Monte Carlo Methods

Random Number Generators

- start with Linear Congruential Generators for $U([0,1])$

1. Inverse Transform Method

$U([0,1])$, $F(x)$ known cumulative density of X

Compute $X_i = F^{-1}(U_i)$

2. Acceptance - Rejection Method

Goal: Generate samples with pdf $f(x)$

Assume: Know how to generate samples with pdf $g(x)$ with $f(x) \le c\,g(x), \forall x \in \mathbb{R}$

Generate sample X from g and accept it w/ probability $\frac{f(X)}{c\,g(X)}$

Step 1: generate X from g

Step 2: generate U from $U([0,1])$

Step 3: if $U \le \frac{f(X)}{c\,g(X)}$

return X $(=Y)$

else

goto Step 1

For 2: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; $g(x) = \frac{1}{2} e^{-|x|}$

3. Box - Muller Method

Z_1, Z_2 indep standard normals $\Rightarrow$ $R = Z_1^2 + Z_2^2$ exponential, w/ mean 2

Generate R first, then choose (Z_1, Z_2) uniformly from $C(0,R$

$U_1, U_2 \sim U([0,1])$

$R = -2 \ln(U_1)$

$V = 2\pi U_2$

$Z_1 = \sqrt{R} \cos V$

$Z_2 = \sqrt{R} \sin V$

Marsaglia - Bray

while $X > 1$

  generate $U_1, U_2 \sim U([0,1])$

  $U_1 = 2U_1 - 1$; $U_2 = 2U_2 - 1$

  $X = U_1^2 + U_2^2$

end

$Y = \sqrt{-2 \frac{\ln X}{X}}$

$Z_1 = U_1 Y$; $Z_2 = U_2 Y$

return Z_1, Z_2

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Variance Reduction Techniques

1. Control Variate Technique

Generate $Y_1, \dots, Y_n$ iid outputs of n replications of Y

In each replication, also generate n other outcomes of a different event, $X_1, \dots, X_n$; look for X strongly correlated with Y

$$\tilde{Y}_i = Y_i - b^*(X_i - E[X]), \quad b^* = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)} \text{ minimizes } \mathrm{var}(Y_{cv}(n))$$

$$Y_{cv}(n) = \frac{1}{n}\sum_{i=1}^{n} \tilde{Y}_i = \hat{Y}(n) - b^*(\hat{X}(n) - E[X])$$

$$\mathrm{var}(Y_{cv}(n)) = (1 - \rho_{XY}^2)\,\mathrm{var}(\hat{Y}(n))$$

Choose $b(n)$ instead of b^*,

$$b(n) = \frac{\sum_{i=1}^{n}(X_i - \hat{X}(n))(Y_i - \hat{Y}(n))}{\sum_{i=1}^{n}(X_i - \hat{X}(n))^2}$$

2. Moment Matching

Match sample moments with known moments of RVs

$$S_i(T) \longrightarrow \tilde{S}_i(T) = S_i(T) \frac{E[S(T)]}{\hat{S}(n)}$$

Use $\tilde{S}_i(T)$ for derivative pricing

3. Antithetic Variables

Reduce variance by introducing negative dependence between pairs of replications

Generate $U_1, U_2, \ldots, U_n \overset{iid}{\sim} U([0,1]) \longrightarrow Z_{1,i} \longrightarrow X_i$

Use also $1-U_1, 1-U_2, \ldots, 1-U_n \longrightarrow Z_{2,i} \longrightarrow Y_i$

$$Y_{AV}(n) = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i + Y_i}{2}$$

$$\mathrm{var}(Y_{AV}(n)) \le \mathrm{var}(\hat{X}(2n)) \Leftrightarrow \mathrm{cov}(X_i, Y_i) < 0$$