Monte Carlo Methods for American Options

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Contents

1 Introduction

There is no analytical formula for valuing American options in the Black-Scholes framework. But some special cases are tractable.

Strictly speaking, we will be pricing (daily) *Bermudan options* today. Pricing true Americans is harder and less practical.

1.1 Tractable special cases

Zero dividend call: when dividend rates are zero, American calls are European. To see this, let C_A, C_E be the price of the American and European calls respectively. By put/call parity, we have

$$\begin{aligned} C_A &\geq C_E \\ &= P_E + S_0 - Ke^{-rT} \\ &> P_E + S_0 - K \\ &> S_0 - K \end{aligned}$$

Note that the last two inequalities aren't strict if $T = 0$.[1] But when time to maturity is non-zero, the European call is strictly more valuable than the exercise value. Thus, it's always better to hold the American than to exercise.

Zero rates put: The same argument as above shows that American puts are European when interest rates are 0.

$$\begin{aligned} P_A &\geq P_E \\ &= C_E + K - S_0 e^{-qT} \\ &> C_E + K - S_0 \\ &> K - S_0 \end{aligned}$$

The perpetual put: an American put option $(K - S_t)^+$ with infinite time to expiry can, surprisingly, be priced in closed form. See [Shr04] Section 8.3.

1.2 Rationale

If they're so hard to price why do American options even exist? Is there an economic reason?

Suppose you think IBM is going to trade up and you buy some IBM calls that expire in three months. Two months later, IBM's up 15% and you're ready to lock in your gains, and reallocate capital.

1. If you bought a European option, you'll sell it.

2. If you bought an American option, you'll exercise it and sell the stock.

[1] In fact, we also need $r \geq 0$.

The underlying stock will be significantly more liquid than the option at your strike. Consequently it will be cheaper to sell, in terms of transaction cost, so the American option seems more attractive.

In the “good” old days of trading, this difference could be significant, and this created a demand for American options.

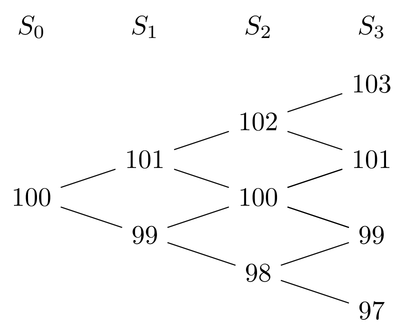
Hopefully people will stop trading them in the future, but they are currently the standard for single name equities, crude oil, etc.

2 Analytical methods

2.1 Binomial tree model

We begin by revisiting the trusty binomial tree pricing model. Consider an American put struck at $K = 101$ and single period discount factor df $= 0.95$.

We model the evolution of the underlying stock using the following binomial tree:



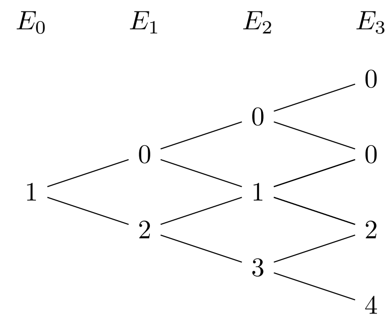
Assume for simplicity that the risk neutral probabilities are 0.5.

More formally, we can write the stock price dynamics as

$$S_{t+1} = S_t + Z_t \tag{1}$$

where Z_t are i.i.d. Rademacher random variables.

We will calculate for each node, three quantities: the exercise value E, the continuation value C, and the option value V. The exercise value is just $E_t(S) = \max(K - S, 0)$, so we readily get:



The value of the option at time t and stock price S is given by

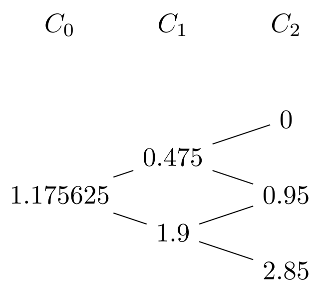
$$V_t(S) = \max(E_t(S), C_t(S)) \tag{2}$$

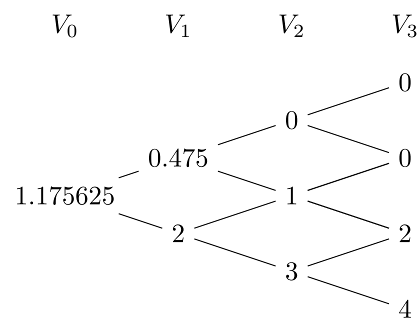
Here $C_t(S)$ is the continuation value, namely, the discounted expected value of the option if it is not exercised at time t. In the binomial model it is given by

$$C_t(S) = \mathrm{df}\,[pV_{t+1}(uS) + (1-p)V_{t+1}(dS)] \tag{3}$$

(While it is suppressed from the notation, p, u and d can depend on t and S also.)

Using equations (2) and (3) we can, starting from $t = 3$ and working backwards inductively, compute all the values C_t and V_t:





We can also express equation (3) in a more concise, model independent way:

$$C_t(S) = \mathrm{df}\,\mathbb{E}[V_{t+1}(S_{t+1})|S_t = S] \tag{4}$$

2.2 Black-Scholes model

In the last section, we used binomial stock price dynamics. Here, we will use geometric Brownian motion, which is underpins the Black-Scholes model:

$$S_{t+\Delta t} = S_t \exp\left\{\left(r - q - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z_t\right\} \tag{5}$$

where Z_t are i.i.d. standard normal random variables.

