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# A Generalization for a Norm Inequality [1]

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**Abstract**: We prove a classical norm inequality by constructing a triangle and generalize it into three vectors. We also discover two norm inequalities by Lagrange multiplier method and solving systems of linear equations.

**Keywords**: System of linear equations, Norm inequality, Lagrange multiplier method

## 1. Introduction

We know that investigating norm inequality is a very active research topic. For instance, Lakey [1] presented a necessary and sufficient condition for the Fourier transform norm inequality, Babeenko and Vakarchuk [2] obtain a strengthened version of the Hrrmander inequality for a functions, Zhang in [3] introduced many basic norm inequality. In section 2 of [3], the author proved the norm inequality

$$\|x-y\| \geq \frac{1}{2}(\|x\|+\|y\|)\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|$$

for any nonzero vectors x and y in $R^2$ by the method of replacement. Motivated by the work mentioned above, we in this paper aim to consider the norm inequality with three vectors. We prove the following two types of norm inequality

$$\|x-y\|^2+\|y-z\|^2+\|z-x\|^2 \geq f, \|x-y\|+\|y-z\|+\|z-x\| \geq g,$$

where $f, g$ are two functions.

## 2. Main results

**Theorem 1**. If any nonzero vectors x and y in $R^2$, then we have

$$\|x-y\| \geq \frac{1}{2}(\|x\|+\|y\|)\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|.$$

**Proof**: Construct a triangle $\Delta ABC$ (see Fig 1). In the triangle $\Delta ABC$, we suppose

$$AB=x, AC=y.$$

Extend line $AB$ to point $D$ and line $AC$ to point $E$ such that

$$\|BD\|=\|y\|, \|CE\|=\|x\|.$$

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# Solvability conditions and general solution for mixed Sylvester equations*

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ABSTRACT

In this paper, we give some necessary and sufficient solvability conditions for the mixed Sylvester matrix equations, and parameterize general solution when it is solvable. Moreover, we investigate the maximal and minimal ranks of the general solution, and maximal and minimal ranks and inertias of Hermitian part of solution, respectively.



## 1. Introduction

The Sylvester matrix equation has found huge applications in $H_2$-optimal control, $H_\infty$-optimal control (Saberi, Stoorvogel, & Sannuti, 2003), robust control (Cavinlii & Bhattacharyya, 1983; Varga, 2000), singular system control (Shahzad, Jones, Kerrigan, & Constantinides, 2011), control theory (Castelan & Gomes da Silva, 2005; Ding & Chen, 2006; Duan & Zhou, 2006; Tsui, 1987; Wimmer, 1994), and neural network (Zhang, Jiang, & Wang, 2002).

Here we consider the mixed Sylvester matrix equations

$$A_1X - YB_1 = C_1, \qquad A_2Z - YB_2 = C_2. \tag{1}$$

Recently, Lee and Vu (2012) proved that the mixed Sylvester matrix Eq. (1) is consistent if and only if there exist invertible matrices $R_1$, $R_2$ and $S$ such that

$$\begin{aligned} \begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} R_1 &= S \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \\ \begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} R_2 &= S \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}. \end{aligned} \tag{2}$$

Liu (2006) derived a solvability condition of (1). However, the expression of the general solution to (1) was not available then. On the other hand, it is difficult to determine whether the solvability condition (2) holds. Naturally, it is interesting to consider new solvability conditions and an expression of the general solution to the mixed Sylvester matrix equations (1).

Our first aim in this paper is to give new necessary and sufficient conditions for the existence of a solution to (1). Then, we present an expression of the general solution to (1) when it is solvable. Finally, we consider some rank and inertia characterizations of the general solution of (1).

The remainder of this paper is organized as follows. In Section 2, we review some known results and lemmas. In Section 3, we consider the solvability conditions and the expression of the general solution to the mixed Sylvester matrix equations (1). In Section 4, we derive the maximal and minimal ranks of the general solution to the mixed Sylvester matrix equations (1). In Section 5, we give the maximal and minimal ranks and inertias of the Hermitian part of the solution of (1).

Throughout this paper, we denote the complex number field by $\mathbb{C}$. The notations $\mathbb{C}^{m\times n}$ and $\mathbb{C}_h^{m\times m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The identity matrix with an appropriate size is denoted by $I$. For a complex matrix $A$, the symbols $A^*$ and $r(A)$ stand for the conjugate transpose and the rank of $A$, respectively. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution $X$ to the following four matrix equations

$$AXA = A, \qquad XAX = X,$$
$$(AX)^* = AX, \qquad (XA)^* = XA.$$

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