CS388L: Introduction to Mathematical Logic
Quiz 6, Due April 15

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The following rule

$$\frac{\Gamma \Rightarrow F \to G \qquad \Delta \Rightarrow \neg G}{\Gamma, \Delta \Rightarrow \neg F}$$

is sound in G_3.

We need to show that if

$$\Delta \to \neg G \tag{1}$$
$$\Gamma \to (F \to G) \tag{2}$$

are tautological in G_3, then

$$\Gamma \land \Delta \to \neg F \tag{3}$$

is also tautological.

Proof. (1) is tautological means that $\forall I$, we have

$$(\Delta \to \neg G)^I = \to (\Delta^I, \neg G^I) = 1$$

From definition, we know that

$$\Delta^I \le \neg G^I \tag{4}$$

(2) is tautological means that $\forall I$, we have

$$(\Gamma \to (F \to G))^I = \to (\Gamma^I, (F \to G)^I) = 1$$

From definition, we know that

$$\Gamma^I \le (F \to G)^I \tag{5}$$

Consider the following two cases:

- if $F^I \leq G^I$, then $(F \to G)^I = \to (F^I, G^I) = 1$. Clearly, (5) holds true. We have:

$$\neg F^I \geq \neg G^I \tag{6}$$

  This is because if both F^I and G^I are greater than 0, then $\neg F^I = \neg G^I = 0$. If both F^I and G^I equal 0, then $\neg F^I = \neg G^I = 1$. If only one of them equals 0, it must be $0 = F^I < G^I$, then $0 = \neg G^I < \neg F^I = 1$.
  Combining (4) and (6), we have:

$$\Delta^I \leq \neg G^I \leq \neg F^I$$

  Thus,

$$\min(\Gamma^I, \Delta^I) \leq \neg F^I$$

- if $F^I > G^I$, then $(F \to G)^I = \to (F^I, G^I) = G^I$. From (4) and (5):

$$\begin{cases} \Delta^I \leq \neg G^I \\ \Gamma^I \leq \quad G^I \end{cases}$$

  Clearly, one of G^I and $\neg G^I$ must be zero. (If $G^I = 0$, the claim holds. If $G^I > 0$, then $\neg G^I = 0$.)
  Thus,

$$\min(\Gamma^I, \Delta^I) = 0 \leq \neg F^I$$

From both cases, we have $\min(\Gamma^I, \Delta^I) \leq \neg F^I$, this is the same as:

$$\begin{aligned} (\Gamma \wedge \Delta)^I &\leq \neg F^I \\ \to ((\Gamma \wedge \Delta)^I, \neg F^I) &= 1 \\ (\Gamma \wedge \Delta \to \neg F)^I &= 1 \end{aligned}$$

Thus, $\Gamma \wedge \Delta \to \neg F$ is tautological in G_3. $\square$