Finite Difference Methods with Backward Euler Schemes for American Options

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1 Introduction

The Finite Difference Method (FDM), when applied with the Backward Euler scheme, offers a robust numerical approach for pricing American options, which are known for their complexity due to the early exercise feature. This method discretizes the option pricing problem into a solvable grid of algebraic equations, addressing the challenges posed by American options that lack closed-form solutions. This introduction serves as a gateway to understanding how FDM and the Backward Euler scheme are implemented to navigate the unique landscape of American option pricing.

2 Transformation to Heat Equation

The standard Black-Scholes PDE for a European call or put option is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where V is the option price, S is the underlying asset price, t is time, σ is volatility, and r is the risk-free interest rate.

We aim to transform this into a heat equation of the form:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

To do this, we make the following substitutions:

- Change the spatial variable from S (the stock price) to x defined as $x = \ln(S/K)$.
- Change the temporal variable from t (time to maturity) to τ defined as $\tau = \frac{1}{2}\sigma^2(T-t)$.
- Transform the option price $V(S,t)$ to a new function $u(x,\tau)$ defined by:

$$u(x,\tau) = e^{\alpha x + \beta \tau} V(S,t) \tag{3}$$

- α and β are defined as

$$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2} \tag{4}$$

$$(\frac{r-q}{\sigma^2} + \frac{1}{2})^2 + \frac{2q}{\sigma^2} \tag{5}$$

3 Boundary conditions for American Options

When we solve the heat equation, we have to limit the problem in a finite domain. The boundary conditions for American options in the context of numerical methods like the Implicit Finite Difference Method need to be carefully chosen to reflect the characteristics of American options, especially the early exercise feature. Here are the appropriate boundary conditions:

- **American Call Option**
  Lower Boundary (x_{min}): Since $V(S,t) \approx 0$, we have $u(x_{min}, \tau) \approx 0$
  Upper Boundary (x_{max}): For high S, $V(S,t) = S - K$, thus $u(x_{max}, \tau) = e^{(\alpha x_{max} + \beta\tau)} K(e^{max} - 1)$
  Initial Conditions (at $\tau = 0$): $u(x, 0) = Ke^{\alpha x} \max(e^x - 1, 0)$

- **American Put Option**
  Upper Boundary (x_{min}): Since $V(S,t) \approx 0$, we have $u(x_{max}, \tau) \approx 0$
  Lower Boundary (x_{max}): For low S, the option may be exercised early, $V(S,t) = K - S$, thus $u(x_{min}, \tau) = e^{(\alpha x_{min} + \beta\tau)} K(1 - e^{min})$
  Initial Conditions (at $\tau = 0$): $u(x, 0) = Ke^{\alpha x} \max(1 - e^x, 0)$

A common approach is to set x_{min} and x_{max} based on a certain number of standard deviations away from the mean of the log price, which can be linked to the asset's volatility σ and time to maturity T. For example, x_{min} and x_{max} can be set at $\pm 3\sigma\sqrt{T}$ from the mean price $\ln(S_0/K) + (r - q - \sigma^2/2)T$ for the option of a non-dividend paying asset.

4 Domain Discretization

Divide the interval $[x_{min}, x_{max}]$ into N equally spaced points, leading to a grid with points $x_0, x_1, \cdots, x_n$.

Similarly, divide the time to maturity into M intervals with uniform spacing Δ_τ, creating points $\tau_0, \tau_1, \cdots, \tau_m$.

5 Backward Euler Scheme

To represent the discretized equations in matrix form for each time step in the finite difference method, we'll use the implicit scheme for the heat equation form of the Black-Scholes equation. Let's assume you have a grid with points $x_0, x_1, \cdots, x_n$ in the spatial domain and $\tau_0, \tau_1, \cdots, \tau_m$ in the time domain.
First derivative in time τ: Using backward differences, the first derivative at a grid point can be approximated as:

$$\left.\frac{\partial u}{\partial \tau}\right|_{i,n} \approx \frac{u_i^{n+1} - u_i^n}{\Delta\tau} \tag{6}$$

Second Derivative in Space x: Using central differences, the second derivative is approximated as:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,n} \approx \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \tag{7}$$

Then, the heat equation comes to the discretized version:

$$\frac{u_i^{n+1} - u_i^n}{\Delta\tau} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \tag{8}$$

Rearranging this equation gives:

$$-\lambda u_{i-1}^{n+1} + (1+2\lambda)u_i^{n+1} - \lambda u_{i+1}^{n+1} = u_i^n \tag{9}$$

where $\lambda = \frac{\Delta\tau}{\Delta x^2}$

When written in matrix form, the system of equations for a time step n solving for time step $n+1$ looks like this:

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \cdots & 0 \\ 0 & -\lambda & 1+2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n + \lambda u_0^{n+1} \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n + \lambda u_{N-1}^{n+1} \end{bmatrix}$$

The first and last rows of the matrix correspond to the boundary conditions.