Can we try the same approach here as in the binomial model?

At maturity, we have

$$V_T(S) = (K - S)^+$$

To find $C_{T-\Delta t}(S)$ we follow equation (4):

$$C_{T-\Delta t}(S) = e^{-r\Delta t}\mathbb{E}[V_T(S_T)|S_{T-\Delta t} = S] \tag{6}$$

The conditional expectation can be found by integration:

$$\mathbb{E}[V_T(S_T)|S_{T-\Delta t} = S] = \int (K - x)^+ p(S_T = x|S_{T-\Delta t} = S)\mathrm{d}x$$

where p is the conditional density function for the stock price process:

$$p(S_T = x|S_{T-\Delta t} = S) = \frac{\mathbb{1}_{x>0}}{x\sqrt{2\pi\sigma^2\Delta t}} \exp\left\{-\frac{[\ln(x/S) - (r - q - \sigma^2/2)\Delta t]^2}{2\sigma^2\Delta t}\right\}$$

We can solve the integral in closed form and get:

$$C_{T-\Delta t}(S) = Ke^{-r\Delta t}N(-d_2) - Se^{-q\Delta t}N(-d_1)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + \left(r - q + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}} \\ d_2 &= d_1 - \sigma\sqrt{\Delta t} \end{aligned}$$

The next step is also easy:

$$\begin{aligned} V_{T-\Delta t}(S) &= \max\left((K - S)^+, C_{T-\Delta t}(S)\right) \\ &= \max\left(K - S, Ke^{-r\Delta t}N(-d_2) - Se^{-q\Delta t}N(-d_1)\right) \end{aligned}$$

But calculating $C_{T-2\Delta t}$ in closed form seems unlikely:

$$C_{T-2\Delta t}(S) = e^{-r\Delta t}\int V_{T-\Delta t}(x)p(S_{T-\Delta t} = x|S_{T-2\Delta t} = S)\mathrm{d}x$$

Question: What is $p(S_{T-\Delta t} = x|S_{T-2\Delta t} = S)$?

In the binomial model, the conditional expectation (4) was just a weighted average

$$pV(uS) + (1 - p)V(dS)$$

and could be computed easily. In the Black-Scholes model, alas, it's a whole integral. And to make matters worse, it's a different integral for each of the infinitely many S, so it's not obvious how to apply quadrature methods.

3 Interlude: Regression

In this section, we derive a way of approximating the conditional expectation (4) in the Black-Scholes model using Monte Carlo simulation.

The method uses a general technique for calculating conditional expectations known as *regression*.

Consider a random variable Y with mean μ_Y and variance σ_Y^2. The expectation μ_Y minimizes the mean squared error:

$$\min_{y \in \mathbb{R}} \mathbb{E}[(Y - y)^2] = \mathbb{E}[(Y - \mu_Y)^2] = \sigma_Y^2$$

The conditional expectation of Y is defined analogously, except instead of a single number μ_Y, the result will be a function $e_Y(x)$.

Let $X : \Omega \to \mathbb{R}^n$ be a random vector. The conditional expectation $e_Y : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that

$$\min_{f \text{ measurable}} \mathbb{E}\left((Y - f(X))^2\right) = \mathbb{E}\left((Y - e_Y(X))^2\right). \tag{7}$$

Note that unlike μ_Y, the conditional expectation $e_Y(x)$ is not generally unique: there may be multiple minimizers of the mean squared error.

The minimization problem in equation (7) is intractable in practice, so instead of minimizing over all measurable functions f, we restrict ourselves to a subset $\mathcal{F}$:

$$\min_{f \in \mathcal{F}} \mathbb{E}\left[(Y - f(X))^2\right]$$

For example $\mathcal{F}$ could be all polynomials of degree at most 2:

$$\min_{(a,b,c) \in \mathbb{R}^3} \mathbb{E}\left[(Y - (aX^2 + bX + c))^2\right] \geq \min_{f \text{ measurable}} \mathbb{E}\left[(Y - f(X))^2\right] \tag{8}$$

The minimizer $a_* X^2 + b_* X + c_*$ won't be as close to Y as the true conditional expectation, but can still be accurate enough.

We can use Monte Carlo to find the optimal coefficients a_*, b_*, c_* by sampling i.i.d. values $\{(x_i, y_i)\}_{i=1}^n$ from the joint distribution (X, Y) and then solving

$$\min_{a,b,c} \frac{1}{N} \sum_{i=1}^{N} (y_i - (a x_i^2 + b x_i + c))^2.$$

By letting F denote the *Vandermonde matrix* for the sample,

$$F = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$$

we can express the minimization problem as:

$$\min_{a,b,c} \frac{1}{N} \left\| y - F \begin{pmatrix} c \\ b \\ a \end{pmatrix} \right\|^2$$

The closed-form solution is then given by:

$$\begin{pmatrix} c_* \\ b_* \\ a_* \end{pmatrix} = (F'F)^{-1}F'y$$

4 Least squares Monte Carlo

We now describe a complete Monte Carlo algorithm to price "American" options. This method may be called Tsitsiklis–Van Roy algorithm [TV01].

First, we simulate paths from the underlying Markov process. We follow the example in [LS01] with strike $K = 1.1$ and interest rate $r = 6\%$. The tables below show the paths and the optimal exercise value at time $t = 3$.

