which completes the proof of the fact that $f(t) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$ is a density function.

8.7 The Box–Muller method

The Box–Muller method is an algorithm for generating samples of the standard normal variable, which are needed for Monte Carlo simulations. In the Box–Muller method, two independent standard normal samples are generated, starting with two independent samples from the uniform distribution. The method is based on the following fact:

If Z_1 and Z_2 are independent standard normal variables, then $R = Z_1^2 + Z_2^2$ is an exponential random variable with mean 2, and, given R, the point (Z_1, Z_2) is uniformly distributed on the circle of center 0 and radius $\sqrt{R}$.

Explaining the Box–Muller method and the Marsaglia polar method for an efficient implementation of the Box–Muller algorithm is beyond the scope of this book. We resume the discussion to applying the polar coordinates change of variable to explain the following part of the Box–Muller method:

Lemma 8.4. *Let Z_1 and Z_2 be independent standard normal variables, and let $R = Z_1^2 + Z_2^2$. Then R is an exponential random variable with mean 2.*

Proof. Recall that the cumulative density function of an exponential random variable X with parameter $\alpha > 0$ is

$$F(x) \;=\; \begin{cases} 1 - e^{-\alpha x}, & \text{if} \quad x \geq 0; \\ 0, & \text{if} \quad x < 0, \end{cases}$$

and its expected value is $E[X] = \frac{1}{\alpha}$.

Let $R = Z_1^2 + Z_2^2$. To prove that R is an exponential random variable with mean $E[R] = 2$, i.e., with parameter $\alpha = \frac{1}{2}$, we will show that the cumulative density of R is

$$P(R \leq x) \;=\; \begin{cases} 1 - e^{-\frac{x}{2}}, & \text{if} \quad x \geq 0; \\ 0, & \text{if} \quad x < 0. \end{cases} \tag{8.34}$$

Let $x < 0$. Since $R = Z_1^2 + Z_2^2 \geq 0$, we find that $P(R \leq x) = 0$.

Let $x \geq 0$. Let $f_1(x_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right)$ and $f_2(x_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2}\right)$ be the density functions of Z_1 and Z_2, respectively, where $\exp(t) = e^t$. Since Z_1 and Z_2 are independent, the joint density function of Z_1 and Z_2 is the product function $f_1(x_1)f_2(x_2)$; cf. Lemma 4.5. We note that $Z_1^2 + Z_2^2 \leq x$ if and only if the point (Z_1, Z_2) is in the disk $D(0, \sqrt{x})$ of center 0 and radius $\sqrt{x}$, i.e., if $(Z_1, Z_2) \in D(0, \sqrt{x})$. Then,

$$P(R \leq x) \;=\; P(Z_1^2 + Z_2^2 \;\leq\; x)$$

$$
\begin{aligned}
&= \int\int_{D(0,\sqrt{x})} f_1(x_1) f_2(x_2)\ dx_1\ dx_2 \\
&= \frac{1}{2\pi} \int\int_{D(0,\sqrt{x})} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right)\ dx_1 dx_2.
\end{aligned}
$$

The polar coordinates change of variables $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ maps any point (x_1, x_2) from the disk $D(0, \sqrt{x})$ into a point (r, θ) from the domain $[0,\ \sqrt{x}] \times [0,\ 2\pi)$. From (8.17), we find that

$$
\begin{aligned}
P(R \le x) &= \frac{1}{2\pi} \int\int_{D(0,\sqrt{x})} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right)\ dx_1 dx_2 \\
&= \frac{1}{2\pi} \int_0^{\sqrt{x}} \int_0^{2\pi} r\ \exp\left(-\frac{(r\cos\theta)^2 + (r\sin\theta)^2}{2}\right)\ d\theta dr \\
&= \frac{1}{2\pi} \int_0^{\sqrt{x}} \int_0^{2\pi} r\ \exp\left(-\frac{r^2}{2}\right)\ d\theta dr \\
&= \frac{1}{2\pi} \cdot 2\pi \int_0^{\sqrt{x}} r\ e^{-\frac{r^2}{2}}\ dr \\
&= \left(-e^{-\frac{r^2}{2}}\right) \Big|_0^{\sqrt{x}} \\
&= 1\ -\ e^{-\frac{x}{2}}.
\end{aligned}
$$

We proved that the cumulative density of the random variable R is equal to $1 - e^{-\frac{x}{2}}$, if $x \ge 0$, and to 0 otherwise. This is equivalent to showing that $R = Z_1^2 + Z_2^2$ is an exponential random variable with mean 2; cf. (8.34). □

8.8 Reducing the Black–Scholes PDE to the heat equation

Let $V(S, t)$ be the value at time t of a European call or put option with maturity T, on a lognormally distributed underlying asset with spot price S and volatility σ, paying dividends continuously at rate q. Recall from section 7.4 that $V(S, t)$ satisfies the Black–Scholes PDE

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0, \quad \forall\ S > 0, \quad \forall\ 0 < t < T. \tag{8.35}
$$

The risk–free interest rate r is assumed to be constant over the lifetime of the option. The boundary condition at maturity is $V(S, T) = \max(S - K, 0)$, for the call option and $V(S, T) = \max(K - S, 0)$, for the put option.

