^I and G^I are greater than 0, then $(\neg F)^I = (\neg G)^I = 0$. If both F^I and G^I equal 0, then $(\neg F)^I = (\neg G)^I = 1$. If only one of them equals 0, it must be $0 = F^I < G^I$, then $0 = (\neg G)^I < (\neg F)^I = 1$.
  Combining (12) and (14), we have:

  $$\Delta^I \leq (\neg G)^I \leq (\neg F)^I$$

  Thus,

  $$\min(\Gamma^I, \Delta^I) \leq (\neg F)^I$$

- if $F^I > G^I$, then $(F \to G)^I = \to (F^I, G^I) = G^I$. From (12) and (13):

  $$\begin{cases} \Delta^I \leq \neg G^I \\ \Gamma^I \leq \quad G^I \end{cases}$$

  Clearly, one of G^I and $(\neg G)^I$ must be zero. (If $G^I = 0$, the claim holds. If $G^I > 0$, then $(\neg G)^I = 0$.)
  Thus,

  $$\min(\Gamma^I, \Delta^I) = 0 \leq (\neg F)^I$$

From both cases, we have $\min(\Gamma^I, \Delta^I) \leq (\neg F)^I$, this is the same as:

$$\begin{aligned} (\Gamma \wedge \Delta)^I &\leq (\neg F)^I \\ \to ((\Gamma \wedge \Delta)^I, (\neg F)^I) &= 1 \\ (\Gamma \wedge \Delta \to \neg F)^I &= 1 \end{aligned}$$

Thus, $\Gamma \wedge \Delta \to \neg F$ is tautological in G_3. $\square$

6 Part 8

6.1 Predicate Formulas: Syntax

The alphabet of predicate logic consists of:

- symbols called *object constants*,
- symbols called *predicate constants*, with a positive integer, called the *arity*, assigned to each of them.
- *object variable* $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \ldots$,
- the propositional connectives,
- the *universal quantifier* $\forall$ and the existential quantifier $\exists$,
- the parentheses and the comma.

In this part of the course, a *string* is a finite string of symbols in the alphabet of predicate logic.

A *term* is an object constant or an object variable. A string is called an *atomic formula* if it has the form

$$P(t_1, \ldots, t_n)$$

where P is a predicate constant of arity n and $t_1, \ldots, t_n$ are terms.

We define when a string is a *(predicate) formula* recursively, as follows:

- every atomic formula is a formula,
- $\top$ and $\bot$ are formulas,
- if F is a formula then $\neg F$ is a formula,
- for any binary connective $\odot$, if F are G are formulas then $(F \odot G)$ is a formula,
- for any quantifier K and any variable v, if F is a formula then KvF is a formula.

When we write predicate formulas, we will use the abbreviations introduced in Part 1 of these lecture notes. A string of the form $\forall v_1 \cdots \forall v_n$ will be written as $\forall v_1 \cdots v_n$ and similarly for the existential quantifier.

An occurrence of a variable v in a formula F is bound if it belongs to a part of F that has the form KvG; otherwise it is free. We say that a variable v is *free (bound)* in F if at least one one occurrence of v in F is free(bound). A formula without free variable is called a *closed formula*, or a *sentence*.

In the following problems, represent the given conditions by formulas that may contain the object constant a and the ternary predicate constants Sum and $Prod$. Think of the object variables as ranging over all nonnegative integers, and interpret the object and predicate constants as follows:

- a represents 0,
- $Sum(x, y, z)$ represents the condition $x + y = z$,
- $Prod(x, y, z)$ represents the condition $xy = z$,

Problem 52 (a) $x = y \quad \Rightarrow \quad Sum(x, a, y)$.

(b) $x = 0 \quad \Rightarrow \quad Sum(x, a, a) \quad Sum(x, x, x)$.

(c) $x = 1 \quad \Rightarrow \quad \forall y\, Prod(x, y, y)$.
$\neg Sum(x, x, x) \wedge \forall z(\neg Sum(z, z, z) \rightarrow \exists y\, Sum(x, y, z))$

Problem 53 (a) $x \leq y \quad \Rightarrow \quad \exists z\, Sum(x, z, y)$.

(b) $x < y \quad \Rightarrow \quad \exists z\, (\neg Sum(z, z, z) \wedge Sum(x, z, y))$.

(c) $x + 1 < y \quad \Rightarrow \quad \exists z\, (\neg Prod(z, z, z) \wedge Sum(x, z, y)$.