#	S_0	S_1	S_2	S_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

#	V_0^*	V_1^*	V_2^*	V_3^*
1	—	—	—	0.00
2	—	—	—	0.00
3	—	—	—	0.07
4	—	—	—	0.18
5	—	—	—	0.00
6	—	—	—	0.20
7	—	—	—	0.09
8	—	—	—	0.00

We now regress the discounted payoffs $e^{-r\delta t}V_3^*$ onto S_2 to calculate the continuation value. Using quadratic regression, this comes out to

$$C_2^*(S_2) = e^{-r\delta t}\mathbb{E}[V_3^*|S_2] \approx 0.82 - 1.14S_2 + 0.39S_2^2$$

#	C_0^*	C_1^*	C_2^*	C_3^*
1	—	—	0.05	0.00
2	—	—	0.01	0.00
3	—	—	0.05	0.00
4	—	—	0.08	0.00
5	—	—	-0.01	0.00
6	—	—	0.18	0.00
7	—	—	0.14	0.00
8	—	—	0.01	0.00

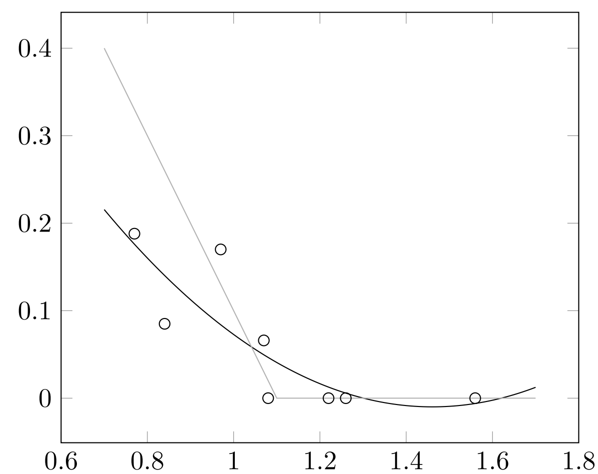


Figure 2: Regression at time $t = 2$. The x-axis is S_2, the stock price at time 2, the circles are $e^{-r\delta t}V_3^*$. These points are interpolated by the quadratic regression curve C_2^*. The grey line is the value of exercising at time 2. Note that the exercise region is disconnected.

To find V_2^*, we take, for each path, the greater of the two curves in Figure 2:

$$V_2^*(S_2) = \max((K - S_2)^+, C_2^*(S_2))$$

We can then expand our table and repeat this process to find V_1^* and V_0^*.

#	S_0	S_1	S_2	S_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

#	V_0^*	V_1^*	V_2^*	V_3^*
1	—	—	0.05	0.00
2	—	—	0.01	0.00
3	—	—	0.05	0.07
4	—	—	0.13	0.18
5	—	—	0.00	0.00
6	—	—	0.33	0.20
7	—	—	0.26	0.09
8	—	—	0.01	0.00

5 Regression II: Getting technical

5.1 The condition number

In the example in section 3 we approximated a generic function $f(X)$ by a quadratic. It is a natural extension to add columns of higher powers of x_i to F. The resulting matrix is always of full rank (see Vandermonde determinant), but $F'F$ is difficult to invert in practice.

To formalize this difficulty, we first extend the notion of relative error to vectors:

$$\varepsilon\text{-rel}(x_{\text{approx}}, x_{\text{true}}) = \frac{||x_{\text{approx}} - x_{\text{true}}||}{||x_{\text{true}}||}$$

5.1.1 Multiplication

Consider an algorithm that calculates

$$y = \text{multiply}(\text{A}, \text{x}) = Ax$$

for a non-singular matrix A. We are interested in the sensitivity of y to small perturbations in x due to numerical error.

Let δy be the sensitivity to a change δx in x:

$$\delta y = \text{multiply}(A, x + \delta x) - \text{multiply}(A, x) = A\delta x$$

The *condition number* $\kappa(A)$ gives an upper bound on the relative error of the output:

$$\frac{||\delta y||}{||y||} \leq \kappa(A)\frac{||\delta x||}{||x||} \tag{9}$$

This inequality is tight, that is, for all A there exist x and δx that make it an equality.

Thus the condition number captures the potential amount of lost precision, when multiplying a vector x by a matrix A.

The condition number can be expressed in terms of matrix norms or using the singular values of A:

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}$$

5.1.2 Linear equations

Now consider solving a non-singular linear system

$$Ax = y$$

The solution is given by

$$x = \text{solve}(A, y) = A^{-1}y.$$

We are interested in the sensitivity of x to small perturbations A and y.

Let δx be the sensitivity to a change δy in y:

$$\delta x = \text{solve}(A, y + \delta y) - \text{solve}(A, y) = A^{-1}\delta y$$

Question: Note that $\text{solve}(A, y) = \text{multiply}(A^{-1}, y)$. Explain using equation (9) why

$$\frac{||\delta x||}{||x||} \leq \kappa(A)\frac{||\delta y||}{||y||} \tag{10}$$

Now let δx be the sensitivity to a change δA in A:

$$\delta x = \text{solve}(A + \delta A, y) - \text{solve}(A, y) = \left[(A + \delta A)^{-1} - A^{-1}\right] y$$

A similar result to equation (10) can be shown in the limit $\delta A \to 0$:

$$\frac{||\delta x||}{||x||} \leq \kappa(A) \frac{||\delta A||}{||A||} \tag{11}$$

As before, for each A, there exist y and δA to make this equality (asymptotically at least).