One way to solve the Black–Scholes PDE (8.35) is to use a change of variables to reduce it to a boundary value problem for the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{8.36}$$

since a closed form solution to this problem is known.

The change of variables is as follows:

$$V(S,t) = \exp(-ax - b\tau)u(x,\tau), \tag{8.37}$$

where

$$x = \ln\left(\frac{S}{K}\right), \tag{8.38}$$

$$\tau = \frac{(T-t)\sigma^2}{2}, \tag{8.39}$$

and the constants a and b are given by

$$a = \frac{r-q}{\sigma^2} - \frac{1}{2}; \tag{8.40}$$

$$b = \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}. \tag{8.41}$$

A brief explanation of how the change of variables (8.37) is chosen is in order.

In general, changing from the variables space (S,t) to the space (x,τ) means choosing functions $\phi(S,t)$ and $\psi(S,t)$ such that $x = \phi(S,t)$ and $\tau = \psi(S,t)$. A simpler version of such change of variables is to have x and τ depend on only one variable each, i.e., $x = \phi(S)$ and $\tau = \psi(t)$.

The Black–Scholes PDE (8.35) has nonconstant coefficients, and therefore is more challenging to solve. However, the Black–Scholes PDE is homogeneous, in the sense that all the terms with nonconstant coefficients are of the form

$$S^2\frac{\partial^2 V}{\partial S^2} \quad \text{and} \quad S\frac{\partial V}{\partial S}.$$

The classical change of variable for ODEs with homogeneous nonconstant coefficients involves the logarithmic function. In our case, we choose the change of variables (8.38), i.e.,

$$x = \ln\left(\frac{S}{K}\right).$$

From a mathematical standpoint, the change of variables (8.38) is equivalent to $x = \ln(S)$. The term $\frac{S}{K}$ is introduced for financial reasons: taking the logarithm of a dollar amount, as would be the case for $\ln(S)$, does not make sense. However, $\frac{S}{K}$ is a non–denomination quantity, i.e., a number, and its logarithm, $\ln\left(\frac{S}{K}\right)$, is well defined.

The Black–Scholes PDE (8.35) is backward in time since the boundary data is given at time T and the solution is required at time 0. We choose the change of variables for τ to obtain a forward PDE in τ, i.e., with boundary data given at $\tau = 0$ and solution required at time $\tau_{final} > 0$. The change of variables (8.39), i.e.,

$$\tau = \frac{(T-t)\sigma^2}{2},$$

accomplishes this; the term $\frac{\sigma^2}{2}$ from the coefficient of the second order partial derivative also cancels out.

We now prove that, if $V(S,t)$ satisfies the Black–Scholes PDE (8.35), then the function $u(x,\tau)$ given by (8.37) satisfies the heat equation (8.36).

We do this in two stages. First, we make the change of variables

$$V(S,t) = w(x,\tau),$$

where

$$x = \ln\left(\frac{S}{K}\right) \quad \text{and} \quad \tau = \frac{(T-t)\sigma^2}{2}.$$

It is easy to see that

$$\frac{\partial x}{\partial S} = \frac{1}{S}; \quad \frac{\partial x}{\partial t} = 0; \tag{8.42}$$

$$\frac{\partial \tau}{\partial S} = 0; \quad \frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \tag{8.43}$$

Using chain rule repeatedly, see (8.2) and (8.3), as well as (8.42) and (8.43) we find that

$$\begin{aligned}
\frac{\partial V}{\partial S} &= \frac{\partial w}{\partial S} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial S} = \frac{1}{S}\frac{\partial w}{\partial x} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial w}{\partial x}\right) \\
&= -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial w}{\partial x}\right) \\
&= -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial S} + \frac{\partial^2 w}{\partial \tau\,\partial x}\frac{\partial \tau}{\partial S}\right) \\
&= -\frac{1}{S^2}\frac{\partial w}{\partial x} + \frac{1}{S^2}\frac{\partial^2 w}{\partial x^2} \\
&= \frac{1}{S^2}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x}\right) \\
\frac{\partial V}{\partial t} &= \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial w}{\partial \tau}
\end{aligned}$$

By substituting in (8.35), we obtain the following PDE satisfied by $w(x,\tau)$:

$$-\frac{\sigma^2}{2}\frac{\partial w}{\partial \tau} + \frac{\sigma^2}{2}S^2\cdot\frac{1}{S^2}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x}\right) + (r-q)S\cdot\frac{1}{S}\frac{\partial w}{\partial x} - rw = 0. \quad (8.44)$$