Problem 54 (a) x is even number;
$\exists y\, Sum(y, y, x)$

(b) the sum of any two odd numbers is even;
$\forall x_1 x_2\, (\neg\exists y\, Sum(y, y, x_1) \wedge \neg\exists y\, Sum(y, y, x_2)) \rightarrow \exists z\, (Sum(x_1, x_2, z) \wedge \exists y\, Sum(y, y, z))$
$\forall x_1 x_2\, (\neg\exists y\, Sum(y, y, x_1) \wedge \neg\exists y\, Sum(y, y, x_2)) \rightarrow \forall z\, (Sum(x_1, x_2, z) \rightarrow \exists y\, Sum(y, y, z))$

(c) addition is commutative.
$\forall xy\, \exists z\, (Sum(x, y, z) \wedge Sum(y, x, z))$

Problem 55 Number x can be represented as the sum of two complete squares.
$\exists y_1 y_2\, (\exists z\, Prod(z, z, y_1) \wedge \exists z\, Prod(z, z, y_2) \wedge Sum(y_1, y_2, x))$

Problem 56 Number x is prime.
$\neg\exists y(\neg(\forall z Prod(y, z, z)) \wedge \neg Sum(y, a, x) \wedge \exists z(Prod(y, z, x))) \wedge \neg Prod(x, x, x)$

6.2 Predicate Formulas: Semantics

The semantics of propositional formulas described in Part 1 of these lecture notes defines which truth value F^I is assigned to a propositional formula F by an interpretation I. Our goal is to extend this definition to predicate logic.

First we need to adapt the definition of an interpretation to predicate formulas. In predicate logic, an *interpretation* I consists of

- a non-empty set $|I|$, called the *universe* of I,
- for every object constant c, an element c^I of $|I|$,
- for every predicate constant P, a function P^I from $|I|^n$ to $\{f, t\}$, where n is the arity of P.

For instance, the sentence before Problem 52 can be viewed as the definition of the interpretation I such that

$$\begin{aligned} |I| &= \mathbf{N}, \\ a^I &= 0, \\ Sum^I(\xi, \eta, \zeta) &= \begin{cases} \text{t}, & \text{if } \xi + \eta = \zeta, \\ \text{f}, & \text{otherwise} \end{cases} \\ Prod^I(\xi, \eta, \zeta) &= \begin{cases} \text{t}, & \text{if } \xi\eta = \zeta, \\ \text{f}, & \text{otherwise} \end{cases} \end{aligned} \tag{15}$$

$(\xi, \eta, \zeta \in \mathbf{N})$.

The result of the *substitution* of a term t for a variable v in a formula F is the formula F^v_t obtained from F by replacing each free occurrence of v by t.

Let I be an interpretation. For any element ξ of its universe $|I|$, select a new object constant ξ^*, called the *name* of ξ. The interpretation I can be extended to the new object constants by defining

$$(\xi^*)^I = \xi$$

for all $\xi \in |I|$.

We define the truth value F^I that is *assgined* to F by I for a sentence F that may contain names recursively, as follows:

- $P(t_1, \ldots, t_n)^I = P^I(t_1^I, \ldots, t_n^I)$,
- $\top^I = \text{t}$, $\bot^I = \text{f}$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$,
- $(\forall v F)^I = \text{t}$ iff for all $\xi \in |I|$, $(F^v_{\xi^*})^I = \text{t}$,

- $(\exists v F)^I = \mathrm{t}$ iff for some $\xi \in |I|$, $(F^v_{\xi^*})^I = \mathrm{t}$.

As in propositional logic, we say that I *satisfies* F, and write $I \models F$ if $F^I = \mathrm{t}$.

Problem 57 Determine which of the sentences

(i) $\exists x Sum(x, x, x)$,

(ii) $\exists x \neg Sum(x, x, x)$,

(iii) $\forall x (Sum(x, x, x) \to \forall y Sum(x, y, y))$

are satisfied by interpretation (15) .

(i) $\exists x Sum(x, x, x)$,
For $0 \in |I|$, $(Sum(x, x, x)^x_{0^*})^I = (Sum(0^*, 0^*, 0^*))^I$
$= Sum^I(0^{*I}, 0^{*I}, 0^{*I}) = Sum^I(0, 0, 0) = \mathrm{t}$

(ii) $\exists x \neg Sum(x, x, x)$,
For $1 \in |I|$, $(\neg Sum(x, x, x)^x_{1^*})^I = (\neg Sum(1^*, 1^*, 1^*))^I$
$= \neg Sum^I(1^{*I}, 1^{*I}, 1^{*I}) = \neg Sum^I(1, 1, 1) = \mathrm{t}$

(iii) $\forall x (Sum(x, x, x) \to \forall y Sum(x, y, y))$
$(Sum(n^*, n^*, n^*))^I = Sum^I(n^{*I}, n^{*I}, n^{*I}) = Sum^I(n, n, n) = \begin{cases} \mathrm{t}, & \text{if } n = 0, \\ \mathrm{f}, & \text{otherwise} \end{cases}$
We only care about $x = 0$,
$(Sum(0^*, m^*, m^*))^I = Sum^I(0^{*I}, m^{*I}, m^{*I}) = Sum^I(0, m, m) = \mathrm{t}$
Thus, $[\forall y Sum(0^*, y, y)]^I = t$, for all $m = |I|$.