Thus, the condition number of A captures the potential amount of precision lost when solving linear systems $Ax = y$.

The problem is that $\kappa(A) \geq 1$ and each of the bounds in equations (9) (10) and (11) are attained.

What this means is that even if your linear solver is numerically stable, errors in A and y may be amplified.

Worse yet, the Vandermonde matrices are particularly bad offenders.

$$F = \begin{pmatrix} 1 & x_1 & \dots & x_1^d \\ 1 & x_2 & \dots & x_2^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^d \end{pmatrix}$$

For example, when $\{x_i\}_{i \in 1, \dots 1000}$ come from a standard normal distribution and $d = 10$, the median condition number is

$$\kappa(F) \approx 90,000 \tag{12}$$

Therefore it's possible to lose up to 5 digits of precision from multiplying with or solving equations of F.

5.1.3 Least squares

Finally, we are interested in the function

$$\begin{aligned} x, \hat{y} &= \text{leastsq}(A, y) \\ &= \text{argmin}_x \, ||Ax - y||, \; A \, \text{argmin}_x \, ||Ax - y|| \\ &= (A'A)^{-1} A'y, \; A(A'A)^{-1} A'y \end{aligned}$$

for a full-rank rectangular matrix A.

The sensitivities of x and $\hat{y}$ to perturbations of A and y are also functions of the condition number:

	$\hat{y}$	x
y	$\frac{1}{R}$	$\frac{\kappa(A)}{\eta R}$
A	$\frac{\kappa(A)}{R}$	$\kappa(A) + \frac{\sqrt{1-R^2}\kappa(A)^2}{\eta R}$

For details and further reading see Part III of [TB97].

5.2 Orthogonal basis functions

In the last section we saw that the higher the condition number $\kappa(A)$, the worse the loss of precision from solving a least squares problem.

To address this issue, we turn to *orthogonal polynomials*.

Example. Laguerre polynomials

$$\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= 1 - x \\
f_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\
f_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \\
&\dots
\end{aligned}$$

Example. Hermite polynomials

$$\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= x \\
f_2(x) &= x^2 - 1 \\
f_3(x) &= x^3 - 3x \\
&\dots
\end{aligned}$$

These polynomials are orthogonal in the sense that

$$\int_{-\infty}^{\infty} f_i(x) f_j(x) p(x) \mathrm{d}x = 0 \text{ if } i \neq j \tag{13}$$

For Laguerre $p(x) = e^{-x}\mathbb{1}_{x\geq 0}$ and for Hermite $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

In fact, the Laguerre polynomials are orthonormal, but the Hermite polynomials aren't:

$$\int_{-\infty}^{\infty} \mathrm{He}_k^2(x) n(x) \mathrm{d}x = k!$$

Question: Evaluate the following integrals using the orthonormality relation:

$$\begin{aligned}
\int_0^{\infty} (1-x)^2 e^{-x} \mathrm{d}x &= \quad ? \\
\int_{-\infty}^{\infty} (x^2 - 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x &= \quad ? \\
\int_{-\infty}^{\infty} (x^4 - 2x^2 + 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x &= \quad ?
\end{aligned}$$

If X is standard exponentially or standard normally distributed (respectively), the random variables $f_0(X), f_1(X) \dots f_d(X)$ will be uncorrelated. As the sample size n increases, the matrix $\Phi'\Phi$ will be closer and closer to diagonal for

$$\Phi = \begin{pmatrix} f_0(x_1) & f_1(x_1) & \dots & f_d(x_1) \\ f_0(x_2) & f_1(x_2) & \dots & f_d(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & \dots & f_d(x_n) \end{pmatrix}$$

Let's take a look at an example with Hermite polynomials of degree 2, assuming all x_i are drawn from $\mathcal{N}(0,1)$. We write $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for conciseness.

$$\begin{aligned}
\mathbb{E}[\Phi'\Phi] &= \mathbb{E}\left[\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 - 1 & x_2^2 - 1 & \dots & x_n^2 - 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 - 1 \\ 1 & x_2 & x_2^2 - 1 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 - 1 \end{pmatrix} \right] \\
&= \mathbb{E}\left[\begin{pmatrix} n & \sum x_i & \sum (x_i^2 - 1) \\ \sum x_i & \sum x_i^2 & \sum (x_i^3 - x_i) \\ \sum (x_i^2 - 1) & \sum (x_i^3 - x_i) & \sum (x_i^2 - 1)^2 \end{pmatrix} \right] \\
&= n \begin{pmatrix} 1 & \int x p(x) \mathrm{d}x & \int (x^2 - 1) p(x) \mathrm{d}x \\ \int x p(x) \mathrm{d}x & \int x^2 p(x) \mathrm{d}x & \int (x^3 - x) p(x) \mathrm{d}x \\ \int (x^2 - 1) p(x) \mathrm{d}x & \int (x^3 - x) p(x) \mathrm{d}x & \int (x^2 - 1)^2 p(x) \mathrm{d}x \end{pmatrix} \\
&= \begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 2n \end{pmatrix}
\end{aligned}$$

Note that the columns of F and Φ have the same span and thus the fitted values will agree[2]:

$$F(F'F)^{-1}F'y = \Phi(\Phi'\Phi)^{-1}\Phi'y$$

We are computing the best fitting polynomial of degree d in both cases after all. The difference is purely numerical.

