MTH 9821 Numerical Methods for Finance

Fall 2023

Homework 9

Assigned: November 6; Due: November 13

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

```
typedef Eigen::VectorXd vec;
typedef Eigen::MatrixXd mat;
typedef Eigen::PermutationMatrix<-1, -1, uint> permutation;
```

(1) Write C++ code for backward and forward substitutions, called forward_subst and backward_subst:

```
vec  forward_subst(const mat & L, const vec & b);
vec backward_subst(const mat & U, const vec & b);
```

(2) Write C++ code called lu_no_pivoting and lu_row_pivoting to compute the LU decomposition without pivoting of a matrix and the LU decomposition with row pivoting of a matrix. The row pivoted LU should satisfy $PA = LU$.

```
std::tuple<mat, mat> lu_no_pivoting (mat A);
std::tuple<permutation, mat, mat> lu_row_pivoting (mat A);
```

(3) Write C++ code for the Cholesky decomposition of an spd matrix. Return U such that $U^tU = A$.

```
mat cholesky(mat A);
```

(4) Write C++ code for Jacobi, Gauss–Siedel, and SOR iterative methods. The input should be a matrix A, a right hand side vector b, a tolerance factor, an initial guess $x^{(0)}$ and a stoping criterion. The output should be an approximate solution x to the linear system $Ax = b$ and the number of iterations performed, n.

When no initial guess $x^{(0)}$ is given, default to the zero vector.

Your function should support two stopping criteria; one based on consecutive iterations being close in norm: $||x^{(n)} - x^{(n-1)}||_2 < \text{tol}$. The other based on checking the residual being small: $||b - Ax||_2 < \text{tol}$.

```
enum StoppingCriterion {consecutive, residual};

std::tuple<vec, uint>
gauss_seidel (const mat & A, const vec & b, const vec & x_0,
              const double tolerance,
              const StoppingCriterion criterion);

std::tuple<vec, uint>
gauss_seidel (const mat & A, const vec & b,
              const double tolerance,
              const StoppingCriterion criterion);
```

Modify the Jacobi, Gauss–Siedel, and SOR codes to work for banded matrices.

(5) If $\rho(R) \geq 1$, then there exists an eigenvalue λ of R with $|\lambda| \geq 1$. Show that if $\rho(R) \geq 1$, then there exist iterations of the form

$$\text{given} \quad x_0, \quad x_{n+1} = Rx_n + c, \ \forall\ n \geq 0,$$

which do not converge.
Hint: Start with the case $c = 0$ and choose x_0 wisely.

(6) Let A be a 14×14 matrix given by

$$\begin{aligned} A(i,i) &= 2, \quad \text{for } i = 0:13 \\ A(i,i-1) &= -1, \quad \text{for } i = 1:13 \\ A(i,i+1) &= -1, \quad \text{for } i = 0:12 \end{aligned}$$

and let b be a column vector given by

$$b(i) = i^2, \quad \forall\ i = 0:13.$$

Our goal is to solve the linear system $Ax = b$ using iterative methods. For all the problems below, use tolerance $tol = 10^{-6}$ and the initial guess vector x_0, with $x_0(0) = 1, \forall\ i = 0:13$.

(i) Use the Jacobi iteration to solve $Ax = b$. Use first the residual-based stopping criterion and then the consecutive approximation stopping criterion. Report the solution and the number of iterations for each algorithm.

(ii) Use the Gauss–Siedel iteration to solve $Ax = b$. Use first the residual-based stopping criterion and then the consecutive approximation stopping criterion. Report the solution and the number of iterations for each algorithm.

(iii) Use the SOR iteration with $\omega = 1.15$ to solve $Ax = b$. Use first the residual-based stopping criterion and then the consecutive approximation stopping criterion. Report the solution and the number of iterations for each algorithm.

(iv) For only this part of the problem, use only the residual-based stopping criterion. Solve $Ax = b$ using the SOR iteration for the following values of ω:

$$\omega = 1.02 : .02 : 1.98.$$

Report the number of iterations to convergence for each value of ω. Comment on the results.