6.3 Logical Validity and Entailment

In predicate logic, we say about a sentence that it is *logically valid* if it is satisfied by all interpretations.

Problem 58 Determine whether the sentences

$$\exists x(P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x)), \tag{16}$$
$$(\exists x P(x) \wedge \exists x Q(x)) \rightarrow \exists x(P(x) \wedge Q(x)) \tag{17}$$

are logically valid.

- (16) is logically valid.
  For any interpretation I, we only need care about the case when $(\exists x(P(x) \wedge Q(x)))^I = \mathrm{t}$, otherwise, the whole sentence is t. We have, for some $\xi \in |I|$,

$$\begin{aligned}(((P(x) \wedge Q(x)))^x_{\xi^*})^I = &\quad (P(\xi^*) \wedge Q(\xi^*))^I \\ = &\quad \wedge(P^I(\xi^{*I}), Q^I(\xi^{*I})) \\ = &\quad \wedge(P^I(\xi), Q^I(\xi)) = \quad \mathrm{t}\end{aligned}$$

  Thus, $P^I(\xi) = Q^I(\xi) = \mathrm{t}$. This is to say:
  $(P(x)^x_{\xi^*})^I = P^I(\xi^{*I}) = P^I(\xi) = \mathrm{t}$ $\qquad (Q(x)^x_{\xi^*})^I = Q^I(\xi^{*I}) = Q^I(\xi) = \mathrm{t}$
  which means $(\exists x P(x))^I = (\exists x Q(x))^I = \mathrm{t}$. Thus,

$$(\exists x P(x) \wedge \exists x Q(x))^I = \mathrm{t}$$

  This is to say, (16) is satisfied by all interpretations. Thus, it is logically valid.
  We can also prove it by natural deduction.

A1.	$\exists x(P(x) \wedge Q(x))$	
1.	$A1 \Rightarrow \exists x(P(x) \wedge Q(x))$	– axiom.
A2.	$P(x) \wedge Q(x)$	
2.	$A2 \Rightarrow P(x) \wedge Q(x)$	– axiom.
3.	$A2 \Rightarrow P(x)$	– $(\wedge E), 2.$
4.	$A2 \Rightarrow \exists x P(x)$	– $(\exists I), 3.$
5.	$A2 \Rightarrow Q(x)$	– $(\wedge E), 2.$
6.	$A2 \Rightarrow \exists x Q(x)$	– $(\exists I), 5.$
7.	$A2 \Rightarrow \exists x P(x) \wedge \exists x Q(x)$	– $(\wedge I), 4, 6.$
8.	$A1 \Rightarrow \exists x P(x) \wedge \exists x Q(x)$	– $(\exists E), 1, 7.$
9.	$\Rightarrow \exists x(P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))$	– $(\rightarrow I), 8.$

- (17) is not logically valid.
  We can find a counter example. Define interpretation I as:
  $$|I| = \{0, 1\},$$
  $$P^I(\xi) = \begin{cases} \mathrm{t}, & \text{if } \xi = 0, \\ \mathrm{f}, & \text{otherwise} \end{cases}$$
  $$Q^I(\xi) = \begin{cases} \mathrm{t}, & \text{if } \xi = 1, \\ \mathrm{f}, & \text{otherwise} \end{cases}$$
  Clearly,
  - for $0 \in |I|, (P(x)^x_{0*})^I = P^I(0^{*I}) = P^I(0) = \mathrm{t}$
  - for $1 \in |I|, (Q(x)^x_{1*})^I = Q^I(1^{*I}) = Q^I(1) = \mathrm{t}$

  Thus, $(\exists x P(x) \wedge \exists x Q(x))^I = \mathrm{t}$. However, $\exists x(P(x) \wedge Q(x))$ can not be satisfied by I.
  - for $0 \in |I|, ((P(x) \wedge Q(x))^x_{0*})^I = \wedge(P^I(0^{*I}), Q^I(0^{*I})) = \wedge(P^I(0), Q^I(0)) = \wedge(\mathrm{t}, \mathrm{f}) = \mathrm{f}$
  - for $1 \in |I|, ((P(x) \wedge Q(x))^x_{1*})^I = \wedge(P^I(1^{*I}), Q^I(1^{*I})) = \wedge(P^I(1), Q^I(1)) = \wedge(\mathrm{f}, \mathrm{t}) = \mathrm{f}$

  This is to say, the above interpretation fails to satisfy (17). In other word, (17) is not logical valid.

The *univeral closure* of a formula F is the sentence $\forall v_1, \cdots, v_n F$, where $v_1, \cdots, v_n$ are all free variables of F. About a formula with free variables we say that it is *logically valid* if its universal closure is logically valid.

Problem 59 For each of the formulas

$$P(x) \to \exists x P(x) \tag{18}$$
$$P(x) \to \forall x P(x) \tag{19}$$

determine whether it is logically valid.