Also note that $\kappa(\Phi'\Phi) \approx d!$, so we're still not well-conditioned, but the situation is already much better than for the monomial basis.

Question: How can the condition number be improved?

In our application, we expect the price to close to lognormal, so

$$X = \frac{S_t - \mathbb{E}[S_t]}{\mathrm{std}(S_t)}$$

will be decently close to a standard normal and we can use Hermite polynomials.

5.3 Discussion

Orthogonal polynomials have a rich theory and are useful in many settings. They are used in the original paper by Longstaff and Schwartz [LS01], but they are by no means the only way to do non-linear regression.

[2] The regression coefficients $(F'F)^{-1}F'y$ and $(\Phi'\Phi)^{-1}\Phi'y$ will of course not.

The authors of [PBS01] use adaptive splines for their regressions.

Orthogonal polynomials can be applied to higher dimesional problems by forming products of the basis functions.

Question: How would you design orthogonal polynomials for a pair of correlated stocks?

6 Dynamic programming

6.1 States and actions

Let us continue with the Black-Scholes model of S_t sampled at discrete times $t = 0, \Delta t, 2\Delta t, \ldots T$.

We will also need an additional state variable x which is 0 is the option has already been exercised and 1 if it's still exercisable:

$$x_{t+\Delta t} = (x_t - a_t)^+ \tag{14}$$

Here $a_t \in \{0, 1\}$ denotes our *action* at time t, 0 meaning "don't exercise" and 1 meaning "exercise".

Question: Explain equation (14) in words.

More formally, an *exercise policy* for a time period $[t, T]$ is a function

$$a : [t, T] \times \mathbb{R}^+ \to \{0, 1\}$$
$$a_t(S) = \begin{cases} 0 & \text{if continue to hold} \\ 1 & \text{if exercise} \end{cases}$$

We will use $\mathcal{A}[t, T]$ denote the set of all admissible policies for period $[t, T]$.

Consider a put option with strike K. For European exercise

$$a_t^{\mathrm{Eur}}(S) = \begin{cases} 0 & \text{if } t < T \\ \mathbb{1}_{K>S} & \text{if } t = T \end{cases}$$

For American exercise we can use the following instead

$$a_t(S) = \mathbb{1}_{b_t > S}$$

Here $b_t \in \mathbb{R}$ is the *exercise boundary.*

For different payoffs, such as the straddle $|S_t - K|$ or higher dimensional problems $\left[\max\left(S_t^{(1)}, S_t^{(2)}\right) - K\right]^+$ the exercise region will have a more complicated form than $S_t \in [0, b_t)$, see Figure 3.[3]

[3] If the dividend rate is zero, the straddle, being worth more than a call will never be optimal to exercise into $S - K$, only into $K - S$.

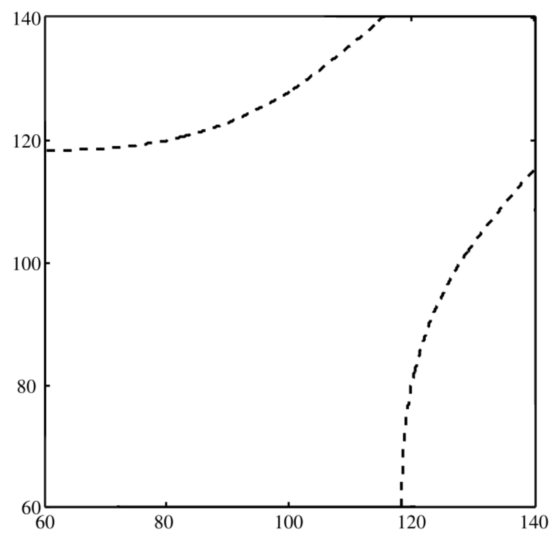


Figure 3: Exercise region for American max option on two independent assets with strike $K = 100$. (Fig. 8.9. from [Gla13])

The value of the American option depends on the exercise policy used. For a specific policy a, we have

$$V_0^{(a)} = \mathbb{E}\left[\sum_{t=0}^{T} e^{-rt}(K - S_t)^+ x_t a_t\right].$$

Example 1. If we use the European exercise a_t^{Eur} as defined above then $x = (1,1,1,1)$ and the value is

$$V_0^{(a^{\mathrm{Eur}})} = Ke^{-rT}N(-d_2) - S_0 e^{-qT}N(-d_1)$$

Example 2. If $a = (0,0,1,1)$ then $x = (1,1,1,0)$, and the value is

$$\begin{aligned} V_0^{(a)} &= \mathbb{E}\left[(K - S_2)^+ e^{-r2\Delta t}\right] \\ &= Ke^{-r2\Delta t}N(-d_2) - S_0 e^{-q2\Delta t}N(-d_1) \end{aligned}$$

6.2 Subproblems and optimality

When you sell someone an American option you have to prepare for the worst case scenario: that your counterparty will exercise the option optimally. Thus the value you will have to charge is

$$V = \sup_{a \in \mathcal{A}[0,T]} V_0^{(a)}$$

We will solve this optimization problem using dynamic programming.[4]

Let us introduce the subproblems

$$V_t^{(a)}(S,x) = \mathbb{E}\left[\sum_{u=t}^{T} e^{-r(u-t)}(K-S_u)^+ x_u a_u(S_u) \middle| S_t = S, x_t = x\right] \tag{15}$$

We assume here that the stock price is a Markov process. In a stochastic volatility model for example, where S_t is non-Markovian, we would have to condition on σ_t as well and work with $V_t^{(a)}(S,\sigma,x)$ and $a_t(S,\sigma)$.