6 Solve the Equation Using Projected SOR method

To solve American option pricing problems using the Successive Over-Relaxation (SOR) method with projection for early exercise, we need to adjust our approach to handle the iterative nature of SOR and ensure that the option value never falls below its intrinsic value (due to early exercise). The SOR method is an iterative technique used to solve a system of linear equations, and it can be particularly effective for the systems arising in finite difference methods for option pricing.

The initial guess for the SOR method at each time step can be the early expiration premium or the option values from the previous time step. This is a reasonable starting point as it accelerates convergence.

The SOR iteration is used to solve the linear system $Ax = b$ where A is the tridiagonal matrix arising from the discretization, x is the vector of unknown option values at the new time level, and b is known from the previous time level.

The SOR update for each element x_i in the vector x is given by:

$$x_i^{(\text{new})} = (1-\omega)x_i^{(\text{old})} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(\text{new})} - \sum_{j>i} a_{ij}x_j^{(\text{old})}\right) \tag{10}$$

After each SOR update, you need to enforce the early exercise feature. This means comparing the updated option value $x_i^{(new)}$ with the intrinsic value. If the intrinsic value is greater, set $x_i^{(new)}$ to the intrinsic value. This step ensures that the option value never falls below the payoff of exercising early. The transformed intrinsic value is:

$$u(x,\tau) = e^{\alpha x + \beta\tau} \max\left(Ke^x - K, 0\right) \tag{11}$$

for an American put option and

$$u(x,\tau) = e^{\alpha x + \beta\tau} \max\left(K - Ke^x, 0\right) \tag{12}$$

for an American put option.

7 Option Value and Greeks

To calculate the option value V using the transformed solution u obtained from the finite difference method, you need to reverse the transformation applied earlier. If x and τ don't exactly align with the grid points in your finite difference method, you will need to interpolate u at (x, τ). Once you have $u(x, \tau)$, reverse the transformation to find the option value $V(S_0, t)$ using

$$V\left(S_0, t\right) = u(x, \tau) \times e^{-\alpha x - \beta \tau} \tag{13}$$

Greeks measure the sensitivity of the option's price to various parameters and are typically derived from the partial derivatives of the option price with respect to these parameters. Greeks of the options using this method is given by:

$$\begin{aligned} \Delta_{fd} &= \frac{V_{i+1} - V_i}{S_{i+1} - S_i} \\ \Gamma_{fd} &= \frac{\frac{V_{i+2} - V_{i+1}}{S_{i+2} - S_{i+1}} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}}{\frac{S_{i+2} + S_{i+1}}{2} - \frac{S_i + S_{i-1}}{2}} \end{aligned} \tag{14}$$

where

$$S_{i-1} = Ke^{x_{i-1}} \qquad S_{i+2} = Ke^{x_{i+2}} \tag{15}$$

and

$$\begin{aligned} V_{i-1} &= \exp\left(-a x_{i-1} - b \tau_{\text{final}}\right) U^M(i-1) \\ V_{i+2} &= \exp\left(-a x_{i+2} - b \tau_{\text{final}}\right) U^M(i+2) \end{aligned} \tag{16}$$

Also we can calculate θ from the equation:

$$\Theta_{fd} = \frac{V_{approx}\left(S_0, \delta t\right) - V_{approx}\left(S_0, 0\right)}{\delta t} \tag{17}$$

where

$$\begin{aligned} V_{i,\delta t} &= \exp\left(-a x_i - b\left(\tau_{\text{final}} - \delta\tau\right)\right) U^{M-1}(i) \\ V_{i+1,\delta t} &= \exp\left(-a x_{i+1} - b\left(\tau_{\text{final}} - \delta\tau\right)\right) U^{M-1}(i+1) \end{aligned} \tag{18}$$

and

$$\delta t = \frac{2\left(\tau_{\text{final}} - \left(\tau_{\text{final}} - \delta\tau\right)\right)}{\sigma^2} = \frac{2\delta\tau}{\sigma^2} \tag{19}$$

8 Further Discussion

I raised a question in class before on why finite difference method can be used to price American option even if it doesn't satisfy the Black-Scholes formula. This is what ChatGPT said: The Finite Difference Method is a powerful tool for pricing American options because it is not restricted by the limitations of analytical formulas like the Black-Scholes equation. Its ability to handle the specific features of American options, particularly the early exercise feature, and its adaptability to various types of boundary conditions, make it a widely used method in option pricing.

I found it make sense since Black-Scholes PDE is actually derived from the dynamics of the market using SDE. It is for European options just because it has a closed form for European options. And finite difference method can be applied to American options since it mocks the early exercise which cannot be caught by Black Scholes formula.