- (18) is logically valid.
- (19) is not logically valid.

A formula F is *equivalent* to a formula G if the formula $F \leftrightarrow G$ is logically valid.

Problem 60 Determine whether the formula

$$\forall x \exists y P(x, y)$$

is equivalent to

$$\exists y \forall x P(x, y)$$

We need to determine whether $\forall x \exists y P(x, y) \leftrightarrow \exists y \forall x P(x, y)$ *is logically valid or not.*

- $\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)$ is not logically valid. We can find a counter example. Define interpretation I as:

$$|I| = \{0, 1\},$$
$$P^I(\xi, \eta) = \begin{cases} \text{t}, & \text{if } \xi = \eta, \\ \text{f}, & \text{otherwise} \end{cases}$$

  There are only two elements 0 and 1 in the universe $|I|$.

  - for $0 \in |I|, \exists y(P(x, y)^x_{0*})^I = (\exists y P(0^*, y))^I$, we can find $0 \in |I|$ that $(P(0^*, y)^y_{0*})^I = P^I(0^{*I}, 0^{*I}) = P^I(0, 0) = \text{t}$,
  - for $1 \in |I|, \exists y(P(x, y)^x_{1*})^I = (\exists y P(1^*, y))^I$, we can find $1 \in |I|$ that $(P(1^*, y)^y_{1*})^I = P^I(1^{*I}, 1^{*I}) = P^I(1, 1) = \text{t}$.

  However, the above interpretation can not satisfy $\exists y \forall x P(x, y)$.

  - for $0 \in |I|, \forall x(P(x, y)^y_{0*})^I = (\forall x P(x, 0^*))^I$, we can find $1 \in |I|$ that $(P(x, 0^*)^x_{1*})^I = P^I(1^{*I}, 0^{*I}) = P^I(1, 0) = \text{f}$,
  - for $1 \in |I|, \forall x(P(x, y)^y_{1*})^I = (\forall x P(x, 1^*))^I$, we can find $0 \in |I|$ that $(P(x, 1^*)^x_{0*})^I = P^I(0^{*I}, 1^{*I}) = P^I(0, 1) = \text{f}$.

- $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$ is logically valid. We can prove by natural deduction.

A1.	$\exists y \forall x P(x, y)$	
1.	$A1 \Rightarrow \exists y \forall x P(x, y)$	– axiom.
A2.	$\forall x P(x, y)$	
2.	$A2 \Rightarrow \forall x P(x, y)$	– axiom.
3.	$A2 \Rightarrow P(x, y)$	– $(\forall E), 2.$
4.	$A2 \Rightarrow \exists y P(x, y)$	– $(\exists I), 3.$
5.	$A2 \Rightarrow \forall x \exists y P(x, y)$	– $(\forall I), 4.$
6.	$A1 \Rightarrow \forall x \exists y P(x, y)$	– $(\exists E), 1, 5.$
7.	$\Rightarrow \exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$	– $(\rightarrow I), 6.$

We say that a set Γ of sentences *entails* a sentence F, or that F is *logical consequence* of Γ, if every interpretation that satisfies all sentences in Γ satisfies F.

Problem 61 Determine whether

$$\exists x P(x,x) \tag{20}$$

is a logical consequence of the sentence

$$\forall x \exists y P(x,y) \tag{21}$$

$$\forall xyz((P(x,y) \wedge P(y,z)) \to P(x,z)). \tag{22}$$

We can find a counter example. Define interpretation I as:

$$|I| = \mathbb{N},$$

$$P^I(\xi,\eta) = \begin{cases} \text{t}, & \text{if } \xi < \eta, \\ \text{f}, & \text{otherwise} \end{cases}$$

I satisfies (21) because for all $\xi = n \in |I|$, we can always have $\eta = n+1$, where

$$((P(x,y)^x_{\xi^*})^y_{\eta^*})^I = P^I(\xi^{*I}, \eta^{*I}) = P^I(n, n+1) = \text{t}.$$

We can show that I also satisfies (22) as follow:
We only care about the case where $(P(\xi^*,\eta^*) \wedge P(\eta^*,\zeta^*))^I = \text{t}$.
From definition, we have:

$$\begin{aligned}(P(\xi^*,\eta^*) \wedge P(\eta^*,\zeta^*))^I &= \wedge(P^I(\xi^{*I},\eta^{*I}), P^I(\eta^{*I},\zeta^{*I})) \\ &= \wedge(P^I(\xi,\eta), P^I(\eta,\zeta)) = \text{t}\end{aligned}$$

We have

$$\left.\begin{array}{lll} P^I(\xi,\eta) = \text{t} & \Rightarrow & \xi < \eta \\ P^I(\eta,\zeta) = \text{t} & \Rightarrow & \eta < \zeta \end{array}\right\} \Rightarrow \xi < \eta < \zeta$$

Clearly,

$$(P(\xi^*,\zeta^*))^I = P^I(\xi^{*I},\zeta^{*I}) = P^I(\xi,\zeta) = \text{t}$$

However, I doesn't satisfy (20).

$$(P(\xi^*,\xi^*))^I = P^I(\xi^{*I},\xi^{*I}) = P^I(\xi,\xi) = \text{f}$$

6.4 Introduction and Elimination Rules for Quantifiers

We say that a term t is *substitutable* for a variable v in a formula F if

- t is a constant, or
- t is a variable w, and no part of F of the form KwG contains an occurrence of v which is free in F.