An option that has been exercised has zero remaining value, i.e.

$$V_t^{(a)}(S,0) = 0,$$

so we will only be interested in $V_t^{(a)}(S,1)$. But both values $x=0$ and $x=1$ are needed to express the equations (16) and (18) below, so we keep the x argument for now.

The subproblems (15) satisfy the recursive equations:

$$\begin{aligned} V_t^{(a)}(S,x) &= (K-S_t)^+ x a_t(S) \\ &\quad + e^{-r\Delta t}\mathbb{E}\left[V_{t+\Delta t}^{(a)}(S_{t+\Delta t},(x-a_t(S))^+ \middle| S_t = S, x_t = x\right] \\ V_T^{(a)}(S,x) &= (K-S)^+ x a_T(S) \end{aligned} \tag{16}$$

where we have written $(x_t - a_t)^+$ instead of just $x_{t+\Delta t}$ to emphasize the dependence on a_t.

Question: Is x_t random? Is $V_t^{(a)}(S,x)$ random?

The conditional expectation in equation (16) is familiar from section 2.1. It is the continuation value:

$$C_t^{(a)}(S) = e^{-r\Delta t}\mathbb{E}\left[V_{t+\Delta t}^{(a)}(S_{t+\Delta t}, x_{t+\Delta t}) \middle| S_t = S, x_t = 1\right] \tag{17}$$

It represents the value of continuing to hold the option instead of immediately exercising.

Theorem 1 (Principle of optimality)**.** *Suppose a^* is an optimal policy, i.e.*

$$V_t^* \stackrel{\text{def}}{=} V_t^{(a^*)} = \sup_{a\in\mathcal{A}[t,T]} V_t^{(a)}$$

The Bellman equations *give a necessary and sufficient condition for optimality:*

$$\begin{aligned} V_t^*(S,x) &= \max_{a\in\{0,1\}} \left\{(K-S)^+ xa \right. \\ &\quad \left. + e^{-r\Delta t}\mathbb{E}\left[V_{t+\Delta t}^*(S_{t+\Delta t},(x-a)^+) \middle| S_t = S, x_t = x\right]\right\} \\ V_T^*(S,x) &= (K-S)^+ x \end{aligned} \tag{18}$$

[4]When dynamic programming was invented, the word *programming* meant something closer to *planning*. Today is means something completely different.

Proof. Technical. For a discussion and a general reference see [Ber12]. □

The Bellman iteration above is written in its general form and it's possible to simplify it. Since we know $V_t^{(a)}(S,0) = 0$,[5] we can substitute this into equation (18)

$$\begin{aligned} V_t^*(S,1) &= \max\left((K-S)^+, e^{-r\Delta t}\mathbb{E}\left[V_{t+\Delta t}^*(S_{t+\Delta t},1)\middle|S_t = S, x_t = 1\right]\right) \\ &= \max\left((K-S)^+, C_t^*(S)\right) \\ V_T^*(S,1) &= (K-S)^+ \end{aligned} \tag{19}$$

The optimal policy is given by

$$a_t^*(S) = \mathbb{1}_{(K-S)^+ > C_t^*(S)}.$$

In what follows we will stop writing $V_t(S,0)$ and just write $V_t(S)$ for short.

Theorem 1 and equation (19) justify the binomial pricing method for American options. They tell us that there doesn't exist a smarter exercise policy a^* that our client could use to extract more value from the option than we priced in.

6.3 High and low bias

In Section 4, we assumed that

$$C_t^*(S_t) \approx \alpha_0 + \alpha_1 S_t + \cdots + \alpha_d S_t^d \tag{20}$$

Even when using a different form of regression, be it splines, kernel regression, etc. it is unusual for the equation (20) to hold with an equality.

Any imperfect interpolation of the data points will constrain the space of value functions V_t explored, and consequently, we will find the best exercise policy $(a_0^*, a_1^*, \ldots a_T^*)$ not in the space of all policies, but in a restricted subset. This, sub-optimal policy, will lead us to underprice the option and which we call *low bias*.

Setting this aside, there is another source of bias. Let $\mathcal{A}$ be a set of exercise policies.

The algorithm above calculates

$$\max_{(a)\in\mathcal{A}} \hat{V}_0^{(a)}$$

This quantity is random; running it on new paths will give a different price. If we run it many times, the price will **on average** be higher than the true value.

$$\mathbb{E}\left[\max_{(a)\in\mathcal{A}} \hat{V}_0^{(a)}\right] \geq V_0^{(a^*)}$$

[5] It's easy to prove by induction.

We call this *high bias*. The proof is by induction, and technical. See [Gla13] Section 8.5.1.

Intuitively, the reason for the high bias is that our algorithm is able to see the future (so some degree) and is able to come up with a better exercise policy than it could from just knowing the present.

In the extreme case that we chose a polynomial degree one less than the number of paths, the resulting regression curve is simply the Lagrange interpolant that goes through each data point.

