5 Adequate sets of connectives: Solutions

1. Show that $\{\neg, \rightarrow\}$ is an adequate set of connectives.

Solution $p \vee q$ is logically equivalent to $\neg p \rightarrow q$; $p \wedge q$ is equivalent to $\neg(p \rightarrow \neg q)$; $p \leftrightarrow q$ is equivalent to $\neg((p \rightarrow q) \rightarrow \neg(q \rightarrow p))$

2. The NOR connective has the truth table shown. Determine whether or not this connective is, on its own, adequate, justifying your answer.

p	q	pNORq
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

Solution Note that pNORq is equivalent to $\neg(p \vee q)$. Hence pNORp is equivalent to $\neg(p \vee p)$, that is, to $\neg p$, and we've got $\neg$. Also $p \wedge q$ is equivalent to $\neg(\neg p \vee \neg q)$ so this is equivalent to $(\neg p)$NOR$(\neg q)$, hence to $(p$NOR$p)$NOR$(q$NOR$q)$ and we've got $\wedge$ as well. Therefore NOR is adequate.

3. Prove that the connective § with truth table shown below is not adequate.

p	q	$p\S q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

Solution We prove by induction on complexity of terms, that for any term built up from a set $\mathcal{L}$ of propositional variables using § only - let us denote the set of such terms by $\mathrm{Term}_{\S}(\mathcal{L})$ - and for any valuation v, if $v(p) = \mathbb{F}$ for every $p \in \mathcal{L}$, then $v(s) = \mathbb{F}$ for every $s \in \mathrm{Term}_{\S}(\mathcal{L})$. That will be enough since then there will be no term in $\mathrm{Term}_{\S}(\mathcal{L})$ which is a tautology.

Base case (s a propositional variable): this is our hypothesis.

Induction step (just one): Suppose that $s = t\S u$ where, by induction, we may assume that $v(t) = \mathbb{F} = v(u)$. Then, consulting the truth table for §, we see that $v(s) = \mathbb{F}$, as required.

4. Determine whether or not the set $\{\wedge, \rightarrow\}$ is adequate, justifying your answer.

Solution No. We prove by induction on complexity of terms, that for any term built up from a set $\mathcal{L}$ of propositional variables using $\rightarrow$ and $\wedge$ only - let us denote the set of such terms by $\mathrm{Term}_{\rightarrow\wedge}(\mathcal{L})$ - and for any valuation v, if $v(p) = \mathbb{T}$ for every $p \in \mathcal{L}$, then $v(s) = \mathbb{T}$ for every $s \in \mathrm{Term}_{\rightarrow\wedge}(\mathcal{L})$.

Base case (s a propositional variable): this is our hypothesis on v.

Induction steps (two of them): Suppose that $s = t \rightarrow u$ where, by induction, we may assume that $v(t) = \mathbb{T} = v(u)$. Then $v(s) = v(t \rightarrow u) = \mathbb{T}$, by the truth table for $\rightarrow$. Similarly $v(t \wedge u) = \mathbb{T}$.

By induction we conclude that $v(s) = \mathbb{T}$ for every term in $\mathrm{Term}_{\rightarrow\wedge}(\mathcal{L})$. In particular there is no term in $\mathrm{Term}_{\rightarrow\wedge}(\mathcal{L})$ which is equivalent to $\neg p$. Hence $\{\rightarrow \wedge\}$ is not adequate.

5. Prove that the connective $\neg$ is not adequate.

Solution Let $\mathcal{L} = \{p, q\}$. We claim that no term in $\mathrm{Term}_{\neg}(\mathcal{L})$ is logically equivalent to $p \wedge q$. To prove this, suppose that $s \in \mathrm{Term}_{\neg}(\mathcal{L})$ is equivalent to $p \wedge q$. Then only one propositional variable appears in s (an easy inductive proof), say p appears. Let v and w be defined by $v(p) = \mathbb{T}$, $v(q) = \mathbb{T}$, $w(p) = \mathbb{T}$, $w(q) = \mathbb{F}$. Then $v(p \wedge q) \neq w(p \wedge q)$ yet $v(s) = w(s)$, so s is not equivalent to $p \wedge q$, contradiction as desired.

6. What does it mean to say that a set of propositional connectives is adequate? Suppose that the propositional connectives $*$ and $\circ$ have the truth tables shown. Show that $\{*, \circ\}$ is adequate. (You may use the fact that the set $\{\wedge, \neg\}$ is adequate.)

p	q	$p * q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

p	q	$p \circ q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

Solution (Very precise (more so than in the notes) answer to first part): A set C of propositional connectives is adequate if for every set $\mathcal{L}$ of propositional variables and every term $s \in S\mathcal{L}$ there is a term $t \in \mathrm{Term}_C\mathcal{L}$ such that, for every function $v_0 : \mathcal{L} \to \{\mathbb{T}, \mathbb{F}\}$, if v is the (unique) extension of v_0 to a valuation on $S\mathcal{L}$ and w is the unique extension of v_0 to a valuation on $\mathrm{Term}(\mathcal{L})$ then $w(t) = v(s)$.

(Perfectly adequate answer to first part): A set C of propositional connectives is adequate if for every propositional term s there is a term built using just the connectives in C to which s is logically equivalent.

(Answer to the rest) Note that $p * q$ is equivalent to $\neg p$ - so we have $\neg$ - and that $p \circ q$ [which we've seen already in Question 3] is equivalent to $\neg p \wedge q$. It will be enough to produce a term involving $*$ and $\circ$ which is equivalent to $p \wedge q$: $(p * q) \circ q$ (or $(p * p) \circ q$) will do.