After canceling out the terms involving S and dividing by $-\frac{\sigma^2}{2}$, the PDE (8.44) becomes

$$\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial x^2} + \left(1 - \frac{2(r-q)}{\sigma^2}\right)\frac{\partial w}{\partial x} + \frac{2r}{\sigma^2}w = 0. \quad (8.45)$$

The PDE (8.45) satisfied by the function $w(x,\tau)$ has constant coefficients and is forward parabolic. To eliminate the lower order terms, i.e., the terms corresponding to $\frac{\partial w}{\partial x}$ and w, let

$$w(x,\tau) = \exp(-ax - b\tau)u(x,\tau),$$

where a and b are constants to be determined later.

It is easy to see that

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \exp(-ax-b\tau)\left(-au + \frac{\partial u}{\partial x}\right);\\
\frac{\partial^2 w}{\partial x^2} &= \exp(-ax-b\tau)\left(a^2 u - 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right);\\
\frac{\partial w}{\partial \tau} &= \exp(-ax-b\tau)\left(-bu + \frac{\partial u}{\partial \tau}\right).
\end{aligned}$$

Therefore, the PDE (8.45) for $w(x,\tau)$ becomes the following PDE for $u(x,\tau)$:

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + \left(2a + 1 - \frac{2(r-q)}{\sigma^2}\right)\frac{\partial u}{\partial x}\\
&- \left(b + a^2 + a\left(1 - \frac{2(r-q)}{\sigma^2}\right) - \frac{2r}{\sigma^2}\right)u = 0.
\end{aligned}$$

We choose the constants a and b is such a way that the coefficients of $\frac{\partial u}{\partial x}$ and u are equal to 0, i.e., such that

$$\left\{\begin{aligned}
2a + 1 - \tfrac{2(r-q)}{\sigma^2} &= 0\\
b + a^2 + a\left(1 - \tfrac{2(r-q)}{\sigma^2}\right) - \tfrac{2r}{\sigma^2} &= 0
\end{aligned}\right.$$

It is easy to see that the solution of this system is given by (8.40–8.41). Thus, using the change of variables (8.37), the Black–Scholes PDE (8.35) for $V(S,t)$ becomes the heat equation for $u(x,\tau)$, i.e.,

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0.$$

While this is all we wanted to show here, we note that we can use the same change of variables to transform the boundary conditions $V(S,T)$ into boundary conditions $u(x,0)$ for the heat equation. A closed formula for solving the heat equation with boundary conditions at time $t = 0$ exists. Therefore, closed formulas for $u(x,\tau)$ can be obtained, and, from (8.37), closed formulas for $V(S,t)$ can then be inferred. If the boundary conditions for $V(S,T)$ are chosen to correspond to those for European call or put options payoffs at time T, then the Black–Scholes formulas (3.63) and (3.64), previously derived in section 4.5 by using risk–neutral pricing, are obtained.

8.9 Barrier options

An exotic option is any option that is not a plain vanilla option. Some exotic options are path–dependent: the value of the option depends not only on the value of the underlying asset at maturity, but also on the path followed by the price of the asset between the inception of the option and maturity.

It rarely happens to have closed formulas for pricing exotic options. However, European barrier options are path–dependent options for which closed form pricing formulas exist. A barrier option is different from a plain vanilla option due to the existence of a *barrier* B: the option either expires worthless (**knock–out options**), or becomes a plain vanilla option (**knock–in options**) if the price of the underlying asset hits the barrier before maturity. Depending on whether the barrier must be hit from below or from above in order to be triggered, an option is called **up** or **down**. Every option can be either a call or a put option, so there are, at the first count, at least eight different types of options:

$$\begin{array}{ccc} \textit{up} & \textit{in} & \textit{put} \\ \textit{down} & \textit{out} & \textit{call} \end{array}$$

This should be read as follows: choose one entry in each column, e.g., up, out, call. This option is an up–and–out call, which expires worthless if the price of the underlying asset hits the barrier B from below, or has the same payoff at maturity as a call option with strike K otherwise.

The position of the strike K relative to the barrier B is also important. If the spot price $S(0)$ of the underlying asset at time 0 is such that the barrier is already triggered, then the option is either worth 0, or it is equivalent to a plain vanilla option, which is priced using the Black–Scholes formula. Therefore, we are interested in the case when the barrier is not triggered already at time 0. There are sixteen different barrier options to price, corresponding to

$$\begin{array}{cccc} \textit{up} & \textit{in} & \textit{put} & B < K \\ \textit{down} & \textit{out} & \textit{call} & B \geq K \end{array}$$

Several barrier options can be priced by using simple no–arbitrage arguments, either in an absolute way, or relative to other barrier options.