As a result each path will have complete clairvoyance of the future and the value of the option will simply be

$$\frac{1}{n}\sum_{i=1}^{n}\max_{t\in[0,T]} e^{-rt}(K-S_t^{(i)})^+$$

This lookback style payoff is significantly more valuable than the American put even when interest rates are zero.

To eliminate the high bias, two sets of paths must be generated. The first set to determine an optimal exercise policy and the second the evaluate the exercise policy out-of-sample.

Suppose we have computed functions C_0^*, C_1^*, C_2^*. using the first set of paths. Then for each path (S_0, S_1, S_2, S_3) we can evaluate the value of the option as follows. If $C_0^*(S_0) < K - S_0$ exercise immediately. If not, look at $C_1^*(S_1) < K - S_1$ and decide whether or not to exercise at time 1. Continue with this process until the option is exercised/expired. Average the resulting discounted payoffs across all paths in the second set. The resulting price will then be a pure lower bound due to the low bias explained above.

6.4 Coming to America

In this section we give a highly informal derivation of the partial differential equation satisfied by $V_t^*(S)$ in the limit where $\Delta t \to 0$. In what follows, let us write $\delta \stackrel{\text{def}}{=} \Delta t$ for conciseness.

We start off with the Bellman equation (19):

$$\max\big((K-S)^+ - V^\delta(t,S), C^\delta(t,S) - V^\delta(t,S)\big) = 0$$

Note that the functions C and V both depend on the exercise frequency $1/\delta$.

Question: What can you say about the signs of $\frac{\partial V}{\partial \delta}$ and $\frac{\partial C}{\partial \delta}$?

The equation holds if and only if

$$(K-S)^+ - V^\delta(t,S) \le 0 \tag{21}$$

$$C^\delta(t,S) - V^\delta(t,S) \le 0 \tag{22}$$

with at least one of the two holding with equality for each t, S.

We can take the limit of the inequality (21) in a straightforward way and get

$$(K-S)^+ \le V(t,S).$$

The left hand side of inequality (22), however is on the order of δ so we have to divide it by $\delta > 0$ before taking the limit.

We will also need the following result from stochastic calculus.

Theorem 2. *Suppose that X_t is a Ito process:*

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

Then we have

$$\lim_{\Delta\to 0} \frac{1}{\Delta t}\mathbb{E}\left[X_{t+\Delta t} - X_t | \mathcal{F}_t\right] = \mu_t \tag{23}$$

$$\lim_{\Delta\to 0} \frac{1}{\Delta t}\operatorname{Var}\left[\left(X_{t+\Delta t} - X_t\right)^2 | \mathcal{F}_t\right] = \sigma_t^2 \tag{24}$$

Proof. A version of this theorem appears as a definition in [Pav14]. $\square$

Returning to the limit:

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\left(C^\delta(t,S) - V^\delta(t,S)\right)\\
=&\lim_{\delta\to 0}\frac{1}{\delta}\left(e^{-r\delta}\mathbb{E}\left[V^\delta(t+\delta, S_{t+\delta}) | S_t = S, x_t = 1\right] - V^\delta(t,S)\right)\\
=&\lim_{\delta\to 0}\frac{1}{\delta}\left(\mathbb{E}\left[V^\delta(t+\delta, S_{t+\delta}) | S_t = S, x_t = 1\right] - V^\delta(t,S)\right)\\
&+\frac{1}{\delta}\left(e^{-r\delta} - 1\right)\mathbb{E}\left[V^\delta(t+\delta, S_{t+\delta}) | S_t = S, x_t = 1\right]\\
=&\text{ drift of } V(t,S_t) - rV(t,S_t)\\
=&\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV
\end{aligned} \tag{25}$$

We conclude that the price $V(t,S)$ of American option in the Black-Scholes model must satisfy

$$(K-S)^+ - V(t,S) \le 0 \tag{26}$$

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV \le 0. \tag{27}$$

with at least one of the two holding with equality for each t, S.

Equivalently, we may write the Hamilton–Jacobi–Bellman style partial differential equation:

$$0 = \max\left((K-S)^+ - V, \frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)$$

7 Longstaff–Schwartz

7.1 Stopping times

A different way of looking at exercise policies is to define the stopping time

$$\tau_0 = \min\{t \ge 0 : a_t(S_t) = 1\}$$

Note that if our option stays out of the money, it doesn't make sense to exercise, we may have $a_t = 0$ for all t. This is just a technical issue and we can choose to set $\tau_0 = T$.

Definition. A random time $\tau : \Omega \to \mathbb{R}^+$ is a called a *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Example. If my wife poisons me, the time when she puts the poison in my dinner is **not** a stopping time. In this case the event $\{\tau \leq t\}$ means "my wife poisoned me before time t". This will not be in the σ-algebra $\mathcal{F}_t$ of events known to me by time t until I start noticing the effects. (The time when I start noticing the symptoms is a stopping time.)

The value of an American put under a specific stopping time τ_0 is

$$V_0^{\tau_0} = \mathbb{E}\left[e^{-r\tau_0}(K - S_{\tau_0})^+\right]$$

Therefore the value of the option is

$$\begin{aligned} V &= \sup_{\tau_0} V_0^{\tau_0} \\ &= \sup_{a \in \mathcal{A}[0,T]} V_0^{(a)} \end{aligned}$$

7.2 The algorithm

The algorithm presented by Longstaff and Schwartz in [LS01] is a variant of the least squares Monte Carlo method we examined at the beginning of Section 4.