Here are the additional inference rules of predicate logic:

$$(\forall I) \quad \frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow \forall v F} \qquad\qquad (\forall E) \quad \frac{\Gamma \Rightarrow \forall v F}{\Gamma \Rightarrow F_t^v}$$

where v is not a free variable of any formula in Γ.

where t is substitutable for v in F

$$(\exists I) \quad \frac{\Gamma \Rightarrow F_t^v}{\Gamma \Rightarrow \exists v F} \qquad\qquad (\exists E) \quad \frac{\Gamma \Rightarrow \exists v F \quad \Delta, F \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Sigma}$$

where t is substitutable for v in F

where v is not a free variable of any formula in Δ, Σ

Prove the given formulas in the natural deduction system.

Problem 62 $(P(a) \wedge \forall x(P(x) \to Q(x))) \to Q(a)$.

A1.	$P(a) \wedge \forall x(P(x) \to Q(x))$	
1.	$A1 \Rightarrow P(a) \wedge \forall x(P(x) \to Q(x))$	– axiom.
2.	$A1 \Rightarrow p(a)$	– $(\wedge E), 1.$
3.	$A1 \Rightarrow \forall x(P(x) \to Q(x))$	– $(\wedge E), 2.$
4.	$A1 \Rightarrow P(a) \to Q(a)$	– $(\forall E), 3.$
5.	$A1 \Rightarrow Q(a)$	– $(\to E), 2, 4.$
6.	$\Rightarrow (P(a) \wedge \forall x(P(x) \to Q(x))) \to Q(a)$	– $(\to I), 5.$

Problem 63 $P(a) \to \neg\forall x \neg P(x)$.

A1.	$P(a)$	
1.	$A1 \Rightarrow P(a)$	– axiom.
A2.	$\forall x \neg P(x)$	
2.	$A2 \Rightarrow \forall x \neg P(x)$	– axiom.
3.	$A2 \Rightarrow \neg P(a)$	– $(\forall E), 2.$
4.	$A1, A2 \Rightarrow \bot$	– $(\neg E), 1, 3.$
5.	$A1 \Rightarrow \neg\forall x \neg P(x)$	– $(\neg I), 4.$
6.	$\Rightarrow P(a) \to \neg\forall x \neg P(x)$	– $(\to I), 5.$

Problem 64 $\forall xy P(x, y) \to \forall x P(x, x)$.

A1.	$\forall xy P(x, y)$	
1.	$A1 \Rightarrow \forall x \forall y P(x, y)$	– axiom.
2.	$A1 \Rightarrow \forall y P(x, y)$	– $(\forall E), 1.$
3.	$A1 \Rightarrow P(x, x)$	– $(\forall E), 2.$
4.	$A1 \Rightarrow \forall x P(x, x)$	– $(\forall I), 3.$
5.	$\Rightarrow \forall xy P(x, y) \to \forall x P(x, x)$	– $(\to I), 4.$

Problem 65 $\forall x P(x) \leftrightarrow \forall y P(y)$.

A1.	$\forall x P(x)$	
1.	$A1 \Rightarrow \forall x P(x)$	– axiom.
2.	$A1 \Rightarrow P(y)$	– $(\forall E), 1.$
3.	$A1 \Rightarrow \forall y P(y)$	– $(\forall I), 2.$
4.	$\Rightarrow \forall x P(x) \to \forall y P(y)$	– $(\to I), 3.$
A2.	$\forall y P(y)$	
5.	$A1 \Rightarrow \forall y P(y)$	– axiom.
6.	$A1 \Rightarrow P(x)$	– $(\forall E), 5.$
7.	$A1 \Rightarrow \forall x P(x)$	– $(\forall I), 6.$
8.	$\Rightarrow \forall y P(y) \to \forall x P(x)$	– $(\to I), 7.$
9.	$\Rightarrow \forall x P(x) \leftrightarrow \forall y P(y)$	– $(\wedge I), 4, 8.$

Problem 66 $\forall x P(x) \wedge \forall x Q(x) \leftrightarrow \forall x (P(x) \wedge Q(x))$.

