(1) (a) $W = \frac{1}{\sigma}\left[\log\frac{S}{S_0} - (r - \frac{\sigma^2}{2})t\right]$

$\tau = T - t$

$V = e^{r(T-t)} C$

(b) $C = e^{-r(T-t)} V(W_t, t)$

$C_t = re^{-r(T-t)} V + e^{-r(T-t)} \cdot \left(\frac{\partial V}{\partial W} \cdot \frac{\partial W}{\partial t} + \frac{\partial V}{\partial t}\right)$

$= e^{-r(T-t)} \cdot \left[rV + V_W \cdot \left(-\frac{r - \frac{\sigma^2}{2}}{\sigma}\right) + V_t\right]$

$C_S = e^{-r(T-t)} \cdot V_W \cdot \frac{\partial W}{\partial S} = e^{-r(T-t)} \cdot V_W \cdot \frac{1}{\sigma S}$

$C_{SS} = e^{-r(T-t)} \cdot \left[V_{WW} \cdot \frac{1}{(\sigma S)^2} + V_W \cdot \frac{-1}{\sigma S^2}\right]$

Therefore, the equation can be rewritten as

$rV + V_W \cdot \frac{-r + \frac{\sigma^2}{2}}{\sigma} + V_t + \frac{1}{2} \cdot [V_{WW} + V_W \cdot (-\sigma)] + \frac{r}{\sigma} V_W - rV = 0$

$\Leftrightarrow V_t + \frac{1}{2} V_{WW} = 0$

Note that $V(W, t) = V(W, T-\tau) \Rightarrow V_\tau = \frac{1}{2} V_{WW}$

(c) $V(W, 0) = \left(K - S_0 \exp\{(r - \frac{\sigma^2}{2})T + \sigma W_T\}\right)^+$

(d) Compare to the method in the lecture, this transformation is more complicated and harder to think about the reason to do this transformation.

(2)

(a) $C(\alpha S, \alpha K) = \alpha C(S, K)$ ①

differentiate both sides of ① by α.

$\Rightarrow \quad S C_S(\alpha S, \alpha K) + K C_K(\alpha S, \alpha K) = C(S, K)$ ②

differentiate both sides of ① by S, K

$\Rightarrow \begin{cases} \alpha C_S(\alpha S, \alpha K) = \alpha C_S(S, K) \\ \alpha C_K(\alpha S, \alpha K) = \alpha C_K(S, K) \end{cases}$ ③

plug ③ into ②

$\Rightarrow \quad S C_S(S, K) + K C_K(S, K) = C(S, K)$ ④

Differentiate both sides of ④ by S, K

$\Rightarrow \begin{cases} C_S(S, K) + S C_{SS}(S, K) + K C_{KS}(S, K) = C_S(S, K) \\ S C_{SK}(S, K) + K C_{KK}(S, K) + C_K(S, K) = C_K(S, K) \end{cases}$

$\Rightarrow \begin{cases} S C_{SS} + K C_{KS} = 0 \\ S C_{SK} + K C_{KK} = 0 \end{cases} \quad \Rightarrow \Gamma = C_{SS} = \frac{K^2}{S^2} C_{KK}$

(b)(c). If $\Gamma \le 0$, which means $C_{KK} < 0$ at $K = K_0$

Then there exists $\Delta k > 0$ where

$C(S, K_0 - \Delta k) + C(S, K_0 + \Delta k) \le 2C(S, K_0)$.

Make a portfolio: long $1 \times C_{K_0 - \Delta k}$, $1 \times C_{K_0 + \Delta k}$, short $2 \times C_{K_0}$.

the gain of making this portfolio is $2C(S, K_0) - C(S, K_0 - \Delta k) - C(S, K_0 - \Delta k) \ge 0$

The P&L of the portfolio is 

so it is an arbitrage.
(since it doesn't lose money and makes money at some points)
which contradicts to the setting. So, $\Gamma > 0$. or there's an butterfly arbitrage

3. $C = S_0 \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$; $P = K \cdot e^{-rT} \cdot N(-d_2) - S_0 \cdot N(-d_1)$

$d_1 = \dfrac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

BS-PDE (non-dividend version): $\dfrac{\partial V}{\partial t} + \dfrac{\sigma^2 S^2}{2} \cdot \dfrac{\partial^2 V}{\partial S^2} + r \cdot S \cdot \dfrac{\partial V}{\partial S} = rV$, $\forall t > 0$, $\forall S > 0$

(a) $dS_t = (\mu - q) S_t dt + \sigma \cdot S_t \, dW_t$

Apply Itô's to $F_t^T = S_t \cdot e^{-q(T-t)}$

$$\frac{\partial F_t^T}{\partial t} = q \cdot S_t \cdot e^{-q(T-t)}$$

$$\frac{\partial F_t^T}{\partial S_t} = e^{-q(T-t)} , \quad \frac{\partial^2 F_t^T}{\partial S_t^2} = 0$$

$$dF_t^T = \left((\mu - q) \cdot S_t \cdot e^{-q(T-t)} + q \cdot S_t \cdot e^{-q(T-t)}\right) dt + \sigma \cdot S_t \cdot e^{-q(T-t)} dW_t$$

$dF_t^T = \mu \cdot F_t^T dt + \sigma \cdot F_t^T dW_t$, the SDE for F_t^T $\square$

(b) $C(S_t, t) = V(F_t^T, t)$, $F_t^T = e^{-q(T-t)} \cdot S_t$.

$$V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} + rF \cdot V_F - rV = 0 ,$$

$$\begin{cases} C_S = \dfrac{\partial C}{\partial S} = \dfrac{\partial C}{\partial F} \cdot \dfrac{\partial F}{\partial S} = V_F \cdot e^{-q(T-t)} \\ C_{SS} = V_{FF} \, e^{-q(T-t) \cdot 2} \end{cases}$$

$$\therefore V_t + \frac{1}{2}\sigma^2 \left(e^{-2q(T-t)} \cdot S_t\right)^2 \cdot C_{SS} \cdot e^{2q(T-t)} + r \cdot e^{-q(T-t)} S_t \cdot C_S \cdot e^{q(T-t)} - rC = 0$$

$$\therefore V_t + \frac{1}{2}\sigma^2 \cdot C_{SS} \cdot S_t^2 \cdot e^{-2q(T-t)} + r \cdot S_t \cdot C_S - rC = 0$$

$\square$