Let us being by defining a sequence of stopping times τ_t

$$\tau_t = \min\{u \geq t : a_u = 1\}$$

If we haven't exercised the option before time t, τ_t is the time we exercise.

The $\{\tau_t\}_{t=0}^T$ satisfy the following recursion

$$\begin{aligned} \tau_T &= T \\ \tau_t &= t a_t + \tau_{t+1}(1 - a_t) \end{aligned}$$

The continutation value is computed by regressing the value function, but we can rewrite it in terms of the stopping time.

$$\begin{aligned} C_t(S_t) &= e^{-r\delta t}\mathbb{E}[V_{t+1}(S_{t+1})|S_t] \\ &= e^{-r\delta t}\mathbb{E}\left[\mathbb{E}\left[e^{-r(\tau_{t+1}-(t+1))}(K - S_{\tau_{t+1}})^+ \middle| S_{t+1}\right] \middle| S_t\right] \\ &= \mathbb{E}\left[e^{-r(\tau_{t+1}-t)}(K - S_{\tau_{t+1}})^+ | S_t\right] \end{aligned}$$

This final conditional expectation is the regression the Longstaff–Schwartz algorithm is based on. We illustrate this with the example seen earlier.

#	S_0	S_1	S_2	S_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

#	τ_0^*	τ_1^*	τ_2^*	τ_3^*
1	—	—	—	3
2	—	—	—	3
3	—	—	—	3
4	—	—	—	3
5	—	—	—	3
6	—	—	—	3
7	—	—	—	3
8	—	—	—	3

The regression to calculate C_2^* will be the same as before:

$$\begin{aligned} C_2^* &= \mathbb{E}[e^{-r(\tau_3-2)}(K - S_{\tau_3})^+ | S_2] \\ &= e^{-r\delta t}\mathbb{E}[(K - S_3)^+ | S_2] \end{aligned}$$

#	C_0^*	C_1^*	C_2^*	C_3^*
1	—	—	0.05	0.00
2	—	—	0.01	0.00
3	—	—	0.05	0.00
4	—	—	0.08	0.00
5	—	—	-0.01	0.00
6	—	—	0.18	0.00
7	—	—	0.14	0.00
8	—	—	0.01	0.00

Choosing the larger of $(K - S_2)^+$ and C_2^*, we can decide if it's optimal to exercise at time 2:

#	τ_0^*	τ_1^*	τ_2^*	τ_3^*
1	—	—	3	3
2	—	—	3	3
3	—	—	3	3
4	—	—	2	3
5	—	—	2	3
6	—	—	2	3
7	—	—	2	3
8	—	—	3	3

To calculate C_1^* we will have to use values from both time $t = 2$ and 3. On paths 1, 2, 3 and 8, we take $(K-S_3)^+$ and on paths 4, 5, 6, 7 we take $(K-S_2)^+$, with the appropriate discounting.

The dependent variable

$$y = \begin{pmatrix} e^{-2r}(1.1-1.34)^+ \\ e^{-2r}(1.1-1.54)^+ \\ e^{-2r}(1.1-1.03)^+ \\ e^{-1r}(1.1-0.97)^+ \\ e^{-1r}(1.1-1.56)^+ \\ e^{-1r}(1.1-0.77)^+ \\ e^{-1r}(1.1-0.84)^+ \\ e^{-2r}(1.1-1.34)^+ \end{pmatrix}$$

will then be regressed onto a basis function expansion of S_1 (i.e. the second column of the path matrix).

This procedure is repeated until the option price at time 0 is obtained. The full stopping time matrix is

#	τ_0^*	τ_1^*	τ_2^*	τ_3^*
1	3	3	3	3
2	3	3	3	3
3	3	3	3	3
4	1	1	2	3
5	2	2	2	3
6	1	1	2	3
7	1	1	2	3
8	1	1	3	3

The original paper includes an additional optimization. Note that unlike in the TVR method in Section 4, the Longstaff-Schwartz algorithm only uses the continuation value $C_t(S)$ to decide whether to exercise or not.

But when the option is out of the money, this decision is trivial: the exercise value is zero, you should never exercise. Therefore the values of $C_t(S)$ for $K \leq S$ are fitted, but never actually need to be looked at.

This suggests the following optimization: only fit $C_t(S)$ for $K > S$. So instead of the approximation

$$C_t(S) = \mathbb{E}[e^{\tau_{t+\delta t}-t}(K - S_{\tau+\delta t})^+ | S_t = S] \approx aS^2 + bS + c$$

we can use

$$\mathbb{E}[e^{\tau_{t+\delta t}-t}(K - S_{\tau+\delta t})^+ | S_t = S, S_t < K] \approx a'S^2 + b'S + c'.$$

Since this only fits $C_t(S)$ for in the money S, it may be possible to use a simpler approximation, e.g. a degree 10 polynomial instead of degree 15, without losing accuracy.

7.3 Discussion

The Longstaff-Schwartz algorithm tends to produce more accurate results than the Tsitsiklis-Van Roy algorithm, so the latter is not used in practice.

For further reading, the best resource is Chapter 8 of Glasserman [Gla13]. The paper [CLP02] is also highly recommended as more rigorous complement to the original Longstaff and Schwartz paper.

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