A1.	$\forall x P(x) \wedge \forall x Q(x)$	
1.	$A1 \Rightarrow \forall x P(x) \wedge \forall x Q(x)$	– axiom.
2.	$A1 \Rightarrow \forall x P(x)$	– $(\wedge E), 1.$
3.	$A1 \Rightarrow P(x)$	– $(\forall E), 2.$
4.	$A1 \Rightarrow \forall x Q(x)$	– $(\wedge E), 1.$
5.	$A1 \Rightarrow Q(x)$	– $(\forall E), 4.$
6.	$A1 \Rightarrow P(x) \wedge Q(x)$	– $(\wedge I), 3, 5.$
7.	$A1 \Rightarrow \forall x (P(x) \wedge Q(x))$	– $(\forall I), 6.$
8.	$\Rightarrow \forall x P(x) \wedge \forall x Q(x) \to \forall x (P(x) \wedge Q(x))$	– $(\to I), 7.$
A2.	$\forall x (P(x) \wedge Q(x))$	
9.	$A2 \Rightarrow \forall x (P(x) \wedge Q(x))$	– $(\forall I), 8.$
10.	$A2 \Rightarrow P(x) \wedge Q(x))$	– $(\forall E), 9.$
11.	$A2 \Rightarrow P(x)$	– $(\wedge E), 10.$
12.	$A2 \Rightarrow \forall x P(x)$	– $(\forall I), 11.$
13.	$A2 \Rightarrow Q(x)$	– $(\wedge E), 9.$
14.	$A2 \Rightarrow \forall x Q(x)$	– $(\forall I), 13.$
15.	$A2 \Rightarrow \forall x P(x) \wedge \forall x Q(x)$	– $(\wedge I), 11, 14.$
16.	$\Rightarrow \forall x (P(x) \wedge Q(x)) \to \forall x P(x) \wedge \forall x Q(x)$	– $(\to I), 15.$
17.	$\Rightarrow \forall x P(x) \wedge \forall x Q(x) \leftrightarrow \forall x (P(x) \wedge Q(x))$	– $(\wedge I), 4, 16.$

Problem 66½ $\forall x P(x) \vee \forall x Q(x) \leftrightarrow \forall x (P(x) \vee Q(x))$.

A1.	$\forall x P(x) \vee \forall x Q(x)$	
1.	$A1 \Rightarrow \forall x P(x) \vee \forall x Q(x)$	– axiom.
A2.	$\forall x P(x)$	
2.	$A2 \Rightarrow \forall x P(x)$	– axiom.
3.	$A2 \Rightarrow P(x)$	– $(\forall E), 2.$
4.	$A2 \Rightarrow P(x) \vee Q(x)$	– $(\vee I), 3.$
A3.	$\forall x Q(x)$	
5.	$A3 \Rightarrow \forall x Q(x)$	– axiom.
6.	$A3 \Rightarrow Q(x)$	– $(\forall E), 5.$
7.	$A3 \Rightarrow P(x) \vee Q(x)$	– $(\vee I), 6.$
8.	$A1 \Rightarrow P(x) \vee Q(x)$	– $(\vee E), 1, 4, 7.$
9.	$A1 \Rightarrow \forall x (P(x) \vee Q(x))$	– $(\forall I), 8.$
10.	$\Rightarrow \forall x P(x) \vee \forall x Q(x) \to \forall x (P(x) \vee Q(x))$	– $(\to I), 9.$

Counter Example for $\leftarrow$:
$|I| = a, b, P^I(a) = \mathrm{t}, P^I(b) = \mathrm{f}, Q^I(a) = \mathrm{f}, Q^I(b) = \mathrm{t}$

$$\begin{aligned} ((P(x) \vee Q(x))^x_{\eta^*})^I = & \quad (P(\eta^*) \vee Q(\eta^*))^I \\ = & \quad \vee(P^I(\eta^{*I}), Q^I(\eta^{*I})) \\ = & \quad \vee(P^I(\eta), Q^I(\eta)) = \begin{cases} \mathrm{t}, & \text{if } \eta = a, \\ \mathrm{t}, & \text{if } \eta = b. \end{cases} \end{aligned}$$

However,

$$\begin{aligned} (P(x)^x_{\eta^*})^I = & \quad (P(\eta^*))^I \\ = & \quad P^I(\eta^{*I}) \\ = & \quad P^I(\eta) = \begin{cases} \mathrm{t}, & \text{if } \eta = a, \\ \mathrm{f}, & \text{if } \eta = b. \end{cases} \end{aligned}$$

Thus, $\forall x(P(x)^x_{\eta^*})^I = \mathrm{f}$. Similarly, we can get $\forall x(Q(x)^x_{\eta^*})^I = \mathrm{f}$. Thus,

$$(\forall x P(x) \vee \forall x Q(x))^I = \mathrm{f}$$

This means that

$$(\forall x(P(x) \vee Q(x)) \to (\forall x P(x) \vee \forall x Q(x)))^I = \mathrm{f}$$

Thus rule doesn't hold for the above interpretation.

