profit with probability 1 should the asset price ever reach the value at which the discontinuity occurred.[1]

Just as in the obstacle problem, we do not know the position of S_f, and we must impose *two* conditions at S_f if the option value is to be uniquely determined. This is one more than if S_f were specified. The second condition at S_f, our fourth constraint above, is that the option delta must also be continuous there. Its derivation is rather more delicate, and we only give an informal financially based argument, for the specific case of the American put.

7.4 The American Put

Consider the American put option, with value $P(S,t)$. We have already argued that this option has an exercise boundary $S = S_f(t)$, where the option should be exercised if $S < S_f(t)$ and held otherwise. Assuming that $S_f(t) < E$, the slope of the payoff function $\max(E - S, 0)$ at the contact point is -1. There are three possibilities[2] for the slope (delta) of the option, $\partial P/\partial S$, at $S = S_f(t)$:

- $\partial P/\partial S < -1$;
- $\partial P/\partial S > -1$;
- $\partial P/\partial S = -1$.

We show that the first two are incorrect.

Suppose first that $\partial P/\partial S < -1$. Then as S increases from $S_f(t)$, $P(S,t)$ drops below the payoff $\max(E - S, 0)$, since its slope is more negative; see Figure 7.3(a). This contradicts our earlier arbitrage bound $P(S,t) \geq \max(E - S, 0)$, and so is impossible.

Now suppose that $\partial P/\partial S > -1$, as in Figure 7.3(b). In this case, we argue that an option value with this slope would be sub-optimal for the holder, in the sense that it does not give the option its maximum value consistent with the Black–Scholes risk-free hedging strategy and the arbitrage constraint $P(S,t) \geq \max(E-S, 0)$. In order to see this, we must discuss the strategy adopted by the holder. There are two aspects to consider. One is the day-to-day arbitrage-based hedging strategy which,

[1] This result does *not* imply that there is a blanket prohibition of discontinuous option prices, caused for example by an instantaneous change in the terms of the contract such as the imposition of a constraint by a change from European to American. Indeed, such discontinuities, or jumps, play an important part in later chapters.

[2] A fourth is that $\partial P/\partial S$ does not exist at $S = S_f(t)$. We assume, as can be shown to be the case, that it does.

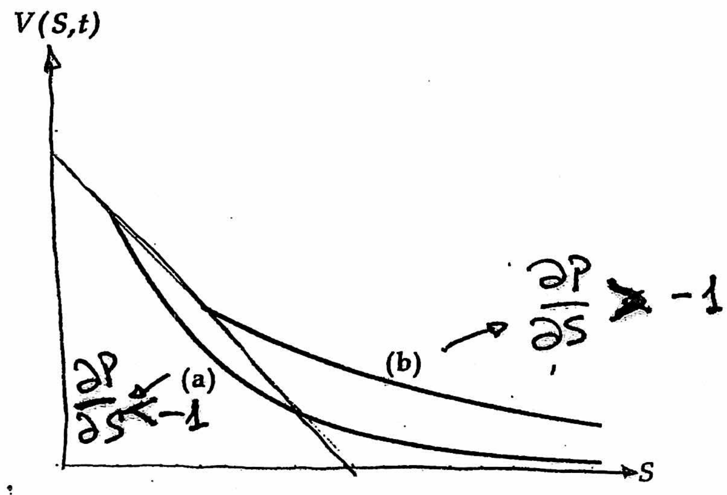

Figure 7.3. Exercise price (a) too low (b) too high.

as above, leads to the Black–Scholes equation. The other is the exercise strategy: the holder must decide, in principle, how far S should fall before he would exercise the option. The basis of this decision is, naturally enough, that the chosen strategy should maximise an appropriate measure of the value of the option to its holder.[3] Because the option satisfies a partial differential equation with $P(S_f(t),t) = E - S_f(t)$ as one of the boundary conditions, the choice of $S_f(t)$ affects the value of $P(S,t)$ for *all* larger values of S. Clearly the case of Figure 7.3(a) corresponds to too low a value of $S_f(t)$, and an arbitrage profit is possible for S just above $S_f(t)$. Conversely, if $\partial P/\partial S > -1$ at $S = S_f(t)$, the value of the option near $S = S_f(t)$ can be increased by choosing a smaller value for S_f: the exercise value then moves up the payoff curve and $\partial P/\partial S$ decreases. The option is thus again misvalued. In fact, the increase in P is passed on by the partial differential equation to all values of S greater than S_f, and by decreasing S_f we arrive at the crossover point between our two incorrect possibilities, which simultaneously maximises the benefit to the holder and avoids arbitrage. This yields the correct free boundary condition $\partial P/\partial S = -1$ at $S = S_f(t)$.

[3] This choice also minimises the benefit to the writer, but since the holder can close the contract by exercising and the writer cannot, the latter's point of view is not relevant to this argument. Of course, the writer requires a greater premium in recompense for the one-sided nature of the contract.