Problem 67 $\forall x P(x) \vee Q(a) \leftrightarrow \forall x(P(x) \vee Q(a))$.

A1.	$\forall x P(x) \vee Q(a)$	
1.	$A1 \Rightarrow \forall x P(x) \vee Q(a)$	– axiom.
A2.	$\forall x P(x)$	
2.	$A2 \Rightarrow \forall x P(x)$	– axiom.
3.	$A2 \Rightarrow P(x)$	– $(\forall E), 2.$
4.	$A2 \Rightarrow P(x) \vee Q(a)$	– $(\vee I), 3.$
5.	$A2 \Rightarrow \forall x(P(x) \vee Q(a))$	– $(\forall I), 4.$
A3.	$Q(a)$	
6.	$A3 \Rightarrow Q(a)$	– axiom.
7.	$A3 \Rightarrow P(x) \vee Q(a)$	– $(\vee I), 6.$
8.	$A3 \Rightarrow \forall x(P(x) \vee Q(a))$	– $(\forall I), 7.$
9.	$A1 \Rightarrow \forall x(P(x) \vee Q(a))$	– $(\vee E), 1, 5, 8.$
10.	$\Rightarrow \forall x P(x) \vee Q(a) \rightarrow \forall x(P(x) \vee Q(a))$	– $(\rightarrow I), 9.$
A4.	$\forall x(P(x) \vee Q(a))$	
11.	$A4 \Rightarrow \forall x(P(x) \vee Q(a))$	– axiom.
12.	$A4 \Rightarrow P(x) \vee Q(a)$	– $(\forall E), 11.$
13.	$\Rightarrow Q(a) \vee \neg Q(a)$	– axiom.
A5.	$Q(a)$	
14.	$A5 \Rightarrow Q(a)$	– axiom.
15.	$A5 \Rightarrow \forall x P(x) \vee Q(a)$	– axiom.
A6.	$\neg Q(a)$	
16.	$A6 \Rightarrow \neg Q(a)$	– axiom.
17.	$A5, A6 \Rightarrow \perp$	– $(\neg E), 16.$
18.	$A5, A6 \Rightarrow P(x)$	– $(C), 17.$
19.	$P(x) \Rightarrow P(x)$	– axiom.
20.	$A4, A6 \Rightarrow P(x)$	– $(\vee E), 12, 18, 19.$
21.	$A4, A6 \Rightarrow \forall x P(x)$	– $(\forall I), 20.$
22.	$A4, A6 \Rightarrow \forall x P(x) \vee Q(a)$	– $(\vee I), 21.$
23.	$A4 \Rightarrow \forall x P(x) \vee Q(a)$	– $(\vee E), 13, 15, 22.$
24.	$\Rightarrow \forall x(P(x) \vee Q(a)) \rightarrow \forall x P(x) \vee Q(a)$	– $(\rightarrow I), 23.$

Problem 68 $(P(a) \vee P(b)) \rightarrow \exists x P(x)$.

A1.	$P(a) \vee P(b)$	
1.	$A1 \Rightarrow P(a) \vee P(b)$	– axiom.
A2.	$P(a)$	
2.	$A2 \Rightarrow P(a)$	– axiom.
3.	$A2 \Rightarrow \exists x P(x)$	– $(\exists I), 2.$
A3.	$P(b)$	
4.	$A3 \Rightarrow P(b)$	– axiom.
5.	$A3 \Rightarrow \exists x P(x)$	– $(\exists I), 4.$
6.	$A1 \Rightarrow \exists x P(x)$	– $(\vee E), 1, 3, 5.$
7.	$\Rightarrow (P(a) \vee P(b)) \rightarrow \exists x P(x)$	– $(\rightarrow I), 6.$

Problem 69 $(\exists x P(x) \wedge \forall x(P(x) \rightarrow Q(x))) \rightarrow \exists x Q(x)$.

A1.	$(\exists x P(x) \wedge \forall x(P(x) \rightarrow Q(x)))$	
1.	$A1 \Rightarrow (\exists x P(x) \wedge \forall x(P(x) \rightarrow Q(x)))$	– axiom.
2.	$A1 \Rightarrow \exists x P(x)$	– $(\wedge E), 1.$
3.	$A1 \Rightarrow \forall x(P(x) \rightarrow Q(x))$	– $(\wedge E), 2.$
4.	$A1 \Rightarrow P(x) \rightarrow Q(x)$	– $(\forall E), 3.$
5.	$P(x) \Rightarrow P(x)$	– axiom.
6.	$A1, P(x) \Rightarrow Q(x)$	– $(\rightarrow E), 5, 4.$
7.	$A1, P(x) \Rightarrow \exists Q(x)$	– $(\exists I), 6.$
8.	$A1 \Rightarrow \exists x Q(x)$	– $(\exists E), 2, 7.$
9.	$\Rightarrow (\exists x P(x) \wedge \forall x(P(x) \rightarrow Q(x)))$	– $(\rightarrow I), 8.$

Problem 70 $\exists x P(x) \leftrightarrow \exists y P(y)$.

A1.	$\exists x P(x)$	
1.	$A1 \Rightarrow \exists x P(x)$	– axiom.
A2.	$P(x)$	
2.	$A2 \Rightarrow P(x)$	– axiom.
3.	$A2 \Rightarrow \exists y P(y)$	– $(\exists I), 2.$
4.	$A1 \Rightarrow \exists y P(y)$	– $(\exists E), 1, 3.$
5.	$\Rightarrow \exists x P(x) \rightarrow \exists y P(y)$	– $(\rightarrow I), 4.$
A3.	$\exists y P(y)$	
6.	$A3 \Rightarrow \exists y P(y)$	– axiom.
A2.	$P(y)$	
7.	$A2 \Rightarrow P(y)$	– axiom.
8.	$A2 \Rightarrow \exists x P(x)$	– $(\exists I), 7.$
9.	$A1 \Rightarrow \exists x P(x)$	– $(\exists E), 1, 8.$
10.	$\Rightarrow \exists y P(y) \rightarrow \exists x P(x)$	– $(\rightarrow I), 9.$
11.	$\Rightarrow \exists x P(x) \leftrightarrow \exists y P(y)$	– $(\wedge I), 4, 10.$

Problem 71 $\neg \exists x P(x) \leftrightarrow \forall x \neg P(x)$.

A1.	$\neg \exists x P(x)$	
1.	$A1 \Rightarrow \neg \exists x P(x)$	– axiom.
A2.	$P(x)$	
2.	$A2 \Rightarrow P(x)$	– axiom.
3.	$A2 \Rightarrow \exists x P(x)$	– $(\exists I), 2.$
4.	$A1, A2 \Rightarrow \perp$	– $(\neg E), 1, 3.$
5.	$A1 \Rightarrow \neg P(x)$	– $(\neg I), 4.$
6.	$A1 \Rightarrow \forall x \neg P(x)$	– $(\forall I), 5.$
7.	$\Rightarrow \neg \exists x P(x) \to \forall x \neg P(x)$	– $(\to I), 6.$
A3.	$\forall x \neg P(x)$	
8.	$A3 \Rightarrow \forall x \neg P(x)$	– axiom.
9.	$A3 \Rightarrow \neg P(x)$	– $(\forall E), 8.$
10.	$P(x) \Rightarrow P(x)$	– axiom.
11.	$A3, P(x) \Rightarrow \perp$	– $(\neg E)$, 10, 9.
A4.	$\exists x P(x)$	
12.	$A4 \Rightarrow \exists x P(x)$	– axiom.
13.	$A3, A4 \Rightarrow \perp$	– $(\exists E), 12, 11.$
14.	$A3 \Rightarrow \neg \exists x P(x)$	– $(\neg I), 13.$
15.	$\Rightarrow \forall x \neg P(x) \to \neg \exists x P(x)$	– $(\to I), 14.$
16.	$\Rightarrow \neg \exists x P(x) \leftrightarrow \forall x \neg P(x)$	– $(\wedge I), 4, 15.$

Problem 71½ $\neg \forall x P(x) \leftrightarrow \exists x \neg P(x)$.

Inference Rules

$$(\wedge I)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G} \qquad (\wedge E)\frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow F} \quad \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow G}$$

$$(\vee I)\frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow F \vee G} \quad \frac{\Gamma \Rightarrow G}{\Gamma \Rightarrow F \vee G} \qquad (\vee E)\frac{\Gamma \Rightarrow F \vee G \quad \Delta, F \Rightarrow H \quad \Sigma, G \Rightarrow H}{\Gamma, \Delta, \Sigma \Rightarrow H}$$

$$(\rightarrow I)\frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad (\rightarrow E)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \rightarrow G}{\Gamma, \Delta \Rightarrow G}$$

$$(\neg I)\frac{\Gamma, F \Rightarrow \bot}{\Gamma \Rightarrow \neg F} \qquad (\neg E)\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow \neg F}{\Gamma, \Delta \Rightarrow \bot}$$

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F} \qquad (W)\ \frac{\Gamma \Rightarrow H}{\Gamma, \Delta \Rightarrow H}$$

$$(\forall I) \quad \frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow \forall v F}$$

where v is not a free variable of any formula in Γ.

$$(\forall E) \quad \frac{\Gamma \Rightarrow \forall v F}{\Gamma \Rightarrow F^v_t}$$

where t is substitutable for v in F

$$(\exists I) \quad \frac{\Gamma \Rightarrow F^v_t}{\Gamma \Rightarrow \exists v F}$$

where t is substitutable for v in F

$$(\exists E) \quad \frac{\Gamma \Rightarrow \exists v F \quad \Delta, F \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Sigma}$$

where v is not a free variable of any formula in Δ, Σ